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Besicovitch almost periodic solutions for a class of second order differential equations involving reflection of the argument

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Abstract. In this paper, using the Fourier series expansion and fixed point methods, we investigate the existence and uniqueness of Besicovitch almost periodic solutions for a class of second order differential equations involving reflection of the argument. Lipschitz nonlinear case is considered.

Keywords: Besicovitch almost periodic solutions, trigonometric polynomials, differential equations, reflection of the argument, fixed point methods.

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1 Introduction

The differential equations involving reflection of the argument have applications in the study of stability of differential-difference equations, see Sharkovskii [14], and such equations have very interesting properties, so many authors worked on them. First-order equations with constant coefficients and reflection have been studied in detail in [1, 11, 13, 15]. There is also an indication ([13, p.169] and [15, p.241]) that "The problem is much more difficult in the case of differential equations with reflection of order greater than one". Wiener and Aftabizadeh [16] initiated the study of boundary value problems for the second order differential equations involving reflection of the argument. Gupta [6, 7] investigated two point boundary value problems for this kind of equations under the Carathéodory conditions. In [11, 12], one of the present authors investigated existence and uniqueness of periodic, almost periodic and pseudo almost periodic solutions of the equations

$$\dot{x}(t) + ax(t) + bx(-t) = g(t), \quad b \neq 0,\ t \in \mathbb{R}$$

and

$$\dot{x}(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), \quad b \neq 0,\ t \in \mathbb{R}.$$

Recently Cabada et al. [2] studied the first order operator $x'(t) + mx(-t)$ coupled with periodic boundary value conditions, and described the eigenvalues of the operator and obtained

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the expression of its related Green's function in the non resonant case. Also Cabada et al. [3], using the theory of fixed point index, established new results for the existence of nonzero solutions of Hammerstein integral equations with reflections. They applied their results to a first-order periodic boundary value problem with reflections. On the other hand, Layton [8] studied the existence and uniqueness of Besicovitch almost periodic solutions for the delay equation

$$\dot{x}(t) + g(x(t), x(t-\tau)) = e(t)$$

under any Besicovitch almost periodic forcing term $e(t)$. But as far as we know, there are no works on the almost periodic solutions for such second-order equations. Motivated by the above references, our present paper is devoted to investigate the existence of a unique Besicovitch almost periodic solution of the second order nonlinear differential equation with reflection of the argument

$$a_0\ddot{x}(t) + b_0\ddot{x}(-t) + a_1\dot{x}(t) + b_1\dot{x}(-t) + a_2x(t) + b_2x(-t) = f(t, x(t), x(-t)), \quad t \in \mathbb{R}. \tag{1.1}$$

Remark 1.1. Mażbic-Kulma [10] investigated firstly the equation

$$\sum_{k=0}^{n} [a_k x^{(k)}(t) + b_k x^{(k)}(-t)] = y(t).$$

The left hand side of equation (1.1) is a special case of this.

In order to develop our results, we review some facts about Bohr almost periodic and Besicovitch almost periodic functions. For further knowledge on almost periodic functions we refer the readers to the books [5, 4, 9].

We denote by $\mathcal{AP}(\mathbb{R})$ the set of all almost periodic functions in the sense of Bohr on $\mathbb{R}$. The Besicovitch space of almost periodic functions, $\mathcal{B}^2(\mathbb{R})$ is the closure of trigonometric polynomials of the form

$$\sum_{s=-n}^{n} a_s e^{i\lambda_s t}, \quad a_s \in \mathbb{C}, \quad a_s = \overline{a_{-s}}, \quad \lambda_{-s} = -\lambda_s \tag{1.2}$$

