Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **49**, 1–21; http://www.math.u-szeged.hu/ejqtde/

A dynamic contact problem between elasto-viscoplastic piezoelectric bodies

Tedjani Hadj ammar ✉ [1], **Benabderrahmane Benyattou** [2] and **Salah Drabla** [3]

[1] Department of Mathematics, University of El-Oued, El-Oued 39000, Algeria
[2] Laboratory of Mathematics and Computer Sciences, University of Laghouat, Laghouat 03000, Algeria
[3] Department of Mathematics, University of Setif 1, Setif 19000, Algeria

Received 13 May 2014, appeared 13 October 2014
Communicated by Michal Fečkan

Abstract. We consider a dynamic contact problem with adhesion between two elastic-viscoplastic piezoelectric bodies. The contact is frictionless and is described with the normal compliance condition. We derive variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field and the adhesion field. We prove the existence of a unique weak solution to the problem. The proof is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

Keywords: elastic-viscoplastic piezoelectric materials, normal compliance, adhesion, evolution equations, fixed point.

2010 Mathematics Subject Classification: 74M15, 74H20, 74H25.

1 Introduction

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [7, 15, 17] and recently in the monographs [18, 19]. The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β, which describes the pointwise fractional density of adhesion of active bonds on the contact surface, and some times referred to as the intensity of adhesion. Following [10], the bonding field satisfies the restriction $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. In this paper we deal with the study of a dynamic frictionless contact problem with adhesion between two

✉ Corresponding author. Email: hadjammart@gmail.com

elastic-viscoplastic piezoelectric materials of the form

$$\begin{aligned} \boldsymbol{\sigma}^\ell = & \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell \\ & + \int_0^t \mathcal{F}^\ell \Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell,\ \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)) \Big)\, ds, \end{aligned} \tag{1.1}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell, \tag{1.2}$$

where $\boldsymbol{D}^\ell$ represents the electric displacement field, $\boldsymbol{u}^\ell$ the displacement field, $\boldsymbol{\sigma}^\ell$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}^\ell$ is a given nonlinear function, $\mathcal{F}^\ell$ is the relaxation tensor, and $\mathcal{G}^\ell$ represents the elasticity operator. $E(\varphi^\ell) = -\nabla \varphi^\ell$ is the electric field, $\mathcal{E}^\ell = (e_{ijk})$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. In (1.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t. It follows from (1.1) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}^\ell_V(t) + \boldsymbol{\sigma}^\ell_E(t) + \boldsymbol{\sigma}^\ell_R(t)$, where $\boldsymbol{\sigma}^\ell_V(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}^\ell_E(t) = (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}^\ell_R(t)$ satisfies a rate-type elastic-viscoplastic relation

$$\boldsymbol{\sigma}^\ell_R(t) = \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + \int_0^t \mathcal{F}^\ell \big(\boldsymbol{\sigma}^\ell_R(s), \varepsilon(\boldsymbol{u}^\ell(s)) \big)\, ds. \tag{1.3}$$

Various results, examples and mechanical interpretations in the study of elastic-viscoplastic materials of the form (1.3) can be found in [8, 11] and references therein. Note also that when $\mathcal{F}^\ell = 0$ the constitutive law (1.1) becomes the Kelvin–Voigt electro-viscoelastic constitutive relation,

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t). \tag{1.4}$$

Dynamic contact problems with Kelvin–Voigt materials of the form (1.4) can be found in [3, 26]. The normal compliance contact condition was first considered in [14] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see e.g. [13, 20]. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities.

In this paper we consider a mathematical frictionless contact problem between two electro-elastic-viscoplastics bodies for rate-type materials of the form (1.1). The contact is modelled with normal compliance and adhesion. The paper is organized as follows. In Section 2 we describe the mathematical models for the frictionless contact problem between two electro-elastic-viscoplastics bodies. The contact is modelled with normal compliance and adhesion. In Section 3 we list the assumption on the data and derive the variational formulation of the problem. In Section 4 we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments.

2 Problem statement

We consider the following physical setting. Let us consider two electro-elastic-viscoplastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d$ $(d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and on two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $\operatorname{meas}\Gamma_1^\ell > 0$, $\operatorname{meas}\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The Ω^ℓ body is submitted to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times (0,T)$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times (0,T)$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times (0,T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The bodies are in adhesive contact with an obstacle, over the contact surface Γ_3. With these assumptions, the classical formulation of the mechanical frictionless contact problem with adhesion between two electro-elastic-viscoplastic bodies is the following.

Problem P. For $\ell = 1,2$, find a displacement field $\boldsymbol{u}^\ell\colon \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell\colon \Omega^\ell \times (0,T) \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell\colon \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}$, a bonding field $\beta\colon \Gamma_3 \times (0,T) \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\ell\colon \Omega^\ell \times (0,T) \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned}\boldsymbol{\sigma}^\ell = {}& \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^*\nabla\varphi^\ell \\ & + \int_0^t \mathcal{F}^\ell\Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell,\, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s))\Big)\,ds\end{aligned} \quad \text{in } \Omega^\ell \times (0,T), \tag{2.1}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell\nabla\varphi^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{2.2}$$

$$\rho^\ell\ddot{\boldsymbol{u}}^\ell = \operatorname{Div}\boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{2.3}$$

$$\operatorname{div}\boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{2.4}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{2.5}$$

$$\boldsymbol{\sigma}^\ell\boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{2.6}$$

$$\begin{cases}\sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = -p_\nu([u_\nu]) + \gamma_\nu\beta^2 R_\nu([u_\nu])\end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$\begin{cases}\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = p_\tau(\beta)\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])\end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.8}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu(R_\nu([u_\nu]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_\tau])|^2\big) - \varepsilon_a\Big)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{2.10}$$

$$\boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{2.11}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \quad \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell \quad \text{in } \Omega^\ell, \tag{2.12}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.13}$$

First, equations (2.1) and (2.2) represent the electro-elastic-viscoplastic constitutive law of the material in which $\varepsilon(\boldsymbol{u}^\ell)$ denotes the linearized strain tensor, $E(\varphi^\ell) = -\nabla\varphi^\ell$ is the electric field, where φ^ℓ is the electric potential, $\mathcal{A}^\ell$ and $\mathcal{G}^\ell$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{F}^\ell$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material. $\mathcal{E}^\ell$ represents the piezoelectric operator, $(\mathcal{E}^\ell)^*$ is its transpose, $\mathcal{B}^\ell$ denotes the electric permittivity operator, and $\boldsymbol{D}^\ell = (D_1^\ell, \dots, D_d^\ell)$ is the electric displacement vector. Equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div"and "div"denote the divergence operator for tensor and vector valued functions, respectively. Next, the equations (2.5) and (2.6) represent the displacement and traction boundary condition, respectively. Condition (2.7) represents the normal compliance conditions with adhesion where γ_ν is a given adhesion coefficient and $[u_\nu] = u_\nu^1 + u_\nu^2$ stands for the displacements in normal direction. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \beta^2 R_\nu([u_\nu])$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu \beta^2 L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (2.8) represents the adhesive contact condition on the tangential plane, where $[\boldsymbol{u}_\tau] = \boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2$ stands for the jump of the displacements in tangential direction. $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \leq L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation (2.9) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [6], see also [22, 23] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from (2.9), $\beta \leq 0$. Equation (2.12) represents the initial displacement field and the initial velocity. Finally, (2.13) represents the initial condition in which β_0 is the given initial bonding field, (2.10) and (2.11) represent the electric boundary conditions.

