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Topological methods on solvability, multiplicity and eigenvalues of a nonlinear fractional boundary value problem

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Abstract. In this paper, we first prove new properties of the (a, q)-stably solvable maps for a class of decomposable operators in the form of LF, where L is a bounded linear operator and F is nonlinear. This class of maps is important in applications as many differential equations can be written as $LF(u) = u$. Secondly, three different approaches, the (a, q)-stably solvable maps, fixed point index and iterative methods are applied to study a nonlinear fractional boundary value problem involving a parameter λ. We obtain intervals of λ that correspond to at least two, one and no positive solutions, respectively. Thirdly, convergence of the eigenvalues and the corresponding eigenvectors for the associated Hammerstein-type integral operator are proved. This paper seems to be the first to apply the theory of (a, q)-stably solvable operators in studying boundary value problems.

Keywords: boundary value problem, cone, eigenvalue, fixed point index, fractional differential equation, linear operator, solvability, stably-solvable maps.

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1 Introduction

In studying existence of positive solutions for boundary value problems, fixed point theory has been widely applied. The common idea is to properly construct a cone and apply techniques such as fixed point theorem of cone expansion and compression in the Banach space [15]. An advantage of this approach is that monotonicity properties of the nonlinear operator are not required. The results ensure existence of a solution but usually do not give much information about the computational aspects of the solution. An alternative approach is to combine fixed point theorem and iterative method. The idea is constructive and related to recursive algorithms in computing. A benefit of iteration is that the solution can be calculated numerically and further properties can also be found. However, as a trade-off, in most cases

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iteration requires the operator to be monotone increasing or decreasing. Both fixed point theory and iterative methods use properties of nonlinear operators to ensure solvability of the equation.

It is interesting to study properties of equations satisfying solvability conditions. In [9], the so-called stably-solvable maps were introduced and used to define the Furi–Martelli–Vignoli nonlinear spectrum. It was shown that this class of operators has some good properties including being invariant under certain operations and satisfying a continuation principle. Later, this concept was generalized in two directions: 1) the L-stably solvable maps in the form of $L+N$, where L is a bounded linear operator and N is nonlinear [12,13], 2) the (a,q)-stably solvable maps with stronger conditions and richer properties [2]. The L-stably solvable map is a key concept in the definition of the semilinear spectrum [12]. The (a,q)-stably solvable map was said to be useful in studying differential equations [2] but we have not seen any examples in the literature. This paper seems to be the first to apply the theory of (a,q)-stably solvable operators to a particular boundary value problem.

We will apply the theory of (a,q)-stably solvable operators, the fixed point index and the iteration technique to the nonlinear fractional boundary value problem (BVP):

$$D^{\alpha}_{0+}u(t)+\lambda h(t)f(u(t))=0, \quad 0<t<1, \quad 2<\alpha<3, \tag{1.1}$$
$$u(0)=u'(0)=u'(1)=0, \tag{1.2}$$

where D^{α}_{0+} denotes the Riemann–Liouvillle fractional derivative, $\lambda>0$ is a parameter, $h\colon (0,1)\to\mathbb{R}^+$ and $f\colon\mathbb{R}\to\mathbb{R}^+$ are nonnegative and continuous and $\int_0^1 h(s)ds>0$. Although the theory of fractional calculus has a long history, new applications have been recently found in many areas including physics, mechanical engineering, electrical engineering, control theory, quantitative finance, econometrics and signal processing. Classical review of fractional differential equations and application examples can be found in [17,18,20] and the references therein.

Existence of a solution for (1.1)–(1.2) is equivalent to existence of a fixed point for a Hammerstein-type integral operator. Fixed point problems for Hammerstein operators have been extensively studied in the past. As an example, a general approach using fixed point index theory can be seen in [26]. However, most of the results are qualitative. For instance, it is often shown that for λ small enough, the Hammerstein operator equation $\lambda T(x)=x$ has two positive solutions. In a number of papers, for instance [19], it is proved that there exists a $\lambda_\star$ such that the equation $\lambda T(x)=x$ has at least two positive solutions, one positive solution and no positive solutions for $0<\lambda<\lambda_\star$, $\lambda=\lambda_\star$ and $\lambda>\lambda_\star$ respectively. In this paper, using three different approaches, we obtain quantitative results that give estimates for the critical value $\lambda_\star$. It is also interesting to compare results obtained by different methods for the same problem.

Existence of solutions for fractional BVPs has been widely studied previously, for instance in [4,5,8,16,21]. On the other hand, bifurcation properties were also discussed, see [7,23] and the references therein. We study the eigenvalues and prove theorems on convergence of the corresponding eigenvectors. The results not only ensure existence of solutions for (1.1)–(1.2), but also provide information on the structure of the solutions in the form of $\|u_n\|\to 0$ or $\|u_n\|\to\infty$ as $n\to\infty$, where u_n is a solution corresponds to λ_n. Some properties of the nonlinear spectrum are also discussed.

In Section 2, we give definitions and preliminary results that will be used in the sequel. New results on (a,q)-stably solvable maps for a class of decomposable nonlinear operators and their application to BVP (1.1)–(1.2) are proved in Section 3. Section 4 obtains the λ-interval for existence of a positive solution when f is not necessarily monotone. Existence of two positive solutions by iteration are proved in Section 5. Finally, results on eigenvalues and eigenvectors are given in Section 6.

