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On positiveness of the fundamental solution for a linear autonomous differential equation with distributed delay

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Abstract. We present necessary and sufficient conditions for the nonoscillation of the fundamental solutions to a linear autonomous differential equation with distributed delay. The conditions are proposed in both the analytic and geometric forms.

Keywords: functional differential equation, distributed delay, nonoscillation, fundamental solution.

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1 Introduction

One of the significant features of differential equations with aftereffect (in contrast to ordinary differential equations) is that solutions to linear equations of first order can have alternating sign. This fact assigns a meaning to the problem of obtaining efficient conditions of oscillation and fixed sign of solutions, to the question on the number of zeros, to the estimation of interval of nonoscillation, and so on. In this paper we establish a number of facts equivalent to the positiveness of the fundamental solution for autonomous equations with aftereffect.

A criterion of nonoscillation of the fundamental solution for $\dot{x}(t) + ax(t-h) = 0$ was obtained in [11, pp. 188–190], and one for $\dot{x}(t) + ax(t) + bx(t-h) = 0$ in [6]. Regions of nonoscillation of the fundamental solution for equations with several concentrated delays were constructed in [9]. The analogous region for $\dot{x}(t) + ax(t) + b\int_{t-\tau-h}^{t-\tau} x(s)\,ds = 0$ was obtained in [8]. Some useful examples illustrating the nonoscillation of autonomous equations with concentrated and distributed delay are considered in [5].

Apparently, there is no known criterion of nonoscillation of the fundamental solution for $\dot{x}(t) + ax(t) + \int_0^t x(t-s)\,dr(s) = 0$ yet. Until now nonautonomous equations have usually been studied. Conditions of nonoscillation for them are only sufficient, and in the case of autonomous equations they are far from sharp conditions.

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Furthermore, authors of many papers (see [4, 14]) obtain conditions of the oscillation of the fundamental solution instead of conditions of nonoscillation. Since our purpose is to obtain necessary and sufficient conditions, obtaining conditions of oscillation and that of nonoscillation are in fact the same problem.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{C}$ be the space of complex numbers, l_1 be an infinite-dimensional vector space with summable components of vectors.

Consider a linear autonomous differential equation with bounded distributed delay

$$\dot{x}(t) + a_0 x(t) + \int_0^{\min\{\omega, t\}} x(t-s)\, dr(s) = f(t), \quad t \geqslant 0, \tag{2.1}$$

where $a_0 \in \mathbb{R}$, $\omega > 0$, the function r is defined on the segment $[0, \omega]$ and does not decrease, $r(0) = 0$, and the function f is locally summable. Without loss of generality it can be assumed [2, pp. 9–10] that $x(\xi) = 0$ for all $\xi < 0$.

Denote the total variation of the function r by $\rho = \int_0^\omega dr(\xi) = r(\omega)$.

We say that a *solution* of equation (2.1) is an absolutely continuous function satisfying equation (2.1) almost everywhere.

Definition 2.1. A function $X\colon \mathbb{R} \to \mathbb{R}$ is called *the fundamental solution* of equation (2.1) if it is a solution of the problem

$$\begin{cases} \dot{X}(t) + a_0 X(t) + \displaystyle\int_0^t X(t-s)\, dr(s) = 0, \\ X(\xi) = 0 \quad \text{for } \xi < 0, \quad X(1) = 1. \end{cases} \tag{2.2}$$

It is known [2, p. 84, Theorem 1.1] that for every given $x_0 \in \mathbb{R}$ there exists a unique solution of equation (2.1) such that $x(0) = x_0$. It has the form

$$x(t) = X(0)x(0) + \int_0^t X(t-s) f(s)\, ds. \tag{2.3}$$

Thus, the fundamental solution is the main subject of our study.

It follows from (2.3) that the fundamental solution of a homogeneous equation is positive if and only if all solutions have fixed sign. For a nonhomogeneous equation, if the kernel of the integral operator is positive, then the operator is isotone. This presents the possibility of fine two-way estimates of a solution.

3 Properties of the characteristic function

According to the statement of the problem and the estimation $|X(t)| \leqslant e^{(|a_0|+\rho)t}$ for $t \in \mathbb{R}_+$ (see [3, p. 94, Property 2]) we see that the conditions from [13, pp. 212–213] hold. Hence, the Laplace transform can be applied to the left and right-hand sides of the equation from problem (2.2).

The Laplace image of the fundamental solution X is $X_0(p) = \frac{1}{g(p)}$, where g is the characteristic function, $g(p) = p + a_0 + \int_0^\omega e^{-p\xi}\, dr(\xi)$, $p \in \mathbb{C}$.

Denote the real part of p by $\Re p$, and the imaginary part by $\Im p$.

Below we present a number of lemmas on properties of the function g.

Lemma 3.1. *The characteristic function g has the following properties:*

(i) g is an analytic function;

(ii) there exists $\lambda_0 \in \mathbb{R}$ such that g has no zeros for $\Re p > \lambda_0$;

(iii) the set of roots of g is finite in every vertical band on the complex plane.

Proof. (i) Denote $\rho_n = \int_0^\omega \xi^n \, dr(\xi)$, $n \in \mathbb{N}$. It is obvious that

$$\int_0^\omega e^{-p\xi} \, dr(\xi) = \int_0^\omega \sum_{n=0}^{\infty} \frac{(-p\xi)^n}{n!} \, dr(\xi) = \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} \int_0^\omega \xi^n \, dr(\xi) = \rho + \sum_{n=1}^{\infty} (-1)^n \frac{\rho_n}{n!} p^n.$$

Thus, $\int_0^\omega e^{-p\zeta} \, dr(\zeta)$ can be represented in the form of a series that converges for all $p \in \mathbb{C}$. Therefore g is an analytic function.

(ii) Note that

$$\Re g(p) = \Re p + a_0 + \int_0^\omega e^{-\Re p \xi} \cos(\Im p \xi) \, dr(\xi) \geqslant \Re p + a_0 - \int_0^\omega e^{-\Re p \xi} \, dr(\xi).$$

If $\Re p < 0$ then $\Re g(p) \geqslant \Re p + a_0 - \rho e^{-\Re p \omega}$. If $\Re p \geqslant 0$ then $\Re g(p) \geqslant \Re p + a_0 - \rho$. Clearly, there exists $\lambda_0 \in \mathbb{R}$ such that $\Re g(p) > 0$ for $\Re p > \lambda_0$.

(iii) Define $g_1(p) = i\Im p$, $g_2(p) = \Re p + a_0 + \int_0^\omega e^{-p\xi} \, dr(\xi)$.

Consider a vertical band: $\mu \leqslant \Re p \leqslant \lambda$.