3 Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represent the space of second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}^\ell.\boldsymbol{v}^\ell = u_i^\ell.v_i^\ell, \quad \left|\boldsymbol{v}^\ell\right| = (\boldsymbol{v}^\ell.\boldsymbol{v}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad \left|\boldsymbol{\tau}^\ell\right| = (\boldsymbol{\tau}^\ell.\boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in L^2(\Omega^\ell)\},\ H_1^\ell = \{\boldsymbol{v}^\ell = (v_i^\ell);\ v_i^\ell \in H^1(\Omega^\ell)\},$$
$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell);\ \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\},\ \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \in \mathcal{H}^\ell;\ \mathrm{div}\boldsymbol{\tau}^\ell \in H^\ell\}.$$

The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx, \quad (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = \int_{\Omega^\ell} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell dx + \int_{\Omega^\ell} \nabla \boldsymbol{u}^\ell.\nabla \boldsymbol{v}^\ell dx,$$
$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell dx + \int_{\Omega^\ell} \mathrm{div}\, \boldsymbol{\sigma}^\ell. \mathrm{Div}\, \boldsymbol{\tau}^\ell dx$$

and the associated norms $\|\cdot\|_{H^\ell}$, $\|\cdot\|_{H_1^\ell}$, $\|\cdot\|_{\mathcal{H}^\ell}$, and $\|\cdot\|_{\mathcal{H}_1^\ell}$ respectively. Here and below we use the notation

$$\nabla \boldsymbol{u}^\ell = (u_{i,j}^\ell),\ \varepsilon(\boldsymbol{u}^\ell) = (\varepsilon_{ij}(\boldsymbol{u}^\ell)), \quad \varepsilon_{ij}(\boldsymbol{u}^\ell) = \frac{1}{2}(u_{i,j}^\ell + u_{j,i}^\ell), \quad \forall u^\ell \in H_1^\ell,$$
$$\mathrm{Div}\, \boldsymbol{\sigma}^\ell = (\sigma_{ij,j}^\ell), \quad \forall \boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell.$$

For every element $\boldsymbol{v}^\ell \in H_1^\ell$, we also use the notation $\boldsymbol{v}^\ell$ for the trace of $\boldsymbol{v}^\ell$ on Γ^ℓ and we denote by v_ν^ℓ and $\boldsymbol{v}_\tau^\ell$ the *normal* and the *tangential* components of $\boldsymbol{v}^\ell$ on the boundary Γ^ℓ given by

$$v_\nu^\ell = \boldsymbol{v}^\ell.\nu^\ell, \quad \boldsymbol{v}_\tau^\ell = \boldsymbol{v}^\ell - v_\nu^\ell \boldsymbol{\nu}^\ell.$$

Let H'_{Γ^ℓ} be the dual of $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell}$ denote the duality pairing between H'_{Γ^ℓ} and H_{Γ^ℓ}. For every element $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$ let $\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell$ be the element of H'_{Γ^ℓ} given by

$$(\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell, \boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = (\boldsymbol{\sigma}^\ell, \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\mathrm{Div}\, \boldsymbol{\sigma}^\ell, \boldsymbol{v}^\ell)_{H^\ell} \quad \forall \boldsymbol{v}^\ell \in H_1^\ell.$$

Denote by σ_ν^ℓ and $\boldsymbol{\sigma}_\tau^\ell$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell$, respectively. If $\boldsymbol{\sigma}^\ell$ is continuously differentiable on $\Omega^\ell \cup \Gamma^\ell$, then

$$\sigma_\nu^\ell = (\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell).\boldsymbol{\nu}^\ell, \quad \boldsymbol{\sigma}_\tau^\ell = \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell - \sigma_\nu^\ell \boldsymbol{\nu}^\ell,$$
$$(\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell, \boldsymbol{v}^\ell)_{-\frac{1}{2},\frac{1}{2},\Gamma^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell.\boldsymbol{v}^\ell da$$

fore all $\boldsymbol{v}^\ell \in H_1^\ell$, where da is the surface measure element.

To obtain the variational formulation of the problem (2.1)–(2.13), we introduce for the bonding field the set

$$\mathcal{Z} = \left\{\theta \in L^\infty\big(0, T; L^2(\Gamma_3)\big);\ 0 \le \theta(t) \le 1 \ \ \forall t \in [0, T],\ \text{a.e. on } \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of H_1^ℓ defined by

$$V^\ell = \left\{\boldsymbol{v}^\ell \in H_1^\ell;\ \boldsymbol{v}^\ell = 0 \text{ on } \Gamma_1^\ell\right\}.$$

Since $\mathrm{meas}\,\Gamma_1^\ell > 0$, the following Korn's inequality holds:

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\mathcal{H}^\ell} \ge c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell, \tag{3.1}$$

where the constant c_K denotes a positive constant which may depend only on Ω^ℓ, Γ_1^ℓ (see [18]). Over the space V^ℓ we consider the inner product given by

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell, \tag{3.2}$$

and let $\|\cdot\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (3.1) that the norms $\|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|\cdot\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.2), there exists a constant $c_0 > 0$, depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}^\ell\|_{V^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{3.3}$$

We also introduce the spaces

$$W^\ell = \left\{ \psi^\ell \in H^1(\Omega^\ell);\ \psi^\ell = 0 \text{ on } \Gamma_a^\ell \right\},$$

$$\mathcal{W}^\ell = \left\{ \boldsymbol{D}^\ell = (D_i^\ell);\ D_i^\ell \in L^2(\Omega^\ell),\ \operatorname{div} \boldsymbol{D}^\ell \in L^2(\Omega^\ell) \right\}.$$

Since meas $\Gamma_a^\ell > 0$, the following Friedrichs–Poincaré inequality holds:

$$\|\nabla \psi^\ell\|_{L^2(\Omega^\ell)^d} \geq c_F \|\psi^\ell\|_{H^1(\Omega^\ell)} \quad \forall \psi^\ell \in W^\ell, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ, Γ_a^ℓ.

Over the space W^ℓ, we consider the inner product given by

$$(\varphi^\ell, \psi^\ell)_{W^\ell} = \int_{\Omega^\ell} \nabla \varphi^\ell . \nabla \psi^\ell dx$$

and let $\|\cdot\|_{W^\ell}$ be the associated norm. It follows from (3.4) that $\|\cdot\|_{H^1(\Omega^\ell)}$ and $\|\cdot\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|\cdot\|_{W^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\mathbf{c}_0$, depending only on Ω^ℓ, Γ_a^ℓ and Γ_3, such that

$$\|\zeta^\ell\|_{L^2(\Omega^\ell)} \leq \mathbf{c}_0 \|\zeta^\ell\|_{W^\ell} \quad \forall \zeta^\ell \in W^\ell. \tag{3.5}$$

The space $\mathcal{W}^\ell$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\ell, \boldsymbol{E}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}^\ell . \boldsymbol{E}^\ell dx + \int_{\Omega^\ell} \operatorname{div} \boldsymbol{D}^\ell . \operatorname{div} \boldsymbol{E}^\ell dx,$$

where $\operatorname{div} \boldsymbol{D}^\ell = (\boldsymbol{D}_{i,i}^\ell)$, and the associated norm $\|\cdot\|_{\mathcal{W}^\ell}$.

