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Note on the binomial partial difference equation

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Abstract. Some formulas for the "general solution" to the binomial partial difference equation

$$c_{m,n} = c_{m-1,n} + c_{m-1,n-1},$$

are known in the literature. However, it seems that there is no such a formula on the most natural domain connected to the equation, that is, on the set $D = \{(m,n) \in \mathbb{N}_0^2 : 0 \le n \le m\}$. By using a connection with the scalar linear first order difference equation we show that the equation on the domain $D \setminus \{(0,0)\}$, can be solved in closed form by presenting a formula for the solution in terms of the "side" values $c_{k,0}$, $c_{k,k}$, $k \in \mathbb{N}$.

Keywords: partial difference equation, binomial coefficients, equation solvable in closed form.

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1 Introduction

As we know, the Newton binomial formula can be written in the following form

$$(a+b)^m = \sum_{n=0}^{m} C_n^m a^{m-n} b^n, \tag{1.1}$$

where m is an arbitrary natural number. The numbers C_n^m, $0 \le n \le m$, are called the binomial coefficients, and from (1.1) and the relation

$$\begin{aligned}(a+b)^m &= (a+b)^{m-1}(a+b) = \left(\sum_{n=0}^{m-1} C_n^{m-1} a^{m-1-n} b^n\right)(a+b) \\ &= C_0^{m-1} a^m + \sum_{n=1}^{m-1} \left(C_n^{m-1} + C_{n-1}^{m-1}\right) a^{m-n} b^n + C_{m-1}^{m-1} b^m,\end{aligned}$$

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where $m \geq 2$, it follows that

$$C_0^m = C_0^{m-1} \quad \text{and} \quad C_m^m = C_{m-1}^{m-1}, \tag{1.2}$$

and

$$C_n^m = C_n^{m-1} + C_{n-1}^{m-1}, \qquad 1 \leq n \leq m-1, \tag{1.3}$$

for every $m \geq 2$. From (1.2) and the obvious fact

$$C_0^1 = C_1^1 = 1,$$

which follows from (1.1) with $m = 1$, we also obtain

$$C_0^m = C_m^m = 1,$$

for every $m \in \mathbb{N}$.

If we "naturally" assume (based, for example, on the combinatorial meanings of the coefficients), that is, introduce by a definition, that

$$C_m^{m-1} = C_{-1}^{m-1} = 0, \quad m \in \mathbb{N}, \quad \text{and} \quad C_0^0 = 1,$$

then we see that such a defined double sequence C_n^m satisfies the relation in (1.3), for all $m, n \in \mathbb{N}_0$ such that $0 \leq n \leq m$. All above mentioned is (or should be) nowadays known to any high-school student. For a good source of some classical things connected to this and related topics, see, for example, the nice problem book [9]. Some more advanced results can be found, for example, in monographs [12] and [14].

Looking at recurrent relation (1.3) it can be seen that it is nothing but a camouflaged partial difference equation, which could be traditionally written in the following form

$$c_{m,n} = c_{m-1,n} + c_{m-1,n-1}, \tag{1.4}$$

more acceptable to the experts on difference equations. This seems one of the first partial difference equations appearing in the literature (it appeared much before than the notion *partial difference equation* was coined), and one of the basic ones (see, for example, [6, p. 1]). Some basic material on partial difference equations, especially related to the methods for solving linear and some related partial difference equations, can be found, for example, in the classical sources [8, Chapter 12] and [10, Chapter 8]. A plenty of classical, as well as recent results on various types of linear and nonlinear partial difference equations, can be found, for example, in the nice monograph [6].

On the other hand, there has been some renewed recent interest in difference equations and systems which can be solved in closed form (see, for example, [1–4, 7, 13, 15, 16, 18–36, 38–47]). For some basic and classical types of solvable difference equations, see, for example, [8, 10, 12] (see, also, [9, Chapter 10], as well as some parts of the book [14]). Our note [15], in which a method for solving the difference equation appearing in [7] was given, triggered the renewed interest. Namely, it turned out that there are numerous nonlinear difference equations and systems of interest which can be transformed into the known solvable ones by using some suitable changes of variables. Many closed form formulas for solutions of the equations and systems seems obtained by using some computer packages, so they need some theoretical explanations. Some explanations for closed form formulas for solutions of such types of equations and systems can be found, for example, in [18, 24, 40]. One of the crucial points in many of the above mentioned papers is the fact that the changes of variables transform the equations into the nonhomogeneous linear first order difference equation, that is, into the equation

