Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **29**, 1–25; http://www.math.u-szeged.hu/ejqtde/

On the existence and properties of three types of solutions of singular IVPs

Jana Burkotová[1], **Martin Rohleder**[✉1] and **Jakub Stryja**[2]

[1]Department of Mathematics, Faculty of Science, Palacký University Olomouc,
17. listopadu 12, 771 46 Olomouc, Czech Republic
[2]Department of Mathematics and Descriptive Geometry, VŠB - Technical University Ostrava,
17. listopadu 15, 708 33 Ostrava, Czech Republic

Received 16 December 2014, appeared 26 May 2015
Communicated by Ivan Kiguradze

Abstract. The paper studies the singular initial value problem

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \qquad t > 0, \qquad u(0) = u_0 \in [L_0, L], \qquad u'(0) = 0.$$

Here, $f \in C(\mathbb{R})$, $f(L_0) = f(0) = f(L) = 0$, $L_0 < 0 < L$ and $xf(x) > 0$ for $x \in (L_0, 0) \cup (0, L)$. Further, $p, q \in C[0, \infty)$ are positive on $(0, \infty)$ and $p(0) = 0$. The integral $\int_0^1 \frac{ds}{p(s)}$ may be divergent which yields the time singularity at $t = 0$. The paper describes a set of all solutions of the given problem. Existence results and properties of oscillatory solutions and increasing solutions are derived. By means of these results, the existence of an increasing solution with $u(\infty) = L$ (a homoclinic solution) playing an important role in applications is proved.

Keywords: second order ODE, time singularity, asymptotic properties, damped oscillatory solution, escape solution, homoclinic solution, unbounded domain.

2010 Mathematics Subject Classification: 34A12, 34C15, 34C37, 34D05.

1 Introduction

We investigate the equation

$$\big(p(t)u'(t)\big)' + q(t)f(u(t)) = 0 \tag{1.1}$$

with the initial conditions

$$u(0) = u_0, \qquad u'(0) = 0, \qquad u_0 \in [L_0, L], \tag{1.2}$$

✉Corresponding author. Email: martin.rohleder@upol.cz

where

$$
\begin{aligned}
&L_0 < 0 < L, && f(L_0) = f(0) = f(L) = 0, && (1.3)\\
&f \in C(\mathbb{R}), && x f(x) > 0 \quad \text{for } x \in (L_0, L) \setminus \{0\}, && (1.4)\\
&p \in C[0,\infty), && p(0) = 0, \quad p(t) > 0 \quad \text{for } t \in (0,\infty), && (1.5)\\
&q \in C[0,\infty), && q(t) > 0 \quad \text{for } t \in (0,\infty). && (1.6)
\end{aligned}
$$

We have been motivated by real world problems from hydrodynamics, nonlinear field theory, population genetics, homogeneous nucleation theory or relativistic cosmology, see [1,7,11–14,20,24,38]. The simplest real model is given by equation (1.1) with $p(t) = q(t) = t^2$ and $f(x) = kx(x - L_0)(L - x)$ with a positive parameter k. Existence and properties of solutions of problem (1.1), (1.2), where $p \equiv q$, have been studied in [3,4,28,29,32–35,37] and their numerical simulations are presented for example in [10,19,23]. Other problems close to (1.1), (1.2) can be found in [2,6,8,16–18,25].

At the beginning, we specify smoothness of solutions that we are interested in. Further, we define different types of solutions according to their asymptotic behaviour.

Definition 1.1. Let $c \in (0,\infty)$. A function $u \in C^1[0,c]$ with $pu' \in C^1[0,c]$ which satisfies equation (1.1) for every $t \in [0,c]$ and which satisfies the initial conditions (1.2) is called a *solution* of problem (1.1), (1.2) on $[0,c]$. If u is solution of problem (1.1), (1.2) on $[0,c]$ for every $c > 0$, then u is called a *solution* of problem (1.1), (1.2).

Definition 1.2. A solution u of problem (1.1), (1.2) is said to be *oscillatory* if $u \not\equiv 0$ in any neighborhood of ∞ and if u has a sequence of zeros tending to ∞. Otherwise, u is called *nonoscillatory*.

Definition 1.3. Consider a solution of problem (1.1), (1.2) with $u_0 \in (L_0, L)$ and denote

$$u_{\sup} = \sup\{u(t) \colon t \in [0,\infty)\}.$$

If $u_{\sup} = L$, then u is called a *homoclinic solution* of problem (1.1), (1.2).
If $u_{\sup} < L$, then u is called a *damped solution* of problem (1.1), (1.2).

Definition 1.4. Let u be a solution of problem (1.1), (1.2) on $[0,c]$, where $c \in (0,\infty)$. If u satisfies

$$u(c) = L, \qquad u'(c) > 0,$$

then u is called an *escape solution* of problem (1.1), (1.2) on $[0,c]$.

The aim of the paper is to find additional conditions for functions f, p and q which guarantee that problem (1.1), (1.2) has all three types of solutions from Definitions 1.3 and 1.4. The existence of damped oscillatory solutions of problem (1.1), (1.2) has been proved in [36]. Here, in our paper, we get such type of solutions under more general assumptions. In addition, we prove the existence of escape and homoclinic solutions. Let us note that the integral $\int_0^1 \frac{\mathrm{d}s}{p(s)}$ may be divergent and, due to the assumption $p(0) = 0$, equation (1.1) has a singularity at $t = 0$. For other problems with such type of singularities, see also [3,4,26,30,31]. In the literature, permanent attention has been devoted to the study of equation (1.1) or to its quasilinear generalizations but in the regular setting ($p > 0$ on $[0,\infty)$). Let us mention the papers [27,39] which deal with Emden–Fowler equations. The papers [5,9,15,21,22,40] investigate more general equations but these equations are regular and their nonlinearities have globally monotonous behavior. According to our basic assumptions (1.3)–(1.6), the results of these papers cannot be applied to problem (1.1), (1.2).

In order to derive the existence of all three types of solutions of problem (1.1), (1.2), we introduce the auxiliary equation

$$\big(p(t)u'(t)\big)' + q(t)\tilde{f}(u(t)) = 0, \tag{1.7}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [L_0, L], \\ 0 & \text{for } x < L_0, \quad x > L. \end{cases} \tag{1.8}$$

By means of results about existence and properties of all three types of solutions of problem (1.7), (1.2), we proceed to the existence of escape and homoclinic solution of problem (1.1), (1.2), which is proved at the end of this paper in Theorem 5.4 and Theorem 5.5.

2 Solvability of problem (1.7), (1.2)

In this section, we generalize and extend results of [36] concerning existence and uniqueness of a solution of problem (1.7), (1.2). Arguments in proofs in this section are similar to those given in [36] but we cannot use these results directly, since some of the assertions made there are violated. Therefore, we need to repeat and prove all assertions under weaker assumptions. Before we state existence and uniqueness results, we provide auxiliary lemmas.

Lemma 2.1. *Let (1.3)–(1.6) hold and let u be a solution of problem (1.7), (1.2).*

a) Assume that there exists $t_1 \geq 0$ such that $u(t_1) \in (0, L)$ and $u'(t_1) = 0$. Then

$$u(t) \geq 0 \Rightarrow u'(t) < 0 \quad \textit{for } t \in (t_1, \theta_1],$$

where θ_1 is the first zero of u on (t_1, ∞). If such θ_1 does not exist, then $u'(t) < 0$ for $t \in (t_1, \infty)$.

b) Assume that there exists $t_2 \geq 0$ such that $u(t_2) \in (L_0, 0)$ and $u'(t_2) = 0$. Then

$$u(t) \leq 0 \Rightarrow u'(t) > 0 \quad \textit{for } t \in (t_2, \theta_2], \tag{2.1}$$

where θ_2 is the first zero of u on (t_2, ∞). If such θ_2 does not exist, then $u'(t) > 0$ for $t \in (t_2, \infty)$.

Proof. a) Let $t_1 \geq 0$ be such that $u(t_1) \in (0, L)$ and $u'(t_1) = 0$. First, we assume that there exists $\theta_1 > t_1$ satisfying $u(t) > 0$ on (t_1, θ_1) and $u(\theta_1) = 0$. Then, by (1.4) and (1.6), $q(t)\tilde{f}(u(t)) > 0$, and hence

$$(pu')'(t) < 0, \qquad t \in (t_1, \theta_1).$$

Since $(pu')(t_1) = 0$ and since pu' is decreasing on (t_1, θ_1), we obtain $pu' < 0$ on (t_1, θ_1), and, due to (1.5), $u' < 0$ on (t_1, θ_1). Furthermore, integrating (1.7) over (t_1, θ_1), we get

$$pu'(\theta_1) = -\int_{t_1}^{\theta_1} q(s)\tilde{f}(u(s))\, \mathrm{d}s < 0.$$

Thus $pu' < 0$ on $(t_1, \theta_1]$. If u is positive on $[t_1, \infty)$, we obtain as before $pu' < 0$ on (t_1, ∞). The inequality $u'(t) < 0$ for $t \in (t_1, \infty)$ follows from (1.5).

b) We argue similarly as in a). □

Further properties can be described by means of the function

$$\tilde{F}(x) = \int_0^x \tilde{f}(z)\,\mathrm{d}z, \qquad x \in \mathbb{R}.$$

Lemma 2.2. *Assume that* (1.3)–(1.6) *hold and that*

$$\text{there exists } \bar{B} \in (L_0, 0) : \tilde{F}(\bar{B}) = \tilde{F}(L), \tag{2.2}$$

$$pq \text{ is nondecreasing on } [0, \infty). \tag{2.3}$$

Let u be a solution of problem (1.7), (1.2) *such that there exist $b \geq 0$, $\theta > b$ satisfying*

$$u(b) \in (\bar{B}, 0), \qquad u'(b) = 0, \qquad u(\theta) = 0, \qquad u(t) < 0 \quad \text{for } t \in [b, \theta).$$

Then u fulfils either

$$u'(t) > 0 \quad \text{for } t \in (b, \infty), \qquad \lim_{t\to\infty} u(t) \in (0, L), \tag{2.4}$$

or

$$\exists\, c \in (\theta, \infty), \qquad u(c) \in (0, L), \qquad u'(c) = 0, \qquad u'(t) > 0 \quad \text{for } t \in (b, c). \tag{2.5}$$

Furthermore, if

$$pq \text{ is increasing on } [0, \infty), \tag{2.6}$$

then the assertion holds also for $u(b) = \bar{B}$, $u'(b) = 0$.

