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Three crossing limit cycles in planar piecewise linear systems with saddle-focus type

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Abstract. This paper presents an analysis on the appearance of limit cycles in planar Filippov system with two linear subsystems separated by a straight line. Under the restriction that the orbits with points in the sliding and escaping regions are not considered, we provide firstly a topologically equivalent canonical form of saddle-focus dynamic with five parameters by using some convenient transformations of variables and parameters. Then, based on a very available fourth-order series expansion of the return map near an invisible parabolic type tangency point, we show that three crossing limit cycles surrounding the sliding set can be bifurcated from generic codimension-three singularities of planar discontinuous saddle-focus system. Our work improves and extends some existing results of other researchers.

Keywords: planar Filippov systems, canonical form, limit cycle, bifurcation.

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1 Introduction

Piecewise linear systems often appear in the descriptions of many real processes such as dry friction in mechanical systems or switches in electronic circuits; for instance, see [3,5,7,19,20]. This kind of systems are generally modeled by ordinary differential equations with discontinuous right-hand sides which can exhibit very complicated dynamics and rich bifurcation phenomenons. The basic methods of qualitative theory are established by Filippov in the book [8]. Up till now, piecewise linear systems have been developed very fast and a large number of books and papers have been published on this topic, see for instance [4,10,15,21].

One of the main problems in qualitative theory of piecewise linear systems is the determination of limit cycles. As we know, the occurrence of limit cycle in smooth systems can be provided through the analysis of Hopf bifurcation surrounding a singular point, however, such an approach fails for piecewise linear systems since the basic requirement of smoothness is not fulfilled here. For this reason, many authors have contributed to develop several valid systematic methods in order to overcome the obstacles in recent years; for instance, see [1,2,6,9,11–14,16–18,22].

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The systems being considered in this paper are planar piecewise linear systems with two linearity regions separated by a straight line, where we will assume that the two linearity regions in the phase plane are the left and right half planes

$$\Sigma^{-} = \{(x,y) \in R^2 \mid x < 0\}), \qquad \Sigma^{+} = \{(x,y) \in R^2 \mid x > 0\},$$

and the line is $\Sigma = \{(x,y) \in R^2 \mid x = 0\}$, for sake of simplicity. Thus the dynamical systems have the following form

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{cases} f^{+}(\mathbf{x}) = (f_1^{+}(\mathbf{x}),\ f_2^{+}(\mathbf{x}))^{\mathrm{T}} = A^{+}\mathbf{x} + \mathbf{u}^{+}, & \text{if } \mathbf{x} \in \Sigma^{+}, \\ f^{-}(\mathbf{x}) = (f_1^{-}(\mathbf{x}),\ f_2^{-}(\mathbf{x}))^{\mathrm{T}} = A^{-}\mathbf{x} + \mathbf{u}^{-}, & \text{if } \mathbf{x} \in \Sigma^{-}, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x,y)^{\mathrm{T}} \in R^2, A^{\pm} = (a_{ij}^{\pm})$ are 2×2 real constant matrices and $\mathbf{u}^{\pm} = (u_1^{\pm}, u_2^{\pm})^{\mathrm{T}}$ are real constant vectors in R^2. Obviously, type (1.1) can be also rewrited as two linear differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^{+} & a_{12}^{+} \\ a_{21}^{+} & a_{22}^{+} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_1^{+} \\ u_2^{+} \end{pmatrix}, \quad (x,y)^{\mathrm{T}} \in \Sigma^{+}, \tag{1.2}$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^{-} & a_{12}^{-} \\ a_{21}^{-} & a_{22}^{-} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_1^{-} \\ u_2^{-} \end{pmatrix}, \quad (x,y)^{\mathrm{T}} \in \Sigma^{-}, \tag{1.3}$$

which are called the *right* and *left subsystems* of (1.1), respectively.

The possible existence of limit cycles of system (1.1) has been considered in many papers; for instance, see [9, 12–14]. Based on the assumption that system (1.1) is a special dynamical case of focus-focus type, i.e., there is a focus point in each subsystem, the paper [9] provides a reduced canonical form of (1.1) with five parameters and proves that two crossing limit cycles can be obtained through a degenerate Hopf bifurcation at infinity and a degenerate pseudo-Hopf bifurcation at the origin. In [12], the Hopf bifurcation of (1.1) is also studied and the appearance of two limit cycles is shown near a focus point of three different cases. Note that at the end of paper [12], the authors conjecture that the maximum number of limit cycles for this class of piecewise linear differential systems is exactly two.

It is easy to see there are several different dynamical types of system (1.1) from the one considered in [9] if we vary the type of singular point in one of the linear regions. For example, the saddle-saddle type and node-node type are studied in [13] and [14], respectively. In these two cases, the authors prove the existence of at least two nested limit cycles and some parameter regions where two nested limit cycles exist are given.

In this paper, we are interesting in the saddle-focus case of (1.1); that is, one subsystem is of saddle type and the other is of focus type. The main purpose of this paper is to investigate the existence and number of crossing period orbits for the saddle-focus dynamics. Note that the linear subsystems (1.2) and (1.3) themselves do not have any limit cycles in the regions Σ^{+} and Σ^{-}, the necessary condition to obtain it is such orbit intersecting transversally the line Σ at least twice. Thus the orbits possessing points in sliding and escaping regions are not needed to study for the existence of crossing limit cycles of (1.1). In the light of the above considerations, a reduced saddle-focus type canonical form with five parameters will be induced firstly by some convenient changes of variables and parameters. Then, we prove that three crossing limit cycles surrounding the sliding set can appear from a codimension-three local bifurcation. This implies that the conjecture in [12] is not correct.