$$x_n = a_n x_{n-1} + b_n, \qquad n \in \mathbb{N}, \tag{1.5}$$

where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are arbitrary sequences of real or complex numbers, and $x_0 \in \mathbb{R}$ or $x_0 \in \mathbb{C}$, respectively. For example, suitable changes of variables transform some of the difference equations in [1,2,4,13,15,16,19,23,34,40,41,43] into equation (1.5) or into its special cases. It should be pointed out that in some cases such obtained equations and systems are not equivalent to the original ones. Such a situation appears for example, in the original source [15], as well as in several later papers [19,23,28,39,45]. Actually, in this or that way, many equations and systems are related to equation (1.5), or to the corresponding difference inequality or linear system of difference equations [5,11,12,36,39].

There are several methods for solving equation (1.5). For instance, by multiplying the equality

$$x_{n-l+1} = a_{n-l+1}x_{n-l} + b_{n-l+1}$$

by $\prod_{j=n-l+2}^{n} a_j$, and summing up such obtained equalities for $1 \le l \le n$, it follows that

$$x_n = x_0 \prod_{j=1}^{n} a_j + \sum_{i=1}^{n} b_i \prod_{j=i+1}^{n} a_j, \qquad n \in \mathbb{N}_0, \tag{1.6}$$

which is the general solution to the equation. For some other methods, see, for example, [12].

For the case of some systems of difference equations the corresponding changes of variables transform them into some solvable linear ones (see, for example, [4,18,20–22,24–30,36,39,40,44,45]). In some other cases, such as in papers [32], [33], [35] and [46], where product-type systems related to the equations and systems in [17] and [37] are considered, or in papers [31,34,38,42], the transformations and methods used therein are more complex, but as a final outcome some solvable linear difference equations of higher order or more complicated solvable linear systems are obtained. If we also note that many of the solvable higher-order difference equations appearing in these papers can be presented as operator products of some linear first-order ones, we see an exceptional importance of equation (1.5).

Now recall another well-known fact regarding the binomial coefficients. Namely, there is a concrete formula for them. It is

$$C_n^m = \frac{m!}{n!(m-n)!}, \qquad 0 \le n \le m, \tag{1.7}$$

which is one of the most basic formulas not only in combinatorics, but in mathematics as whole.

The fact that the double sequence C_n^m is a solution to equation (1.4) suggests that the equation could be "solvable" in closed form. This is, in a way, true. Namely, for the partial difference equation (1.4) it is possible to find its "general solution" (the notion is more obscure than the corresponding one for the scalar difference equation and we will give some additional comments on it below). Indeed, if the following two operators are defined as

$$Eu_{m,n} = u_{m+1,n} \quad \text{and} \quad Fu_{m,n} = u_{m,n+1},$$

(see [10, p. 239]) then equation (1.4) can be written in the form

$$c_{m,n} = (I + F^{-1})c_{m-1,n},$$

from which it follows that

$$c_{m,n} = (I + F^{-1})^m c_{0,n} = \sum_{j=0}^{m} C_j^m F^{-j} c_{0,n} = \sum_{j=0}^{m} C_j^m c_{0,n-j},$$

which is the "general" solution to equation (1.4), since $c_{0,n}$, $n \in \mathbb{Z}$, is an arbitrary sequence (i.e. function) on $\mathbb{Z}$. To find a concrete solution to equation (1.4), it is clear that the initial values should be given at the points $(0,n)_{n\in\mathbb{Z}}$ on the y-axis. This means that the "general" solution to equation (1.4) corresponds to the right-half plane. This clearly shows that beside a partial difference equation, the types of domains involved considerably influence on the "general" solution to the equation.

The particular solution to equation (1.4) given in (1.7) is obtained from the "general" one

$$c_{m,n} = C_0^m c_{0,n} + C_1^m c_{0,n-1} + \cdots + C_{m-1}^m c_{0,n-m+1} + C_m^m c_{0,n-m},$$

by choosing the boundary values $c_{0,k}$ as follows

$$c_{0,0} = 1 \quad \text{and} \quad c_{0,k} = 0,\ k \neq 0.$$

However, the initial values appearing on y-axis seem less natural for the binomial coefficients and for getting formula (1.7), due to the fact that the natural domain for them is the set

$$D = \{(m,n) \in \mathbb{N}_0^2 : 0 \leq n \leq m\},$$

since C_0^m and C_m^m, $m \in \mathbb{N}_0$, are specified.

