Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **77**, 1–10; doi: 10.14232/ejqtde.2015.1.77 http://www.math.u-szeged.hu/ejqtde/

Periodic solutions for a class of second-order Hamiltonian systems of prescribed energy

Dong-Lun Wu ✉ [1], **Chun Li** [2] and **Pengfei Yuan** [2]

[1]College of Science, Southwest Petroleum University, Chengdu, 610500, P.R. China
[2]School of Mathematics and Statistics, Southwest University, Chongqing 400715, P.R. China

Received 20 April 2015, appeared 14 November 2015
Communicated by Gabriele Bonanno

Abstract. In this paper, the existence of non-constant periodic solutions for a class of conservative Hamiltonian systems with prescribed energy is obtained by the saddle point theorem.

Keywords: periodic solutions, prescribed energy, Hamiltonian systems, saddle point theorem.

2010 Mathematics Subject Classification: 34C25, 58E05.

1 Introduction and main results

Consider the second order Hamiltonian system

$$\ddot{u}(t) + \nabla V(u) = 0, \tag{1.1}$$

such that

$$\frac{1}{2}|\dot{u}(t)|^2 + V(u) = h, \tag{1.2}$$

where $V\colon R^N \to R$ is a C^1-map and $\nabla V(x)$ denotes the gradient with respect to the x variable, $(\cdot,\cdot)\colon R^N \times R^N \to R$ denotes the standard inner product in R^N and $|\cdot|$ is the induced norm. Furthermore, h stands for the total energy of system (1.1).

Hamiltonian systems have many applications in applied science. There are many papers [1–8, 10–12, 14, 15] which obtained the existence of periodic and connected orbits for (1.1). As we know, along with a classical solution of (1.1), the total energy is a constant. In 1978, under some constraint on the energy sphere, Rabinowitz [10] used variational methods to prove the existence of periodic solutions for a class of first order Hamiltonian systems with prescribed energy. After then, the prescribed energy problems have been studied by many mathematicians [1–4, 6, 7, 11] using geometric, topological or variational methods. In 1984, Benci [4] obtained the following theorem.

✉Corresponding author. Email: wudl2008@163.com

Theorem A ([4]). *Suppose that $V \in C^2(R^N, R)$ satisfies:*

(A_1) $\Omega := \{x \in R^N : V(x) < h\}$ *is non-empty and bounded.*

Then system (1.1)–(1.2) *has at least one periodic solution.*

As shown in [4], condition (A_1) is necessary for the existence of periodic solutions of system (1.1)–(1.2). However, the periodic solution may be constant in Theorem A. The author needed the following condition to obtain the existence of non-constant periodic solutions, which is

(A_2) $\nabla V(x) \neq 0$ for every $x \in \partial\Omega$.

Furthermore, it is assumed that V is of C^2 class in Theorem A. Recently, Zhang [15] has proved the existence of non-constant periodic solutions for system (1.1)–(1.2) with V being only required to be of C^1 class. He got the following theorem.

Theorem B ([15]). *Suppose that $V \in C^1(R^N, R)$ satisfies:*

(B_1) *there are constants $\mu_1 > 0$ and $\mu_2 > 0$ such that*

$$(x, \nabla V(x)) \geq \mu_1 V(x) - \mu_2, \qquad \forall x \in R^N$$

(B_2) $V(x) \geq h$ *and* $\nabla V(x) \to 0$, *as* $|x| \to +\infty$,

(B_3) $V(x) \geq a|x|^{\mu_1} + b$, $a > 0$, $b \in R$,

(B_4) $\limsup_{|x|\to 0} V(x) < h$.

Then for any $h > \mu_2/\mu_1$, system (1.1)–(1.2) *has at least a non-constant C^2-periodic solution. This result can be obtained by the saddle point theorem of Benci–Rabinowitz.*

In 2012, Che and Xue [6] proved the existence of periodic solutions for system (1.1)–(1.2) under some weaker assumptions. They considered the energy h to be a parameter and used monotonicity method to obtain the existence of periodic solutions. Then they obtained the following theorem. Subsequently, let $V_\infty = \liminf_{|x|\to+\infty} V(x)$.

