



# Existence of an antisymmetric solution of a boundary value problem with antiperiodic boundary conditions

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**Abstract.** In this work, an application is made of a recent extension of the Leggett–Williams fixed point theorem, commonly referred to as an Avery type fixed point theorem, to a second order boundary value problem with antiperiodic boundary conditions. Under certain conditions and with the use of concavity, an antisymmetric solution to the boundary value problem is shown to exist. In conclusion, a non-trivial example is provided.

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## 1 Introduction

Recently, Avery et.al. have extended the original Leggett–Williams fixed point theorem [14] by generalizing several of the conditions to include convex and concave functionals. For some examples of this work, see [3, 5–7]. In this paper, we will utilize the Avery type fixed point theorem found in [4] which differs from the original Leggett–Williams fixed point theorem in that it does not require either of the functional boundaries to be invariant with respect to the functional wedge.

This particular fixed point theorem has been employed to show the existence of positive solutions to a second order right focal boundary value problem and second [1], fourth [8], and  $2n$ th order [2] conjugate boundary value problems.

In this work, we apply the theorem to the second order boundary value problem


$$x'' + f(x) = 0, \quad t \in (0, T) \quad (1.1)$$

with antiperiodic boundary conditions

$$x(0) + x(T) = 0, \quad x'(0) + x'(T) = 0, \quad (1.2)$$

where  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $n \in \mathbb{R}^+$ . We will show that if  $f$  satisfies certain conditions, (1.1), (1.2) has an antisymmetric solution  $x(t)$  on  $[0, T]$  in the sense that  $x(T - t) = -x(t)$ .

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We remark that it may seem natural to consider the boundary conditions  $x(0) + x(T) = 0$ ,  $x'(0) - x'(T) = 0$ , when using fixed point theory to show the existence of an antisymmetric solution to a second order boundary value problem. However, the boundary value problem  $-x'' = 0$ ,  $x(0) + x(T) = 0$ ,  $x'(0) - x'(T) = 0$ , has infinitely many nontrivial solutions of the form  $x(t) = mt - mT/2$ , where  $m \in \mathbb{R} \setminus \{0\}$ , implying no Green's function for this boundary value problem exists.

Also note that if  $x$  is antisymmetric and satisfies (1.2), then  $x(T/2) = 0$  and  $x'(0) = x'(T) = 0$ . Hence,  $x$  is forced to have a zero at  $T/2$  and  $x'$  is forced to have zeros at 0 and  $T$ . This is similar to the approach taken by Altwayt and Eloë in [1], where they forced  $x$  to have zeros at 0 and  $T$  and  $x'$  to have a zero at  $T/2$ .

In each of the previous papers, a concavity like property of the Green's function is obtained and is key in proving the existence of solutions. In the papers studying boundary value problems with conjugate boundary conditions, solutions are shown to be symmetric. Here, we take a similar approach. However, since our solutions are not strictly positive, the concavity property is somewhat different than in other papers of interest. Of particular note is the dependence of the property upon the midpoint of the interval  $[0, T]$  instead of the zero. The study of antiperiodic boundary value problems has been one of much interest. For more work on the existence of solutions of boundary problems with antiperiodic conditions, see [9–11] and references therein.

In Section 2, we introduce the Green's function and crucial concavity property. Section 3 is where one will find important definitions and the Avery type fixed point theorem. In Section 4, we impose conditions upon  $f$  and demonstrate that these conditions lead to a fixed point of the operator. Finally, in Section 5, we provide a nontrivial example.

## 2 The Green's function

Throughout this paper, we will utilize the Banach space  $C[0, T]$  endowed with the supremum norm.