So, it is a natural question whether there is a closed form formula for solutions to equation (1.4) which reconstructs its solutions on domain D by using the given "side" values $c_{0,k}$ and $c_{k,k}$, when $k \in \mathbb{N}$.

Our aim here is to present a closed form formula for solutions to equation (1.4) in domain $D \setminus \{(0,0)\}$. The formula could be known, but we could not locate it in the literature. Beside this, the formula seems quite unknown to a wide audience, so deserves publication in a visible place. Another aim is to point out a strong connection of a partial difference equation and its solvability with the solvability of the linear difference equation of first order.

2 Main result

Here, we first give a motivation for the main formula obtained in this paper, and show its connection with a special case of equation (1.5). The method presented here is half-constructive. Namely, we will first solve equation (1.4) in some special cases and then based on the obtained formulas we will assume the form of the general solution to the equation and confirm it by induction. The main idea is to note that for a fixed m equation (1.4) is actually equation (1.5) with

$$a_n = 1 \quad \text{and} \quad b_n = c_{m-1,n}, \qquad n \in \mathbb{N},$$

which is, among others, why we paid some attention to equation (1.5) in the introduction, as well as to its usefulness in various applications.

Although, in this case the corresponding linear equation is much simpler than the one in (1.5) and its solution is essentially obtained by the telescoping method, we want to point out the connection since we highly expect that for some related partial difference equations we might arrive at a position to use more general form of equation (1.5). We also want to emphasize that, because of the specificity of the domain, we will use its natural division by the lines

$$y = x + k, \quad \text{where} \quad x \in \mathbb{N}_0,$$

and where k is a fixed natural number. Namely, we will find a formula for the solution on the line with coefficient k, and then by using the formula on the line we will find a formula for the solution on the line with coefficient $k+1$. Since k is an arbitrary natural number, this decomposition will produce the solution on the whole domain D.

To demonstrate the method, and to point out clearly connection of equation (1.4) with a scalar linear difference equation of first-order, we will first find the solution to equation (1.4) for the case when $k = 1, 2, 3$.

First assume that $k = 1$, that is, $m = n + 1$. Then in this case equation (1.4) becomes

$$c_{n+1,n} = c_{n,n} + c_{n,n-1}, \qquad n \in \mathbb{N}. \tag{2.1}$$

Summing up the equations in (2.1) from 1 to n, or equivalently solving it by using the change of variables

$$x_n = c_{n+1,n}, \qquad n \in \mathbb{N},$$

which transforms equation (2.1) into the following linear difference equation of first-order

$$x_n = x_{n-1} + c_{n,n}, \qquad n \in \mathbb{N},$$

and using formula (1.6) with $a_n = 1$ and $b_n = c_{n,n}$, we get

$$c_{n+1,n} = \sum_{i=1}^{n} c_{i,i} + c_{1,0}, \qquad n \in \mathbb{N}_0. \tag{2.2}$$

If $k = 2$, then $m = n + 2$. If we put it into (1.4), we get

$$c_{n+2,n} = c_{n+1,n} + c_{n+1,n-1}, \qquad n \in \mathbb{N}. \tag{2.3}$$

Similarly to the previous case, we get

$$c_{n+2,n} = \sum_{j=1}^{n} c_{j+1,j} + c_{2,0}, \tag{2.4}$$

for every $n \in \mathbb{N}_0$.

Using (2.2) in (2.4), change the order of summation and by some calculation, we get

$$\begin{aligned} c_{n+2,n} &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{j} c_{i,i} + c_{1,0} \Big) + c_{2,0} \\ &= \sum_{i=1}^{n} c_{i,i} \sum_{j=i}^{n} 1 + \sum_{j=1}^{n} c_{1,0} + c_{2,0} \\ &= \sum_{i=1}^{n} (n-i+1) c_{i,i} + n c_{1,0} + c_{2,0}, \end{aligned} \tag{2.5}$$

for every $n \in \mathbb{N}_0$.

If $k = 3$, then $m = n + 3$, and if we put it in (1.4), we get

$$c_{n+3,n} = c_{n+2,n} + c_{n+2,n-1}, \qquad n \in \mathbb{N}. \tag{2.6}$$

Similar to the case $k = 1$, we get

$$c_{n+3,n} = \sum_{j=1}^{n} c_{j+2,j} + c_{3,0}, \tag{2.7}$$

for every $n \in \mathbb{N}_0$.