2 Preliminaries

Let X, Y be Banach spaces, if $M \subset X$ is bounded, the Kuratowski measure of noncompactness $\alpha(M)$ is defined as the following, see for example [6]:

$$\alpha(M) = \inf\{\epsilon > 0 : M \text{ can be covered by finitely many sets with diameter } \leq \epsilon\}.$$

Let $C(X, Y)$ denote all continuous maps from X to Y. For $F \in C(X, Y)$, the upper and lower measure of noncompactness are defined by, see for example [3]:

$$[F]_A = \inf\{k > 0 : \alpha(F(M)) \leq k\alpha(M), \text{ for every bounded set } M \subset X\},$$
$$[F]_a = \sup\{k > 0 : \alpha(F(M)) \geq k\alpha(M), \text{ for every bounded set } M \subset X\}.$$

The upper and lower quasi-norms are defined by

$$[F]_Q = \limsup_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|}, \qquad [F]_q = \liminf_{\|x\|\to\infty} \frac{\|F(x)\|}{\|x\|}.$$

Definition 2.1. An operator $F\colon X \to Y$ is called *stably-solvable* if and only if for any given compact map $h\colon X \to Y$ with zero upper quasi-norm ($[h]_Q = 0$), the equation $F(x) = h(x)$ has a solution.

The class of stably-solvable operators corresponds to the property of surjectivity when specialized to linear operators. In [12], it was generalized to the L-stably solvable operators for semilinear maps in the form of $L + N$, where L is a linear operator and N is nonlinear. Later it was generalized to the following (a, q)-stably solvable maps by Appell, Giorgieri and Väth [2].

Definition 2.2. Given $a \geq 0$ and $q \geq 0$, a map $F \in C(X, Y)$ is called (a, q)-stably-solvable if for any $h \in C(X, Y)$ with $[h]_A \leq a$ and $[h]_Q \leq q$, the equation

$$F(x) = h(x)$$

has a solution $x \in X$.

The stably-solvable maps are special case of the (a, q)-stably-solvable maps when $a = q = 0$. As a and q become larger, the class gets smaller since the condition is stronger.

We will use the following definitions of fractional calculus.

Definition 2.3. The standard Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $u\colon (0, \infty) \to \mathbb{R}$ is defined as

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s) ds$$

provided that the right side is point-wise defined on $(0, \infty)$. Here Γ denotes the Gamma function.

Definition 2.4. The Riemann–Liouvillle fractional derivative of order $\alpha > 0$ of a continuous function $u\colon [0,\infty) \to \mathbb{R}$ is defined to be

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}ds, \qquad n = \lceil\alpha\rceil,$$

where $\lceil\alpha\rceil$ denotes the ceiling function, returning the smallest integer greater than or equal to α.

It is known [8] that $u \in C(0,1)$ is a solution of (1.1)–(1.2) if and only if

$$u(t) = \lambda\int_0^1 G(t,s)h(s)f(u(s))ds, \qquad 0 \le t \le 1, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \le t \le s \le 1, \\ \dfrac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.5. *For* $0 \le t,s \le 1$,

$$q(t)G(1,s) \le G(t,s) \le G(1,s) \le H(\alpha) := \frac{(\alpha-2)^{\alpha-2}}{\Gamma(\alpha)(\alpha-1)^{\alpha-1}} < 1,$$

where $q(t) = t^{\alpha-1}$.

Proof. It is easy to see that

$$G(1,s) = \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}.$$

Let $g(s) = s(1-s)^{\alpha-2}$. Then $g'(s) = 0$ has the solution $s_0 = \frac{1}{\alpha-1}$, which is the maximum point of $g(s)$ for $0 \le s \le 1$. So

$$G(1,s) \le \frac{g(s_0)}{\Gamma(\alpha)} = \frac{(\alpha-2)^{\alpha-2}}{\Gamma(\alpha)(\alpha-1)^{\alpha-1}} = H(\alpha).$$

For $\alpha > 2, \Gamma(\alpha) > 1$ and so $H(\alpha) < 1$.

The inequality $q(t)G(1,s) \le G(t,s)$ is shown in Lemma 2.8 [8]. □

3 Decomposable (a,q)-stably solvable maps and BVP (1.1)–(1.2)

In applications, many differential equations can be written as the operator equation

$$LF(u) = u, \qquad u \in X,$$

where X is a Banach space, L is a bounded linear operator and F is nonlinear. The Hammerstein integral equation given in (3.1) is a typical example. Lemma 3.1 and Theorem 3.2 extend the continuation principle for (a,q)-stably solvable maps [2] to nonlinear maps in the form of LF.

Lemma 3.1. *Let F be (a,q)-stably solvable, L be a bounded linear operator. Assume that L is invertible. Then LF is $(a/[L^{-1}]_A, q/\|L^{-1}\|)$-stably solvable.*

Proof. Assume that $G\colon X \to X$ satisfies the conditions $[G]_A \le a/[L^{-1}]_A$, and $[G]_Q \le q/\|L^{-1}\|$. Then

$$[L^{-1}G]_A \le [L^{-1}]_A[G]_A \le a, \qquad [L^{-1}G]_Q \le \|L^{-1}\|[G]_Q \le q.$$

Since F is (a,q)-stably solvable, the equation

$$F(x) = L^{-1}G(x)$$

has a solution. Therefore, $LF(x) = G(x)$ has a solution, so that LF is $(a/[L^{-1}]_A, q/\|L^{-1}\|)$-stably solvable. □

Theorem 3.2. *Let F be (a,q)-stably solvable, L be linear and invertible. Assume that $H\colon X \times [0,1] \to X$ satisfies $[H(\cdot,0)]_q < 1$, and*

$$\alpha(H(M \times [0,1])) \le a_1\alpha(M), \quad \textit{for any bounded } M \subset X,$$

where $a_1 = \frac{a}{[L^{-1}]_A}$. Let

$$S = \{x:\ x \in X, LF(x) = H(x,t), \quad \textit{for } t \in [0,1]\}.$$

If $F(S)$ is bounded, then the equation

$$LF(x) = H(x,1)$$

has a solution.

