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Infinitely many solutions for elliptic problems in $\mathbb{R}^N$ involving the $p(x)$-Laplacian

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Abstract. We consider the $p(x)$-Laplacian equations in $\mathbb{R}^N$. The potential function does not satisfy the coercive condition. We obtain the existence of infinitely many solutions of the equations, improving a recent result of Duan–Huang [L. Duan, L. H. Huang, *Electron. J. Qual. Theory Differ. Equ.* 2014, No. 28, 1–13].

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1 Introduction and main results

Let us consider the following nonlinear elliptic problem:

$$-\triangle_{p(x)}u + V(x)|u|^{p(x)-2}u = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and $1 < p^- := \inf_{\mathbb{R}^N} p(x) \le \sup_{\mathbb{R}^N} p(x) := p^+ < N$, V is the new potential function, and the nonlinear term f is sublinear with some precise assumptions that we state below.

We emphasize that the operator $-\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous and in general, it does not have the first eigenvalue. The study of various mathematical problems with variable exponent growth condition has received considerable attention in recent years. These problems appear in a lot of applications, such as image processing models (see e.g. [6,21]), stationary thermorheological viscous flows (see [2]) and the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium (see [3]). We refer to [4,14–18,23–25] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

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It is well known, the main difficulty in treating problem (P) in $\mathbb{R}^N$ arises from the lack of compactness of the Sobolev embeddings, which prevents from checking directly that the energy functional associated with (P) satisfies the C-condition. To overcome the difficulty of the noncompact embedding, Dai in [7,8] proves that the subspace of radially symmetric functions of $W^{1,p(x)}(\mathbb{R}^N)$, denoted further by $W_r^{1,p(x)}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $2 \le N < p^- \le p^+ < +\infty$. Later, Alves–Liu [1] (who use the conditions (V_1) and (V_2)), Ge–Zhou–Xue [19, 20] (who assume that conditions (V_1) and (V_3) hold) also establish new compact embedding theorems for the subspace of $W^{1,p(x)}(\mathbb{R}^N)$ when $1 < p^- \le p^+ < N$. Furthermore, the authors make the following assumptions on the potential function V.

(V_1) $V \in C(\mathbb{R}^N)$ and $\inf V > 0$.

(V_2) For any $M > 0$, $\{x \in \mathbb{R}^N : V(x) \le M\}$ is a bounded set.

(V_3) There exists $r > 0$ such that for any $b > 0$

$$\lim_{|y|\to\infty} \mu\big(\{x \in \mathbb{R}^N : V(x) \le b\} \cap B_r(y)\big) = 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^N$.

We emphasize that in our approach, no coerciveness hypothesis (V_2) and not necessarily radial symmetry will be required on the potential V. To the best of our knowledge, there are only a few works concerning on this case up to now. Inspired by the above facts and the aforementioned papers, the main purpose of this paper is to study the existence of infinitely many solutions for problem (P) when $F(x,u)$ is sublinear in u at infinity. Our tool used here is a variational method combined with the theory of variable exponent Sobolev spaces.

We are now in the position to state our main results.

Theorem 1.1. *Suppose that (V_1) and the following condition $H(f)$ holds,*

$H(f)$ *$f(x,u) = \frac{b(x)}{q(x)}|u|^{q(x)-2}u$, $b\colon \mathbb{R}^N \to \mathbb{R}^+$ is a positive continuous function such that $b \in L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)$ and $1 < q^- \le q^+ < p^-$, where $p(x) \le s(x) \ll p^*(x)$, $p^*(x) = \frac{Np(x)}{N-p(x)}$, and $s(x) \ll p^*(x)$ means that $_{x\in\mathbb{R}^N}(p^*(x) - s(x)) > 0$.*

Then problem (P) possesses infinitely many solutions $\{u_k\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{1}{p(x)}\Big(|\nabla u_k|^{p(x)} + V(x)|u_k|^{p(x)}\Big)dx - \int_{\mathbb{R}^N} F(x,u_k)dx \to 0^-, \quad \text{as } k \to \infty,$$

where $F(x,u_k) = \int_0^{u_k} f(x,t)dt$.

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces and the nonsmooth critical point theory of the locally Lipschitz functionals. In Section 3, the proof of the main results is given.