Estimate $g_2(p)$:

$$|g_2(p)| \leqslant |\Re p + a_0| + \int_0^\omega \left| e^{-p\xi} \right| \, dr(\xi) \leqslant \max\{|\mu + a_0|, |\lambda + a_0|\} + \rho \max\{e^{-\mu\omega}, 1\}.$$

For $|\Im p| > \max\{|\mu + a_0|, |\lambda + a_0|\} + \rho \max\{e^{-\mu\omega}, 1\}$ there holds $|g(p)| \geqslant |g_1(p)| - |g_2(p)| > 0$. Hence the function g has a finite set of roots in every vertical band. □

Lemma 3.2. *Let $g(p)$ have no zeros for $\Re p = \mu$. Then for all $t \geqslant 0$ there holds the estimation*

$$\left| \int\limits_{\mu - i\infty}^{\mu + i\infty} \frac{e^{pt}}{g(p)} \, dp \right| \leqslant N e^{\mu t},$$

where N is a positive real number.

Proof. Add the function $\frac{e^{pt}}{p}$ to and subtract it from the subintegral function, and estimate the integral

$$\left| \int_{\mu - i\infty}^{\mu + i\infty} \left(\frac{e^{pt}}{g(p)} + \frac{e^{pt}}{p} - \frac{e^{pt}}{p} \right) dp \right| \leqslant \left| \int_{\mu - i\infty}^{\mu + i\infty} \frac{e^{pt}}{p} \, dp \right| + e^{\mu t} \int_{\mu - i\infty}^{\mu + i\infty} \left| \frac{1}{g(p)} - \frac{1}{p} \right| |dp|.$$

The function $g(p) - p$ is bounded for $\Re p = \mu$. Further,

$$\begin{aligned} \int_{\mu - i\infty}^{\mu + i\infty} \left| \frac{1}{g(p)} - \frac{1}{p} \right| |dp| &= \int_{\mu - i\infty}^{\mu + i\infty} \left| \frac{g(p) - p}{p g(p)} \right| |dp| \\ &\leqslant c \int_{-\infty}^{+\infty} \left| \frac{1}{(\mu + iy) g(\mu + iy)} \right| dy = \int_{-\infty}^{+\infty} \left| O\left(\frac{1}{y^2} \right) \right| dy \leqslant A. \end{aligned}$$

Then

$$\left| \int_{\mu - i\infty}^{\mu + i\infty} \frac{e^{pt}}{g(p)} \, dp \right| \leqslant \left| \int_{\mu - i\infty}^{\mu + i\infty} \frac{e^{pt}}{p} \, dp \right| + A e^{\mu t}.$$

Note that for all $\mu \in \mathbb{R}$ the integral $\int_{\mu - i\infty}^{\mu + i\infty} \frac{e^{pt}}{p} \, dp$ converges [7, pp. 464–465] (equals 0 for $\mu < 0$ and 1 for $\mu \geqslant 0$). Therefore the lemma is true. □

Lemma 3.3. *For any $\lambda, \mu \in \mathbb{R}$ such that $\mu < \lambda$ and for all $t \geqslant 0$ it holds that*

$$\lim_{y\to\pm\infty}\left|\int\limits_{\mu+iy}^{\lambda+iy}\frac{e^{pt}}{g(p)}\,dp\right| = 0.$$

Proof. For sufficiently large y we put $p = x + iy$, and estimate

$$\left|\frac{ce^{pt}}{p}\right| = \left|ce^{iyt}\frac{e^{xt}}{x+iy}\right| = |c|e^{\lambda t}\frac{\sqrt{x^2+y^2}}{|x^2-y^2|} \leqslant |c|e^{\lambda t}\frac{\sqrt{(\max(\lambda,\mu))^2+y^2}}{|(\min(\lambda,\mu))^2-y^2|}.$$

We have

$$\lim_{y\to\pm\infty}\left|\int\limits_{\mu+iy}^{\lambda+iy}\frac{ce^{pt}}{p}\,dp\right| \leqslant |c|e^{\lambda t}\lim_{y\to\pm\infty}\int\limits_{\mu}^{\lambda}\frac{\sqrt{(\max(\lambda,\mu))^2+y^2}}{|(\min(\lambda,\mu))^2-y^2|}\,dx = 0.$$

Now, add the function $\frac{ce^{pt}}{p}$ to and subtract it from the subintegral function $\frac{e^{pt}}{g(p)}$, put $p = x + iy$, and estimate the modulus of the expression for sufficiently large $|y|$. □

Consider the function $G(p) = \frac{1}{g(p)} - \frac{1}{p}$. It is easily shown that $G(p)$ is the Laplace image of some function [13, p. 231, Theorem 8.5]. Therefore $\frac{1}{g(p)}$ is the Laplace image of some function as it is the sum of analytic functions. Thus one can apply the inverse Laplace transform for X.

Lemma 3.4. *For the fundamental solution X of equation (2.1) there exists the estimation $|X(t)| \leqslant Mt^n e^{\zeta_0 t}$ for all $t \geqslant 0$, where $M \in \mathbb{R}_+$, ζ_0 is the maximal real part of zeros of the function g, n is the maximal multiplicity of zeros with real part ζ_0.*

Proof. Consider a rectangle $ABCD$ in $\mathbb{C}$ containing zeros of the function g with the maximal real part. Here the leg AB is to the right of these zeros and lies on the line $\Re p = \lambda > \zeta_0$. The leg CD is on the line $\Re p = \mu < \zeta_0 < \lambda$. Real parts of other zeros are less than μ. The legs AD and BC are on the lines $\Im p = -y < 0$ and $\Im p = y > 0$ respectively, where y is a positive real number.

The integral $\int\limits_{ABCD}\frac{e^{pt}}{g(p)}\,dp$ is represented as the sum of the integrals

$$I_1 = \int\limits_{AB}\frac{e^{pt}}{g(p)}\,dp,\quad I_2 = \int\limits_{BC}\frac{e^{pt}}{g(p)}\,dp,\quad I_3 = \int\limits_{CD}\frac{e^{pt}}{g(p)}\,dp,\quad I_4 = \int\limits_{DA}\frac{e^{pt}}{g(p)}\,dp.$$

Using Lemma 3.3 for I_2 and I_4 we obtain $\lim\limits_{y\to+\infty}|I_2| = 0$ and $\lim\limits_{y\to+\infty}|I_4| = 0$. For I_3 Lemma 3.2 holds. In view of [7, p. 79] we get

$$I_1 + I_2 + I_3 + I_4 = \int\limits_{ABCD}\frac{e^{pt}}{g(p)}\,dp = 2\pi i\sum_{j=1}^{s}\operatorname{res}\frac{e^{z_j t}}{g(z_j)},$$

where z_j are zeros of g inside $ABCD$, and s is the number of zeros.

Further,

$$\left|X(t) - \sum_{j=1}^{s}M_j(t)e^{\Re z_j t}\right| \leqslant Ne^{\mu t} + |I_2| + |I_4|.$$

For $y \to +\infty$ we have

$$\left|X(t) - \sum_{j=1}^{s}M_j(t)e^{\Re z_j t}\right| \leqslant Ne^{\mu t}.$$

Thus,

$$|X(t)| \leqslant Mt^n e^{\zeta_0 t}.$$

□

Lemma 3.5. *If all zeros of the function g with the maximal real part are not real then the fundamental solution of equation (2.1) oscillates.*

Proof. The characteristic function g has a finite number of zeros p_i with the maximal real parts $\Re p_i = \alpha_{\max}$.

