Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **73**, 1–10; http://www.math.u-szeged.hu/ejqtde/

Asymptotic behavior and uniqueness of boundary blow-up solutions to elliptic equations

Qiaoyu Tian ✉ and **Yonglin Xu**

School of Mathematics and Computer Science, Northwest University for Nationalities,
Lanzhou 730030, P.R. China

Received 10 July 2014, appeared 11 January 2015
Communicated by Gabriele Villari

Abstract. In this paper, under some structural assumptions of weight function $b(x)$ and nonlinear term $f(u)$, we establish the asymptotic behavior and uniqueness of boundary blow-up solutions to semilinear elliptic equations

$$\begin{cases} \Delta u = b(x) f(u), & x \in \Omega, \\ u(x) = \infty, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. Our analysis is based on the Karamata regular variation theory and the López-Gómez localization method.

Keywords: boundary blow-up solutions, asymptotic behavior, López-Gómez localization method, Karamata regular variation theory.

2010 Mathematics Subject Classification: 35J75, 35J66.

1 Introduction and main results

In this paper, we deal with the asymptotic behavior of boundary blow-up solutions to semilinear elliptic equations

$$\begin{aligned} \Delta u &= b(x) f(u), & x \in \Omega, \\ u(x) &= \infty, & x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain, weight function $b(x)$ satisfies

(b_1) there exists a positive nondecreasing function $a(x) \in C([0,\delta])$ such that

$$\lim_{d(x)\to 0} \frac{b(x)}{a(d(x))} = 1, \tag{1.2}$$

where

$$\frac{1}{a(r)} \int_0^r a(s)\, ds \in C^1([0,\delta]);$$

✉ Corresponding author. Email: tianqiaoyu2004@163.com

(b_2) for any $x_0 \in \partial\Omega$, there exists $\tau > 0$, such that $b(x) \in C^1(\overline{\Omega}_\tau(x_0) \cap \overline{\Omega})$ satisfies

$$b_{x_0}(r) \in C^1((0,\tau)), \qquad b'_{x_0}(r) > 0 \quad \text{for each } r \in (0,\tau), \tag{1.3}$$

and

$$\lim_{\substack{x\in\partial\Omega,\, x\to x_0,\\ r\to 0}} \frac{b_x(r)}{b_{x_0}(r)} = 1, \tag{1.4}$$

where $\Omega_\tau(x_0)$ is a ball in $\mathbb{R}^N$ of radius τ centered at x_0, boundary normal sections $b_x(r)$ defined as

$$b_x(r) = b(x - r\mathbf{n}_x), \qquad r > 0,\ r \sim 0, \tag{1.5}$$

where $\mathbf{n}_x$ stands for the outward unit normal vector at $x \in \partial\Omega$.

The nonlinear term $f(u)$ satisfies

(f_1) $f \geq 0$ is locally Lipschitz continuous on $[0,\infty)$ and $f(u)/u$ is increasing on $(0,\infty)$;

(f_2) there exist some $\mathcal{L} \in C^2([A,\infty))$ satisfying $\lim_{u\to\infty} \mathcal{L}(u) = \infty$ if $r = 1$ and $\mathcal{L}' \in NRV_{-r}(0 \leq r \leq 1)$, a slowly varying function $\mathscr{L}$ and $p \geq 0$ such that

$$\lim_{u\to\infty} \frac{f(\mathcal{L}(u))}{\mathscr{L}'(u)u^{p+r}} = 1, \tag{1.6}$$

where $p > 1 - r$ if $0 \leq r < 1$ and $p \geq 0$ if $r = 1$.

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that $b(x)$ satisfies (b_1)–(b_2) and $f(u)$ satisfies (f_1)–(f_2). Then, problem (1.1) possesses a unique positive solution $u(x)$. Moreover, for each $x_0 \in \partial\Omega$, any positive solution $u(x)$ satisfies*

$$\lim_{r\to 0} \frac{u(x_0 - r\boldsymbol{n}_{x_0})}{I(x_0)^{-\frac{p}{p-1}} \mathcal{L}(\Phi_{x_0}(d(x)))} = \left(\frac{p+1}{p-1}\right)^{\frac{p+1}{p-1}}, \tag{1.7}$$

where

$$\Phi_{x_0}(t) = \int_t^\infty \left[\int_0^s \left(\frac{\mathscr{L}'(\Phi_{x_0})b_{x_0}}{\mathcal{L}'(\Phi_{x_0})}\right)^{\frac{1}{p+r+1}}\right]^{-\frac{p+r+1}{p+r-1}} ds, \tag{1.8}$$

$$I(x_0) = \lim_{t\to 0} \frac{\Phi_{x_0}(t)\Phi''_{x_0}(t)}{[\Phi'_{x_0}(t)]^2}, \tag{1.9}$$

$\mathscr{L}, \mathcal{L}$ appear in (1.6) and b_{x_0} is defined by (1.5).