**Lemma 2.1.** *For  $h \in C[0, T]$ ,  $x$  is the unique solution to the boundary value problem*

$$x'' + h(t) = 0, \tag{2.1}$$

*satisfying boundary conditions (1.2) if and only if*

$$x(t) = \int_0^T G(t, s)h(s)ds,$$

*where  $G(t, s)$ , defined on  $[0, T] \times [0, T]$  by*

$$G(t, s) = \frac{1}{2} \begin{cases} s - t + \frac{T}{2}, & 0 \leq s \leq t \leq T, \\ t - s + \frac{T}{2}, & 0 \leq t \leq s \leq T, \end{cases}$$

*is the Green's function for  $-x'' = 0$  satisfying boundary conditions (1.2).*

*Proof.*  $x''(t) = -h(t)$  if and only if

$$x(t) = c_0 + c_1t - \int_0^t (t - s)h(s)ds.$$

The function  $x$  satisfies the boundary condition  $x(0) + x(T) = 0$  if and only if

$$c_0 = \frac{1}{2} \left( -c_1 T + \int_0^T (T-s)h(s)ds \right),$$

and  $x$  satisfies the boundary condition  $x'(0) + x'(T) = 0$  if and only if

$$c_1 = \frac{1}{2} \int_0^T h(s)ds.$$

Therefore,  $x$  solves (2.1), (1.2) if and only if

$$\begin{aligned} x(t) &= \frac{1}{2} \left( -\frac{1}{2} \int_0^T Th(s)ds + \int_0^T (T-s)h(s)ds \right) + \frac{1}{2} \int_0^T th(s)ds - \int_0^t (t-s)h(s)ds \\ &= \int_0^t G(t,s)h(s)ds, \end{aligned}$$

where

$$G(t,s) = \frac{1}{2} \begin{cases} s-t + \frac{T}{2}, & 0 \leq s \leq t \leq T, \\ t-s + \frac{T}{2}, & 0 \leq t \leq s \leq T, \end{cases}$$

□

Thus, if  $x$  is a fixed point of the operator  $H: C[0, T] \rightarrow C[0, T]$  defined by

$$Hx(t) := \int_0^T G(t,s)f(x(s))ds, \quad t \in [0, T],$$

then  $x$  is a solution of the boundary value problem (1.1), (1.2).

**Lemma 2.2.** *If  $s \in [0, T]$ ,  $G(t, s)$  has the property that  $yG(T/2 - w, s) \leq wG(T/2 - y, s)$  for all  $w, y \in [0, T/2]$  with  $y \leq w$ .*

*Proof.* If  $y = 0$ ,  $0 \leq wG(T/2, s) = w \min\{s, T-s\}$ , so we assume  $y \neq 0$ . We consider three cases. Case 1: Let  $0 \leq T/2 - w \leq T/2 - y \leq s$ . Now,

$$yG(T/2 - w, s) = \frac{1}{2}(-yw + y(T-s)) \leq \frac{1}{2}(-yw + w(T-s)) = wG(T/2 - y, s).$$

Case 2: Let  $0 \leq T/2 - w \leq s \leq T/2 - y$ . Then,

$$\frac{G(T/2 - w, s)}{G(T/2 - y, s)} = \frac{1/2(T - (s+w))}{1/2(s+y)} = \frac{w((T-s)/w - 1)}{y(s/y + 1)}.$$

So,

$$\frac{G(T/2 - w, s)}{G(T/2 - y, s)} \leq \frac{w}{y}$$

is equivalent to

$$\frac{T-s}{w} - 1 \leq \frac{s}{y} + 1,$$

or

$$s \geq \frac{y(T-2w)}{w+y}.$$

Well,

$$\frac{y(T-2w)}{w+y} = \frac{2y}{w+y}(T/2-w) \leq T/2-w \leq s.$$

Thus,

$$\frac{G(T/2-w,s)}{G(T/2-y,s)} \leq \frac{w}{y}.$$

Case 3: Let  $0 \leq s \leq T/2-w \leq T/2-y$ . Then,

$$\frac{G(T/2-w,s)}{G(T/2-y,s)} = \frac{1/2(s+w)}{1/2(s+y)} = \frac{w(s/w+1)}{y(s/y+1)} \leq \frac{w}{y}.$$

Notice that in Cases 2 and 3, both  $y$  and  $G(T/2-y,s) \geq 0$  so the inequalities may be cross multiplied to obtain the desired result.  $\square$

### 3 The fixed point theorem

**Definition 3.1.** Let  $\mathbb{E}$  be a real Banach space. A nonempty closed convex set  $\mathbb{P} \subset \mathbb{E}$  is called a cone provided:

- (i)  $x \in \mathbb{P}$ ,  $\lambda \geq 0$  implies  $\lambda x \in \mathbb{P}$ ;
- (ii)  $x \in \mathbb{P}$ ,  $-x \in \mathbb{P}$  implies  $x = 0$ .