2 Preliminary definitions and notations

Due to the discontinuity of the vector field of system (1.1), the usual concepts of flow (or orbit in phase space) and singularity need to be generalized from the classical smooth ones. In this section, we will use the techniques and approaches presented by Filippov in [8] to establish these notations.

Clearly, for any point $p(x,y) \in \Sigma^{\pm}$, the local flow of (1.1), denoted by $\varphi(t,p) = \varphi^{\pm}(t,p)$, can be defined by the vector fields $(f_1^{\pm}, f_2^{\pm})$ as usual. In particular, a point $p(x,y)$ satisfying $\varphi(t,p) = p$ for all $t \in R$ is called a *real (virtual) singular point* of the right subsystem (1.2) if this point locates in the region $x > 0$ $(x < 0)$. A similar definition can be done for the left subsystem (1.3).

If $p(0,y) \in \Sigma$ satisfying $f_1^+(p)f_1^-(p) > 0$, then both vector fields point towards the same direction from one side of Σ to the other. Thus, the local flow $\varphi(t,p)$ can be connected by matching the flows of (f_1^+, f_2^+) and (f_1^-, f_2^-) through the point p. This point is called a *crossing point*, and the *cross region* is defined as follows

$$\Sigma^c = \{p \in \Sigma \mid f_1^+(p)f_1^-(p) > 0\}.$$

If $p(0,y) \in \Sigma$ satisfying $f_1^+(p)f_1^-(p) < 0$, then the local flow $\varphi(t,p)$ is given by the vector field

$$(f_1^0(p),\ f_2^0(p)) = \left(0,\ \frac{f_1^+(p)f_2^-(p) - f_1^-(p)f_2^+(p)}{f_1^+(p) - f_1^-(p)}\right),$$

which is the linear convex combination of (f_1^+, f_2^+) and (f_1^-, f_2^-) tangent to Σ. Here we speak of p as a *sliding (escaping) point* when $f_1^+(p) < 0$ and $f_1^-(p) > 0$ $(f_1^+(p) > 0$ and $f_1^-(p) < 0)$. The corresponding *sliding and escaping regions* are denoted by

$$\Sigma^s = \{p \in \Sigma \mid f_1^+(p) < 0,\ f_1^-(p) > 0\},$$

and

$$\Sigma^e = \{p \in \Sigma \mid f_1^+(p) > 0,\ f_1^-(p) < 0\},$$

respectively.

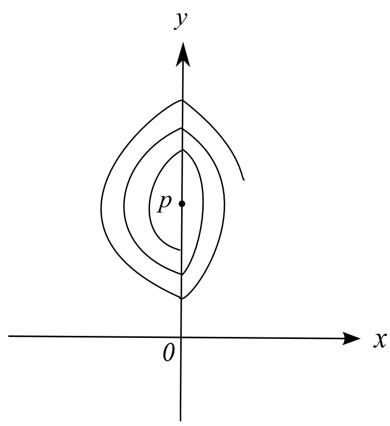

Figure 2.1: A pseudo-focus point p of system (1.1).

If one of the vectors $(f_1^{\pm}, f_2^{\pm})$ is tangent to Σ at a point $p(0,y)$, that is $f_1^+(p) = 0$ or $f_1^-(p) = 0$, then this point is called a *tangency point*. Furthermore, we say that the point p is a *visible (invisible) tangency point* of (1.2) if the local flow of (f_1^+, f_2^+) passing through p at time

$t = t_p$ remains in the region $x > 0$ ($x < 0$). Analogously, an equivalent definition can be given for the subsystem (1.3).

A point $p(0, y) \in \Sigma$ is called pseudo-focus point (or focus for simplification) if $f_1^+(p) = f_1^-(p) = 0$ and the orbits of (1.1) spiral around p in its neighborhood. In this case, both vector fields $(f_1^\pm, f_2^\pm)$ vanish at the point p (see Figure 2.1). From [6,12], we know that there are four possible types of focus, denoted by FF, FP, PF and PP, where F stands for the word "focus" and P the word "parabolic".

3 Canonical form of the saddle-focus dynamics

The establishment of the canonical form is a very important task in the study of planar piecewise linear systems related to complex behavior such as bifurcation and stability of limit cycles. On this issue one usually needs to adopt the idea of topological equivalence to reduce the number of parameters of model (1.1). In [9], the Liénard-like canonical form of (1.1) with seven parameters is obtained by making a continuous piecewise linear change of variables, which is invariant on the discontinuous line Σ and is homeomorphic in the open half-planes Σ^+ and Σ^-.

Denote the traces and determinants of $A^\pm$ by $T^\pm$ and $D^\pm$, respectively. We now proceed as in [9, Proposition 3.1], which we repeat here for completeness.

