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Existence of solutions for fractional differential equations with three-point boundary conditions at resonance in $\mathbb{R}^n$

Fu-Dong Ge[1] and **Hua-Cheng Zhou**[✉2]

[1]College of Information Science and Technology, Donghua University, Shanghai 201620, P. R. China
[2]Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, P. R. China

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Abstract. In this paper, by applying the coincidence degree theory which was first introduced by Mawhin, we obtain an existence result for the fractional three-point boundary value problems in $\mathbb{R}^n$, where the dimension of the kernel of fractional differential operator with the boundary conditions can take any value in $\{1, 2, \dots, n\}$. This is our novelty. Several examples are presented to illustrate the result.

Keywords: fractional differential equations, resonance, coincidence degree theory.

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1 Introduction

In this paper, we are concerned with the existence of solutions for the following fractional three-point boundary value problems (BVPs) at resonance in $\mathbb{R}^n$:

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f(t, x(t), D_{0^+}^{\alpha-1} x(t)), \quad 1 < \alpha \leq 2, \ t \in (0,1), \\ x(0) = \theta, \quad D_{0^+}^{\alpha-1} x(1) = A D_{0^+}^{\alpha-1} x(\xi), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ and $I_{0^+}^{\alpha}$ are the Riemann–Liouville differentiation and integration; θ is the zero vector in $\mathbb{R}^n$; A is a square matrix of order n satisfying $\operatorname{rank}(I - A) < n$; $\xi \in (0,1)$ is a fixed constant; $f\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, that is,

(i) for each $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \mapsto f(t, u, v)$ is measurable on $[0,1]$;

(ii) for a.e. $t \in [0,1]$, $(u, v) \mapsto f(t, u, v)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$;

(iii) for every compact set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$, the function $\varphi_\Omega(t) = \sup\{|f(t, u, v)| : (u, v) \in \Omega\} \in L^1[0,1]$, where $|x| = \max\{|x_i|, \ i = 1, 2, \dots, n\}$, the norm of $x = (x_1, x_2, \dots, x_n)^\top$ in $\mathbb{R}^n$.

✉Corresponding author. Email: hczhou@amss.ac.cn

The system (1.1) is said to be at resonance in $\mathbb{R}^n$ if $\det(I-A)=0$, i.e., $\dim\ker(I-A)\geq 1$, otherwise, it is said to be non-resonant. In the past three decades, many authors investigated the existence of solutions for the fractional differential equations with the boundary value conditions. The attempts on $\det(I-A)\neq 0$, non-resonance case, for fractional differential equations are available in [1–3, 10, 11, 17, 21–23], and the attempts on $\det(I-A)=0$ and $n\leq 2$, resonance case, can be found in [4–6, 8, 9, 13, 14, 18–20], and the references therein. However, to the best of our knowledge, almost all results derived in these papers are for the case $n=1$ with $\dim\ker L=0$ or 1 and for the case $n=2$ with $\dim\ker L=2$. It is still open for the case $n\geq 3$. So we study this issue in this paper.

For instance, when $n=1$, consider the following problems

$$\begin{cases} D^{\alpha}_{0^+}u(t)=f(t,u(t),D^{\alpha-1}_{0^+}u(t)), \quad 0<t<1, \\ I^{2-\alpha}_{0^+}u(t)|_{t=0}=0, \quad D^{\alpha-1}_{0^+}u\,(1)=\sum\limits_{i=1}^{m-2}\beta_i D^{\alpha-1}_{0^+}u\,(\xi_i) \end{cases} \tag{1.2}$$

where $1<\alpha\leq 2$, $\xi_i\in(0,1)$, $\beta_i\in\mathbb{R}$ $i=1,2,\dots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_i<1$, $f\colon[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. It follows from the argument above that (1.2) is resonant when $\sum_{i=1}^{m-2}\beta_i=1$ and it is non-resonant when $\sum_{i=1}^{m-2}\beta_i\neq 1$.

Further, in order to apply the coincidence degree theory of Mawhin [15], we suppose additionally that A satisfies $\operatorname{rank}(I-A)<n$ and one of the following conditions

(a_1) A is idempotent, that is, $A^2=A$, or;

(a_2) $A^2=I$, where I stands for the identity matrix of size n.

It is also obvious that $\dim\ker(I-A)$ can take any value in $\{1,2\dots,n\}$ for suitable A, which is surely generalize the previous efforts. However, we point out that without the above assumptions, it will be difficult to construct the projector Q as (3.1) below. This is the reason why we only choose the two special cases of A. Removing such an assumption, for the general A satisfying $\operatorname{rank}(I-A)<n$, the problem (1.1) may be a challenging problem, which is also an issue of our further research.

In particular, when $A=I$, it is clear that A satisfies (a_1) and (a_2). It is also obvious that $\det(I-A)=0$, so under this boundary condition, the system (1.1) is at resonance. Besides, $\ker L=\{(c_1,c_2,\dots,c_n)^{\top}t^{\alpha-1}:c_i\in\mathbb{R}, i=1,2,\dots,n\}$ and $\dim\ker L=n$, where L is defined by (2.2) below. For $A=0$, it is clear that $\det(I-A)=1$, $\ker L=\{0\}$, so, in this case, the system (1.1) is non-resonant.

In paper[16], the authors investigated the following second differential system

$$\begin{cases} u''(t)=f(t,u(t),u'(t)), \quad 0<t<1, \\ u'(0)=\theta, \quad u(1)=Au(\eta), \end{cases} \tag{1.3}$$

where $f\colon[0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathéodory function and the square matrix A satisfies the condition (a_1) or (a_2). Therefore, it is more natural to ask whether there exists a solution when the order of the derivative is fractional. In this paper, we offer an answer by considering the system (1.1).

The goal of this paper is to study the existence of solutions for the fractional differential equations with boundary value conditions when $n\geq 3$. The layout of this paper will be as follows: in Section 2, we give some necessary background and some preparations for our consideration. The statement and the proof of our main result will be given in Section 3 by the coincidence degree theory of Mawhin [15]. In Section 4, we present several examples to illustrate the main result.

2 Background materials and preliminaries

In this section, we introduce some necessary definitions and lemmas which will be used later. For more details, we refer the reader to [7,12,15], and the references therein.

Definition 2.1 ([12])**.** The fractional integral of order $\alpha > 0$ of a function $x\colon (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Remark 2.2. The notation $I_{0+}^{\alpha} x(t)|_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0,\varepsilon)$ $(\varepsilon > 0)$ of 0 as follows:

$$I_{0+}^{\alpha} x(t)|_{t=0} = \lim_{t\to 0^+} I_{0+}^{\alpha} x(t).$$

Generally, $I_{0+}^{\alpha} x(t)|_{t=0}$ is not necessarily zero. For instance, let $\alpha \in (0,1)$, $x(t) = t^{-\alpha}$. Then

$$I_{0+}^{\alpha} t^{-\alpha}|_{t=0} = \lim_{t\to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha}\, ds = \lim_{t\to 0^+} \Gamma(1-\alpha) = \Gamma(1-\alpha).$$

Definition 2.3 ([12])**.** The fractional derivative of order $\alpha > 0$ of a function $x\colon (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha]+1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.4 ([12])**.** *Assume that $x \in C(0,+\infty) \cap L_{loc}(0,+\infty)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,+\infty) \cap L_{loc}(0,+\infty)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-N},$$

for some $c_i \in \mathbb{R}$, $i = 1,\dots,N$, where N is the smallest integer greater than or equal to α.