The interest in these problems goes back to the pioneering works of López-Gómez. Precisely, López-Gómez [11], used the so-called López-Gómez's localization method, ascertained asymptotic behavior of boundary blow-up solutions to problem (1.1) with $f(u) = u^p$ and $b(x)$ vanishing on the boundary of the underlying domain at different rates according to the point of boundary. This results was developed by López-Gómez [12], Cano-Casanova and López-Gómez [1, 2], Wei and Zhu [18], Wang and Wang [19] and Xie [20]. In particular, Huang et. al. [10] obtained asymptotic behavior of boundary blow-up solutions to problem (1.1) with nonlinear term f satisfying

(f_3) there exists a slowly varying function H and $p > 1$ such that

$$\lim_{u\to\infty} \frac{f(u)}{H(u)u^p} = 1. \tag{1.10}$$

Remark 1.2. Note that (f_3) implies that $f(u) \in RV_p$, see Remark 1.1 of [10]. It can easily be seen that $f(\mathcal{L}(u)) \in RV_p$ if (f_2) holds. Thus f is a normalized varying function at infinity with index $p/(1-r)$ if $0 \le r < 1$ and f is rapidly varying with index ∞ if $r = 1$, for more details see [9]. Consequently, the main results of this paper give a unified asymptotic behavior of boundary blow-up solutions to problem (1.1).

Remark 1.3. Based on the results of López-Gómez [1,2,11,12], Ouyang and Xie [13,14], Xie [20], Xie and Zhao [21] established some similar asymptotic behavior of boundary blow-up solutions to problem (1.1). Recently, Huang et. al. [8], using the Karamata regular variation theory approach introduced by Cîrstea and Rădulescu [5,6], established asymptotic behavior and uniqueness of boundary blow-up solutions to problem (1.1) with f satisfying (f_3), extended the main results of [13,14,20]. Similarly, we can obtain similar asymptotic behavior of boundary blow-up solutions to problem (1.1) with f satisfying (f_2).

Remark 1.4. For the existence of boundary blow-up solutions to problem (1.1), see Theorem 1.1 of [4].

Remark 1.5. One easily sees that $\Phi_{x_0}(t)$, defined by (1.8), is a decreasing C^2-function on some interval $(0,\varsigma)$, for some $\varsigma > 0$. Consequently, taking into account Lemma 3.1 in [3], $I(x_0) \ge 1$. Furthermore, $-\Phi'_{x_0}(t)$ is normalized regularly varying at zero of index $I(x_0)/(1-I(x_0))$ if $I(x_0) > 1$ and $\Phi_{x_0}(t)$ has a representation formula if $I(x_0) = 1$.

Remark 1.6. By (1.8), we know that

$$\Phi'_{x_0}(t) = -\left[\int_0^t \left(\frac{\mathscr{L}'(\Phi_{x_0})b_{x_0}}{\mathcal{L}'(\Phi_{x_0})}\right)^{\frac{1}{p+r+1}}\right]^{-\frac{p+r+1}{p+r-1}}, \tag{1.11}$$

and

$$\Phi''_{x_0}(t) = \frac{p+r+1}{p+r-1}\left[\int_0^t \left(\frac{\mathscr{L}'(\Phi_{x_0})b_{x_0}}{\mathcal{L}'(\Phi_{x_0})}\right)^{\frac{1}{p+r+1}}\right]^{-\frac{2(p+r)}{p+r-1}} \left(\frac{\mathscr{L}'(\Phi_{x_0})b_{x_0}}{\mathcal{L}'(\Phi_{x_0})}\right)^{\frac{1}{p+r+1}}. \tag{1.12}$$

Thus, taking into account (f_2) and $\lim_{t\to 0}\Phi_{x_0}(t) = \infty$, we obtain