Proposition 3.1. *Assume that $a_{12}^+ a_{12}^- > 0$ in system (1.1). Then the homeomorphism $\tilde{\mathbf{x}} = h(\mathbf{x})$ given by*

$$\tilde{\mathbf{x}} = M^-(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ -a_{11}^- & -a_{12}^- \end{bmatrix} \mathbf{x} - \begin{bmatrix} 0 \\ u_1^- \end{bmatrix}, \quad \text{if } \mathbf{x} \in \Sigma^- \cup \Sigma,$$

and

$$\tilde{\mathbf{x}} = M^+(\mathbf{x}) = \frac{a_{12}^-}{a_{12}^+} \begin{bmatrix} 1 & 0 \\ -a_{11}^+ & -a_{12}^+ \end{bmatrix} \mathbf{x} - \begin{bmatrix} 0 \\ u_1^- \end{bmatrix}, \quad \text{if } \mathbf{x} \in \Sigma^+,$$

after dropping tildes, transforms system (1.1) into the canonical forms

$$\dot{\mathbf{x}} = g(\mathbf{x}) = \begin{cases} g^-(\mathbf{x}) = \begin{bmatrix} g_1^-(\mathbf{x}) \\ g_2^-(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ D^- & T^- \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ v_2^- \end{bmatrix}, & \text{if } \mathbf{x} \in \mathbf{\Sigma}^-, \\ \\ g^+(\mathbf{x}) = \begin{bmatrix} g_1^+(\mathbf{x}) \\ g_2^+(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ D^+ & T^+ \end{bmatrix} \mathbf{x} + \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix}, & \text{if } \mathbf{x} \in \mathbf{\Sigma}^+, \end{cases} \tag{3.1}$$

where

$$v_2^- = a_{22}^- u_1^- - a_{12}^- u_2^-, \quad v_1^+ = \frac{a_{12}^-}{a_{12}^+} u_1^+ - u_1^-,$$

$$v_2^+ = -\frac{a_{12}^-}{a_{12}^+}(a_{11}^+ u_1^+ + a_{12}^+ u_2^+) + (a_{11}^+ + a_{22}^+) u_1^-.$$

Besides the invariance of the discontinuity line Σ, the crossing, sliding and escaping regions of system (1.1) are transformed by the homeomorphism h into line and regions of the same type for system (3.1).

Moreover, there is a topological equivalence between systems (1.1) and (3.1) for all their orbits not having points in the common with the sliding and escaping regions. However, the homeomorphism h preserves the attractive or repulsive character of the sliding and escaping regions.

Remark 3.2. The assumption $a^+_{12}a^-_{12} > 0$ in Proposition 3.1 is a sufficient condition such that the vector fields of original system (1.1) and induced system (3.1) preserve the same sign of x-component on the discontinuous line. In fact, by above change $\tilde{\mathbf{x}} = h(\mathbf{x})$, we have

$$g^-_1(h(0,y)) = a^-_{12}y + u^-_1 = f^-_1(0,y),$$

and

$$g^+_1(h(0,y)) = a^-_{12}y + u^-_1 + v^+_1 = \frac{a^-_{12}}{a^+_{12}}(a^+_{12}y + u^+_1) = \frac{a^-_{12}}{a^+_{12}}f^+_1(0,y).$$

Remark 3.3. The assumption $a^+_{12}a^-_{12} > 0$ in Proposition 3.1 is also a necessary condition for the existence of crossing limit cycles of system (1.1). Effectively, if the crossing region of (1.1) exists with $a^+_{12}a^-_{12} \leq 0$, then the inequality

$$f^+_1(0,y)f^-_1(0,y) = (a^+_{12}y + u^+_1)(a^-_{12}y + u^-_1) > 0$$

implies that the crossing region is an open interval (or a point) of the line Σ; that is, the x-component of both vector fields (f^+_1, f^+_2) and (f^-_1, f^-_2) has constant sign at the crossing region, and so elementary qualitative arguments preclude the existence of crossing limit cycles.

Remark 3.4. The forms of the left and right subsystem of (3.1) can be transformed from each other by the change of variables $(x,y) \to (x, y - v^+_1)$. Geometrically, it is enough to translate vertically the horizontal coordinate axes in the amount v^+_1.

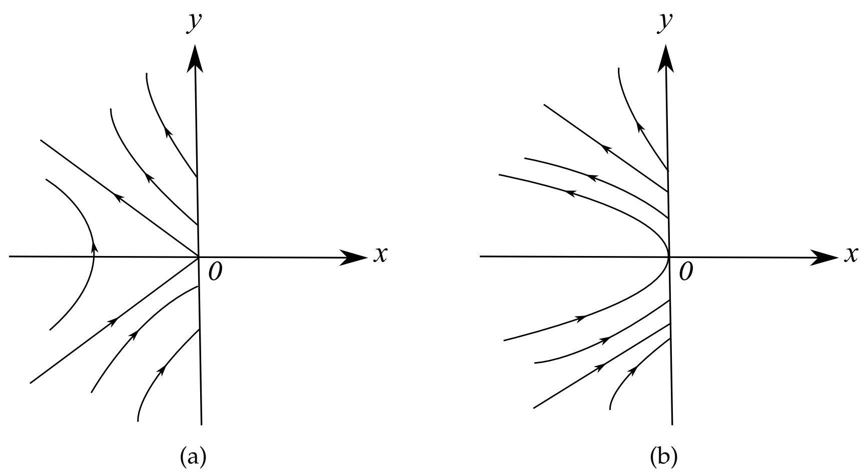

Figure 3.1: From left to right, phase portraits of the left subsystem of (3.1) when $v^-_2 = 0$ and $v^-_2 > 0$, respectively.