For any $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top \in \mathbb{R}^n$, the fractional derivative of order $\alpha > 0$ of x is defined by

$$D_{0+}^{\alpha} x(t) = (D_{0+}^{\alpha} x_1(t), D_{0+}^{\alpha} x_2(t), \dots, D_{0+}^{\alpha} x_n(t))^\top \in \mathbb{R}^n.$$

The following definitions and coincidence degree theory are fundamental in the proof of our main result. One can refer to [7,15].

Definition 2.5. Let X and Y be normed spaces. A linear operator $L\colon \operatorname{dom}(L) \subset X \to Y$ is said to be a Fredholm operator of index zero provided that

(i) $\operatorname{im} L$ is a closed subset of Y, and

(ii) $\dim\ker L = \operatorname{codim}\operatorname{im} L < +\infty$.

It follows from Definition 2.5 that there exist continuous projectors $P\colon X \to X$ and $Q\colon Y \to Y$ such that

$$\operatorname{im} P = \ker L, \quad \ker Q = \operatorname{im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{im} L \oplus \operatorname{im} Q$$

and the mapping $L|_{\operatorname{dom} L \cap \ker P}\colon \operatorname{dom} L \cap \ker P \to \operatorname{im} L$ is invertible. We denote the inverse of $L|_{\operatorname{dom} L \cap \ker P}$ by $K_P\colon \operatorname{im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q}\colon Y \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I-Q)$. Furthermore, for every isomorphism $J\colon \operatorname{im} Q \to \ker L$, we can obtain that the mapping $K_{P,Q} + JQ\colon Y \to \operatorname{dom} L$ is also an isomorphism and for all $x \in \operatorname{dom} L$, we know that

$$(K_{P,Q} + JQ)^{-1}x = (L + J^{-1}P)x. \tag{2.1}$$

Definition 2.6. Let L be a Fredholm operator of index zero, let $\Omega \subseteq X$ be a bounded subset and $\operatorname{dom} L \cap \Omega \neq \emptyset$. Then the operator $N\colon \overline{\Omega} \to Y$ is called to be L-compact in $\overline{\Omega}$ if

(i) the mapping $QN\colon \overline{\Omega} \to Y$ is continuous and $QN(\overline{\Omega}) \subseteq Y$ is bounded, and

(ii) the mapping $K_{P,Q}N\colon \overline{\Omega} \to X$ is completely continuous.

Assume that L is defined in Definition 2.6 and $N\colon \overline{\Omega} \to Y$ is L-compact. For any $x \in \overline{\Omega}$, by (2.1), we shall see that

$$\begin{aligned} Lx = (K_{P,Q} + JQ)^{-1}x - J^{-1}Px &= (K_{P,Q} + JQ)^{-1}\left[Ix - K_{P,Q}J^{-1}Px - JQJ^{-1}Px\right] \\ &= (K_{P,Q} + JQ)^{-1}(Ix - Px). \end{aligned}$$

Then we can equivalently transform the existence problem of the equation $Lx = Nx$, $x \in \overline{\Omega}$ into a fixed point problem of the operator $P + (K_{P,Q} + JQ)N$ in $\overline{\Omega}$.

This can be guaranteed by the following lemma, which is also the main tool in this paper.

Lemma 2.7 ([15]). *Let $\Omega \subset X$ be bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0,1)$;

(ii) $Nx \notin \operatorname{im} L$ for every $x \in \ker L \cap \partial\Omega$;

(iii) $\deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q\colon Y \to Y$ a continuous projector such that $\ker Q = \operatorname{im} L$ and $J\colon \operatorname{im} Q \to \ker L$ is an isomorphism.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

In this paper, we utilize spaces X, Y introduced as

$$X = \left\{ x(t) \in \mathbb{R}^n : x(t) = I_{0+}^{\alpha-1}u(t),\ u \in C([0,1],\mathbb{R}^n),\ t \in [0,1] \right\}$$

with the norm $\|x\| = \max\{\|x\|_\infty, \|D_{0+}^{\alpha-1}x\|_\infty\}$ and $Y = L^1([0,1],\mathbb{R}^n)$ with the norm $\|y\|_1 = \int_0^1 |y(s)|\,ds$, respectively, where $\|\cdot\|_\infty$ represents the sup-norm.

We have the following compactness criterion on subset F of X (see, e.g., [19]).

Lemma 2.8. *$F \subset X$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous which are understood in the following sense:*

(1) there exists an $M > 0$ such that for every $x \in F$, $\|x\| \leq M$;

(2) for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x(t_1) - x(t_2)| < \varepsilon, \quad |D_{0+}^{\alpha-1}x(t_1) - D_{0+}^{\alpha-1}x(t_2)| < \varepsilon, \quad \text{for } t_1, t_2 \in [0,1], |t_1 - t_2| < \delta,\ \forall x \in F.$$

Now we define the linear operator $L\colon \operatorname{dom} L \subseteq X \to Y$ by

$$Lx := D^{\alpha}_{0+}x, \tag{2.2}$$

where $\operatorname{dom} L = \left\{x \in X : D^{\alpha}_{0+}x \in Y,\ x(0) = \theta,\ D^{\alpha-1}_{0+}x(1) = AD^{\alpha-1}_{0+}x(\xi)\right\}$. Define $N\colon X \to Y$ by

$$Nx(t) := f(t, x(t), D^{\alpha-1}_{0+}x(t)), \quad t \in [0,1]. \tag{2.3}$$

Then the problem can be equivalently rewritten as $Lx = Nx$.

Lemma 2.9. *The operator L defined above is a Fredholm operator of index zero.*

Proof. For any $x \in \operatorname{dom} L$, by Lemma 2.4 and $x(0) = \theta$, we obtain

$$x(t) = I^{\alpha}_{0+}Lx(t) + ct^{\alpha-1}, \quad c \in \mathbb{R}^n,\ t \in [0,1], \tag{2.4}$$

which, together with $D^{\alpha-1}_{0+}x(1) = AD^{\alpha-1}_{0+}x(\xi)$, yields

$$\ker L = \{x \in X : x(t) = ct^{\alpha-1},\ t \in [0,1],\ c \in \ker(I-A)\} \backsimeq \ker(I-A)t^{\alpha-1}. \tag{2.5}$$

Now we claim that

$$\operatorname{im} L = \{y \in Y : g(y) \in \operatorname{im}(I-A)\}, \tag{2.6}$$

where $g\colon Y \to \mathbb{R}^n$ is a continuous linear operator defined by

$$g(y) := \frac{A}{\Gamma(\alpha)}\int_0^{\xi} y(s)\,ds - \frac{I}{\Gamma(\alpha)}\int_0^1 y(s)\,ds. \tag{2.7}$$

Actually, for any $y \in \operatorname{im} L$, there exists a function $x \in \operatorname{dom} L$ such that $y = Lx$. It follows from (2.4) that $x(t) = I^{\alpha}_{0+}y(t) + ct^{\alpha-1}$. Together with $D^{\alpha-1}_{0+}x(1) = AD^{\alpha-1}_{0+}x(\xi)$, we obtain

$$\frac{A}{\Gamma(\alpha)}\int_0^{\xi} y(s)\,ds - \frac{I}{\Gamma(\alpha)}\int_0^1 y(s)\,ds = (I-A)c, \quad c \in \mathbb{R}^n,$$

which means that $g(y) \in \operatorname{im}(I-A)$.