$$\begin{aligned}
&\lim_{t\to 0} \frac{\Phi''_{x_0}(t)\mathcal{L}'(\Phi_{x_0}(t))}{b_{x_0}(t)f(\mathcal{L}(\Phi_{x_0}(t)))} \\
&= \lim_{t\to 0} \frac{\mathscr{L}'(\Phi_{x_0}(t))\Phi^{p+r}_{x_0}(t)}{f(\mathcal{L}(\Phi_{x_0}(t)))}\left[\frac{\Phi_{x_0}(t)\Phi''_{x_0}(t)}{[\Phi'_{x_0}(t)]^2}\right]^{-(p+r)} \frac{[\Phi''_{x_0}(t)]^{p+r+1}}{[\Phi'_{x_0}(t)]^{2(p+r)}}\frac{\mathcal{L}'(\Phi_{x_0}(t))}{\mathscr{L}'(\Phi_{x_0}(t))b_{x_0}(t)} \\
&= [I(x_0)]^{-(p+r)}\left(\frac{p+r+1}{p+r-1}\right)^{p+r+1},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\to 0}\frac{[\Phi'_{x_0}(t)]^2\mathcal{L}''(\Phi_{x_0}(t))}{b_{x_0}(t)f(\mathcal{L}(\Phi_{x_0}(t)))}\\
&\quad=\lim_{t\to 0}\frac{\Phi_{x_0}(t)\mathcal{L}''(\Phi_{x_0}(t))}{\mathcal{L}'(\Phi_{x_0}(t))}\frac{\mathscr{L}'(\Phi_{x_0}(t))\Phi_{x_0}^{p+r}(t)}{f(\mathcal{L}(\Phi_{x_0}(t)))}\left[\frac{\Phi_{x_0}(t)\Phi''_{x_0}(t)}{[\Phi'_{x_0}(t)]^2}\right]^{-(p+r+1)}\\
&\quad\quad\times\frac{[\Phi''_{x_0}(t)]^{p+r+1}}{[\Phi'_{x_0}(t)]^{2(p+r)}}\frac{\mathcal{L}'(\Phi_{x_0}(t))}{\mathscr{L}'(\Phi_{x_0}(t))b_{x_0}(t)}\\
&\quad=-r[I(x_0)]^{-(p+r+1)}\left(\frac{p+r+1}{p+r-1}\right)^{p+r+1}.
\end{aligned}$$

The structure of this paper is as follows. In Section 2, we collect some preliminary results of Karamata regular variation theory. In Section 3 we prove some auxiliary results. Theorem 1.1 will be proved in Section 4.

2 Auxiliary results

The main purpose of this section is to provide some concepts from the theory of regular variation. For detailed accounts of the theory of regular variation, its extensions and many of its applications, we refer the interested reader to [7,15–17]. When the regular variation occurs at infinity and there is no possibility of confusion, we omit "at infinity".

Definition 2.1. A positive measurable function f defined on $[D,\infty)$ for some $D>0$ is called regularly varying (at infinity) with index $p\in\mathbb{R}$ (written $f\in RV_p$) if for all $\xi>0$

$$\lim_{u\to\infty}\frac{f(\xi u)}{f(u)}=\xi^p.$$

When the index of regular variation p is zero, we say that the function is slowly varying. The transformation $f(u)=u^pL(u)$ reduces regular variation to slow variation.

Proposition 2.2. *Assume that L is slowly varying. Then the convergence $L(\xi u)/L(u)\to 1$ as $u\to\infty$ holds uniformly on each compact ε-set in $(0,\infty)$.*

Proposition 2.3. *If L is slowly varying, then*

(i) $\ln L(u)/\ln u\to 0$ *as* $u\to\infty$;

(ii) for any $\alpha>0$, $u^\alpha L(u)\to\infty$, $u^{-\alpha}L(u)\to 0$ *as* $u\to\infty$;

(iii) $(L(u))^\alpha$ *varies slowly for every* $\alpha\in\mathbb{R}$;

(iv) if L_1 *varies slowly, so do* $L(u)L_1(u)$ *and* $L(u)+L_1(u)$.

Now we collect some important results which will be used in the proof of Theorem 1.1.

Definition 2.4. A function $\underline{u}\in C^2(\Omega)$ is a (classical) subsolution to problem (1.1), if $\underline{u}=+\infty$ on $\partial\Omega$ and

$$\Delta\underline{u}\ge b(x)f(\underline{u}),\qquad x\in\Omega.$$

Similarly, $\overline{u}$ is a (classical) supersolution to problem (1.1), if $\overline{u}=+\infty$ on $\partial\Omega$ and

$$\Delta\overline{u}\le b(x)f(\overline{u}),\qquad x\in\Omega.$$

The following comparison principle which plays an important role in the proof of Theorem 1.1 will be used in later sections.

Proposition 2.5. *Let f be continuous on $(0,\infty)$ such that $f(u)/u$ is increasing for $u > 0$, Let $b(x) \in C(\Omega)$ be a nonnegative function. Assume that $u_1, u_2 \in C^2(\Omega)$ are positive such that*

$$\begin{cases} \Delta u_1 - b(x)f(u_1) \leq 0 \leq \Delta u_2 - b(x)f(u_2), & x \in \Omega, \\ \limsup_{d(x,\partial\Omega)\to 0}(u_2 - u_1)(x) \leq 0. \end{cases}$$

Then we have $u_1 \geq u_2$ in Ω.

3 Auxiliary results

To prove Theorem 1.1 by the López-Gómez localization method, firstly consider the corresponding singular problem with radial weight function $b(x)$ in a ball or an annular domain. Note that, in this case, (3.2) is uniformly satisfied on $\partial\Omega$.