Theorem C ([6]). *Suppose that $V \in C^1(R^N, R)$ satisfies (B_1) and the following conditions*

(C_1) *V achieves a global minimum V_0 at x_0;*

(C_2) $V_\infty > V_0$.

Then for all $h \in \left(\frac{\mu_2}{\mu_1}, V_\infty\right)$, there exists a non-constant periodic solution of energy h.

But condition (B_1) is still needed for proving the compactness condition. Motivated by these papers, we will obtain the existence of periodic solutions for system (1.1)–(1.2) under some different conditions. The following theorem is our main result.

Theorem 1.1. *Suppose that $V_\infty = +\infty$ and $V \in C^1(R^N, R)$ satisfies*

(V_1) $(x, \nabla V(x)) > 0$ *for any* $x \in R^N \setminus \{0\}$;

(V_2) $\liminf_{|x|\to+\infty}(x, \nabla V(x)) > 0$.

Then for any $h > V(0)$, system (1.1)–(1.2) *possesses at least one non-constant periodic solution.*

Remark 1.2. In Theorem 1.1, the total energy could be negative if $V(0)$ is smaller than zero which is different from Theorem B and Theorem C. Furthermore, there are functions satisfying (V_1), (V_2) but not the conditions (B_1) and (B_3). For example, let

$$V(x) = \ln(|x|^2 + 1) - 1.$$

2 Variational settings

Let us set $H^1 = W^{1,2}(R/Z, R^N)$. And we define the equivalent norm in H^1 as follows.

$$\|u\| = \left(\int_0^1 |\dot{u}(t)|^2 dt\right)^{1/2} + |u(0)|.$$

The maximum norm is defined by

$$\|u\|_\infty := \max_{t\in[0,1]} |u(t)|.$$

In order to deal with the prescribed energy situation, let $f\colon H^1 \to R$ be the functional defined by

$$f(u) = \frac{1}{2}\int_0^1 |\dot{u}(t)|^2 dt \int_0^1 (h - V(u(t)))dt. \tag{2.1}$$

This functional has been used by van Groesen [14] to study the existence of brake orbits for smooth Hamiltonian systems with prescribed energy and by A. Ambrosetti and V. Coti Zelati [1,2] to study the existence of periodic solutions of singular Hamiltonian systems. It can easily be checked that $f \in C^1(H^1, R)$ and

$$\langle f'(u), u\rangle = \int_0^1 |\dot{u}(t)|^2 dt \int_0^1 \left(h - V(u(t)) - \frac{1}{2}(\nabla V(u(t)), u(t))\right) dt. \tag{2.2}$$

In this paper, we still make use of the saddle point theorem introduced by Benci and Rabinowitz in [5] to look for the critical points of f. First, we recall that a functional I is said to satisfy the $(PS)^+$ condition, if any sequence $\{u_n\} \subset H^1$ satisfying

$$f(u_n) \to C \quad \text{and} \quad f'(u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

with any $C > 0$, implies a convergent subsequence.

Lemma 2.1 ([5]). *Let X be a Banach space and let $f \in C(X, R)$ satisfy $(PS)^+$ condition. Let $X = X_1 \bigoplus X_2$, $\dim X_1 < \infty$,*

$$B_a = \{x \in X | \|x\| \le a\},$$
$$S = \partial B_\rho \bigcap X_2, \rho > 0,$$
$$\partial Q = \left(B_L \bigcap X_1\right) \bigcup \left(\partial B_L \bigcap \left(X_1 \bigoplus R^+ e\right)\right), \qquad L > \rho,$$

where $e \in X_2$, $\|e\| = 1$,

$$\partial B_L \bigcap \left(X_1 \bigoplus R^+ e\right) = \{x_1 + se \mid (x_1, s) \in X_1 \times R^+, \|x_1\|^2 + s^2 = L^2\},$$
$$Q = \{x_1 + se \mid (x_1, s) \in X_1 \times R, s \ge 0, \|x_1\|^2 + s^2 \le L^2\}.$$

If

$$f|_S \ge \alpha > 0$$

and

$$f|_{\partial Q} \le 0,$$

then f possesses a critical value $c \ge \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{x\in Q} f(g(x)),$$

where

$$\Gamma = \{g \in C(Q, X), g|_{\partial Q} = id\}.$$

The following lemma shows that the critical points of f are non-constant periodic solutions after being scaled.