under the norm

$$\|f\|^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2\, dt.$$

Here $\|\cdot\|$ on $\mathcal{B}^2(\mathbb{R})$ is induced by the inner product

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\overline{g(t)}\, dt.$$

Alternatively, $\mathcal{B}^2(\mathbb{R})$ could be defined as the set of all $f(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\lambda_j t}$ with $\lambda_{-j} = -\lambda_j, a_{-j} = \overline{a_j}$, and $||f||^2 = \sum_{-\infty}^{\infty} |a_j|^2 < \infty$. For $\Lambda \subset \mathbb{R}$ the closed subspace $\mathcal{B}^2_\Lambda$ of $\mathcal{B}^2$ is defined as

$$\mathcal{B}^2_\Lambda = \left\{ f(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\lambda_j t} \;\middle|\; \lambda_j \in \Lambda,\ \lambda_{-j} = -\lambda_j,\ a_{-j} = \overline{a_j},\ \sum_{-\infty}^{\infty} |a_j|^2 < \infty \right\}.$$

The space $\mathcal{B}^{2,1}(\mathbb{R})$ is defined to be the closure of the trigonometric polynomials (1.2) in the norm

$$f(t) = \sum_{j=-\infty}^{\infty} e_n e^{i\lambda_j t}, \quad \|f\|_1^2 = \sum_{j=-\infty}^{\infty} (1 + |\lambda_j|^2)|a_j|^2 < \infty. \tag{1.3}$$

For $\Lambda \subset \mathbb{R}$, $\mathcal{B}^{2,1}_\Lambda$ is defined as $\mathcal{B}^2_\Lambda \cap \mathcal{B}^{2,1}$. For details on some notations see Layton [8].

2 The linear problem

For $e(t) \in \mathcal{B}^2$,

$$e(t) = \sum_{n=-\infty}^{\infty} e_n e^{i\lambda_n t},$$

consider the problem of finding a solution $x(t) \in \mathcal{B}^2$ to the linear equation

$$L_{a,b}x \equiv a_0\ddot{x}(t) + b_0\ddot{x}(-t) + a_1\dot{x}(t) + b_1\dot{x}(-t) + a_2x(t) + b_2x(-t) = e(t), \quad t \in \mathbb{R}. \tag{2.1}$$

Let Λ be the Fourier exponents of $e(t)$, then formally, the solution $x(t) \in \mathcal{B}^2_\Lambda$ to (2.1) is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{i\lambda_n t}. \tag{2.2}$$

Putting it into equation (2.1), we obtain

$$-a_0\lambda_n^2 x_n - b_0\lambda_n^2\overline{x}_n + ia_1\lambda_n x_n - ib_1\lambda_n\overline{x}_n + a_2x_n + b_2\overline{x}_n = e_n. \tag{2.3}$$

Let $x_n = \alpha_n + i\beta_n$ and $e_n = \xi_n + i\eta_n$, comparing the coefficients of $e^{i\lambda_n t}$, we have

$$\begin{aligned} (-a_0\lambda_n^2 - b_0\lambda_n^2 + a_2 + b_2)\alpha_n + (-a_1\lambda_n + b_1\lambda_n)\beta_n &= \xi_n, \\ (a_1\lambda_n - b_1\lambda_n)\alpha_n + (-a_0\lambda_n^2 + b_0\lambda_n^2 + a_2 - b_2)\beta_n &= \eta_n. \end{aligned} \tag{2.4}$$

We denote the coefficient determinant of the system (2.4) by $d(\lambda_n)$, then

$$d(\lambda_n) = (a_0^2 - b_0^2)\lambda_n^4 - [(a_1 - b_1)^2 + 2a_0a_2 - 2b_0b_2]\lambda_n^2 + (a_2^2 - b_2^2) \tag{2.5}$$

Lemma 2.1. *If $(a_1 - b_1)^4 + 4[(a_0b_2 - a_2b_0)^2 + (a_0a_2 - b_0b_2)(a_1 - b_1)^2] < 0$, then $d(\lambda_n) \neq 0$ and is bounded away from zero.*