In the following, we will deduce the canonical form of saddle-focus dynamics. According to Remark 3.4, we assume without loss of generality that the left subsystem of (3.1) is of saddle type and the right one of focus type, that is $D^- < 0$ in Σ^- and $(T^+)^2 - 4D^+ < 0$ in Σ^+, respectively. Thus, the left subsystem of (3.1) has a saddle point at $(-v^-_2/D^-, 0)$ which is a real singular point for $v^-_2 < 0$, the origin for $v^-_2 = 0$ and a virtual singular point for $v^-_2 > 0$. Note that for the last two cases, no orbit of the left subsystem of (3.1) can touch Σ twice in the half plane $x < 0$ (see Figure 3.1), i.e., there is not any crossing periodic orbit in system (3.1). Therefore, from now on we suppose that $v^-_2 < 0$.

By a direct computation, the simplified saddle-focus model of system (1.1) is obtained.

Proposition 3.5. *Assume that*

$$\frac{T^-}{\sqrt{-D^-}} = \alpha, \qquad \frac{-v_2^-}{\sqrt{-D^-}} = k, \qquad T^+ = 2\gamma, \qquad D^+ = \gamma^2 + \omega^2$$

with $\omega > 0$ in the canonical form (3.1), and let

$$\beta = \frac{\gamma}{\omega}, \qquad a = \frac{v_1^+}{\omega}, \qquad b = \frac{v_2^+}{\omega}.$$

Then the changes of variables

$$(x, t, y) \to \left(\frac{x}{\sqrt{-D^-}}, \frac{t}{\sqrt{-D^-}}, y \right)$$

for the left half plane and

$$(x, t, y) \to \left(\frac{x}{\omega}, \frac{t}{\omega}, y \right)$$

for the right half plane transform the canonical form (3.1) into the form

$$\dot{\mathbf{x}} = \begin{cases} \begin{bmatrix} 0 & -1 \\ -1 & \alpha \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -k \end{bmatrix}, & \text{if } \mathbf{x} \in \Sigma^-, \\[2ex] \begin{bmatrix} 0 & -1 \\ 1+\beta^2 & 2\beta \end{bmatrix} \mathbf{x} + \begin{bmatrix} a \\ b \end{bmatrix}, & \text{if } \mathbf{x} \in \Sigma^+. \end{cases} \tag{3.2}$$

Remark 3.6. The assumption $v_2^- < 0$ implies that $k > 0$ in (3.2). Meanwhile, under the change of variables and time

$$(x, y, t) \to (x, -y, -t),$$

as well the change of parameters

$$(\alpha, \beta, k, a, b) \to (-\alpha, -\beta, k, -a, b),$$

the canonical form (3.2) is invariant, so we only need to consider $a \geq 0$ in the study of system (3.2).

As far as the equilibrium and tangency points of (3.2) are considered, the next proposition is obvious.

Proposition 3.7. *For system (3.2) the following statements hold.*

(i) The left subsystem of (3.2) has a real saddle point at $(-k, 0)$ and an invisible tangency point at the origin.

(ii) If $a > 0$, then the right subsystem of (3.2) has a focus point at $\left(-\frac{2a\beta+b}{1+\beta^2}, a\right)$ and a tangency point at $(0, a)$, which are real and visible when $2a\beta + b < 0$, but virtual and invisible when $2a\beta + b > 0$, respectively.

(iii) If $a = 0$, then the origin is a pseudo-focus point of (3.2) with PF-type for $b = 0$, and PP-type for $b > 0$. If $a = 0$ and $b < 0$, then the origin is an invisible-visible tangency point behaving as a regular point.

4 Limit cycles in saddle-focus canonical form

In this section, the main results about existence and number of crossing period orbits are given for the saddle-focus dynamics. Clearly, the left half system of (3.2) at the saddle point $(-k, 0)$ is determined by a matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & \alpha \end{pmatrix},$$

whose corresponding eigenvalues are

$$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2+4}}{2} > 0 \qquad \text{and} \qquad \lambda_2 = \frac{\alpha - \sqrt{\alpha^2+4}}{2} < 0.$$

Generated by the eigenvalues λ_1 and λ_2 associating with the eigenvectors $(1, -\lambda_1)^T$ and $(1, -\lambda_2)^T$, the stable and unstable manifolds of the saddle point contain half lines

$$W_-^s = \{(x, y) \mid y = -\lambda_2(x + k),\ x \le 0\}$$

and

$$W_-^u = \{(x, y) \mid y = -\lambda_1(x + k),\ x \le 0\},$$

which intersect the switched line Σ at the points $(0, -k\lambda_2)$ and $(0, -k\lambda_1)$, respectively. Thus for any point $A = (0, r) \in \Sigma$ satisfying $r > -k\lambda_2 > 0$, as time $t > 0$ increases, the flow $\varphi^-(t, A)$ will enter the half plane Σ^- and not reach Σ again. Similarly, for any point $A' = (0, r') \in \Sigma$ satisfying $r' < -k\lambda_1 < 0$, the flow $\varphi^-(t, A')$ also remains in the half plane Σ^- and does not touch Σ again for $t \in (-\infty, 0)$. Therefore, it is enough to consider the existence of crossing period orbits for system (3.2) in a bounded region including the origin.