**Definition 3.2.** A map  $\alpha$  is a nonnegative continuous concave functional on a cone  $\mathbb{P}$  of a real Banach space  $\mathbb{E}$  if  $\alpha: \mathbb{P} \rightarrow [0, \infty)$  is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all  $x, y \in \mathbb{P}$  and  $t \in [0, 1]$ . Similarly, a map  $\beta$  is a nonnegative continuous convex functional on a cone  $\mathbb{P}$  of a real Banach space  $\mathbb{E}$  if  $\beta: \mathbb{P} \rightarrow [0, \infty)$  is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all  $x, y \in \mathbb{P}$  and  $t \in [0, 1]$ .

We now define sets that are integral to the fixed point theorem. Let  $\alpha$  and  $\psi$  be nonnegative continuous concave functions on  $\mathbb{P}$ , and let  $\delta$  and  $\beta$  be nonnegative continuous convex functions on  $\mathbb{P}$ . We define the sets

$$A = A(\alpha, \beta, a, d) = \{x \in \mathbb{P} : a \leq \alpha(x) \text{ and } \beta(x) \leq d\},$$

$$B = B(\delta, b) = \{x \in A : \delta(x) \leq b\},$$

and

$$C = C(\psi, c) = \{x \in A : c \leq \psi(x)\}.$$

The following fixed point theorem is due to Anderson, Avery, and Henderson [4] and is an extension of the original Leggett–Williams fixed point theorem [14].

**Theorem 3.3.** Suppose  $\mathbb{P}$  is a cone in a real Banach space  $\mathbb{E}$ ,  $\alpha$  and  $\psi$  are nonnegative continuous concave functionals on  $\mathbb{P}$ ,  $\delta$  and  $\beta$  are nonnegative continuous convex functionals on  $\mathbb{P}$ , and for nonnegative real numbers  $a, b, c$ , and  $d$ , the sets  $A, B$ , and  $C$  are defined as above. Furthermore, suppose  $A$  is a bounded subset of  $\mathbb{P}$ ,  $H: A \rightarrow \mathbb{P}$  is a completely continuous operator, and that the following conditions hold:

(A1)  $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset, \{x \in \mathbb{P} : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset;$

(A2)  $\alpha(Hx) \geq a$  for all  $x \in B$ ;

(A3)  $\alpha(Hx) \geq a$  for all  $x \in A$  with  $\delta(Hx) > b$ ;

(A4)  $\beta(Hx) \leq d$  for all  $x \in C$ ; and

(A5)  $\beta(Hx) \leq d$  for all  $x \in A$  with  $\psi(Hx) < c$ .

Then  $H$  has a fixed point  $x^* \in A$ .

## 4 Antisymmetric solutions of the BVP

Now, we show the existence of an antisymmetric solution of (1.1), (1.2) in the sense that  $x(T-t) = -x(t)$ . We assume that  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f$  is odd, and  $f([0, \infty)) \subset [0, \infty)$ . Define the cone  $\mathbb{P}$  by

$\mathbb{P} = \{x \in C[0, T] : x(T-t) = -x(t), x \text{ is nonnegative, nonincreasing, and concave on } [0, \frac{T}{2}]\}$ .