Proof. If $(a_1 - b_1)^4 + 4[(a_0b_2 - a_2b_0)^2 + (a_0a_2 - b_0b_2)(a_1 - b_1)^2] < 0$, then

$$\begin{aligned} \Delta &\equiv [(a_1 - b_1)^2 + 2a_0a_2 - 2b_0b_2]^2 - 4(a_0^2 - b_0^2)(a_2^2 - b_2^2) \\ &= (a_1 - b_1)^4 + 4[(a_0b_2 - a_2b_0)^2 + (a_0a_2 - b_0b_2)(a_1 - b_1)^2] \\ &< 0. \end{aligned}$$

And this implies $a_0^2 - b_0^2 \neq 0$.

$$\begin{aligned} d(\lambda_n) &= (a_0^2 - b_0^2)\left[\lambda_n^2 + \frac{(a_1 - b_1)^2 + 2a_0a_2 - 2b_0b_2}{2(a_0^2 - b_0^2)}\right]^2 \\ &\quad + \frac{4(a_0^2 - b_0^2)(a_2^2 - b_2^2) - [(a_1 - b_1)^2 + 2a_0a_2 - 2b_0b_2]^2}{4(a_0^2 - b_0^2)} \\ &= (a_0^2 - b_0^2)\left[\lambda_n^2 + \frac{(a_1 - b_1)^2 + 2a_0a_2 - 2b_0b_2}{2(a_0^2 - b_0^2)}\right]^2 - \frac{\Delta}{4(a_0^2 - b_0^2)}. \end{aligned}$$

It is easy to see that

$$d(\lambda_n) \geq -\frac{\Delta}{4(a_0^2 - b_0^2)} > 0,$$

provided $a_0^2 - b_0^2 > 0$. While

$$d(\lambda_n) \leq -\frac{\Delta}{4(a_0^2 - b_0^2)} < 0,$$

provided $a_0^2 - b_0^2 < 0$. So $|d(\lambda_n)| \geq |\Delta/(4(a_0^2 - b_0^2))| > 0$. That is $d(\lambda_n)$ is bounded away from zero. $\square$

Remark 2.2.

(i) The condition of Lemma 2.1 is possible, for example $a_0 = 1$, $b_0 = 2$, $a_1 = 2$, $b_1 = 1$, $a_2 = 0$, $b_2 = 1$, and $\Delta = -3 < 0$.

(ii) From the proof of Lemma 2.1, we see that $(a_1 - b_1)^4 + 4[(a_0 b_2 - a_2 b_0)^2 + (a_0 a_2 - b_0 b_2) \times (a_1 - b_1)^2] < 0$ implies $|a_0| \neq |b_0|$ and $|a_2| \neq |b_2|$.

If $(a_1 - b_1)^4 + 4[(a_0 b_2 - a_2 b_0)^2 + (a_0 a_2 - b_0 b_2)(a_1 - b_1)]^2 < 0$, then

$$\begin{aligned}\alpha_n &= \frac{1}{d(\lambda_n)} \begin{vmatrix} \xi_n & (b_1 - a_1)\lambda_n \\ \eta_n & (b_0 - a_0)\lambda_n^2 + a_2 - b_2 \end{vmatrix} \\ &= \frac{1}{d(\lambda_n)} \{[(b_0 - a_0)\lambda_n^2 + a_2 - b_2]\xi_n + (a_1 - b_1)\lambda_n \eta_n\},\end{aligned}$$

$$\begin{aligned}\beta_n &= \frac{1}{d(\lambda_n)} \begin{vmatrix} -(a_0 + b_0)\lambda_n^2 + a_2 + b_2 & \xi_n \\ (a_1 - b_1)\lambda_n & \eta_n \end{vmatrix} \\ &= \frac{1}{d(\lambda_n)} \{[-(a_0 + b_0)\lambda_n^2 + a_2 + b_2]\eta_n + (b_1 - a_1)\lambda_n \xi_n\}\end{aligned}$$

and

$$\begin{aligned}d^2(\lambda_n)(\alpha_n^2 + \beta_n^2) = {} & [(b_0 - a_0)\lambda_n^2 + a_2 - b_2]^2 \xi_n^2 + (a_1 - b_1)^2 \lambda_n^2 \eta_n^2 \\ & + [-(a_0 + b_0)\lambda_n^2 + a_2 + b_2]^2 \eta_n^2 + (a_1 - b_1)^2 \lambda_n^2 \xi_n^2 \\ & + 2[(b_0 - a_0)\lambda_n^2 + a_2 - b_2](a_1 - b_1)\lambda_n \xi_n \eta_n \\ & + 2[-(a_0 + b_0)\lambda_n^2 + a_2 + b_2](b_1 - a_1)\lambda_n \eta_n \xi_n.\end{aligned}$$