Lemma 2.2. *Let f be defined as in (2.1) and $\tilde{q} \in H^1$ such that $f'(\tilde{q}) = 0$, $f(\tilde{q}) > 0$. Set*

$$T^2 = \frac{\frac{1}{2}\int_0^1 |\dot{\tilde{q}}(t)|^2 dt}{\int_0^1 (h - V(\tilde{q}(t))dt}.$$

Then $\tilde{u}(t) = \tilde{q}(t/T)$ is a non-constant T-periodic solution for (1.1)–(1.2).

Proof. The proof of this lemma is similar to Lemma 3.1 of [2]. Here we sketch the proof for the readers' convenience. Since $f'(\tilde{q}) = 0$, we can deduce that $\langle f'(\tilde{q}), v\rangle = 0$ for all $v \in H^1$ which can be written as

$$\int_0^1 (\dot{\tilde{q}}(t), \dot{v}(t))dt \int_0^1 (h - V(\tilde{q}(t)))dt = \frac{1}{2}\int_0^1 |\dot{\tilde{q}}(t)|^2 dt \int_0^1 (\nabla V(\tilde{q}(t)), v(t))dt. \tag{2.3}$$

Then we divide equation (2.3) by $\int_0^1 (h - V(\tilde{q}(t)))dt$ which is positive since $f(\tilde{q}) > 0$ and obtain that

$$\int_0^1 (\dot{\tilde{q}}(t), \dot{v}(t))dt = T^2 \int_0^1 (\nabla V(\tilde{q}(t)), v(t))dt \quad \text{for all} \quad v \in H^1,$$

which implies that

$$\frac{1}{T^2}\ddot{\tilde{q}}(t) + \nabla V(\tilde{q}(t)) = 0. \tag{2.4}$$

This shows $\tilde{u}(t) = \tilde{q}(t/T)$ satisfies (1.1). The conservation of energy for (2.4) shows that there exists a constant K such that

$$\frac{1}{2T^2}|\dot{\tilde{q}}(t)|^2 + V(\tilde{q}(t)) = K. \tag{2.5}$$

By the definition of T, we integrate (2.5) on [0,1] and get that

$$K = \frac{1}{2T^2}\int_0^1 |\dot{\tilde{q}}(t)|^2 dt + \int_0^1 V(\tilde{q}(t))dt = h.$$

We finish the proof of this lemma. □

3 Proof of Theorem 1.1

It is known that the deformation lemma can be proved when the usual $(PS)^+$ condition is replaced by $(CPS)_C$ condition (see Lemma 3.1 for the definition of $(CPS)_C$) which means that Lemma 2.1 holds under $(CPS)_C$ condition with positive level. Subsequently, we apply Lemma 2.1 to obtain the critical points of f under $(CPS)_C$ condition for any $C > 0$.

Lemma 3.1. *Suppose that the conditions of Theorem 1.1 hold, then f satisfies $(CPS)_C$ condition which means that for all $C > 0$, and $\{u_j\}_{j\in N} \subset H^1$ such that*

$$f(u_j) \to C, \qquad \|f'(u_j)\|(1 + \|u_j\|) \to 0 \quad \text{as} \quad j \to \infty, \tag{3.1}$$

then sequence $\{u_j\}_{j\in N}$ has a strongly convergent subsequence.