Proof. According to Lemma 2.1, $u'(t) > 0$ for $t \in (b, \theta]$. Assume that there exists $c > \theta$ such that $u'(c) = 0$ and $u'(t) > 0$ for $t \in (b, c)$. Let $u(c) \geq L$. Then there exists $b_1 \in (\theta, c]$ such that $u(b_1) = L$, $u' > 0$ on (b, b_1). Multiplying equation (1.7) by pu', integrating over (b, b_1), we obtain

$$\int_b^{b_1} \left(p(t)u'(t)\right)' p(t)u'(t)\,\mathrm{d}t = -\int_b^{\theta} (pq)(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t - \int_{\theta}^{b_1} (pq)(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t.$$

By (2.3), we get

$$\begin{aligned} 0 \leq \frac{\left(p(b_1)u'(b_1)\right)^2}{2} &\leq -(pq)(\theta)\int_b^{\theta} \tilde{f}(u(t))u'(t)\,\mathrm{d}t - (pq)(\theta)\int_{\theta}^{b_1} \tilde{f}(u(t))u'(t)\,\mathrm{d}t \\ &\leq (pq)(\theta)\left(\tilde{F}(u(b)) - \tilde{F}(u(b_1))\right) = (pq)(\theta)\left(\tilde{F}(u(b)) - \tilde{F}(L)\right). \end{aligned}$$

Hence,

$$\tilde{F}(u(b)) \geq \tilde{F}(L). \tag{2.7}$$

On the other hand, since $\bar{B} < u(b) < 0$, we get by (2.2)

$$\tilde{F}(L) = \tilde{F}(\bar{B}) > \tilde{F}(u(b)). \tag{2.8}$$

This contradicts (2.7). Consequently, $u(c) \in (0, L)$ and (2.5) holds.

Let $u'(t) > 0$ on (b, ∞). Then u is increasing and it has a limit for $t \to \infty$. Let $\lim_{t\to\infty} u(t) > L$. Then there exists $b_1 > \theta$ such that $u(b_1) = L$, $u'(b_1) > 0$, which yields a contradiction as before. Let $\lim_{t\to\infty} u(t) = L$. Then

$$\lim_{t\to\infty} \tilde{F}(u(t)) = \tilde{F}(L),$$

and, by (2.8), there exists $T > b$ such that $\tilde{F}(u(T)) > \tilde{F}(u(b))$. Hence, multiplying (1.7) by pu' and integrating over (b, T), we get

$$0 < \frac{(p(T)u'(T))^2}{2} \leq (pq)(\theta)\left(\tilde{F}(u(b)) - \tilde{F}(u(T))\right) < 0.$$

This contradiction yields $\lim_{t\to\infty} u(t) \in (0, L)$ and (2.4) holds.

Let us assume that (2.6) is fulfilled and $u(b) = \bar{B}$, $u'(b) = 0$. We follow the steps in the first part of this proof. If there exists b_1 such that $u(b_1) = L, u' > 0$ on (b, b_1), then, by multiplying equation (1.7) by pu' and integrating over (b, b_1), we obtain the contradiction

$$0 \leq \frac{(p(b_1)u'(b_1))^2}{2} < (pq)(\theta)\left(\tilde{F}(\bar{B}) - \tilde{F}(L)\right) = 0.$$

Consequently, if there exists $c \in (0, \infty)$ such that $u'(c) = 0$, $u'(t) > 0$ for $t \in (b, c)$, then $u(c) \in (0, L)$.

Let $u'(t) > 0$ for $t \in (b, \infty)$. Due to the above arguments, $\lim_{t\to\infty} u(t) \leq L$. Assume that $\lim_{t\to\infty} u(t) = L$. Then

$$\tilde{F}(u(b)) = \tilde{F}\left(\bar{B}\right) = \tilde{F}(L) = \lim_{t\to\infty} \tilde{F}(u(t)).$$

Multiplying equation (1.7) by pu', integrating over (b, θ) and over (θ, t) for $t > \theta$, we get due to (2.1)

$$0 < \frac{(p(\theta)u'(\theta))^2}{2} < (pq)(\theta)\tilde{F}\left(\bar{B}\right),$$
$$-\frac{(p(\theta)u'(\theta))^2}{2} < \frac{(p(t)u'(t))^2}{2} - \frac{(p(\theta)u'(\theta))^2}{2} < (pq)(\theta)\left(-\tilde{F}(u(t))\right).$$

Therefore,

$$(pq)(\theta)\tilde{F}\left(\bar{B}\right) > \frac{(p(\theta)u'(\theta))^2}{2} > (pq)(\theta)\tilde{F}\left(u(t)\right).$$

Letting $t \to \infty$, we get

$$(pq)(\theta)\tilde{F}\left(\bar{B}\right) > \frac{(p(\theta)u'(\theta))^2}{2} \geq (pq)(\theta)\tilde{F}\left(\bar{B}\right).$$

This contradiction completes the proof. □

The following lemma can be proved analogously.

Lemma 2.3. *Assume (1.3)–(1.6), (2.2), (2.3). Let u be a solution of problem (1.7), (1.2) such that there exist $a \geq 0$, $\theta > a$ satisfying*

$$u(a) \in (0, L), \qquad u'(a) = 0, \qquad u(\theta) = 0, \qquad u(t) > 0 \quad \text{for } t \in [a, \theta). \tag{2.9}$$

Then u fulfils either

$$u'(t) < 0 \quad \text{for } t \in (a, \infty), \qquad \lim_{t\to\infty} u(t) \in \left(\bar{B}, 0\right), \tag{2.10}$$

or

$$\exists\, b \in (\theta, \infty) \colon u(b) \in \left(\bar{B}, 0\right), \qquad u'(b) = 0, \qquad u'(t) < 0 \quad \text{for } t \in (a, b). \tag{2.11}$$

Remark 2.4. Let assumptions (1.3)–(1.6) hold. Both equations (1.1) and (1.7) have a constant solution $u(t) \equiv L$ (resp. $u(t) \equiv 0$ resp. $u(t) \equiv L_0$). By Lemma 2.1, the solution $u(t) \equiv 0$ is the only solution satisfying for some $t_0 > 0$ conditions $u(t_0) = 0$, $u'(t_0) = 0$. Assume moreover

$$f \in \mathrm{Lip}_{\mathrm{loc}}([L_0, L] \setminus \{0\}). \tag{2.12}$$

Then, by (1.8), (2.12), the solution $u(t) \equiv L$ (resp. $u(t) \equiv L_0$) is the only solution satisfying for some $t_0 > 0$ conditions $u(t_0) = L$, $u'(t_0) = 0$ (resp. $u(t_0) = L_0$, $u'(t_0) = 0$).

Lemma 2.5. *Assume (1.3)–(1.6) and (2.12). Let u be a solution of problem (1.7), (1.2) with $u_0 \in (L_0, \bar{B}]$. Assume that there exist $\theta > 0$, $a > \theta$ such that*

$$\begin{aligned} u(\theta) &= 0, & u(t) &< 0 \quad \text{for } t \in [0, \theta), \\ u'(a) &= 0, & u'(t) &> 0 \quad \text{for } t \in (\theta, a). \end{aligned}$$

Then

$$u(a) \in (0, L), \qquad u'(t) > 0 \quad \text{for } t \in (0, a). \tag{2.13}$$

Proof. Directly from Lemma 2.1, we have $u' > 0$ on $(0, a)$. Therefore,

$$pu'(t) > 0, \qquad t \in (0, a). \tag{2.14}$$

On contrary to (2.13), assume that $u(a) \geq L$. Then, by (2.12) and Remark 2.4, we have $u(a) > L$. Therefore, there exists $a_0 \in (\theta, a)$ such that $u(t) > L$ on $(a_0, a]$. Integrating equation (1.7) over (a_0, a), we get

$$pu'(a) - pu'(a_0) = \int_{a_0}^{a} q(s)\tilde{f}(u(s))\,\mathrm{d}s = 0.$$

By virtue of (1.8), $pu'(a_0) = 0$, contrary to (2.14). □

The next theorem generalizes the existence results from [36] (Theorem 2.5), where the Banach fixed point theorem was used. Here, we prove the existence of solutions of auxiliary problem (1.7), (1.2) by virtue of the Schauder fixed point theorem.

Theorem 2.6 (Existence of solution of problem (1.7), (1.2))**.** *Let assumptions (1.3)–(1.6) and*

$$\lim_{t \to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0 \tag{2.15}$$

hold. Then for each $u_0 \in [L_0, L]$ problem (1.7), (1.2) has a solution u. If in addition conditions (2.2), (2.3) and (2.12) hold, then the solution u satisfies:

$$\text{if } u_0 \in [\bar{B}, L], \quad \text{then } u(t) > \bar{B}, \qquad t \in (0, \infty), \tag{2.16}$$

$$\text{if } u_0 \in (L_0, \bar{B}), \quad \text{then } u(t) > u_0, \qquad t \in (0, \infty). \tag{2.17}$$

Proof. Clearly, for $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$ there exists a solution by Remark 2.4. Assume that $u_0 \in (L_0, 0) \cup (0, L)$. Integrating equation (1.7), we get an equivalent form

$$u(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \qquad t \in [0, \infty).$$

By (1.4), (1.8), there exists $M > 0$ such that $|\tilde{f}(x)| \leq M$, $x \in \mathbb{R}$. Put

$$\varphi(t) = \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s, \qquad t > 0. \tag{2.18}$$

Choose an arbitrary $b > 0$. By (2.15), there exists $\varphi_b > 0$ such that $|\varphi(t)| \le \varphi_b$ for each $t \in (0, b]$. Consider the Banach space $C[0,b]$ with the maximum norm and define an operator $\mathcal{F}\colon C[0,b] \to C[0,b]$,

$$(\mathcal{F}u)(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Put $\Lambda = \max\{|L_0|, L\}$ and consider the ball $\mathcal{B}(0,R) = \{u \in C[0,b] : \|u\|_{C[0,b]} \le R\}$, where $R = \Lambda + M\varphi_b$. The norm of operator $\mathcal{F}$ is estimated as follows

$$\|\mathcal{F}u\|_{C[0,b]} = \max_{t\in[0,b]} \left| u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \le \Lambda + M\varphi_b = R,$$

which yields that $\mathcal{F}$ maps $\mathcal{B}(0,R)$ on itself. Choose a sequence $\{u_n\} \subset C[0,b]$ such that $\lim_{n\to\infty} \|u_n - u\|_{C[0,b]} = 0$. Since the function $\tilde{f}$ is continuous, we get

$$\lim_{n\to\infty} \|\mathcal{F}u_n - \mathcal{F}u\|_{C[0,b]} \le \lim_{n\to\infty} \|\tilde{f}(u_n) - \tilde{f}(u)\|_{C[0,b]} \left(\int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \right) = 0,$$

that is the operator $\mathcal{F}$ is continuous. Choose an arbitrary $\varepsilon > 0$ and put $\delta = \frac{\varepsilon}{M\varphi_b}$. Then, for $t_1, t_2 \in [0,b]$ and for $u \in \mathcal{B}(0,R)$, we have

$$|t_1 - t_2| < \delta \Rightarrow |(\mathcal{F}u)(t_1) - (\mathcal{F}u)(t_2)| = \left| \int_{t_1}^{t_2} \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \le M\varphi_b|t_2 - t_1| < \varepsilon.$$

Hence, functions in $\mathcal{F}(\mathcal{B}(0,R))$ are equicontinuous, and, by the Arzelà–Ascoli theorem, the set $\mathcal{F}(\mathcal{B}(0,R))$ is relatively compact. Consequently, the operator $\mathcal{F}$ is compact on $\mathcal{B}(0,R)$.