Simple fact of $\xi_n \eta_n \leq (\xi_n^2 + \eta_n^2)/2$ implies

$$\begin{aligned}d^2(\lambda_n)(\alpha_n^2 + \beta_n^2) \leq {} & [(b_0 - a_0)\lambda_n^2 + a_2 - b_2]^2 \xi_n^2 + (a_1 - b_1)^2 \lambda_n^2 \eta_n^2 \\ & + [-(a_0 + b_0)\lambda_n^2 + a_2 + b_2]^2 \eta_n^2 + (a_1 - b_1)^2 \lambda_n^2 \xi_n^2 \\ & + [(b_0 - a_0)\lambda_n^2 + a_2 - b_2](a_1 - b_1)\lambda_n (\xi_n^2 + \eta_n^2) \\ & + [-(a_0 + b_0)\lambda_n^2 + a_2 + b_2](b_1 - a_1)\lambda_n (\xi_n^2 + \eta_n^2) \\ = {} & P_1(\lambda_n)\xi_n^2 + P_2(\lambda_n)\eta_n^2,\end{aligned}$$

where $P_1(\lambda)$ and $P_2(\lambda)$ are polynomials of λ with degree 4. Since $d^2(\lambda)$ is a polynomial of λ with degree 8, $\lim_{\lambda \to \infty} P_k(\lambda)/d^2(\lambda) = 0$, $k = 1, 2$. On the other hand, by the proof of Lemma 2.1, $|d^2(\lambda)| \geq |\Delta/(4(a_0^2 - b_0^2))|^2 > 0$. So there exists a constant $M > 0$, such that

$$\max\{|P_1(\lambda)|, |P_2(\lambda)|\} \leq M^2$$

and

$$\alpha_n^2 + \beta_n^2 \leq M^2(\xi_n^2 + \eta_n^2),$$

so

$$|\alpha_n + i\beta_n| \leq M|e_n|.$$

Hence, the infinite series $\sum_{n=-\infty}^{\infty} (\alpha_n + i\beta_n) e^{i\lambda_n t}$ is absolutely convergent, so we have next theorem.

Theorem 2.3. *If $(a_1-b_1)^4+4[(a_0b_2-a_2b_0)^2+(a_0a_2-b_0b_2)(a_1-b_1)]^2<0$, then $L_{a,b}^{-1}$ exists on $\mathcal{B}^2$. The solution $x(t)$ to (2.1) exists, is an element of $\mathcal{B}^2_\Lambda$, is unique (up to a function with $\mathcal{B}^2$-norm zero) and is given by (2.2). Furthermore, there exists a constant $M>0$, such that*

$$\|L_{a,b}^{-1}e\|\le M\|e\|, \tag{2.6}$$

and so $\|L_{a,b}^{-1}\|\le M$.