Proof. By (3.1), we can deduce that

$$\frac{C}{2} \le f(u_j) \le C+1, \qquad \|f'(u_j)\|(1+\|u_j\|) \le C \tag{3.2}$$

for j large enough. Then it follows from (2.1), (2.2) and (3.2) that

$$\begin{aligned} 3C+2 &\ge 2f(u_j) + \|f'(u_j)\|(1+\|u_j\|) \\ &\ge 2f(u_j) - \langle f'(u_j), u_j\rangle \\ &= \frac{1}{2}\|\dot{u}_j\|_{L^2}^2 \int_0^1 (\nabla V(u_j(t)), u_j(t))dt. \end{aligned} \tag{3.3}$$

If $\|\dot{u}_j\|_{L^2}$ is unbounded, then we can choose a subsequence, still denoted by $\{\dot{u}_j\}$, such that $\|\dot{u}_j\|_{L^2} \to \infty$ as $j \to \infty$. Then it follows from (3.3) that

$$\int_0^1 (\nabla V(u_j(t)), u_j(t))dt \to 0 \quad \text{as } j \to \infty.$$

By (V_1), we can see that $(\nabla V(u_j(t)), u_j(t)) \to 0$ as $j \to \infty$ for a.e. $t \in [0,1]$. There exists a set $\Lambda \subset [0,1]$ such that $(\nabla V(u_j(t)), u_j(t)) \to 0$ as $j \to \infty$ for all $t \in \Lambda$ with meas $\Lambda = 1$, where meas denotes the Lebesgue measure. Combining (V_1) and (V_2), we deduce that $(\nabla V(x), x) = 0$ if and only if $x = 0$ which implies that

$$|u_j(t)| \to 0 \quad \text{as } j \to \infty, \quad \text{for all } t \in \Lambda. \tag{3.4}$$

Otherwise, there exists $\beta_1 > 0$ such that $\forall\ N > 0$, there exists $j_N > N$ and $t_N \in \Lambda$ such that

$$|u_{j_N}(t_N)| \ge \beta_1. \tag{3.5}$$

It follows from (V_1) and (V_2) that there exists $\theta > 0$ such that

$$(\nabla V(x), x) \ge \theta \quad \text{for all } |x| \ge \beta_1.$$

Since $(\nabla V(u_j(t)), u_j(t)) \to 0$ as $j \to \infty$ for all $t \in \Lambda$, there exists $\eta > 0$ such that for any $j > \eta$ and $t \in \Lambda$ we have

$$(\nabla V(u_j(t)), u_j(t)) \le \frac{1}{2}\theta. \tag{3.6}$$

Let $N > \eta$ in (3.5), we can obtain

$$(\nabla V(u_{j_N}(t_N)), u_{j_N}(t_N)) \ge \theta,$$

which contradicts (3.6). Then we obtain (3.4). By Egorov's theorem, we can see that there exists $\Lambda_1 \subset \Lambda$ such that

$$|u_j(t)| \to 0 \quad \text{as } j \to \infty \quad \text{uniformly in } \Lambda_1 \tag{3.7}$$

with meas $\Lambda_1 \in (\frac{1}{4}, \frac{1}{2})$. By $V \in C^1(R^N, R)$, $h > V(0)$ and (3.7), we can deduce that there exists $l > 0$ such that

$$V(u_j(t)) \le V(0) + \varepsilon_0 \quad \text{for } j > l \text{ and } t \in \Lambda_1,$$

where $\varepsilon_0 = \frac{h-V(0)}{2} > 0$, which implies that

$$\int_0^1 h - V(u_j(t))dt = \int_{\Lambda_1} h - V(u_j(t))dt \ge \int_{\Lambda_1} h - V(0) - \varepsilon_0 dt \ge \frac{1}{4}\varepsilon_0$$

for $j > l$. By $\|\dot{u}_j\|_{L^2} \to \infty$ as $j \to \infty$ and the definition of f, we can deduce that

$$f(u_j) \to +\infty \quad \text{as } j \to \infty,$$

which contradicts (3.1). Then we get that $\|\dot{u}_j\|_{L^2}$ is bounded.