Theorem 3.1. *Suppose $\Omega_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$, $f(u)$ satisfies (f_1)–(f_2), and $b(x) = b(r - \|x - x_0\|)$, $b \in C([0,r] : [0,\infty))$ satisfies (b_1). Then, problem (1.1) possesses a unique positive solution $u(x)$. Moreover, any positive solution $u(x)$ satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)}{\Phi(d(x))} = I^{-\frac{p}{p-1}} \left(\frac{p+1}{p-1}\right)^{\frac{p+1}{p-1}}, \tag{3.1}$$

where

$$I = \lim_{t\to 0} \frac{\Phi(t)\Phi''(t)}{[\Phi'(t)]^2}, \qquad \Phi(t) = \int_t^\infty \left[\int_0^s \left(\frac{\mathscr{L}(\Phi)a}{\mathcal{L}(\Phi)}\right)^{\frac{1}{p+1}}\right]^{-\frac{p+1}{p-1}} ds, \tag{3.2}$$

$\mathscr{L}, \mathcal{L}$ appear in (1.6) and a appears in (b_1) .

Similarly, we have the following corresponding results when $\Omega = \Omega_{r_1,r_2}(x_0) = \{x \in \mathbb{R}^N : r_1 < |x - x_0| < r_2\}$.

Theorem 3.2. *Suppose $\Omega = \Omega_{r_1,r_2}(x_0)$, $f(u)$ satisfies (f_1)–(f_2), and $b(x) = b(r_2 - \|x - x_0\|)$, $b \in C([0,r] : [0,\infty))$ satisfies (b_1). Then problem (1.1) possesses a unique positive solution $u(x)$ and (3.1) holds.*

Note that, when the domain is an annular domain,

$$d(x) = \begin{cases} r_2 - |x - x_0|, & (r_1 + r_2)/2 \leq |x - x_0| < r_2, \\ |x - x_0| - r_1, & r_1 \leq |x - x_0| < (r_1 + r_2)/2. \end{cases}$$

In the following, the proof of Theorem 3.1 will be given. Theorem 3.2 can be proved by similar arguments, more details are omitted here.

Proof of Theorem 3.1. It is interesting to note that (1.2) holds uniformly, for each $\varepsilon > 0$; choose $\delta > 0$ sufficiently small such that,

$$(1-\varepsilon)a(d(x) - \beta) < b(x) < (1+\varepsilon)a(d(x) + \beta), \qquad 0 < \beta < d(x) < \delta.$$

For fixed $\beta \in (0,\delta)$, define $u_\pm(x) = \mathcal{L}(\xi^\pm \Phi(d(x) \pm \beta))$, $x \in \Omega_\beta^\pm$, where $u(x)$ is the solution to problem (1.1), $\Phi(t)$ is defined by (3.2), $\Omega_\delta = \{x \in \Omega,\ 0 < d(x) < \delta\}$, $\Omega_\beta^- = \Omega_{2\delta} \backslash \bar{\Omega}_\beta$, $\Omega_\beta^+ = \Omega_{2\delta-\beta}$ and

$$\xi^\pm = \left[\frac{1 \pm \varepsilon}{1 \mp \varepsilon}\left(\frac{p+1}{p-1}\right)^{p+1} I^p\right]^{\frac{1}{p-1}}.$$

Consequently,

$$\begin{aligned}
\nabla u_\pm(x) &= \xi^\pm \mathcal{L}'(\xi^\pm \Phi(d(x) \pm \beta))\Phi'(d(x) \pm \beta)\nabla d(x),\\
\Delta u_\pm(x) &= (\xi^\pm)^2 \mathcal{L}''(\xi^\pm \Phi(d(x) \pm \beta))[\Phi'(d(x) \pm \beta)]^2\\
&\quad + \xi^\pm \mathcal{L}'(\xi^\pm \Phi(d(x) \pm \beta))\Phi''(d(x) \pm \beta)\\
&\quad + \xi^\pm \mathcal{L}'(\xi^\pm \Phi(d(x) \pm \beta))\Phi'(d(x) \pm \beta)\Delta d(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Delta u_+(x) - b(x)f(u_+(x))\\
&\quad \geq (\xi^+)^2 \mathcal{L}''(\xi^\pm \Phi(d(x)+\beta))[\Phi'(d(x)+\beta)]^2 + \xi^+ \mathcal{L}'(\xi^\pm \Phi(d(x)+\beta))\Phi''(d(x)+\beta)\\
&\qquad + \xi^+ \mathcal{L}'(\xi^+ \Phi(d(x)+\beta))\Phi'(d(x)+\beta)\Delta d(x) - (1+\varepsilon)a(d(x)+\beta)f(u_+(x))\\
&\quad = a(d(x)+\beta)f(u_+(x))[\mathscr{A}_1^+(d(x)+\beta) + \mathscr{A}_2^+(d(x)+\beta)\Delta d(x) - (1+\varepsilon)],
\end{aligned}$$