Remark 2.4. The assumption $(a_1-b_1)^4+4[(a_0b_2-a_2b_0)^2+(a_0a_2-b_0b_2)(a_1-b_1)]^2<0$ is a sufficient condition for the existence of a unique $\mathcal{B}^2$-solution of the equation (2.1), but not a necessary one for just the existence of $\mathcal{B}^2$-solutions. As an example, let a_1 be any real number, and set $\lambda_0=1/(a_1^2+2)^{1/2}$. Then the function $x(t)$ defined by $x(t)=(1+a_1\lambda_0 i)e^{i\lambda_0 t}$ $+(1-a_1\lambda_0 i)e^{-i\lambda_0 t}$ belongs to the space $\mathcal{B}^2$ with $\|x\|^2=2(1+a_1^2\lambda_0^2)>0$. We can easily check that $\ddot{x}(t)+\ddot{x}(-t)+a_1\dot{x}(t)+x(-t)=0$ and hence $x(t)$ is a solution of equation (2.1) with $a_0=b_0=b_2=1$, $b_1=0$, $a_2=0$ and $e(t)=0$. But if $|a_1|\ge 2$, then $(a_1-b_1)^4$ $+4[(a_0b_2-a_2b_0)^2+(a_0a_2-b_0b_2)(a_1-b_1)]^2=a_1^2(a_1^2-4)+4\ge 4>0$.

Lemma 2.5. *If $(a_1-b_1)^4+4[(a_0b_2-a_2b_0)^2+(a_0a_2-b_0b_2)(a_1-b_1)]^2<0$, then $L_{a,b}^{-1}$ maps $\mathcal{B}^2$ into $\mathcal{B}^{2,1}$ continuously.*

Proof.

$$\begin{aligned}\|L_{a,b}^{-1}e\|_1^2 &= \sum_{j=-\infty}^{\infty}(1+|\lambda_j|)^2(\alpha_j^2+\beta_j^2)\\ &\le \sum_{j=-\infty}^{\infty}\frac{1}{d^2(\lambda_j)}(1+|\lambda_j|)^2\Big\{\{[(b_0-a_0)\lambda_j^2+a_2-b_2]\xi_j+(a_1-b_1)\lambda_j\eta_j\}^2\\ &\quad+\{[-(a_0+b_0)\lambda_j^2+a_2+b_2]\eta_j+(b_1-a_1)\lambda_j\xi_j\}^2\Big\}\\ &\le \sum_{j=-\infty}^{\infty}\frac{1}{d^2(\lambda_j)}(1+|\lambda_j|)^2[P_1(\lambda_j)\xi_j^2+P_2(\lambda_j)\eta_j^2].\end{aligned}$$

Since $d^2(\lambda)$ is a polynomial of λ with degree 8, $\lim_{\lambda\to\infty}(1+|\lambda|)^2P_k(\lambda)/d^2(\lambda)=0$, $k=1,2$. Utilizing the fact $d^2(\lambda)|\ge|\Delta/(4(a_0^2-b_0^2))|^2>0$ we conclude that there exists a constant $C>0$ such that

$$\|L_{a,b}^{-1}e\|_1^2\le\sum_{j=-\infty}^{\infty}C(\xi_j^2+\eta_j^2)=C\sum_{j=-\infty}^{\infty}|e_j|^2.$$

□

3 The nonlinear equation

Now let us consider the nonlinear equation

$$a_0\ddot{x}(t)+b_0\ddot{x}(-t)+a_1\dot{x}(t)+b_1\dot{x}(-t)+a_2x(t)+b_2x(-t)=f(t,x(t),x(-t)),\quad t\in\mathbb{R}. \tag{3.1}$$

The next lemma is an extension of Lemma 4.1 of [8].

Lemma 3.1. *If $f(t,x,y)$ is uniformly almost periodic in t in the sense of Bohr, and satisfies Lipschitz condition*

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le L(|x_1-x_2|+|y_1-y_2|)$$

for some constant $L>0$, then $f(t,\cdot,\cdot)\colon\mathcal{B}^2\times\mathcal{B}^2\to\mathcal{B}^2$ is continuous.

Proof. Every $x, y \in \mathcal{B}^2$ is the $\mathcal{B}^2$-limit of a sequence $\{p_n\}$, $\{q_n\}$ of trigonometric polynomials. Each $p_n, q_n \in \mathcal{AP}(\mathbb{R})$ (i.e. almost periodic in the sense of Bohr). Since $f(t, \cdot, \cdot)$ is uniformly continuous and uniformly almost periodic in t in the sense of Bohr, $f(t, p_n(t), q_n(t)) \in \mathcal{AP}(\mathbb{R})$. Hence $f(t, p_n(t), q_n(t)) \in \mathcal{B}^2$.

