Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **34**, 1–6; http://www.math.u-szeged.hu/ejqtde/

A general Lipschitz uniqueness criterion for scalar ordinary differential equations

Josef Diblík✉[1], **Christine Nowak**[2] and **Stefan Siegmund**[3]

[1]Brno University of Technology, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, 602 00 Brno, Czech Republic, and Brno University of Technology, Department of Mathematics, Faculty of Electrical Engineering and communication, 616 00 Brno, Czech Republic
[2]Institute for Mathematics, University of Klagenfurt, 9020 Klagenfurt, Austria
[3]Institute for Analysis & Center for Dynamics, Institute for Mathematics, Technische Universität Dresden, 01062 Dresden, Germany

Received 18 February 2014, appeared 29 July 2014
Communicated by Stevo Stević

Abstract. The classical Lipschitz-type criteria guarantee unique solvability of the scalar initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, by putting restrictions on $|f(t,x) - f(t,y)|$ in dependence of $|x - y|$. Geometrically it means that the field differences are estimated in the direction of the x-axis. In 1989, Stettner and the second author could establish a generalized Lipschitz condition in both arguments by showing that the field differences can be measured in a suitably chosen direction $v = (d_t, d_x)$, provided that it does not coincide with the directional vector $(1, f(t_0, x_0))$.
Considering the vector v depending on t, a new general uniqueness result is derived and a short proof based on the implicit function theorem is developed. The advantage of the new criterion is shown by an example. A comparison with known results is given as well.

Keywords: fundamental theory of ordinary differential equations, initial value problems, uniqueness, Lipschitz type conditions.

2010 Mathematics Subject Classification: 34A12.

1 Introduction

We consider the scalar initial value problem

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0, \tag{1.1}$$

and assume throughout the paper that $f\colon D \to \mathbb{R}$ is a continuous function on an open neighborhood D of the point $(t_0, x_0) \in \mathbb{R}^2$. Problem (1.1) is called *locally uniquely solvable* if there exists an open interval I containing t_0 such that (1.1) has exactly one solution on I.

✉Corresponding author. Email: diblik.j@fce.vutbr.cz, diblik@feec.vutbr.cz

The uniqueness problem of (1.1) attracts permanent attention because it is not really solved up to now as simple examples show. The classical Lipschitz condition and its generalizations [1], including the results by Nagumo, Osgood, Perron and Kamke, consider $|f(t,x) - f(t,y)|$ in dependence of $|x - y|$ and thus measure the field differences in the direction of the x-axis. In 1989, Stettner and Nowak [9] could establish a generalized Lipschitz condition in both arguments. The field differences can be measured in a suitably chosen direction $v = (d_t, d_x)$, provided that it does not coincide with the directional vector $(1, f(t_0, x_0))$. The particular case with the t-axis as direction, thus requiring a Lipschitz condition with respect to the first argument of f, if $f(t_0, x_0) \neq 0$, was independently published first by Mortici [6] and then by Cid and López Pouso [2, 4]. Stettner and Nowak's paper is written in German, and therefore it is maybe non-accessible by not German-speaking colleagues as it is also remarked by Cid and López Pouso [3]. Hoag [5] extends the approach of a Lipschitz condition in the first argument including cases when $f(t_0, x_0) = 0$.

In Section 2, considering the vector v depending on t, a new general uniqueness result is derived. We give a rather short proof based on the implicit function theorem. In Section 3 we compare our criterion with known results and show the advantage by an example.

2 A general Lipschitz uniqueness criterion

Theorem 2.1. *Let $v(t) = (\varphi(t), \psi(t))$ be a continuously differentiable vector on an open neighborhood of t_0 with real entries φ and ψ such that*

(i) $\psi(t_0) \neq f(t_0, x_0)\varphi(t_0)$,

(ii) for a constant $L \geq 0$ and every $k \in \mathbb{R}$

$$|f(t,x) - f(t + k\varphi(t), x + k\psi(t))| \leq L|k| \tag{2.1}$$

whenever the arguments of f are well-defined and belong to D.

