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Asymptotic character of non-oscillatory solutions to functional differential systems

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Abstract. In this paper the behaviour of solutions to systems of three functional differential equations is investigated. We are interested in the acquirement of conditions which ensure that certain of four possible non-oscillatory types holds. A sub-linear as well as a super-linear system is studied.

Keywords: neutral differential equation, system of functional differential equation, non-oscillatory solution, asymptotic properties of solutions.

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1 Introduction

We consider the system of three functional differential equations with deviating arguments

$$\begin{aligned} \big[y_1(t) + a(t)y_1(g(t))\big]' &= p_1(t)y_2(t) \\ y_2'(t) &= p_2(t)f_2(y_3(h_3(t))) \\ y_3'(t) &= f_3(t, y_1(h_1(t))), \qquad t \ge t_0 \ge 0, \end{aligned} \tag{1.1}$$

where the following assumptions are given:

(a) $a \in C([t_0,\infty),[0,\infty))$;

(b) $g \in C([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty} g(t) = \infty$;

(c) $p_i \in C([t_0,\infty),[0,\infty))$, $p_i(t) \not\equiv 0$ on any interval $[T,\infty) \subset [t_0,\infty)$, $\int_{t_0}^{\infty} p_i(t)\,dt < \infty$ for $i = 1,2$;

(d) $h_i \in C([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty} h_i(t) = \infty$, $i = 1,3$ and $h_3(t) \le t$ for $t \ge t_0$;

(e) $f_2 \in C(\mathbb{R},\mathbb{R})$, $|f_2(u)| \le K|u|^\beta$ for $u \in \mathbb{R}$, constants K, β satisfy $K > 0$, $0 < \beta \le 1$;

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(f) $f_3 \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$, $|f_3(t,v)| \le \omega(t,|v|)$ for $(t,v) \in [t_0,\infty) \times \mathbb{R}$, $\omega \in C([t_0,\infty) \times \mathbb{R}_0^+, \mathbb{R}_0^+)$, where $\mathbb{R}_0^+$ is the set of all nonnegative real numbers and $\omega(t,z)$ is non-decreasing with respect to z for any $t \in [t_0,\infty)$.

Functional differential equations with deviating arguments and their systems have been studied by many authors. The asymptotic behaviour of solutions to functional differential equations and systems is studied for example in [3, 10, 11] and to equations of neutral type in [4, 5, 7]. The classification of non-oscillatory solutions to systems of neutral differential equations is given in [12–14] and to systems of neutral dynamic equations on time scales in [1]. For nonlinear equations some comparison theorems were introduced in [9] and existence of positive solutions is investigated in [2, 6].

This paper brings a generalization to results for asymptotic properties presented in [8] for systems of three equations if one of the equations is of neutral type. The system (1.1) can be transformed neither to third-order neutral differential equation nor to differential equation of neutral type with quasi-derivatives.

A function $y = (y_1, y_2, y_3)$ is a solution to (1.1) if

1. there exists $t_1 \ge t_0$ such that y is continuous for
$$t \ge \min\left\{ t_1, \inf_{t\ge t_1} h_1(t), \inf_{t\ge t_1} h_3(t), \inf_{t\ge t_1} g(t) \right\};$$
2. functions $y_i(t)$, $i = 2,3$ and $z_1(t)$, which is defined as $z_1(t) = y_1(t) + a(t)y_1(g(t))$ for $t \ge t_1$, are continuously differentiable on $[t_1,\infty)$;
3. y satisfies (1.1) on $[t_1,\infty)$.

The set of solutions y to (1.1) that satisfy the condition
$$\sup_{t\ge T} \left\{ \sum_{i=1}^{3} |y_i(t)| \right\} > 0 \quad \text{for any } T \ge t_1$$
is denoted as W. A solution $y \in W$ is considered to be non-oscillatory if there exists a $T_y \ge t_1$ such that every component is different from zero for $t \ge T_y$. Otherwise a solution $y \in W$ is said to be oscillatory.

2 Main results

In this section we establish conditions under which one of four possible types of asymptotic properties holds.

The system (1.1) is super-linear [sub-linear] if $\frac{\omega(t,z)}{z}$, $z > 0$ is non-decreasing [non-increasing] with respect to z for any $t \ge t_0$.