In order to simplify the notations, we define the product spaces

$$\begin{gathered} \boldsymbol{V} = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \\ \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \quad W = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{gathered} \tag{3.6}$$

The spaces $\boldsymbol{V}$, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_{\boldsymbol{V}}$, $(\cdot,\cdot)_W$, and $(\cdot,\cdot)_{\mathcal{W}}$. The associate norms will be denoted by $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_W$, and $\|\cdot\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$, $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X,$$
$$\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0, r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy–Lipschitz (see, [23, p. 48]).

Theorem 3.1. *Assume that $(X, \|\cdot\|_X)$ is a real Banach space and $T > 0$. Let $F(t,\cdot)\colon X \to X$ be an operator defined a.e. on $(0,T)$ satisfying the following conditions:*

1. *There exists a constant $L_F > 0$ such that*

$$\|F(t,x) - F(t,y)\|_X \le L_F \|x-y\|_X \quad \forall x,y \in X, \ \text{ a.e. } \ t \in (0,T).$$

2. *There exists $p \ge 1$ such that $t \mapsto F(t,x) \in L^p(0,T;X) \quad \forall x \in X$.*

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,p}(0,T;X)$ such that

$$\dot{x}(t) = F(t,x(t)), \ \text{ a.e. } t \in (0,T),$$
$$x(0) = x_0.$$

Theorem 3.1 will be used in Section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

In the study of the Problem **P**, we consider the following assumptions:

we assume that the *viscosity operator* $\mathcal{A}^\ell\colon \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that} \\ & |\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)| \le L_{\mathcal{A}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\ & \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b)} & \text{There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} \\ & (\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \\ & \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(c)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ & \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(d)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{0}) \text{ is continuous on } \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \end{array}\right. \tag{3.7}$$

The *elasticity operator* $\mathcal{G}^\ell\colon \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{ll} \text{(a)} & \text{There exists } L_{\mathcal{G}^\ell} > 0 \text{ such that} \\ & |\mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)| \le L_{\mathcal{G}^\ell} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\ & \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ & \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(c)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } \mathcal{H}^\ell. \end{array}\right. \tag{3.8}$$

The *viscoplasticity operator* $\mathcal{F}^\ell\colon \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{(a) There exists } L_{\mathcal{F}^\ell} > 0 \text{ such that} \\ \quad |\mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{\eta}_1,\boldsymbol{\xi}_1) - \mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{\eta}_2,\boldsymbol{\xi}_2)| \le L_{\mathcal{F}^\ell}\big(|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|\big) \\ \quad \forall\, \boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{\eta},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \quad \text{for any } \boldsymbol{\eta},\boldsymbol{\xi}\in\mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x}\mapsto \mathcal{F}^\ell(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } \mathcal{H}^\ell. \end{cases} \tag{3.9}$$

The *piezoelectric tensor* $\mathcal{E}^\ell\colon \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\begin{cases} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x},\tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x}\in\Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell),\ 1 \le i,j,k \le d. \end{cases} \tag{3.10}$$

Recall also that the transposed operator $(\mathcal{E}^\ell)^*$ is given by $(\mathcal{E}^\ell)^* = (e^{\ell,*}_{ijk})$ where $e^{\ell,*}_{ijk} = e^\ell_{kij}$ and the following equality holds

$$\mathcal{E}^\ell\sigma.\boldsymbol{v} = \sigma.(\mathcal{E}^\ell)^*\boldsymbol{v} \quad \forall\sigma\in\mathbb{S}^d,\ \forall \boldsymbol{v}\in\mathbb{R}^d.$$

The *electric permittivity operator* $\mathcal{B}^\ell = (b^\ell_{ij})\colon \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

$$\begin{cases} \text{(a) } \mathcal{B}^\ell(\boldsymbol{x},\mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j) \quad \forall \mathbf{E} = (E_i)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega^\ell. \\ \text{(b) } b^\ell_{ij} = b^\ell_{ji},\ b^\ell_{ij}\in L^\infty(\Omega^\ell), \quad 1\le i,j\le d. \\ \text{(c) There exists } m_{\mathcal{B}^\ell} > 0 \text{ such that } \mathcal{B}^\ell\mathbf{E}.\mathbf{E} \ge m_{\mathcal{B}^\ell}|\mathbf{E}|^2 \\ \quad \forall\,\mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega^\ell. \end{cases} \tag{3.11}$$

The *normal compliance functions* $p_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists\, L_\nu > 0 \text{ such that } |p_\nu(\boldsymbol{x},r_1) - p_\nu(\boldsymbol{x},r_2)| \le L_\nu|r_1-r_2| \\ \quad \forall\, r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\ \forall r\in\mathbb{R}. \\ \text{(c) } p_\nu(\boldsymbol{x},r) = 0, \text{ for all } r\le 0, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases} \tag{3.12}$$

The *tangential compliance functions* $p_\tau\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a) } \exists\, L_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x},d_1) - p_\tau(\boldsymbol{x},d_2)| \le L_\tau|d_1-d_2| \\ \quad \forall\, d_1,d_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(b) } \exists\, M_\tau > 0 \text{ such that } |p_\tau(\boldsymbol{x},d)| \le M_\tau\ \forall\, d\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x}\mapsto p_\tau(\boldsymbol{x},d) \text{ is measurable on } \Gamma_3,\ \forall d\in\mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x}\mapsto p_\tau(\boldsymbol{x},0)\in L^2(\Gamma_3). \end{cases} \tag{3.13}$$

We suppose that the mass density satisfies

$$\rho^\ell \in L^\infty(\Omega^\ell) \text{ and } \exists\rho_0 > 0 \text{ such that } \rho^\ell(x) \ge \rho_0 \text{ a.e. } x\in\Omega^\ell,\ \ell = 1,2. \tag{3.14}$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{aligned} \mathbf{f}^\ell_0 &\in L^2(0,T;L^2(\Omega^\ell)^d), \quad \mathbf{f}^\ell_2 \in L^2(0,T;L^2(\Gamma^\ell_2)^d), \\ q^\ell_0 &\in C(0,T;L^2(\Omega^\ell)), \quad q^\ell_2 \in C(0,T;L^2(\Gamma^\ell_b)), \end{aligned} \tag{3.15}$$

$$q_2^\ell(t) = 0 \quad \text{on } \Gamma_3 \;\; \forall t \in [0,T]. \tag{3.16}$$

The adhesion coefficients γ_ν, γ_τ and ε_a satisfy the conditions

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \;\; \varepsilon_a \in L^2(\Gamma_3), \;\; \gamma_\nu, \gamma_\tau, \varepsilon_a \geq 0, \text{ a.e. on } \Gamma_3, \tag{3.17}$$

and, finally, the initial data satisfy

$$\boldsymbol{u}_0 \in \boldsymbol{V}, \quad \boldsymbol{v}_0 \in H, \quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1, \text{ a.e. on } \Gamma_3. \tag{3.18}$$