On the other hand, from the statement (ii) of Proposition 3.7, we know that the focus point of the right subsystem of (3.2) is real when $2a\beta + b < 0$. This condition also implies that the right tangency point $(0, a)$ is visible, namely, the local flow $\varphi^+(t)$ passing through $(0, a)$ at $t = t_0$ remains in the region $x > 0$ and spirals around the real focus point as time t increases or decreases depending on the stability of right focus point. As a result, a sliding periodic orbit surrounding the real focus point may occur; see [11]. Here we point out that such an issue is not considered in this paper and will appear elsewhere, and so in the remainder of this paper we assume $2a\beta + b \ge 0$.

Under the above restrictions, we know that both the left tangency point $(0, 0)$ and the right one $(0, a)$ of (3.2) are invisible. In order to obtain the existence of crossing periodic orbits, it is very necessary to analyse the return map of (3.2) near these tangency points. For this aim, we give an important lemma as follows.

Lemma 4.1. *Let $x = \Phi(y)$ be the solution of initial value problem*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}y} = ly(1 + (my + nx) + (my + nx)^2 + (my + nx)^3 + \cdots), \\ \Phi(-r) = 0, \end{cases} \tag{4.1}$$

where $l \ne 0$ and $r > 0$. If there is a point $(0, \rho)$ with $\rho > 0$ such that $\Phi(\rho) = 0$ and

$$l\Phi(y) < 0 \quad \text{for } -r < y < \rho,$$

then for $r > 0$ and sufficiently small,

$$\rho = r - \frac{2}{3}mr^2 + \frac{4}{9}m^2r^3 - \left(\frac{44}{135}m^3 - \frac{2}{15}lmn\right)r^4 + o(r^4).$$

Proof. For $0 < r \ll 1$, the solution $x = \Phi(y)$ of initial value problem (4.1) can be written as a power series expansion with respect to time t, i.e.,

$$y(t) = t - r, \qquad x(t) = a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots \tag{4.2}$$

with $y(0) = -r$ and $x(0) = 0$, where a_i are undetermined coefficients for $i \in Z^+$. Substituting (4.2) into the first expression of (4.1) and comparing coefficients of t from t^0 to t^5, we obtain

$$\begin{aligned}
a_1 &= l(-r + mr^2 - m^2r^3 + m^3r^4 + o(r^4)),\\
a_2 &= \frac{l}{2}(1 - 2mr + (3m^2 + ln)r^2 - (4m^3 + 3lmn)r^3 + o(r^3)),\\
a_3 &= \frac{l}{6}(2m - (6m^2 + 3ln)r + (12m^3 + 14lmn)r^2 + o(r^2)),\\
a_4 &= \frac{l}{24}(6m^2 + 3ln - (24m^3 + 32lmn)r + o(r)),\\
a_5 &= \frac{l}{120}(24m^3 + 32lmn + o(1)).
\end{aligned}$$

Denote by t^* the minimum positive time such that $x(t) = 0$, then by inverting series (4.2) we have

$$t^* = 2r - \frac{2}{3}mr^2 + \frac{4}{9}m^2r^3 - \left(\frac{44}{135}m^3 - \frac{2}{15}lmn\right) r^4 + o(r^4),$$

and so

$$y(t^*) = r - \frac{2}{3}mr^2 + \frac{4}{9}m^2r^3 - \left(\frac{44}{135}m^3 - \frac{2}{15}lmn\right) r^4 + o(r^4)$$

completes the proof. □

Theorem 4.2. *Assuming $a = b = 0$ in system (3.2), the following statements hold.*

(i) If $\alpha = \beta = 0$, then the origin is a nonlinear center.

(ii) If $\beta = 0$, then the origin is asymptotically stable for $\alpha < 0$ and unstable for $\alpha > 0$.

(iii) If $\alpha < 0 < \beta \ll 1$, then the the origin is unstable and it is surrounded by a stable limit cycle Γ_1^s.

(iv) If $\alpha > 0$ and $0 < -\beta \ll 1$, then the the origin is asymptotically stable and it is surrounded by an unstable limit cycle Γ_1^u.

Proof. (i) It is easy to see when $a = b = \alpha = \beta = 0$, the orbits of both the left and the right subsystem of (3.2) are symmetric with respect to the line $y = 0$, hence any orbit surrounding the origin is closed and statement (i) follows.

(ii) From Proposition 3.7 (iii), the origin is a PF-type pseudo-focus point for $a = b = 0$. It means stating at any point $(0, y_0)$ with $0 < y_0 \ll 1$, the orbits of the left subsystem will go into the zones Σ^- in a counterclockwise direction until they reach Σ at a point $(0, y_1)$ with $y_1 < 0$ after a time t^-. Now we can define a left return map P_L as $y_1 = P_L(y_0) < 0$ with $P_L(0) = 0$. and a right return map P_R as $z_1 = P_R(z_0) > 0$ with $P_R(0) = 0$ and $0 < -z_0 \ll 1$. Subsequently, the Poincaré map of (3.2) near the origin is constructed by P_L and P_R as

$$\tilde{r} = P(\sigma) = P_R(P_L(\sigma)) \quad \text{for } 0 < \sigma \ll 1 \tag{4.3}$$

satisfying $P(0) = 0$. In what follows, we give the detailed calculations for P_R and P_L.

