Existence of monotonic L_φ-solutions for quadratic Volterra functional-integral equations

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Abstract. We study the quadratic integral equation in the space of Orlicz space E_φ in the most important case when φ satisfies the Δ_2-condition. Considered operators are not compact and then we use the technique of measure of noncompactness associated with the Darbo fixed point theorem to prove the existence of a monotonic, but discontinuous solution. Our present work allows to generalize both previously proved results for quadratic integral equations, as well as, that for classical equations. Due to different continuity properties of considered operators in Orlicz spaces, we distinguish different cases and we study the problem in the most important case – in such a way to cover all Lebesgue spaces L_p ($p \geq 1$).

Keywords: quadratic integral equation, monotonic solutions, Orlicz spaces, Δ_2-condition, superposition operators.

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1 Introduction

This paper is devoted to study the following quadratic functional-integral equation

$$x(t) = g(t) + G(x)(t) \cdot \int_0^t K(t,s) f(s, x(\eta(s))) \, ds, \qquad t \in [0, d]. \tag{1.1}$$

The quadratic integral equations are often applicable for instance in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, in traffic theory and in numerous branches of mathematical physics (cf. [8, 14, 20, 21]). Moreover, from the mathematical point of view, this is also an interesting problem because of lack of the possibility to use the Schauder fixed point theorem. In an important case, when G is the superposition operator we cannot expect its compactness and we will consider the compactness in measure. This is sufficient to apply the Darbo fixed point theorem.

Usually, such integral equations are investigated in the space of continuous functions $C[0,1]$ (in Banach algebras) or in the Lebesgue spaces $L_p[0,1]$ with $p \geq 1$. In particular,

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this leads to many restrictions on the growth of considered functions. Another motivation for solutions in Orlicz spaces is the result of some physical problems (see [22]) with exponential nonlinearities or with rapidly growing kernels ([29], for instance). In such a case, we cannot expect that the solutions are still continuous and it seems to be important from the applications point of view (cf. [18, 19, 22, 38]). In such a case, it is worthwhile to consider the Nemytskii superposition operator as acting on some Orlicz spaces [29], which makes this equation more applicable.

Depending on conditions for G, K and f we can try to find solutions in arbitrary Orlicz spaces. Unfortunately, this leads to several restrictions on the considered operators. When we study not so general case, we are able to weaken the assumptions making our results more applicable. Here we study the case of discontinuous solutions being regular in some sense.

Let us recall that if the space of solutions is smaller with respect to the inclusion, then we obtain their additional properties, but usually it require stronger assumptions for existence results. Both directions: as weak as possible assumptions and the "optimal" set of assumptions for a prescribed solution space seem to be interesting and worthwhile to be investigated. In the first case we usually investigate L_1-solutions, in the second L_p- or L_φ-solutions are expected. On the other hand, similar problems are investigated for non-quadratic integral equations (for both types of results). We try to obtain some new results for quadratic integral equations in such a way to extend all earlier results for these equations and simultaneously at least to cover the special case of non-quadratic equations.

In our earlier papers, some special cases were investigated. In [23] the problem is solved in the case of Banach–Orlicz algebras. It means that we have some extra properties of solutions, but under conditions stronger that in this paper. The paper [24] is devoted to studying the case of L_p-solutions, so it is not the case of algebras with respect to the pointwise multiplication. If we fix a space of solutions, then it is possible to associate some intermediate spaces in such a way to find a solution in L_p and they are either of the L_∞ type [24, Theorem 3.1] or L_q type ([24, Theorem 3.3]). In this paper we extend both ideas – we fix a space of (possible) solutions and then we indicate the intermediate spaces (cf. also Corollary 5.2).

Recall that in the non-quadratic case it was proved by Orlicz and Szufla in [35] then there are three independent cases (i.e. Δ_3, Δ' and Δ_2 conditions separately) when studying L_φ-solutions. We make efforts to extend the results for quadratic equations as well as simultaneously fully cover all the results for non-quadratic equations. The main goal of our paper is to unify the study of both problems in the considered case.

In this paper we study a particularly interesting case – the most important and widely studied in the context of classical (i.e. non-quadratic) case of Orlicz spaces L_φ for φ satisfying Δ_2-condition (cf. [2, 34, 35, 37, 39] for non-quadratic equations). In this case some additional properties of solutions are also investigated (like constant-sign solutions of classical integral equations, see [1, 2], for instance). We will discuss the monotonicity property of solutions. It is an important property of solutions considered in recent papers (see [13, 15, 26], for example). The considered class of Orlicz spaces allows us to cover the case of Lebesgue spaces L_p for $p > 1$ as a particular case.

The theorems proved by us extend, in particular, that presented in [4, 7, 15, 16] considered in the space $C(I)$ or in Banach algebras (cf. [17]). However, such a class of solutions seems to be inadequate for integral problems and leads to several restrictions on functions. We solve the problem of the existence and monotonicity properties of solutions for some important classes of functions. The key point is to control the acting and continuity conditions for considered operators, but they are depending on the choice of L_φ.

2 Basic notations

Let $\mathbb{R}$ be the field of real numbers. In the paper we will denote by I a compact interval $[0, a] \subset \mathbb{R}$. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ. Denote by $B_r(x)$ the closed ball with center at the point x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. When necessary, we will also indicate the space by using the notation $B_r(E)$. If X is a subset of E, then $\bar{X}$ and conv X denote the closure and convex hull of X, respectively.