On the other hand, for any $y \in Y$ satisfying $g(y) \in \operatorname{im}(I-A)$, there exists a constant c^* such that $g(y) = (I-A)c^*$. Let $x^*(t) = I^{\alpha}_{0+}y(t) + c^*t^{\alpha-1}$. A straightforward computation shows that $x^*(0) = \theta$ and $D^{\alpha-1}_{0+}x^*(1) = AD^{\alpha-1}_{0+}x^*(\xi)$. Hence, $x^* \in \operatorname{dom} L$ and $y(t) = D^{\alpha}_{0+}x^*(t)$, which implies that $y \in \operatorname{im} L$.

Next, we set $\rho_A = k(I-A)$, where

$$k = \begin{cases} 1, & \text{if the hypothesis } (a_1) \text{ holds, i.e., } A^2 = A; \\ \frac{1}{2}, & \text{if the hypothesis } (a_2) \text{ holds, i.e., } A^2 = I. \end{cases} \tag{2.8}$$

For $A^2 = A$, we have

$$\begin{aligned} &\rho_A^2 = (I-A)^2 = I - 2A + A^2 = I - A = \rho_A, \\ &(I-\rho_A)(\xi^{\alpha}A - I) = A(\xi^{\alpha}A - I) = \xi^{\alpha}A^2 - A = (\xi^{\alpha}-1)A = (\xi^{\alpha}-1)(I-\rho_A). \end{aligned} \tag{2.9}$$

For $A^2 = I$, we have

$$\begin{aligned} \rho_A^2 &= \frac{1}{4}(I-A)^2 = \frac{1}{4}(I - 2A + A^2) = \frac{1}{2}(I-A) = \rho_A, \\ (I-\rho_A)(\xi^{\alpha}A - I) &= \frac{1}{2}(I+A)(\xi^{\alpha}A - I) \\ &= \frac{1}{2}[\xi^{\alpha} - I + \xi^{\alpha}A^2 - A] = \frac{1}{2}(\xi^{\alpha}-1)(I+A) = (\xi^{\alpha}-1)(I-\rho_A). \end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10) that ρ_A satisfies the following properties

$$\rho_A^2 = \rho_A, \qquad (I-\rho_A)(\xi^\alpha A - I) = (\xi^\alpha - 1)(I-\rho_A). \tag{2.11}$$

Furthermore, we note that if $y = ct^{\alpha-1}$, $c \in \mathbb{R}^n$, then

$$g(y) = \frac{A}{\Gamma(\alpha)}\int_0^\xi y(s)\,ds - \frac{I}{\Gamma(\alpha)}\int_0^1 y(s)\,ds = \frac{(\xi^\alpha A - I)c}{\Gamma(\alpha+1)}. \tag{2.12}$$

Define the continuous linear mapping $Q\colon Y \to Y$ by

$$Qy(t) := \frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}(I-\rho_A)g(y)t^{\alpha-1}, \quad t \in [0,1],\ y \in Y. \tag{2.13}$$

By (2.11), it is easy to verify $Q^2y = Qy$, that is, Q is a projection operator. The equality $\ker Q = \operatorname{im} L$ follows from the trivial fact that

$$y \in \ker Q \Leftrightarrow g(y) \in \ker(I-\rho_A) \Leftrightarrow g(y) \in \operatorname{im}\rho_A \Leftrightarrow g(y) \in \operatorname{im}(I-A) \Leftrightarrow y \in \operatorname{im} L.$$

Therefore, we get $Y = \ker Q \oplus \operatorname{im} Q = \operatorname{im} L \oplus \operatorname{im} Q$.

Finally, we shall prove that $\operatorname{im} Q = \ker L$. Indeed, for any $z \in \operatorname{im} Q$, let $z = Qy$, $y \in Y$. By (2.11), we have

$$k(I-A)z(t) = \rho_A z(t) = \rho_A Qy(t) = \frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}\rho_A(I-\rho_A)g(y)t^{\alpha-1} = \theta,$$

which implies $z \in \ker L$. Conversely, for each $z \in \ker L$, there exists a constant $c^* \in \ker(I-A)$ such that $z = c^*t^{\alpha-1}$ for $t \in [0,1]$. By (2.11) and (2.12), we derive

$$Qz(t) = \frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}(I-\rho_A)g(c^*t^{\alpha-1})t^{\alpha-1} = c^*t^{\alpha-1} = z(t), \quad t \in [0,1],$$

which implies that $z \in \operatorname{im} Q$. Hence we know that $\operatorname{im} Q = \ker L$. Then the operator L is a Fredholm operator of index zero. The proof is complete. □

Define the operator $P\colon X \to X$ as follows:

$$Px(t) = \frac{1}{\Gamma(\alpha)}(I-\rho_A)D_{0+}^{\alpha-1}x(0)t^{\alpha-1}. \tag{2.14}$$

Lemma 2.10. *The mapping $P\colon X \to X$ defined as above is a continuous projector such that*

$$\operatorname{im} P = \ker L, \qquad X = \ker L \oplus \ker P$$

and the linear operator $K_P\colon \operatorname{im} L \to \operatorname{dom} L \cap \ker P$ can be written as

$$K_P y(t) = I_{0+}^\alpha y(t),$$

also

$$K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1} \quad \textit{and} \quad \|K_P y\| \le 1/\Gamma(\alpha)\|y\|_1.$$

Proof. The continuity of P is obvious. By the first identity of (2.11), we have $(I-\rho_A)^2 = (I-\rho_A)$, which implies that the mapping P is a projector. Moreover, if $v \in \operatorname{im} P$, there exists an $x \in X$ such that $v = Px$. By the first identity of (2.11) again, we see that

$$\frac{1}{\Gamma(\alpha)}(I-A)(I-\rho_A)D_{0+}^{\alpha-1}x(0) = \frac{1}{k\Gamma(\alpha)}\rho_A(I-\rho_A)D_{0+}^{\alpha-1}x(0) = 0,$$

which gives us $v \in \ker L$. Conversely, if $v \in \ker L$, then $v(t) = c_* t^{\alpha-1}$ for some $c_* \in \ker(I-A)$, and we deduce that

$$Pv(t) = \frac{1}{\Gamma(\alpha)}(I-\rho_A)D_{0+}^{\alpha-1}v(0)t^{\alpha-1} = (I-\rho_A)c_* t^{\alpha-1} = c_* t^{\alpha-1} = v(t), \quad t \in [0,1],$$

which gives us $v \in \operatorname{im} P$. Thus, we get that $\ker L = \operatorname{im} P$ and consequently $X = \ker L \oplus \ker P$.