2 Preliminaries

In order to discuss problem (P), we need some facts on space $W^{1,p(x)}(\mathbb{R}^N)$ which are called variable exponent Sobolev spaces. For a deeper treatment on these spaces, we refer to [10–12, 22]. Write

$$C_+(\mathbb{R}^N) = \Big\{p \in C(\mathbb{R}^N) : p(x) > 1 \text{ for any } x \in \mathbb{R}^N\Big\},$$

$$p^- = \inf_{x\in\mathbb{R}^N} p(x), \quad p^+ = \sup_{x\in\mathbb{R}^N} p(x) \text{ for any } p \in C_+(\mathbb{R}^N).$$

Denote by $S(\mathbb{R}^N)$ the set of all measurable real-valued functions defined on $\mathbb{R}^N$. Note that two measurable functions in $S(\mathbb{R}^N)$ are considered as the same element of $\mathcal{S}(\mathbb{R}^N)$ when they are equal almost everywhere.

Let $p \in C_+(\mathbb{R}^N)$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u \in S(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{p(x)} dx < +\infty \right\},$$

endowed with the norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Then we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{ u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\}$$

with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

With these norms, the spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable reflexive Banach spaces; see [12] for the details.

Proposition 2.1 ([13])**.** *Set $\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)} dx$. For $u \in L^{p(x)}(\mathbb{R}^N)$, we have*

(i) for $u \neq 0, | u |_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(ii) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(iii) if $|u|_{p(x)} \ge 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;

(iv) if $|u|_{p(x)} \le 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$.

Next, we consider the case that V satisfies (V_1). On the linear subspace

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx < +\infty \right\},$$

we equip with the norm

$$\|u\|_E = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left(\left| \frac{\nabla u}{\lambda} \right|^{p(x)} + V(x) \left| \frac{u}{\lambda} \right|^{p(x)} \right) dx \le 1 \right\}.$$

Then $(E, \|\cdot\|_E)$ is continuously embedded into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|_E)$ is also a separable reflexive Banach space. It is easy to see that with the norm $\|\cdot\|_E$, Proposition 2.1 remains valid, that is,

Proposition 2.2. *Set $I(u) = \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)}) dx$. If $u \in W^{1,p(x)}(\mathbb{R}^N)$, then*

(i) for $u \neq 0, \|u\|_E = \lambda \Leftrightarrow I(\frac{u}{\lambda}) = 1$;

(ii) $\|u\|_E < 1(= 1; > 1) \Leftrightarrow I(u) < 1(= 1; > 1)$;

(iii) *if* $\|u\|_E \geq 1$, *then* $\|u\|_E^{p^-} \leq I(u) \leq \|u\|_E^{p^+}$;

(iv) *if* $\|u\|_E \leq 1$, *then* $\|u\|_E^{p^+} \leq I(u) \leq \|u\|_E^{p^-}$.

Proposition 2.3 ([13]). *The conjugate space of* $L^{p(x)}(\mathbb{R}^N)$ *is* $L^{q(x)}(\mathbb{R}^N)$, *where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. *For any* $u \in L^{p(x)}(\mathbb{R}^N)$ *and* $v \in L^{q(x)}(\mathbb{R}^N)$, *we have*

$$\left| \int_{\mathbb{R}^N} uv dx \right| \leq 2|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.4 ([11]). *Let* $p : \mathbb{R}^N \to \mathbb{R}$ *be Lipschitz continuous and satisfy* $1 < p^- \leq p^+ < N$, *and* $q : \mathbb{R}^N \to \mathbb{R}$ *be a measurable function.*

(i) If $p \leq q \leq p^*$, *then there is a continuous embedding* $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.

(ii) If $p \leq q \ll p^*$, *then there is a compact embedding* $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{loc}(\mathbb{R}^N)$.

Proposition 2.5 ([10]). *Let* $p(x)$ *and* $q(x)$ *be measurable functions such that* $p(x) \in L^\infty(\mathbb{R}^N)$ *and* $1 \leq p(x)q(x) \leq \infty$ *almost everywhere in* $\mathbb{R}^N$. *Let* $u \in L^{q(x)}(\mathbb{R}^N)$, $u \neq 0$. *Then*

$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ *is a constant, then* $\left| |u|^p \right|_{q(x)} = |u|^p_{pq(x)}$.