Proof. Since $F\colon X \to X$ is (a,q)-stably solvable, by Lemma 3.1, $LF\colon X \to X$ is (a_1,q_1)-stably solvable, where

$$a_1 = a/[L^{-1}]_A, \qquad q_1 = q/\|L^{-1}\|.$$

Since $F(S)$ is bounded $LF(S)$ is also bounded. Applying the continuation principle for (a,q)-stably solvable maps [2], there exists $x \in X$ such that

$$LF(x) = H(x,1).$$ □

We now apply Theorem 3.2 to prove existence of a solution for the BVP (1.1)–(1.2). We use the Banach space $X = C[0,1]$ with the standard norm

$$\|u\| = \max_{0 \le t \le 1} |u(t)|, \qquad u \in X.$$

Define the Hammerstein-type operator $N\colon \mathbb{R} \times X \to X$:

$$N(\lambda,u)(t) = \lambda \int_0^1 G(t,s)h(s)f(u(s))ds, \qquad t \in [0,1],\ u \in X. \tag{3.1}$$

For $u \in X$, u is a solution of (1.1)–(1.2) if and only if $N(\lambda,u) = u$.

Theorem 3.3. *Assume that $h \in C[0,1]$, $h(t) \ge 0$ for $t \in [0,1]$, $\|h\| > 0$, $f\colon [0,\infty) \to (0,\infty)$ is non-decreasing. Denote*

$$\overline{f}_\infty := \limsup_{x \to \infty} \frac{f(x)}{x}.$$

If $\overline{f}_\infty < +\infty$, the BVP (1.1)–(1.2) has at least one positive solution for

$$\lambda \in \left(0, \frac{1}{\|h\|(2 + H(\alpha))\overline{f}_\infty}\right).$$

Proof. Let $f(x) = f(0)$ for $x \in (-\infty, 0)$. Considering equation (2.1), define

$$F(u)(t) = \lambda h(t) f(u), \qquad u \in C[0,1], \tag{3.2}$$

$$L(u)(t) = \int_0^1 G(t,s)u(s)ds, \qquad u \in C[0,1]. \tag{3.3}$$

By Lemma 2.5, for $u \in C[0,1]$, we have

$$\|L(u)\| = \max_{t\in[0,1]} \left| \int_0^1 G(t,s)u(s)ds \right| \le \max_{t\in[0,1]} \int_0^1 G(t,s)ds\|u\| \le H(\alpha)\|u\|.$$

Therefore, L is bounded and $\|L\| \le H(\alpha) < 1$, which implies that $(I - L)$ is invertible and

$$\|(I-L)^{-1}\| \le \frac{1}{1-H(\alpha)}.$$

Clearly the operator equation $LF(u) = u$ is equivalent to the following:

$$(I-L)^{-1}(I-F)u = -F(u). \tag{3.4}$$

It is known that identity map I is (a,q)-stably solvable for $a, q \in [0,1)$ but not $(1,1)$-stably solvable. $(I-L)^{-1}$ is (a,q)-stably solvable for $a, q < 1 - H(\alpha)$ [2]. Consider the nonlinear map $F\colon C[0,1] \to C[0,1]$ defined by (3.2),

$$\begin{aligned}
[F]_Q &= \limsup_{\|u\|\to\infty} \frac{\|F(u)\|}{\|u\|} \\
&\le \limsup_{\|u\|\to\infty} \frac{\lambda\|h\|f(\|u\|)}{\|u\|} \\
&= \lambda\|h\|\overline{f}_\infty \\
&\le \lambda\|h\|(2+H(\alpha))\overline{f}_\infty < 1.
\end{aligned}$$

Next, suppose $D \in C[0,1]$ is a arbitrary bounded set. There exists $M > 0$ such that $\|u\| \le M$ for any $u \in D$.

$$\|F(u)\| \le \lambda\|h\|f(\|u\|) \le \lambda\|h\|f(M).$$

This implies that $F(u),\ u \in D$ is uniformly bounded. Moreover, for $t_1, t_2 \in [0,1]$ and $u \in D$,

$$\begin{aligned}
|F(u)(t_1) - F(u)(t_2)| = {}& \lambda|h(t_1)f(u(t_1)) - h(t_2)f(u(t_2))| \\
\le {}& \lambda|h(t_1) - h(t_2)|f(M) + h(t_2)|f(u(t_1)) - f(u(t_2))| \\
\to {}& 0 \quad \text{as } |t_2 - t_1| \to 0.
\end{aligned}$$

Hence $F(D)$ is equicontinuous. By the Ascoli–Arzelà theorem, we get that $F(D)$ is relatively compact. Therefore, F is compact, which implies that $[F]_A = 0$. By the Rouché type perturbation result for (a,q)-stably solvable maps ([2], Proposition 5), $(I-F)$ is (a,q)-stably solvable for $a < 1,\ q = 1 - [F]_Q > 0$.

Let $H(u,t) = -tF(u)$ and

$$S = \{u:\ u \in C[0,1],\ (I-L)^{-1}(I-F)u = -tF(u),\ t \in [0,1]\}.$$

By Theorem 3.2, if $(I-F)(S)$ is bounded, then equation (3.4) has a solution which implies the BVP (1.1)–(1.2) has a solution.