Moreover, let $y \in \operatorname{im} L$. There exists $x \in \operatorname{dom} L$ such that $y = Lx$, and we obtain

$$K_P y(t) = x(t) + ct^{\alpha-1},$$

where $c \in \mathbb{R}^n$ satisfies $c = Ac$. It is easy to see that $K_P y \in \operatorname{dom} L$ and $K_P y \in \ker P$. Therefore, K_P is well defined. Further, for $y \in \operatorname{im} L$, we have

$$L(K_P y(t)) = D_{0+}^{\alpha}(K_P y(t)) = y(t)$$

and for $x \in \operatorname{dom} L \cap \ker P$, we obtain that

$$K_P(Lx(t)) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-1},$$

for some $c_1, c_2 \in \mathbb{R}^n$. In view of $x \in \operatorname{dom} L \cap \ker P$, we know that $c_1 = c_2 = 0$. Therefore, $(K_P L)x(t) = x(t)$. This shows that $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. Finally, by the definition of K_P, we derive

$$\|D_{0+}^{\alpha-1}K_P y\|_\infty = \left\| \int_0^t y(s)\,ds \right\|_\infty \le \|y\|_1 \tag{2.15}$$

and

$$\|K_P y\|_\infty = \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds \right\|_\infty \le \frac{1}{\Gamma(\alpha)}\|y\|_1. \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\|K_P y\| = \max\{\|D_{0+}^{\alpha-1}K_P y\|_\infty, \|K_P y\|_\infty\} \le \max\left\{\|y\|_1, \frac{1}{\Gamma(\alpha)}\|y\|_1\right\} = \frac{1}{\Gamma(\alpha)}\|y\|_1. \tag{2.17}$$

Then Lemma 2.10 is proved. □

Remark 2.11. The constant $\frac{1}{\Gamma(\alpha)}$ in (2.16) is sharp, and its value can not be improved. Actually, one can prove the following proposition.

Proposition 2.12. *Let $\alpha \in (1,2]$ and define the linear mapping $T\colon L^1[0,1] \to C[0,1]$ by*

$$(Ty)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds.$$

Then $\|T\| = \frac{1}{\Gamma(\alpha)}$.

Proof. Indeed, from $\|(Ty)(t)\|_\infty \le \frac{1}{\Gamma(\alpha)}\|y\|_1$, we have

$$\|T\| \le \frac{1}{\Gamma(\alpha)}. \tag{2.18}$$

On the other hand, for $\varepsilon \in (0,1)$, let

$$y(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \le t \le \varepsilon, \\ 0, & \varepsilon < t \le 1. \end{cases}$$

A direct computation shows that $\|y\|_1 = 1$, and

$$|Ty(1)| = \frac{1}{\Gamma(\alpha)}[1-\delta(\varepsilon)],$$

where $\delta(\varepsilon) = 1 - \frac{1-(1-\varepsilon)^\alpha}{\alpha\varepsilon} > 0$. It is easy to verify that δ is an increasing function with respect to ε and $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$. Thus, $\|T\| \ge \frac{1}{\Gamma(\alpha)}[1-2\delta(\varepsilon)]$. As ε was chosen arbitrarily, we have $\|T\| \ge \frac{1}{\Gamma(\alpha)}$. This together with (2.18) leads to conclusion. □

Lemma 2.13. *Let f be a Carathéodory function. Then N defined by (2.3) is L-compact.*

Proof. Let Ω be a bounded subset in X. By the hypothesis (iii) on the function f, there exists a function $\varphi_\Omega(t) \in L^1[0,1]$ such that for all $x \in \Omega$,

$$|f(t, x(t), D_{0^+}^{\alpha-1}x(t))| \le \varphi_\Omega(t), \quad \text{a.e. } t \in [0,1], \tag{2.19}$$

which, along with (2.7) and (2.13), implies

$$\begin{aligned} \|Qy\|_1 &= \left|\frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}(I-\rho_A)g(y)\right| \int_0^1 s^{\alpha-1}\,ds \\ &\le \frac{(\|A\|+1)\|I-\rho_A\|}{|1-\xi^\alpha|}\|\varphi_\Omega\|_1 < \infty. \end{aligned} \tag{2.20}$$

This shows that $QN(\overline{\Omega}) \subseteq Y$ is bounded. The continuity of QN follows from the hypothesis on f and the Lebesgue dominated convergence theorem.

Next, we shall show that $K_{P,Q}N$ is completely continuous. First, for any $x \in \Omega$, we have

$$\begin{aligned} K_{P,Q}Nx(t) &= K_P(I-Q)Nx(t) = K_PNx(t) - K_PQNx(t) \\ &= I_{0+}^\alpha Nx(t) - \frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}(I-\rho_A)g(Nx(t))I_{0+}^\alpha t^{\alpha-1}. \end{aligned}$$

By Lemma 2.8, it is easy to know that $K_{P,Q}N$ is continuous.

From (2.19) and (2.7), we derive that

$$\begin{aligned} |g(Nx(t))| &= \left|\frac{A}{\Gamma(\alpha)}\int_0^\xi f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds - \frac{I}{\Gamma(\alpha)}\int_0^1 f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds\right| \\ &\le \frac{\|A\|+1}{\Gamma(\alpha)}\|\varphi_\Omega\|_1. \end{aligned} \tag{2.21}$$

From (2.21), we obtain

$$\begin{aligned}\|K_{P,Q}Nx\| &= I_{0+}^{\alpha}Nx(t) - \frac{\Gamma(\alpha+1)}{\xi^{\alpha}-1}(I-\rho_A)g(Nx(t))I_{0+}^{\alpha}t^{\alpha-1} \\ &\le \|\varphi_{\Omega}\|_1 + \frac{\Gamma(\alpha+1)\|I-\rho_A\|}{\Gamma(\alpha)|\xi^{\alpha}-1|}|g(Nx(t))|\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,ds \\ &\le \|\varphi_{\Omega}\|_1 + \frac{\Gamma(\alpha+1)\|I-\rho_A\|(\|A\|+1)}{\Gamma(2\alpha)|\xi^{\alpha}-1|}\|\varphi_{\Omega}\|_1 < \infty,\end{aligned}$$

which shows that $K_{P,Q}N\overline{\Omega}$ is uniformly bounded in X. Noting that

$$b^p - a^p \le (b-a)^p \quad \text{for any } b \ge a > 0,\ 0 < p \le 1, \tag{2.22}$$

for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we shall see that

$$\begin{aligned}&|K_{P,Q}Nx(t_2) - K_{P,Q}Nx(t_1)| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]Nx(s)\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Nx(s)\,ds\right. \\ &\quad \left. - \frac{\Gamma(\alpha+1)}{\xi^{\alpha}-1}(I-\rho_A)g(Nx(t))[I_{0+}^{\alpha}t_2^{\alpha-1} - I_{0+}^{\alpha}t_1^{\alpha-1}]\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-t_1)^{\alpha-1}\varphi_{\Omega}(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\varphi_{\Omega}(s)\,ds \\ &\quad + \frac{\Gamma(\alpha+1)\|I-\rho_A\|(\|A\|+1)}{\Gamma(2\alpha)|\xi^{\alpha}-1|}\|\varphi_{\Omega}\|_1|t_2^{2\alpha-1}-t_1^{2\alpha-1}| \\ &\to 0 \text{ as } t_2 \to t_1\end{aligned}$$

and

$$\left|D_{0^+}^{\alpha-1}K_{P,Q}Nx(t_2) - D_{0^+}^{\alpha-1}K_{P,Q}Nx(t_1)\right| = \left|\int_{t_1}^{t_2}Nx(s)\,ds\right| \le \int_{t_1}^{t_2}\varphi_{\Omega}(s)\,ds \to 0 \quad \text{as } t_2 \to t_1.$$

Then we get that $K_{P,Q}N\overline{\Omega}$ is equicontinuous in X. By Lemma (2.8), $K_{P,Q}N\overline{\Omega} \subseteq X$ is relatively compact. Thus we can conclude that the operator N is L-compact continuous in $\overline{\Omega}$. The proof is complete. □

3 Main result

In this section, we shall present and prove our main result.