**Lemma 4.1.**  $L : A \rightarrow \mathbb{P}$

*Proof.* Let  $x \in A \subset \mathbb{P}$ . First, we show  $Hx(T-t) = -Hx(t)$  for  $t \in [0, T]$ . Notice for  $(t, s) \in [0, T] \times [0, T]$ ,  $G(T-t, T-s) = G(t, s)$ . Now,

$$Hx(T-t) = \int_0^T G(T-t, s)f(x(s))ds.$$

Substitute  $s = T-r$ . Then,

$$\begin{aligned} Hx(T-t) &= - \int_T^0 G(T-t, T-r)f(x(T-r))dr \\ &= \int_0^T G(t, r)f(-x(r))dr \\ &= - \int_0^T G(t, r)f(x(r))dr \\ &= -Hx(t). \end{aligned}$$

Now,  $(Hx)''(t) = -f(x(t))$ . Since  $x \in A$ ,  $x(t) \geq 0$  for  $t \in [0, T/2]$ , and so,  $(Hx)''(t) \leq 0$  for  $t \in [0, T/2]$ . Thus,  $Hx$  is concave on  $[0, T/2]$ , and therefore,  $(Hx)'(t)$  is decreasing on  $[0, T/2]$ . Additionally, since  $\frac{\partial}{\partial t}G(t, s)|_{t=0} = 1/2$ ,  $f$  is odd, and  $x \in A$ , we have  $(Hx)'(0) = 1/2 \int_0^T f(x(s))ds = 0$ . Thus,  $(Hx)'(t) \leq 0$  for  $t \in [0, T/2]$  meaning  $Hx$  is nonincreasing on  $[0, T/2]$ . Lastly, due to antisymmetry,  $Hx(T/2) = 0$  implying that  $Hx \geq 0$  on  $[0, T/2]$ .  $\square$

**Remark 4.2.** Notice that if  $x \in \mathbb{P}$ , then for  $y, w \in [0, T/2]$  with  $y \leq w$ ,

$$\frac{x(T/2-w) - x(T/2)}{(T/2-w) - T/2} \geq \frac{x(T/2-y) - x(T/2)}{(T/2-y) - T/2}$$

due to the fact that  $x$  is nonnegative, nonincreasing, and concave. Since  $x(T/2) = 0$ ,

$$yx(T/2-w) \leq wx(T/2-y).$$

**Theorem 4.3.** Let  $\tau, \mu, \nu \in (0, T/2]$  with  $0 < \tau \leq \mu < \nu \leq T/2$ . Let  $d$  and  $m$  be positive real numbers with  $0 < m \leq \frac{d\mu}{T/2}$  and suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, odd,  $f([0, \infty)) \subset [0, \infty)$ , and satisfies:

- (a)  $f(w) \geq \frac{4d}{\nu^2 - \tau^2}$  for  $w \in \left[ \frac{\tau d}{T/2}, \frac{\nu d}{T/2} \right]$ ,
- (b)  $f(m) \geq f(w)$  for  $w \in [0, m]$  with  $f(w)$  decreasing for  $w \in [m, d]$ , and
- (c)  $\int_0^{T/2-\mu} (T/2-s)f\left(\frac{ms}{T/2-\mu}\right) ds \leq \frac{2d - f(m)\mu^2}{2}$ .

Then, (1.1), (1.2) has at least one antisymmetric solution  $x^* \in A(\alpha, \beta, \frac{\tau d}{T/2}, d)$ .

*Proof.* Define  $a := \frac{\tau d}{T/2}$ ,  $b := \frac{\nu d}{T/2}$ , and  $c := \frac{\mu d}{T/2}$ . For  $x \in \mathbb{P}$ , define the concave functionals  $\alpha$  and  $\psi$  on  $\mathbb{P}$  by

$$\begin{aligned}\alpha(x) &:= \min_{t \in [0, T/2-\tau]} x(t) = x(T/2 - \tau), \\ \psi(x) &:= \min_{t \in [0, T/2-\mu]} x(t) = x(T/2 - \mu),\end{aligned}$$

and the convex functionals  $\delta$  and  $\beta$  on  $\mathbb{P}$  by

$$\begin{aligned}\delta(x) &:= \max_{t \in [T/2-\nu, T/2]} x(t) = x(T/2 - \nu), \\ \beta(x) &:= \max_{t \in [0, T/2]} x(t) = x(0).\end{aligned}$$

By definition,  $A \subset \mathbb{P}$ , and for all  $x \in A$ ,  $d \geq \beta(x) = \max_{t \in [0, T/2]} x(t) = x(0)$ , and so  $A$  is bounded. By Lemma 4.1, we have  $H: A \rightarrow \mathbb{P}$ , and a subsequent standard application of the Arzelà–Ascoli theorem may be used to show that  $H$  is completely continuous.