Assume that there exists $u_n \in S$, $\|u_n\| \to \infty$, then we have

$$(I-F)u_n = -t_n(I-L)F(u_n), \qquad t_n \in [0,1]$$

and

$$\|u_n\| - \|F(u_n)\| \le \|(I-F)u_n\| = t_n\|(I-L)F(u_n)\| \le \|I-L\|\|F(u_n)\|.$$

Further calculation leads to the following:

$$\begin{aligned} 1 &\le \|I-L\|\frac{\|F(u_n)\|}{\|u_n\|} + \frac{\|F(u_n)\|}{\|u_n\|} \\ &\le \lambda\|h\|(2+\|L\|)\frac{f(\|u_n\|)}{\|u_n\|} \\ &\le \lambda\|h\|(2+H(\alpha))\overline{f}_\infty. \end{aligned}$$

This contradicts the assumption

$$\lambda < \frac{1}{\|h\|(2+H(\alpha))\overline{f}_\infty}.$$

So S is bounded and the equation $LF(u) = u$ has a solution. Assume $u_0 \in C[0,1]$ is a solution. Since G, h, f are all non-negative, u_0 is non-negative. Since $f(0) > 0$ it is clear that 0 is not a solution and, by Lemma 2.5 it follows that $u_0(t) \ge t^{\alpha-1}\|u_0\|$, so u_0 is positive on $(0,1]$. The proof is complete. □

Remark 3.4. If f is non-negative and $f(0) = 0$ then the theorem proves existence of a non-negative solution, but in this case 0 is obviously a solution.

Theorem 3.3 can be easily generalized to the case of $F(t,x)\colon [0,1]\times[0,\infty) \to (0,\infty)$ instead of $h(t)f(x)$ to obtain existence of a solution for the equation

$$D^{\alpha}_{0+}u(t) + \lambda F(t,u(t)) = 0, \quad 0<t<1, \quad 2<\alpha<3 \tag{3.5}$$

subject to the boundary condition (1.2).

Theorem 3.5. *Let $F(t,x)\colon [0,1]\times[0,\infty)\to(0,\infty)$ be non-decreasing in x. Denote*

$$\overline{F}_\infty := \limsup_{x\to\infty} \max_{t\in[0,1]} \frac{F(t,x)}{x}.$$

If $\overline{F}_\infty < +\infty$, the BVP (3.5)–(1.2) has at least one positive solution for

$$\lambda \in \left(0, \frac{1}{(2+H(\alpha))\overline{F}_\infty}\right).$$

4 Existence of a positive solution by fixed point index

We work in the space $X = C[0,1]$ as in Section 3. Existence of a solution for BVP (1.1)–(1.2) is equivalent to the existence of a fixed point for the Hammerstein operator N defined by (3.1). To use fixed point index, a proper cone needs to be constructed. We use a well-known type of cone, see for example [5,8,26]. Define the cone K as

$$K = \{u \in X\colon\ u(t) \ge q(t)\|u\|,\ t\in[0,1]\}, \tag{4.1}$$

where $q(t) = t^{\alpha-1}$ (see Lemma 2.5). For $\lambda \in (0, \infty)$, it follows from

$$q(t)G(1,s) \le G(t,s) \le G(1,s)$$

that $N(\lambda, K) \subset K$. If $u \in K$, then $\|u\| = u(1)$. Let $K_r = \{u \in K,\ \|u\| < r\}$ and $\partial K_r = \{u \in K,\ \|u\| = r\}$. We will use the following lemmas for fixed point index, see, for example, [15].

Lemma 4.1. *Let $N\colon K \to K$ be a completely continuous mapping. If*

$$Nu \ne \mu u, \quad \text{for all } u \in \partial K_r, \quad \text{and all } \mu \ge 1,$$

then the fixed point index $i(N, K_r, K) = 1$.

Lemma 4.2. *Let $N\colon K \to K$ be a completely continuous mapping and satisfy $Nu \ne u$ for $u \in \partial K_r$. If $\|Nu\| \ge \|u\|$, for $u \in \partial K_r$, then the fixed point index $i(N, K_r, K) = 0$.*

Define the linear operator $T\colon C[0,1] \to C[0,1]$:

$$T(u)(t) = \int_0^1 G(t,s)h(s)u(s)ds. \tag{4.2}$$

Then $T(K) \subset K$ and using the Ascoli–Arzelà theorem, it can be shown that T is a completely continuous. Let the spectral radius of the operator T be denoted $r(T)$. Under our hypotheses it is known that $r(T) > 0$, see for example [26]. By the well-known Krein–Rutman theorem [15], $r(T)$ is an eigenvalue of T with a positive eigenvector (in K), the so-called principal eigenvalue. Then $\mu_1 = \frac{1}{r(T)}$ is called the principal characteristic value of T.

Theorem 4.3. *Assume that $h(s) \ge 0$ for $s > 0$ and $f(x) > 0$ for $x > 0$. Denote*

$$f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}, \qquad \overline{f}_0 = \limsup_{x\to 0^+} \frac{f(x)}{x}, \qquad l = \min_{x\in(0,\infty)} \frac{f(x)}{x}. \tag{4.3}$$

If $f_\infty = \infty$, and $0 < \overline{f}_0 < \infty$, then the BVP (1.1)–(1.2) has at least one positive solution for $0 < \lambda < 1/(\overline{f}_0 r(T))$ and has no positive solution for $\lambda > 1/(lr(T))$.

Proof. Let $\lambda < \frac{1}{\overline{f}_0 r(T)}$. Select $\varepsilon > 0$ small enough such that $\lambda(\overline{f}_0 + \varepsilon)r(T) < 1$. Assume that $\delta > 0$ is such that $\frac{f(x)}{x} < \overline{f}_0 + \varepsilon$ for $x \in (0, 2\delta)$. We claim that $N(\lambda, u) \ne \mu u$ for $u \in \partial K_\delta$, and $\mu \ge 1$.