Theorem 3.1. *Let f be a Carathéodory function and the following conditions hold:*

(H_1) *There exist three nonnegative functions $a, b, c \in L^1[0,1]$ such that*

$$|f(t,u,v)| \le a(t)|u| + b(t)|v| + c(t), \quad \textit{for all } t \in [0,1],\ u,v \in \mathbb{R}^n$$

(H_2) *There exists a constant $A_1 > 0$ such that for $x \in \operatorname{dom} L$, if $|D_{0^+}^{\alpha-1}x(t)| > A_1$ for all $t \in [0,1]$, then*

$$\int_{\xi}^{1} f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds \notin \operatorname{im}(I-A).$$

(H_3) *There exists a constant* $A_2 > 0$ *such that for any* $e \in \mathbb{R}^n$ *satisfying* $e = Ae$ *and* $\min_{1\le i\le n}\{|e_i|\} > A_2$*, either*

$$\langle e, QNe\rangle \le 0, \quad \min_{1\le i\le n}\{|e_i|\} > A_2,$$

or else

$$\langle e, QNe\rangle \ge 0, \quad \min_{1\le i\le n}\{|e_i|\} > A_2,$$

where $\langle\cdot,\cdot\rangle$ *is the scalar product in* $\mathbb{R}^n$.

Then the BVPs (1.1) *have at least one solution in space* X *provided that*

$$(\|I-\rho_A\|+1)(\|a\|_1+\|b\|_1) < \Gamma(\alpha). \tag{3.1}$$

Proof. We shall construct an open bounded subset Ω in X satisfying all assumption of Lemma 2.7. Let

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}. \tag{3.2}$$

For any $x \in \Omega_1$, $x \notin \ker L$, we get that $\lambda \ne 0$. Since $Nx \in \operatorname{im} L = \ker Q$, by the definition of $\operatorname{im} L$, we have $g(Nx) \in \operatorname{im}(I-A)$, where

$$g(Nx) = \frac{A}{\Gamma(\alpha)}\int_0^{\xi} f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds - \frac{I}{\Gamma(\alpha)}\int_0^1 f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds.$$

Hence

$$\begin{aligned}&\int_{\xi}^1 f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds\\ &\quad= -\Gamma(\alpha)g(Nx) - (A-I)\int_0^{\xi} f(s, x(s), D_{0^+}^{\alpha-1}x(s))\,ds \in \operatorname{im}(I-A).\end{aligned} \tag{3.3}$$

It follows from hypothesis (H_2) and (3.3) that there exists $t_0 \in [0,1]$ such that $|D_{0^+}^{\alpha-1}x(t_0)| \le A_1$. Then by $D_{0^+}^{\alpha-1}x(0) = D_{0^+}^{\alpha-1}x(t_0) - \int_0^t D_{0+}^{\alpha}x(s)\,ds$, we deduce that

$$|D_{0^+}^{\alpha-1}x(0)| \le A_1 + \|D_{0+}^{\alpha}x\|_1 = A_1 + \|Lx\|_1 \le A_1 + \|Nx\|_1,$$

which implies

$$\|Px\| = \left\|\frac{1}{\Gamma(\alpha)}(I-\rho_A)D_{0^+}^{\alpha-1}x(0)t^{\alpha-1}\right\| \le \frac{\|I-\rho_A\|}{\Gamma(\alpha)}(A_1 + \|Nx\|_1). \tag{3.4}$$

Further, again for $x \in \Omega_1$, since $\operatorname{im} P = \ker L$, $X = \ker L \oplus \ker P$, we have $(I-P)x \in \operatorname{dom} L \cap \ker P$ and $LPx = \theta$. Then

$$\|(I-P)x\| = \|K_P L(I-P)x\| \le \|K_P Lx\| \le \frac{1}{\Gamma(\alpha)}\|Lx\|_1 \le \frac{1}{\Gamma(\alpha)}\|Nx\|_1. \tag{3.5}$$

From (3.4) and (3.5), we can conclude that

$$\|x\| = \|Px + (I-P)x\| \le \|Px\| + \|(I-P)x\| \le \frac{\|I-\rho_A\|}{\Gamma(\alpha)}A_1 + \frac{\|I-\rho_A\|+1}{\Gamma(\alpha)}\|Nx\|_1. \tag{3.6}$$

Moreover, by the definition of N and the hypothesis (H_1), we see that

$$\|Nx\|_1 = \int_0^1 |f(s, x(s), D_{0^+}^{\alpha-1}x(s))|\,dt \le \|a\|_1\|x\|_\infty + \|b\|_1\|D_{0^+}^{\alpha-1}x\|_\infty + \|c\|_1. \tag{3.7}$$

Then

$$\|x\| \leq \frac{\|I-\rho_A\|}{\Gamma(\alpha)} A_1 + \frac{\|I-\rho_A\|+1}{\Gamma(\alpha)} (\|a\|_1 \|x\|_\infty + \|b\|_1 \|D_{0^+}^{\alpha-1} x\|_\infty + \|c\|_1). \tag{3.8}$$

From (3.8) and $\|x\|_\infty \leq \|x\|$, we derive

$$\|x\|_\infty \leq \frac{\frac{\|I-\rho_A\|}{\Gamma(\alpha)} A_1 + \frac{\|I-\rho_A\|+1}{\Gamma(\alpha)} (\|b\|_1 \|D_{0^+}^{\alpha-1} x\|_\infty + \|c\|_1)}{1 - \frac{\|I-\rho_A\|+1}{\Gamma(\alpha)} \|a\|_1}, \tag{3.9}$$

which, together with $\|D_{0^+}^{\alpha-1} x\|_\infty \leq \|x\|$, (3.8) and (3.9), gives us

$$\|D_{0^+}^{\alpha-1} x\|_\infty \leq \frac{\frac{\|I-\rho_A\|}{\Gamma(\alpha)} A_1 + \frac{\|I-\rho_A\|+1}{\Gamma(\alpha)} \|c\|_1}{1 - \frac{\|I-\rho_A\|+1}{\Gamma(\alpha)} (\|a\|_1 + \|b\|_1)} = \frac{\|I-\rho_A\| A_1 + (\|I-\rho_A\|+1)\|c\|_1}{\Gamma(\alpha) - (\|I-\rho_A\|+1)(\|a\|_1 + \|b\|_1)}. \tag{3.10}$$

It follows from (3.9) and (3.10) that Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{im} L\}. \tag{3.11}$$

For any $x \in \Omega_2$, it follows from $x \in \ker L$ that $x = et^{\alpha-1}$ for some $e \in \ker(I-A)$, and it follows from $Nx \in \operatorname{im} L$ that $g(Nx) \in \operatorname{im}(I-A)$. By a similar argument as above, by hypothesis (H_2), we arrive at $|D_{0^+}^{\alpha-1} x(t_0)| = |e|\Gamma(\alpha) \leq A_1$. Thus we get that

$$\|x\| \leq |e|\Gamma(\alpha) \leq A_1.$$

That is, Ω_2 is bounded in X. If the first part of (H_3) holds, denote

$$\Omega_3 = \{x \in \ker L : -\lambda x + (1-\lambda) QNx = \theta, \quad t \in [0,1]\},$$

then for any $x \in \Omega_3$, we know that

$$x = et^{\alpha-1} \text{ with } e \in \ker(I-A) \text{ and } \lambda x = (1-\lambda) QNx.$$

If $\lambda = 0$, we have $Nx \in \ker Q = \operatorname{im} L$, then $x \in \Omega_2$, by the argument above, we get that $\|x\| \leq A_1$. Moreover, if $\lambda \in (0,1]$ and if $|e| > A_2$, by the hypothesis (H_3), we deduce that

$$0 < \lambda |e|^2 = \lambda \langle e, e\rangle = (1-\lambda)\langle e, QNe\rangle \leq 0,$$

which is a contradiction. Then $\|x\| = \|et^{\alpha-1}\| \leq \max\{|e|, \Gamma(\alpha)|e|\}$. That is to say, Ω_3 is bounded. If other part of (H_3) holds, we take

$$\Omega_3 = \{x \in \ker L : \lambda x + (1-\lambda) QNx = \theta, \ t \in [0,1]\}.$$

By using the same arguments as above, we can conclude that Ω_3 is also bounded.