Under the condition $a = b = 0$, the solutions of right subsystem of (3.2) have the forms

$$\begin{cases} x(t) = e^{\beta t}[(\cos t - \beta \sin t) \cdot x(0) - \sin t \cdot y(0)], \\ y(t) = e^{\beta t}[(1+\beta^2) \sin t \cdot x(0) + (\cos t + \beta \sin t) \cdot y(0)]. \end{cases}$$

Let the initial value $y(0) < 0$ and denote by t^+ the minimum positive time such that $x(t^+) = x(0) = 0$, then $t^+ = \pi$. For this case, the right return map P_R and its inverse P_R^{-1} are

$$P_R(y(0)) = y(\pi) = -e^{\beta\pi} y(0) > 0,$$

and

$$P_R^{-1}(y(\pi)) = y(0) = -e^{-\beta\pi} y(\pi) < 0, \tag{4.4}$$

respectively.

In order to determine P_L near the origin, we recast the left subsystem of (3.2) as

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{k - (\alpha y - x)} = \frac{y}{k}\left(1 + \frac{\alpha y - x}{k} + \frac{(\alpha y - x)^2}{k^2} + \frac{(\alpha y - x)^3}{k^3} + \cdots\right), \quad \text{for } x < 0.$$

Suppose that $x = \Phi_L(y)$ is the solution of above equation with initial value $\Phi_L(-r) = 0$ for $0 < r \ll 1$, then $\Phi_L(P_L^{-1}(-r)) = 0$ with $P_L^{-1}(-r) > 0$. By Lemma 4.1, we have

$$P_L^{-1}(-r) = r - \frac{2\alpha}{3k} r^2 + \frac{4\alpha^2}{9k^2} r^3 - \frac{44\alpha^3 + 18\alpha}{135k^3} r^4 + o(r^4). \tag{4.5}$$

Now setting $y(0) = -r < 0$ and $y(\pi) = \tilde{r} > 0$, by combining (4.4) and (4.5), we have

$$\begin{aligned} P^{-1}(\tilde{r}) &= P_L^{-1}(P_R^{-1}(\tilde{r})) = P_L^{-1}(-e^{-\beta\pi}\tilde{r}) \\ &= e^{-\beta\pi}\tilde{r} - \frac{2\alpha}{3k} e^{-2\beta\pi}\tilde{r}^2 + \frac{4\alpha^2}{9k^2} e^{-3\beta\pi}\tilde{r}^3 - \frac{44\alpha^3 + 18\alpha}{135k^3} e^{-4\beta\pi}\tilde{r}^4 + o(\tilde{r}^4), \end{aligned}$$

which yields

$$P^{-1}(\tilde{r}) - \tilde{r} = (e^{-\beta\pi} - 1)\tilde{r} - \frac{2\alpha}{3k} e^{-2\beta\pi}\tilde{r}^2 + o(\tilde{r^2}). \tag{4.6}$$

Therefore by (4.3), we obtain

$$P(\sigma) = \tilde{r} < P^{-1}(\tilde{r}) = \sigma$$

for $\beta = 0$ and $\alpha < 0$, and

$$P(\sigma) = \tilde{r} > P^{-1}(\tilde{r}) = \sigma$$

for $\beta = 0$ and $\alpha > 0$. This completes the proof of statement *(ii)*.

(iii) If $\alpha < 0 < \beta \ll 1$, by (4.6) there exists

$$\tilde{r}^* = -\frac{3k\pi}{2\alpha}\beta + o(\beta) > 0$$

satisfying $P^{-1}(\tilde{r}^*) - \tilde{r}^* = 0$, which leads to the existence of a limit cycle Γ_1^s surrounding the origin. Moreover, by using (4.6) again,

$$\left(\frac{\mathrm{d}P^{-1}}{\mathrm{d}\tilde{r}}\right)_{\tilde{r}=\tilde{r}^*} = e^{-\beta\pi} - \frac{4\alpha}{3k} e^{-2\beta\pi}\tilde{r}^* + o(\tilde{r}^*) = 1 + \pi\beta + o(\beta) > 1,$$

so the obtained crossing limit cycle Γ_1^s is stable. The statement *(iii)* holds.

(iv) For $\alpha > 0$ and $0 < -\beta \ll 1$, a similar argument as the proof of case *(iii)* guarantees the final statement of Theorem 4.2. □

Next, we will study the existence and number of crossing limit cycles for $a = 0$ and $b \neq 0$. As pointed out in the former part of this section that $2a\beta + b \geq 0$ is assumed, it is sufficient to consider the case $a = 0$ and $b > 0$.

Theorem 4.3. *Assuming $a = 0$ and $b > 0$ in system (3.2), the following statements hold.*

(i) If $\alpha < 0 < b < -\frac{2k\beta}{\alpha} \ll 1$, then the origin is unstable and it is surrounded by a stable limit cycle Γ_1^s, while for $\alpha < 0 < -\frac{2k\beta}{\alpha} \leq b \ll 1$, the origin is asymptotically stable and it is surrounded by a stable limit cycle Γ_1^s and an unstable limit cycle Γ_2^u.

(ii) If $\alpha > 0$ and $0 < b < -\frac{2k\beta}{\alpha} \ll 1$, then the origin is asymptotically stable and it is surrounded by an unstable limit cycle Γ_1^u, while for $\alpha > 0$ and $0 < -\frac{2k\beta}{\alpha} \leq b \ll 1$, the origin is unstable and it is surrounded by an unstable limit cycle Γ_1^u and a stable limit cycle Γ_2^s.