Now, we show (A1) holds. If  $x \in \mathbb{P}$  and  $\beta(x) > d$ , then

$$\alpha(x) = x(T/2 - \tau) \geq \frac{\tau}{T/2} x(0) > \frac{\tau d}{T/2} = a.$$

Thus,  $\{x : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$ . Choose

$$K \in \left( \frac{4d}{T(T-\mu)}, \frac{4d}{T(T-\nu)} \right).$$

For  $t \in [0, T/2]$ , define

$$x_K(t) := \int_0^T KG(T/2-t, s) ds = \frac{K}{8}(T-2t)(T+2t),$$

and for  $t \in [T/2, T]$ , define  $x_K(t) := -x_K(T-t)$ . Note that  $x_K \in \mathbb{P}$ . Therefore,

$$\alpha(x_K) = x_K(T/2 - \tau) = \frac{K}{8}(2\tau)(2T - 2\tau) > \frac{16d\tau(T-\tau)}{8T(T-\mu)} \geq \frac{d\tau}{T/2} = a,$$

and

$$\beta(x_K) = x_K(0) = \frac{KT^2}{8} < \frac{4dT^2}{8T(T-\nu)} \leq \frac{dT}{2(T/2)} = d.$$

Thus,  $x_K \in A$ . Also,

$$\psi(x_K) = x_K(T/2 - \mu) = \frac{K}{8}(2\mu)(2T - 2\mu) > \frac{16d\mu(T - \mu)}{8T(T - \mu)} = \frac{d\mu}{T/2} = c,$$

and

$$\delta(x_K) = x_K(T/2 - \nu) = \frac{K}{8}(2\nu)(2T - 2\nu) < \frac{16d\nu(T - \nu)}{8T(T - \nu)} = \frac{d\nu}{T/2} = b.$$

Hence,  $x_K \in \{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$ .

Next, we move to (A2). Let  $x \in B$ , and note that for  $s \in [T/2, T]$ , we have  $G(0, s)$  and  $f(x(s))$  are both nonpositive implying  $\int_{T/2}^T G(0, s)f(x(s))ds > 0$ . From condition (a), we have

$$\begin{aligned} \alpha(Hx) &= \int_0^T G(T/2 - \tau, s)f(x(s))ds \\ &\geq \frac{\tau}{T/2} \int_0^T G(0, s)f(x(s))ds \\ &= \frac{\tau}{T/2} \int_0^{T/2} G(0, s)f(x(s))ds + \frac{\tau}{T/2} \int_{T/2}^T G(0, s)f(x(s))ds \\ &\geq \frac{\tau}{T/2} \int_0^{T/2} G(0, s)f(x(s))ds \\ &\geq \frac{\tau}{T/2} \int_{T/2-\nu}^{T/2-\tau} G(0, s)f(x(s))ds \\ &\geq \frac{4d}{\nu^2 - \tau^2} \cdot \frac{\tau}{T/2} \int_{T/2-\nu}^{T/2-\tau} \frac{1}{2}(T/2 - s)ds \\ &= \frac{4d}{\nu^2 - \tau^2} \cdot \frac{\tau}{T/2} \cdot \frac{\nu^2 - \tau^2}{4} \\ &= \frac{\tau d}{T/2} = a. \end{aligned}$$