Also since f is Lipschtz,

$$\begin{aligned} &\frac{1}{2T}\int_{-T}^{T} |f(t, p_n, q_n) - f(t, x, y)|^2 \, dt \\ &\quad \le L^2 \frac{1}{2T}\int_{-T}^{T} (|p_n - x| + |q_n - y|)^2 \, dt \\ &\quad \le 2L^2 \frac{1}{2T}\int_{-T}^{T} (|p_n - x|^2 + |q_n - y|^2) \, dt, \end{aligned}$$

therefore

$$\|f(t, p_n, q_n) - f(t, x, y)\|^2 \le 2L^2(\|p_n - x\|^2 + \|q_n - y\|^2),$$

and $f(t, x(t), y(t)) \in \mathcal{B}^2$. □

It is easy to see that the following lemma holds.

Lemma 3.2. *If $x(t) \in \mathcal{B}^2$, then $x(-t) \in \mathcal{B}^2$.*

Theorem 3.3. *Suppose $(a_1 - b_1)^4 + 4[(a_0 b_2 - a_2 b_0)^2 + (a_0 a_2 - b_0 b_2)(a_1 - b_1)]^2 < 0$. If $f(t, x, y)$ is uniformly almost periodic in t in the sense of Bohr, and satisfies Lipschitz condition*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|)$$

for some constant L and $2ML < 1$, then the equation (3.1) has a unique Besicovitch almost periodic solution $x(t)$, and $x(t) \in \mathcal{B}^{2,1}$.

Proof. For every $\phi \in \mathcal{B}^2$, $f(t, \phi(t), \phi(-t)) \in \mathcal{B}^2$ by Lemma 3.1 and 3.2. From Theorem 2.3, the equation

$$a_0\ddot{x}(t) + b_0\ddot{x}(-t) + a_1\dot{x}(t) + b_1\dot{x}(-t) + a_2 x(t) + b_2 x(-t) = f(t, \phi(t), \phi(-t)) \tag{3.2}$$

has a unique solution $T\phi \in \mathcal{B}^2$. So $T\colon \mathcal{B}^2 \to \mathcal{B}^2$. For every $\phi, \psi \in \mathcal{B}^2$, $T\phi - T\psi$ is a solution of

$$\begin{aligned} &a_0\ddot{x}(t) + b_0\ddot{x}(-t) + a_1\dot{x}(t) + b_1\dot{x}(-t) + a_2 x(t) + b_2 x(-t) \\ &\quad = f(t, \phi(t), \phi(-t) - f(t, \psi(t), \psi(-t)). \end{aligned} \tag{3.3}$$

According to Theorem 2.3, we have

$$\begin{aligned} &\|T\phi - T\psi\| \\ &\quad = \|L_{a,b}^{-1}[f(t, \phi(t), \phi(-t)) - f(t, \psi(t), \psi(-t))]\| \\ &\quad \le M\|f(t, \phi(t), \phi(-t))) - f(t, \psi(t), \psi(-t))\| \\ &\quad \le 2ML\|\phi - \psi\|. \end{aligned}$$

Since $2ML < 1$, T is a contraction mapping. So T has a unique fixed point in $\mathcal{B}^2$. Since $f(t, x(t), x(-t)) \in \mathcal{B}^2$ and $x = L_{a,b}^{-1} f(t, x(t), x(-t))$, $x \in \mathcal{B}^{2,1}$ by Lemma 2.5. □

Remark 3.4. If the condition $(a_1 - b_1)^4 + 4[(a_0 b_2 - a_2 b_0)^2 + (a_0 a_2 - b_0 b_2)(a_1 - b_1)]^2 < 0$ is not satisfied, then $d(\lambda_n)$ may become arbitrarily close to zero. That means we will meet the so-called small denominator problem. We will consider this case in future by means of KAM theory.

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