We define the functions h_*, r_* as
$$h_*(t) = \min\{h_1(t), t\}, \qquad r_*(t) = \inf_{s\ge t} h_*(s).$$

For $t \ge t_0$ the following integrals are defined
$$P_i(t) = \int_t^\infty p_i(s)\, ds, \qquad i = 1,2;$$
$$Q(t) = \int_t^\infty p_1(s)P_2(s)\, ds.$$

It is obvious that the inequality $Q(t) \le P_1(t)P_2(t)$ holds for $t \ge t_0$. Functions $P_1(t), P_2(t)$ and $Q(t)$ are non-increasing and $\lim_{t\to\infty} P_i(t) = 0,\ i = 1,2$ and $\lim_{t\to\infty} Q(t) = 0$.

Theorem 2.1. *We suppose that (1.1) is either*

(A) a super-linear one and

$$\int_{t_0}^{\infty} P_1(h_*(s))P_2(h_*(s))\omega(s,c)\,ds < \infty \tag{2.1}$$

for all $c > 0$;

or

(B) the system (1.1) is sub-linear and

$$\int_{t_0}^{\infty} \frac{P_i(h_*(s))\omega(s, cP_1(h_*(s))P_2(h_*(s)))\,ds}{P_i(h_1(s))} < \infty \tag{2.2}$$

for $i = 1,2$ and all $c > 0$,

then for any non-oscillatory $y \in W$, one of the following cases (I)–(IV) holds:

(I)

$$\lim_{t\to\infty} |z_1(t)| = \lim_{t\to\infty} |y_2(t)| = \lim_{t\to\infty} |y_3(t)| = \infty;$$

(II) there exists a nonzero constant α_1 that

$$\lim_{t\to\infty} z_1(t) = \alpha_1, \qquad \lim_{t\to\infty} y_2(t)P_1(t) = \lim_{t\to\infty} y_3(t)Q(t) = 0;$$

(III) there exists a nonzero constant α_2 that

$$\lim_{t\to\infty} \frac{-z_1(t)}{P_1(t)} = \lim_{t\to\infty} y_2(t) = \alpha_2, \qquad \lim_{t\to\infty} y_3(t)P_2(t) = 0;$$

(IV) there exists a constant α_3 that

$$\lim_{t\to\infty} y_3(t) = \alpha_3, \qquad \lim_{t\to\infty} \frac{z_1(t)}{Q(t)} = \lim_{t\to\infty} \frac{-y_2(t)}{P_2(t)} = f_2(\alpha_3).$$

Proof. Let $y \in W$ be a non-oscillatory solution to (1.1). Let $t_2 \ge t_1$, such that for $t \ge t_2$ the functions $y_1(t), y_1(g(t)), y_2(t), y_3(t), z_1(t)$ are of a constant sign and the inequality (2.3) holds. From the definition of $z_1(t)$, the first equation of (1.1), (a) and (c) we conclude that $z_1(t)$ is monotonous and fulfills

$$|z_1(t)| \ge |y_1(t)| \quad \text{for } t \ge t_2. \tag{2.3}$$

Case (A) We suppose that (1.1) is super-linear and (2.1) holds. Let $T \ge t_2$. We consider T in such a way that $r_*(T) \ge t_2$ and for $P_i(T)$ hold

$$P_i(T) \le 1, \qquad i = 1,2. \tag{2.4}$$

By integrating the first equations of (1.1) from T to t we have

$$|z_1(t)| \le |z_1(T)| + \int_T^t p_1(x_1)|y_2(x_1)|\,dx_1, \qquad t \ge T, \tag{2.5}$$

$$|y_2(t)| \le |y_2(T)| + \int_T^t p_2(x_2)|f_2(y_3(h_3(x_2)))|\,dx_2, \qquad t \ge T \tag{2.6}$$

and a combination of (2.5) and (2.6) yields

$$
\begin{aligned}
|z_1(t)| &\le |z_1(T)| + |y_2(T)| \int_T^t p_1(x_1)\,dx_1 \\
&\quad + \int_T^t p_1(x_1) \int_T^{x_1} p_2(x_2)|f_2(y_3(h_3(x_2)))|\,dx_2\,dx_1, \qquad t \ge T.
\end{aligned}
\tag{2.7}
$$

By integrating the third equation of (1.1) from T to t with using (f) and (2.3) we obtain

$$
|y_3(t)| \le |y_3(T)| + \int_T^t \omega(s, |z_1(h_1(s))|)\,ds, \qquad t \ge T. \tag{2.8}
$$

Considering (d), (e), (2.8) and Taylor's theorem we have

$$
\begin{aligned}
|f_2(y_3(h_3(t)))| &\le K|y_3(h_3(t))|^\beta \le K\left(|y_3(T)| + \int_T^{h_3(t)} \omega(s, |z_1(h_1(s))|)\,ds\right)^\beta \\
&\le M + N \int_T^t \omega(s, |z_1(h_1(s))|)\,ds, \qquad t \ge \overline{T} > T,
\end{aligned}
\tag{2.9}
$$

where $M = K|y_3(T)|^\beta$ and $N = K\beta|y_3(T)|^{\beta-1}$ and $\overline{T}$ fulfill a condition, that $h_3(t) \ge T$ for $t \ge \overline{T}$.