Otherwise, there exist $u_0 \in \partial K_\delta$ and $\mu_0 \ge 1$ such that $N(\lambda, u_0) = \mu_0 u_0$. Note that $0 \le \delta q(t) \le u_0(t) \le \|u_0\| = \delta$. Then

$$\begin{aligned} \mu_0 u_0(t) = N(\lambda, u_0)(t) &= \int_0^1 G(t,s)h(s)f(u_0(s))\,ds \\ &\le \lambda(\overline{f}_0 + \varepsilon) \int_0^1 G(t,s)h(s)u_0(s)\,ds \\ &= \lambda(\overline{f}_0 + \varepsilon)Tu_0(t). \end{aligned}$$

Thus $Tu_0(t) \ge \frac{\mu_0}{\lambda(\overline{f}_0+\varepsilon)} u_0(t)$. By an old known result, see for example [25, Theorem 2.7], this implies $r(T) \ge \mu_0/(\lambda(\overline{f}_0 + \varepsilon))$, a contradiction. By Lemma 4.1, $i(N, K_\delta, K) = 1$.

We now show that $i(N, K_R, K) = 0$ for R sufficiently large. Choose $c > 0$ such that $\int_c^1 G(1,s)h(s)ds > 0$. Select $M > 0$ large enough such that

$$\lambda M \int_c^1 G(1,s)h(s)q(s)ds > 1.$$

There exists $M_1 > 0$, such that $\frac{f(x)}{x} > M$ for $x \geq M_1$. We may take $M_1 > \max\{c^{\alpha-1}, 2\delta\}$ and then we let $R = \frac{M_1}{c^{\alpha-1}}$. For $u \in \partial K_R$, we have

$$u(t) \geq q(t)\|u\| \geq c^{\alpha-1}\|u\| = M_1, \quad \text{for } t \in [c,1].$$

Therefore

$$\begin{aligned}\|N(\lambda,u)\| = N(\lambda,u)(1) &= \lambda \int_0^1 G(1,s)h(s)f(u(s))ds \\ &\geq \lambda M \int_c^1 G(1,s)h(s)u(s)ds \\ &\geq \lambda M\|u\| \int_c^1 G(1,s)h(s)q(s)ds > \|u\|.\end{aligned}$$

By Lemma 4.2, $i(N, K_R, K) = 0$. From the additivity property of fixed point index,

$$i(N, K_R \setminus \overline{K}_\delta, K) = i(N, K_R, K) - i(N, K_\delta, K) = -1.$$

So N has a fixed point in $K_R \setminus K_\delta$. It is a positive solution of (1.1)–(1.2).

For $\lambda > \frac{1}{lr(T)}$, nonexistence of a positive solution can be obtained by [26, Theorem 8]. □

Remark 4.4. Assume the conditions of Theorem 4.3 are satisfied. If $l = \overline{f}_0$, then the BVP (1.1)–(1.2) has at least one positive solution for $\lambda \in \left(0, \frac{1}{lr(T)}\right)$ and has no positive solution for $\lambda \in \left(\frac{1}{lr(T)}, +\infty\right)$. It would be interesting to know whether or not there is a positive solution when $\lambda = \frac{1}{lr(T)}$.

Remark 4.5. It is easy to construct functions satisfying the conditions of Theorem 4.3.

Remark 4.6. A parameter $\mu \in \mathbb{R}$ is an eigenvalue of an nonlinear operator $N\colon X \to X$ if there exists $u \in X, u \neq 0$ such that $\mu u = Nu$, and u is called the corresponding eigenvector. A principal eigenvalue is an eigenvalue with a positive eigenvector [24]. Theorem 4.3 implies that for $\mu > \overline{f}_0 r(T)$, μ is a principle eigenvalue of the Hammerstein integral operator:

$$Nu = \int_0^1 G(t,s)h(s)f(u)ds. \tag{4.4}$$

By the results of [11], all eigenvalues of an nonlinear operator N are in the nonlinear spectrum of N. Since the spectrum is a closed set, we obtain that $[\overline{f}_0 r(T), \infty) \subset \sigma(N)$ [11].

5 Two positive solutions by iteration

Using iteration techniques, existence of two positive solutions can be obtained when f is non-decreasing. The results extend the previous work [14] on an algebraic system. We again use the notations of (4.1)–(4.3). In addition, let

$$f_0 = \lim_{x\to 0} \frac{f(x)}{x}, \qquad A = \int_0^1 G(1,s)h(s)ds. \tag{5.1}$$

Lemma 5.1. *Assume f is non-decreasing for $x \in (0, +\infty)$, $f_0 = \infty$ and $l = \min_{x \in (0,\infty)} \frac{f(x)}{x} > 0$. If $0 < \lambda_1 < \lambda_2 < \frac{1}{lA}$, then there exists $u_1 \leq u_2$, $u_1, u_2 \in K \setminus \{\theta\}$, such that $N(\lambda_1, u_1)(t) = u_1(t)$ and $N(\lambda_2, u_2)(t) = u_2(t)$.*