We will use a modified inner product on H, given by

$$((\boldsymbol{u}, \boldsymbol{v}))_H = \sum_{\ell=1}^{2} (\rho^\ell \boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H^\ell}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H,$$

that is, it is weighted with ρ^ℓ, and we let $|||\cdot|||_H$ be the associated norm, i.e.,

$$|||\boldsymbol{v}|||_H = ((\boldsymbol{v}, \boldsymbol{v}))_H^{\frac{1}{2}}, \quad \forall v \in H.$$

It follows from assumption (3.14) that $|||\cdot|||_H$ and $\|\cdot\|_H$ are equivalent norms on H, and the inclusion mapping of $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ into $(H, |||\cdot|||_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying H with its own dual, we can write the Gelfand triple

$$\boldsymbol{V} \subset H \subset \boldsymbol{V}'.$$

Using the notation $(\cdot,\cdot)_{\boldsymbol{V}' \times \boldsymbol{V}}$ to represent the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$ we have

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} = ((\boldsymbol{u}, \boldsymbol{v}))_H, \quad \forall \boldsymbol{u} \in H, \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Finally, we denote by $\|\cdot\|_{\boldsymbol{V}'}$ the norm on $\boldsymbol{V}'$. Using the Riesz representation theorem, we define the linear mappings $\mathbf{f}\colon [0,T] \to \boldsymbol{V}'$ and $q\colon [0,T] \to W$ as follows:

$$(\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \cdot \boldsymbol{v}^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2} \mathbf{f}_2^\ell(t) \cdot \boldsymbol{v}^\ell \, da \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{3.19}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \zeta^\ell \, dx - \sum_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t) \zeta^\ell \, da \quad \forall \zeta \in W. \tag{3.20}$$

Next, we denote by $j_{ad}\colon L^\infty(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ the adhesion functional defined by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \Big(-\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + p_\tau(\beta) \boldsymbol{R}_\tau([\boldsymbol{u}_\tau])[\boldsymbol{v}_\tau] \Big) \, da. \tag{3.21}$$

In addition to the functional (3.21), we need the normal compliance functional

$$j_{\nu c}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu])[v_\nu] \, da. \tag{3.22}$$

Keeping in mind (3.12)–(3.13), we observe that the integrals (3.21) and (3.22) are well defined and we note that conditions (3.15) imply

$$\mathbf{f} \in L^2(0,T; \boldsymbol{V}'), \quad q \in C(0,T; W). \tag{3.23}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (2.1)–(2.13).

Problem PV. Find a displacement field $\boldsymbol{u}\colon [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma}\colon [0,T] \to \mathcal{H}$, an electric potential field $\varphi\colon [0,T] \to W$, a bonding field $\beta\colon [0,T] \to L^\infty(\Gamma_3)$ and a electric displacement field $\boldsymbol{D}\colon [0,T] \to \mathcal{W}$ such that

$$\begin{aligned}\boldsymbol{\sigma}^\ell &= \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell \\ &+ \int_0^t \mathcal{F}^\ell \Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)) \Big)\, ds \end{aligned} \quad \text{in } \Omega^\ell \times (0,T), \tag{3.24}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{3.25}$$

$$\begin{aligned}(\ddot{u}, v)_{\boldsymbol{V}' \times \boldsymbol{V}} &+ \sum_{\ell=1}^{2} (\sigma^\ell,\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) + j_{vc}(\boldsymbol{u}(t), \boldsymbol{v}) \\ &= (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in (0,T),\end{aligned} \tag{3.26}$$

$$\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi^\ell(t), \nabla \phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)), \nabla \phi^\ell)_{H^\ell} = (q(t), \phi)_W \quad \forall \phi \in W,\ t \in (0,T), \tag{3.27}$$

$$\dot{\beta}(t) = -\Big(\beta(t) \big(\gamma_\nu (R_\nu([u_\nu(t)]))^2 + \gamma_\tau \, |\boldsymbol{R}_\tau([\boldsymbol{u}_\tau(t)])|^2 \big) - \varepsilon_a \Big)_+ \text{ in a.e. } (0,T), \tag{3.28}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \beta(0) = \beta_0. \tag{3.29}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a bonding field and a electric displacement field. The existence of the unique solution to Problem **PV** is stated and proved in the next section.

Remark 3.2. We note that, in Problem **P** and in Problem **PV**, we do not need to impose explicitly the restriction $0 \le \beta \le 1$. Indeed, equation (3.28) guarantees that $\beta(x,t) \le \beta_0(x)$ and, therefore, assumption (3.18) shows that $\beta(x,t) \le 1$ for $t \ge 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\beta(x,t_0) = 0$ at time t_0, then it follows from (3.28) that $\dot{\beta}(x,t) = 0$ for all $t \ge t_0$ and therefore, $\beta(x,t) = 0$ for all $t \ge t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \le \beta(x,t) \le 1$ for all $t \in [0,T]$, a.e. $x \in \Gamma_3$.

Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \le \beta, \beta_1, \beta_2 \le 1$ a.e. $x \in \Gamma_3$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{v}$ represent elements of $\boldsymbol{V}$ and $C > 0$ represents generic constants which may depend on Ω^ℓ, Γ_3, p_ν, p_τ, γ_ν, γ_τ and L. First, we note that the functional j_{ad} and j_{vc} are linear with respect to the last argument and, therefore,

$$\begin{aligned} j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) &= -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}), \\ j_{vc}(\boldsymbol{u}, -\boldsymbol{v}) &= -j_{vc}(\boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{3.30}$$

Next, using (3.21), the properties of the truncation operators R_ν and $\boldsymbol{R}_\tau$ as well as assumption (3.13) on the function p_τ, after some calculus we find

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \le C \int_{\Gamma_3} |\beta_1 - \beta_2| |\boldsymbol{u}_1 - \boldsymbol{u}_2| da,$$

and, by (3.20), we obtain

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \le C |\beta_1 - \beta_2|_{L^2(\Gamma_3)} |\boldsymbol{u}_1 - \boldsymbol{u}_2|_{\boldsymbol{V}}. \tag{3.31}$$

Similar computations, based on the Lipschitz continuity of R_ν, $\boldsymbol{R}_\tau$ and p_τ show that the following inequality also holds:

$$|j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{v})| \leq C\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V. \tag{3.32}$$

We take now $\beta_1 = \beta_2 = \beta$ in (3.31) to deduce

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0. \tag{3.33}$$

Also, we take $\boldsymbol{u}_1 = \boldsymbol{v}$ and $\boldsymbol{u}_2 = 0$ in (3.32) then we use the equalities $R_\nu(0) = 0$, $\boldsymbol{R}_\tau(0) = 0$ and (3.30) to obtain

$$j_{ad}(\beta, \boldsymbol{v}, \boldsymbol{v}) \geq 0. \tag{3.34}$$

Now, we use (3.22) to see that

$$j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{v}) \leq \int_{\Gamma_3} |p_\nu([u_{1\nu}]) - p_\nu([u_{2\nu}])||[v_\nu]|da,$$

and therefore (3.12.b) and (3.3) imply

$$|j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{v})| \leq C\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V. \tag{3.35}$$

We use again (3.22) to see that

$$j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = -\int_{\Gamma_3} (p_\nu([u_{1\nu}]) - p_\nu([u_{2\nu}]))([u_{1\nu} - u_{2\nu}])da$$

and therefore (3.12.c) implies

$$j_{\nu c}(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{\nu c}(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0. \tag{3.36}$$

We take $\boldsymbol{u}_1 = \boldsymbol{v}$ and $\boldsymbol{u}_2 = 0$ in the previous in equality and use (3.22) and (3.36) to obtain

$$j_{\nu c}(\boldsymbol{v}, \boldsymbol{v}) \geq 0. \tag{3.37}$$

Inequalities (3.31)–(3.37) and equality (3.30) will be used in various places in the rest of the paper.