Using (2.5) in (2.7), changing the order of summation and by some calculation, we get

$$\begin{aligned} c_{n+3,n} &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{j} (j-i+1)c_{i,i} + jc_{1,0} + c_{2,0} \Big) + c_{3,0} \\ &= \sum_{i=1}^{n} c_{i,i} \sum_{j=i}^{n} (j-i+1) + \sum_{j=1}^{n} jc_{1,0} + \sum_{j=1}^{n} c_{2,0} + c_{3,0} \\ &= \sum_{i=1}^{n} c_{i,i} \sum_{j=1}^{n-i+1} s + \frac{n(n+1)}{2} c_{1,0} + nc_{2,0} + c_{3,0} \\ &= \sum_{i=1}^{n} \frac{(n-i+1)(n-i+2)}{2} c_{i,i} + \frac{n(n+1)}{2} c_{1,0} + nc_{2,0} + c_{3,0} \\ &= \sum_{i=1}^{n} C_2^{n-i+2} c_{i,i} + C_2^{n+1} c_{1,0} + C_1^{n} c_{2,0} + C_0^{n-1} c_{3,0}, \qquad n \in \mathbb{N}_0, \end{aligned} \tag{2.8}$$

where we have also used the well-known formula

$$\sum_{j=1}^{l} j = \frac{l(l+1)}{2}, \qquad l \in \mathbb{N},$$

for $l = n - i + 1$ and $l = n$.

Hence, on the lines $m = n + k$, for $k \in \{1, 2, 3\}$, we have found the solution to equation (1.4) by constructing it. The procedure can be continued for $k = 4$ and other small values of k. However, since we need a closed form formula to solutions of equation (1.4) which holds for every $m \in \mathbb{N}$, we need a formula for $c_{n+k,n}$ which holds for every $k \in \mathbb{N}$.

Formulas (2.2), (2.5) and (2.8) suggest that the following formula holds

$$c_{n+k,n} = \sum_{i=1}^{n} C_{k-1}^{n-i+k-1} c_{i,i} + \sum_{j=1}^{k} C_{k-j}^{n+k-j-1} c_{j,0}, \tag{2.9}$$

for every $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

We apply the method of induction. According to the previous consideration equality (2.9) holds for $k = 1$ since (2.2) can be written in the form

$$c_{n+1,n} = \sum_{i=1}^{n} C_0^{n-i} c_{i,i} + c_{1,0}, \qquad n \in \mathbb{N}_0.$$

Assume that (2.9) hold for some $k \in \mathbb{N}$.

If we put $m = n + k + 1$ into (1.4), we get

$$c_{n+k+1,n} = c_{n+k,n} + c_{n+k,n-1}, \qquad n \in \mathbb{N}. \tag{2.10}$$

Summing up the equations in (2.10) from 1 to n, we get

$$c_{n+k+1,n} = \sum_{j=1}^{n} c_{j+k,j} + c_{k+1,0}, \qquad n \in \mathbb{N}_0. \tag{2.11}$$

Using the hypothesis (2.9) in (2.11), and by some simple calculations, we get

$$\begin{aligned} c_{n+k+1,n} &= \sum_{j=1}^{n}\left(\sum_{i=1}^{j} C_{k-1}^{j-i+k-1} c_{i,i} + C_{k-1}^{j+k-2} c_{1,0} + C_{k-2}^{j+k-3} c_{2,0} + \cdots + C_{0}^{j-1} c_{k,0}\right) + c_{k+1,0} \\ &= \sum_{i=1}^{n} c_{i,i} \sum_{j=i}^{n} C_{k-1}^{j-i+k-1} + \sum_{r=1}^{k} c_{r,0} \sum_{j=1}^{n} C_{k-r}^{j+k-r-1} + c_{k+1,0}, \end{aligned} \tag{2.12}$$

for every $n \in \mathbb{N}_0$.