From (2.7) and (2.9) for $z_1(t)$ the following inequality holds

$$
\begin{aligned}
|z_1(t)| &\le |z_1(T)| + |y_2(T)| \int_T^t p_1(x_1)\,dx_1 \\
&\quad + M \int_T^t p_1(x_1) \int_T^{x_1} p_2(x_2)\,dx_2\,dx_1 \\
&\quad + N \int_T^t p_1(x_1) \int_T^{x_1} p_2(x_2) \int_T^{x_2} \omega(s, |z_1(h_1(s))|)\,ds\,dx_2\,dx_1, \qquad t \ge \overline{T}.
\end{aligned}
\tag{2.10}
$$

From (2.6) and (2.9) by changing of the order of integration we have

$$
|y_2(t)| \le |y_2(T)| + M \int_T^t p_2(x_2)\,dx_2 + N \int_T^t \omega(s, |z_1(h_1(s))|) P_2(s)\,ds, \qquad t \ge \overline{T}. \tag{2.11}
$$

Since there exists $\lim_{t\to\infty} |z_1(t)|$, there are two possibilities: either $\lim_{t\to\infty} |z_1(t)| = \infty$ or $\lim_{t\to\infty} |z_1(t)| < \infty$. Let us assume the first possibility, thus

$$
\lim_{t\to\infty} |z_1(t)| = \infty. \tag{2.12}
$$

We will prove by contrapositive that the case (I) stands.

Let $\limsup_{t\to\infty} |y_2(t)| < \infty$, then from (2.5) we have a contradiction to (2.12).

Let $\limsup_{t\to\infty} |y_3(t)| < \infty$. Then from (2.7) and (e) we obtain a contradiction to (2.12).

Hence if $\lim_{t\to\infty} |z_1(t)| = \infty$, then $\limsup_{t\to\infty} |y_3(t)| = \limsup_{t\to\infty} |y_2(t)| = \infty$ hold and the case (I) stands.

Let $\lim_{t\to\infty} |z_1(t)| < \infty$. The relation (2.1) implies that the function $P_1(t)P_2(t)\omega(t,c)$ is integrable on $[T,\infty)$ for any constant $c > 0$. We will prove that also the function $p_1(t)y_2(t)$ is integrable on $[T,\infty)$. Because of (2.11), by changing of the order of integration we have

$$
\begin{aligned}
\int_{\overline{T}}^\infty p_1(t)|y_2(t)|\,dt &\le |y_2(T)| \int_{\overline{T}}^\infty p_1(t)\,dt + M \int_{\overline{T}}^\infty p_1(t) \int_T^t p_2(x_2)\,dx_2\,dt \\
&\quad + N \int_T^\infty P_1(s)P_2(s)\omega(s, |z_1(h_1(s))|)\,ds.
\end{aligned}
$$

The first equation of (1.1) gives

$$z_1(t) = \alpha_1 - \int_t^\infty p_1(s) y_2(s)\, ds, \qquad t \geq T, \tag{2.13}$$

where $\alpha_1 = z_1(T) + \int_T^\infty p_1(s) y_2(s)\, ds$, $\alpha_1 \in \mathbb{R}$.

The relation (2.13) ensures that $\lim_{t\to\infty} z_1(t) = \alpha_1$. From (2.11) for $t \geq \overline{T}$ we have

$$\begin{aligned} P_1(t)|y_2(t)| \leq P_1(t) \left[|y_2(T)| + MP_2(T) + N \int_T^{t_1} \omega(s, |z_1(h_1(s))|) P_2(s)\, ds \right] \\ + N \int_{t_1}^t \omega(s, |z_1(h_1(s))|) P_1(s) P_2(s)\, ds. \end{aligned}$$

From (2.8) for $t \geq T$ we have

$$\begin{aligned} Q(t)|y_3(t)| \leq Q(t) \left[|y_3(T)| + \int_T^{t_1} \omega(s, |z_1(h_1(s))|)\, ds \right] \\ + \int_{t_1}^t \omega(s, |z_1(h_1(s))|) P_1(s) P_2(s)\, ds. \end{aligned}$$

The formulae $P_1(t)|y_2(t)|$ and $Q(t)|y_3(t)|$ can be made arbitrarily small by choosing t_1 sufficiently large and then letting t tend to ∞. Consequently

$$\lim_{t\to\infty} P_1(t) y_2(t) = 0 = \lim_{t\to\infty} Q(t) y_3(t)$$

and if $\alpha_1 \neq 0$ the case (II) holds.