Let $S = S(I)$ denote the set of measurable (in the Lebesgue sense) functions on I and let meas stand for the Lebesgue measure on the real line $\mathbb{R}$. Identifying the functions that are equal almost everywhere, the set S furnished with the metric $\rho(x, y) = \inf_{\epsilon > 0}[a + \text{meas}\{s : |x(s) - y(s)| \geq \epsilon\}]$ becomes a complete space. Moreover, the space S with the topology convergence in measure on ρ is a metric space, because the convergence in measure is equivalent to convergence with respect to ρ (cf. Proposition 2.14 in [40]). It is well known (Lebesgue's theorem) that convergence a.e. implies convergence in measure; the converse is true only if the measure is discrete. Nevertheless (by the Riesz theorem), each sequence which is convergent in measure admits an a.e. convergent subsequence. The compactness in such a topology we will call a "compactness in measure" and such sets have important properties when considered as subsets of some Orlicz spaces.

In order to make the paper self-contained we need to recall some basic notions and facts in the theory of Orlicz spaces.

Let M and N be complementary N-functions, i.e. $N(x) = \sup_{y \geq 0}(xy - M(x))$, where $N \colon [0, +\infty) \to [0, +\infty)$ is continuous, even and convex with $\lim_{x \to 0} \frac{N(x)}{x} = 0$, $\lim_{x \to \infty} \frac{N(x)}{x} = \infty$ and $N(x) > 0$ if $x > 0$ ($N(x) = 0 \iff x = 0$). The Orlicz class, denoted by $\mathcal{O}_P$, consists of measurable functions $x \colon I \to \mathbb{R}$ for which $\rho(x; M) = \int_I M(x(t))\, dt < \infty$. We shall denote by $L_M(I)$ the Orlicz space of all measurable functions $x \colon I \to \mathbb{R}$ for which

$$\|x\|_M = \inf_{\lambda > 0} \left\{ \int_I M\left(\frac{x(s)}{\lambda}\right) ds \leq 1 \right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Note that $E_M(I) \subseteq L_M(I) \subseteq \mathcal{O}_M(I)$. The inclusion $L_M(I) \subset L_P(I)$ holds if, and only if, there exist positive constants u_0 and a such that $P(u) \leq aM(u)$ for $u \geq u_0$.

An important property of $E_M(I)$ spaces lies in the fact that this is a class of functions from $L_M(I)$ having absolutely continuous norms.
Moreover, we have $E_M(I) = L_M(I) = \mathcal{O}_M(I)$ if M satisfies the Δ_2-condition, i.e.

(Δ_2) there exist ω, $t_0 \geq 0$ such that for $t \geq t_0$, we have $M(2t) \leq \omega M(t)$.

Let us observe, that the N-functions $M_1(u) = \frac{u^p}{p}$ and $M_2(u) = |u|^\alpha(|\ln|u|| + 1)$ for $\alpha \geq \frac{3 + \sqrt{5}}{2}$ satisfy this condition, while the function $M_3(u) = \exp|u| - |u| - 1$ does not. Moreover, the complement functions to $M_4(u) = \exp u^2 - 1$ and $M_5(u) = \exp|u| - |u| - 1$ satisfy this condition while the original functions M_4 and M_5 do not.

Sometimes, we will use a more general concept of function spaces, i.e. ideal spaces. A normed space $(X, \|\cdot\|)$ of (classes of) measurable functions $x \colon I \to U$ (U is a normed space) is called pre-ideal if for each $x \in X$ and each measurable $y \colon I \to U$ the relation $|y(s)| \leq |x(s)|$ (for almost all $s \in I$) implies $y \in X$ and $\|y\| \leq \|x\|$. If X is also complete, it is called an ideal space (see [41]). The class of Orlicz spaces stands for an important example of ideal spaces.

By $\mathcal{M}_E$ we denote the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Definition 2.1 ([12]). A mapping $\mu \colon \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(i) $\mu(X) = 0 \Rightarrow X \in \mathcal{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(iii) $\mu(\bar{X}) = \mu(\mathrm{conv}\, X) = \mu(X)$.

(iv) $\mu(\lambda X) = |\lambda| \mu(X)$, for $\lambda \in \mathbb{R}$.

(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

(vi) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An important example of such a kind of mappings is the following.

Definition 2.2 ([12]). Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as

$$\beta_H(X) = \inf\{r > 0 : \text{there exists a finite subset Y of E such that } x \subset Y + B_r\}.$$

For any $\varepsilon > 0$, let $c(X)$ be a measure of equiintegrability of the set X in $L_M(I)$ (cf. Definition 3.9 in [40] or [27, 28]):

$$c(X) = \lim_{\varepsilon \to 0} \sup_{\mathrm{meas}\, D \leq \varepsilon} \sup_{x \in X} \|x \cdot \chi_D\|_{L_M(I)},$$

where χ_D denotes the characteristic function of D.

The following theorem clarifies the connections between different coefficients in Orlicz spaces. Thus [28, Theorem 1] for the case of Orlicz spaces can be read as follows.

Proposition 2.3. *Let X be a nonempty, bounded and compact in measure subset of an Orlicz space $L_\varphi(I)$, where φ satisfies the Δ_2-condition. Then*

$$\beta_H(X) = c(X).$$

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in $L_M(I)$ iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms cf. [3], in particular $X \subset E_M(I)$).

The importance of such a kind of functions can be clarified by using the contraction property with respect to this measure instead of compactness in the Schauder fixed point theorem. Namely, we have the Darbo theorem ([12]).

Theorem 2.4. *Let Q be a nonempty, bounded, closed and convex subset of E and let $V \colon Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $k \in [0, 1)$ such that*

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty subset X of E. Then V has at least one fixed point in the set Q and the set $\mathrm{Fix}\, V$ of all fixed points of V satisfies $\mu(\mathrm{Fix}\, V) = 0$.