4 Existence and uniqueness result

Now, we propose our existence and uniqueness result.

Theorem 4.1. *Assume that* (3.8)–(3.18) *hold. Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \beta, \boldsymbol{D}\}$ *to Problem* ***PV****. Moreover, the solution satisfies*

$$\boldsymbol{u} \in H^1(0, T; \boldsymbol{V}) \cap C^1(0, T; H), \quad \ddot{\boldsymbol{u}} \in L^2(0, T; \boldsymbol{V}'), \tag{4.1}$$

$$\varphi \in C(0, T; W), \tag{4.2}$$

$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Z}. \tag{4.3}$$

The functions $\boldsymbol{u}, \varphi, \boldsymbol{\sigma}, \boldsymbol{D}$ and β which satisfy (3.24)–(3.29) are called a weak solution to the contact Problem **P**. We conclude that, under the assumptions (3.7)–(3.18), the mechanical problem (2.1)–(2.13) has a unique weak solution satisfying (4.1)–(4.3). The regularity of the weak solution is given by (4.1)–(4.3) and, in term of stresses,

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \tag{4.4}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{4.5}$$

Indeed, it follows from (3.26) and (3.27) that $\rho^\ell \ddot{\boldsymbol{u}}^\ell = \operatorname{Div} \boldsymbol{\sigma}^\ell(t) + \mathbf{f}_0^\ell(t)$, $\operatorname{div} \boldsymbol{D}^\ell(t) - q_0^\ell(t) = 0$ for all $t \in [0,T]$ and therefore the regularity (4.1) and (4.2) of $\boldsymbol{u}$ and φ, combined with (3.14)–(3.16) implies (4.4)–(4.5).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows. Everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, γ_ν, γ_τ and L and may change from place to place. Let $\eta \in L^2(0,T;\boldsymbol{V}')$ be given. In the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta^u$. Find a displacement field $\boldsymbol{u}_\eta \colon [0,T] \to \boldsymbol{V}$ such that

$$\begin{aligned}(\ddot{u}_\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} \\ = (\mathbf{f}(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ \text{a.e. } t \in (0,T),\end{aligned} \tag{4.6}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \ \ \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell \quad \text{in } \Omega^\ell, \tag{4.7}$$

To solve Problem $\mathbf{PV}_\eta^u$, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. Let $\boldsymbol{V}$ and H denote real Hilbert spaces such that $\boldsymbol{V}$ is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of the dual $\boldsymbol{V}'$ of $\boldsymbol{V}$, i.e. $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$, and we say that the inclusions above define a Gelfand triple. The notations $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{\boldsymbol{V}'}$ and $(\cdot,\cdot)_{\boldsymbol{V}'\times\boldsymbol{V}}$ represent the norms on $\boldsymbol{V}$ and on $\boldsymbol{V}'$ and the duality pairing between $\boldsymbol{V}'$ them, respectively. The following abstract result may be found in [23, p. 48].

Theorem 4.2. *Let $\boldsymbol{V}$, H be as above, and let $A\colon \boldsymbol{V} \to \boldsymbol{V}'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq w\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \lambda \ \ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{4.8}$$

$$\|A\boldsymbol{v}\|_{\boldsymbol{V}'} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{V}} + 1) \ \ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{4.9}$$

for some constants $w > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\boldsymbol{u}_0 \in H$ and $f \in L^2(0,T;\boldsymbol{V}')$, there exists a unique function $\boldsymbol{u}$ which satisfies

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap C^1(0,T;H), \ \ \dot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'),$$
$$\dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{f}(t) \text{ a.e. } t \in (0,T),$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0$$

We have the following result for the problem.

Lemma 4.3. *There exists a unique solution to Problem $\mathbf{PV}_\eta^u$ and it has its regularity expressed in (4.1).*

Proof. We define the operator $A\colon \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2}(\mathcal{A}^{\ell}\varepsilon(\boldsymbol{u}^{\ell}),\ \varepsilon(\boldsymbol{v}^{\ell}))_{\mathcal{H}^{\ell}} \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}. \tag{4.10}$$

Using (4.10), (3.2) and (3.7) it follows that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|^2_{\boldsymbol{V}'} \leq \sum_{\ell=1}^{2} \|\mathcal{A}^{\ell}\varepsilon(\boldsymbol{u}^{\ell}) - \mathcal{A}^{\ell}\varepsilon(\boldsymbol{v}^{\ell})\|^2_{\mathcal{H}^{\ell}} \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V},$$

and keeping in mind the Krasnoselski theorem (see [12, p. 60]), we deduce that $A\colon \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator. Now, by (4.10), (3.2) and (3.7), we find

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq m\|\boldsymbol{u} - \boldsymbol{v}\|^2_{\boldsymbol{V}} \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}, \tag{4.11}$$

where the positive constant $m = \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{v} = 0$ in (4.11) we obtain

$$\begin{aligned}(A\boldsymbol{u},\boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} &\geq m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \|Ao\|^2_{\boldsymbol{V}'}\|\boldsymbol{u}\|_{\boldsymbol{V}} \\ &\geq \frac{1}{2}m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'} \quad \forall \boldsymbol{u} \in \boldsymbol{V},\end{aligned}$$

which implies that A satisfies condition (4.8) with $\omega = \frac{m}{2}$ and $\lambda = -\frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'}$. Moreover, by (4.10) and (3.7) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \leq C^1\|u\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V}.$$

where $C^1 = \max\{C^1_{\mathcal{A}^1}, C^1_{\mathcal{A}^2}\}$ and $C^2 = \max\{C^2_{\mathcal{A}^1}, C^2_{\mathcal{A}^2}\}$. This inequality and (3.2) imply that A satisfies condition (4.9). Finally, we recall that by (3.15) and (3.18) we have $\mathbf{f} - \eta \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$.

It follows now from Theorem 4.2 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \ \ \dot{\boldsymbol{v}}_\eta \in L^2(0,T;\boldsymbol{V}'), \tag{4.12}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = \mathbf{f}(t), \ \text{ a.e. } t \in [0,T] \tag{4.13}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{4.14}$$

Let $\boldsymbol{u}_\eta\colon [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{4.15}$$

It follows from (4.10) and (4.12)–(4.15) that $\boldsymbol{u}_\eta$ is a unique solution to the variational problem $\mathbf{PV}^u_\eta$ and it satisfies the regularity expressed in (4.1). □

In the second step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3 to construct the following Cauchy problem for the stress field.

Problem $\mathbf{PV}^\sigma_\eta$. *Find a stress field* $\boldsymbol{\sigma}_\eta = (\boldsymbol{\sigma}^1_\eta, \boldsymbol{\sigma}^2_\eta)\colon [0,T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}^\ell_\eta(t) = \mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(t)) + \int_0^t \mathcal{F}^\ell(\boldsymbol{\sigma}^\ell_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(s)))\,ds, \quad \ell = 1,2, \tag{4.16}$$

for all $t \in [0,T]$.