and

$$\begin{aligned}
&\Delta u_-(x) - b(x)f(u(x))\\
&\quad \leq (\xi^-)^2 \mathcal{L}''(\xi^- \Phi(d(x)-\beta))[\Phi'(d(x)-\beta)]^2 + \xi^- \mathcal{L}'(\xi^- \Phi(d(x)-\beta))\Phi''(d(x)-\beta)\\
&\qquad + \xi^- \mathcal{L}'(\xi^- \Phi(d(x)-\beta))\Phi'(d(x)-\beta)\Delta d(x) - (1-\varepsilon)a(d(x)-\beta)f(u_-(x))\\
&\quad = a(d(x)-\beta)f(u_-(x))[\mathscr{A}_1^-(d(x)-\beta) + \mathscr{A}_2^-(d(x)-\beta)\Delta d(x) - (1-\varepsilon)],
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{A}_1^\pm(t) &= \frac{(\xi^\pm)^2 \mathcal{L}''(\xi^\pm \Phi(t))[\Phi'(t)]^2}{a(t)f(\mathcal{L}(\xi^\pm \Phi(t)))} + \frac{\xi^\pm \mathcal{L}'(\xi^\pm \Phi(t))\Phi''(t)}{a(t)f(\mathcal{L}(\xi^\pm \Phi(t)))},\\
\mathscr{A}_2^\pm(t) &= \frac{\xi^\pm \mathcal{L}'(\xi^\pm \Phi(t))\Phi'(t)}{a(t)f(\mathcal{L}(\xi^\pm \Phi(t)))}.
\end{aligned}$$

Similar computations as in Remark 1.6 show that

$$\lim_{t\to 0} \mathscr{A}_1^\pm(t) = \left[(\xi^\pm)^2 I^{-(p+r)} - r\xi^\pm I^{-(p+r+1)}\right]\left(\frac{p+r+1}{p+r-1}\right)^{p+r+1},$$

and

$$\lim_{t\to 0} \mathscr{A}_2^\pm(t) = \lim_{t\to 0} \frac{\xi^\pm \mathcal{L}'(\xi^\pm \Phi(t))\Phi''(t)}{a(t)f(\mathcal{L}(\xi^\pm \Phi(t)))}\,\frac{\Phi'(t)}{\Phi''(t)} = 0.$$

Consequently,

$$\lim_{d(x)\pm\beta\to 0} \left[\mathscr{A}_1^\pm(d(x)\pm\beta) + \mathscr{A}_2^\pm(d(x)\pm\beta)\Delta d(x) - (1\pm\varepsilon)\right] = \pm\varepsilon,$$

which implies that we can choose $\delta > 0$ such that

$$\begin{cases}\Delta u_\beta^+ - b(x)f(u_\beta^+) \geq 0, & x \in \Omega_\beta^+,\\ \Delta u_\beta^- - b(x)f(u_\beta^-) \leq 0, & x \in \Omega_\beta^-.\end{cases}$$

Define $M(2\delta) = \max_{d(x)\geq 2\delta} u(x)$, where $u(x)$ is a nonnegative solution of problem (1.1). Obviously, $u(x) \leq M(2\delta) + u_\beta^-$, $x \in \{x \in \Omega : d(x) = 2\delta\}$ and $\lim_{d\to\beta}[M(2\delta) + u_\beta^-] = \infty$. Namely, $u(x) \leq M(2\delta) + u_\beta^-$, $x \in \partial\Omega_\beta^-$. On the other hand, $\Delta(M(2\delta) + u_\beta^-) = \Delta u_\beta^- \leq b(x)f(u_\beta^-) \leq b(x)f(M(2\delta) + u_\beta^-)$, $x \in \Omega_\beta^-$, the comparison principle of elliptic equations leads to

$$u(x) \leq M(2\delta) + u_\beta^-, \qquad x \in \Omega_\beta^-. \tag{3.3}$$

Define $u_\beta^+(x) = \mathcal{L}(\xi^+\Phi(2\delta))$, $x \in \{x \in \Omega : d(x) = 2\delta - \beta\}$ and $N(2\delta) = \mathcal{L}(\xi^+\Phi(2\delta))$. It is easy to see that

$$u_\beta^+(x) \leq N(2\delta) + u(x), \quad x \in \{x \in \Omega : d(x) = 2\delta - \beta\}, \quad \lim_{d\to 0}[u_\beta^+(x) - N(2\delta) - u(x)] = -\infty.$$

That is, $u_\beta^+(x) \leq N(2\delta) + u(x)$, $x \in \partial\Omega_\beta^+$. Note that $\Delta(u_\beta^+(x) - N(2\delta)) = \Delta u_\beta^+(x) \geq b(x)f(u_\beta^+) \geq b(x)f(u_\beta^+(x) - N(2\delta))$. This fact, combined with the comparison principle shows that

$$u_\beta^+(x) \leq N(2\delta) + u(x), \qquad x \in \Omega_\beta^+. \tag{3.4}$$

According to (3.3) and (3.4), we find

$$u_\beta^+(x) - N(2\delta) \leq u(x) \leq M(2\delta) + u_\beta^-, \qquad x \in \Omega_\beta^- \cap \Omega_\beta^+.$$

This yields

$$\frac{u_\beta^+(x) - N(2\delta)}{\mathcal{L}(\xi^\pm\Phi(d(x)))} \leq \frac{u(x)}{\mathcal{L}(\xi^\pm\Phi(d(x)))} \leq \frac{M(2\delta) + u_\beta^-}{\mathcal{L}(\xi^\pm\Phi(d(x)))}, \qquad x \in \Omega_\beta^- \cap \Omega_\beta^+. \tag{3.5}$$

Letting $\varepsilon \to 0$ and $d(x) \to 0$ in (3.5) leads to (3.1), here we use the fact that $d(x) \to 0$ implies $\beta \to 0$ if $x \in \Omega_\beta^- \cap \Omega_\beta^+$. □

4 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by the localization method introduced in [11,12].