Proof. Assume $x_0 \in (0, \infty)$ such that $f(x_0) = lx_0$. Note that

$$l < \frac{1}{\lambda_2 A} < \frac{1}{\lambda_1 A}.$$

Let

$$u_0(t) = \frac{x_0}{A} \int_0^1 G(t,s)h(s)ds \quad \text{for } t \in [0,1].$$

It is clear that $u_0 \in K \setminus \{\theta\}$ and $\|u_0\| = x_0$. For $t \in [0,1]$,

$$\begin{aligned} N(\lambda_1, u_0)(t) &= \lambda_1 \int_0^1 G(t,s)h(s)f(u_0(s))ds \\ &\leq \lambda_1 \int_0^1 G(t,s)h(s)f(\|u_0\|)ds \\ &= \lambda_1 l x_0 \int_0^1 G(t,s)h(s)ds \\ &< \frac{x_0}{A} \int_0^1 G(t,s)h(s)ds = u_0(t). \end{aligned}$$

Let $u_1^1(t) = N(\lambda_1, u_0)(t)$ and $u_1^j(t) = N(\lambda_1, u_1^{j-1})(t) = N^j(\lambda_1, u_0)(t), j = 2, 3, \ldots,$ for $t \in [0,1]$. Then

$$u_0 > u_1^1 > u_1^2 > \cdots > u_1^j > u_1^{j+1} > \cdots \geq \theta.$$

Since the sequence $\{u_1^j\}_{j=1}^{\infty}$ is decreasing and has a lower bound, for any $t \in [0,1]$, $\lim_{j\to\infty} u_1^j(t)$ exists and the convergence is uniform. Assume that $\lim_{j\to\infty} u_1^j = u_1$, we show that $u_1(t) > 0$ for $t \in (0,1]$. Otherwise, since $u_1 \in K$, we would have $u_1(t) = 0$ for $t \in (0,1]$ and then $\lim_{j\to\infty} u_1^j(t) = 0$ for $t \in (0,1]$, and $u_1^j \in K$ implies that $\|u_1^j\| \to 0$. Since $\lim_{x\to 0} \frac{f(x)}{x} = \infty$, for any $H > 0$, there exists J such that for $j > J$ we have

$$\frac{f(u_1^j(t))}{u_1^j(t)} > H, \qquad t \in [0,1].$$

Select H large enough such that $\lambda_1 H A > 1$. For $j > J$,

$$\begin{aligned} u_1^{j+1}(1) &= N(\lambda_1, u_1^j(1)) \\ &= \lambda_1 \int_0^1 G(1,s)h(s)f(u_1^j(s))ds \\ &> \lambda_1 H \int_0^1 G(1,s)h(s)q(s)\|u_1^j\|ds \\ &\geq u_1^j(1)\lambda_1 H \int_0^1 G(1,s)h(s)q(s)ds \\ &\geq u_1^j(1). \end{aligned}$$

The contradiction shows that $u_1 \in K \setminus \theta$ and u_1 is a fixed point of $N(\lambda_1, u)$.

Similarly, from $u_2^1(t) = N(\lambda_2, u_0)(t)$ and $u_2^j(t) = N(\lambda_2, u_2^{j-1})(t)$, $j = 2, 3, \ldots,$ we can construct a sequence

$$u_0 > u_2^1 > u_2^2 > \cdots > u_2^j > u_2^{j+1} > \cdots \geq \theta$$

such that $\lim u_2^j \to u_2 \in K \setminus \theta$ as $j \to \infty$, and u_2 is a fixed point of $N(\lambda_2, u)$. It is easy to see that

$$u_1^1 = N(\lambda_1, u_0) < N(\lambda_2, u_0) = u_2^1.$$

Since f is non-decreasing, we have $u_1^j \le u_2^j$ for $j = 2, 3, \ldots$. Therefore, $u_1 \le u_2$. The proof is complete. □

Lemma 5.2. *Suppose f is non-decreasing on $(0, \infty)$, $f(x) > 0$ for $x > 0$ and $f_\infty = \infty$. For any $b > 0$, let*

$$S_b = \{u \in K \; : N(\lambda, u) = u, \; for \; \lambda \in [b, \infty)\}.$$

Then S_b is bounded.

Proof. Assume there exist $u_n \in S_b$ and $\lambda_n \in [b, \infty)$ such that

$$N(\lambda_n, u_n) = u_n, \quad \text{and} \quad \lim_{n\to\infty} \|u_n\| = \infty.$$

Select H large enough such that

$$Hbc^{\alpha-1} \int_c^1 G(1, s)h(s)ds > 1,$$

where $1 > c > 0$ is a constant. There exist $M > 0$ such that for $n > M$, $f(c^{\alpha-1}\|u_n\|) > Hc^{\alpha-1}\|u_n\|$. Since $u_n \in K$, $\|u_n\| = u_n(1)$ and $u_n(s) \ge q(s)\|u_n\|$, $n = 1, 2, \ldots$ and for $n > M$, we have

$$\begin{aligned}
u_n(1) &= \lambda_n \int_0^1 G(1, s)h(s)f(u_n(s))ds \\
&\ge \lambda_n \int_c^1 G(1, s)h(s)f(q(s)u_n(1))ds \\
&\ge \lambda_n \int_c^1 G(1, s)h(s)f(c^{\alpha-1}u_n(1))ds \\
&> u_n(1)bHc^{\alpha-1} \int_c^1 G(1, s)h(s)ds > u_n(1).
\end{aligned}$$

This contradiction shows that S_b is bounded. □

Lemma 5.3. *Assume that $f_0 = f_\infty = \infty$ and f is also non-decreasing for $x \in (0, +\infty)$. Let*

$$A = \int_0^1 G(1, s)h(s)ds, \qquad l = \min_{x\in(0,\infty)} \frac{f(x)}{x}.$$

Then the Hammerstein integral operator (2.1) has a fixed point for $\lambda = \frac{1}{lA}$.