Let $\alpha_1 = 0$. The super-linearity of (1.1) and (2.1), (2.4) imply that the functions

$$P_1(h_1(t)) P_2(t) \omega(t, 1), \qquad P_1(t) P_2(t) \omega(t, 1), \qquad P_2(t) \omega(t, c P_1(h_1(t)))$$

are integrable on $[T, \infty)$ for any $c > 0$.

We can choose $T_1 \geq \overline{T}$ in such a way that not only $T_1^* = r_*(T_1) \geq \overline{T}$ but also

$$|z_1(h_1(t))| \leq 1, \qquad t \geq T_1, \tag{2.14}$$

$$N \int_{T_1}^\infty P_1(h_1(s)) P_2(s) \omega(s, 1)\, ds \leq \frac{1}{3}, \tag{2.15}$$

$$N \int_{T_1}^\infty P_1(s) P_2(s) \omega(s, 1))\, ds \leq \frac{1}{3}. \tag{2.16}$$

Combining (2.11), (2.13) and by changing of the order of integration we get

$$\begin{aligned} |z_1(t)| \leq P_1(t) \left[|y_2(T)| + MP_2(T) + N \int_T^t P_2(s) \omega(s, |z_1(h_1(s))|)\, ds \right] \\ + N \int_t^\infty P_1(s) P_2(s) \omega(s, |z_1(h_1(s))|) ds, \quad t \geq \overline{T}. \end{aligned} \tag{2.17}$$

The inequality above may be rearranged to the form

$$\begin{aligned} \frac{|z_1(t)|}{P_1(t)} \leq K_1 + N \int_{T_1}^t P_2(s) \omega(s, |z_1(h_1(s))|)\, ds \\ + \frac{N}{P_1(t)} \int_t^\infty P_1(s) P_2(s) \omega(s, |z_1(h_1(s))|)\, ds, \qquad t \geq T_1, \end{aligned} \tag{2.18}$$

where
$$K_1 \geq |y_2(T)| + MP_2(T) + N\int_T^{T_1} P_2(s)\omega(s,|z_1(h_1(s))|)\,ds$$
is a positive constant.

Denote for $t \geq T_1$ two types of sets
$$I_t^1 = \{s \in [T_1,\infty),\ h_1(s) \leq t\} \quad \text{and} \quad J_t^1 = \{s \in [T_1,\infty),\ h_1(s) > t\}.$$
Then for $s \in I_t^1$ or $s \in J_t^1$ respectively hold
$$\frac{|z_1(h_1(s))|}{P_1(h_1(s))} \leq \sup_{T_1^* \leq \sigma \leq t} \frac{|z_1(\sigma)|}{P_1(\sigma)} \quad \text{for } s \in I_t^1$$
and since $|z_1(t)|$ is a non-increasing function on $[t_2,\infty)$, we obtain
$$|z_1(h_1(s))| \leq |z_1(t)| \quad \text{for } s \in J_t^1.$$
The super-linearity of (1.1) implies
$$\omega(s,ab) \leq a\omega(s,b) \quad \text{for } 0 < a \leq 1,\ b > 0. \tag{2.19}$$
The inequality (2.18) may be modified based on (2.14)–(2.16) to
$$\begin{aligned}
\frac{|z_1(t)|}{P_1(t)} &\leq K_1 + N\sup_{T_1^* \leq s \leq t} \frac{|z_1(s)|}{P_1(s)} \Bigg[\int_{I_t^1 \cap [T_1,t)} P_1(h_1(s))P_2(s)\omega(s,1)\,ds \\
&\qquad\qquad + \frac{1}{P_1(t)}\int_{I_t^1 \cap [t,\infty)} P_1(h_1(s))P_1(s)P_2(s)\omega(s,1)\,ds\Bigg] \\
&\quad + N\frac{|z_1(t)|}{P_1(t)}\left[P_1(t)\int_{J_t^1 \cap [T_1,t)} P_2(s)\omega(s,1)\,ds + \int_{J_t^1 \cap [t,\infty)} P_1(s)P_2(s)\omega(s,1)\,ds\right] \\
&\leq K_1 + \sup_{T_1^* \leq s \leq t} \frac{|z_1(s)|}{P_1(s)} N\int_{T_1}^{\infty} P_1(h_1(s))P_2(s)\omega(s,1)\,ds \\
&\quad + \frac{|z_1(t)|}{P_1(t)} N\int_{T_1}^{\infty} P_1(s)P_2(s)\omega(s,1)\,ds \\
&\leq K_1 + \frac{1}{3}\sup_{T_1^* \leq s \leq t} \frac{|z_1(s)|}{P_1(s)} + \frac{1}{3}\frac{|z_1(t)|}{P_1(t)} \quad \text{for } t \geq T_1,
\end{aligned}$$
and
$$\frac{|z_1(t)|}{P_1(t)} \leq \overline{K}_1 + \frac{1}{2}\sup_{T_1 \leq s \leq t} \frac{|z_1(s)|}{P_1(s)} \quad \text{for } t \geq T_1,$$
where
$$\overline{K}_1 = \frac{3}{2}K_1 + \frac{1}{2}\sup_{T_1^* \leq s \leq T_1} \frac{|z_1(s)|}{P_1(s)}.$$
Thus we have the estimation
$$\frac{|z_1(t)|}{P_1(t)} \leq \sup_{T_1 \leq s \leq t} \frac{|z_1(s)|}{P_1(s)} \leq 2\overline{K}_1 \quad \text{for } t \geq T_1.$$
The inequality above leads to
$$|z_1(h_1(t))| \leq K_1^* P_1(h_1(t)) \quad \text{for } t \geq T, \tag{2.20}$$