Set

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |u|^{p(x)} dx.$$

We know that (see [5]), $I \in C^1(E, \mathbb{R})$ and $p(x)$-Laplacian operator $-\Delta_{p(x)} u$ is the derivative operator of J in the weak sense. We denote $\mathcal{L}= I' : E \to E^*$, then

$$\langle \mathcal{L}u, v \rangle = \int_{\mathbb{R}^N} (|\nabla u(x)|^{p(x)-2} \nabla u \cdot \nabla v + V(x)|u|^{p(x)-2} uv) dx, \quad \forall u, v \in E.$$

Proposition 2.6 ([13]). *Set* E *and* $\mathcal{L}$ *as above, then*

(i) $\mathcal{L} : E \to E^*$ *is a continuous, bounded and strictly monotone operator;*

(ii) $\mathcal{L}$ *is a mapping of type* (S_+), *if* $u_n \rightharpoonup u$ *(weak) in* E *and* $\limsup_{n\to\infty} (\mathcal{L}(u_n) - \mathcal{L}(u), u_n - u) \leq 0$, *then* $u_n \to u$ *in* E;

(iii) $\mathcal{L} : E \to E^*$ *is a homeomorphism.*

In order to assure the existence of infinitely many solutions for the problem (P), our main tool will be the variant fountain theorem [26, Theorem 2.2], which will be used in our proof.

Let X be a Banach space with the norm $\| \cdot \|$ and $X = \overline{\bigoplus_{i\in\mathbb{N}}^\infty X_i}$ with $\dim X_i < \infty$ for any $i \in \mathbb{N}$. Set

$$Y_k = \bigoplus_{i=0}^{k} X_i, \qquad Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}. \tag{2.1}$$

Consider the following C^1-functional $\varphi_\lambda : E \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in [1, 2],$$

where $A, B : X \to \mathbb{R}$ are two functionals.

Lemma 2.7. *Suppose that the functional φ_λ defined above, and satisfies the following conditions.*

(1) φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$.

(2) $B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X.

(3) There exist $\rho_k > r_k > 0$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u)$$

for all $\lambda \in [1,2]$, $d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \varphi_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$.

Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that

$$\varphi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \qquad \varphi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty.$$

Particularly, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k, then φ_1 has infinitely many nontrivial critical points $\{u_k\} \in X \backslash \{0\}$ satisfying $\varphi_1(u_k) \to 0^-$ as $k \to \infty$.

In order to discuss the problem (P), we need to consider the energy functional $\varphi : E \to \mathbb{R}$ defined by

$$\varphi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left[|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right] dx - \int_{\mathbb{R}^N} F(x,u) dx.$$

Under our hypotheses, it follows from a Hölder-type inequality and Sobolev's embedding theorem that the energy functional φ is well-defined. It is well known that $\varphi \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle \varphi'(u), v \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + V(x)|u|^{p(x)-2} uv \right) dx - \int_{\mathbb{R}^N} f(x,u) v dx$$

for $v \in E$. It is standard to verify that the weak solutions of problem (P) correspond to the critical points of the functional φ.

3 Proof of the main results

In this section, for the notation in Lemma 2.7, the space $X = E$, and related functionals on E are

$$A(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left[|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right] dx, \qquad B(u) = \int_{\mathbb{R}^N} F(x,u) dx. \tag{3.1}$$

So the perturbed functional which we will study is

$$\varphi_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left[|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right] dx - \lambda \int_{\mathbb{R}^N} F(x,u) dx.$$

Clearly, $\varphi_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda \in [1,2]$. We choose a completely orthogonal basis $\{e_i\}$ of E and define $X_i := \mathbb{R} e_i$, and Z_k, Y_k defined as (2.1). We shall prove that φ_λ satisfies the conditions of Lemma 2.7. Following along the same lines as in [9], we can obtain that

- $B(u) \geq 0$, and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E (see [9, Lemma 3.1]).

- There exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that
$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \varphi_\lambda(u) \geq 0$$
and
$$d_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} \varphi_\lambda(u) \to 0$$
as $k \to \infty$ uniformly for $\lambda \in [1,2]$. For further details, we refer the reader to Duan–Huang [9, Lemma 3.2],

- There exists a sequence $\{r_k\}$ with $0 < r_k < \rho_k$ for all $k \in \mathbb{N}$ such that
$$b_k(\lambda) := \inf_{u\in Y_k, \|u\|=r_k} \varphi_\lambda(u) < 0, \qquad \forall\lambda \in [1,2],$$
for details, see [9, Lemma 3.3].