Proof. We only consider statement (i) and similar treatments can be done for statement (ii).

From Proposition 3.7 (iii), the origin is a PP-type pseudo-focus point of (3.2) when $a = 0$ and $b > 0$. Note that the left return map P_L near the origin does not depend upon the parameter b, so the expression (4.5) remains unchanged here. To determine the right return map P_R, we rewrite the right subsystem of (3.2) as a power series expansion

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{-y}{(1+\beta^2)x + 2\beta y + b} \\ &= -\frac{y}{b}\left(1 + \left(-\frac{2\beta}{b}y - \frac{1+\beta^2}{b}x\right) + \left(-\frac{2\beta}{b}y - \frac{1+\beta^2}{b}x\right)^2 + \cdots\right), \quad \text{for } x > 0.\end{aligned}$$

Let $x = \Phi_R(y)$ be the solution of the above equation with initial value $\Phi_R(-r) = 0$ for $0 < r \ll 1$, then $\Phi_R(P_R(-r)) = 0$ with $P_R(-r) > 0$. By Lemma 4.1, it follows that

$$P_R(-r) = r + \frac{4\beta}{3b}r^2 + \frac{16\beta^2}{9b^2}r^3 + \frac{316\beta^3 - 36\beta}{135b^3}r^4 + o(r^4). \tag{4.7}$$

Setting $P_L^{-1}(-r) = \sigma$, we can get from (4.5) and (4.7)

$$\begin{aligned}P(\sigma) - \sigma &= P_R(-r) - P_L^{-1}(-r) \\ &= V_2 r^2 + V_3 r^3 + V_4 r^4 + o(r^4),\end{aligned} \tag{4.8}$$

where

$$V_2 = \frac{2}{3kb}(2k\beta + b\alpha), \qquad V_3 = \frac{4}{9k^2b^2}(4k^2\beta^2 - b^2\alpha^2),$$
$$V_4 = \frac{2}{135k^3b^3}(158k^3\beta^3 - 18k^3\beta + 22\alpha^3b^3 + 9\alpha b^3).$$

Now, we distinguish two cases to consider.

(a) If $\alpha < 0 < b < -\frac{2k\beta}{\alpha} \ll 1$, then $V_2 > 0$. By (4.8) we have $P(\sigma) - \sigma > 0$ for $0 < r \ll 1$, and so the origin is unstable. Meanwhile, according to the continuous dependence of solution with respect to parameters, the stable limit cycle Γ_1^s obtained in Theorem 4.2 (iii) always persist for $0 < b \ll 1$.

(b) If $\alpha < 0 < -\frac{2k\beta}{\alpha} < b \ll 1$, then $V_2 < 0$. In addition, for $\alpha < 0$ and $b = -\frac{2k\beta}{\alpha} > 0$, the equality $V_2 = V_3 = 0$ and the inequality $V_4 < 0$ hold. Thus by (4.8), we have $P(\sigma) - \sigma < 0$ for $\alpha < 0 < -\frac{2k\beta}{\alpha} \leq b$ and $0 < r \ll 1$, and so the origin is asymptotically stable. Furthermore, the fact of case (a) shows that the Poincaré map P changes from unstable to stable under the transition of parameter b from $0 < b < -\frac{2k\beta}{\alpha}$ to $b \geq -\frac{2k\beta}{\alpha}$, respectively. Therefore a non-smooth Hopf-like bifurcation must occurs and so the existence of an unstable limit cycle Γ_2^u is obtained for $\alpha < 0 < -\frac{2k\beta}{\alpha} < b$. By applying Theorem 4.2 (iii) again, the conclusion of Theorem 4.3 (i) is shown directly. □

Finally, the existence and number of crossing limit cycles of system (3.2) will be investigated under $a > 0$ and $b > 0$. In this case, we have the following result.

Theorem 4.4. *Assuming $a > 0$ and $b > 0$ in system (3.2), the following statements hold.*

(i) If $\alpha < 0 < -\frac{2k\beta}{\alpha} < b \ll 1$ and $0 < a \ll 1$, then near the origin system (3.2) has two stable limit cycles Γ_1^s and Γ_3^s, and an unstable limit cycle Γ_2^u.

(ii) If $\alpha > 0$, $0 < -\frac{2k\beta}{\alpha} < b \ll 1$ and $0 < a \ll 1$, then near the origin system (3.2) has two unstable limit cycles Γ_1^u and Γ_3^u, and a stable limit cycle Γ_2^s.

Proof. In the following, we just prove statement (i) since these arguments are also valid for case (ii). According to Theorem 4.3 (i), both the stable limit cycle Γ_1^s and the unstable limit cycle Γ_2^u always persist for $\alpha < 0 < -\frac{2k\beta}{\alpha} < b \ll 1$ and $0 < a \ll 1$. Therefore, it is sufficient to show the existence of a stable limit cycle Γ_3^s.

From statement (ii) of Proposition 3.7, the right subsystem of (3.2) has an invisible tangency point at $(0, a)$ when $a > 0$ and $b > 0$. Obviously, the right return map P_R surrounding this tangency point is always dependent on parameter a, and here we denote it by $P_R(y; a)$ with $y \leq a$.