Proof. Fixed $\varepsilon > 0$, according to (1.4), there exist $\rho = \rho(\varepsilon) \in (0, \eta)$ and $\mu = \mu(\varepsilon)$ such that for each $x \in \partial\Omega \cap \overline{\Omega}_\rho(x_0)$, $r \in (0, \mu)$,

$$1 - \varepsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{b(x - r\mathbf{n}_x)}{b(x_0 - r\mathbf{n}_{x_0})} < 1 + \varepsilon. \tag{4.1}$$

Define $\mathscr{B} = \{x - r\mathbf{n}_x : x \in \partial\Omega \cap \overline{\Omega}_\rho(x_0),\ r \in [0, \mu]\}$. Note that for each $y \in \mathscr{B}$ (ρ, μ can be shortened if necessary), there exists a unique $y_0 \in \partial\Omega \cap \overline{\Omega}_\rho(x_0)$, and $r(y) \in [0, \mu]$, such that $y = y_0 - r(y)\mathbf{n}_{y_0}$, $r(y) = |y - y_0| = \mathrm{dist}(y, \partial\Omega)$. Furthermore, there exists $r_0 \in (0, \min\{\rho/2, \mu/2\})$, such that $\Omega_{r_0}(x_0 - r_0\mathbf{n}_{x_0}) \subset \Omega$, and $\overline{\Omega}_{r_0}(x_0 - r_0\mathbf{n}_{x_0}) \cap \partial\Omega = \{x_0\}$. Thus there exists $\sigma_0 > 0$ such that for $\sigma \in (0, \sigma_0]$, $\overline{\Omega}_{r_0}(x_0 - (r_0 + \sigma)\mathbf{n}_{x_0}) \subset \Omega \cap \mathrm{Int}\,\mathscr{B}$. Consequently, for $\sigma \in [0, \sigma_0]$ and $y \in \overline{\Omega}_{r_0}(x_0 - (r_0 + \sigma)\mathbf{n}_{x_0})$,

$$\begin{aligned} b(y) &= b(y_0 - r(y)\mathbf{n}_{y_0}) \geq (1 - \varepsilon)b(x_0 - r(y)\mathbf{n}_{x_0}) = (1 - \varepsilon)b_{x_0}(r(y)) \\ &\geq (1 - \varepsilon)b_{x_0}(\mathrm{dist}(y, \partial\Omega_{r_0}(x_0 - (r_0 + \sigma)\mathbf{n}_{x_0}))), \end{aligned}$$

which shows that $b(y) \geq (1 - \varepsilon)b_{x_0}(r_\sigma)$, where $r_\sigma = \mathrm{dist}(y, \partial\Omega_{r_0}(x_0 - (r_0 + \sigma)\mathbf{n}_{x_0}))$.

Let $\mathcal{U}$ be the unique solution to problem

$$\begin{cases} \Delta u = (1-\varepsilon) b_{x_0}(r_\sigma) f(u), & x \in \Omega_{r_0}(x_0 - (r_0+\sigma)\mathbf{n}_{x_0}), \\ u(x) = +\infty, & x \in \partial\Omega_{r_0}(x_0 - (r_0+\sigma)\mathbf{n}_{x_0}), \end{cases} \tag{4.2}$$

where $\sigma \in [0, \sigma_0]$. Equation (3.1) shows that

$$\lim_{x \to \partial\Omega_{r_0}(x_0-(r_0+\sigma)\mathbf{n}_{x_0})} \frac{\mathcal{U}(x)}{K_1(x_0)\mathcal{B}_1(r_\sigma)} = (1-\varepsilon)^\beta,$$

where

$$\mathcal{B}_1(t) = \int_0^r \int_0^s (H \circ \mathcal{B}_1^{-\beta}(t)) b_{x_0}(t)\, dt\, ds, \qquad K_1(x_0) = [\beta(\beta+1)C_{x_0} - \beta]^\beta, \qquad \beta = \frac{1}{p-1},$$
$$C_{x_0} = \lim_{t \to 0} \frac{[\mathcal{B}_1'(t)]^2}{\mathcal{B}_1(t) b_{x_0}(t) H \circ \mathcal{B}_1^{-\beta}(t)}.$$