By Lemma 3.1 there exists $\alpha_0 < \alpha_{\max}$ such that the function g has a finite number (denote it by s) of zeros p_i with $\Re p_i \geqslant \alpha_0$. Then

$$x(t) = \sum_{i=1}^{s} \left(A_i(t)e^{\alpha_{\max}t}\cos(\beta_i t) + B_i(t)e^{\alpha_{\max}t}\sin(\beta_i t)\right) + \varepsilon(t), \tag{3.1}$$

where $i, i_0 \in \mathbb{N}$, $\alpha_i, \beta_i \in \mathbb{R}$, A_i, B_i are polynomials. From Lemma 3.2 we obtain $|\varepsilon(t)| \leqslant Ne^{\alpha_0 t}$, $N \in \mathbb{R}$. From (3.1) we have

$$\frac{x(t)}{e^{\alpha_{\max}t}} = \sum_{i=1}^{s} (A_i(t)\cos(\beta_i t) + B_i(t)\sin(\beta_i t)) + \varepsilon(t)e^{-\alpha_{\max}t}.$$

Assume that the function g has no real zeros. Then $\beta_i \neq 0$, $i \in \mathbb{N}$.

Denote the power of the polynomials A_i, B_i by m_0. Denote coefficients at t^{m_0} in the polynomials A_i, B_i by a_i and b_i respectively. Then

$$y(t) = \frac{x(t)}{t^{m_0}e^{\alpha_{\max}t}} = \psi(t) + \varepsilon_{ab}(t),$$

where

$$\psi(t) = \sum_{i=1}^{s} (a_i\cos(\beta_i t) + b_i\sin(\beta_i t)), \quad \varepsilon_{ab}(t) = \sum_{i=1}^{s} (\varepsilon_{ai}(t)\cos(\beta_i t) + \varepsilon_{bi}(t)\sin(\beta_i t)) + \frac{\varepsilon(t)}{t^{m_0}e^{\alpha_{\max}t}}.$$

Clearly, $\frac{\varepsilon(t)}{t^{m_0}e^{\alpha_{\max}t}}, \varepsilon_{ai}(t), \varepsilon_{bi}(t) \to 0$ as $t \to +\infty$. Hence $\varepsilon_{ab}(t) \to 0$ as $t \to +\infty$. Without loss of generality we can assume that $R_i = \sqrt{a_i^2 + b_i^2} > 0$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_s$.

The function ψ is infinitely differentiable. Write k-th derivatives, where k is a multiple of 4:

$$\psi^{(k)}(t) = \sum_{i=1}^{s} \left(\frac{a_i}{\beta_i^k}\cos(\beta_i t) + \frac{b_i}{\beta_i^k}\sin(\beta_i t)\right) = \sum_{i=1}^{s} \frac{R_i}{\beta_i^k}\cos\left(\beta_i t - \arccos\frac{a_i}{R_i}\right).$$

For sufficiently great k we have $\frac{R_1}{\beta_1^k} > \sum_{i=2}^{i_0}\frac{R_i}{\beta_i^k}$. Then for any $n \in \mathbb{N}$ we get

$$\psi^{(k)}(\theta_n) = \frac{R_1}{\beta_1^k} + \sum_{i=2}^{s}\frac{R_i}{\beta_i^k}\cos\left(\beta_i\theta_n - \arccos\frac{a_i}{R_i}\right) \geqslant \frac{R_1}{\beta_1^k} - \sum_{i=2}^{s}\frac{R_i}{\beta_i^k} > 0,$$

$$\psi^{(k)}(\theta_n + \pi) = -\frac{R_1}{\beta_1^k} + \sum_{i=2}^{s}\frac{R_i}{\beta_i^k}\cos\left(\beta_i(\theta_n + \pi) - \arccos\frac{a_i}{R_i}\right) \leqslant -\frac{R_1}{\beta_1^k} + \sum_{i=2}^{s}\frac{R_i}{\beta_i^k} < 0,$$

where $\theta_n = \frac{1}{\beta_1}\left(\arccos\frac{a_1}{R_1} + 2\pi n\right)$. Hence the function $\psi^{(k)}$ has a sequence of zeros t_n, $n \in \mathbb{N}$, such that $\lim_{n\to+\infty} t_n = +\infty$.

Consider $\psi^{(k-1)}$. From Lagrange's theorem there exist t_n^* in the intervals $[\theta_n, \theta_n + \pi]$, $n \in \mathbb{N}$, such that $\psi^{(k-1)}(t_n^*) = \frac{\psi^{(k)}(\theta_n+\pi) - \psi^{(k)}(\theta_n)}{\pi} < 0$. Similarly, there exist t_n^{**} in $[\theta_n - \pi, \theta_n]$ such that $\psi^{(k-1)}(t_n^{**}) = \frac{\psi^{(k)}(\theta_n) - \psi^{(k)}(\theta_n - \pi)}{\pi} > 0$. Hence the function $\psi^{(k-1)}$ has a sequence of zeros t'_n, $n \in \mathbb{N}$, such that $\lim_{n\to+\infty} t'_n = +\infty$. Likewise, we obtain that ψ oscillates. Hence y oscillates. Therefore x oscillates. □

4 A theorem on differential inequalities and the positiveness of the fundamental solution

In this section we apply a theorem on differential inequalities (see [3, p. 57, Lemma 2.4.3], [1, p. 356, Theorem 15.6]) to obtain a criterion of the positiveness of the fundamental solution. Below we formulate this theorem in the form suitable for us, and prove it.

Lemma 4.1. *Suppose there exists a positive absolutely continuous function v defined on the axis $\mathbb{R}$ such that $(Lv)(t) = \dot{v}(t) + a_0 v(t) + \int_0^{+\infty} v(t-s)\,dr(s) \leqslant 0$ for almost all $t \in \mathbb{R}_+$. Then the fundamental solution of equation (2.1) is positive.*

Proof. Consider an equation

$$(Lv)(t) = \varphi(t).$$

Using (2.3), write

$$v(t) = X(t)v(0) + \int_0^t X(t-s)\varphi(s)\,ds - \int_0^t X(t-s)\int_s^{+\infty} v(s-\xi)\,dr(\xi)\,ds. \tag{4.1}$$

Assume that there exists t_0 such that $X(t_0) = 0$. Then

$$v(t_0) = \int_0^{t_0} X(t_0-s)\varphi(s)\,ds - \int_0^{t_0} X(t_0-s)\int_s^{+\infty} v(s-\xi)\,dr(\xi)\,ds \leqslant 0.$$

This fact is impossible because v is positive. Hence the fundamental solution of equation (2.1) is positive. □

Corollary 4.2. *If the conditions of Lemma 4.1 hold then $X(t) \geqslant \frac{v(t)}{v(0)}$.*

Proof. Since X is positive, and φ is nonpositive, the inequality $X(t) \geqslant \frac{v(t)}{v(0)}$ follows from equality (4.1). □

Define a function $P\colon \mathbb{R} \to \mathbb{R}$ by $P(\zeta) = -\zeta + a_0 + \int_0^{\omega} e^{\zeta\xi}\,dr(\xi)$.