The Schauder fixed point theorem yields a fixed point $u^\star$ of $\mathcal{F}$ in $\mathcal{B}(0,R)$. Therefore,

$$u^\star(t) = u_0^\star - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u^\star(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Hence, $u^\star(0) = u_0^\star$,

$$\left(p(t)(u^\star)'(t)\right)' = -q(t)\tilde{f}\left(u^\star(t)\right), \qquad t \in [0,b].$$

Since $|(u^\star)'(t)| \le M\varphi(t)$ and, by (2.15), $\lim_{t\to 0^+}(u^\star)'(t) = 0 = (u^\star)'(0)$. According to (1.8), $\tilde{f}(u^\star(t))$ is bounded on $[0,\infty)$ and hence, by Theorem 11.5 in [17], $u^\star$ can be extended to interval $[0,\infty)$ as a solution of equation (1.7).

Now, let (2.2), (2.3) and (2.12) hold. Using Lemmas 2.2, 2.3 and 2.5 we get estimates (2.16) and (2.17). For more details, see the proof of Theorem 2.5 in [36]. □

Remark 2.7. Under assumptions (1.3)–(1.6) and (2.15), each solution of problem (1.7), (1.2) is defined on the half-line $[0,\infty)$. In addition, the set of these solutions with $u_0 \in (L_0, 0) \cup (0, L)$ is composed of three disjoint classes $\mathcal{S}_d$ (damped solutions), $\mathcal{S}_h$ (homoclinic solutions), $\mathcal{S}_e$ (escape solutions). Here

1. $u \in \mathcal{S}_d$ if and only if $u_{\sup} < L$,
2. $u \in \mathcal{S}_h$ if and only if $u_{\sup} = L$,
3. $u \in \mathcal{S}_e$ if and only if $u_{\sup} > L$.

The uniqueness result from Theorem 2.5 in [36] is extended in the following theorem, where weaker assumptions are considered.

Theorem 2.8 (Uniqueness and continuous dependence on initial values)**.** *Let assumptions* (1.3)–(1.6), (2.15) *hold and let*

$$f \in \operatorname{Lip}[L_0, L]. \tag{2.19}$$

Then for each $u_0 \in [L_0, L]$, *problem* (1.7), (1.2) *has a unique solution. Further, for each* $b > 0$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $B_1, B_2 \in [L_0, L]$

$$|B_1 - B_2| < \delta \Rightarrow \|u_1 - u_2\|_{C^1[0,b]} < \varepsilon.$$

Here, u_i *is a solution of problem* (1.7), (1.2) *with* $u_0 = B_i$, $i = 1, 2$.

Proof. For $i \in \{1,2\}$ choose $B_i \in [L_0, L]$. By Theorem 2.6, there exists a solution u_i of problem (1.7), (1.2) with $u_0 = B_i$. We integrate (1.7) where $u = u_i$, and get by (1.2)

$$u_i(\xi) = B_i - \int_0^\xi \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u_i(\tau))\, \mathrm{d}\tau\, \mathrm{d}s, \qquad \xi \in [0,\infty). \tag{2.20}$$

Denote

$$\varrho(t) = \max\{|u_1(\xi) - u_2(\xi)| : \xi \in [0,t]\}, \qquad t \in [0,\infty).$$

Then (2.20) yields

$$\varrho(t) \le |B_1 - B_2| + \int_0^t \frac{1}{p(s)} \int_0^s q(\tau) \left|\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))\right| \mathrm{d}\tau\, \mathrm{d}s, \qquad t \in [0,\infty).$$

By (2.19), there exists a Lipschitz constant $K \in (0,\infty)$ for f on $[L_0, L]$. Then K is the Lipschitz constant for $\tilde{f}$ on $\mathbb{R}$ and

$$\varrho(t) \le |B_1 - B_2| + K \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\varrho(\tau)\, \mathrm{d}\tau\, \mathrm{d}s, \qquad t \in [0,\infty). \tag{2.21}$$

Denote (*cf.* (2.15) and (2.18))

$$\varphi(s) = \frac{1}{p(s)} \int_0^s q(\tau)\, \mathrm{d}\tau, \qquad s \in (0,\infty), \qquad \varphi(0) = 0.$$

Choose $b > 0$. Then, due to (2.15), there exists $\varphi_b \in (0,\infty)$ such that

$$|\varphi(s)| \le \varphi_b, \qquad s \in [0,b]. \tag{2.22}$$

Since ϱ is nondecreasing on $[0,b]$, we get by (2.21)

$$\varrho(t) \le |B_1 - B_2| + K \int_0^t \varrho(s)\varphi(s)\, \mathrm{d}s, \qquad t \in [0,b],$$

and, using the Gronwall lemma, we arrive at

$$\varrho(t) \le |B_1 - B_2| e^{Kb\varphi_b}, \qquad t \in [0,b]. \tag{2.23}$$

Similarly, by (2.20), we get for $i \in \{1,2\}$

$$u_i'(t) = -\frac{1}{p(t)} \int_0^t q(s)\tilde{f}(u_i(s))\, \mathrm{d}s, \qquad t \in (0,\infty), \qquad u_i'(0) = 0.$$

Therefore,

$$\left|u_1'(t) - u_2'(t)\right| \le \frac{K}{p(t)} \int_0^t q(s)|u_1(s) - u_2(s)|\,\mathrm{d}s, \qquad t \in [0,\infty)\,.$$

Applying (2.22) and (2.23), we get

$$\max\{\left|u_1'(t) - u_2'(t)\right| : t \in [0,b]\} \le |B_1 - B_2| K\varphi_b e^{Kb\varphi_b}.$$

Consequently,

$$\|u_1 - u_2\|_{C^1[0,b]} \le |B_1 - B_2|(1 + K\varphi_b)e^{Kb\varphi_b} < \varepsilon,$$

provided $|B_1 - B_2| < \delta$, where

$$\delta = \frac{\varepsilon}{(1 + K\varphi_b)e^{Kb\varphi_b}}.$$

If $B_1 = B_2$, then $u_1(t) = u_2(t)$ on each $[0,b] \subset \mathbb{R}$ which yields the uniqueness of a solution of problem (1.7), (1.2). □

Example 2.9. Let us put

$$p(t) = t^\alpha, \qquad q(t) = t^\beta, \qquad t \in [0,\infty)\,. \tag{2.24}$$

Assume that $\alpha > 0$, $\beta \ge 0$, $\beta > \alpha - 1$. Then p and q satisfy (1.5), (1.6), (2.6) (consequently (2.3)) and (2.15). Further, let $f \in C(\mathbb{R})$ be such that

$$f(x) = k|x|^\gamma \operatorname{sgn} x(x - L_0)(L - x), \qquad x \in [L_0, L]\,, \tag{2.25}$$

where $0 < L < -L_0$, $\gamma > 0$, $k > 0$. Then f fulfils (1.3), (1.4), (2.2) and (2.12). Therefore, the assertions of Theorem 2.6 are valid. Now, assume that the constants in (2.25) fulfil $0 < L < -L_0$, $\gamma \ge 1$, $k > 0$. Then f fulfils in addition (2.19) and the assumptions of Theorem 2.8 as well as (2.16) and (2.17) are valid.

3 Existence and properties of damped solutions of problem (1.1), (1.2)

Now, we specify an interval for starting values u_0 where the existence of damped solutions is guaranteed. Moreover, we provide conditions under which each damped solution is oscillatory. Note that by virtue of Definition 1.3 and (1.8), all results of this section are proved for the original problem (1.1), (1.2), since function f coincides with function $\tilde{f}$ on $[L_0, L]$ and $L_0 \le u(t) \le L$ holds for $t \in (0,\infty]$ if u is a damped solution. In addition, all lemmas of Section 2 are valid for damped solutions of problem (1.1), (1.2).

Theorem 3.1 (Existence of damped solutions of problem (1.1), (1.2))**.** *Assume that the assumptions (1.3)–(1.6), (2.2), (2.3), (2.12) and (2.15) are fulfilled. Then for each $u_0 \in (\bar{B}, L)$, problem (1.1), (1.2) has a solution u. The solution u is damped and satisfies (2.16).*

Proof. Choose $u_0 \in (\bar{B}, L)$. By Theorem 2.6, there exists a solution u of problem (1.7), (1.2) satisfying (2.16). Using Lemmas 2.1–2.3, we get $u_{\sup} < L$, and a solution u is damped. For more details, see the proof of Theorem 2.6 in [36]. By (1.8), $f(u(t)) = \tilde{f}(u(t))$ for $t \in [0,\infty)$ and then u is a solution of problem (1.1), (1.2). □

Remark 3.2. If moreover (2.6) is fulfilled, then, by Lemma 2.2, the assertion of Theorem 3.1 holds for $u_0 = \bar{B}$, too. Functions satisfying (2.6) are presented in Example 2.9.

In order to obtain conditions under which every damped solution is oscillatory, we distinguish two cases according to the convergence or divergence of the integral $\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s$.

I. We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s < \infty. \tag{3.1}$$

II. We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s = \infty. \tag{3.2}$$

Definition 3.3. A function u is called *eventually positive* (resp. *eventually negative*), if there exists $t_0 > 0$ such that $u(t) > 0$ (resp. $u(t) < 0$) for $t \in (t_0, \infty)$.

Lemma 3.4. *Assume* (1.3)–(1.6), (2.2), (2.3), (3.1) *and*

$$\lim_{t\to\infty} \int_1^t \frac{1}{p(s)} \int_1^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \infty. \tag{3.3}$$

Let u be a damped solution of problem (1.1), (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$ which is nonoscillatory. Then*

$$\lim_{t\to\infty} u(t) = 0. \tag{3.4}$$

Proof. Assume that u is a damped nonoscillatory solution of problem (1.1), (1.2) with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is either eventually positive or eventually negative.