where K_1^* is an appropriate positive constant.

The function $p_2(t)f_2(y_3(h_3(t)))$ is integrable on $[T,\infty)$ which means that from (2.9) and (2.20) by changing of the order of integration we have

$$\int_{\overline{T}}^{\infty} p_2(t)|f_2(y_3(h_3(t)))|\,dt \le M\int_{\overline{T}}^{\infty} p_2(t)\,dt + N\int_{T}^{\infty} P_2(s)\omega(s,K_1^*P_1(h_1(s)))\,ds.$$

Then for $y_2(t)$ the equality

$$y_2(t) = \alpha_2 - \int_t^{\infty} p_2(s)f_2(y_3(h_3(s)))\,ds, \qquad t \ge T, \tag{2.21}$$

holds, where

$$\alpha_2 = y_2(T) + \int_T^{\infty} p_2(s)f_2(y_3(h_3(s)))\,ds.$$

Since from (2.21) we have that $\lim_{t\to\infty} y_2(t) = \alpha_2$, thus (2.13) (where $\alpha_1 = 0$) by L'Hôpital's rule implies

$$\lim_{t\to\infty}\frac{z_1(t)}{P_1(t)} = -\alpha_2.$$

The condition (f), and (2.8), (2.20) give

$$\begin{aligned} P_2(t)|y_3(t)| \le P_2(t)&\left[|y_3(T)| + \int_T^{t_1}\omega(s,K_1^*P_1(h_1(s)))\,ds\right] \\ &+ \int_{t_1}^{t} P_2(s)\omega(s,K_1^*P_1(h_1(s)))\,ds, \qquad t \ge T. \end{aligned}$$

The formula $P_2(t)|y_3(t)|$ can be made arbitrarily small by choosing t_1 sufficiently large and then letting t tend to ∞. Consequently $\lim_{t\to\infty} P_2(t)|y_3(t)| = 0$. If $\alpha_2 \ne 0$ the case (III) comes into being.

Let $\alpha_1 = \alpha_2 = 0$.

The super-linearity of (1.1), (2.1) and (2.4) imply that the functions $P_2(h_1(t))\omega(t,cP_1(h_1(t)))$, $P_2(t)\omega(t,cP_1(h_1(t)))$ and $\omega(t,cP_1(h_1(t))P_2(h_1(t)))$ are integrable on the interval $[T,\infty)$ for any constant $c > 0$.

We choose T_2 in such a manner that $T_2^* = r_*(T_2) \ge \overline{T}$ and moreover,

$$\frac{|z_1(t)|}{P_1(t)} \le 1, \qquad t \ge T_2, \tag{2.22}$$

$$N\int_{T_2}^{\infty} P_2(h_1(s))\omega(s,P_1(h_1(s)))\,ds \le \frac{1}{3}, \tag{2.23}$$

$$N\int_{T_2}^{\infty} P_2(s)\omega(s,P_1(h_1(s)))\,ds \le \frac{1}{3} \tag{2.24}$$

are fulfilled.