3 Considered operators

In this paper we propose to reduce the considered problem to the operator form. This means that the properties of operators on selected domains will form the main problem in our proofs. In particular, we will investigate many properties of operators acting on different function spaces. Let us recall some basic lemmas.

One of the most important operators studied in nonlinear functional analysis is the so-called superposition (Nemytskii) operator [6]. Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$F(x)(t) = f(t, x(t)), \qquad t \in I.$$

The operator F in such a way is called the superposition operator generated by the function f. We will be interested in the case when F acts between some Orlicz spaces.

A full discussion of necessary and sufficient conditions for continuity and boundedness of such a type of operators can be found in [6]. The following property will be useful in our proofs.

Lemma 3.1. *Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then the superposition operator F transforms measurable functions into measurable functions.*

We will utilize the fact, that Carathéodory mappings transforming measurable functions into the same space are (sequentially) continuous with respect to the topology of convergence in measure.

Lemma 3.2 ([30, Lemma 17.5] in S and [36] in $L_M(I)$)**.** *Assume that a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F maps a sequence of functions convergent in measure into a sequence of functions convergent in measure.*

In Orlicz spaces there is no automatic continuity of superposition operators like in L^p spaces, but the following lemma can be helpful in our problem (remember, that the Orlicz space L_M is ideal and if M satisfies Δ_2-condition it is also regular cf. [5, Theorem 1]):

Lemma 3.3 ([29, Theorem 17.6])**.** *Suppose the function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and*
$$|f(t, x)| \leq b(t) + kM_2^{-1}\left[M_1\left(\frac{x}{r}\right)\right], \qquad t \in I \ \text{and} \ x \in \mathbb{R},$$
where $b \in L_{M_2}$ and $r, k \geq 0$. If the N-function M_2 satisfies Δ_2-condition, then the superposition operator F generated by f acts from $B_r(E_{M_1}(I))$ into the space $L_{M_2}(I) = E_{M_2}(I)$ and is continuous.

Let us note that in a special case of functions of the form $f(t, x) = g(t)h(x)$, the superposition operator F is continuous from the space of continuous functions $C(I)$ into $L_M(I)$ even when M does not satisfies Δ_2-condition [5]. Since $E_M(I)$ is a regular part of an Orlicz space $L_M(I)$ (cf. [40, p. 72]), in the context of arbitrary Orlicz spaces, we will use the following (see also Lemma 3.3).

Lemma 3.4. *Let f be a Carathéodory function. If the superposition operator F acts from $L_{M_1}(I)$ into $E_{M_2}(I)$, then it is continuous.*

Let us introduce two more operators playing an important role in this paper, namely the linear integral operator

$$H(x) = \int_0^t K(t,s)x(s)\,ds$$

and the pointwise multiplication operator. The first one is well-known and all necessary results concerning the properties of such a kind of operators in Orlicz spaces can be found in [29, Lemma 16.3], so here we omit the details and important results will be pointed out in the proofs of our main results.

Now, we need to describe the second one. By $U(x)(t)$ we will denote the operator of the form:

$$U(x)(t) = G(x)(t) \cdot A(x)(t),$$

where $A = H \circ F$ is a Volterra–Hammerstein operator and F is a superposition operator.

Generally speaking, the product of two functions $x, y \in L_M(I)$ is not in $L_M(I)$. However, if x and y belong to some particular Orlicz spaces, then the product $x \cdot y$ belongs to a third Orlicz space. Let us note that one can find two functions belonging to Orlicz spaces: $u \in L_U(I)$ and $v \in L_V(I)$ such that the product uv does not belong to any Orlicz space (this product is not integrable). Nevertheless, to clarify the applicability of our results, we recall the following lemma.

Lemma 3.5 ([29, Lemma 13.5], [33, Theorem 10.2]). *Let φ_1, φ_2 and φ be arbitrary N-functions. The following conditions are equivalent:*

1. *For every functions $u \in L_{\varphi_1}(I)$, $w \in L_{\varphi_2}$ and $u \cdot w \in L_\varphi(I)$.*

2. *There exists a constant $k > 0$ such that for all measurable u, w on I we have $\|uw\|_\varphi \le k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.*

3. *There exists numbers $C > 0$, $u_0 \ge 0$ such that for all $s, t \ge u_0$, $\varphi\left(\frac{st}{C}\right) \le \varphi_1(s) + \varphi_2(t)$.*

4. *$\limsup_{t \to \infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.*

We are able also to remind the following simple sufficient condition for the above statements hold true.

Lemma 3.6 ([29, p. 223]). *If there exist complementary N-functions Q_1 and Q_2 such that the inequalities*

$$Q_1(\alpha u) < \varphi^{-1}[\varphi_1(u)]$$

$$Q_2(\alpha u) < \varphi^{-1}[\varphi_2(u)]$$

hold, then for every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_\varphi(I)$. If, moreover, φ satisfies the Δ_2-condition, then it is sufficient that the inequalities

$$Q_1(\alpha u) < \varphi_1[\varphi^{-1}(u)]$$

$$Q_2(\alpha u) < \varphi_2[\varphi^{-1}(u)]$$

hold.

An interesting discussion about necessary and sufficient conditions for product operators can be found in [29, 33]. A simplest case leads to well-known inequality $\|x \cdot y\|_1 \leq \|x\|_p \cdot \|y\|_q$ for conjugated p and q, i.e. $1/p + 1/q = 1$.