In the sequel, we will show that all conditions of Lemma 2.7 are satisfied.

Assume that Ω is a bounded open subset of X such that $\cup_{i=1}^3 \overline{\Omega}_i \subseteq \Omega$. It follows from Lemmas 2.9 and 2.13 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω and the argument above, in order to complete the theorem, we only need to prove that the condition (iii) of Lemma 2.7 is also satisfied. For this purpose, let

$$H(x, \lambda) = \pm\lambda x + (1-\lambda) QNx, \tag{3.12}$$

where we let the isomorphism $J\colon \operatorname{im} Q \to \ker L$ be the identical operator. Since $\Omega_3 \subseteq \Omega$, $H(x,\lambda) \neq 0$ for $(x,\lambda) \in \ker L \cap \partial\Omega \times [0,1]$, then by the homotopy property of degree, we obtain

$$\begin{aligned} &\deg\left(JQN|_{\ker L\cap\partial\Omega}, \Omega\cap\ker L, 0\right)\\ &\quad= \deg\left(H(\cdot,0), \Omega\cap\ker L, 0\right)\\ &\quad= \deg\left(H(\cdot,1), \Omega\cap\ker L, 0\right)\\ &\quad= \deg\left(\pm\mathrm{Id}, \Omega\cap\ker L, 0\right) = \pm 1 \neq 0. \end{aligned} \tag{3.13}$$

Thus (H_3) of Lemma 2.7 is fulfilled and Theorem 3.1 is proved. The proof is complete. □

4 Examples

In this section, we shall present three examples to illustrate our main result in $\mathbb{R}^3$ with $\dim\ker L = 1$, $\dim\ker L = 2$, $\dim\ker L = 3$, respectively, which surely generalize the previous results [4–6, 8, 9, 13, 14, 18–20], where the dimension of $\dim\ker L$ is only 1 or 2.

Example 4.1. Let us consider the following system with $\dim\ker L = 1$ in $\mathbb{R}^3$.

$$\begin{cases} D_{0^+}^{\frac{3}{2}} x_1(t) = \dfrac{1}{148}\left(4t^{\frac{1}{2}}x_3(t) + \dfrac{1}{\sqrt{\pi}} D_{0^+}^{\frac{1}{2}} x_1(t) - 4\right), \quad t\in(0,1),\\ D_{0^+}^{\frac{3}{2}} x_2(t) = \dfrac{1}{148}\left(t^{-\frac{1}{2}}x_2(t) + \dfrac{2}{\sqrt{\pi}} D_{0^+}^{\frac{1}{2}} x_1(t)\right), \quad t\in(0,1),\\ D_{0^+}^{\frac{3}{2}} x_3(t) = \dfrac{4t+1}{148}, \quad t\in(0,1),\\ x_1(0) = x_2(0) = x_3(0) = 0,\\ D_{0^+}^{\frac{1}{2}} x_1(1) = -3D_{0^+}^{\frac{1}{2}} x_1\left(\frac{1}{2}\right) + 3D_{0^+}^{\frac{1}{2}} x_2\left(\frac{1}{2}\right) - 3D_{0^+}^{\frac{1}{2}} x_3\left(\frac{1}{2}\right),\\ D_{0^+}^{\frac{1}{2}} x_2(1) = -5D_{0^+}^{\frac{1}{2}} x_1\left(\frac{1}{2}\right) + 5D_{0^+}^{\frac{1}{2}} x_2\left(\frac{1}{2}\right) - 5D_{0^+}^{\frac{1}{2}} x_3\left(\frac{1}{2}\right),\\ D_{0^+}^{\frac{1}{2}} x_3(1) = -D_{0^+}^{\frac{1}{2}} x_1\left(\frac{1}{2}\right) + D_{0^+}^{\frac{1}{2}} x_2\left(\frac{1}{2}\right) - D_{0^+}^{\frac{1}{2}} x_3\left(\frac{1}{2}\right). \end{cases} \tag{4.1}$$

Let $\alpha = 3/2$, $\xi = 1/2$ and

$$A = \begin{bmatrix} -3 & 3 & -3\\ -5 & 5 & -5\\ -1 & 1 & -1 \end{bmatrix}. \tag{4.2}$$

It is clear that $A^2 = A$ and $\dim\ker(I-A) = 1$. Define the function $f\colon [0,1]\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ by

$$f(t,u,v) = \frac{1}{148}\begin{pmatrix} 4t^{\frac{1}{2}}x_3 + \dfrac{1}{\sqrt{\pi}}y_1 - 4\\ t^{-\frac{1}{2}}x_2 + \dfrac{2}{\sqrt{\pi}}y_1\\ 4t+1 \end{pmatrix} \tag{4.3}$$

for all $t \in [0,1]$ and $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then problem (4.1) has one solution if and only if problem (1.1), with A and f defined by (4.2), (4.3), has one solution. Hence we need only to verify that the conditions of Theorem 3.1 are satisfied.

Check (H_1) of Theorem 3.1: for some $r \in \mathbb{R}$, where $|u| = |(u_1, u_2, u_3)| = \max\{|u_1|, |u_2|, |u_3|\}$ let $\Omega = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon |u| \le r, |v| \le r\}$ and let $\varphi_\Omega(t) = \frac{1}{148}(4rt^{\frac{1}{2}} + rt^{-\frac{1}{2}} + \frac{3}{\sqrt{\pi}}r + 4t + 4) \in L^1[0,1]$. Since $\|A\| = \max\{\sum_{j=1}^n |a_{ij}|, j = 1, 2, \dots, n\} = 15$, let

$$a(t) = \frac{1}{148}(4t^{\frac{1}{2}} + t^{-\frac{1}{2}}), \quad b(t) = \frac{3}{148\sqrt{\pi}}, \quad c(t) = \frac{4t+4}{148}. \tag{4.4}$$

It is easy to see that (H_1) of Theorem 3.1 and the condition $(\|I - \rho_A\| + 1)(\|a\|_1 + \|b\|_1) < \Gamma(\alpha)$ hold.