From (2.13) (with $\alpha_1 = 0$), (2.21) (with $\alpha_2 = 0$) and (2.9) by changing of the order of integration we have

$$\begin{aligned} |z_1(t)| \le P_1(t)P_2(t)&\left[M + N\int_T^t \omega(s,|z_1(h_1(s))|)\,ds\right] \\ &+ NP_1(t)\int_t^{\infty} P_2(s)\omega(s,|z_1(h_1(s))|)\,ds, \qquad t \ge \overline{T}. \end{aligned}$$

The inequality above may be rearranged to

$$\begin{aligned}\frac{|z_1(t)|}{P_1(t)} \le P_2(t)\left[M + N\int_T^t \omega(s, |z_1(h_1(s))|)\,ds\right] \\ + N\int_t^\infty P_2(s)\omega(s, |z_1(h_1(s))|)\,ds, \qquad t \ge \overline{T}.\end{aligned} \tag{2.25}$$

We define a function $u(t)$ in the following way $u(t) = \sup_{s\ge t} \frac{|z_1(s)|}{P_1(s)}$. It is evident, that $u(t)$ is non-increasing and $\lim_{t\to\infty} u(t) = 0$. Since the right-hand side of (2.25) is non-increasing with respect to t we have

$$\begin{aligned}\frac{u(t)}{P_2(t)} \le K_2 + N\int_{T_2}^t \omega(s, |z_1(h_1(s))|)\,ds \\ + \frac{N}{P_2(t)}\int_t^\infty P_2(s)\omega(s, |z_1(h_1(s))|)\,ds, \qquad t \ge T_2,\end{aligned} \tag{2.26}$$

where $K_2 \ge M + N\int_T^{T_2} \omega(s, |z_1(h_1(s))|)\,ds$ is a positive constant.

Denote for $t \ge T_2$

$$I_t^2 = \{s \in [T_2, \infty);\ h_1(s) \le t\} \quad \text{and} \quad J_t^2 = \{s \in [T_2, \infty);\ h_1(s) > t\}.$$

Then we have

$$\frac{u(h_1(s))}{P_2(h_1(s))} \le \sup_{T_2^* \le \sigma \le t} \frac{u(\sigma)}{P_2(\sigma)} \quad \text{for } s \in I_t^2$$

and

$$u(h_1(s)) \le u(t) \quad \text{for } s \in J_t^2.$$

The super-linearity of system given by (2.19) implies that we may rearrange (2.26) on the basis of (2.22)–(2.24) to

$$\begin{aligned}\frac{u(t)}{P_2(t)} &\le K_2 + N\sup_{T_2^*\le s\le t}\frac{u(s)}{P_2(s)}\left[\int_{I_t^2\cap[T_2,t)} P_2(h_1(s))\omega(s, P_1(h_1(s)))\,ds\right. \\ &\quad \left. + \frac{1}{P_2(t)}\int_{I_t^2\cap[t,\infty)} P_2(s)P_2(h_1(s))\omega(s, P_1(h_1(s)))\,ds\right] \\ &\quad + N\frac{u(t)}{P_2(t)}\left[\int_{J_t^2\cap[T_2,t)} P_2(s)\omega(s, P_1(h_1(s)))\,ds + \int_{J_t^2\cap[t,\infty)} P_2(s)\omega(s, P_1(h_1(s)))\,ds\right] \\ &\le K_2 + N\sup_{T_2^*\le s\le t}\frac{u(s)}{P_2(s)}\int_{T_2}^\infty P_2(h_1(s))\omega(s, P_1(h_1(s)))\,ds \\ &\quad + \frac{u(t)}{P_2(t)}N\int_{T_2}^\infty P_2(s)\omega(s, P_1(h_1(s)))\,ds \\ &\le K_2 + \frac{1}{3}\sup_{T_2^*\le s\le t}\frac{u(s)}{P_2(s)} + \frac{1}{3}\frac{u(t)}{P_2(t)}, \qquad t \ge T_2\end{aligned}$$

and we have

$$\frac{u(t)}{P_2(t)} \le \overline{K}_2 + \frac{1}{2}\sup_{T_2\le s\le t}\frac{u(s)}{P_2(s)} \quad \text{for } t \ge T_2,$$

where

$$\overline{K}_2 = \frac{3}{2}K_2 + \frac{1}{2}\sup_{T_2^*\le s\le T_2}\frac{u(s)}{P_2(s)}.$$

The initial estimation can be refined

$$\frac{|z_1(t)|}{P_1(t)P_2(t)} \le \frac{u(t)}{P_2(t)} \le \sup_{T_2\le s\le t} \frac{u(s)}{P_2(s)} \le 2\overline{K}_2 \quad \text{for } t \ge T_2.$$

The inequality above gives

$$|z_1(h_1(t))| \le K_2^* P_1(h_1(t))P_2(h_1(t)) \quad \text{for } t \ge T, \tag{2.27}$$

where K_2^* is an adequate positive constant.

