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Aronszajn–Hukuhara type theorem for semilinear differential inclusions with nonlocal conditions

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Abstract. In this note we investigate the topological structure of the mild solution set of nonlocal Cauchy problems governed by semilinear differential inclusions in separable Banach spaces. We show that the mild solution set is a compact absolute retract (and then a continuum and R_δ-set). As a particular case, the topological structure of the periodic mild solution set is deduced. An illustrating example is supplied.

Keywords: nonlocal conditions, semilinear differential inclusions, mild solutions, periodic solutions, R_δ-sets, absolute retract.

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1 Introduction

The problem of studying the topological properties of the solution set (also known as Peano funnel) arose in 1890 when Peano [22] showed that, under the only assumption of continuity, the uniqueness of solutions for the classic Cauchy problem does not hold. Peano himself proved that the fibers of the solution set are connected and compact in $\mathbb{R}$. In 1923 this result was extended to differential equations in $\mathbb{R}^n$ by Kneser and, five years later, Hukuhara proved in [17] that the solution set is a *continuum* (i.e. a nonempty compact and connected set) in the Banach space of continuous functions. Later on, Aronszajn [1] succeeded in defining a new topological concept for more precisely describing the structure of this set. By introducing the notion of R_δ-set (in particular it is an acyclic set, i.e. it has the cohomology of a single point), he obtained a more precise characterization of the solution set. Therefore, without a Lipschitz assumption, the solution set may not consist of a unique element but, from the point of view of algebraic topology, it is equivalent to a point (in the sense that it has the same cohomology). In the literature the results showing that the Peano funnel is a continuum are called "Hukuhara type theorems", while the results establishing the R_δ-property are known as "Aronszajn type theorems".

The first papers devoted to Cauchy problems involving differential inclusions studied the finite dimensional case (see, e.g. [10, 13]); then Tolstonogov [25], Papageorgiou [20] and others

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deal with this research in abstract spaces. Of course, the study in abstract spaces presents several additional difficulties compared to the finite dimensional case.

Our aim is to analyze in a Banach separable space E the topological structure of the solution set for the following Cauchy problem driven by a semilinear differential inclusion under a nonlocal condition

$$\begin{cases} \dot{x} \in A(t)x + F(t,x) \\ x(0) + \theta(x) = x_0. \end{cases} \tag{P}$$

Here $\{A(t)\}_{t\in[0,b]}$ is a family of linear operators in E, $x_0 \in E$, $F\colon [0,b] \times E \to \mathcal{P}(E)$ is a multimap and $\theta\colon C([0,b];E) \to E$ is a given function.

The research on nonlocal Cauchy problems in Banach spaces, which are more general than the initial ones, is only twenty years old and results concerning the existence of mild solutions are mainly presented. Byszewski [4] emphasizes the importance of nonlocal conditions in order to describe physical problems which cannot be studied by means of classical Cauchy problems. Successively, several mathematicians obtained existence theorems for nonlocal problems governed by ordinary differential equations or inclusions either with autonomous and nonautonomous linear part (we refer for instance to the recent papers [3, 6, 7, 9, 15, 19]).

In Section 3, we prove a new "Aronszajn–Hukuhara type theorem" for the nonlocal problem (P) by requiring the nonlinearity to be measurable in the first variable and Lipschitzian in the second one, while on the linear part usual conditions are assumed. We note that under our assumptions the topological properties of the solution set are non trivial, even in the more restrictive case when the differential inclusion does not present a linear part and the nonlocal condition comes down to a classical initial one (see Example 3.5).

Our approach is based on a very interesting result proved by Ricceri [23] for compact-valued multimaps, together with a Saint-Raymond theorem [24] for convex-valued multimaps.

Thanks to our previous result we deduce an Aronszajn–Hukuhara type theorem for periodic problems governed by the same semilinear differential inclusion. This result extends in a broad sense an existence theorem due to Bader (see [2, Theorem 8]).

Finally, in Section 4 we present an example as an application of our Theorem 3.1.