Lemma 4.3. *The fundamental solution of equation (2.1) is positive on the semiaxis $[0, +\infty)$ if and only if the function P has at least one real zero.*

Proof. Necessity. The converse contradicts Lemma 3.5.

Sufficiency. Let $P(\zeta_0) = 0$ for some $\zeta_0 \in \mathbb{R}$. Let $v(t) = e^{-\zeta_0 t}$, $t \in \mathbb{R}$. We get

$$(Lv)(t) = e^{-\zeta_0 t}\left(-\zeta_0 + a_0 + \int_0^{\omega} e^{\zeta_0\xi}\,dr(\xi)\right) = e^{\zeta_0 t}P(\zeta_0).$$

From here by Lemma 4.1 we obtain that the fundamental solution of equation (2.1) is positive. □

The following lemmas provide some properties of the function P.

Lemma 4.4. $P'(\zeta) = -1 + \int_0^{\omega} \xi e^{\zeta\xi}\,dr(\xi)$, $P''(\zeta) = \int_0^{\omega} \xi^2 e^{\zeta\xi}\,dr(\xi)$.

Proof. This follows immediately from the definition of a derivative. □

Lemma 4.5. *The function P has at least one real zero if and only if it follows from $P'(\zeta^*) = 0$, where $\zeta^* \in \mathbb{R}$, that $P(\zeta^*) \leqslant 0$.*

Proof. Note that $P''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$. Hence the function P' increases on the whole axis. Moreover, $\lim_{\zeta\to-\infty} P'(\zeta) = -1$ and $\lim_{\zeta\to+\infty} P'(\zeta) = +\infty$. Therefore the function P has a unique minimum point ζ^*. If $P(\zeta^*) > 0$ then the function P has no zeros. If $P(\zeta^*) = 0$ then ζ^* is the unique zero of P. If $P(\zeta^*) < 0$ then the function P has exactly two zeros (to the right and to the left from ζ^*). □

By Lemma 4.3 and Lemma 4.5, we obtain the following theorem.

Theorem 4.6. *The following conditions are equivalent.*

1. *The fundamental solution of equation (2.1) is positive on the semiaxis $[0, +\infty)$.*
2. *The function P has at least one real zero.*
3. *If $P'(\zeta^*) = 0$ for $\zeta^* \in \mathbb{R}$, then $P(\zeta^*) \leqslant 0$.*

From Lemma 3.4 and Theorem 4.6 we obtain the following.

Theorem 4.7. *Let the solution of problem (2.2) be positive on the semiaxis $[0, +\infty)$. Then $e^{\zeta_0 t} \leqslant X(t) \leqslant Mt^n e^{\zeta_0 t}$, where $M \in \mathbb{R}_+$, ζ_0 is the maximal real zero of the function g, n is the maximal multiplicity of a root ζ such that $\Re\zeta = \zeta_0$.*

The following examples illustrate the use of Theorem 4.6.

Example 4.8. Consider an equation

$$\dot{x}(t) + k\int_0^{\omega} e^{\alpha s} x(t-s)\, ds = f(t), \tag{4.2}$$

where $\alpha, k \in \mathbb{R}$. Clearly, for $k \leqslant 0$ the fundamental solution of equation (4.2) is positive on the semiaxis $[0, +\infty)$. Let $k > 0$. Using Theorem 4.6 we obtain that the fundamental solution of equation (4.2) is positive on the semiaxis $[0, +\infty)$ if and only if $\alpha\omega \leqslant q(k\omega^2)$, where the function $u = q(v)$ is defined by

$$\left\{ u = \frac{\xi^2}{e^{\xi}(\xi-1)+1},\ v = \xi + \frac{\xi(1-e^{\xi})}{e^{\xi}(\xi-1)+1} \right\}.$$

The region of positiveness contains all points below and on the curve on Fig. 4.1.

For $\alpha\omega = 0$ we obtain the known result $k\omega^2 \leqslant \xi_0(2-\xi_0)$, where ξ_0 is the positive root of the equation $e^{-\zeta} = 1 - \frac{\xi}{2}$, that is $k\omega^2 \approx 0.65$ [8].

Example 4.9. Consider

$$\dot{x}(t) + k\int_0^{1} x(t-s)\, dc(s) = f(t), \tag{4.3}$$

where the function c is the Cantor function [12, p. 21].

Denote $\varphi(\zeta) = \int_0^1 e^{\zeta\xi}\, dc(\xi)$.

From item (3) of Theorem 4.6 we obtain that the fundamental solution of equation (4.3) is positive on the semiaxis $[0, +\infty)$ if and only if $k \leqslant k_0 = \frac{1}{\varphi'(\zeta_0)}$, where ζ_0 is the unique root of the equation $\frac{1}{\zeta} = w(\zeta)$.

Obviously, $c\left(\frac{s}{3}\right) = \frac{1}{2}c(s)$ for $s \in \left[0, \frac{1}{3}\right]$, $c\left(\frac{2}{3} + \frac{s}{3}\right) = \frac{1}{2} + \frac{1}{2}c(s)$ for $s \in \left[\frac{2}{3}, 1\right]$.

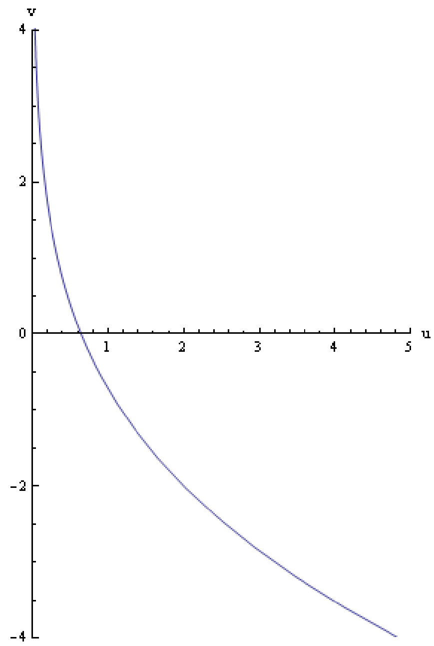

Figure 4.1: The function $u = q(v)$.

The function φ is a solution of the functional equation $\varphi(\zeta) = \frac{1+e^{\zeta\frac{2}{3}}}{2}\varphi\left(\frac{\zeta}{3}\right)$, $\zeta \in \mathbb{R}$. Therefore,

$$\varphi(\zeta) = \prod_{k=1}^{\infty} \frac{1+e^{\frac{2\zeta}{3^k}}}{2}, \qquad \varphi'(\zeta) = \varphi(\zeta) \sum_{k=1}^{\infty} \frac{2e^{\frac{2\zeta}{3^k}}}{3^k\left(e^{\frac{2\zeta}{3^k}}+1\right)}.$$

Hence,

$$w(\zeta) = \frac{\varphi'(\zeta)}{\varphi(\zeta)} = 2\sum_{k=1}^{\infty} \frac{1}{3^k\left(1+e^{-\frac{2\zeta}{3^k}}\right)}.$$

The product and sums in the expressions for the functions φ, φ', w converge for all ζ.