We will prove that $\lim_{t\to\infty} u(t) = 0$. Since u is nonoscillatory, Lemma 2.1 guarantees the existence of $t_0 > 1$ such that u is either increasing or decreasing on $[t_0, \infty)$. Therefore, there exists $\lim_{t\to\infty} u(t) = c$. Since $u_{\sup} < L$, we have $c < L$. Integrating equation (1.1) from t_0 to t and dividing this by $p(t)$, we get

$$\begin{aligned} u'(t) &= \frac{p(t_0)u'(t_0)}{p(t)} - \frac{1}{p(t)} \int_{t_0}^t q(s) f(u(s))\,\mathrm{d}s, \\ u(t) &= u(t_0) + \int_{t_0}^t \frac{p(t_0)u'(t_0)}{p(s)}\,\mathrm{d}s - \int_{t_0}^t \frac{1}{p(s)} \int_{t_0}^s q(\tau) f(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s. \end{aligned} \tag{3.5}$$

Let u be eventually positive. Then $c \in [0, L)$. Assume $c \in (0, L)$. Then there exists $M > 0$ such that $f(u(t)) \ge M$ for $t \ge t_0$. From (3.5), we obtain

$$\begin{aligned} u(t) &\le u(t_0) + p(t_0)u'(t_0) \int_{t_0}^t \frac{1}{p(s)}\,\mathrm{d}s - M \int_{t_0}^t \frac{1}{p(s)} \int_{t_0}^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \\ \lim_{t\to\infty} u(t) &\le u(t_0) + p(t_0)u'(t_0) \int_{t_0}^\infty \frac{1}{p(s)}\,\mathrm{d}s - M \lim_{t\to\infty} \int_{t_0}^t \frac{1}{p(s)} \int_{t_0}^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = -\infty, \end{aligned}$$

which contradicts $c \in (0, L)$. Hence $c = 0$.

Let u be eventually negative. If u is negative on $[0, \infty)$, then, by Lemma 2.1 b), we get $u'(t) > 0$ for $t \in (0, \infty)$ and thus $c \in (L_0, 0]$. Now, assume that there exist $a \ge 0$ and $\theta > a$ satisfying (2.9) and $u(t) < 0$ for $t > \theta$. By Lemma 2.3, it occurs either (2.10) or (2.11). If (2.10) holds, then $c \in (\bar{B}, 0)$. If (2.11) holds, then, by Lemma 2.1 b), $c \in (\bar{B}, 0]$. Assume that $c \in (L_0, 0)$. Then there exists $M > 0$ such that $-f(u(t)) \ge M$ for $t \ge t_0$ and, similarly as in the eventually positive case, we derive a contradiction. Therefore, $c = 0$ and (3.4) is proved. □

Theorem 3.5 (Damped solution is oscillatory, Case I.)**.** *Assume* (1.3)–(1.6), (2.2), (2.3), (3.1),

$$\liminf_{x\to 0^+} \frac{f(x)}{x} > 0, \qquad \liminf_{x\to 0^-} \frac{f(x)}{x} > 0, \tag{3.6}$$

$$\int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s = \infty, \quad \textit{where } \ell(t) = \int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s. \tag{3.7}$$

Let u be a damped solution of problem (1.1), (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is oscillatory.*

Proof. Step 1. We show that (3.7) implies (3.3). Let us put

$$h(t) = \int_1^t \frac{1}{p(s)} \left(\int_1^s q(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s.$$

We accomplish the proof indirectly. Let

$$\lim_{t\to\infty} h(t) =: K < \infty. \tag{3.8}$$

Then integration by parts yields for every $\tau > 1$

$$\begin{aligned}\int_1^\tau \ell^2(t)q(t)\,\mathrm{d}t &= \ell(\tau) \int_\tau^\infty \frac{1}{p(s)}\,\mathrm{d}s \int_1^\tau q(\xi)\,\mathrm{d}\xi + 2\int_1^\tau \ell(t)h'(t)\,\mathrm{d}t \\ &\le \ell(\tau) \int_\tau^\infty \frac{1}{p(s)} \left(\int_1^s q(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s + 2\ell(\tau)h(\tau) + 2h(\tau) \int_1^\tau \frac{1}{p(t)}\,\mathrm{d}t.\end{aligned}$$

Since (3.1) yields $\lim_{\tau\to\infty} \ell(\tau) = 0$, we get by (3.8) for $\tau \to \infty$

$$\int_1^\infty \ell^2(t)q(t)\,\mathrm{d}t \le \lim_{\tau\to\infty} \ell(\tau) \int_\tau^\infty \frac{1}{p(s)} \left(\int_1^s q(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s + 2\lim_{\tau\to\infty} \ell(\tau)K + 2K\ell(1) = 2K\ell(1) < \infty,$$

and (3.7) is not fulfilled.

Step 2. Let u be a damped solution which is nonoscillatory. By Step 1, (3.3) holds and, by Lemma 3.4, we have $\lim_{t\to\infty} u(t) = 0$, which together with (3.6) gives

$$\liminf_{t\to\infty} \frac{f(u(t))}{u(t)} > 0.$$

Consequently, there exist $\alpha > 0$ and $t_1 > 0$ such that

$$u(t) \ne 0, \qquad \frac{f(u(t))}{u(t)} \ge \alpha, \qquad t \in [t_1, \infty). \tag{3.9}$$

Put $\rho(t) = \frac{p(t)u'(t)}{u(t)}$ for $t \ge t_1$. By (1.7) and (3.9), we have

$$\rho'(t) = -q(t)\frac{f(u(t))}{u(t)} - \frac{1}{p(t)}\rho^2(t) \le -\alpha q(t) - \frac{1}{p(t)}\rho^2(t), \qquad t \ge t_1.$$

Multiplying this inequality by ℓ^2 and integrating from t_1 to t, we get

$$\int_{t_1}^t \ell^2(s)\rho'(s)\,\mathrm{d}s \le -\alpha \int_{t_1}^t \ell^2(s)q(s)\,\mathrm{d}s - \int_{t_1}^t \frac{1}{p(s)}\ell^2(s)\rho^2(s)\,\mathrm{d}s, \qquad t \ge t_1.$$

Integrating it by parts, we obtain

$$\begin{aligned}\ell^2(t)\rho(t) - \ell^2(t_1)\rho(t_1) \le & -\alpha \int_{t_1}^t \ell^2(s)q(s)\,\mathrm{d}s \\ & - \int_{t_1}^t \frac{1}{p(s)}\big(\ell^2(s)\rho^2(s) + 2\ell(s)\rho(s) + 1\big)\,\mathrm{d}s \\ & + \int_{t_1}^t \frac{1}{p(s)}\,\mathrm{d}s, \qquad t \in [t_1,\infty).\end{aligned}$$

Further,

$$\begin{aligned}\ell(t)(\ell(t)\rho(t)+1) - \ell(t) \le \ell^2(t_1)\rho(t_1) - \alpha \int_{t_1}^t \ell^2(s)q(s)\,\mathrm{d}s \\ - \int_{t_1}^t \frac{1}{p(s)}(\ell(s)\rho(s)+1)^2\,\mathrm{d}s + \int_{t_1}^\infty \frac{1}{p(s)}\,\mathrm{d}s, \qquad t \in [t_1,\infty),\end{aligned}$$

and finally,

$$\ell(t)(\ell(t)\rho(t)+1) \le \ell(t_1)(\ell(t_1)\rho(t_1)+1) - \alpha \int_{t_1}^t \ell^2(s)q(s)\,\mathrm{d}s - \int_{t_1}^t \frac{1}{p(s)}(\ell(s)\rho(s)+1)^2\,\mathrm{d}s,$$

$t \in [t_1,\infty)$. By (3.7), there exist $t_0 \ge t_1$ such that

$$\int_{t_1}^t \ell^2(s)q(s)\,\mathrm{d}s \ge \frac{1}{\alpha}\ell(t_1)(\ell(t_1)\rho(t_1)+1), \qquad t \in [t_0,\infty),$$

and hence, we get

$$0 < \int_{t_1}^t \frac{1}{p(s)}(\ell(s)\rho(s)+1)^2\,\mathrm{d}s \le -\ell(t)(\ell(t)\rho(t)+1), \qquad t \in [t_0,\infty). \tag{3.10}$$

Put

$$x(t) = \int_{t_1}^t \frac{1}{p(s)}(\ell(s)\rho(s)+1)^2\,\mathrm{d}s, \qquad t \in [t_0,\infty).$$

Then

$$x'(t) = \frac{1}{p(t)}(\ell(t)\rho(t)+1)^2, \qquad t \in [t_0,\infty),$$

and by (3.10)

$$x^2(t) \le \ell^2(t)(\ell(t)\rho(t)+1)^2, \qquad t \in [t_0,\infty).$$

Therefore, x fulfils the differential inequality

$$x^2(t) \le p(t)\ell^2(t)x'(t), \qquad t \in [t_0,\infty).$$

Integrating it over $[t_1,t]$, we derive

$$\frac{1}{\ell(t)} \le \frac{1}{x(t_1)} + \frac{1}{\ell(t_1)}, \qquad t \in [t_1,\infty),$$

and for $t \to \infty$ we have by (3.1)

$$\infty = \lim_{t\to\infty} \frac{1}{\ell(t)} \le \frac{1}{x(t_1)} + \frac{1}{\ell(t_1)} < \infty.$$

This contradiction yields that u is oscillatory. □

Example 3.6. Let p and q be given by (2.24) and assume that α and β fulfil $\alpha \in (1,2]$, $\alpha - 1 < \beta$ or $2 < \alpha$, $2\alpha - 3 \le \beta$. Then

$$\ell(t) = \int_t^\infty \frac{\mathrm{d}s}{s^\alpha} = \frac{t^{1-\alpha}}{\alpha - 1},$$

and

$$\int_1^\infty \ell^2(s) q(s)\,\mathrm{d}s = \frac{1}{(\alpha-1)^2}\int_1^\infty s^{2-2\alpha+\beta}\,\mathrm{d}s = \infty,$$

provided $\beta \ge 2\alpha - 3$. Since the implications

$$\alpha \in (1,2] \Rightarrow 2\alpha - 3 \le \alpha - 1,$$
$$2 < \alpha \Rightarrow \alpha - 1 < 2\alpha - 3$$

are valid, we deduce that p and q satisfy (1.5), (1.6), (2.6) (consequently (2.3)), (2.15), (3.1) and (3.7). Further, let $f \in C(\mathbb{R})$ be such that

$$f(x) = \begin{cases} -|x|^a(x+2), & x \in [-2,0], \\ x^b(1-x), & x \in [0,1], \end{cases} \tag{3.11}$$

where $0 < a \le b \le 1$. Then $L_0 = -2$, $L = 1$ and f fulfils (1.3), (1.4), (2.2), (2.12) and (3.6). Therefore, the assertion of Theorem 3.5 is valid.