2 Preliminaries

Let Y be a topological space. We will use the following notations:

$$\mathcal{P}(Y) = \{H \subset Y \,:\, H \neq \emptyset\};$$
$$\mathcal{P}_f(Y) = \{H \in \mathcal{P}(Y) \,:\, H \text{ closed}\};$$
$$\mathcal{P}_k(Y) = \{H \in \mathcal{P}(Y) \,:\, H \text{ compact}\};$$

moreover, if Y is a linear topological space, we mean

$$\mathcal{P}_c(Y) = \{H \in \mathcal{P}(Y) \,:\, H \text{ convex}\}; \qquad \mathcal{P}_{fc}(Y) = \mathcal{P}_f(Y) \cap \mathcal{P}_c(Y); \quad \text{etc.}$$

Let X, Y be Hausdorff topological spaces, we introduce the following definitions for multimaps (see e.g. [11, Definition 4.1.3]). A map $F\colon X \to \mathcal{P}(Y)$ is said to be

upper semicontinuous at $x_0 \in X$ if, for every open set $\Omega \subseteq Y$ with $F(x_0) \subseteq \Omega$, there exists a neighborhood V of x_0 such that $F(x) \subseteq \Omega$ for every $x \in V$;

lower semicontinuous at $x_0 \in X$ if, for every open set $\Omega \subseteq Y$ with $F(x_0) \cap \Omega \neq \emptyset$, there exists a neighborhood V of x_0 such that $F(x) \cap \Omega \neq \emptyset$ for every $x \in V$;

continuous at $x_0 \in X$ if F is both lower and upper semicontinuous at $x_0 \in X$.

From now on we consider the real interval $[0, b]$ endowed with the usual Lebesgue measure. A multimap $F\colon [0, b] \times X \to \mathcal{P}(Y)$ satisfies the *Scorza-Dragoni property* [*lower Scorza-Dragoni property*] if

(SD) [(l-SD)] for every $\epsilon > 0$ there exists a compact $K_\epsilon \subset [0, b]$ such that $\mu([0, b] \setminus K_\epsilon) < \epsilon$ and $F_{|K_\epsilon \times X}$ is continuous [lower semicontinuous].

Moreover if X, Y are metric spaces, endowed respectively by the metric d and d', a multimap $F\colon X \to \mathcal{P}_b(Y)$, where $\mathcal{P}_b(Y) = \{H \in \mathcal{P}(Y) : H \text{ bounded}\}$, is said to be a *contraction* if there exists a constant $\alpha \in [0, 1[$ such that

$$H(F(x), F(y)) \leq \alpha d(x, y), \quad \text{for all } x, y \in X,$$

where H is the usual Hausdorff distance, i.e. $H(A, B) = \max\{e(A, B), e(B, A)\}$ for A, B bounded subsets of Y, being $e(A, B)$ the excess of A over B (see [11, Definition 4.1.40]).

In this framework, if F takes compact values, then the above definitions of upper semicontinuity, lower semicontinuity and continuity respectively coincide (see [11, Proposition 4.1.51]) with the definitions of H-upper semicontinuity, H-lower semicontinuity, H-continuity (see [11, Definition 4.1.45]) .

Let $(\Omega, \mathcal{S}_\Omega)$ be a measurable space, i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{S}_\Omega$ and Y be a separable metric space. A map $F\colon \Omega \to \mathcal{P}(Y)$ is said to be *measurable* [*strongly measurable*] if $F^-(A) \in \mathcal{S}_\Omega$, for each open [closed] set $A \subset Y$, where $F^-(A) = \{x \in \Omega : F(x) \cap A \neq \emptyset\}$.

In the sequel, by E we will denote a real Banach space endowed with the norm $\|\cdot\|_E$, by $C([0, b], E)$ the space of E-valued continuous functions on $[0, b]$ with the usual norm $\|\cdot\|_C$ and by $L^1([0, b], E)$ the space of E-valued Bochner integrable functions on $[0, b]$ with norm $\|u\|_1 = \int_0^b \|u(t)\|_E \, dt$; moreover $L^1_+([0, b]) = \{f \in L^1([0, b], \mathbb{R}) : f(t) > 0, \text{ for all } t \in [0, b]\}$.