Next, we claim that $|u_j(0)|$ is still bounded. Otherwise, there is a subsequence, still denoted by $\{u_j\}$, such that $|u_j(0)| \to +\infty$ as $j \to +\infty$. Since $\|\dot{u}_j\|_{L^2}$ is bounded, by Hölder's inequality, we can deduce that

$$\min_{0\leq t\leq 1} |u_j(t)| \geq |u_j(0)| - \|\dot{u}_j\|_{L^2}^2 \to +\infty \quad \text{as } j \to \infty.$$

Then it follows from $\liminf_{|x|\to\infty} V(x) = +\infty$ that there exist $\zeta > h$ and $r > 0$ such that

$$V(x) \geq \zeta \tag{3.8}$$

for all $|x| \geq r$. By the definition of f, it follows from (3.8) that

$$\begin{aligned} f(u_j) &= \frac{1}{2}\int_0^1 |\dot{u}_j(t)|^2 dt \int_0^1 (h - V(u_j(t)))dt \\ &\leq \frac{h-\zeta}{2}\|\dot{u}_j\|_{L^2}^2 \\ &\leq 0 \quad \text{for } j \text{ large enough.} \end{aligned}$$

which contradicts (3.2). Hence $|u_j(0)|$ is bounded, which implies that $\|u_j\|$ is bounded. Then there is a weakly convergent subsequence, still denoted by $\{u_j\}$, such that $u_j \rightharpoonup u_0$ in H^1. The following proof is similar to that in [15]. Then we have $u_j \to u$ strongly in H^1. Hence f satisfies $(CPS)_C$ condition. □

Subsequently, we use Lemma 2.1 to prove that the functional f possesses at least one critical point.

Lemma 3.2. *Suppose that the conditions of Theorem 1.1 hold, then functional f possesses at least one critical point in H^1.*

Proof. We set that

$$\begin{aligned} &X_1 = R^N, \qquad X_2 = \left\{u \in W^{1,2}(R/Z, R^N), \int_0^1 u(t)dt = 0\right\}, \\ &S = \left\{u \in X_2 \,\middle|\, \left(\int_0^1 |\dot{u}(t)|^2 dt\right)^{1/2} = \rho\right\}, \\ &P = \{u(t) = u_1 + se(t),\ u_1 \in X_1,\ e \in X_2,\ \|e\| = 1,\ s \in R^+, \|u\| = (|u_1|^2 + s^2)^{1/2} = L > \rho\}, \\ &\partial Q = \{u_1 \in R^N \mid |u_1| = L\} \bigcup P. \end{aligned}$$

For all $u \in X_2$, by Poincaré–Wirtinger's inequality, we obtain that there exists a constant $C_1 > 0$ such that

$$\int_0^1 |\dot{u}(t)|^2 dt \geq C_1\|u\|^2. \tag{3.9}$$

Moreover, if $u \in X_2$, the Sobolev's inequality shows that

$$\sqrt{12}\|u\|_\infty \leq \|u\|. \tag{3.10}$$

Since $h > V(0)$, there exists $\delta > 0$ such that $h - V(u(t)) \geq \frac{h-V(0)}{2}$ for $|u| \leq \delta$. For any $u \in S \subseteq X_2$, we can choose $\left(\int_0^1 |\dot{u}(t)|^2 dt\right)^{1/2} = \rho \leq \sqrt{12C_1}\delta$, then we can deduce from (3.9) and (3.10) that $\|u\|_\infty \leq \delta$. Thus we have

$$\begin{aligned} f(u) &= \frac{1}{2}\int_0^1 |\dot{u}(t)|^2 dt \int_0^1 (h - V(u(t)))dt \\ &\geq \frac{h-V(0)}{4}\int_0^1 |\dot{u}(t)|^2 dt \end{aligned}$$

and

$$f|_S \geq \frac{h-V(0)}{4}\rho^2 > 0.$$

When $u \in \partial Q$, there are two cases needed to be discussed.