Obviously, condition (1) in Lemma 2.7 have been satisfied. Thus, conditions (1), (2) and (3) in Lemma 2.7 have been verified. Therefore, we know from Lemma 2.7 that there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that
$$\varphi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \qquad \varphi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty. \tag{3.2}$$
For simplicity, we denote $u(\lambda_n)$ by u_n for all $n \in \mathbb{N}$.

Claim: The sequence $\{u_n\}$ is bounded in E.

By virtue of hypothesis $H(f)$, Proposition 2.2, Proposition 2.3 and Proposition 2.5, we have
$$\begin{aligned}
\frac{1}{p^+}\min\{\|u_n\|^{p^+}, \|u_n\|^{p^-}\} &\leq \int_{\mathbb{R}^N} \frac{1}{p(x)}\left[|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right]dx \\
&= \varphi_{\lambda_n}(u_n) + \lambda_n \int_{\mathbb{R}^N} F(x,u_n)dx \\
&= \varphi_{\lambda_n}(u_n) + \lambda_n \int_{\mathbb{R}^N} b(x)|u_n|^{q(x)}dx \\
&\leq M_0 + 2|b|_{\frac{s(x)}{s(x)-q(x)}} \left||u_n|^{q(x)}\right|_{\frac{s(x)}{q(x)}} \\
&\leq \begin{cases} M_0 + 2|b|_{\frac{s(x)}{s(x)-q(x)}} |u_n|^{q^+}_{s(x)}, & \text{if } |u_n|_{s(x)} \geq 1, \\ M_0 + 2|b|_{\frac{s(x)}{s(x)-q(x)}} |u_n|^{q^-}_{s(x)}, & \text{if } |u_n|_{s(x)} \leq 1, \end{cases}
\end{aligned} \tag{3.3}$$
for some $M_0 > 0$. Since $1 < q^- \leq q^+ < p^-$, (3.3) implies that $\{u_n\}$ is bounded in E. Next, we show that there is a strongly convergent subsequence of $\{u_n\}$ in E. Indeed, in view of the boundedness of $\{u_n\}$, passing to a subsequence if necessary, still denoted by $\{u_n\}$, we may assume that $u_n \rightharpoonup u$ weakly in E.

From the choice of the function $b \in L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)$, for any given number $\varepsilon > 0$, we can choose $R_\varepsilon > 0$ such that
$$\int_{|x|>R_\varepsilon} |b(x)|^{\frac{s(x)}{s(x)-q(x)}} dx < \varepsilon^{\frac{s^- - q^+}{s^-}}. \tag{3.4}$$

Since the embedding $E \hookrightarrow L^{s(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ is compact, $u_n \rightharpoonup u_0$ in E implies $u_n \to u_0$ in $L^{s(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, and hence,
$$\lim_{n\to+\infty} \int_{|x|\leq R_\varepsilon} |u_n - u|^{s(x)}dx = 0. \tag{3.5}$$

Let $B_\varepsilon = \{x \in \mathbb{R}^N : |x| \leq R_\varepsilon\}$ and $B_\varepsilon^c = \mathbb{R}^N \setminus B_\varepsilon$. Using Proposition 2.1 and (3.4), we have

$$|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(B_\varepsilon^c)} < \varepsilon. \tag{3.6}$$

Also from Proposition 2.1 and (3.5), there exists $n_0 \in N$ such that

$$|u_n - u_0|_{L^{s(x)}(B_\varepsilon)} < \varepsilon, \quad \text{for } n \geq n_0. \tag{3.7}$$

Now it is easily seen that

$$\begin{aligned}\langle \mathcal{L}u_n - \mathcal{L}u_0, u_n - u_0\rangle = {}& \langle \varphi'_{\lambda_n}(u_n) - \varphi'_{\lambda_1}(u_0), u_n - u_0\rangle \\ &+ \int_{\mathbb{R}^N} (\lambda_n f(x,u_n) - f(x,u_n))(u_n - u_0)dx.\end{aligned} \tag{3.8}$$

We will estimate the right-hand side of (3.8). One clearly has

$$\langle \varphi'_{\lambda_n}(u_n) - \varphi'_{\lambda_1}(u_0), u_n - u_0\rangle = \langle \varphi'_{\lambda_n}(u_n), u_n - u_0\rangle - \langle \varphi'_{\lambda_1}(u_0), u_n - u_0\rangle \to 0. \tag{3.9}$$