Given a multimap $G\colon [0, b] \to \mathcal{P}(E)$, we put

$$\mathcal{S}^1_G = \{g \in L^1([0, b]; E) \, : \, g(t) \in G(t), \text{ a.e. } t \in [0, b]\}.$$

Further, we will need the following two important theorems. Let us recall some topological notions involved in Aronszajn type theorems.

A subset A of a metric space X is an *R_δ-set* if it is the intersection of a decreasing sequence of nonempty compact absolute retracts. Recall that a set $D \subset X$ is an absolute retract if, for every metric space Y and closed $C \subset Y$, every continuous $f\colon C \to D$ has a continuous extension $\hat{f}\colon Y \to D$ (see [12, Definition 2.3.15]).

Remark 2.1. *The following statements hold for a nonempty subset A of a metric space X (cf. [12, Remark 2.3.16]):*

if A is an R_δ-set, then A is a continuum*;*

if A is a closed convex subset of a normed space, then A is an absolute retract.

Theorem 2.2 (cf. [24])**.** *Let X be a complete metric space and $\Phi\colon X \to \mathcal{P}_k(X)$ be a contraction. Then* Fix(Φ)*, i.e. the set of all fixed points of Φ, is nonempty and compact in X.*

Theorem 2.3 (cf. [23])**.** *Let H be a closed convex subset of a Banach space E and $\Phi\colon H \to \mathcal{P}_{fc}(H)$ be a contraction. Then* $\mathrm{Fix}(\Phi)$ *is a nonempty absolute retract.*

Finally, we relate the following version of the strongly measurable selection theorem (see [5, Lemma 3.1]).

Theorem 2.4. *Let E be a separable Banach space, $x \in C([0,b];E)$ and $F\colon [0,b]\times E \to \mathcal{P}_k(E)$ be a map satisfying the lower Scorza-Dragoni property. Then the multimap $F(\cdot, x(\cdot))$ has a strongly measurable selection.*

3 Main results

In this section we investigate the very general problem (P). We study the existence of mild solutions as well as the topological structure of the mild solution set.

On the family of linear operators $\{A(t)\}_{t\in[0,b]}$ we assume the next hypothesis:

(HA) $A(t) : \mathcal{D}(A) \subseteq E \to E$, with $D(A)$ not depending on $t \in [0,b]$ and dense in E, and $\{A(t)\}_{t\in[0,b]}$ generates an *evolution system* $\{T(t,s)\}_{(t,s)\in\Delta}$, $\Delta = \{(t,s) \in [0,b]\times[0,b] : 0 \le s \le t \le b\}$,

Recall that (see, e.g. [21]) a two parameter family $\{T(t,s)\}_{(t,s)\in\Delta}$ is called an *evolution system* if $T(t,s)\colon E \to E$, for every $(t,s) \in \Delta$, is a bounded linear operator and the following conditions are satisfied.

(i) $T(s,s) = I, \quad s \in [0,b]; \quad T(t,r)T(r,s) = T(t,s)$ for $0 \le s \le r \le t \le b$;

(ii) $(t,s) \mapsto T(t,s)$ is strongly continuous on Δ.