In the study of Problem $\mathbf{PV}^\sigma_\eta$ we have the following result.

Lemma 4.4. *There exists a unique solution of Problem $\boldsymbol{PV}^{\sigma}_{\eta}$ and it satisfies $\boldsymbol{\sigma}_{\eta} \in W^{1,2}(0,T;\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_i$ and $\boldsymbol{u}_i$ represent the solutions of problem $\boldsymbol{PV}^{\sigma}_{\eta}$ and $\boldsymbol{PV}^{u}_{\eta}$, respectively, for $\boldsymbol{\eta}_i \in L^2(0,T;\boldsymbol{V}')$, $i = 1,2$, then there exists $c > 0$ such that*

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le c\left(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\right) \tag{4.17}$$

for all $t \in [0,T]$.

Proof. Let $\Lambda_\eta = (\Lambda^1_\eta, \Lambda^2_\eta) \colon L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda^{\ell}_{\eta}\boldsymbol{\sigma}(t) = \mathcal{G}^{\ell}\boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}{}_{\eta}(t)) + \int_0^t \mathcal{F}^{\ell}(\boldsymbol{\sigma}^{\ell}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_{\eta}(s)))\, ds, \quad \ell = 1,2 \tag{4.18}$$

for all $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in L^2(0,T;\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$ we use (4.18) and (3.9) to obtain

$$\|\Lambda_\eta\, \boldsymbol{\sigma}_1(t) - \Lambda_\eta\, \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le \max(L_{\mathcal{F}^1}, L_{\mathcal{F}^2}) \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for p large enough, a power Λ^p_η of the operator Λ_η is a contraction on the Banach space $L^2(0,T;\mathbf{V})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_\eta \in L^2(0,T;\mathcal{H})$ such that $\Lambda_\eta \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of Problem $\mathbf{PV}^{\sigma}_{\eta}$ and, using (4.16), the regularity of $\boldsymbol{u}_\eta$ and the properties of the operators $\mathcal{G}^{\ell}$ and $\mathcal{F}^{\ell}$, it follows that $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$.

Consider now $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L^2(0,T;\boldsymbol{V}')$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$. We have

$$\boldsymbol{\sigma}^{\ell}_i(t) = \mathcal{G}^{\ell}\boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_i(t)) + \int_0^t \mathcal{F}^{\ell}(\boldsymbol{\sigma}^{\ell}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^{\ell}_i(s)))\, ds, \quad \ell = 1,2 \ \ \forall t \in [0,T],$$

and, using the properties (3.8) and (3.9) of $\mathcal{F}^{\ell}$, and $\mathcal{G}^{\ell}$ we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le c\Big(&\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathbf{V}} + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds \\ &+ \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\Big) \quad \forall t \in [0,T].\end{aligned}$$

Using now a Gronwall argument in the previous inequality we deduce (4.17), which concludes the proof. □

In the third step, let $\eta \in L^2(0,T;\boldsymbol{V}')$, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3 and we consider the following variational problem.

Problem $\mathbf{PV}^{\varphi}_{\eta}$. Find the electric potential field $\varphi_\eta \colon [0,T] \to W$ such that

$$\sum_{\ell=1}^{2}(\mathcal{B}^{\ell}\nabla\varphi^{\ell}_{\eta}(t), \nabla\phi^{\ell})_{H^{\ell}} - \sum_{\ell=1}^{2}(\mathcal{E}^{\ell}\varepsilon(\boldsymbol{u}^{\ell}_{\eta}(t)), \nabla\phi^{\ell})_{H^{\ell}} = (q(t), \phi)_W \tag{4.19}$$
$$\forall \phi \in W, \ \text{a.e. } t \in (0,T).$$

We have the following result.

Lemma 4.5. *Problem $\boldsymbol{PV}^{\varphi}_{\eta}$ has a unique solution φ_η which satisfies the regularity (4.2).*

Proof. We define a bilinear form: $b(\cdot,\cdot)\colon W\times W\to\mathbb{R}$ such that

$$b(\varphi,\phi)=\sum_{\ell=1}^{2}(\mathcal{B}^{\ell}\nabla\varphi^{\ell},\nabla\phi^{\ell})_{H^{\ell}}\quad\forall\varphi,\phi\in W. \tag{4.20}$$

We use (4.20), (3.4) and (3.11) to show that the bilinear form $b(\cdot,\cdot)$ is continuous, symmetric and coercive on W. Moreover, using the Riesz representation theorem we may define an element $q_\eta\colon[0,T]\to W$ such that

$$(q_\eta(t),\phi)_W=\sum_{\ell=1}^{2}(q_\eta^{\ell}(t),\phi^{\ell})_{W^{\ell}}+\sum_{\ell=1}^{2}(\mathcal{E}^{\ell}\varepsilon(\boldsymbol{u}_\eta^{\ell}(t)),\nabla\phi^{\ell})_{H^{\ell}}\quad\forall\phi\in W,t\in(0,T).$$

We apply the Lax–Milgram theorem to deduce that there exists a unique element $\varphi_\eta(t)\in W$ such that

$$b(\varphi_\eta(t),\phi)=(q_\eta(t),\phi)_W\quad\forall\phi\in W. \tag{4.21}$$

We conclude that $\varphi_\eta(t)$ is a solution to Problem $\mathbf{P}_\eta^{\varphi}$. Let $t_1,t_2\in[0,T]$, it follows from (4.19) that

$$\|\varphi_\eta(t_1)-\varphi_\eta(t_2)\|_W\le C\big(\|\boldsymbol{u}_\eta(t_1)-\boldsymbol{u}_\eta(t_2)\|_V+\|q(t_1)-q(t_2)\|_W\big),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\eta$ and q imply that $\varphi_\eta\in C(0,T;W)$. □

Now we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3 and we consider the following initial-value problem.

Problem $\mathbf{PV}_\eta^{\beta}$. Find the adhesion field $\beta_\eta\colon[0,T]\to L^2(\Gamma_3)$ such that

$$\dot\beta_\eta(t)=-\Big(\beta_\eta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2+\gamma_\tau\left|\boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}(t)])\right|^2\big)-\varepsilon_a\Big)_+,\ \text{a.e. } t\in(0,T), \tag{4.22}$$

$$\beta_\eta(0)=\beta_0. \tag{4.23}$$

We have the following result.