Thus $u|_{\Omega_{r_0}(x_0-(r_0+\sigma)\mathbf{n}_{x_0})}$ is a bounded subsolution of (4.2), hence, for each $\sigma \in [0, \sigma_0]$ and $x \in \Omega_{r_0}(x_0 - (r_0+\sigma)\mathbf{n}_{x_0})$, $\underline{u}_\sigma = u|_{\Omega_{r_0}(x_0-(r_0+\sigma)\mathbf{n}_{x_0})} \le \mathcal{U}$, and

$$\limsup_{x \to \partial\Omega_{r_0}(x_0-(r_0+\sigma)\mathbf{n}_{x_0})} \frac{\underline{u}_\sigma}{K_1(x_0)\mathcal{B}_1(r_\sigma)} \le (1-\varepsilon)^\beta.$$

Letting $\sigma \to 0$ gives

$$\lim_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K_1(x_0)\mathcal{B}_1(r)} \le (1-\varepsilon)^\beta.$$

This is valid for any sufficiently small $\varepsilon > 0$, then

$$\lim_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K_1(x_0)\mathcal{B}_1(r)} \le 1. \tag{4.3}$$

For any $x_0 \in \partial\Omega$, there exist $0 < r_1 < r_2$ and σ_0, such that

$$\Omega \subset \bigcap_{0 \le \sigma \le \sigma_0} \Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\mathbf{n}_{x_0}), \qquad \partial\Omega \cap \overline{\Omega}_{r_1,r_2}(x_0 + r_1\mathbf{n}_{x_0}) = \{x_0\}$$

and r_1 is small enough, r_2 is large enough such that $\Omega \subset \Omega_{r_1,r_2/3}(x_0 + r_1\mathbf{n}_{x_0})$.

By (4.1), we find that for each $y \in \Omega_{2\eta}(x_0) \cap \overline{\Omega}$, where $\eta \in \min\{\rho, \mu\}$ is small, $b(y) = b(y_0 - r(y)\mathbf{n}_{y_0}) \le (1+\varepsilon) b_{x_0}(r(y)) \le (1+\varepsilon) b_{x_0}(\mathrm{dist}(y, \partial\Omega_{r_1}(x_0 + r_1\mathbf{n}_{x_0})))$. Define the function $\widetilde{b}\colon \Omega_{r_1,r_2}(x_0 + r_1\mathbf{n}_{x_0}) \to [0, \infty)$ as $\widetilde{b}(y) = \widetilde{b}(r) = (1+\varepsilon) b_{x_0}(r)$, where $y \in \Omega_{2\eta}(x_0) \cap \overline{\Omega}$ and $r = \mathrm{dist}(y, \partial\Omega_{r_1,r_2}(x_0 + r_1\mathbf{n}_{x_0}))$. Moreover, $\widetilde{b}(\mathrm{dist}(y, \partial\Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\mathbf{n}_{x_0})) \ge b(y)$, for each $y \in \overline{\Omega}$, $\sigma \in [0, \sigma_0]$,

Let $\mathfrak{U}$ be the unique solution to

$$\begin{cases} \Delta u = \widetilde{b}(r) f(u), & x \in \Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\mathbf{n}_{x_0}), \\ u(x) = +\infty, & x \in \partial\Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\mathbf{n}_{x_0}), \end{cases}$$

where $r = \mathrm{dist}(y, \partial\Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\mathbf{n}_{x_0}))$, and

$$\lim_{x \to \partial\Omega_{r_1,r_2}(x_0+(r_1+\sigma)\mathbf{n}_{x_0})} \frac{\mathfrak{U}(x)}{K_2(x_0)\mathcal{B}_2(r)} = (1+\varepsilon)^\beta,$$

where

$$\mathcal{B}_2(t) = \int_0^r \int_0^s (H \circ \mathcal{B}_2^{-\beta}(r)) b_{x_0}(r)\, ds\, dt,$$
$$K_2(x_0) = [\beta(\beta+1)C_{x_0} - \beta]^{\beta}, \qquad \beta = \frac{1}{p-1},$$
$$C_{x_0} = \lim_{t\to 0} \frac{[\mathcal{B}_2'(t)]^2}{\mathcal{B}_2(t) b_{x_0}(r) H \circ \mathcal{B}_2^{-\beta}(t)}.$$

Moreover, $\mathfrak{U}|_{\Omega}$ is a subsolution of (1.1), this implies that $\mathfrak{U}(x) \le u(x)$, for each $\sigma \in [0, \sigma_0]$ and $x \in \Omega_{r_1,r_2}(x_0 + (r_1+\sigma)\mathbf{n}_{x_0}) \cap \Omega$. This yields

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K_2(x_0)\mathcal{B}_2(r)} \ge (1+\varepsilon)^{\beta}.$$

Letting $\sigma \to 0$, we derive that

$$\liminf_{\substack{x\to x_0,\\ x\in\Omega_{r_1,r_2}(x_0+r_1\mathbf{n}_{x_0})}} \frac{u(x)}{K_2(x_0)\mathcal{B}_2(r)} \ge 1. \tag{4.4}$$

It can easily be seen that $\mathcal{B}_1(r) = \mathcal{B}_2(r)$ and $K_1(x_0) = K_2(x_0)$. Using (4.3) and (4.4), we obtain (1.7).