In the following by $\mathcal{L}(E)$ we denote the space of all bounded linear operators from E in E endowed with the usual norm $\|\cdot\|_{\mathcal{L}(E)}$. It is easy to see that by (ii) there exists a positive constant M such that

$$\|T(t,s)\|_{\mathcal{L}(E)} \le M, \qquad (t,s) \in \Delta. \tag{3.1}$$

On the map $F\colon [0,b]\times E \to \mathcal{P}_{kc}(E)$ we assume the following hypotheses:

(F1) for every $x \in E$, the map $F(\cdot, x)$ is measurable;

(F2) there exists $\alpha \in L^1_+([0,b])$ with $\int_0^b \alpha(s)\,ds < \frac{1}{2M}$ such that

$$H(F(t,x),F(t,y)) \le \alpha(t)\|x-y\|_E, \quad \text{for a.a. } t \in [0,b] \text{ and } x,y \in E;$$

(F3) there exist $m \in L^1_+([0,b])$ and a nondecreasing function $\rho\colon \mathbb{R}^+_0 \to \mathbb{R}^+$ such that

$$\|F(t,x)\| \le m(t)\rho(\|x\|_E), \quad \text{for a.e. } t \in [0,b] \text{ and all } x \in E,$$

where $\|F(t,x)\| = \sup_{z\in F(t,x)} \|z\|_E$.

On the operator $\theta\colon C([0,b],E) \to E$ we assume that

(Hθ) there exists $\gamma > 2$ such that

$$\|\theta(x) - \theta(y)\|_E \le \frac{1}{\gamma M}\|x-y\|_C, \quad \text{for all } x,y \in C([0,b],E).$$

The constant M in (F2) and (Hθ) has been introduced in (3.1).

Recall that a function $x \in C([0,b];E)$ is said to be a *mild solution* for (P) if

$$x(t)=T(t,0)\,(x_0-\theta(x))+\int_0^t T(t,s)f(s)\,ds, \quad \text{for all } t\in[0,b]$$

where $f \in S^1_{F(\cdot,x(\cdot))} = \{g \in L^1([0,b],E)\,:\, g(t)\in F(t,x(t)) \text{ for a.e. } t\in[0,b]\}$.

Now, we state and prove our main result.

Theorem 3.1. *Let E be a separable Banach space and assume that $\{A(t)\}_{t\in[0,b]}$, $F\colon [0,b]\times E \to \mathcal{P}_{kc}(E)$ and $\theta\colon C([0,b],E)\to E$ satisfy respectively (HA), (F1)–(F3) and (Hθ).*

Then the set of all mild solutions of problem (P) is a nonempty compact absolute retract in $C([0,b],E)$.

Proof. First of all, we consider the operator $\Gamma : C([0,b],E) \to \mathcal{P}(C([0,b],E))$ defined, for every $x\in C([0,b],E)$, as

$$\Gamma(x)=\Big\{h\in C([0,b],E)\,:\, h(t)=T(t,0)(x_0-\theta(x))+\int_0^t T(t,s)g(s)\,ds, \quad \text{for all } t\in[0,b],\ \text{ where } g\in S^1_{F(\cdot,x(\cdot))}\Big\}.$$

Let us show that the multioperator Γ fulfills all the hypotheses of Theorems 2.2 and 2.3. We proceed by steps.

Step 1. We prove that $\Gamma(x)\neq\emptyset$, for all $x\in C([0,b],E)$.

Fix $x\in C([0,b],E)$. By (F2), since F takes compact values, we can say that $F(t,\cdot)$ is continuous in E, for every $t\in[0,b]$. Further, (F1) is satisfied and the Banach space E is separable, so from [11, Proposition 4.4.29] we deduce that F has the Scorza-Dragoni property. Now Theorem 2.4 implies that the multimap $F(\cdot,x(\cdot))$ has a strongly measurable selection g such that $g(t)\in F(t,x(t))$ for all $t\in[0,b]$.

From (F3) we have

$$\|g(t)\|_E \le m(t)\rho(\|x\|_C), \qquad \text{for a.e. } t\in[0,b]$$

and so $g\in L^1([0,b],E)$. Clearly, the map $h\colon[0,b]\to E$ defined by

$$h(t)=T(t,0)(x_0-\theta(x))+\int_0^t T(t,s)g(s)\,ds, \quad \text{for all } t\in[0,b]$$

belongs to $\Gamma(x)$ and then $\Gamma(x)$ is nonempty.

Step 2. Γ has convex and compact values.