Since the function $f_3(t, y_1(h_1(t)))$ is integrable on $[T,\infty)$ because of (2.3), (2.27) and (f) we get

$$\int_T^\infty |f_3(t, y_1(h_1(t)))|\,dt \le \int_T^\infty \omega(t, |z_1(h_1(t))|)\,dt$$
$$\le \int_T^\infty \omega(t, K_2^* P_1(h_1(t))P_2(h_1(t)))\,dt.$$

Integrating the third equation of (1.1) we gain

$$y_3(t) = \alpha_3 - \int_t^\infty f_3(s, y_1(h_1(s)))\,ds, \qquad t \ge T, \tag{2.28}$$

where $\alpha_3 = y_3(T) + \int_T^\infty f_3(s, y_1(h_1(s)))\,ds$.

The relation (2.28) shows that $\lim_{t\to\infty} y_3(t) = \alpha_3$ and from (2.13) and (2.21) we obtain (by L'Hôpital's rule)

$$\lim_{t\to\infty} \frac{z_1(t)}{Q(t)} = \lim_{t\to\infty} \frac{\int_t^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2) f_2(y_3(h_3(x_2)))\,dx_2\,dx_1}{\int_t^\infty p_1(x_1)\int_{x_1}^\infty p_2(x_2)\,dx_2\,dx_1} = f_2(\alpha_3),$$
$$\lim_{t\to\infty} \frac{y_2(t)}{P_2(t)} = \lim_{t\to\infty} \frac{-\int_t^\infty p_2(s) f_2(y_3(h_3(s)))\,ds}{\int_t^\infty p_2(s)\,ds} = -f_2(\alpha_3).$$

The case (IV) holds. The proof of case (A) of Theorem 2.1 is completed.

Case (B)

We suppose that (1.1) is sub-linear and (2.2) holds. This implies that the function $P_1(t)P_2(t)\omega(t,c)$ is integrable on $[T,\infty)$.

The cases (I) and (II) we prove similarly to the previous case (A). Let $\alpha_1 = 0$. The relation (2.2) and the sub-linearity of (1.1) imply that the functions $P_2(t)\omega(t, cP_1(h_1(t)))$ and

$$\frac{P_1(t)P_2(t)\omega(t, P_1(h_1(t)))}{P_1(h_1(t))}$$

are integrable on $[T,\infty)$ for any $c > 0$.

We will prove that the function $\frac{|z_1(t)|}{P_1(t)}$ is bounded on $[T,\infty)$. For the sake of contradiction we estimate T_3, T_4 and T_5 in such a manner that $\overline{T} < T_3 < T_4 < T_5$ where $T_3^* = r_*(T_3) \ge \overline{T}$ and moreover we have

$$|z_1(T_3^*)| \ge P_1(T_3^*), \tag{2.29}$$

$$\sup_{T_3^*\le s\le t} \frac{|z_1(s)|}{P_1(s)} = \sup_{T_4\le s\le t} \frac{|z_1(s)|}{P_1(s)}, \qquad t \ge T_4, \tag{2.30}$$

$$N\int_{T_4}^\infty P_2(s)\omega(s, P_1(h_1(s)))\,ds \le \frac{1}{4}, \tag{2.31}$$

$$N\int_{T_4}^{\infty}\frac{P_1(s)P_2(s)\omega(s,P_1(h_1(s)))\,ds}{P_1(h_1(s))}\le\frac{1}{4}, \tag{2.32}$$

$$|y_2(T)|+MP_2(T)+N\int_{T}^{T_4}P_2(s)\omega(s,|z_1(h_1(s))|)\,ds\le\frac{|z_1(T_5)|}{4P_1(T_5)}. \tag{2.33}$$

We rearrange the inequality (2.17) to the form

$$\begin{aligned}\frac{|z_1(t)|}{P_1(t)}\le{}&|y_2(T)|+MP_2(T)+N\int_{T}^{T_4}P_2(s)\omega(s,|z_1(h_1(s))|)\,ds\\&+N\int_{T_4}^{t}P_2(s)\omega(s,|z_1(h_1(s))|)\,ds+\frac{N}{P_1(t)}\int_t^{\infty}P_1(s)P_2(s)\omega(s,|z_1(h_1(s))|)\,ds\end{aligned} \tag{2.34}$$

for $t\ge T_4$.