For (A3), let  $x \in A$  with  $\delta(Hx) > b$ . Then,

$$\begin{aligned} \alpha(Hx) &= \int_0^T G(T/2 - \tau, s)f(x(s))ds \\ &\geq \frac{\tau}{\nu} \int_0^T G(T/2 - \nu, s)f(x(s))ds \\ &= \frac{\tau}{\nu} \delta(Lx) \\ &> \frac{\tau}{\nu} b \\ &= \frac{\tau \nu d}{\nu(T/2)} = a. \end{aligned}$$

Penultimately, we look at (A4). To that end, let  $x \in C$ . Recall that  $0 < m \leq \frac{d\mu}{T/2} = c$ . Thus, for  $s \in [0, T/2 - \mu]$ ,

$$x(s) \geq \frac{cs}{T/2 - \mu} \geq \frac{ms}{T/2 - \mu}.$$

Also, using antisymmetry and the substitution  $u = T - s$ , we have

$$\begin{aligned}
\int_{T/2}^T G(0, s) f(x(s)) ds &= \int_{T/2}^T \frac{1}{2} (T/2 - s) f(x(s)) ds \\
&= \int_{T/2}^T \frac{1}{2} (T/2 - s) f(-x(T - s)) ds \\
&= - \int_{T/2}^T \frac{1}{2} (T/2 - s) f(x(T - s)) ds \\
&= \int_0^{T/2} \frac{1}{2} (T/2 - u) f(x(u)) du \\
&= \int_0^{T/2} G(0, u) f(x(u)) du.
\end{aligned}$$

Apply properties (b) and (c) to the work above to find,

$$\begin{aligned}
\beta(Hx) &= \int_0^T G(0, s) f(x(s)) ds \\
&= 2 \int_0^{T/2} G(0, s) f(x(s)) ds \\
&= \int_0^{T/2} (T/2 - s) f(x(s)) ds \\
&= \int_0^{T/2-\mu} (T/2 - s) f(x(s)) ds + \int_{T/2-\mu}^{T/2} (T/2 - s) f(x(s)) ds \\
&\leq \int_0^{T/2-\mu} (T/2 - s) f\left(\frac{ms}{T/2 - \mu}\right) ds + f(m) \int_{T/2-\mu}^{T/2} (T/2 - s) ds \\
&\leq \frac{2d - f(m)\mu^2}{2} + \frac{f(m)\mu^2}{2} = d.
\end{aligned}$$

Finally, we show (A5) holds. Let  $x \in A$  with  $\psi(Hx) < c$ . Then, we have

$$\begin{aligned}
\beta(Hx) &= \int_0^T G(0, s) f(x(s)) ds \\
&\leq \frac{T/2}{\mu} \int_0^T G(T/2 - \mu, s) f(x(s)) ds \\
&= \frac{T/2}{\mu} \psi(Lx) \\
&< \frac{T/2}{\mu} c = d.
\end{aligned}$$

Since all of the conditions of the Theorem 3.3 hold, we have that there exists at least one antisymmetric solution  $x^* \in A(\alpha, \beta, \frac{\tau d}{T/2}, d)$ .  $\square$

This application of Theorem 3.3 has its advantages. First, notice unlike some other applications of fixed point theorems, the upper and lower bounds on  $f$  are relaxed. For example, an application of Krasnosel'skii's fixed point theorem [13] involving an assumption where  $f(x) \geq Br > 0$  on  $[0, r]$ , or similar fixed point theorems involving a similar assumption, could not be applied, since  $f$  is an odd function. Also, since the fixed point  $x^* \in A(\alpha, \beta, \frac{\tau d}{T/2}, d)$ , we obtain location information on the solution, namely, that  $\frac{\tau d}{T/2} \leq x^*(T/2 - \tau)$  and  $x(0) \leq d$ . Since the solution is antisymmetric, we also have that  $x^*(T/2 + \tau) \leq -\frac{\tau d}{T/2}$  and  $-d \geq x^*(T)$ .



Lastly, since  $A \subset \mathbb{P}$ ,  $x^*$  is nonincreasing on  $[0, T/2]$  and nondecreasing on  $[T/2, T]$ , giving  $\frac{\tau d}{T/2} \leq x^*(t) \leq d$  for  $t \in [0, \tau - T/2]$ ,  $-d \leq x^*(t) \leq -\frac{\tau d}{T/2}$  for  $t \in [\tau + T/2, T]$ , and  $|x^*(t)| \leq d$  for  $t \in [0, T]$ .