Proof. Choose a sequence $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots < \frac{1}{lA}$ satisfying

$$\lim_{n\to\infty} \lambda_n = \frac{1}{lA}.$$

By Lemma 5.1, there exists a non-decreasing sequence $\{u_n\}_{n=1}^\infty \subset \{K \setminus \theta\}$ such that

$$u_n(t) = N(\lambda_n, u_n)(t) = \lambda_n \int_0^1 G(t, s)h(s)f(u_n(s))ds. \tag{5.2}$$

By Lemma 5.2, $\{u_n\}_{n=1}^{\infty}$ is equicontinuous and uniformly bounded. Let $n \to \infty$ in equation (5.2) and let $\lim_{n\to\infty} u_n(t) = u_\star(t)$ for $t \in [0,1]$. Using Lebesgue's dominated convergence theorem, we have

$$u_\star(t) = \frac{1}{lA}\int_0^1 G(t,s)h(s)f(u_\star(s))ds.$$

Therefore $u_\star$ is the fixed point associated with $\lambda = \frac{1}{lA}$. The proof is complete. □

Theorem 5.4. *Assume $f_0 = f_\infty = \infty$ and f is also non-decreasing for $x \in (0,+\infty)$. Then BVP (1.1)–(1.2) has at least two, one and no positive solution for $\lambda \in \left(0, \frac{1}{lA}\right)$, $\lambda = \frac{1}{lA}$ and $\lambda \in \left(\frac{1}{lr(T)}, \infty\right)$ respectively.*

Proof. Assume that $\lambda \in \left(0, \frac{1}{lA}\right)$. By Lemmas 5.1 and 5.3, there exist $u_*, u_\lambda \in \{K \setminus \theta\}$, $u_\lambda \le u_\star$ such that

$$N\left(\frac{1}{lA}, u_\star\right)(t) = u_\star(t) \quad \text{and} \quad N(\lambda, u_\lambda)(t) = u_\lambda(t), \qquad t \in [0,1].$$

If $u_\lambda = u_\star$, we would have the contradiction:

$$N(\lambda, u_\lambda) = u_\lambda = u_\star = N\left(\frac{1}{lA}, u_\star\right) = N\left(\frac{1}{lA}, u_\lambda\right).$$

Hence $u_\lambda < u_\star$. In the following, we will construct two open sets $\Omega_1 \subset \Omega_2 \subset C[0,1]$, where $\Omega_2 = \{u \in C[0,1],\ \|u\| \le R\}$, R is same as in the proof of Theorem 4.3 for K_R with the extra condition that M_1 is large enough such that $\frac{M_1}{c^{\alpha-1}} > \|u_*\| + 1$. Following the proof of Theorem 4.3, we have

$$N(\lambda, u)\| \ge \|u\|,\ u \in K \cap \partial\Omega_2.$$

Now, let

$$\Omega_1 = \{u \in C[0,1],\ -\delta < u(t) < u_\star(t)\}.$$

For $u \in K \cap \partial\Omega_1$, we have $\|u\| = u(1) = u_\star(1)$ and

$$\begin{aligned}\|N(\lambda u)(1)\| &= \lambda \int_0^1 G(1,s)h(s)f(u(s))ds \\ &< \frac{1}{lA}\int_0^1 G(1,s)h(s)f(u_\star(s))ds = u_\star(1).\end{aligned}$$

So $\|N(\lambda, u)\| < \|u\|$ for $u \in K \cap \partial\Omega_1$. By the well-known Guo–Krasnoselskii fixed point theorem, N has a fixed point $\overline{u}_\lambda \in K \cap (\Omega_2 \setminus \Omega_1)$. It is a positive solution of (1.1)–(1.2). Since $u_\lambda \in \Omega_1$, $u_\lambda \ne \overline{u}_\lambda$. The BVP (1.1)–(1.2) has two positive solutions. □

6 Eigenvalues and eigenvectors

In the literature, existence of positive solutions for the Hammerstein integral equation (4.4) has been extensively studied, for example, see [19] and the references therein. In this section, we obtain results on the convergence of eigenvalues and their corresponding eigenvectors for the operator N.

Recall that μ is an eigenvalue of N if there exists $u \in C[0,1]$, $u \ne 0$ such that $N(u) = \mu u$ (see Remark 4.6). Define the following conditions on $f\colon \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0,\infty)$.

H_1: $f(0) > 0$;

H_2: There exist $L \geq l > 0$ such that $L \geq \frac{f(x)}{x} \geq l$;

H_3: $\inf_{x \geq 0} f(x) = d > 0$ and $\sup_{x \geq 0} f(x) = D < +\infty$.

Theorem 6.1. *Assume that* $h\colon (0,1) \to \mathbb{R}^+$ *and denote*

$$A = \int_0^1 G(1,s)h(s)ds, \qquad B = \int_0^1 G(1,s)q(s)h(s)ds.$$

Then

a) *if* H_1 *holds, the Hammerstein operator* N *defined by* (4.4) *has a sequence of eigenvalues* μ_n *such that* $\mu_n \to \infty$*, and the corresponding eigenvectors uniformly converges to zero;*

b) *if* H_2 *holds,* N *has a sequence of eigenvalues* μ_n *such that* $\mu_n \to \mu_0 \in [lB, LA]$*. The corresponding eigenvectors* $\|y_n\| \to \infty$*;*

c) *if* H_3 *is satisfied, then (1)* N *has a sequence of eigenvalues* ν_n *such that* $\nu_n \to \infty$*, and the corresponding eigenvectors uniformly converges to zero; (2)* N *also has a sequence of eigenvalues* $\mu_n \to 0$ *such that the corresponding eigenvectors* $\|y_n\| \to \infty$*.*