After making a translation $(\tilde{x}, \tilde{y}) = (x, y - a)$, the right subsystem of (3.2) is changed into

$$\begin{aligned} \dot{\tilde{x}} &= -\tilde{y}, \\ \dot{\tilde{y}} &= (1+\beta^2)\tilde{x} + 2\beta\tilde{y} + 2a\beta + b. \end{aligned} \tag{4.9}$$

By the same derivation as for (4.7), the right return map of (4.9) near the origin is

$$\tilde{P}_R(-\tilde{r}) = \tilde{r} + \frac{4\beta}{3(b+2a\beta)}\tilde{r}^2 + \frac{16\beta^2}{9(b+2a\beta)^2}\tilde{r}^3 + \frac{316\beta^3 - 36\beta}{135(b+2a\beta)^3}\tilde{r}^4 + o(\tilde{r}^4),$$

for $0 < \tilde{r} \ll 1$. Thanks to $P_R(y; a) = \tilde{P}_R(y - a) + a$, we have

$$\begin{aligned} P_R(y; a) = 2a - y &+ \frac{4\beta}{3(b+2a\beta)}(a-y)^2 + \frac{16\beta^2}{9(b+2a\beta)^2}(a-y)^3 \\ &+ \frac{316\beta^3 - 36\beta}{135(b+2a\beta)^3}(a-y)^4 + o((a-y)^4), \end{aligned} \tag{4.10}$$

where $0 < a - y \ll 1$.

Now, it is easy to see that the existence of a crossing limit cycle is equivalent to the existence of a positive $\bar{r}$ satisfying $P_R(-\bar{r}; a) = P_L^{-1}(-\bar{r})$, in other words, the existence of zero for the function

$$\Psi(r, a) = P_R(-r; a) - P_L^{-1}(-r) \tag{4.11}$$

with respect to variable r. Because of this, we pay our attention to study the existence of zeros for (4.11) by applying the implicit function theorem.

From (4.11) we know that $\Psi(0,0) = 0$. And for $0 < a, r \ll 1$, by using (4.5) and (4.10),

$$\begin{aligned}\Psi(r,a) = 2a &+ \left(\frac{4\beta}{3(b+2a\beta)}(a+r)^2 + \frac{2\alpha}{3k}r^2\right)\\ &+ \left(\frac{16\beta^2}{9(b+2a\beta)^2}(a+r)^3 - \frac{4\alpha^2}{9k^2}r^3\right)\\ &+ \left(\frac{316\beta^3 - 36\beta}{135(b+2a\beta)^3}(a+r)^4 + \frac{44\alpha^3+18\alpha}{135k^3}r^4\right) + o((a+r)^4, r^4),\end{aligned} \tag{4.12}$$

which results in

$$\frac{\partial \Psi}{\partial a}(0,0) = 2.$$

Thus by the implicit function theorem, there exists a smooth function $a = \delta(r)$, defined in a neighborhood of 0 with $\delta(0) = 0$, such that $\Psi(r, \delta(r)) = 0$.

Next, we need to show that $\delta(r) > 0$ for $0 < r \ll 1$. From (4.12), we have

$$\frac{\partial \Psi}{\partial r}(0,0) = 0, \qquad \frac{\partial^2 \Psi}{\partial r^2}(0,0) = \frac{4}{3b}(2k\beta + b\alpha),$$

and so

$$\delta'(0) = -\frac{\Psi_r(0,0)}{\Psi_a(0,0)} = 0.$$

Furthermore, after neglecting some vanishing terms, we have

$$\delta''(0) = -\frac{\Psi_{rr}(0,0)}{\Psi_a(0,0)} = -\frac{2}{3b}(2k\beta + b\alpha),$$

from which $\delta''(0) > 0$ follows for $\alpha < 0 < -\frac{2k\beta}{\alpha} < b \ll 1$. According to the smoothness of $\delta(r)$, we conclude that $a = \delta(r) > 0$ for $0 < r \ll 1$, i.e., the existence of a crossing limit cycle Γ_3^s is proven. It is worthwhile to note that the occurrence of Γ_3^s is dependent on the positive parameter a, however, the other two limit cycles Γ_1^s or Γ_2^u are not the case. This shows Γ_3^s is different from them.

Finally, for $\alpha < 0 < -\frac{2k\beta}{\alpha} < b \ll 1$ we have

$$\frac{\partial^2 \Psi}{\partial r^2}(0,0) = \frac{4}{3b}(2k\beta + b\alpha) < 0,$$

which leads to $\frac{\partial \Psi}{\partial r}(r,a) < 0$ for $0 < a \ll 1$, hence the deduced limit cycle Γ_3^s is stable. This completes the proof of the statement (i). □

5 Conclusion

In this paper we have studied the existence and number of limit cycles for planar Filippov system (1.1), whose right-hand side depends on twelve parameters. By using linear variable transformations and time rescaling, we have transformed system (1.1) into the topologically equivalent saddle-focus canonical form (3.2) containing only five parameters. It is proved, under additional assumptions, that system (3.2) has three limit cycles surrounding the sliding set. This improves and extends the results and conjecture provided by M. Han and W. Zhang in [12]. Our proof is based on the change of stability of a singular point and the existence of fix points of the Poincaré map. To use this method we have deduced a very available fourth-order series expansion of return map near an invisible parabolic type tangency point. We believe this expression can be used widely in the analysis of Hopf bifurcation for planar non-smooth systems.

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