Case 1. If $u \in \{u_1 \in R^N \mid |u_1| = L\}$, it follows from $V_\infty = +\infty$ that

$$\int_0^1 h - V(u(t))dt \leq 0, \quad \text{as } L \text{ large enough,} \tag{3.11}$$

which implies that

$$f|_{\partial Q} \leq 0 \quad \text{for } L \text{ large enough.}$$

Case 2. If $u \in P$. For $\sigma > 0$, set

$$\Gamma_\sigma(u) = \{t \in [0,1] : |u(t)| \geq \sigma\|u\|\}.$$

Then there exists $\varepsilon_1 > 0$ such that

$$\text{meas}\,(\Gamma_{\varepsilon_1}(u)) \geq \varepsilon_1 \tag{3.12}$$

for all $u \in P$. Otherwise, there exists a sequence $\{u_n\}_{n\in N} \subset P$ such that

$$\text{meas}\left(\Gamma_{\frac{1}{n}}(u_n)\right) \leq \frac{1}{n}. \tag{3.13}$$

Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ for all $n \in N$. Then there exists a $v_0 \in X_1 \bigoplus \text{span}\{e\}$ such that $\|v_0\| = 1$ and $v_n \to v_0$ in $L^2([0,1], R^N)$. Then we have

$$\int_0^1 |v_n(t) - v_0(t)|^2 dt \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Furthermore, we claim that there exist constants τ_1, $\tau_2 > 0$ such that

$$\text{meas}(\Gamma_{\tau_1}(v_0)) \geq \tau_2. \tag{3.15}$$

If not, we have $\text{meas}\left(\Gamma_{\frac{1}{n}}(v_0)\right) = 0$ for all $n \in N$. Then by Sobolev's embedding theorem, we have

$$0 \leq \int_0^1 |v_0(t)|^3 dt \leq \|v_0\|_{L^\infty}\|v_0\|_{L^2}^2 \leq C_2^2\|v_0\|_{L^\infty} \leq \frac{C_2^2}{n} \to 0$$

as $n \to \infty$, for some $C_2 > 0$, which contradicts $\|v_0\| = 1$. Then (3.15) holds. By (3.13) and (3.15), we obtain

$$\begin{aligned}\text{meas}\left(\Gamma^c_{\frac{1}{n}}(v_n)\bigcap\Gamma_{\tau_1}(v_0)\right) &= \text{meas}\left(\Gamma_{\tau_1}(v_0)\setminus\left(\Gamma_{\frac{1}{n}}(v_n)\bigcap\Gamma_{\tau_1}(v_0)\right)\right)\\ &\geq \text{meas}\left(\Gamma_{\tau_1}(v_0)\right) - \text{meas}\left(\Gamma_{\frac{1}{n}}(v_n)\bigcap\Gamma_{\tau_1}(v_0)\right)\\ &\geq \tau_2 - \frac{1}{n},\end{aligned}$$

where $\Gamma^c_{\frac{1}{n}}(v_n) = [0,1]\setminus\Gamma_{\frac{1}{n}}(v_n)$. For n large enough, we can deduce that

$$|v_n(t)-v_0(t)|^2 \geq \big||v_n(t)|-|v_0(t)|\big|^2 \geq \left(\tau_1 - \frac{1}{n}\right)^2 \geq \frac{1}{4}\tau_1^2$$

for all $t \in \Gamma^c_{\frac{1}{n}}(v_n)\bigcap\Gamma_{\tau_1}(v_0)$. Consequently, for n large enough, we have

$$\begin{aligned}\int_0^1 |v_n(t)-v_0(t)|^2dt &\geq \int_{\Gamma^c_{\frac{1}{n}}(v_n)\bigcap\Gamma_{\tau_1}(v_0)} |v_n(t)-v_0(t)|^2dt\\ &\geq \frac{1}{4}\tau_1^2\left(\tau_2-\frac{1}{n}\right)\\ &\geq \frac{1}{8}\tau_1^2\tau_2\\ &> 0,\end{aligned}$$

which contradicts (3.14). Then we obtain (3.12). By the definition of $\Gamma_{\varepsilon_1}(u)$, we conclude that for any $u \in P$, we have

$$\inf_{t\in\Gamma_{\varepsilon_1}(u)} |u(t)| \geq \varepsilon_1\|u\| = \varepsilon_1 L \to +\infty \quad \text{as } L \to +\infty. \tag{3.16}$$