Check (H_2) of Theorem 3.1: noting that for any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we have

$$f_3(t, x_1, x_2, x_3, y_1, y_2, y_3) = \frac{4t+1}{148} \ge \frac{1}{148} > 0.$$

This together with $\operatorname{im}(I - A) = \{\tau(1,0,0) + \zeta(0,1,0) : \tau, \zeta \in \mathbb{R}\}$ yields

$$\int_{\frac{1}{2}}^{1} f(s, x(s), D_{0^+}^{\frac{1}{2}} x(s))\, ds \notin \operatorname{im}(I - A).$$

Check (H_3) of Theorem 3.1: for any $y \in L^1([0,1], \mathbb{R}^3)$, by (2.13), we have $\rho_A = I - A$ and

$$Qy(t) = \frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}(I - \rho_A) g(y) t^{\alpha-1} = \frac{3\sqrt{\pi}}{\sqrt{2} - 4} A g(y) t^{\alpha-1}, \tag{4.5}$$

where

$$g(y) = \frac{2A}{\sqrt{\pi}} \int_0^{1/2} y(s)\, ds - \frac{2I}{\sqrt{\pi}} \int_0^1 y(s)\, ds.$$

For any $e \in \mathbb{R}^3$ satisfying $e = Ae$, e can be written as

$$e = \sigma(3,5,1)^\top, \quad \text{for } \sigma \in \mathbb{R}.$$

By (2.3), we have

$$N\left(et^{\frac{1}{2}}\right)(t) = \frac{1}{148}\left(4\sigma t + 1.5\sigma - 4, 8\sigma, 4t + 1\right)^\top \tag{4.6}$$

and

$$g(N(et^{\alpha-1})(t)) = \frac{A}{148\sqrt{\pi}}\left(2.5\sigma - 4, 8\sigma, 2\right)^\top - \frac{I}{148\sqrt{\pi}}\left(7\sigma - 8, 16\sigma, 6\right)^\top. \tag{4.7}$$

It follows from (4.5), (4.6) and (4.7) that

$$Q\left(Net^{\frac{1}{2}}\right) = \frac{3\sqrt{\pi}}{\sqrt{2} - 4} A g\left(Net^{\frac{1}{2}}\right) t^{\frac{1}{2}} = \frac{3t^{\frac{1}{2}}}{148(\sqrt{2} - 4)}\left(-10.5\sigma, -17.5\sigma, -3.5\sigma\right)^\top$$

and

$$\left\langle e, QNet^{\frac{1}{2}} \right\rangle = -\frac{367.5}{148(\sqrt{2} - 4)} \sigma^2 t^{\frac{1}{2}} \ge 0.$$

Therefore, (4.1) admits at least one solution.

Example 4.2. Consider the following system with $\dim\ker L = 2$ in $\mathbb{R}^3$.

$$
\begin{cases}
D_{0^+}^{\frac{3}{2}} x_1(t) = \begin{cases} \dfrac{1}{36}, & |D_{0^+}^{1/2} x_1(t)| < 1; \\ \dfrac{D_{0^+}^{\frac{1}{2}} x_1(t) + \left[D_{0^+}^{1/2} x_1(t)\right]^{-1} - 1}{36}, & |D_{0^+}^{1/2} x_1(t)| \geq 1, \end{cases} \\
D_{0^+}^{\frac{3}{2}} x_2(t) = \dfrac{|x_2(t)| + |x_3(t)|}{36}, \\
D_{0^+}^{\frac{3}{2}} x_3(t) = -\dfrac{x_3(t)}{36}, \\
x_1(0) = x_2(0) = x_3(0) = 0, \\
D_{0^+}^{\frac{1}{2}} x_1(1) = D_{0^+}^{\frac{1}{2}} x_1\left(\dfrac{1}{2}\right), \\
D_{0^+}^{\frac{1}{2}} x_2(1) = -D_{0^+}^{\frac{1}{2}} x_2\left(\dfrac{1}{2}\right) + 2D_{0^+}^{\frac{1}{2}} x_3\left(\dfrac{1}{2}\right), \\
D_{0^+}^{\frac{1}{2}} x_3(1) = D_{0^+}^{\frac{1}{2}} x_3\left(\dfrac{1}{2}\right).
\end{cases}
\tag{4.8}
$$

Let $\alpha = 3/2$, $\xi = 1/2$, for all $t \in [0,1]$, $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3) \in \mathbb{R}^3$,

$$
f(t,u,v) = \frac{1}{36} \begin{pmatrix} \begin{cases} 1, & |y_1| < 1; \\ y_1 + 1/y_1 - 1, & |y_1| \geq 1 \end{cases} \\ |x_2| + |x_3| \\ -x_3 \end{pmatrix}
\tag{4.9}
$$

and

$$
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.
\tag{4.10}
$$

It is not difficult to see that $A^2 = I$ and $\dim\ker(I - A) = 2$. Then problem (4.8), with A and f defined by (4.10) and (4.9), has one solution if and only if problem (1.1) has one solution.

Check (H_1) of Theorem 3.1: for some $r \in \mathbb{R}$, $\Omega = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| \leq r, |v| \leq r\}$, let $\varphi_\Omega(t) = \frac{1}{12}r + \frac{1}{36r} + \frac{1}{36} \in L^1[0,1]$. Since $\|A\| = 3$, let

$$
a(t) = \frac{1}{18}, \quad b(t) = \frac{1}{36}, \quad c(t) = \frac{1}{36}.
\tag{4.11}
$$

One can see that the condition (H_1) of Theorem 3.1 is satisfied.

Check (H_2) of Theorem 3.1: it follows from the definition of f that $|f_1| > \frac{1}{36} > 0$. This together with $\operatorname{im}(I - A) = \{\sigma_0(0,0,1) : \sigma_0 \in \mathbb{R}\}$ implies that the condition (H_2) of Theorem 3.1 is satisfied.

Check (H_3) of Theorem 3.1: since $\dim\ker(I - A) = 2$, for any $e \in \mathbb{R}^3$ satisfying $e = Ae$, e can be written as

$$
e = \sigma_1(1,0,0)^T + \sigma_2(0,1,1)^T, \quad \text{for } \sigma_i \in \mathbb{R},\ i = 1,2.
$$

For any $y \in L^1([0,1], \mathbb{R}^3)$, by (2.13) and $\rho_A = \frac{1}{2}(I - A)$, we have

$$
Qy(t) = \frac{\Gamma(\alpha+1)}{\xi^\alpha - 1}(I - \rho_A)g(y)t^{\alpha-1} = \frac{3\sqrt{\pi}}{\sqrt{2} - 4}\,\frac{I + A}{2}\,g(y)t^{\alpha-1},
\tag{4.12}
$$

where

$$g(y) = \frac{2A}{\sqrt{\pi}} \int_0^{1/2} y(s)\,ds - \frac{2I}{\sqrt{\pi}} \int_0^1 y(s)\,ds.$$

By (2.3), we have

$$N\left(et^{\frac{1}{2}}\right)(t) = \begin{cases} \frac{1}{36}\left(1, 2|\sigma_2|t^{\frac{1}{2}}, -\sigma_2 t^{\frac{1}{2}}\right)^{\top}, & |\sigma_1| < 1, \\ \frac{1}{36}\left(\sigma_1 + \frac{1}{\sigma_1} - 1, 2|\sigma_2|t^{\frac{1}{2}}, -\sigma_2 t^{\frac{1}{2}}\right)^{\top}, & |\sigma_1| \geq 1. \end{cases} \tag{4.13}$$

Let $A_2 = 1$, for $\min\{|\sigma_1|, |\sigma_2|, |\sigma_3|\} > 1$, it follows from (4.12) and (4.13) that

$$Q\left(Net^{\frac{1}{2}}\right) = \frac{3\sqrt{\pi}}{\sqrt{2}-4}\frac{I+A}{2} g\left(Net^{\frac{1}{2}}\right) t^{\frac{1}{2}} = \frac{t^{\frac{1}{2}}}{8\sqrt{2}-32}\left(\sigma_1 + \frac{1}{\sigma_1} - 1, \frac{4-\sqrt{2}}{6}\sigma_2, \frac{4-\sqrt{2}}{6}\sigma_2\right)^{\top}$$

and

$$\left\langle e, QNet^{\frac{1}{2}}\right\rangle = \frac{t^{\frac{1}{2}}}{8\sqrt{2}-32}\left[(\sigma_3 - 1/2)^2 + 3/4 + \frac{4-\sqrt{2}}{3}\sigma_2^2\right] < 0.$$

Therefore, (4.8) admits at least one solution.