Lemma 4.6. *There exists a unique solution $\beta_\eta\in W^{1,\infty}(0,T;L^2(\Gamma_3))$ to Problem $\mathbf{PV}_\eta^{\beta}$. Moreover, $\beta_\eta(t)\in\mathcal{Z}$ for all $t\in[0,T]$.*

Proof. For the simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta\colon[0,T]\times L^2(\Gamma_3)\to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta)=-\Big(\beta\big[\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2+\gamma_\tau\left|\boldsymbol{R}_\tau([\boldsymbol{u}_{\eta\tau}(t)])\right|^2\big]-\varepsilon_a\Big)_+, \tag{4.24}$$

for all $t\in[0,T]$ and $\beta\in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta\in L^2(\Gamma_3)$, the mapping $t\to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy–Lipschitz theorem given in Theorem 3.1 we deduce that there exists a unique function $\beta_\eta\in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to the Problem $\mathbf{PV}_\eta^{\beta}$. Also, the arguments used in Remark 3.2 show that $0\le\beta_\eta(t)\le1$ for all $t\in[0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Z}$, we find that $\beta_\eta(t)\in\mathcal{Z}$, which concludes the proof of the lemma. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{G}^\ell$, the operator $\mathcal{E}^\ell$, the functional j_{ad} and the function j_{vc}, for $t \in [0,T]$, we consider the operator

$$\Lambda\colon L^2(0,T;\boldsymbol{V}') \longrightarrow L^2(0,T;\boldsymbol{V}')$$

which maps every element $\eta \in L^2(0,T;\boldsymbol{V}')$ to the element $\Lambda(\eta) \in L^2(0,T;\boldsymbol{V}')$ defined by

$$\begin{aligned}(\Lambda(\eta)(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = &\sum_{\ell=1}^{2}\big(\mathcal{G}^\ell\varepsilon(\boldsymbol{u}_\eta^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\varphi_\eta^\ell\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}\\ &+\sum_{\ell=1}^{2}\bigg(\int_0^t \mathcal{F}^\ell\big(\boldsymbol{\sigma}_\eta^\ell,\varepsilon(\boldsymbol{u}_\eta^\ell(s))\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\bigg)_{\mathcal{H}^\ell} + j_{ad}(\beta_\eta(t),\boldsymbol{u}_\eta(t),\boldsymbol{v})\\ &+ j_{vc}(\boldsymbol{u}_\eta(t),\boldsymbol{v}),\quad \forall \boldsymbol{v}\in\boldsymbol{V}.\end{aligned} \tag{4.25}$$

Here, for every $\eta \in L^2(0,T;\boldsymbol{V}')$, $\boldsymbol{u}_\eta$, $\boldsymbol{\sigma}_\eta$, φ_η and β_η represent the displacement field, the stress field, the the potential electric field and bonding field obtained in Lemmas 4.3, 4.4, 4.5 and 4.6 respectively. We have the following result.

Lemma 4.7. *The operator Λ has a unique fixed point $\eta^* \in L^2(0,T;\boldsymbol{V}')$.*

Proof. We show that, for a positive integer n, the mapping Λ^n is a contraction on $L^2(0,T;\boldsymbol{V}')$. To this end, we suppose that η_1 and η_2 are two functions in $L^2(0,T;\boldsymbol{V}')$ and denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$, $\varphi_{\eta_i} = \varphi_i$ and $\beta_{\eta_i} = \beta_i$ for $i = 1,2$. We use (3.8), (3.10), (3.12), (3.13) and (3.9), the definition of R_ν, $\boldsymbol{R}_\tau$ and Remark 3.2, we have

$$\begin{aligned}\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|_{\boldsymbol{V}'}^2 \le &\sum_{\ell=1}^{2}\|\mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}_1^\ell(t)) - \mathcal{G}^\ell\boldsymbol{\varepsilon}(\boldsymbol{u}_2^\ell(t))\|_{\mathcal{H}^\ell}^2\\ &+\sum_{\ell=1}^{2}\int_0^t \big\|\mathcal{F}^\ell\big(\boldsymbol{\sigma}_1^\ell(s),\varepsilon(\boldsymbol{u}_1^\ell(s))\big) - \mathcal{F}^\ell\big(\boldsymbol{\sigma}_2^\ell(s),\varepsilon(\boldsymbol{u}_2^\ell(s))\big)\big\|_{\mathcal{H}^\ell}^2\,ds\\ &+\sum_{\ell=1}^{2}\|(\mathcal{E}^\ell)^*\nabla\varphi_1^\ell(t) - (\mathcal{E}^\ell)^*\nabla\varphi_2^\ell(t)\|_{\mathcal{H}^\ell}^2\\ &+C\|p_\nu([u_{1\nu}(t)]) - p_\nu([u_{2\nu}(t)])\|_{L^2(\Gamma_3)}^2\\ &+C\|\beta_1^2(t)R_\nu([u_{1\nu}(t)]) - \beta_2^2(t)R_\nu([u_{2\nu}(t)])\|_{L^2(\Gamma_3)}^2\\ &+C\|p_\tau(\beta_1(t))\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(t)]) - p_\tau(\beta_2(t))\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(t)])\|_{L^2(\Gamma_3)}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|_{\boldsymbol{V}'}^2\\ &\quad\le \bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2 + \int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds\\ &\qquad+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s))\|_{\mathcal{H}}^2\,ds + \|\varphi_1(t)-\varphi_2(t)\|_W^2 + \|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2\bigg).\end{aligned} \tag{4.26}$$

We use estimate (4.17) to obtain

$$\begin{aligned}&\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|_{\boldsymbol{V}'}^2\\ &\quad\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2\\ &\qquad+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds + \|\varphi_1(t)-\varphi_2(t)\|_W^2 + \|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2\bigg).\end{aligned} \tag{4.27}$$

Moreover, from (4.6) we obtain

$$(\dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{v}}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_{V'\times V} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\boldsymbol{v}_1^\ell) - \mathcal{A}^\ell \varepsilon(\boldsymbol{v}_2^\ell), \varepsilon(\boldsymbol{v}_1^\ell - \boldsymbol{v}_2^\ell))_{\mathcal{H}^\ell} + (\eta_1 - \eta_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_{V'\times V} = 0.$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$ and condition (3.7) to find

$$m \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))\|_V^2 \, ds \le - \int_0^t (\eta_1(s) - \eta_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))_{V'\times V} \, ds,$$

for all $t \in [0, T]$. Then, using the inequality $2ab \le \frac{a^2}{m} + mb^2$ we obtain

$$\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))\|_V^2 \, ds \le C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 \, ds \quad \forall t \in [0, T]. \tag{4.28}$$

On the other hand, from the Cauchy problem (4.22)–(4.23) we can write

$$\beta_i(t) = \beta_0 - \int_0^t \Big(\beta_i(s) \big(\gamma_\nu (R_\nu([u_{i\nu}(s)]))^2 + \gamma_\tau \, |\boldsymbol{R}_\tau([\boldsymbol{u}_{i\tau}(s)])|^2 \big) - \varepsilon_a \Big)_+ ds$$

and then

$$\begin{aligned} \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le C \int_0^t \|\beta_1(s) R_\nu([u_{1\nu}(s)])^2 - \beta_2(s) R_\nu([u_{2\nu}(s)])^2\|_{L^2(\Gamma_3)} ds \\ + \; C \int_0^t \|\beta_1(s) \, |\boldsymbol{R}_\tau([\boldsymbol{u}_{1\tau}(s)])|^2 - \beta_2(s) \, |\boldsymbol{R}_\tau([\boldsymbol{u}_{2\tau}(s)])|^2 \, \|_{L^2(\Gamma_3)} ds. \end{aligned}$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le C \Big(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d} ds \Big). \tag{4.29}$$

Next, we apply Gronwall's inequality to deduce

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \le C \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3)^d} \, ds.$$

and from the relation (3.3) we obtain

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 \le C \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 \, ds. \tag{4.30}$$

We use now (4.19), (3.4), (3.10), and (3.11) to find

$$\|\varphi_1(t) - \varphi_2(t)\|_W^2 \le C \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V^2. \tag{4.31}$$