Theorem 3.7 (Damped solution is oscillatory, Case II.). *Assume* (1.3)–(1.6), (2.2), (2.3), (3.2) *and*

$$\int_1^\infty q(s)\,\mathrm{d}s = \infty. \tag{3.12}$$

Let u be a damped solution of problem (1.1), (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is oscillatory.*

Proof. Step 1. Let u be a damped solution of problem (1.1), (1.2) which is eventually positive. Then there exists $t_0 \ge 1$ such that $u(t) > 0$ for $t \in [t_0, \infty)$. Assume that $u' > 0$ on $[t_0, \infty)$. Then u is increasing on interval $[t_0, \infty)$ and there exists a limit $\lim_{t\to\infty} u(t) =: \ell_0 \in (u(t_0), L)$. Put $m_0 = \min\{f(x) \colon x \in [u(t_0), \ell_0]\} > 0$. By (1.1), we have

$$\big(p(t)u'(t)\big)' = -q(t)f(u(t)) \le -q(t)m_0, \qquad t \in [t_0, \infty).$$

Integrating this inequality over (t_0, t) and dividing by $p(t)$, we get

$$u'(t) \le \frac{p(t_0)u'(t_0)}{p(t)} - \frac{m_0}{p(t)}\int_{t_0}^t q(s)\,\mathrm{d}s, \qquad t \in [t_0, \infty),$$

$$0 < u(t) \le u(t_0) + p(t_0)u'(t_0)\int_{t_0}^t \frac{1}{p(s)}\,\mathrm{d}s - m_0 \int_{t_0}^t \frac{1}{p(s)}\left(\int_{t_0}^s q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s, \qquad t \in [t_0, \infty).$$

We divide this inequality by $m_0 \int_{t_0}^t \frac{1}{p(s)}\,\mathrm{d}s$ and we get

$$\frac{\int_{t_0}^t \frac{1}{p(s)}\left(\int_{t_0}^s q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s}{\int_{t_0}^t \frac{1}{p(s)}\,\mathrm{d}s} < \frac{u(t_0)}{m_0\int_{t_0}^t \frac{1}{p(s)}\,\mathrm{d}s} + \frac{p(t_0)u'(t_0)}{m_0}, \qquad t \in [t_0, \infty),$$

$$\lim_{t\to\infty} \frac{\int_{t_0}^t \frac{1}{p(s)}\left(\int_{t_0}^s q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s}{\int_{t_0}^t \frac{1}{p(s)}\,\mathrm{d}s} = \lim_{t\to\infty}\frac{\frac{1}{p(t)}\int_{t_0}^t q(\xi)\,\mathrm{d}\xi}{\frac{1}{p(t)}} = \lim_{t\to\infty}\int_{t_0}^t q(\xi)\,\mathrm{d}\xi = \infty.$$

On the other hand,

$$\lim_{t\to\infty} \frac{u(t_0)}{m_0 \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s} + \frac{p(t_0)u'(t_0)}{m_0} = \frac{p(t_0)u'(t_0)}{m_0} < \infty.$$

We have $\infty \le \frac{p(t_0)u'(t_0)}{m_0} < \infty$. This is a contradiction. Therefore, there exists $t_1 \ge t_0$ such that $u(t_1) \in (0, L)$, $u'(t_1) \le 0$. Since u is eventually positive, equation (1.1) together with (1.4), (1.6) yields that pu' is decreasing and, from $p(t_1)u'(t_1) \le 0$, we get that pu' is negative on (t_1, ∞). Therefore, there exists $K > 0$ and $t_2 > t_1$ such that

$$\begin{aligned} pu'(t) &< -K, & t &\in (t_2, \infty), \\ u'(t) &< -K\frac{1}{p(t)}, & t &\in (t_2, \infty). \end{aligned}$$

Integrating this inequality over (t_2, t), we obtain

$$u(t) - u(t_2) < -K \int_{t_2}^{t} \frac{\mathrm{d}s}{p(s)}.$$

Letting $t \to \infty$ and using (3.2), we get

$$\lim_{t\to\infty} u(t) \le u(t_2) - K \int_{t_2}^{\infty} \frac{\mathrm{d}s}{p(s)} = -\infty,$$

contrary to the assumption that u is eventually positive.

Step 2. Let u be a damped solution of problem (1.1), (1.2) which is eventually negative. Then there exists $t_0 \ge 1$ such that $u(t) < 0$ for $t \in [t_0, \infty)$. We show that $u(t) > L_0$ for $t \in [t_0, \infty)$. If $u(t) < 0$ for $t \in [0, \infty)$, then, from Lemma 2.1 b), we have $u(t) \in (L_0, 0)$, $u'(t) > 0$ for $t \in (0, \infty)$. Assume that there exist $a \ge 0$, $\theta \in (a, t_0)$ such that u fulfils (2.9), $u(t) < 0$ for $t \in (\theta, \infty)$. By Lemma 2.3, either (2.10) or (2.11) holds. If (2.10) is valid, then $u(t) \in (\bar{B}, 0)$ for $t \in (\theta, \infty)$. If (2.11) is fulfilled, then, by Lemma 2.1 b), we have $u(t) \in (\bar{B}, 0)$ for $t \in (\theta, \infty)$. We have shown that $u(t) \in (L_0, 0)$ for $t \in [t_0, \infty)$ and that there exists $\lim_{t\to\infty} u(t) > L_0$. (Solution u is increasing in a neighbourhood of ∞). Analogously as in Step 1, we can derive that u cannot be eventually negative.

Consequently, u is oscillatory. □

Example 3.8. Let p and q be given by (2.24), where $\alpha \in (0, 1)$, $\beta \ge 0$. Then p and q satisfy (1.5), (1.6), (2.6) (consequently (2.3)), (2.15), (3.2) and (3.12). Let $f \in C(\mathbb{R})$ be given either by (2.25), where L_0, L, γ and k fulfil $0 < L < -L_0$, $\gamma > 0$, $k > 0$, or by (3.11), where a and b fulfil $0 < b \le a$. Then the assertion of Theorem 3.7 is valid.

4 Properties of escape and homoclinic solutions

In this section, we prove some important properties of escape and homoclinic solutions. In order to obtain existence results, the monotonicity of escape and homoclinic solutions is needed, see Lemma 4.1 and Lemma 4.2. Moreover, we specify asymptotic behaviour of homoclinic solutions in Lemma 4.3. Note that, by Theorem 3.1, a solution of problem (1.1), (1.2) is damped if $u_0 \in (\bar{B}, L)$, $\bar{B} < 0$. Therefore, we can restrict our consideration about escape and homoclinic solutions on $u_0 \in (L_0, 0)$.

Lemma 4.1 (Escape solution is increasing)**.** *Let assumptions* (1.3)–(1.6) *hold. If a solution u of problem* (1.7), (1.2) *with $u_0 \in (L_0, 0)$ is an escape solution, then*

$$\exists c \in (0, \infty): \; u(c) = L, \quad u'(t) > 0 \quad \textit{for } t \in (0, \infty). \tag{4.1}$$

Proof. Let u be an escape solution of problem (1.7), (1.2) with $u_0 \in (L_0, 0)$. Then there exists a constant $c \in (0, \infty)$ such that $u(c) = L$, $u'(c) > 0$. Let $c_1 > c$ be such that $u'(c_1) = 0$ and $u(t) > L$, $u'(t) > 0$ for $t \in (c, c_1)$. Due to (1.8), we have

$$u'(t) = \frac{p(c)u'(c)}{p(t)} > 0 \quad \text{for } t \in (c, c_1],$$

a contradiction. Therefore, $u'(t) > 0$ for $t > c$. Now, we prove that $u'(t) > 0$ for $t \in (0, c)$. Since $u_0 \in (L_0, 0)$, Lemma 2.1 b) yields that there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$, $u'(t) > 0$ for $t \in (0, \theta_0]$. We integrate (1.7) from $t \in (\theta_0, c)$ to c and get

$$0 < p(c)u'(c) + \int_t^c q(s)\tilde{f}(u(s))\,\mathrm{d}s = p(t)u'(t), \qquad t \in (\theta_0, c).$$

To summarize, $u'(t) > 0$ for $t > 0$. $\square$

Lemma 4.2 (Homoclinic solution is increasing)**.** *Let assumptions* (1.3)–(1.6), (2.2), (2.3) *and* (2.12) *hold. If a solution u of problem* (1.7), (1.2) *with $u_0 \in (L_0, 0)$ is homoclinic, then*

$$\lim_{t\to\infty} u(t) = L, \qquad u'(t) > 0 \quad \textit{for } t \in (0, \infty). \tag{4.2}$$

Proof. Let u be a homoclinic solution of problem (1.7), (1.2) with $u_0 \in (L_0, 0)$. Then, by Lemma 2.1 b), there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$, $u'(t) > 0$ for $t \in (0, \theta_0]$.

Assume on the contrary, that there exists $t_1 > \theta_0$ such that $u'(t_1) = 0$, $u'(t) > 0$ for $t \in (0, t_1)$. Since u is homoclinic and (2.12) holds, $u(t_1) \in (0, L)$. By Lemma 2.1 a), there exists $\theta_1 > t_1$ such that $u(\theta_1) = 0$, $u'(t) < 0$ for $t \in (t_1, \theta_1]$. (Since $u_{\sup} = L$, the case $u(t) \in (0, u(t_1))$ for $t > t_1$ cannot occur.) By Lemma 2.3, there exists $t_2 > \theta_1$ such that $u(t_2) \in (\bar{B}, 0)$, $u'(t_2) = 0$, $u'(t) < 0$ for $t \in [\theta_1, t_2)$. (Since $u_{\sup} = L$, u cannot fulfil (2.10).) Repeating this procedure, we obtain a sequence of zeros $\{\theta_n\}_{n=0}^{\infty}$ of u and a sequence of local maxima $\{u(t_{2n+1})\}_{n=0}^{\infty}$ of u. Therefore, u is oscillatory.