By using recurrent relation (1.3), we have

$$\begin{aligned} \sum_{j=1}^{n} C_{k-r}^{j+k-r-1} &= \sum_{j=1}^{n}\left(C_{k-r+1}^{j+k-r} - C_{k-r+1}^{j+k-r-1}\right) = C_{k-r+1}^{n+k-r} - C_{k-r+1}^{k-r} \\ &= C_{k-r+1}^{n+k-r}, \end{aligned} \tag{2.13}$$

for every $1 \leq r \leq k$, and

$$\sum_{j=i}^{n} C_{k-1}^{j-i+k-1} = \sum_{j=i}^{n}\left(C_{k}^{j-i+k} - C_{k}^{j-i+k-1}\right) = C_{k}^{n-i+k} - C_{k}^{k-1} = C_{k}^{n-i+k}, \tag{2.14}$$

for every $1 \leq i \leq n$.

Using (2.13) and (2.14) in (2.12), it follows that

$$c_{n+k+1,n} = \sum_{i=1}^{n} C_{k}^{n-i+k} c_{i,i} + \sum_{j=1}^{k+1} C_{k-j+1}^{n+k-j} c_{j,0},$$

from which along with the method of induction it follows that formula (2.9) holds for every $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

Due to the above considerations, we are now in a position to formulate and prove the main result in this note.

Theorem 2.1. *If $(u_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$, are given sequences of real numbers. Then the solution to partial difference equation (1.4) on domain $D \setminus \{(0,0)\}$, with the boundary value conditions given by*

$$c_{k,0} = u_k \quad \text{and} \quad c_{k,k} = v_k, \qquad k \in \mathbb{N}, \tag{2.15}$$

is given by

$$c_{m,n} = \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} v_i + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} u_j. \tag{2.16}$$

Proof. If we put $k = m - n$ in formula (2.9) (note that $m > n$ so $k \in \mathbb{N}$), use the "side" conditions given in (2.15), and by some simple calculations, we obtain formula (2.16). □

Remark 2.2. Note that Theorem 2.1 actually says that the general solution to partial difference equation (1.4) on domain $D \setminus \{(0,0)\}$ is given by the formula

$$c_{m,n} = \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} c_{i,i} + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} c_{j,0}, \tag{2.17}$$

which is the closed form formula that we wanted to obtain (the formula which reconstructs the solutions to equation (1.4) by given "side" values $c_{k,0}$, $c_{k,k}$, $k \in \mathbb{N}$).

Remark 2.3. Note that formula (2.17) does not contain value $c_{0,0}$, which is why instead of domain D we consider domain $D \setminus \{(0,0)\}$.

Corollary 2.4. *The solution to partial difference equation (1.4) on domain $D \setminus \{(0,0)\}$, with the boundary value conditions given by*

$$c_{k,0} = 1 \quad \text{and} \quad c_{k,k} = 1, \qquad k \in \mathbb{N}, \tag{2.18}$$

is given by

$$c_{m,n} = C_n^m. \tag{2.19}$$

Proof. If we put the conditions in (2.18) in formula (2.16), we get

$$c_{m,n} = \sum_{i=1}^{n} C_{m-n-1}^{m-i-1} + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j}. \tag{2.20}$$

We have

$$\sum_{i=1}^{n} C_{m-n-1}^{m-i-1} = \sum_{i=1}^{n} \left(C_{m-n}^{m-i} - C_{m-n}^{m-i-1} \right) = C_{m-n}^{m-1} - C_{m-n}^{m-n-1} = C_{m-n}^{m-1}. \tag{2.21}$$

By using (2.21) in (2.20), noticing that

$$\sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} = \sum_{r=0}^{m-n-1} C_r^{n-1+r},$$

and then applying recurrent relation (1.3), we get

$$\begin{aligned} c_{m,n} &= C_{m-n}^{m-1} + \sum_{r=0}^{m-n-1} C_r^{n-1+r} \\ &= \sum_{r=0}^{m-n} C_r^{n-1+r} = C_0^{n-1} + \sum_{r=1}^{m-n} \left(C_r^{n+r} - C_{r-1}^{n+r-1} \right) \\ &= C_0^{n-1} + C_{m-n}^m - C_0^n = C_{m-n}^m. \end{aligned} \tag{2.22}$$

From (2.22) and since $C_{m-n}^m = C_n^m$, formula (2.19) follows, as desired. □

Remark 2.5. The formula

$$\sum_{r=0}^{m-n} C_r^{n-1+r} = C_{m-n}^m,$$

is well-known and can be found in many books on combinatorics or problem books on elementary mathematics, in this or some equivalent forms (see, e.g., [9] or [12]). We have added the proof for its simplicity, benefit of the reader and for the completeness of the note.

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