Fixed $x\in C([0,b],E)$, the convexity of $\Gamma(x)$ immediately follows from the convexity of the values of F and from the linearity of the operator $T(t,s)\colon E\to E$, for every $(t,s)\in\Delta$.

We show that the set $\Gamma(x)$ is compact.

First of all, we prove that $\Gamma(x)$ is relatively compact.

Let $(z_n)_n$ be a sequence such that $z_n\in\Gamma(x)$, $n\in\mathbb{N}$, and $(f_n)_n$ be a sequence such that, for every $n\in\mathbb{N}$, $f_n\in S^1_{F(\cdot,x(\cdot))}$ and

$$z_n(t)=T(t,0)(x_0-\theta(x))+\int_0^t T(t,s)f_n(s)\,ds, \quad \text{for all } t\in[0,b].$$

Let us note that, by (F3), the set $\{f_n\}_n$ is integrably bounded, i.e.

$$\|f_n(s)\|_E \le \|F(s,x(s))\| \le m(s)\rho(\|x\|_C) := \tilde{m}(s), \quad \text{for a.e. } s\in[0,b],$$

where $\tilde{m} \in L^1_+([0,b])$.

On the other hand, for all $t \in [0,b]$, the set $\{f_n(t)\}_n$ is relatively compact in E being a subset of the compact $F(t,x(t))$.

Therefore the set $\{f_n\}_n$ satisfies the hypotheses of [18, Proposition 4.2.1], so that it is weakly compact in $L^1([0,b],E)$; hence w.l.o.g. we can assume $f_n \rightharpoonup \bar{f}$ in $L^1([0,b],E)$. Now, by using [8, Theorem 2] we can apply [18, Theorem 5.1.1], so that

$$\int_0^{\cdot} T(\cdot,s) f_n(s)\, ds \to \int_0^{\cdot} T(\cdot,s)\bar{f}(s)\, ds \quad \text{in } C([0,b],E).$$

Hence the sequence $(z_n)_n$ converges in $C([0,b],E)$ and, therefore, $\Gamma(x)$ is relatively compact in $C([0,b],E)$.

Now, we have to prove that $\Gamma(x)$ is closed.

Let $(y_n)_n$ be a sequence in $\Gamma(x)$ such that $y_n \to \bar{y}$ in $C([0,b],E)$. Let $(f_n)_n$ be a sequence in $S^1_{F(\cdot,x(\cdot))}$ such that

$$y_n(t) = T(t,0)(x_0 - \theta(x)) + \int_0^t T(t,s) f_n(s)\, ds, \quad \text{for all } t \in [0,b].$$

By means of the same arguments as above, we can claim that passing to a subsequence, if necessary, one has

$$y_n \to T(\cdot,0)(x_0 - \theta(x)) + \int_0^{\cdot} T(\cdot,s)\bar{f}(s)\, ds \quad \text{in } C([0,b],E).$$

The uniqueness of the limit algorithm guarantees that

$$\bar{y}(t) = T(t,0)(x_0 - \theta(x)) + \int_0^t T(t,s)\bar{f}(s)\, ds, \quad \text{for every } t \in [0,b].$$

By [18, Lemma 5.1.1], it is $\bar{f} \in S^1_{F(\cdot,x(\cdot))}$; hence, we can conclude that $\bar{y} \in \Gamma(x)$.

So $\Gamma(x)$ is closed and, hence, compact.

Step 3. Γ is a contraction.

Let us fix $x, y \in C([0,b],E)$ and $h \in \Gamma(x)$. We have

$$h(t) = T(t,0)(x_0 - \theta(x)) + \int_0^t T(t,s) g(s)\, ds, \quad \text{for all } t \in [0,b],$$

where $g \in S^1_{F(\cdot,x(\cdot))}$.