Since V is of C^1 class and $V_\infty = +\infty$, there exists a global minimum $V_{\min} \in R$ such that $V(x) \geq V_{\min}$ for any $x \in R^N$. It follows from (3.16) and $\liminf_{|x|\to+\infty} V(x) = +\infty$ that

$$\begin{aligned}\int_0^1 h - V(u(t))dt &= h - \int_{[0,1]\setminus\Gamma_{\varepsilon_1}(u)} V(u(t))dt - \int_{\Gamma_{\varepsilon_1}(u)} V(u(t))dt\\ &\leq h - (1-\text{meas}(\Gamma_{\varepsilon_1}(u)))V_{\min} - \int_{\Gamma_{\varepsilon_1}(u)} V(u(t))dt \to -\infty\end{aligned}$$

as $L \to +\infty$, which implies (3.11). Together with Lemma 3.1, we can deduce from Lemma 2.1 that f possesses a critical value c. Hence there exists a $u_0 \in H^1$ such that

$$c = f(u_0) \geq \frac{h-V(0)}{4}\rho^2 > 0, \qquad f'(u_0) = 0.$$

Then we finish the proof of this lemma. □

Finally, it follows from Lemma 2.2 that system (1.1)–(1.2) possesses at least one nonconstant periodic solution. Then we finish the proof of Theorem 1.1.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 11471267), the Young scholars development fund of Southwest Petroleum University (SWPU) (Grant No. 201599010116) and the Fundamental Research Funds for the Central Universities (No. XDJK2014B041).

References

[1] A. Ambrosetti, V. Coti Zelati, Closed orbits of fixed energy for singular Hamiltonian systems, *Arch. Rational Mech. Anal.* **112**(1990), 339–362. MR1077264; url

[2] A. Ambrosetti, V. Coti Zelati, Closed orbits of fixed energy for a class of N-body problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9**(1992), 187–200. MR1160848

[3] A. Ambrosetti, Critical points and nonlinear variational problems, *Mém. Soc. Math. France (N.S.)* **49**(1992), 1–139. MR1164129

[4] V. Benci, Closed geodesics for the Jacobi metric and periodic solutions of prescribed energy of natural Hamiltonian systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1**(1984), 401–412. MR779876

[5] V. Benci, P. H. Rabinowitz, Critical point theorems for indefinite functionals, *Invent. Math.* **52**(1979), 241–273. MR537061; url

[6] C. Che, X. Xue, Periodic solutions for second order Hamiltonian systems on an arbitrary energy surface, *Ann. Polon. Math.* **105**(2012), 1–12. MR2924238; url

[7] H. Gluck, W. Ziller, Existence of periodic motions of conservative systems, in: *Seminar on minimal submanifolds*, Ann. of Math. Stud., Vol. 103, Princeton Univ. Press, Princeton, NJ, 1983, pp. 65–98. MR795229

[8] K. Hayashi, Periodic solutions of classical Hamiltonian systems, *Tokyo J. Math.* **6**(1983), 473–486. MR732099; url

[9] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Vol. 74, Springer-Verlag, New York, 1989. MR982267; url

[10] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **31**(1978), 157–184. MR0467823

[11] P. H. Rabinowitz, Periodic and heteroclinic orbits for a periodic Hamiltonian system, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **6**(1989), 331–346. MR1030854

[12] P. H. Rabinowitz, Periodic solutions of a Hamiltonian systems on a prescribed energy surface, *J. Differential Equations* **33**(1979), 336–52. MR543703; url

[13] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS, Regional Conf. Ser. in Math., Vol. 65, Amer. Math. Soc., Providence, RI, 1986. MR845785

[14] E. W. C. van Groesen, Analytical mini-max methods for Hamiltonian brake orbits of prescribed energy, *J. Math. Anal. Appl.* **132**(1988), 1–12. MR942350; url

[15] S. Q. Zhang, Periodic solutions for some second order Hamiltonian systems, *Nonlinearity* **22**(2009), 2141–2150. MR2534297; url