However it is known, that for arbitrary Orlicz space L_φ we have: $x \in L_\varphi$, $y \in L_\infty$ implies that $x \cdot y \in L_\varphi$. This fact can be useful in our theory, but for quadratic problems this leads to some restriction on an operator A (as claimed in [24]). If we try to preserve the property of arbitrariness of L_φ we are unable to formulate some natural assumptions guaranteeing the continuity of considered operators. Thus, we propose to put some restriction for Orlicz spaces covering the most applicable cases, but still allowing to prove some important properties of operators (continuity, for instance). As claimed we will consider Orlicz spaces generated by φ satisfying the Δ_2-condition.

Since the superposition operator is not compact, in general, we will consider the case when our operators are neither Lipschitz nor compact. Recall that the quadratic integral equations stand for classical examples of problems when the Schauder fixed point theorem cannot be applied. We will show that the Darbo fixed point theorem based on contraction property with respect to a measure of noncompactness is still available.

We are interested in studying the functional-integral equations, so we need to check the properties of the composition operators in Orlicz spaces $C_\tau(x(t)) = x(\tau(t))$. Although for the case of continuous solutions it is a trivial problem, we would like to emphasize the differences in the case of Orlicz spaces. The composition operator in Orlicz spaces was investigated by many authors (see [31, 25], for instance). Let us present some basic results.

Lemma 3.7 ([25, Theorem 2.2], [31, Theorem 2.1]). *Let $\tau\colon I \to I$ be a measurable mapping. Then it induces a composition operator C_τ on $L_\varphi(I)$ iff*

(A) there is a constant $K > 1$ such that $meas(\tau^{-1}(E)) \leq K \cdot meas(E)$, for all measurable sets $E \subset I$.

It is also a bounded linear operator, i.e.

(B) there exists a constant $M > 0$ independent on $x \in L_\varphi(I)$ such that $\|C_\tau(x)\|_\varphi \leq M\|x\|_\varphi$.

If, in addition, φ satisfies the Δ_2-condition for all $u > 0$, then the two conditions (A) and (B) are equivalent.

Some exact dependencies between K and M can be found in [25]. We are interested in solving some problems on a compact interval I and then the condition (A) just means nonsingularity of τ. As a consequence, we get the following

Lemma 3.8. *Let $\tau\colon I \to I$ be a measurable mapping such that there is a constant $K > 1$ with $\mathrm{meas}(\tau^{-1}(E)) \leq K \cdot \mathrm{meas}(E)$, for all measurable $E \subset I$. Then $C_\tau\colon E_\varphi(I) \to E_\varphi(I)$.*

Proof. The condition (A) is expressed in terms of τ, but the condition (B) is sufficient. Namely, $\sup_{x \in X} \|C_\tau x \cdot \chi_D\|_{L_M(I)} \leq M \cdot \sup_{x \in X} \|x \cdot \chi_D\|_{L_M(I)}$ and then $c(C_\tau(X)) \leq M \cdot c(X)$. For arbitrary bounded subset $X \subset E_\varphi(I)$ we have $c(X) = 0$ and then $c(C_\tau(X)) = 0$. Thus $C_\tau(X) \subset E_\varphi(I)$. $\qquad\square$

4 Monotonic functions

Let us recall that in metric spaces the set U_0 is compact if and only if each sequence taken from U_0 has a subsequence that converges in U_0 (i.e. sequentially compact). In particular, we need

to use this simple fact in the space S. We will try to find some monotonic solutions for the considered problem. However, in the case of discontinuous functions we define the concept of monotonicity for equivalence classes of functions equal almost everywhere. We follow some ideas from [30].

Let X be a bounded subset of measurable functions. Assume that there is a family of subsets $(\Omega_c)_{0 \leq c \leq a}$ of the interval I such that meas $\Omega_c = c$ for every $c \in I$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$. It is clear that by putting $\Omega_c = [0, c) \cup Z$ or $\Omega_c = [0, c) \setminus Z$, where Z is a set with measure zero, this family contains nonincreasing functions (possibly except for a set Z). We will call the functions from this family "a.e. nonincreasing" functions. This is the case when we choose a measurable and nonincreasing function y and all functions equal a.e. to y satisfy the above condition. This means that such a notion can be also considered in the space S. Thus we can write that elements from $L_M(I)$ belong to this class of functions.

Further, let Q_r $(r > 0)$ stand for the subset of the ball B_r consisting of all functions which are a.e. nonincreasing on I. Functions a.e. nondecreasing are defined in a similar way. It is known that such a family constitutes a set which is compact in measure in S (cf. [30, section 19.8]). We are interested if the set is still compact in measure as a subset of some subspaces of S. In general, it is not true, but we are able to prove that for the case of Orlicz spaces, we have the following.

Lemma 4.1 ([23])**.** *Let X be a bounded subset of $L_M(I)$ consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval I. Then X is compact in measure in $L_M(I)$.*

We are interested in studying if the operator G takes this set into itself. We need the following useful lemma for superposition operators.

Lemma 4.2 ([11, Lemma 4.2])**.** *Suppose the function $t \to f(t, x)$ is a.e. nondecreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \to f(t, x)$ is a.e. nondecreasing on $\mathbb{R}$ for any $t \in I$. Then the superposition operator F generated by f transforms functions being a.e. nondecreasing on I into functions having the same property.*

For an abstract operator G we will need to *assume* the above property. Note that the superposition operator takes the bounded sets compact in measure into the sets with the same property. Namely, we have (see Lemma 3.3 for an acting condition below) the following proposition.