Notice that the function w has the following properties:

- $w(0) = \frac{1}{2}$,
- $\lim_{\zeta\to+\infty} w(\zeta) = 1$, $\lim_{\zeta\to-\infty} w(\zeta) = 0$,
- $w(\zeta) + w(-\zeta) = 1$,
- $w'(\zeta) > 0$,
- $w''(\zeta) < 0$ for $\zeta > 0$, $w''(\zeta) > 0$ for $\zeta < 0$, $w''(0) = 0$.

The approximate behavior of the function w is represented on Fig. 4.2.

Obviously, equation $\frac{1}{\zeta} = w(\zeta)$ has the unique root ζ_0. Since $\zeta_0 \approx 1.48$, we have $k_0 \approx 0.618$.

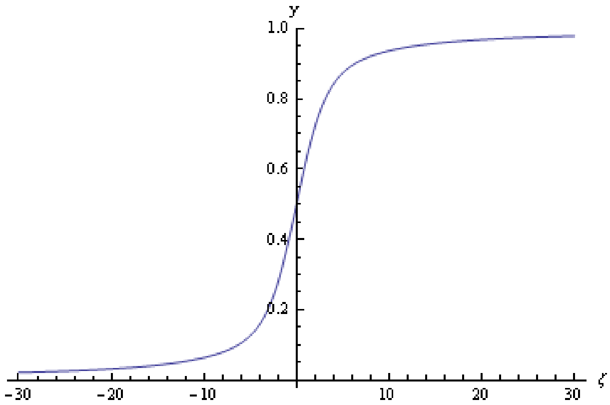

Figure 4.2: The function $y = w(\zeta)$.

5 Geometric representation

It is known [12, p. 22, Theorem I.14] that the function r is represented as a sum of a jump function, a continuous component, and a singular component. That is

$$\int_0^t x(t-s)\,dr(s) = \sum_{i=1}^{\infty} a_i x(t-r_i) + b_1 \int_0^t \tilde{\kappa}(s)x(t-s)\,ds + b_2 \int_0^t x(t-s)\,d\sigma(s).$$

We shall obtain the domain of positivity of the fundamental solution X in terms of the parameters $(a_0, b_1, b_2, a_1, a_2, \dots)$ for fixed κ, σ, r_i, $i \in \mathbb{N}$.

Let F be a set of points $(u_0, u_1, u_2, \dots) \in l_1$ such that

$$\begin{aligned} -\zeta + u_0 + \sum_{i=1}^{\infty} u_{i+2} e^{\zeta r_i} + u_1 \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta\xi}\,d\xi + u_2 \int_0^{\omega} e^{\zeta\xi}\,d\sigma(\xi) = 0, \\ -1 + \sum_{i=1}^{\infty} u_{i+2} r_i e^{\zeta r_i} + u_1 \int_0^{\omega} \tilde{\kappa}(\xi)\xi e^{\zeta\xi}\,d\xi + u_2 \int_0^{\omega} \xi e^{\zeta\xi}\,d\sigma(\xi) = 0. \end{aligned} \tag{5.1}$$

Take $A = (a_0, 0, \dots) \in l_1$, $M = (a_0, b_1, b_2, a_1, a_2, a_3, \dots) \in l_1$. Then parametric equations

$$u_0 = a_0, \quad u_{i+2} = a_i\tau, \quad i \in \mathbb{N}, \quad u_1 = b_1\tau, \quad u_2 = b_2\tau, \quad \tau \geqslant 0, \tag{5.2}$$

define the ray AM. Combining (5.2) and (5.1), we get

$$\begin{aligned} -\zeta + a_0 + \tau \sum_{i=1}^{\infty} a_i e^{\zeta r_i} + b_1\tau \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta\xi}\,d\xi + b_2\tau \int_0^{\omega} e^{\zeta\xi}\,d\sigma(\xi) = 0, \\ -1 + \tau \sum_{i=1}^{\infty} a_i r_i e^{\zeta r_i} + b_1\tau \int_0^{\omega} \tilde{\kappa}(\xi)\xi e^{\zeta\xi}\,d\xi + b_2\tau \int_0^{\omega} \xi e^{\zeta\xi}\,d\sigma(\xi) = 0. \end{aligned}$$

Clearly, the ray AM intersects the set F if and only if the equation

$$\begin{aligned} -(\zeta - a_0)\left(\sum_{i=1}^{\infty} a_i r_i e^{\zeta r_i} + b_1 \int_0^{\omega} \tilde{\kappa}(\xi)\xi e^{\zeta\xi}\,d\xi + b_2 \int_0^{\omega} \xi e^{\zeta\xi}\,d\sigma(\xi)\right) \\ + \sum_{i=1}^{\infty} a_i e^{\zeta r_i} + b_1 \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta\xi}\,d\xi + b_2 \int_0^{\omega} e^{\zeta\xi}\,d\sigma(\xi) = 0 \end{aligned} \tag{5.3}$$

is solvable for ζ.

By ψ we denote the left side of (5.3). Differentiating ψ with respect to ζ, we obtain

$$\psi'(\zeta) = -(\zeta - a_0)\left(\sum_{i=1}^{\infty} a_i r_i^2 e^{\zeta r_i} + b_1 \int_0^{\omega} \tilde{\kappa}(\xi)\xi^2 e^{\zeta\xi}\, d\xi + b_2 \int_0^{\omega} \xi^2 e^{\zeta\xi}\, d\sigma(\xi)\right).$$

The function ψ' has a unique zero, $\psi'(\zeta) > 0$ for $\zeta < a_0$, and $\psi'(\zeta) < 0$ for $\zeta > a_0$. Hence the function ψ has the unique maximum $\zeta = a_0$. The function ψ is positive on $(-\infty, a_0)$. If $\zeta > a_0$ then ψ decreases towards $-\infty$. It follows that ψ has a unique root ζ_0. Therefore the ray AM intersects the set F in the unique point ζ_0, and

$$\tau = \tau_0 = \left(\sum_{i=1}^{\infty} a_i r_i e^{\zeta_0 r_i} + b_1 \int_0^{\omega} \tilde{\kappa}(\xi)\xi e^{\zeta_0\xi}\, d\xi + b_2 \int_0^{\omega} \xi e^{\zeta_0\xi}\, d\sigma(\xi)\right)^{-1}.$$

We say that the set F is a *surface*, and the point M is *below* the surface F if $\tau_0 > 1$, the point M is *on* the surface F if $\tau_0 = 1$, the point M is *above* the surface F if $\tau_0 < 1$.

Lemma 5.1. *The function P has at least one real zero if and only if the point M is not above the surface F.*

Proof. Necessity. Assume the converse. Then the point M is above the surface F, but the function P has at least one real zero. This means that there exist parameters ζ_0 and $\tau_0 < 1$ such that $-\zeta_0 + a_0 + \tau_0 \sum_{i=1}^{\infty} a_i e^{\zeta_0 r_i} + b_1\tau_0 \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta_0\xi}\, d\xi + b_2\tau_0 \int_0^{\omega} e^{\zeta_0\xi}\, d\sigma(\xi) = 0$.