We conclude this section by remarking that if the cone  $\mathbb{P}$  is defined by

$$\mathbb{P} = \{x \in C[0, T] : x(T-t) = x(t), x \text{ is nonnegative, nondecreasing, and concave on } [0, \frac{T}{2}]\},$$

Theorem 3.3 can be applied to give the existence of a symmetric solution of (1.1), (1.2). This result is similar to the result in [1] and is therefore omitted.

## 5 Example

To conclude the paper, we present an example demonstrating an application of Theorem 4.3. Let  $T = 1$ ,  $\tau = \frac{2}{23}$ ,  $\mu = \frac{1}{8}$ ,  $\nu = \frac{1}{2}$ ,  $d = 2$ , and  $m = \frac{3}{8}$ . First note,  $0 < \tau \leq \mu < \nu \leq T/2$ , and  $\frac{d\mu}{T/2} = \frac{1}{2}$  so that  $0 < m \leq \frac{d\mu}{T/2}$ . Define a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(w) = \begin{cases} 96w & : w \in [0, \frac{3}{8}] \\ -\frac{24}{13}(w-2) + 33 & : w \in [\frac{3}{8}, 2] \\ 33 & : w \in [2, \infty), \end{cases}$$

and let  $f(w) = -f(-w)$  for  $w \in (-\infty, 0)$ . Then,

- (a) for  $w \in [\frac{\tau d}{T/2}, m] = [\frac{8}{23}, \frac{3}{8}]$ ,  $f(w) \geq f(\frac{8}{23}) \approx 33.39130 > 32.9980 \approx \frac{4d}{\nu^2 - \tau^2}$  and for  $w \in [m, \frac{\nu d}{T/2}] = [\frac{3}{8}, 2]$ ,  $f(w) \geq f(2) = 33 > 32.9980 \approx \frac{4d}{\nu^2 - \tau^2}$ ,
- (b)  $f(m) \geq f(w)$  for  $w \in [0, m] = [0, \frac{3}{8}]$  and  $f(w)$  is decreasing for  $w \in [m, d] = [\frac{3}{8}, 2]$ ,
- (c)  $\int_0^{T/2-\mu} (T/2-s)f(\frac{ms}{T/2-\mu})ds = \int_0^{3/8} (1/2-s)f(s)ds = \int_0^{3/8} (1/2-s)(96s)ds = 1.6875 \leq 1.71875 = \frac{2 \cdot 2 - 96 \cdot \frac{3}{8} \cdot (\frac{1}{8})^2}{2} = \frac{2d - f(m)\mu^2}{2}$ .

Thus, the hypotheses of Theorem 4.3 are satisfied. This gives that the antiperiodic boundary value problem

$$x'' + f(x) = 0, \quad t \in (0, 1), \quad x(0) + x(1) = 0, \quad x'(0) + x'(1) = 0$$

has at least one antisymmetric solution  $x^*$  with location information

$$\frac{8}{23} \leq x^*\left(\frac{19}{46}\right) \quad \text{and} \quad x^*(0) \leq 2.$$

Additionally, due to the antisymmetry of and antiperiodic boundary conditions of  $x^*$ , we have

$$-2 \leq x^*(1), \quad x^*\left(\frac{27}{46}\right) \leq -\frac{8}{23}, \quad x^*\left(\frac{1}{2}\right) = 0, \quad \text{and} \quad (x^*(0))' = (x^*(1))' = 0.$$

For the example above, existence of a nonzero solution positive on  $(0, 1/2)$  follows from known fixed point index results and can then be extended by antisymmetry to  $[0, 1]$ . For example, some sharp conditions related to eigenvalues of the associated linear operator are applicable. The methods apply whenever there is a suitable Green's function, see [16, 17] for examples. However, these results do not give good information on the location of the solution.

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