Proposition 4.3. *Let M be an N-function satisfying the Δ_2-condition. Assume that a function $f \colon I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and the function $t \to f(t, x)$ is a.e. nondecreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \to f(t, x)$ is a.e. nondecreasing on $\mathbb{R}$ for any $t \in I$. Assume moreover, that $F \colon L_M(I) \to L_M(I)$. Then $F(V)$ is compact in measure for arbitrary bounded and compact in measure subset V of $L_M(I)$.*

Proof. Let V be a bounded and compact in measure subset of $L_M(I)$. By our assumption $L_M(I) = E_M(I)$ and then $F(V) \subset L_M(I) = E_M(I)$. As a subset of S the set $F(V)$ is compact in measure (cf. [9]). The topology of convergence in measure is metrizable, so the compactness of this set is equivalent with its sequential compactness.

Take an arbitrary sequence $(y_n) \subset F(V) \subset E_M(I)$, then we get a sequence (x_n) in V such that $y_n = F(x_n)$. Since $(x_n) \subset V$ (as follows from Lemma 3.2), the operator F transforms this sequence into the sequence convergent in measure. Thus $(y_n) = (F(x_n))$ is compact in measure, so is $F(V)$. Recall that an important property of Orlicz spaces $L_M(I)$ is that of being

continuously embedded into the space S. This means that every convergent sequence in $L_M(I)$ is also convergent in S. Finally $F(V)$ is compact in measure in $L_M(I)$. $\qquad\square$

5 Main results

Let $J = [0, d]$ and let B denote the operator associated with the right-hand side of the equation (1.1) which takes the form $x = Bx$, where $B(x) = g + U(x)$ and $U(x)(t) = G(x)(t) \cdot \int_0^t K(t, s) f(s, x(\eta(s)))\, ds$.

Thus $B = g + G \cdot A = g + G \cdot H \circ F \circ C_\eta$ and then equation (1.1) is of the form

$$x = g + G \cdot H \circ F \circ C_\eta. \tag{5.1}$$

We will try to choose the domains of operators defined above in such a way to obtain the existence of solutions in a prescribed Orlicz space $L_\varphi(J)$. We formulate some conditions allowing us to consider strongly nonlinear operators. Since we consider an abstract form for the operator G we need to describe its properties. Related results for the superposition operator are described in the first part of our paper.

Let us discuss the choice of domains and ranges for considered operators. Let $G\colon L_\varphi(J) \to L_{\varphi_1}(J)$ and $F\colon L_\varphi(J) \to L_N(J)$. Recall that C_η does not change the target space for F. We need also to describe some assumptions on "intermediate" spaces being the images of $L_\varphi(J)$ for G and F and the range for H (i.e. $L_{\varphi_2}(J)$). This approach is based on a classical (non-quadratic) case as in [35, 37, 39] and seems to be important in view of optimality of assumptions. Recall that for quadratic problems all the spaces considered in previous papers were Banach algebras (most of all $C(I)$, some Banach–Orlicz algebras in [23]) or $L_\infty(J)$ in place of $L_{\varphi_2}(J)$ (cf. [24]).

Theorem 5.1. *Assume that $\varphi, \varphi_1, \varphi_2$ are N-functions and that M and N are complementary N-functions. Moreover, put the following set of assumptions:*

(N1) *(the choice of spaces) there exists a constant $k_1 > 0$ such that for every $u \in L_{\varphi_1}(J)$ and $w \in L_{\varphi_2}(J)$ we have $\|uw\|_\varphi \le k_1 \|u\|_{\varphi_1} \|w\|_{\varphi_2}$,*

(C1) *$g \in E_\varphi(J)$ is nondecreasing a.e. on J,*

(C2) *$f\colon J \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and $f(t, x)$ is assumed to be nondecreasing with respect to both variable t and x separately,*

(C3) *(the growth condition) $|f(t, x)| \le b(t) + R(|x|)$ for $t \in J$ and $x \in \mathbb{R}$, where $b \in E_N(J)$ and R is non-negative, nondecreasing, continuous function defined on $\mathbb{R}^+$,*

(C4) *(relationships between the choice of spaces and growth conditions) φ is an N-function and the function N satisfies the Δ_2-condition and suppose that there exist $\gamma \ge 0$ such that*

$$R(u) \le \gamma N^{-1}(\varphi(u)) \quad \text{for } u \ge 0.$$

(G1) *(the operator G) $G\colon L_\varphi(J) \to L_{\varphi_1}(J)$ takes continuously $E_\varphi(J)$ into $E_{\varphi_1}(J)$ and there exists a positive function $G_0 \in L_\varphi(J)$ such that for $t \in J$ $|G(x)(t)| \le G_0(t)\|x\|_\varphi$ and that G takes the set of all a.e. nondecreasing functions into itself. Moreover, assume that for any $x \in E_\varphi(J)$ we get $G(x) \in E_{\varphi_1}(J)$.*

(K1) *(the kernel K) $s \to K(t, s) \in L_M(J)$ for a.e. $t \in J$ and $p(t) = \|K(t, \cdot)\|_M \in E_{\varphi_2}(J)$. Moreover, assume the linear operator H with the kernel K maps the set of all a.e. nondecreasing functions into itself.*

(K2) (the functional dependence) $\eta: J \to J$ *is an increasing absolutely continuous function and there are positive constants* Z *such that* $\eta' \geq Z$ *a.e. on* $(0, d)$.

Assume that for some $q > 0$ *the following inequality holds true on an interval* $J_0 = [0, a] \subset J = [0, d]$

$$\int_{J_0} \varphi \left(|g(t)| + G_0(t) \cdot q \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z}(q-1) \right) \right] \right) dt \leq q.$$

If moreover, $\left(k_1 \cdot \|G_0\|_{\varphi_1} \cdot \|p\|_{\varphi_2} \cdot [\|b\|_N + \gamma + \frac{\gamma}{Z}(q-1)] \right) < 1$ *for* q *satisfying the above inequality, then there exists an a.e. nondecreasing solution* $x \in E_\varphi(J_0)$ *of* (1.1) *on* $J_0 \subset J$.