We prove that the sequence $\{u(t_{2n+1})\}_{n=0}^{\infty}$ is nonincreasing. Choose $j = 2n+1$, $n \in \mathbb{N}_0$. Multiplying equation (1.7) by pu', integrating this from t_j to t_{j+2} and using (2.3) and the mean value theorem, we get $\xi_1 \in [t_j, \theta_j]$, $\xi_2 \in [\theta_j, t_{j+1}]$, $\xi_3 \in [t_{j+1}, \theta_{j+1}]$, $\xi_4 \in [\theta_{j+1}, t_{j+2}]$ such that

$$\begin{aligned} 0 &= \int_{t_j}^{t_{j+2}} (p(t)u'(t))' p(t)u'(t)\,\mathrm{d}t = (pq)(\xi_1)\left(\tilde{F}(u(t_j)) - \tilde{F}(u(\theta_j))\right) \\ &\quad + (pq)(\xi_2)\left(\tilde{F}(u(\theta_j)) - \tilde{F}(u(t_{j+1}))\right) + (pq)(\xi_3)\left(\tilde{F}(u(t_{j+1})) - \tilde{F}(u(\theta_{j+1}))\right) \\ &\quad + (pq)(\xi_4)\left(\tilde{F}(u(\theta_{j+1})) - \tilde{F}(u(t_{j+2}))\right) \\ &\le (pq)(\xi_4)\left(\tilde{F}(u(t_j)) - \tilde{F}(u(t_{j+2}))\right). \end{aligned}$$

Hence $\tilde{F}(u(t_j)) \ge \tilde{F}(u(t_{j+2}))$. Since function $\tilde{F}$ is increasing on $[0, L]$, we get $u(t_j) \ge u(t_{j+2})$. The sequence $\{u(t_{2n+1})\}_{n=0}^{\infty}$ is nonincreasing because j is chosen arbitrarily. Thus $u_{\sup} < L$, which cannot be fulfilled because u is homoclinic. We have proved that $u'(t) > 0$ for $t \in (0, \infty)$. Since $u_{\sup} = L$, then $\lim_{t\to\infty} u(t) = L$. $\square$

In order to prove further asymptotic properties of homoclinic solutions, we will use the condition

$$\liminf_{t\to\infty} p(t) > 0. \tag{4.3}$$

Lemma 4.3. *Assume that* (1.3)–(1.6), (2.2), (2.3) *and* (2.12) *hold. Further, assume that either condition* (3.1) *is valid or conditions* (3.2) *and* (4.3) *are fulfilled. If a solution u of problem* (1.7), (1.2) *with $u_0 \in (L_0, 0)$ is homoclinic, then u fulfils*

$$\lim_{t\to\infty} u'(t) = 0. \tag{4.4}$$

Proof. According to Lemma 4.2, u fulfils (4.2). Hence, there exists $t_0 > 0$ such that $u(t_0) = 0$, $u > 0$ and $\tilde{f}(u) > 0$ on (t_0, ∞). We have $(pu')' < 0$ and function pu' is decreasing on (t_0, ∞). Since $p > 0$, $u' \geq 0$ on $[0, \infty)$, there exists

$$\lim_{t\to\infty} p(t)u'(t) \geq 0. \tag{4.5}$$

Assume (3.1), then we have $\lim_{t\to\infty} \frac{1}{p(t)} = 0$ and $\lim_{t\to\infty} p(t) = \infty$. Since pu' is decreasing, we obtain from (4.5)

$$0 \leq \lim_{t\to\infty} p(t)u'(t) < p(t_0)u'(t_0) < \infty,$$

and (4.4) follows.

Assume (3.2) and (4.3). By (4.5), we have

$$\lim_{t\to\infty} p(t)u'(t) = K \geq 0.$$

Let $K > 0$. Then $p(t)u'(t) \geq K$ for $t \geq t_0$ and

$$u'(t) \geq \frac{K}{p(t)}, \qquad t \geq t_0,$$
$$u(t) - u(t_0) \geq K \int_{t_0}^{t} \frac{\mathrm{d}s}{p(s)}, \qquad t \geq t_0.$$

Letting $t \to \infty$, we get, by (3.2) and (4.2), that $L \geq K \cdot \infty$, a contradiction. Therefore, $K = 0$ and, due to (4.3), we have (4.4). □

Remark 4.4. If we add condition (2.12) in Theorem 3.5 or Theorem 3.7, then also a reverse statement is valid: If solution u is oscillatory, then u is damped. Really, if u is oscillatory, then u is not monotonous. Thus, by Lemma 4.1 and Lemma 4.2, u can be neither escape nor homoclinic. Since the classes of solutions from Remark 2.7 are disjoint, solution u has to be damped.

5 Existence of escape and homoclinic solutions

Here, the main results of this paper are derived. The goal is to give sufficient conditions for the existence of escape and homoclinic solutions of problem (1.1), (1.2). First, we analyse problem (1.7), (1.2) and we proceed by generalizing these results to problem (1.1), (1.2), provided that each damped solution is oscillatory.

The following lemma is essential for the existence of escape solutions.

Lemma 5.1. *Let* (1.3)–(1.6), (2.2), (2.12) *and either* (3.1), (3.6), (3.7) *or* (3.2), (4.3) *be satisfied. Further we assume*

$$(pq)' > 0 \quad \text{on } (0,\infty), \tag{5.1}$$

$$\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = 0, \tag{5.2}$$

$$\liminf_{t\to\infty} \frac{p(t)}{q(t)} > 0, \tag{5.3}$$

$$\liminf_{t\to\infty} q(t) > 0. \tag{5.4}$$

Choose $C \in (L_0, \bar{B})$ *and* $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. *Let for each* $n \in \mathbb{N}$, u_n *be a solution of problem* (1.7), (1.2) *with* $u_0 = B_n$ *and let* $(0, b_n)$ *be the maximal interval such that*

$$u_n(t) < L, \qquad u_n'(t) > 0, \qquad t \in (0, b_n). \tag{5.5}$$

Finally, assume that for $n \in \mathbb{N}$ *there exist* $\gamma_n \in (0, b_n)$ *such that*

$$u_n(\gamma_n) = C \quad \text{and} \quad \{\gamma_n\}_{n=1}^{\infty} \quad \text{is unbounded.} \tag{5.6}$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ *contains an escape solution of problem* (1.7), (1.2).

Proof. Since the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, there exists a subsequence going to ∞ as $n \to \infty$. For simplicity, let us denote it by $\{\gamma_n\}_{n=1}^{\infty}$. Then we have

$$\lim_{n\to\infty} \gamma_n = \infty, \qquad \gamma_n < b_n, \qquad n \in \mathbb{N}. \tag{5.7}$$

Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (1.7), (1.2).

Step 1. Choose $n \in \mathbb{N}$. Then we have two possibilities:

1. u_n is a damped solution. Then, if (3.1), (3.6) and (3.7) hold, we get, by Theorem 3.5, that u_n is oscillatory. If (3.2) and (4.3) hold, we can use Theorem 3.7, because (5.4) yields (3.12), and we get again that u_n is oscillatory.
2. u_n is a homoclinic solution, which yields $b_n = \infty$ (*cf.* Lemma 4.2) and we write $u_n(b_n) = \lim_{t\to\infty} u_n(t) = L$. By Lemma 4.3, u_n fulfils (4.4) and hence $u_n'(b_n) = 0$.

Therefore, we have

$$u_n(b_n) \in (0, L], \qquad u_n'(b_n) = 0, \tag{5.8}$$

for both $b_n < \infty$ and $b_n = \infty$. In addition,

$$\exists \bar{\gamma}_n \in [\gamma_n, b_n) : u_n'(\bar{\gamma}_n) = \max\{u_n'(t) : t \in [\gamma_n, b_n)\}. \tag{5.9}$$

Due to (1.7), u_n fulfils

$$\tilde{f}(u_n(t))u_n'(t) = -\frac{p(t)u_n'(t)(p(t)u_n'(t))'}{p(t)q(t)}, \qquad t \in (0, b_n). \tag{5.10}$$

Further, we put

$$E_n(t) = \frac{(p(t)u_n'(t))^2}{2} \frac{1}{p(t)q(t)} + \tilde{F}(u_n(t)), \qquad t \in (0, b_n). \tag{5.11}$$

Then, by (5.10),

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = \frac{(p(t)u_n'(t))\,(p(t)u_n'(t))'}{p(t)q(t)} + \frac{(p(t)u_n'(t))^2}{2}\left(\frac{1}{p(t)q(t)}\right)' + \tilde{f}(u_n(t))u_n'(t)$$
$$= \frac{(p(t)u_n'(t))^2}{2}\left(\frac{1}{p(t)q(t)}\right)' = -\frac{(p(t)u_n'(t))^2}{2}\frac{(p(t)q(t))'}{(p(t)q(t))^2}, \qquad t \in (0, b_n).$$

Having in mind (1.6), (5.1) and (5.5), we have

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = -\frac{u_n'^2(t)}{2q^2(t)}\,(p(t)q(t))' < 0, \qquad t \in (0, b_n). \tag{5.12}$$

Integrating (5.12) over (γ_n, b_n) and using (5.5), (5.9), we obtain

$$E_n(\gamma_n) - E_n(b_n) = \int_{\gamma_n}^{b_n} \frac{u_n'^2(t)\,(p(t)q(t))'}{2q^2(t)}\,\mathrm{d}t \le u'(\overline{\gamma}_n)\int_{\gamma_n}^{b_n} \frac{u_n'(t)\,(p(t)q(t))'}{2q^2(t)}\,\mathrm{d}t$$
$$\le u'(\overline{\gamma}_n)K_n \int_{\gamma_n}^{b_n} u_n'(t)\,\mathrm{d}t,$$

where

$$K_n = \sup\left\{\frac{(p(t)q(t))'}{2q^2(t)} : t \in (\gamma_n, b_n)\right\} \in (0, \infty).$$

Hence, we have

$$E_n(\gamma_n) \le E_n(b_n) + u_n'(\overline{\gamma}_n)K_n(L - C). \tag{5.13}$$

Having in mind (1.5), (1.6), (5.5) and (5.6), we get from (5.11)

$$E_n(\gamma_n) > \tilde{F}(u_n(\gamma_n)) = \tilde{F}(C). \tag{5.14}$$

Since $\tilde{F}$ is increasing on $[0, L]$, (5.8) and (5.11) give for $b_n < \infty$

$$E_n(b_n) \le \tilde{F}(u_n(b_n)) \le \tilde{F}(L). \tag{5.15}$$

Let $b_n = \infty$, which means that u_n is homoclinic and $\lim_{t\to\infty} u_n(t) = L$. Then there exists $t_0 > 0$ such that $u_n(t) > 0$ and $\tilde{f}(u_n(t)) > 0$ for $t \in [t_0, \infty)$. Therefore, pu_n' is decreasing on $[t_0, \infty)$. Due to (1.5) and (5.5), $p > 0, u_n' > 0$ on $(0, \infty)$, and hence

$$0 \le \lim_{t\to\infty} p(t)u_n'(t) < p(t_0)u_n'(t_0) < \infty.$$

Therefore, using (5.4), (5.8), we get

$$0 \le \limsup_{t\to\infty} \frac{p(t)}{q(t)}u_n'(t) < \infty$$

and

$$\lim_{t\to\infty} \frac{p(t)}{q(t)}u_n'^2(t) = 0.$$

Consequently, (5.15) is valid also for $b_n = \infty$. Further, using (5.13)–(5.15), we derive

$$\tilde{F}(C) < E_n(\gamma_n) \le \tilde{F}(L) + u_n'(\bar{\gamma}_n)K_n(L - C), \tag{5.16}$$

and hence,

$$\frac{\tilde{F}(C)-\tilde{F}(L)}{L-C}\frac{1}{K_n} < u_n'(\bar{\gamma}_n). \tag{5.17}$$