Then (1.1) *is locally uniquely solvable.*

Proof. Peano's theorem guarantees that (1.1) has at least one solution $x\colon [t_0 - \alpha_0, t_0 + \alpha_0] \to \mathbb{R}$ for some $\alpha_0 > 0$. By assumption (i) there exists $\alpha \in (0, \alpha_0]$ with $\psi(t) \neq f(t, x(t))\varphi(t)$ for all $t \in (t_0 - \alpha, t_0 + \alpha)$. To prove that (1.1) is locally uniquely solvable with solution x on $I := (t_0 - \alpha, t_0 + \alpha)$ assume to the contrary that there exists a solution $y\colon I \to \mathbb{R}$ of (1.1) and $x \not\equiv y$ on $[t_0, t_0 + \alpha)$ (the case $x \not\equiv y$ on $(t_0 - \alpha, t_0]$ is treated similarly). For $t_1 := \sup\{t \in [t_0, t_0 + \alpha) : x(s) = y(s) \text{ for } s \in [t_0, t]\}$ we have $t_1 \in [t_0, t_0 + \alpha)$, $x(t_1) = y(t_1) =: x_1$ by continuity and also

$$\psi(t_1) \neq f(t_1, x_1)\varphi(t_1). \tag{2.2}$$

We show that the equation

$$y(t + k(t)\varphi(t)) = x(t) + k(t)\psi(t) \tag{2.3}$$

is uniquely solvable with respect to $k = k(t)$ on a subinterval of I. The problem suggests to apply the implicit function theorem. Let

$$F(t,k) := y(t + k\varphi(t)) - x(t) - k\psi(t).$$

This function is defined in an open set containing $(t_1, 0)$ with the property

$$F(t_1, 0) = y(t_1) - x(t_1) = 0.$$

As

$$\frac{\partial F}{\partial k}(t,k) = f(t + k\varphi(t), y(t + k\varphi(t)))\varphi(t) - \psi(t),$$

we get with assumption (2.2)

$$\frac{\partial F}{\partial k}(t_1, 0) = f(t_1, x_1)\varphi(t_1) - \psi(t_1) \neq 0\,.$$

The implicit function theorem (cf., e.g., [8, Theorem 9.28]) now yields that there exists a unique continuously differentiable function $k = k(t)$ on an open interval $I_1 \subset I$ containing t_1 such that $k(t_1) = 0$ and $F(t, k(t)) = 0$ for all $t \in I_1$.

We show that $k(t) \equiv 0$ on a subinterval of I_1 with $t_1 \in I_1$. Due to (2.2), there exist a constant $\eta > 0$ and an open interval $I_2 \subset I_1$ containing t_1 such that

$$|f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\varphi(t) - \psi(t)| \geq \eta \quad \text{for} \quad t \in I_2.$$

Moreover, there exists a constant M such that

$$|f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))| \leq M, \quad |\varphi'(t)| \leq M, \quad |\psi'(t)| \leq M, \quad t \in I_2.$$

Now we consider $u(t) := k^2(t)$ on I_2. Using the derivative of the function $k(t)$, relation (2.3) and inequality (2.1) we get for $t \in I_2$

$$\begin{aligned}
\dot{u}(t) = 2k(t)\dot{k}(t) &= 2k(t)\frac{\dot{x}(t) - \dot{y}(t + k(t)\varphi(t))(1 + k(t)\varphi'(t)) + k(t)\psi'(t)}{\dot{y}(t + k(t)\varphi(t))\varphi(t) - \psi(t)} \\
&= 2k(t)\frac{f(t, x(t)) - f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))(1 + k(t)\varphi'(t)) + k(t)\psi'(t)}{f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\varphi(t) - \psi(t)} \\
&= 2k(t)\frac{f(t, x(t)) - f(t + k(t)\varphi(t), x(t) + k(t)\psi(t))(1 + k(t)\varphi'(t)) + k(t)\psi'(t)}{f(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\varphi(t) - \psi(t)} \\
&\leq \frac{2(L + M^2 + M)}{\eta}k^2(t) \;=\; \frac{2(L + M^2 + M)}{\eta}u(t)
\end{aligned}$$

which is equivalent to

$$\frac{d}{dt}\left[u(t)\exp\left(-\frac{2(L + M^2 + M)}{\eta}(t - t_1)\right)\right] \leq 0\,.$$