Now, by noting that the multimap $F(\cdot,y(\cdot))$ and the function g satisfy the hypotheses of [26, Lemma 3.9], we have that there exists a measurable selection $w\colon [0,b] \to E$ of the multimap $F(\cdot,y(\cdot))$, such that

$$\|g(t) - w(t)\|_E = \delta(g(t), F(t,y(t))), \quad \text{for all } t \in [0,b], \tag{3.2}$$

where $\delta(g(t),F(t,y(t))) = \inf_{z\in F(t,y(t))} \|g(t) - z\|_E$.

By (F3), the map w is Bochner integrable. We associate to this map the function $p\colon [0,b] \to E$ defined as

$$p(t) = T(t,0)(x_0 - \theta(y)) + \int_0^t T(t,s) w(s)\, ds, \quad \text{for all } t \in [0,b].$$

Clearly $p \in \Gamma(y)$.

We are now in the position to estimate $e(\Gamma(x), \Gamma(y))$, i.e. the excess of $\Gamma(x)$ over $\Gamma(y)$. Indeed, by using (Hθ), (3.2) and (F2), we get the following inequality:

$$\begin{aligned}\|h(t)-p(t)\|_E &\le \|T(t,0)(\theta(y)-\theta(x))\|_E + \int_0^t \|T(t,s)(g(s)-w(s))\|_E\,ds\\ &\le M\|\theta(y)-\theta(x)\|_E + M\int_0^b \|g(s)-w(s)\|_E\,ds\\ &\le \frac{1}{\gamma}\|x-y\|_C + M\int_0^b \delta(g(s),F(s,y(s)))\,ds\\ &\le \frac{1}{\gamma}\|x-y\|_C + M\int_0^b H(F(s,x(s)),F(s,y(s)))\,ds\\ &\le \frac{1}{\gamma}\|x-y\|_C + M\int_0^b \alpha(s)\|x-y\|_C\,ds\\ &\le \frac{1}{\gamma}\|x-y\|_C + \frac{1}{2}\|x-y\|_C \;=\; L\|x-y\|_C,\end{aligned}$$

for all $t \in [0,b]$, where $L = \frac{1}{\gamma} + \frac{1}{2}$. Hence

$$\delta(h,\Gamma(y)) = \inf_{z\in\Gamma(y)} \|h-z\|_C \le \|h-p\|_C \le L\|x-y\|_C.$$

We deduce that

$$e(\Gamma(x),\Gamma(y)) = \sup_{h\in\Gamma(x)} \delta(h,\Gamma(y)) \le L\|x-y\|_C. \tag{3.3}$$

Of course, analogously it is

$$e(\Gamma(y),\Gamma(x)) \le L\|x-y\|_C. \tag{3.4}$$

From (3.3) and (3.4), we get $H(\Gamma(x),\Gamma(y)) \le L\|x-y\|_C$; being $L < 1$ (see (Hθ)), we can say that Γ is a contraction.

Step 4. We can now apply Theorems 2.2 and 2.3, so that the set $\mathrm{Fix}(\Gamma)$ of all mild solutions of problem (P) is a nonempty compact absolute retract in $C([0,b],E)$. □

Remark 3.2. Obviously the set of all mild solutions of problem (P) is an R_δ-set and, according to Remark 2.1, a continuum too. For this reason we say that our result is an Aronszajn–Hukuhara type theorem.

From Theorem 3.1 we deduce the topological properties of the classical solution set for a nonlinear differential inclusion with nonlocal condition.

Corollary 3.3. *Let E be a separable Banach space and $x_0 \in E$. Let $F\colon [0,b]\times E \to \mathcal{P}_{kc}(E)$ and $\theta\colon C([0,b],E) \to E$ satisfy respectively (F1)–(F3) and (Hθ), by assuming $M = 1$ both in (F2) and in (Hθ). Then the solution set of problem*

$$\begin{cases} \dot{x} \in F(t,x)\\ x(0)+\theta(x) = x_0\end{cases} \tag{NP}$$

is a nonempty compact absolute retract in $C([0,b],E)$.