□

Acknowledgements

This works was partially supported by the National Natural Science Foundation of China (No. 11401473, 11326100), Natural Science Foundation of Gansu Province (No. 145RJZA214), Fundamental Research Funds for the Central Universities (No. 31920140058, 31920130001, 31920130006), Talent Introduction Scientific Research Foundation of Northwest University for Nationalities (No. xbmuyjrc201305, xbmuyjrc201121), Research and Innovation Teams of Northwest University for Nationalities.

References

[1] S. Cano-Casanova, J. López-Gómez, Existence, uniqueness and blow-up rate of large solutions for a canonical class of one-dimensional problems on the half-line. *J. Differential Equations* **224**(2008), 3180–3203. MR2420518

[2] S. Cano-Casanova, J. López-Gómez, Blow-up rates of radially symmetric large solutions. *J. Math. Anal. Appl.* **352**(2009), 166–174. MR2499895

[3] F. Cîrstea, Elliptic equations with competing rapidly varying nonlinearities and boundary blow-up, *Adv. Differential Equations* **12**(2007), 995–1030. MR2351836

[4] F. Cîrstea, V. Rădulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, *Comm. Contemp. Math.* **4**(2002), 559–586. MR1918760

[5] F. Cîrstea, V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, *C. R. Acad. Sci. Paris Ser. I.* **335**(2002),447–452. MR1937111

[6] F. Cîrstea, V. Rădulescu, Asymptotics for the blow-up boundary solution of the logistic equation with absorption, *C. R. Acad. Sci. Paris, Ser. I.* **336**(2003), 231–236. MR1968264

[7] M. Ghergu, V. Rădulescu, *Singular elliptic problems: bifurcation and asymptotic analysis*, Oxford Lecture Series in Mathematics and its Applications, Vol. 37, Oxford University Press, 2008. MR2488149

[8] S. Huang, W-T. Li, Q. Tian, Y. Mi, General uniqueness results and blow-up rates for large solutions of elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*. **142**(2012), 825–837. MR2966115

[9] S. Huang, W-T. Li, M. Wang, A unified asymptotic behavior of boundary blow-up solutions to elliptic equations, *Differential Integral Equations* **26**(2013), 675–692. MR3098021

[10] S. Huang, , Q. Tian, S. Zhang, J. Xi, Z. Fan, The exact blow-up rates of large solutions for semilinear elliptic equations. *Nonlinear Anal*. **73**(2010), 3489–3501. MR2718153

[11] J. López-Gómez, The boundary blow-up rate of large solutions, *J. Differential Equations* **195**(2003), 25–45. MR2019241

[12] J. López-Gómez, Optimal uniqueness theorems and exact blow-up rates of large solutions, *J. Differential Equations* **224**(2006), 385–439. MR2223723

[13] T. Ouyang, Z. Xie, The uniqueness of blow-up solution for radially symmetric semilinear elliptic equation. *Nonlinear Anal*. **64**(2006), 2129–2142. MR2211204

[14] T. Ouyang, Z. Xie, The exact boundary blow-up rate of large solutions for semilinear elliptic problems. *Nonlinear Anal*. **68**(2008), 2791–2800. MR2397762

[15] H. Rademacher, Einige besondere probleme partieller Differentialgleichungen (in German), in: *Die Differential und Integralgleichungen der Mechanik und Physik, I*, Rosenberg, New York, second edition, 1943, pp. 838–845. MR0150433

[16] S. I. Resnick, *Extreme values, regular variation, and point processes*, Springer Verlag, Berlin/New York, 1987. MR900810

[17] E. Seneta, *Regularly varying functions*, Lecture Notes in Math. Vol. 508, Springer Verlag, Berlin/New York, 1976. MR0453936

[18] L. Wei, J. Zhu, The existence and blow-up rate of large solutions of one-dimensional p-Laplacian equations, *Nonlinear Anal. Real World Appl*. **13**(2012), 665–676. MR2846871

[19] Y. Wang, M. Wang, The blow-up rate and uniqueness of large solution for a porous media logistic equation, *Nonlinear Anal. Real World Appl*. **11**(2010), 1572–1580. MR2646570

[20] Z. Xie, Uniqueness and blow-up rate of large solutions for elliptic equation $-\Delta u = \lambda u - b(x)f(u)$, *J. Differential Equations* **247**(2009), 344–363. MR2523682

[21] Z. Xie, C. Zhao, Blow-up rate and uniqueness of singular radial solutions for a class of quasi-linear elliptic equations, *J. Differential Equations* **252**(2012), 1776–1788. MR2853560