Step 2. Now, consider the sequence $\{u_n\}_{n=1}^{\infty}$. Assumptions (5.2) and (5.7) imply

$$\lim_{n\to\infty} K_n = 0, \tag{5.18}$$

which, by (5.17), yields

$$\lim_{n\to\infty} u_n'(\bar{\gamma}_n) = \infty. \tag{5.19}$$

Since $\tilde{F} \geq 0$ on $[l_0, L]$, we get from (5.11)

$$E_n(\bar{\gamma}_n) \geq \frac{p(\bar{\gamma}_n)u_n'^2(\bar{\gamma}_n)}{2q(\bar{\gamma}_n)}, \qquad n \in \mathbb{N}.$$

Further, since E_n is decreasing on $(0, b_n)$ according to (5.12), we derive from (5.16)

$$\frac{p(\overline{\gamma}_n)u_n'^2(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)} \leq E_n(\overline{\gamma}_n) \leq E_n(\gamma_n) \leq \tilde{F}(L) + u_n'(\overline{\gamma}_n)K_n(L-C), \qquad n \in \mathbb{N}.$$

Consequently,

$$u_n'(\overline{\gamma}_n)\left(\frac{p(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)}u_n'(\overline{\gamma}_n) - K_n(L-C)\right) \leq \tilde{F}(L) < \infty, \qquad n \in \mathbb{N}. \tag{5.20}$$

Due to (5.3), (5.18) and (5.19),

$$\lim_{n\to\infty}\left(\frac{p(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)}u_n'(\overline{\gamma}_n) - K_n(L-C)\right) = \infty. \tag{5.21}$$

Conditions (5.19)–(5.21) yield a contradiction. Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (1.7), (1.2). □

Now, we are ready to prove our main results about the existence of escape and homoclinic solutions. All next existence theorems have the following common assumptions

$$\text{(1.3)–(1.6), (2.2), (2.15), (2.19) and (5.1)–(5.4).} \tag{5.22}$$

We provide the existence results for two cases which are characterized by conditions (3.1) and (3.2). Therefore, we will use in addition either assumptions

$$\text{(3.1), (3.6) and (3.7)} \tag{5.23}$$

or assumptions

$$\text{(3.2) and (4.3).} \tag{5.24}$$

Under these assumptions, we prove that problem (1.7), (1.2) has at least one escape solution.

Theorem 5.2 (Existence of an escape solution of problem (1.7), (1.2))**.** *Assume that* (5.22) *and either* (5.23) *or* (5.24) *hold. Then there exists at least one escape solution of problem* (1.7), (1.2).

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 2.8, there exists a unique solution u_n of problem (1.7), (1.2) with $u_0 = B_n$. By Lemma 2.1 b), there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then (5.5) holds.

If u_n is damped, then, by Theorem 3.5 or Theorem 3.7, u_n is oscillatory (*cf.* Step 1. in the proof of Lemma 5.1), and hence there exists $\gamma_n \in (0, b_n)$ such that $u_n(\gamma_n) = C$. If u_n is not damped, then it is either a homoclinic or an escape solution (*cf.* Remark 2.7), and clearly, there exists $\gamma_n \in (0, b_n)$ satisfying $u_n(\gamma_n) = C$.

Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get the sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (1.7), (1.2) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$. Then, by Theorem 2.8, the sequence $\{u_n\}_{n=1}^{\infty}$ converges locally uniformly on $[0, \infty)$ to the constant function $u \equiv L_0$. Therefore, $\lim_{n\to\infty} \gamma_n = \infty$ and (5.6) is valid. Consequently, according to Lemma 5.1, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (1.7), (1.2). □

The following theorem provides the existence of a homoclinic solution of problem (1.7), (1.2). The proof is based on a description of sets of initial values of damped and escape solutions.

Theorem 5.3 (Existence of a homoclinic solution of problem (1.7), (1.2))**.** *Assume that* (5.22) *and either* (5.23) *or* (5.24) *hold. Then there exists a homoclinic solution of problem* (1.7), (1.2)*.*

Proof. Step 1. Let $\mathcal{M}_d \subset (L_0, 0)$ be the set of all $u_0 \in (L_0, 0)$ such that the corresponding solutions of problem (1.7), (1.2) are damped. By Theorem 3.1 and its proof, $\mathcal{M}_d$ is nonempty. Let us choose $u_0 \in \mathcal{M}_d$ and let u be the corresponding solution of problem (1.7), (1.2). Then, according to Theorem 3.5 or Theorem 3.7, u is oscillatory. Therefore, there exist $0 < a_1 < b_1$ such that

$$u(a_1) = A_1 > 0, \qquad u(b_1) = B_1 < 0.$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{2}\min\{A_1, |B_1|\}.$$

Let v be the solution of equation (1.7) satisfying $v(0) = v_0 \in (L_0, 0)$. By Theorem 2.8, there exists $\delta > 0$ such that

$$|v_0 - u_0| < \delta \Rightarrow \|u - v\|_{C^1[0,b_1]} < \varepsilon.$$

Consequently,

$$u(t) - \varepsilon < v(t) < u(t) + \varepsilon \quad \text{for } t \in [0, b_1],$$

and

$$v(a_1) > \frac{A_1}{2} > 0, \qquad v(b_1) < \frac{B_1}{2} < 0.$$

Therefore, if $|v_0 - u_0| < \delta$, then v is not an increasing function, and so v is damped (*cf.* Lemma 4.1, Lemma 4.2 and Remark 2.7) . We have proved, that if $u_0 \in \mathcal{M}_d$, then $(u_0 - \delta, u_0 + \delta) \subset \mathcal{M}_d$, that is $\mathcal{M}_d$ is open in $(L_0, 0)$.

Step 2. Let $\mathcal{M}_e \subset (L_0, 0)$ be the set of all $u_0 \in (L_0, 0)$ such that the corresponding solutions of problem (1.7), (1.2) are escape solutions. By Theorem 5.2, $\mathcal{M}_e$ is nonempty. Let us choose $u_0 \in \mathcal{M}_e$ and let u be the corresponding escape solution of problem (1.7), (1.2). Then u fulfils (4.1). Hence, there exists $c_1 > c$ such that

$$u(c_1) = L_1 > L.$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{2}(L_1 - L).$$

Let v be the solution of equation (1.7) satisfying $v(0) = v_0 \in (L_0, 0)$. By Theorem 2.8, there exists $\delta > 0$ such that

$$|v_0 - u_0| < \delta \Rightarrow \|u - v\|_{C^1[0,c_1]} < \varepsilon.$$

Consequently,

$$u(t) - \varepsilon < v(t) < u(t) + \varepsilon \quad \text{for } t \in [0, c_1]$$

and

$$v(c_1) > \frac{1}{2}(L + L_1) > L.$$

Therefore, due to Remark 2.7, if $|v_0 - u_0| < \delta$, then v is an escape solution. We have proved, that if $u_0 \in \mathcal{M}_e$, then $(u_0 - \delta, u_0 + \delta) \subset \mathcal{M}_e$, that is $\mathcal{M}_e$ is open in $(L_0, 0)$.

Step 3. Let $\mathcal{M}_h \subset (L_0, 0)$ be defined by

$$\mathcal{M}_h = (L_0, 0) \setminus (\mathcal{M}_d \cup \mathcal{M}_e).$$

Since $\mathcal{M}_d \cup \mathcal{M}_e$ is nonempty and open set in $(L_0, 0)$, $\mathcal{M}_h$ has to be nonempty and closed in $(L_0, 0)$. In addition, if we choose $u_0 \in \mathcal{M}_h$, then the corresponding solution of problem (1.7), (1.2) fulfils $u_{\sup} = L$, and, due to Remark 2.7, u is a homoclinic solution of problem (1.7), (1.2). □

Finally, we extend the existence results from Theorem 5.2 and Theorem 5.3 to problem (1.1), (1.2) and reach the main purpose of this paper.

Theorem 5.4 (Existence of an escape solution of problem (1.1), (1.2))**.** *Assume that* (5.22) *and either* (5.23) *or* (5.24) *hold. Then there exist constant $c \in (0, \infty)$ and function u such that u is an escape solution of problem* (1.1), (1.2) *on* $[0, c]$.

Proof. By Theorem 5.2, there exists an escape solution u of problem (1.7), (1.2). By Lemma 4.1, u fulfils (4.1). Due to (1.8) u, is an escape solution of problem (1.1), (1.2) on $[0, c]$. □

Theorem 5.5 (Existence of a homoclinic solution of problem (1.1), (1.2))**.** *Assume that* (5.22) *and either* (5.23) *or* (5.24) *hold. Then there exists a homoclinic solution of problem* (1.1), (1.2).

Proof. By Theorem 5.3, there exists a homoclinic solution u of problem (1.7), (1.2). Due to (1.8), u is a homoclinic solution of problem (1.1), (1.2), as well. □

We conclude with examples where functions p, q and f were chosen in such a way that problem (1.1), (1.2) has at least one homoclinic solution.

Example 5.6. Let p and q be given by (2.24), where α and β fulfil $\alpha \in (1, 2]$, $\alpha - 1 < \beta \leq \alpha$ or $\alpha \in (2, 3]$, $2\alpha - 3 \leq \beta \leq \alpha$. Then p and q satisfy (1.5), (1.6), (2.15), (3.1), (3.7) and (5.1)–(5.4). Note that (5.3) follows from $\beta \leq \alpha$ and that the inequality $2\alpha - 3 \leq \alpha$ yields $\alpha \leq 3$. Let $f \in C(\mathbb{R})$ be given by (3.11) where $a = b = 1$. Then $L_0 = -2$, $L = 1$ and f fulfils (1.3), (1.4), (2.2), (2.19) and (3.6). Therefore, assumptions (5.22) and (5.23) are fulfilled and assertions of Theorems 5.4 and 5.5 are valid.

Example 5.7. Let p and q be given by (2.24), where α and β fulfil $\alpha \in (0, 1]$, $\beta \geq 0$, $\alpha - 1 < \beta \leq \alpha$. Then p and q satisfy (1.5), (1.6), (2.15), (3.2), (4.3) and (5.1)–(5.4). Let $f \in C(\mathbb{R})$ be given by (2.25) with $0 < L < -L_0$, $\gamma \geq 1$, $k > 0$. Then f satisfies (1.3), (1.4), (2.2) and (2.19). Therefore, the assumptions (5.22) and (5.24) are fulfilled and assertions of Theorems 5.4 and 5.5 are valid.