Consider the following function $Q\colon \mathbb{R} \to \mathbb{R}$ and its derivatives

$$Q(\zeta) = -\zeta + a_0 + \tau_0 \sum_{i=1}^{\infty} a_i e^{\zeta r_i} + b_1\tau_0 \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta\xi}\, d\xi + b_2\tau_0 \int_0^{\omega} e^{\zeta\xi}\, d\sigma(\xi),$$
$$Q'(\zeta) = -1 + \tau_0 \sum_{i=1}^{\infty} a_i r_i e^{\zeta r_i} + b_1\tau_0 \int_0^{\omega} \tilde{\kappa}(\xi)\xi e^{\zeta\xi}\, d\xi + b_2\tau_0 \int_0^{\omega} \xi e^{\zeta\xi}\, d\sigma(\xi),$$
$$Q''(\zeta) = \tau_0 \sum_{i=1}^{\infty} a_i r_i^2 e^{\zeta r_i} + b_1\tau_0 \int_0^{\omega} \tilde{\kappa}(\xi)\xi^2 e^{\zeta\xi}\, d\xi + b_2\tau_0 \int_0^{\omega} \xi^2 e^{\zeta\xi}\, d\sigma(\xi).$$

Clearly, $Q''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$. Hence the function Q' increases. In addition, $Q'(\zeta_0) = 0$ and the function Q' changes sign from $-$ to $+$. Therefore ζ_0 is the unique minimum of the function Q.

Let ζ^* be the minimum of the function P. Then taking into account the inequality $\tau_0 < 1$ we have $P(\zeta^*) > Q(\zeta^*) \geqslant Q(\zeta_0) = 0$. Hence the continuous function P is positive at the minimum. Thus it has no zeros. This contradiction concludes the proof.

Sufficiency. If the point M is not above the surface F then there exist $\zeta_0 > 0$ and $\tau_0 \geqslant 1$ such that $-\zeta_0 + a_0 + \tau_0 \sum_{i=1}^{\infty} a_i r_i e^{\zeta_0 r_i} + b_1\tau_0 \int_0^{\omega} \tilde{\kappa}(\xi)\xi e^{\zeta_0\xi}\, d\xi + b_2\tau_0 \int_0^{\omega} \xi e^{\zeta_0\xi}\, d\sigma(\xi) = 0$. Hence

$$P(\zeta_0) = -\zeta_0 + a_0 + \sum_{i=1}^{\infty} a_i e^{\zeta_0 r_i} + b_1 \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta_0\xi}\, d\xi + b_2 \int_0^{\omega} e^{\zeta_0\xi}\, d\sigma(\xi)$$
$$\leqslant -\zeta_0 + a_0 + \tau_0 \sum_{i=1}^{\infty} a_i e^{\zeta_0 r_i} + b_1\tau_0 \int_0^{\omega} \tilde{\kappa}(\xi) e^{\zeta_0\xi}\, d\xi + b_2\tau_0 \int_0^{\omega} e^{\zeta_0\xi}\, d\sigma(\xi) = 0.$$

That is $P(\zeta_0) \leqslant 0$. Also, $\lim_{\zeta\to+\infty} P(\zeta) = +\infty$. Therefore the function P has at least one real zero. $\square$

Theorem 5.2. *The following conditions are equivalent.*

1. *The solution of problem* (2.2) *is positive on the semiaxis* $[0, +\infty)$.
2. *The function* P *has at least one real zero.*
3. *If* $P'(\zeta^*) = 0$ *for* $\zeta^* \in \mathbb{R}$, *then* $P(\zeta^*) \leqslant 0$.
4. *The point* M *is not above the surface* F.

Proof. Theorem 5.2 follows from Theorem 4.6 and Lemma 5.1. □

6 Monotonicity of the fundamental solution

In this section we study the monotonicity of the fundamental solution making use of its positiveness.

Lemma 6.1. *Let* x *and* y *be solutions of homogeneous equation* (2.1) *(and may have different initial points). If* $x(t_0) = y(t_0)$, *and* $x(t) \leq y(t)$ *for all* $t \in [t_0 - \omega, t_0)$, *then for any* $\varepsilon > 0$ *there exists* $t_1 \in (t_0, t_0 + \varepsilon)$ *such that* $x(t_1) \geqslant y(t_1)$.

Proof. Assume the converse. That is, suppose the conditions of Lemma 6.1 hold, however for some $\delta > 0$ for all $t \in (t_0, t_0 + \delta)$ we have $x(t) < y(t)$.

Denote $z = y - x$. We have $z(t_0) = 0$, and $z(\tau) \geq 0$ for all $\tau \in [t_0 - \omega, t_0 + \delta)$.

For every $t \in (t_0, t_0 + \delta)$ we obtain

$$z(t) = z(t) - z(t_0) = \int_{t_0}^{t} \dot{z}(s)ds = a_0 \int_{t_0}^{t} z(s)ds - \int_{t_0}^{t} \int_{0}^{\omega} z(t-\tau)\, dr(\tau)\, ds \leqslant a_0 \int_{t_0}^{t} z(s)ds.$$

Hence, if $a_0 \geqslant 0$ then $z(t) \leqslant z(t_0)e^{a_0(t-t_0)} = 0$; if $a_0 < 0$ then $z(t) \leqslant a_0 \int_{t_0}^{t} z(s)ds < 0$. This contradicts the assumption and concludes the proof. □

Lemma 6.2. *If* $X(t) > 0$ *for all* $t \geqslant 0$ *then for every interval* $[\alpha, \beta]$, *where* $0 \leqslant \alpha < \beta$, *the function* X *may have a strict minimum in* $[\alpha, \beta]$ *only at points* α *or* β.

Proof. Assume the fundamental solution X has minimum in $[\alpha, \beta]$ at a point $t_1 \in (\alpha, \beta)$. Fix $\delta \in (0, \beta - t_1)$ such that $X(t) > X(t_1)$ for all $t \in (t_1, t_1 + \delta)$. Fix also $t_0 \in \{t \in [0, t_1) \mid \forall s \in [0, t_1)\ (X(t) \geq X(s))\}$ (this set is not empty since X is continuous).

Put $\gamma = X(t_1)/X(t_0)$. Obviously, $\gamma \in (0, 1)$.

Put $Y(t) = \gamma X(t - (t_1 - t_0))$. Since homogeneous equation (2.1) is autonomous, the function Y is its solution (with initial condition $Y(t_1 - t_0) = \gamma$).

Note that $Y(t_1) = X(t_1)$, and $Y(t) < X(t)$ for $t \in [t_1 - \omega, t_1) \cup (t_1, t_1 + \delta)$ (assuming $X(t) = 0$ for $t < 0$, and $Y(t) = 0$ for $t < t_1 - t_0$). Indeed,

$$Y(t) = X(t_1)\frac{X(t - (t_1 - t_0))}{X(t_0)} \leq X(t_1) < X(t).$$

Hence, by Lemma 6.1 there exists $t_2 \in (t_1, t_1 + \delta)$ such that $Y(t_2) \geqslant X(t_2)$. This contradiction concludes the proof. □

From Lemma 6.2 we obtain the following theorem.