Proof. It is enough to observe that (NP) is a particular case of (P), just by taking $A(t) = 0$ for every $t \in [0,b]$, where 0 is the null-operator in $\mathcal{L}(E)$. So $T(t,s) = I$ for every $(t,s) \in \Delta$ and $M = 1$ (see (3.1)).

Therefore, Theorem 3.1 immediately implies that the set of solutions (which in this setting are absolutely continuous)

$$\left\{x \in C([0,b],E) : x(t) = x_0 - \theta(x) + \int_0^t g(s)\,ds, \text{ for all } t \in [0,b], \text{ where } g \in S^1_{F(\cdot,x(\cdot))}\right\}$$

is a nonempty compact absolute retract in $C([0,b],E)$. □

Remark 3.4. We wish to note that under our hypotheses the topological structure of the solution set can be not trivial, since in general the uniqueness of the solution for a Cauchy problem governed by a differential inclusion is not guaranteed, even in very classical settings. We show it by means of the following example.

Example 3.5. Consider the Cauchy problem

$$\begin{cases} \dot{x} \in F(t,x), \\ x(0) = 1, \end{cases}$$

where $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}_{kc}(\mathbb{R})$ is defined by

$$F(t,x) = \frac{1}{4}[1,x] := \left\{y \in \mathbb{R} : y = \frac{\lambda + (1-\lambda)x}{4}, \lambda \in [0,1]\right\}.$$

Here $A(t) \equiv 0$ for all $t \in [0,1]$ and $\theta \equiv 0$.

It is easy to see that the problem has infinitely many solutions.

On the other hand, the multimap F takes compact convex values and satisfies hypotheses (F1). Further, put $\alpha(t) = \frac{1}{4}$ for all $t \in [0,1]$, we have that (F2) holds (with $M = 1$). Also, if we consider $\rho(s) = \frac{1+s}{4}$ for all $s \in [0,+\infty)$ and $m(t) = 1$ for all $t \in [0,1]$, we can say that F satisfies (F3). Therefore we can use Corollary 3.3 and obtain that the solution set of the given problem is a nonempty compact absolute retract in $C([0,1],\mathbb{R})$.

3.1 Semilinear differential inclusions under periodic conditions

Very important nonlocal Cauchy problems are the periodic ones, namely

$$\begin{cases} \dot{x} \in A(t)x + F(t,x), \\ x(0) = x(b). \end{cases} \tag{PP}$$

In this setting, each mild solution is a function $x \in C([0,b],E)$ such that

$$x(t) = T(t,0)x(b) + \int_0^t T(t,s)f(s)\,ds, \quad \text{for all } t \in [0,b],$$

where $f \in S^1_{F(\cdot,x(\cdot))}$.

In order to let our Theorem 3.1 be easily usable in the periodic setting, we deduce the following result.

Corollary 3.6. *Let E be a separable Banach space and $\{A(t)\}_{t\in[0,b]}$ be a family of linear operators satisfying* (HA) *and $M < \frac{1}{2}$, where M is defined in* (3.1)*. Assume that $F\colon [0,b] \times E \to \mathcal{P}_{kc}(E)$ satisfies properties* (F1)–(F3).

Then the mild solution set of problem (PP) *is a nonempty compact absolute retract in $C([0,b],E)$.*

Proof. Of course, problem (PP) is a particular case of problem (P) by taking $x_0 = 0$ and $\theta\colon C([0,b],E) \to E$ as $\theta(x) = -x(b)$, for all $x \in C([0,b],E)$. This map is trivially Lipschitzian with $L = 1$; hence, put $L = \frac{1}{M\gamma}$, we have $\gamma = \frac{1}{M} > 2$, as required in (Hθ). From Theorem 3.1 it immediately follows that the mild solution set for the periodic problem (PP) is a nonempty compact absolute retract in $C([0,b],E)$. □