Moreover, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} (\lambda_n f(x,u_n) - f(x,u_n))(u_n - u_0)dx \\
&\quad\leq q^+ \int_{\mathbb{R}^N} b(x)(\lambda_n |u_n|^{q(x)-1} + |u_0|^{q(x)-1})|u_n - u_0|dx \\
&\quad\leq q^+ \int_{\mathbb{R}^N} b(x)|u_n|^{q(x)-1}|u_n - u_0| + q^+ \int_{\mathbb{R}^N} b(x)|u_0|^{q(x)-1}|u_n - u_0|dx \\
&\quad= q^+ \int_{\mathbb{R}^N} \frac{b(x)}{V^{\frac{q(x)-1}{s(x)}}} V^{\frac{q(x)-1}{s(x)}} |u_n|^{q(x)-1}|u_n - u_0|dx \\
&\qquad+ q^+ \int_{\mathbb{R}^N} \frac{b(x)}{V^{\frac{q(x)-1}{s(x)}}} V^{\frac{q(x)-1}{s(x)}} |u_0|^{q(x)-1}|u_n - u_0|dx \\
&\quad\leq \frac{q^+}{K_0} \int_{\mathbb{R}^N} b(x) V^{\frac{q(x)-1}{s(x)}} |u_n|^{q(x)-1}|u_n - u_0|dx \\
&\qquad+ \frac{q^+}{K_0} \int_{\mathbb{R}^N} b(x) V^{\frac{q(x)-1}{s(x)}} |u_0|^{q(x)-1}|u_n - u_0|dx\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{q^+}{K_0}\Big[\int_{B_\varepsilon} b(x)V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}|u_n-u_0|dx \\
&\qquad + \int_{B_\varepsilon^c} b(x)V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}|u_n-u_0|dx\Big] \\
&\quad + \frac{q^+}{K_0}\Big[\int_{B_\varepsilon} b(x)V^{\frac{q(x)-1}{s(x)}}|u_0|^{q(x)-1}|u_n-u_0|dx \\
&\qquad + \int_{B_\varepsilon^c} b(x)V^{\frac{q(x)-1}{s(x)}}|u_0|^{q(x)-1}|u_n-u_0|dx\Big] \\
&=: \frac{q^+}{K_0}[I_1+I_2] + \frac{q^+}{K_0}[I_3+I_4],
\end{aligned} \tag{3.10}
$$

where

$$
K_0 := \min\left\{V_0^{\frac{q^- -1}{s^+}}, V_0^{\frac{q^+ -1}{s^-}}\right\}.
$$

Using the Proposition 2.5, Hölder's inequality and (3.7), we have

$$
\begin{aligned}
I_1 &= \int_{B_\varepsilon} b(x)V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}|u_n-u_0|dx \\
&\leq 3|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(B_\varepsilon)}\left|V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}\right|_{L^{\frac{s(x)}{q(x)-1}}(B_\varepsilon)}|u_n-u_0|_{L^{s(x)}(B_\varepsilon)} \\
&\leq 3\varepsilon|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(B_\varepsilon)}\left|V^{\frac{s(x)-1}{q(x)}}|u_n|^{q(x)-1}\right|_{L^{\frac{s(x)}{q(x)-1}}(B_\varepsilon)} \\
&\leq 3\varepsilon|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)}\left|V^{\frac{s(x)-1}{q(x)}}|u_n|^{q(x)-1}\right|_{L^{\frac{s(x)}{q(x)-1}}(\mathbb{R}^N)} \\
&\leq 3\varepsilon|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)}
\begin{cases}
\left|V(x)|u_n|^{s(x)}\right|_1^{\frac{q^+-1}{s^-}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \leq 1, \\
\left|V(x)|u_n|^{s(x)}\right|_1^{\frac{q^--1}{s^+}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \geq 1,
\end{cases} \\
&\leq 3\varepsilon|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)}
\begin{cases}
\|u_n\|_E^{\frac{(q^+-1)s^+}{s^-}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \leq 1, \|u_n\|_E \geq 1, \\
\|u_n\|_E^{q^+-1}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \leq 1, \|u_n\|_E \leq 1, \\
\|u_n\|_E^{q^--1}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \geq 1, \|u_n\|_E \geq 1, \\
\|u_n\|_E^{\frac{(q^--1)s^-}{s^+}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \geq 1, \|u_n\|_E \leq 1.
\end{cases}
\end{aligned} \tag{3.11}
$$