Theorem 6.3. *If $X(t) > 0$ for all $t \geqslant 0$ then one of the following propositions holds:*

- *X increases on $(0, +\infty)$;*
- *X decreases on $(0, +\infty)$;*
- *there exists $t_0 \in (0, +\infty)$ such that X increases on $(0, t_0)$ and decreases on $(t_0, +\infty)$.*

We apply Theorem 6.3 to study the fundamental solution of equation (2.1).

If $a_0 \geqslant 0$, $a_0 + \rho \neq 0$, and $X(t) > 0$ for all $t \geqslant 0$, then $\dot{X}(t) < 0$. Hence the fundamental solution X strictly decreases and has a nonnegative limit at infinity. Denote it by c.

Rewrite the equation from problem (2.2) in the form:

$$X(t) - X(0) = -\int_0^t \left(a_0 X(\xi) + \int_0^\xi X(\xi - s)\, dr(s) \right) d\xi.$$

Since $X(\tau) > c$ for some $c \geqslant 0$ and for every $\tau \in [0, t]$, we get

$$X(0) - X(t) \geqslant c \int_0^t \left(a_0 + \int_0^\xi dr(s) \right) d\xi.$$

Assume that $c > 0$. Then the function at the left-hand side of the last inequality is bounded, and the one at the right-hand side is unbounded. We have come to contradiction. Thus, X converges to 0. By Theorem 6.3 there exists $t_0 \in [0, +\infty)$ such that X decreases on $(t_0, +\infty)$.

Consider now the case $a_0 < 0$.

Suppose $a_0 < -\rho$. Then $P(0) = a_0 + \int_0^\omega dr(\xi) < 0$, $\lim_{\zeta \to -\infty} P(\zeta) = +\infty$. Hence P has a real zero ζ_0. Applying Theorem 5.2, we get that $X(t) > 0$ for all $t \geqslant 0$. Since $\zeta_0 < 0$, from Theorem 6.3 X increases on $(0, +\infty)$.

Suppose $a_0 < 0$, $a_0 > -\rho$, $-1 + \int_0^\omega \xi\, dr(\xi) < 0$, and $X(t) > 0$ for all $t \geqslant 0$. Then $P(0) > 0$, $P'(0) < 0$. Hence the real zeros of P are positive. Therefore the real zeros of g are negative. By Lemma 3.5 we get that all zeros of the function g are in the left half-plane. Thus, by Lemma 3.4 $X(t)$ converges to 0. By Theorem 6.3 there exists $t_0 \in (0, +\infty)$ such that X increases on $(0, t_0)$ and decreases on $(t_0, +\infty)$.

Suppose $a_0 < 0$, $a_0 > -\rho$, $-1 + \int_0^\omega \xi\, dr(\xi) > 0$, and $X(t) > 0$ for all $t \geqslant 0$. Then $P(0) > 0$ and $P'(0) > 0$. Hence the real zeros of P are negative. Therefore the real zeros of g are positive. Thus, from Lemma 3.4 X is unstable. From Theorem 6.3 X increases on $(0, +\infty)$.

Suppose $a_0 = -\rho$, $-1 + \int_0^\omega \xi\, dr(\xi) < 0$. Then the function P has two real zeros, one of them is 0 and the other is positive. From Theorem 5.2 we get that $X(t) > 0$ for all $t \geqslant 0$. The real zeros of g are nonpositive. From Lemma 3.5 we get that all zeros of the function g have nonpositive real part. Thus, from Lemma 3.4 X is bounded.

Suppose $a_0 = -\rho$, $-1 + \int_0^\omega \xi\, dr(\xi) > 0$. Then the function P has two real zeros, one of them is 0 and the other is negative. From Theorem 5.2 we get that $X(t) > 0$ for all $t \geqslant 0$. The real zeros of g are nonnegative. Thus, from Lemma 3.4 X is unstable. From Theorem 6.3 X increases on $(0, +\infty)$.

Suppose $a_0 = -\rho$, $\int_0^\omega \xi\, dr(\xi) = 1$. Then the function P has a unique real zero. From Theorem 5.2 we get that $X(t) > 0$ for all $t \geqslant 0$. From Lemma 3.5 we get that all zeros of the function g have nonpositive real part. However, X is not bounded. For example, the fundamental solution of $\dot{x}(t) + ax(t) + bx(t-1) = 0$ increases linearly (see [10]).

7 Example

Consider an equation

$$\dot{x}(t) + a_0 x(t) + a_1 x(t-1) + a_2 x(t-3) + k\int_{t-2}^{t} x(s)\,ds = f(t). \tag{7.1}$$

In this case the surface F has the form

$$F = \left\{ \begin{array}{l} (u_0, u_1, u_2, u_3) \in \mathbb{R} \times \mathbb{R}^3_+ : \\ u_2 = \frac{u_1}{2\zeta^2}\left(e^{-\zeta}(3\zeta+1) - e^{\zeta}(\zeta+1)\right) + \frac{1}{2}\left(3(\zeta - u_0) - 1\right) e^{-\zeta}, \\ u_3 = -\frac{u_1}{2\zeta^2}\left(e^{-3\zeta}(\zeta+1) + e^{-\zeta}(\zeta-1)\right) - \frac{1}{2}\left(\zeta - 1 - u_0\right) e^{-3\zeta}. \end{array} \right\}$$

Let us represent the four-parameter set F as a family of three-parameter ones with u_0 fixed. Denote cut sets for fixed $u_0 = a_0$ by F_{a_0}. The surfaces F_{a_0} for different a_0 are colored blue on Fig. 7.1 and on Fig. 7.2. In addition, the meet of the hyperplanes $u_0 + 2u_1 + u_2 + u_3 = 0$ and $u_0 = a_0$ is a three-dimensional plane (it is painted green on Fig. 7.2).

The fundamental solution of equation (7.1) is positive on the semiaxis $[0, +\infty)$ if and only if the point $M(k, a_1, a_2)$ is not above the surface F_{a_0}.

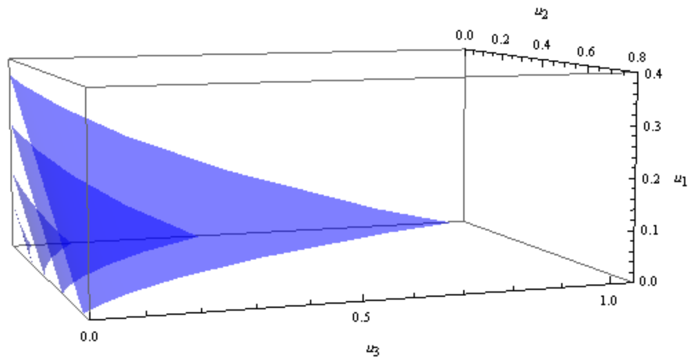

Figure 7.1: The surfaces F_{a_0} for $a_0 = 0.4; 0; -0.4; -0.7$.

Using the results of section 6 one can obtain more accurate information on the behavior of the fundamental solution of equation (7.1).

If $a_0 \geqslant 0$, $a_0 + 2k + a_1 + a_2 \neq 0$, and the point $M(k, a_1, a_2)$ is not above the surface F_{a_0}, then the fundamental solution of equation (7.1) is positive and strictly decreasing on the semiaxis $[0, +\infty)$.