Proof. We need to divide the proof into a few steps.

I. We need to prove that the operator B is well-defined from $L_\varphi(J)$ into itself and continuous on a constructed domain. Note that the assumption (K2) allows us to use Lemma 3.7 and $x(\eta(\cdot)) \in L_\varphi(J)$. Since η is strictly increasing, it is nonsingular and for all measurable subsets $X \subset J$ with $\operatorname{meas}(\eta^{-1}(X)) \leq d \operatorname{meas}(X)$.

First of all observe that the assumptions (C2)–(C4) and Lemma 3.3 implies that the superposition operator F is continuous mappings from $E_\varphi(J)$ into $E_N(J)$. In this case we will prove that U is a continuous mapping from the unit ball in $E_\varphi(J)$ into the space $E_\varphi(J)$.

Let us recall that $x \in E_\varphi(J)$ iff for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\chi T\|_\varphi < \varepsilon$ for every measurable subset T of J with the Lebesgue measure smaller that δ (i.e. x has absolutely continuous norm).

Assumption (K1) and Theorem 16.3 and Lemma 16.3 (with $M_1 = N$, $M_2 = \varphi_2$ and $N_1 = M$) of [29] imply that the operator H maps $E_N(J)$ into $E_{\varphi_2}(J)$ and is continuous. Then A is a continuous mapping from $B_1(E_\varphi(J))$ into $E_{\varphi_2}(J)$. By our assumption (G1) the operator G is continuous from $B_1(E_\varphi(J))$ into $E_{\varphi_1}(J)$ and then by (N1) and Proposition 3.5 the operator U has the same property and then U is a continuous mapping from $B_1(E_\varphi(J))$ into the space $E_\varphi(J)$. Finally, by the assumption (C1) B maps $B_1(E_\varphi(J))$ into $E_\varphi(J)$ continuously.

Since the composition operator C_η is linear and continuous, we are able to repeat the above consideration for $x(\eta(\cdot))$ instead of $x(\cdot)$ (cf. Lemma 3.8).

II. We will prove the boundedness of the operator B, namely we will construct an invariant set $V \subset B_1(E_\varphi(J))$ for B in $L_\varphi(J)$.

Denote by Q the set of all positive numbers q for which

$$\int_{J_0} \varphi \left(|g(t)| + G_0(t) \cdot q \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z}(q-1) \right) \right] \right) dt \leq q.$$

By r we will denote $\sup Q$. Recall that $J_0 = [0, a] \subset J$. Clearly, for a sufficiently small number a this set is nonempty due to our assumption. It should be noted that the assumption (N1) implies that $p \in L_{\varphi_2}(J)$ implies $p \in L_\varphi(J)$ (by putting $u = \operatorname{const.}$ and $w = p$). The same holds true for the function G_0 with values in $L_{\varphi_1}(J)$.

Let V denote the closure of the set $\{x \in E_\varphi(J_0) : \int_0^a \varphi(|x(s)|) \, ds \leq r - 1\}$. Clearly V is not a ball in $E_\varphi(J_0)$, but $V \subset B_r(E_\varphi(J_0))$ (cf. [29, p. 222]). Notice that $\overline{V}$ is a bounded closed and convex subset of $E_\varphi(J_0)$.

Take an arbitrary $x \in V$. By using [29, Theorem 10.5 with $k = 1$], we obtain that for any $t \in J_0$

$$\begin{aligned} \|R(|x(\eta)\chi_{[0,t]}|)\|_N &\leq \gamma \left\| N^{-1} \left(\varphi \left(|x(\eta)\chi_{[0,t]}| \right) \right) \right\|_N \\ &\leq \left(\gamma + \gamma \int_0^t \varphi \left(|x(\eta(s))| \right) ds \right) \end{aligned}$$

$$\leq \left(\gamma + \gamma \int_0^t \varphi\left(|x(\eta(s))| \right) \frac{\eta'(s)}{Z} \, ds \right)$$

$$= \left(\gamma + \frac{\gamma}{Z} \int_{\eta(0)}^{\eta(t)} \varphi\left(|x(u)| \right) du \right)$$

$$\leq \left(\gamma + \frac{\gamma}{Z} \int_{J_0} \varphi\left(|x(u)| \right) du \right) \tag{5.2}$$

and then by the Hölder inequality and our assumptions we get

$$|A(x)(t)| \leq |p(t)| \left[\|b\|_N + \|R(|x\chi_{[0,t]}|)\|_N \right].$$

Thus for any measurable subset T of J. For arbitrary $x \in V$ and $t \in J_0$

$$
\begin{aligned}
|B(x)(t)| &\leq |g(t)| + |Ux(t)| \\
&\leq |g(t)| + |G(x)(t)| \cdot |A(x)(t)| \\
&\leq |g(t)| + |G(x)(t)| \cdot \int_0^t |K(t,s)| \cdot |f(s, x(\eta(s)))| \, ds \\
&\leq |g(t)| + |G(x)(t)| \cdot |p(t)| \left[\|b\|_N + \|R(|x\chi_{[0,t]}|)\|_N \right] \\
&\leq |g(t)| + G_0(t) \cdot \|x\|_\varphi \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z} \int_{J_0} \varphi(|x(u)|) \, du \right) \right] \\
&\leq |g(t)| + G_0(t) \cdot \left(1 + \int_J \varphi(|x(t)|) \, dt \right) \cdot |p(t)| \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z} \int_{J_0} \varphi(|x(u)|) \, du \right) \right] \\
&\leq |g(t)| + G_0(t) \cdot r \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z}(r-1) \right) \right].
\end{aligned}
$$

Finally

$$\int_{J_0} \varphi(B(x)(t)) \, dt \leq \int_{J_0} \varphi \left(|g(t)| + G_0(t) \cdot r \cdot |p(t)| \cdot \left[\|b\|_N + \left(\gamma + \frac{\gamma}{Z}(r-1) \right) \right] \right) dt.$$

By the definition of r we get $\int_{J_0} \varphi(B(x)(t)) \, dt \leq r$ and then $B(V) \subset V$. Consequently $B(\overline{V}) \subset \overline{B(V)} \subset \overline{V} = V$.