We substitute (4.30) and (4.31) in (4.27) to obtain

$$\begin{aligned} \|\Lambda(\eta_1)(t) - \Lambda(\eta_2)(t)\|_{V'}^2 &\le C \left(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V^2 + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|_V^2 \, ds \right) \\ &\le C \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))\|_V^2 \, ds. \end{aligned}$$

It follows now from the previous inequality and the estimate (4.28) that

$$\|\Lambda(\eta_1)(t) - \Lambda(\eta_2)(t)\|_{V'}^2 \le C \int_0^t \|\eta_1(s) - \eta_2(s))\|_{V'}^2 \, ds.$$

Reiterating this inequality n times we obtain

$$\|\Lambda^n(\eta_1) - \Lambda^n(\eta_2)\|^2_{L^2(0,T;\boldsymbol{V}')} \le \frac{C^n T^n}{n!}\|\eta_1 - \eta_2\|^2_{L^2(0,T;\boldsymbol{V}')}. \tag{4.32}$$

Thus, for n sufficiently large, Λ^n is a contraction on the Banach space $L^2(0,T;\boldsymbol{V}')$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. **Existence.** Let $\eta^* \in L^2(0,T;\boldsymbol{V}')$ be the fixed point of Λ and denote

$$\boldsymbol{u}_* = \boldsymbol{u}_{\eta^*}, \quad \varphi_* = \varphi_{\eta^*}, \quad \beta_* = \beta_{\eta^*}, \tag{4.33}$$

$$\boldsymbol{\sigma}^\ell_* = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*) + (\mathcal{E}^\ell)^*\nabla\varphi^\ell_* + \boldsymbol{\sigma}^\ell_{\eta^*} \quad \forall t \in [0,T], \tag{4.34}$$

$$\boldsymbol{D}^\ell_* = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell_*) - \mathcal{B}^\ell\nabla\varphi^\ell_*, \quad \forall t \in [0,T]. \tag{4.35}$$

We prove that $(\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \beta_*, \boldsymbol{D}_*)$ satisfies (3.24)–(3.29) and the regularities (4.1)–(4.3). Indeed, we write (4.6) for $\eta = \eta^*$ and use (4.33) to find

$$\begin{aligned}(\ddot{u}_*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} &+ \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + (\eta^*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} \\ &= (\mathbf{f}(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{a.e. } t \in [0,T].\end{aligned} \tag{4.36}$$

Using equality $\Lambda\eta^* = \eta^*$ it follows that

$$\begin{aligned}&(\eta^*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} \\ &= \sum_{\ell=1}^{2}\left(\mathcal{G}^\ell\varepsilon(\boldsymbol{u}^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}\left((\mathcal{E}^\ell)^*\nabla\varphi^\ell_*,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} \\ &\quad + \sum_{\ell=1}^{2}\left(\int_0^t \mathcal{F}^\ell\Big(\boldsymbol{\sigma}^\ell_*(s) - \mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell_*(s)) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell, \varepsilon(\boldsymbol{u}^\ell_*(s))\Big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} \\ &\quad + j_{ad}(\beta_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) + j_{vc}(\boldsymbol{u}_*(t), \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}.\end{aligned} \tag{4.37}$$

We now substitute (4.37) into (4.36) to obtain

$$\begin{aligned}&(\ddot{\boldsymbol{u}}_*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} \\ &+ \sum_{\ell=1}^{2}\left(\mathcal{G}^\ell\varepsilon(\boldsymbol{u}^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}\left((\mathcal{E}^\ell)^*\nabla\varphi^\ell_*,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} \\ &+ \sum_{\ell=1}^{2}\left(\int_0^t \mathcal{F}^\ell\Big(\boldsymbol{\sigma}^\ell_*(s) - \mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell_*(s)) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell, \varepsilon(\boldsymbol{u}^\ell_*(s))\Big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\right)_{\mathcal{H}^\ell} \\ &+ j_{ad}(\beta_*(t), \boldsymbol{u}_*(t), \boldsymbol{v}) + j_{vc}(\boldsymbol{u}_*(t), \boldsymbol{v}) = (\mathbf{f}(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}.\end{aligned} \tag{4.38}$$

Using $\boldsymbol{u}_{\eta^*}$ in (4.19), by (4.33) we have:

$$\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi^\ell_*(t), \nabla\phi^\ell)_{H^\ell} - \sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}^\ell_*(t)), \nabla\phi^\ell)_{H^\ell} = (q(t), \phi)_W \quad \forall \phi \in W,\ t \in [0,T]. \tag{4.39}$$

Additionally, we use $\boldsymbol{u}_{\eta^*}$ in (4.22) and (4.33) to find

$$\dot{\beta}_*(t) = -\Big(\beta_*(t)\Big(\gamma_\nu(R_\nu([u_{*\nu}(t)]))^2 + \gamma_\tau\,|\boldsymbol{R}_\tau([\boldsymbol{u}_{*\tau}(t)])|^2\Big) - \varepsilon_a\Big)_+, \quad \text{a.e. } t \in [0,T]. \tag{4.40}$$

The relations (4.33), (4.34), (4.38)–(4.40) allow us to conclude now that $(\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \beta_*, \boldsymbol{D}_*)$ satisfies (3.24)–(3.28). Next, (3.29) and the regularity (4.1)–(4.3) follow from Lemmas 4.3, 4.4, 4.5 and 4.6. Since $\boldsymbol{u}_*$ and φ_* satisfy (4.1) and (4.3), it follows from Lemma 4.4 and (4.34) that

$$\sigma_* \in L^2(0,T;\mathcal{H}). \tag{4.41}$$

We choose $v = (v^1, v^2)$ with $v^\ell = \omega^\ell \in D(\Omega^\ell)^d$ and $v^{3-\ell} = 0$ in (4.38) and by (4.33) and (3.19):

$$\rho^\ell \ddot{\boldsymbol{u}}_*^\ell = \operatorname{Div} \boldsymbol{\sigma}_*^\ell + \boldsymbol{f}_0^\ell, \text{ a.e. } t \in [0,T], \quad \ell = 1,2.$$

Also, by (3.14), (3.15), and (4.41) we have:

$$(\operatorname{Div} \boldsymbol{\sigma}_*^1, \operatorname{Div} \boldsymbol{\sigma}_*^2) \in L^2(0,T;\boldsymbol{V}').$$

Let $t_1, t_2 \in [0,T]$, by (3.10), (3.11), (3.4) and (4.35), we deduce that

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \le C\big(\|\varphi_*(t_1) - \varphi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\big).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (4.1) and (4.2) implies

$$\boldsymbol{D}_* \in C(0,T;\mathcal{H}). \tag{4.42}$$

We choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (4.39) and using (3.20) we find

$$\operatorname{div} \boldsymbol{D}_*^\ell(t) = q_0^\ell(t) \quad \forall t \in [0,T], \quad \ell = 1,2.$$

By (3.15) and (4.42) we obtain

$$\boldsymbol{D}_* \in C(0,T;\mathcal{W}).$$

Finally we conclude that the weak solution $(\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \beta_*, \boldsymbol{D}_*)$ of the piezoelectric contact problem **P** has the regularity (4.1)–(4.5), which concludes the existence part of Theorem 4.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.25). □

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