Suppose $a_0 < 0$. Let $D_{a_0} \in \mathbb{R}^3_+$ be a region with fixed a_0 such that for every point $M(k, a_1, a_2)$ belonging to D_{a_0} there exists $t_0 \in (0, +\infty)$ such that the fundamental solution of equation (7.1) increases for $t \in (0, t_0)$ and decreases for $t \in (t_0, +\infty)$. The dynamics of D_{a_0} is shown on Fig. 7.2, its bounds have more saturated color. Note that F_{a_0} belongs to D_{a_0}, and the plane does not belong to D_{a_0}. The surface F_{a_0} and the plane touch with the line $\{a_0 + 2u_1 + u_2 + u_3 = 0, 4u_1 + u_2 + 3u_3 = 1\}$ for $-\frac{1}{3} \leqslant a_0 \leqslant -1$ (the blue line on Fig. 7.2). If $a_0 \geqslant -\frac{1}{3}$, then F_{a_0} and the plane do not intersect. For $a_0 < -1$ D_{a_0} does not exist.

If $M(k, a_1, a_2)$ belongs to the part of the plane painted the saturated color, then the fundamental solution of equation (7.1) is bounded. If $M(k, a_1, a_2)$ belongs to the blue line, then the fundamental solution of equation (7.1) increases linearly. The fundamental solution of equation (7.1) strictly increases for other a_0, a_1, a_2, k.

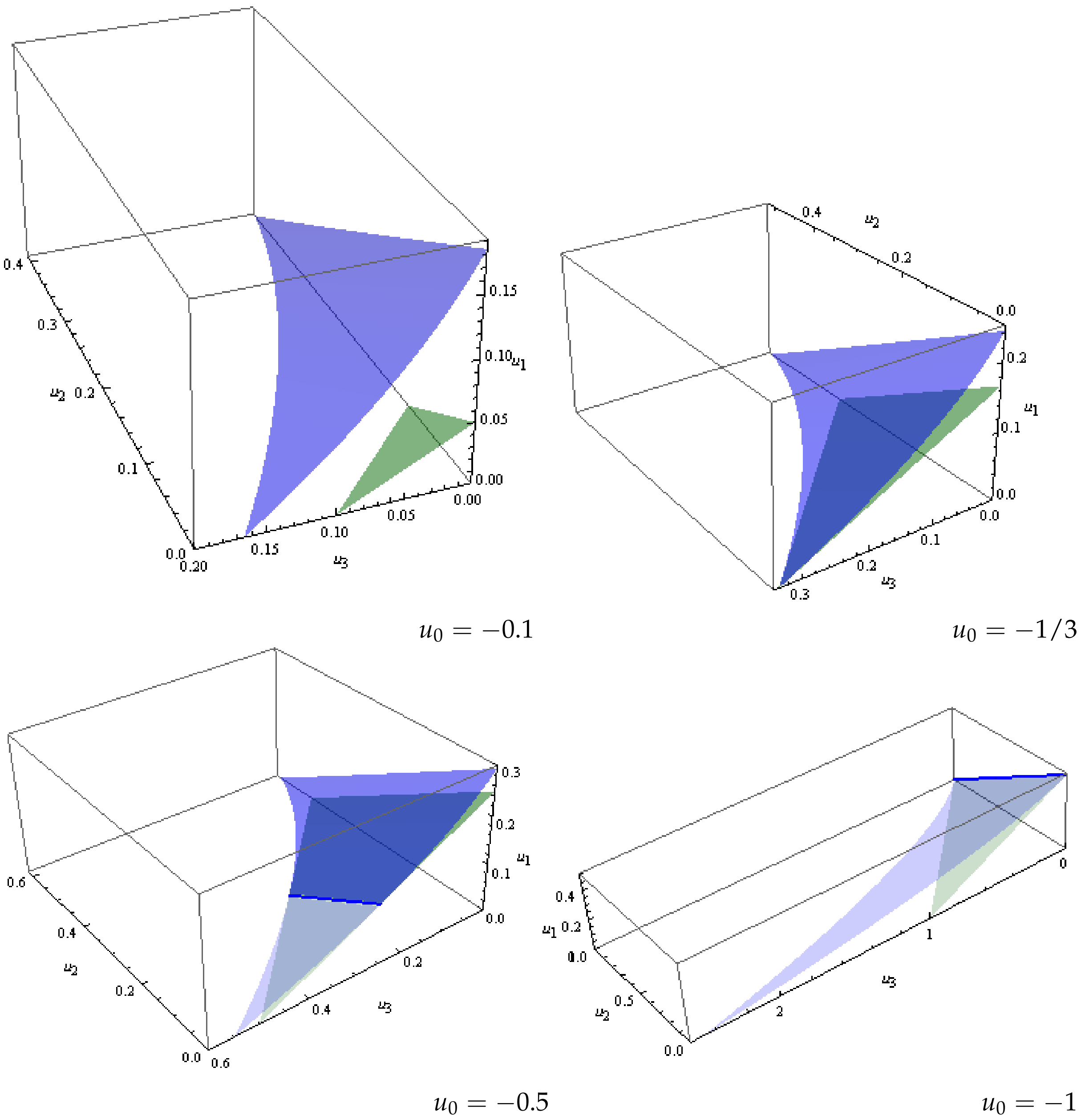

Figure 7.2: Dynamics of the regions D_{a_0}.

8 Generalization

Denote $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geqslant s\}$.

Consider an equation

$$\dot{x}(t) + a_0(t)x(t) + \sum_{i=1}^{\infty} a_i(t)x(t - r_i(t)) + \int_0^t k(t,s)x(t-s)\,ds = f(t), \tag{8.1}$$

where $a_0\colon \mathbb{R}_+ \to \mathbb{R}$, $a_i\colon \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \mathbb{N}$, the functions a_0, a_i, $i \in \mathbb{N}$, are measurable, $k\colon \Delta \to \mathbb{R}_+$, the function $k(\cdot,s)$ is measurable, the function $k(t,\cdot)$ is locally summable, the function f is locally summable. The delay is bounded, i.e. there exists $\omega \in \mathbb{R}_+$ such that $k(t,s) = 0$ for all $s > \omega$ and $r_i(t) \leqslant \omega$ for all $i \in \mathbb{N}$. For every $\xi < 0$ we suppose $x(\xi) = 0$.

For example, let $a_0 = \sup_t a_0(t)$, $a_i = \sup_t a_i(t)$, $r_i = \sup_t r_i(t)$, $b\tilde{\kappa}(s) = \sup_t k(t,s)$, $a_0 \in \mathbb{R}$, $a_i, r_i \in \mathbb{R}_+$, $i \in \mathbb{N}$, the series $\sum_{i=1}^{\infty} a_i$ be convergent. Then using the Theorem on a differential inequality we come again to the analysis of the function P. Therefore if the point $M(a_0, b, 0, a_1, a_2, a_3, \ldots)$ is not above the surface F, then the Cauchy function of equation (8.1) is positive on Δ. Note that this condition of positiveness is sufficient, but is not necessary and sufficient like the analogous condition for autonomous equation (2.1).

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