Then $B \colon V \to V$. Moreover, B is continuous on $V \subset B_r(E_\varphi(J_0))$ (see the part I of the proof).

III. Now, let a subset Q_r of V consist of a.e. nondecreasing functions. We need to investigate the properties of this set. We follow the idea from [10]. As claimed in [10], this set is nonempty, bounded (by r) and convex.

As a subset of $L_\varphi(J_0)$ it is a (sequentially) closed set. Indeed, let (y_n) be a sequence of elements in Q_r convergent in $L_\varphi(J_0)$ to y. Then the sequence is also convergent in measure and as a consequence of the Vitali convergence theorem for Orlicz spaces and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on J_0 (cf. [35]). Moreover, y is still nondecreasing a.e. on J_0 which means that $y \in Q_r$ and so the set Q_r is closed. Now, in view of Lemma 4.1 the set Q_r is compact in measure.

IV. We check the continuity and monotonicity properties of B in Q_r, so $U \colon Q_r \to Q_r$. The first property is essentially depending on the choice of φ and we need to use its properties.

We begin by demonstrating that B preserve the monotonicity of functions. Take $x \in Q_r$, then x and $x(\eta)$ are a.e. nondecreasing on J and consequently $F(x(\eta))$ is also of the same type in virtue of the assumption (C2) and Lemma 4.2. Further, $A(x) = H \circ F(x(\eta))$ is a.e.

nondecreasing on J_0 thanks for the assumption (K1). Since the pointwise product of a.e. monotonic functions is still of the same type and by (G1), the operator U is a.e. nondecreasing on J_0.

Moreover, the assumption (C1) permits us to deduce that $Bx(t) = g(t) + U(x)(t)$ is also a.e. nondecreasing on J_0. This fact, together with the assertion that $B\colon V \to V$ gives us that B is also a self-mapping of the set Q_r. From the above considerations it follows that B maps continuously Q_r into Q_r.

V. Now we will prove that B is a contraction with respect to a Hausdorff measure of noncompactness and we use the Darbo fixed point theorem to find a solution in Q_r.

Assume that X is a nonempty subset of Q_r and let the fixed constant $\varepsilon > 0$ be arbitrary. Since $L_{\varphi_1}(J)$ is an ideal space, our assumption (G1) implies that $\|G(x)\|_{\varphi_1} \le \|G_0\|_{\varphi_1}\|x\|_\varphi$. Then for an arbitrary $x \in X$ and for a set $D \subset J_0$, $\operatorname{meas} D \le \varepsilon$ we obtain

$$
\begin{aligned}
\|B(x) \cdot \chi_D\|_\varphi &\le \|g\chi_D\|_\varphi + \|U(x) \cdot \chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + \|G(x) \cdot A(x)\chi_D\|_\varphi \\
&\le \|g\chi_D\|_\varphi + k_1\|G(x)\chi_D\|_{\varphi_1} \cdot \|A(x)\|_{\varphi_2} \\
&= \|g\chi_D\|_\varphi + k_1\|G(x)\chi_D\|_{\varphi_1} \left\| \int_{J_0} K(\cdot, s)f(s, x(\eta(s))) \, ds \right\|_{\varphi_2} \\
&\le \|g\chi_D\|_\varphi + k_1\|G_0\|_{\varphi_1}\|x\chi_D\|_\varphi \left\| \int_{J_0} |K(\cdot, s)|(b(s) + R(|x(\eta(s))|)) \, ds \right\|_{\varphi_2} \\
&\le \|g\chi_D\|_\varphi + k_1\|G_0\|_{\varphi_1}\|x\chi_D\|_\varphi\|p\|_{\varphi_2}[\|b\|_N + \|R(|x(\eta)|)\|_N] \\
&\le \|g\chi_D\|_\varphi + k_1\|G_0\|_{\varphi_1}\|x\chi_D\|_\varphi\|p\|_{\varphi_2}[\|b\|_N + \gamma\|N^{-1}(\varphi(|x(\eta)|))\|_N] \\
&\le \|g\chi_D\|_\varphi + k_1\|G_0\|_{\varphi_1}\|x\chi_D\|_\varphi\|p\|_{\varphi_2}\left[\|b\|_N + \gamma\left(1 + \int_{J_0} \varphi(|x(\eta(s))|) \, dt\right)\right] \\
&\le \|g\chi_D\|_\varphi + k_1\|G_0\|_{\varphi_1}\|x\chi_D\|_\varphi\|p\|_{\varphi_2}\left[\|b\|_N + \gamma + \frac{\gamma}{Z}(r-1)\right].
\end{aligned}
$$

Hence, taking into account that $g \in E_\varphi$

$$
\lim_{\varepsilon \to 0}\left\{ \sup_{\operatorname{meas} D \le \varepsilon} [\{\|g\chi_D\|_\varphi\}] \right\} = 0.
$$