On the other hand, using Proposition 2.4, Proposition 2.5, Hölder's inequality and (3.6), we have

$$
\begin{aligned}
I_2 &= \int_{B_\varepsilon^c} b(x)V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}|u_n-u_0|dx \\
&\leq 3|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(B_\varepsilon^c)}\left|V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}\right|_{L^{\frac{s(x)}{q(x)-1}}(B_\varepsilon^c)}|u_n-u_0|_{L^{s(x)}(B_\varepsilon^c)} \\
&\leq 3|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(B_\varepsilon^c)}\left|V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}\right|_{L^{\frac{s(x)}{q(x)-1}}(\mathbb{R}^N)}|u_n-u_0|_{L^{s(x)}(\mathbb{R}^N)} \\
&\leq 3C\varepsilon\|u_n-u_0\|_E\left|V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}\right|_{L^{\frac{s(x)}{q(x)-1}}(\mathbb{R}^N)}
\end{aligned}
$$

$$\leq 3C\varepsilon\|u_n-u_0\|_E \begin{cases} \left|V(x)|u_n|^{s(x)}\right|_1^{\frac{q^+-1}{s^-}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \leq 1, \\ \left|V(x)|u_n|^{s(x)}\right|_1^{\frac{q^--1}{s^+}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \geq 1, \end{cases}$$

$$\leq 3C\varepsilon(\|u_n\|_E+\|u_0\|_E) \begin{cases} \|u_n\|_E^{\frac{(q^+-1)s^+}{s^-}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \leq 1,\ \|u_n\|_E \geq 1, \\ \|u_n\|_E^{q^+-1}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \leq 1,\ \|u_n\|_E \leq 1, \\ \|u_n\|_E^{q^--1}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \geq 1,\ \|u_n\|_E \geq 1, \\ \|u_n\|_E^{\frac{(q^--1)s^-}{s^+}}, & \text{if } \left|V(x)|u_n|^{s(x)}\right|_1 \geq 1,\ \|u_n\|_E \leq 1. \end{cases} \tag{3.12}$$

Similarly, we also have that

$$I_3 = \int_{B_\varepsilon} b(x)V^{\frac{q(x)-1}{s(x)}}|u_0|^{q(x)-1}|u_n-u_0|dx$$

$$\leq 3\varepsilon|b|_{L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)} \begin{cases} \|u_0\|_E^{\frac{(q^+-1)s^+}{s^-}}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \leq 1,\ \|u_0\|_E \geq 1, \\ \|u_0\|_E^{q^+-1}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \leq 1,\ \|u_0\|_E \leq 1, \\ \|u_0\|_E^{q^--1}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \geq 1,\ \|u_0\|_E \geq 1, \\ \|u_0\|_E^{\frac{(q^--1)s^-}{s^+}}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \geq 1,\ \|u_0\|_E \leq 1, \end{cases} \tag{3.13}$$

and

$$I_4 = \int_{B_\varepsilon^c} b(x)V^{\frac{q(x)-1}{s(x)}}|u_n|^{q(x)-1}|u_n-u_0|dx$$

$$\leq 3C\varepsilon(\|u_n\|_E+\|u_0\|_E) \begin{cases} \|u_0\|_E^{\frac{(q^+-1)s^+}{s^-}}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \leq 1,\ \|u_0\|_E \geq 1, \\ \|u_0\|_E^{q^+-1}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \leq 1,\ \|u_0\|_E \leq 1, \\ \|u_0\|_E^{q^--1}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \geq 1,\ \|u_0\|_E \geq 1, \\ \|u_0\|_E^{\frac{(q^--1)s^-}{s^+}}, & \text{if } \left|V(x)|u_0|^{s(x)}\right|_1 \geq 1,\ \|u_0\|_E \leq 1. \end{cases} \tag{3.14}$$

Since ε is arbitrary, it follows from (3.10)–(3.14) that

$$\int_{\mathbb{R}^N}(\lambda_n f(x,u_n)-f(x,u_n))(u_n-u_0)dx \to 0 \quad \text{as } n \to +\infty. \tag{3.15}$$

According to (3.8), (3.9) and (3.15) we obtain

$$\langle \mathcal{L}u_n - \mathcal{L}u_0, u_n-u_0\rangle \to 0 \quad \text{as } n \to +\infty, \tag{3.16}$$

which implies $u_n \to u_0$ in E from Proposition 2.6 (ii). Thus, from the last assertion of Lemma 2.7, we know that $\varphi = \varphi_1$ has infinitely many nontrivial critical points. Therefore, problem (P) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is completed.

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