4 An example

We investigate the following nonlocal Cauchy problem driven by a partial differential inclusion

$$\begin{cases} \frac{\partial}{\partial t} y(t,z) \in \frac{\partial^2}{\partial z^2} y(t,z) + \frac{1}{4}[1, y(t,z)], & (t,z) \in [0,1]\times[0,1] \\ y(t,0) = y(t,1), & t \in [0,1] \\ y(0,z) = \sum_{j=1}^{q} k_j y(s_j,z), & z \in [0,1],\ s_j \in [0,1],\ k_j \in \mathbb{R},\ j = 1,\dots,q. \end{cases} \tag{4.1}$$

Put $E = L^2([0,1],\mathbb{R})$, we consider the family of operators $\{A(t)\}_{t\in[0,1]} = \{A\}$, being $A\colon \mathcal{D}(A) \subset E \to E$ is the Laplace operator

$$A(\omega) = \frac{\partial^2}{\partial z^2}\omega, \quad \text{for all } \omega \in \mathcal{D}(A),$$

where $\mathcal{D}(A) = \{\omega \in E : \omega, \omega' \text{ are absolutely continuous}, \omega'' \in E, \omega(0) = \omega(1)\}$. Further, we can consider the evolution operator T defined in $\Delta = \{(t,s) : 0 \le s \le t \le 1\}$ by $\{T(t,s)\}_{(t,s)\in\Delta} = \{U(t-s)\}_{(t,s)\in\Delta}$, being $\{U(t)\}_{t\in[0,+\infty[}$ the C_0-semigroup generated by A.

We assume that $\sum_{j=1}^{q} |k_j| = \frac{1}{4M}$, where $M > 0$ is the constant defined in (3.1). Moreover, we put

$$F(t,x) = \frac{1}{4M}[1,x] := \left\{ y \in E : y = \frac{\lambda + (1-\lambda)x}{4M},\ \lambda \in [0,1] \right\}$$

for all $(t,x) \in [0,1] \times E$, and

$$\theta(x) = -\sum_{j=1}^{q} k_j x(s_j), \quad \text{for all } x \in C([0,1],E).$$

Now, problem (4.1) can be rewritten as a nonlocal Cauchy problem where the differential inclusion presents autonomous linear term, that is

$$\begin{cases} \dot{x} \in Ax + F(t,x), \\ x(0) + \theta(x) = 0. \end{cases} \tag{4.2}$$

Note that the operator θ satisfies condition $(H\theta)$, since

$$\|\theta(x) - \theta(y)\|_E \le \sum_{j=1}^{q} |k_j| \|x(s_j) - y(s_j)\|_E \le \frac{1}{4M}\|x - y\|_C,$$

for all $x, y \in C([0,1],E)$, being $\gamma = 4 > 2$. Moreover, it is easy to see that (F1) is satisfied. Further, put $\alpha(t) = \frac{1}{4M}$ for all $t \in [0,1]$, we have $\alpha \in L^1_+([0,1])$, $\int_0^1 \alpha(t)\,dt = \frac{1}{4M} < \frac{1}{2M}$ and the following inequality holds

$$H(F(t,x), F(t,y)) \le \frac{1}{4M}\|x - y\|_E, \quad \text{for all } x, y \in E,$$

so (F2) is satisfied. We also have

$$\|F(t,x)\| = \sup_{h\in F(t,x)} \|h\|_E = \frac{1}{4M} \sup_{\lambda\in[0,1]} \|\lambda + (1-\lambda)x\|_E$$
$$\leq \frac{1+\|x\|_E}{4M} \leq m(t)\rho(\|x\|_E), \quad \text{for all } t \in [0,1] \text{ and } x \in E,$$

where we have chosen the nondecreasing function $\rho\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ as

$$\rho(s) = \frac{1+s}{4M}, \text{ for all } s \in \mathbb{R}_0^+$$

and the Lebesgue integrable function $m(t) = 1$ for all $t \in [0,1]$. Therefore, (F3) is satisfied too.

Then, by applying our Theorem 3.1, we claim the existence of a least one mild solution for (4.2) and, as a consequence, for (4.1); moreover, we have characterized the topological structure of the solution set.

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