Since $u(t_1) = k^2(t_1) = 0$, we get $u(t) = k^2(t) \equiv 0$ and hence from (2.3), $x(t) \equiv y(t)$ on I_2, which contradicts the definition of t_1. □

3 Concluding remarks and comparison with known results

The function $k(t)$ in the proof of Theorem 2.1 measures in the case when $v(t)$ is a unit vector the distance between the points $(t, x(t))$ and $(t + k(t)\varphi(t), y(t + k(t)\varphi(t)))$ on the graphs of the solutions x and y because

$$\operatorname{dist}\big((t, x(t)), (t + k(t)\varphi(t), y(t + k(t)\varphi(t)))\big) = |k(t)|\sqrt{\varphi^2(t) + \psi^2(t)} = |k(t)|.$$

By the specification $v(t) = (\varphi(t), \psi(t)) = (0, 1)$ we get the well-known Lipschitz condition. The specification $v(t) = (\varphi(t), \psi(t)) = (1, 0)$ yields the result by Mortici cited above. The latter case contains the following special uniqueness criterion which is given in [7]. It was already known by Peano.

Corollary 3.1. *If $f: \mathbb{R} \to \mathbb{R}^+$ is continuous and positive then the equation $\dot{x} = f(x)$ has uniqueness, i.e. exactly one solution passes through every point of $\mathbb{R}^2$.*

Finally, the choice $v(t) = (\varphi(t), \psi(t)) = (d_t, d_x)$ turns our result into the following criterion published in German by Stettner and Nowak [9].

Theorem 3.2. *Let D be an open neighborhood of the point (t_0, x_0) and $f: D \to \mathbb{R}$ be continuous on D. Let d_t, d_x be real numbers such that*

i) $d_t^2 + d_x^2 > 0$,

ii) $d_x \neq f(t, x)d_t$ on D,

iii) for a constant $L \geq 0$ and every $k \in \mathbb{R}$ the inequality

$$|f(t, x) - f(t + kd_t, x + kd_x)| \leq L|k|$$

is satisfied whenever the arguments of f are in D.

Then (1.1) has at most one solution.

Now we illustrate the advantage of Theorem 2.1.

Example 1. Consider the initial value problem

$$\frac{dx}{dt} = f(t, x), \quad x(0) = 0, \tag{3.1}$$

where

$$f(t, x) := \begin{cases} 1 + x, & x < t^2, \\ 1 + x + \sqrt{x - t^2}, & x \geq t^2. \end{cases}$$

It is easy to check that f is not Lipschitz continuous with respect to x in any neighborhood of $(0, 0)$, and the problem cannot be treated by Theorem 3.2 using a constant vector $v = (d_t, d_x)$. Nevertheless, problem (3.1) is locally unique which can be shown by Theorem 2.1 using the vector $v(t) = (\varphi(t), \psi(t)) = (1, 2t)$. As $0 = \psi(0) \neq f(0, 0)\varphi(0) = 1$, assumption (i) is fulfilled. We briefly explain that assumption (ii) also holds on an arbitrary open and bounded neighbourhood $D \subset \mathbb{R} \times \mathbb{R}$ of $(0, 0)$. Let $M_1 := \sup\{|t| : (t, x) \in D\} < \infty$ and $L := 2M_1 + 1$. Consider the theoretically possible cases