Thus by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $\operatorname{meas} D \le \varepsilon$ we get

$$
c(B(X)) \le k_1 \cdot \|G_0\|_{\varphi_1} \cdot \|p\|_{\varphi_2} \cdot \left[\|b\|_N + \gamma + \frac{\gamma}{Z}(r-1)\right] \cdot c(X).
$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of a regular part E_φ of L_φ, we can use Proposition 2.3 and get

$$
\beta_H(B(X)) \le k_1 \cdot \|G_0\|_{\varphi_1} \cdot \|p\|_{\varphi_2} \cdot \left[\|b\|_N + \gamma + \frac{\gamma}{Z}(r-1)\right] \cdot \beta_H(X).
$$

The inequality obtained above together with the properties of the operator B and the set Q_r established before and the inequality

$$
k_1 \cdot \|G_0\|_\varphi \cdot \|p\|_{\varphi_2} \cdot \left[\|b\|_N + \gamma + \frac{\gamma}{Z}(r-1)\right] < 1
$$

allow us to apply the Darbo fixed point theorem 2.4, which completes the proof. $\qquad\square$

The Lebesgue spaces L_p can be treated as Orlicz spaces L_{M_p} for $M_p(x) = \frac{x^p}{p}$, $p > 1$. It is clear that in this space M_p satisfies Δ_2-condition and, therefore, the case of L_p-solutions is covered by our result. Thus, let us present a special case for this class of solutions, which will form still more general result than the earlier ones. To simplify the set of assumptions let us restrict to the case of the superposition operator G.

Assume that $p > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Denote by q the value $\min(p_1, p_2)$ and by y the value $\max(p_1, p_2)$. This implies, in particular, that $q \leq 2p$.

Let us consider an interesting case, when the operator $G(x)(t) = h(t, x(t))$. Then equation (1.1) takes the form

$$x(t) = g(t) + h(t, x(t)) \cdot \int_0^t K(t, s) f(s, x(\eta(s))) \, ds, \qquad t \in [0, d]. \tag{5.3}$$

We shall treat equation (5.3) under the following (less abstract) set of assumptions presented below.

(i) $g \in L_p(J)$ is nondecreasing a.e. on J.

(ii) Assume that functions $f, h \colon J \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and there are positive constants b_i ($i = 1, 2$) and positive functions $a_i \in L_q$ ($i = 1, 2$) such that

$$|h(t, x)| \leq a_1(t) + b_1|x|^{\frac{p}{q}} \quad \text{and} \quad |f(t, x)| \leq a_2(t) + b_2|x|^{\frac{p}{q}}$$

for all $t \in J$ and $x \in \mathbb{R}$. Moreover, the functions f, h are assumed to be nondecreasing with respect to both variables t and x separately.

(iii) Let the function K be measurable in (t, s). Moreover, assume that the function $t \to \|K(t, \cdot)\|_{q'} \in L_y(J)$, where $\frac{1}{q} + \frac{1}{q'} = 1$ and that the linear integral operator with the kernel K maps the set of all a.e. nondecreasing functions into itself.

(iv) $\eta \colon J \to J$ is increasing absolutely continuous function and there is a positive constant Z such that $\eta' \geq Z$ a.e. on $(0, d)$.

In addition, let r be a positive number such that

$$\|g\|_p + \left[\|a_1\|_q + b_1 \cdot r^{\frac{p}{q}} \right] \cdot \|K_0\| \cdot \left[\|a_2\|_q + \frac{b_2}{Z^{\frac{1}{q}}} \cdot r^{\frac{p}{q}} \right] \leq r,$$

where $\|K_0\| = \|t \to \|K(t, \cdot)\|_{q'}\|_y$.

Corollary 5.2. *Let the assumptions (i)–(iv) be satisfied. If $b_1 b_2 \|K_0\| r^{\frac{2p}{q} - 1} < Z^{\frac{1}{q}}$, then equation (5.3) has at least one $L_p(J)$-solution a.e nondecreasing on some subinterval $[0, a] \subset J$.*

For the case of classical Volterra equations (non-quadratic) in L_p treated as a special case of Orlicz spaces see also [1], but in the case of completely continuous integral operator (not applicable in the case of quadratic equations).

Remark 5.3. Let us note that if the operator G takes the simple form $G(x)(t) = q(t) \cdot x(t)$, then our assumptions referred to quadratic integral equations

$$x(t) = g(t) + q(t) \cdot x(t) \cdot \int_0^t K(t, s) f(s, x(s)) \, ds, \qquad t \in [0, d]. \tag{5.4}$$

Since we are motivated by some study on quadratic integral equations, this is of our particular interest. Note, that a full description for acting and continuity conditions for $G(x) = a(t)x(t)$ can be found in [29, Theorem 18.2] (cf. assumption (G1)).

As a particular case we solve the following form of the equation (1.1):

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \psi(s) (\log\left(1 + |x(\sqrt{s})|\right)) \, ds. \tag{5.5}$$

The equation (5.5) is the quadratic integral equation of generalized Chandrasekhar type (cf. [7, 14, 21] for the classical case of this equation and its applications). It arose in connection with scattering through a homogeneous semi-infinite plane atmosphere (see [20, 21]) and discontinuous solutions for this equation can be used as good description of non-homogeneous atmosphere (cf. [4]).

In this case we have $K(t,s) = \frac{t}{t+s} \psi(s)$ and then for some sufficiently well-chosen functions ψ our result applies ($\psi(s) = (1/2) \cdot e^{-s}$, for instance).

More examples of interesting equations can be found in recent papers of Banaś and co-authors [14, 16], in the book [29, Chapter III, sec. 16] or in [32, 34].

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