Proof. a) Let K be the cone defined by (4.1). For $r > 0$, define

$$(N_r u)(t) = \begin{cases} \|u\| \displaystyle\int_0^1 G(t,s)h(s)f\left(r\frac{u(s)}{\|u\|}\right) ds & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

N_r is a positively homogeneous, compact operator. From $u \in K$, $\|u\| = u(1)$ and $q(t)G(1,s) \leq G(t,s) \leq G(1,s)$, it can be shown that $N_r\colon K \to K$. Since $f(0) > 0$, there exists $\delta > 0$ such that $f(x) > \frac{f(0)}{2}$ for $|x| < \delta$. Let $0 < r < \delta$, then for $u \in K$ and $\|u\| = 1$,

$$\begin{aligned} \|N_r u\| &= \max_{t \in [0,1]} \int_0^1 G(t,s)h(s)f(ru(s))ds \\ &\geq \frac{f(0)}{2} \max_{t \in [0,1]} \int_0^1 G(t,s)h(s)ds \\ &\geq \frac{f(0)}{2} \max_{t \in [0,1]} \int_0^1 q(t)G(1,s)h(s)ds \\ &= \frac{f(0)}{2} \int_0^1 G(1,s)h(s)ds \\ &= \frac{f(0)A}{2} > 0. \end{aligned}$$

We have $\inf\{\|N_r u\| : u \in K,\ \|u\| = 1\} > 0$. Since N_r is compact, there exists $\lambda_r > 0$ and $u_r \in K$ such that $N_r u_r = \lambda_r u_r$ [6]. Thus

$$\frac{1}{\lambda_r} \int_0^1 G(t,s)h(s)f(ru_r(s))ds = u_r(t).$$

In addition,

$$\lambda_r = \lambda_r \max_{t \in [0,1]} u_r(t) = \max_{t \in [0,1]} \int_0^1 G(t,s)h(s)f(ru_r(s))ds \geq \frac{f(0)A}{2}. \tag{6.1}$$

Let $y_r = ru_r$, then $\|y_r\| = r$ and

$$\int_0^1 G(t,s)h(s)f(ru_r(s))ds = \frac{\lambda_r}{r} y_r(t).$$

Let $r \to 0$, we can obtain the eigenvalue sequence $\mu_r = \frac{\lambda_r}{r} \to \infty$ and $\|y_r\| \to 0$.

b) Let N_r be defined in a) and select $r > 1$. For $u \in K$ and $\|u\| = 1$, we have

$$\begin{aligned}
\|N_r u\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(ru(s))ds \\
&\geq lr \max_{t\in[0,1]} q(t) \int_0^1 G(1,s)h(s)u(s)ds \\
&\geq lr \int_0^1 G(1,s)q(s)h(s)ds \geq lrB.
\end{aligned}$$

By a similar argument to that of a), there exist λ_r and $u_r \in K$, $\|u_r\| = 1$ such that $N_r u_r = \lambda_r u_r$ and $\lambda_r \geq lrB$. In addition,

$$\begin{aligned}
\lambda_r &= \lambda_r \max_{t\in[0,1]} u_r(t) \\
&= \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(ru_r(s))ds \\
&\leq Lr \int_0^1 G(1,s)h(s)ds = LrA.
\end{aligned}$$

Let $r_n \to \infty$ $(n \to \infty)$, we have

$$\frac{1}{LA} \leq \frac{r_n}{\lambda_{r_n}} \leq \frac{1}{lB}.$$

Let $\mu_0 = \lim_{n\to\infty} \frac{\lambda_{r_n}}{r_n}$ and $y_n = r_n u_n$, then $\|y_n\| \to \infty$ as $n \to \infty$, $\mu_0 \in [lB, LA]$.

c) The conclusion (1) follows directly from a) since H_1 is satisfied. (2) As in the proof of b), there exist λ_r and $u_r \in P$, $\|u_r\| = 1$ such that $N_r u_r = \lambda_r u_r$ and $\lambda_r \geq lB$. We also have

$$\lambda_r = \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(rx(s))ds \leq DA. \tag{6.2}$$

Again, let $y_n = r_n u_n$, as $r_n \to \infty$, we have $\|y_n\| \to \infty$ and $\mu_n = \frac{\lambda_{r_n}}{r_n} \to 0$. The proof is complete. □

Remark 6.2. Consider the Hammerstein-type operator with a parameter $N(\lambda, u)$ defined by (3.1). According to the definitions of bifurcation point and asymptotic bifurcation point given in [22], Theorem 6.1 can be stated as the following:

a) if H_1 holds, then 0 is a bifurcation point of $N(\lambda, u) = u$;

b) if H_2 holds, then $N(\lambda, u) = u$ has an asymptotic bifurcation point $\lambda_0 \in \left[\frac{1}{LA}, \frac{1}{lB}\right]$;

c) in case of H_3 is satisfied, 0 is a bifurcation point and ∞ is an asymptotic bifurcation point of $N(\lambda, u) = u$.

Remark 6.3. Theorem 6.1 not only proves that there exists a sequence $\{\lambda_n\}_1^\infty$ with corresponding solutions of (1.1)–(1.2), but also gives some the properties of the set of solutions. In case a), there exist $\lambda_n \to 0$ with corresponding solutions u_n. In addition, the set of solutions $\|u_n\| \to 0$. Case b) ensures (1.1)–(1.2) has solutions for $\frac{1}{LA} \leq \lambda_n \leq \frac{1}{lB}$ and the corresponding solutions $u_n \to \infty$. At the last, case c) provides existence of solutions with $\lambda_n \to \infty$ and the corresponding solutions $\|u_n\| \to \infty$.

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