Example 5.8. Consider the initial problem

$$\begin{aligned}(t^2u'(t))' + \sqrt{t^3}u(t)(1-u(t))(u(t)+4) = 0, \qquad t > 0,\\ u(0) = u_0 \in [-4,1], \qquad u'(0) = 0.\end{aligned} \tag{5.25}$$

Here $L_0 = -4$, $L = 1$, $\bar{B} = \sqrt{6} - 3$, $f(x) = x(1-x)(x+4)$, $p(t) = t^2$, $q(t) = \sqrt{t^3}$. According to Example 2.9, we see that (2.6) and all assumptions of Theorems 2.6, 2.8 and 3.1 are satisfied. By Remark 3.2, for each $u_0 \in [\bar{B}, L)$ there exists a unique damped solution u of problem (5.25). Since all assumptions of Theorem 3.5 are valid, we get that if $u_0 \neq 0$, then u is oscillatory. In addition, assumptions (5.22) and (5.23) of Theorems 5.4 and 5.5 are fulfilled. Therefore, there exists at least one homoclinic solution of problem (5.25). Further, having in mind that $\mathcal{M}_e \subset (L_0, 0)$ is open (*cf*. Step 2 in the proof of Theorem 5.3), we get infinitely many escape solutions u of problem (5.25) on $[0, c]$, where c can be different for different solutions. By Theorem 3.1, we have $\mathcal{M}_e \subset (L_0, \bar{B})$.

Example 5.9. Consider problem (1.1), (1.2) with $p(t) = \sqrt[4]{t^7}$, $q(t) = \sqrt[4]{t^5} + \arctan t$, $t \in [0, \infty)$, and

$$f(x) = \begin{cases} x(1-x)(x+3) & \text{for } x > 0,\\ \frac{7}{13}x(1-x)(x+2) & \text{for } x \le 0.\end{cases}$$

Here $L_0 = -2$, $L = 1$, $\bar{B} = -1$ and we can verify that (2.6) and all assumptions of Theorems 2.6, 2.8, 3.1, 3.5 as well as assumptions (5.22) and (5.23) of Theorems 5.4 and 5.5 are satisfied. Therefore, the same conclusions for problem (1.1), (1.2) as in Example 5.8 are valid.

Example 5.10. Consider problem (1.1), (1.2) with

$$p(t) = \sqrt{t}, \qquad q(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \qquad t \in [0, \infty).$$

Then p and q satisfy (1.5), (1.6), (2.15), (3.2), (4.3) and (5.1)–(5.4). Further, let $f \in C(\mathbb{R})$ be such that

$$f(x) = \begin{cases} -(x + 2^\gamma + 2) & \text{for } x \le -2,\\ |x|^\gamma \operatorname{sgn} x & \text{for } x \in (-2, 1),\\ 2 - x & \text{for } x \ge 1,\end{cases} \qquad \gamma \ge 1.$$

Then $L_0 = -2 - 2^\gamma$, $L = 2$ and f fulfils (1.3), (1.4), (2.2) and (2.19). Therefore, assumptions (5.22), (5.24) of Theorems 5.4 and 5.5 are fulfilled and the same conclusions for problem (1.1), (1.2) as in Example 5.8 are valid.

Acknowledgements

The first two authors gratefully acknowledge support received from the grant Mathematical models, IGA_PrF_2015_013 and from the grant No. 14-06958S of the Grant Agency of the Czech Republic.

The third author was supported by the European Development Fund in the IT4Innovations Centre of Excellence project CZ.1.05/1.1.00/02.0070.

References

[1] F. F. Abraham, *Homogeneous nucleation theory*, Academic Press, New York 1974.

[2] R. P. Agarwal, D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Kluwer, Dordrecht 2001. MR1845855

[3] R. P. Agarwal, D. O'Regan, *Singular differential and integral equations with applications*, Kluwer, Dordrecht 2003. MR2011127

[4] R. P. Agarwal, D. O'Regan, A survey of recent results for initial and boundary value problems singular in the dependent variable, in: *Handbook of differential equations, ordinary differential equations*, Vol. 1, A. Cañada, P. Drábek, A. Fonda (eds.), Elsevier, North Holland, Amsterdam, 2004, pp. 1–68. MR2166489

[5] M. Bartušek, M. Cecchi, Z. Došlá, M. Marini, On oscillatory solutions of quasilinear differential equations, *J. Math. Anal. Appl.* **320**(2006), 108–120. MR2230460

[6] H. Berestycki, P. L. Lions, L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbf{R}^N$, *Indiana Univ. Math. J.* **30**(1981), 141–157. MR0600039

[7] V. Bongiorno, L. E. Scriven, H. T. Davis, Molecular theory of fluid interfaces, *J. Colloid Interface Sci.* **57**(1976), 462–475. url

[8] D. Bonheure, J. M. Gomes, L. Sanchez, Positive solutions of a second-order singular ordinary differential equation, *Nonlinear Anal.* **61**(2005), 1383–1399. MR2135816

[9] M. Cecchi, M. Marini, G. Villar, On some classes of continuable solutions of a nonlinear differential equation, *J. Differential Equations* **118**(1995), No. 2, 403–419. MR1330834

[10] F. Dell'Isola, H. Gouin, G. Rotoli, Nucleation of spherical shell-like interfaces by second gradient theory: numerical simulations, *Eur. J. Mech. B/Fluids* **15**(1996), 545–568.

[11] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, *J. Math. Physics* **5**(1965), 1252–1254. MR0174304

[12] P. C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture notes in Biomathematics, Vol. 28, Springer-Verlag, Berlin-New York, 1979. MR0527914

[13] R. A. Fischer, The wave of advance of advantegeous genes, *Annals of Eugenics* **7**(1937), 355-369. url

[14] H. Gouin, G. Rotoli, An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids, *Mech. Research Communic.* **24**(1997), 255–260. url

[15] L. F. Ho, Asymptotic behavior of radial oscillatory solutions of a quasilinear elliptic equation, *Nonlinear Anal* **41**(2000), 573–589. MR1780633

[16] I. T. Kiguradze, *Some singular boundary value problems for ordinary differential equations*, Izdat. Tbilis. Univ., Tbilisi, 1975. MR0499402

[17] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluver Academic, Dordrecht, 1993. MR1220223

[18] I. T. Kiguradze, B. Shekhter, *Singular boundary value problems for second-order ordinary differential equations*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987. MR925830

[19] G. Kitzhofer, O. Koch, P. Lima, E. Weinmüller, Efficient numerical solution of the density profile equation in hydrodynamics, *J. Sci. Comput.* **32**(2007), 411–424. MR2335787

[20] N. B. Konyukhova, P. M. Lima, M. L. Morgado, M. B. Soloviev, Bubbles and droplets in nonlinear physics models: analysis and numerical simulation of singular nonlinear boundary value problems, *Comp. Maths. Math. Phys.* **48**(2008), No. 11, 2018–2058. MR2528876

[21] M. R. S. Kulenović, Ć. Ljubović, All solutions of the equilibrium capillary surface equation are oscillatory, *Appl. Math. Lett.* **13**(2000), 107–110. MR1760271

[22] W. T. Li, Oscillation of certain second-order nonlinear differential equations, *J. Math. Anal. Appl.* **217**(1998), 1–14. MR1492076

[23] P. M. Lima, N. V. Chemetov, N. B. Konyukhova, A. I. Sukov, Analytical-numerical investigation of bubble-type solutions of nonlinear singular problems, *J. Comp. Appl. Math.* **189**(2006), 260–273. MR2202978

[24] A. P. Linde, *Particle physics and inflationary cosmology*, Harwood Academic, Chur, Switzerland, 1990.

[25] K. McLeod, W. C. Troy, F. B. Weissler, Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros, *J. Differential Equations* **83**(1990), 368–378. MR1033193; url

[26] D. O'Regan, *Theory of singular boundary value problems*, World Scientific Publishing Co., Inc., River Edge, NJ, 1994. MR1286741

[27] C. H. Ou, J. S. W. Wong, On existence of oscillatory solutions of second order Emden–Fowler equations, *J. Math. Anal. Appl.* **277**(2003), 670–680. MR1961253

[28] I. Rachůnková, L. Rachůnek, Asymptotic formula for oscillatory solutions of some singular nonlinear differential equation, *Abstr. Appl. Anal.*, **2011**, Art. ID 981401, 9 pp. MR2802847

[29] I. Rachůnková, L. Rachůnek, J. Tomeček, Existence of oscillatory solutions of singular nonlinear differential equations, *Abstr. Appl. Anal.* **2011**, Art. ID 408525, 20 pp. MR2795071

[30] I. Rachůnková, S. Staněk, M. Tvrdý, Singularities and Laplacians in boundary value problems for nonlinear ordinary differential equations, in: *Handbook of Differential Equations, Ordinary Differential Equations*, Vol. 3, A. Cañada, P. Drábek, A. Fonda (eds.), Elsevier, 2006, pp. 607–723. MR2457638

[31] I. Rachůnková, S. Staněk, M. Tvrdý, *Solvability of nonlinear singular problems for ordinary differential equations*, Hindawi Publishing Corporation, New York, USA, 2009. MR2572243

[32] I. Rachůnková, J. Tomeček, Homoclinic solutions of singular nonautonomous second order differential equations, *Bound. Value Probl.* **2009**, Art. ID 959636, 21 pp. MR2552066

[33] I. Rachůnková, J. Tomeček, Bubble-type solutions of nonlinear singular problem, *Math. Comp. Modelling* **51**(2010), 658–669. MR2594716

[34] I. Rachůnková, J. Tomeček, Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics, *Nonliner Anal.* **72**(2010), 2114–2118. MR2577608

[35] I. Rachůnková, J. Tomeček, J. Stryja, Oscillatory solutions of singular equations arising in hydrodynamics, *Adv. Difference Equ.* **2010**, Art. ID 872160, 13 pp. MR2652448

[36] M. Rohleder, On the existence of oscillatory solutions of the second order nonlinear ODE, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **51**(2012), No. 2, 107–127. MR3058877

[37] J. Vampolová, On existence and asymptotic properties of Kneser solutions to singular second order ODE, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **52**(2013), No. 1, 135–152. MR3202755

[38] J. D. van der Waals, R. Kohnstamm, *Lehrbuch der Thermodynamik*, Vol. 1, Leipzig, 1908.

[39] J. S. W. Wong, Second-order nonlinear oscillations: a case history, in: *Differential & difference equations and applications*, Hindawi Publishing Corporation, New York, NY, USA, 2006, pp. 1131–1138. MR2309447

[40] J. S. W. Wong, R. P. Agarwal, Oscillatory behavior of solutions of certain second order nonlinear differential equations, *J. Math. Anal. Appl.* **198**(1996), 337–354. MR1376268