$$\begin{aligned} &\alpha) \quad x < t^2 \wedge x + 2tk < (t + k)^2, \\ &\beta) \quad x < t^2 \wedge x + 2tk \geq (t + k)^2, \\ &\gamma) \quad x \geq t^2 \wedge x + 2tk < (t + k)^2, \\ &\delta) \quad x \geq t^2 \wedge x + 2tk \geq (t + k)^2, \end{aligned}$$

and note that $\beta)$ is impossible. Then condition (2.1) of the form

$$|f(t, x) - f(t + k, x + 2tk)| \leq L|k|$$

is also fulfilled, since in the case α)

$$|f(t,x) - f(t+k, x+2tk)| = |1+x-(1+x+2tk)| = 2|t||k| \le 2M_1|k| \le L|k|,$$

in the case γ), regarding that $\sqrt{x-t^2} < |k|$,

$$\begin{aligned}|f(t,x) - f(t+k, x+2tk)| &= |1+x+\sqrt{x-t^2} - (1+x+2tk)| \\ &\le |k| + 2|t||k| \le |k| + 2M_1|k| = L|k|\end{aligned}$$

and in the case δ), regarding that $\sqrt{x-t^2} \ge |k|$,

$$\begin{aligned}&|f(t,x) - f(t+k, x+2tk)| \\ &\quad = \left|1+x+\sqrt{x-t^2} - \left(1+x+2tk+\sqrt{x+2tk-(t+k)^2}\right)\right| \\ &\quad \le 2|t||k| + \left|\sqrt{x-t^2} - \sqrt{x-t^2-k^2}\right| \le 2M_1|k| + \left|\frac{k^2}{\sqrt{x-t^2}+\sqrt{x-t^2-k^2}}\right| \\ &\quad \le 2M_1|k| + \left|\frac{k^2}{\sqrt{x-t^2}}\right| \le 2M_1|k| + \left|\frac{k^2}{k}\right| = 2M_1|k| + |k| = L|k|,\end{aligned}$$

where without loss of generality we can assume $k \neq 0$.

Acknowledgements

The first author is supported by the Operational Programme Research and Development for Innovations, No. CZ.1.05/2.1.00/03.0097 (AdMaS). The authors would like to thank the editor and referees for the careful reading of the manuscript and valuable suggestions.

References

[1] R. P. Agarwal, V. Lakshmikantham, *Uniqueness and nonuniqueness criteria for ordinary differential equations*, World Scientific Publishing, 1993. MR1336820; url

[2] J. Á. Cid, R. López Pouso, Does Lipschitz with respect to x imply uniqueness for the differential equation $y' = f(x,y)$?, *Amer. Math. Monthly* **116**(2009), No. 1, 61–66. MR2478753; url

[3] J. Á. Cid, R. López Pouso, Addendum to [2], *Amer. Math. Monthly* **117**(2010), Editor's Endnotes, p. 754.

[4] J. Á. Cid, R. López Pouso, On first-order ordinary differential equations with nonnegative right-hand sides, *Nonlinear Anal.* **52**(2003), 1961–1977. MR1954592; url

[5] J. T. Hoag, Existence and uniqueness of a local solution for $x' = f(t,x)$ using inverse functions, *Electron. J. Differential Equations* **2013**, No. 124, 1–3. MR3065077

[6] C. Mortici, On the solvability of the Cauchy problem, *Nieuw Arch. Wiskd. IV. Ser.* **17**(1999), 21–23. MR1702578

[7] W. Rudin, Nonuniqueness and growth in first-order differential equations, *Amer. Math. Monthly* **89**(1982), 241–244. MR0650671; url

[8] W. Rudin, *Principles of mathematical analysis*, third edition, McGraw-Hill Book Co., 1976. MR0385023

[9] H. Stettner, Chr. Nowak, Eine verallgemeinerte Lipschitzbedingung als Eindeutigkeitskriterium bei gewöhnlichen Differentialgleichungen (in German) [A generalized Lipschitz condition as criterion of uniqueness in ordinary differential equations], *Math. Nachr.* **141**(1989), 33–35. MR1014412; url