Example 4.3. Consider the following system with $\dim\ker L = 3$ in $\mathbb{R}^3$.

$$\begin{cases} D_{0^+}^{\frac{3}{2}} x_1(t) = \frac{x_2(t)}{8}, \\ D_{0^+}^{\frac{3}{2}} x_2(t) = -\frac{x_1(t)}{8}, \\ D_{0^+}^{\frac{3}{2}} x_3(t) = \begin{cases} \frac{1}{8}, & |D_{0^+}^{1/2} x_3(t)| < 1; \\ \frac{D_{0^+}^{\frac{1}{2}} x_3(t) + \left[D_{0^+}^{1/2} x_3(t)\right]^{-1} - 1}{8}, & |D_{0^+}^{1/2} x_3(t)| \geq 1, \end{cases} \\ x_1(0) = x_2(0) = x_3(0) = 0, \\ D_{0^+}^{\frac{1}{2}} x_1(1) = D_{0^+}^{\frac{1}{2}} x_1\left(\frac{1}{2}\right), \quad D_{0^+}^{\frac{1}{2}} x_2(1) = D_{0^+}^{\frac{1}{2}} x_2\left(\frac{1}{2}\right), \quad D_{0^+}^{\frac{1}{2}} x_3(1) = D_{0^+}^{\frac{1}{2}} x_3\left(\frac{1}{2}\right). \end{cases} \tag{4.14}$$

Let $\alpha = 3/2, \xi = 1/2$, and $A = I$. It is clear that $\dim\ker(I-A) = 3$. Then problem (4.1) has one solution if and only if problem (1.1) has one solution.

Check (H_1) of Theorem 3.1: for some $r \in \mathbb{R}$, $\Omega = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| \leq r, |v| \leq r\}$, let $\varphi_\Omega(t) = \frac{3}{8}r + \frac{1}{8r} + \frac{1}{8} \in L^1[0,1]$. Since $\|A\| = 1$, let

$$a(t) = \frac{1}{8}, \quad b(t) = 0, \quad c(t) = \frac{r}{8} + \frac{1}{8r} + \frac{1}{8}. \tag{4.15}$$

One can see that the condition (H_1) of Theorem 3.1 is satisfied.

Check (H_2) of Theorem 3.1: corresponding to the system (1.1), we get that

$$f_3(t, x_1, x_2, x_3, y_1, y_2, y_3) = \begin{cases} \frac{1}{8}, & |y_3| < 1; \\ \frac{y_3 + 1/y_3 - 1}{8}, & |y_3| \geq 1 \end{cases}$$

and $|f_3| > 0$ for any $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$. This together with $\mathrm{im}(I-A) = \{(0,0,0)\}$ implies that the condition (H_2) of Theorem 3.1 is satisfied.

Check (H_3) of Theorem 3.1: since $\dim\ker(I-A)=3$, for any $e\in\mathbb{R}^3$ satisfying $e=Ae$, e can be written as

$$e=\sigma_1(1,0,0)^T+\sigma_2(0,1,0)^T+\sigma_3(0,0,1)^T,\quad \text{for } \sigma_i\in\mathbb{R},\ i=1,2,3.$$

For any $y\in L^1([0,1],\mathbb{R}^3)$, by (2.13), we have $\rho_A=\theta_{3\times 3}$ and

$$Qy(t)=\frac{\Gamma(\alpha+1)}{\xi^\alpha-1}(I-\rho_A)g(y)t^{\alpha-1}=\frac{3\sqrt{\pi}}{\sqrt{2}-4}g(y)t^{\alpha-1}, \tag{4.16}$$

where

$$g(y)=\frac{2A}{\sqrt{\pi}}\int_0^{1/2}y(s)\,ds-\frac{2I}{\sqrt{\pi}}\int_0^1 y(s)\,ds=\frac{2I}{\sqrt{\pi}}\int_{\frac{1}{2}}^1 y(s)\,ds.$$

By (2.3), we have

$$N(et^{\frac{1}{2}})(t)=\begin{cases}\frac{1}{8}\left(\sigma_2t^{\frac{1}{2}},-\sigma_1t^{\frac{1}{2}},1\right)^\top, & |\sigma_3|<1,\\ \frac{1}{8}\left(\sigma_2t^{\frac{1}{2}},-\sigma_1t^{\frac{1}{2}},\sigma_3+\frac{1}{\sigma_3}-1\right)^\top, & |\sigma_3|\geq 1.\end{cases} \tag{4.17}$$

Let $A_2=1$, for $\min\{|\sigma_1|,|\sigma_2|,|\sigma_3|\}>1$, it follows from (4.16) and (4.17) that

$$Q\left(Net^{\frac{1}{2}}\right)=\frac{3\sqrt{\pi}}{\sqrt{2}-4}Ag\left(Net^{\frac{1}{2}}\right)t^{\frac{1}{2}}=\frac{3t^{\frac{1}{2}}}{8\sqrt{2}-32}\left(\frac{4-\sqrt{2}}{3}\sigma_2,\frac{\sqrt{2}-4}{3}\sigma_1,\sigma_3+\frac{1}{\sigma_3}-1\right)^\top$$

and

$$\left\langle e,QNet^{\frac{1}{2}}\right\rangle=\frac{3t^{\frac{1}{2}}}{8\sqrt{2}-32}\left[(\sigma_3-1/2)^2+3/4\right]<0.$$

Therefore, (4.14) admits at least one solution.

5 Concluding remarks

In this paper, we consider fractional BVPs at resonance in $\mathbb{R}^n$. The dimension of the kernel of fractional differential operator with the boundary conditions can take any value in $\{1,2\ldots,n\}$, which generalizes the existing literature [4–6, 8, 9, 13, 14, 18–20], where the $\dim\ker L=1$ for $n=1$, or $\dim\ker L=2$ for $n=2$. The illustrative examples validate the applicability of Theorem (2.7). Note that only the two particular cases: $A^2=A$, $A^2=I$ are considered. For the general A satisfying $\operatorname{rank}(I-A)<n$, the system is still resonant. However, we do not know for the system if there are solutions for (1.1) due to some difficulty in constructing the projector Q. We shall investigate this problem in our forthcoming paper. Finally, our result can also be easily generalized to other fractional BVPs, for instance,

$$\begin{cases}D_{0+}^\alpha x(t)=f(t,x(t),D_{0+}^{\alpha-1}x(t)), & 1<\alpha\leq 2,\ t\in(0,1),\\ I_{0+}^{2-\alpha}x(t)|_{t=0}=\theta, & x(1)=Ax(\xi),\end{cases} \tag{5.1}$$

where the matrix A satisfies $\operatorname{rank}(I-A\xi^{\alpha-1})<n$ which means this system is resonant. Particularly, when $\alpha=2$, the system (1.1) and (5.1) become a system of second order differential equations, which can be regarded as a generalization of the results in [16], where a system of second order differential equations was considered.

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