We define v_1 as follows

$$v_1(t)=\sup_{T_3^*\le s\le t}\frac{|z_1(s)|}{P_1(s)},\qquad t\ge T_3^*.$$

The function $v_1(t)$ is non-decreasing, $\lim_{t\to\infty}v_1(t)=\infty$ and $v_1(T_3^*)\ge 1$. It is obvious that the right-hand side of (2.34) is nondecreasing with respect to t. Since (1.1) is sub-linear for ω we have

$$\omega(s,ab)\le a\omega(s,b)\quad\text{for } a\ge 1,\ b>0. \tag{2.35}$$

We may convert the inequality (2.34) to

$$\begin{aligned}P_1(t)v_1(t)\le{}&\frac{|z_1(T_5)|P_1(t)}{4P_1(T_5)}+NP_1(t)\int_{T_4}^{t}P_1(s)v_1(h_1(s))\omega(s,P_1(h_1(s)))\,ds\\&+N\int_t^{\infty}P_1(s)P_2(s)v_1(h_1(s))\omega(s,P_1(h_1(s)))\,ds,\qquad t\ge T_5\end{aligned}$$

and since

$$\frac{|z_1(T_5)|}{P_1(T_5)}\le v_1(t),\qquad t\ge T_5$$

we have

$$\begin{aligned}\frac{3}{4}P_1(t)v_1(t)\le{}&NP_1(t)\int_{T_4}^{t}P_2(s)v_1(h_1(s))\omega(s,P_1(h_1(s)))\,ds\\&+N\int_t^{\infty}P_1(s)P_2(s)v_1(h_1(s))\omega(s,P_1(h_1(s)))\,ds,\qquad t\ge T_5.\end{aligned} \tag{2.36}$$

Denote for $t\ge T_3$

$$\widetilde{I}_t^1=\{s\in[T_3,\infty),\ h_1(s)\le t\}\quad\text{and}\quad\widetilde{J}_t^1=\{s\in[T_3,\infty),\ h_1(s)>t\}.$$

It follows that

$$v_1(h_1(s))\le v_1(t)\quad\text{for } s\in\widetilde{I}_t^1$$

and

$$P_1(h_1(s))v_1(h_1(s))\le\sup_{\sigma\ge t}(P_1(\sigma)v_1(\sigma))\quad\text{for } s\in\widetilde{J}_t^1.$$

It is obvious that $0<\sup_{\sigma\ge t}(P_1(\sigma)v_1(\sigma))<\infty$. From (2.36), (2.31) and (2.32) we have

$$
\begin{aligned}
\frac{3}{4}P_1(t)v_1(t) \le NP_1(t)v_1(t) &\left[\int_{\tilde{J}_t^1 \cap [T_4,t)} P_1(s)\omega(s, P_1(h_1(s)))\,ds \right.\\
&\qquad \left. + \frac{1}{P_1(t)} \int_{\tilde{J}_t^1 \cap [t,\infty)} P_1(s)P_2(s)\omega(s, P_1(h_1(s)))\,ds\right]\\
&+ N \sup_{s\ge t}\left(P_1(s)v_1(s)\right) \left[P_1(t)\int_{\tilde{J}_t^1 \cap [T_4,t)} \frac{P_2(s)\omega(s,P_1(h_1(s)))\,ds}{P_1(h_1(s))}\right.\\
&\qquad \left. + \int_{\tilde{J}_t^1 \cap [t,\infty)} \frac{P_1(s)P_2(s)\omega(s,P_1(h_1(s)))\,ds}{P_1(h_1(s))}\right]\\
\le P_1(t)v_1(t)N &\int_{T_4}^{\infty} P_2(s)\omega(s, P_1(h_1(s)))\,ds\\
&+ \sup_{s\ge t}\left(P_1(s)v_1(s)\right) N \int_{T_4}^{\infty} \frac{P_1(s)P_2(s)\omega(s,P_1(h_1(s)))\,ds}{P_1(h_1(s))}\\
\le \frac{1}{4}P_1(t)v_1(t) &+ \frac{1}{4}\sup_{s\ge t}(P_1(s)v_1(s)), \qquad t \ge T_5.
\end{aligned}
$$

Since it is evident that $0 < \sup_{s\ge t}(P_1(s)v_1(s)) < \infty$, it implies

$$
P_1(t)v_1(t) \le \frac{1}{2}\sup_{s\ge t}\left(P_1(s)v_1(s)\right), \qquad t \ge T_5
$$

and there is the contradiction.

The function $\frac{|z_1(t)|}{P_1(t)}$ is bounded on $[T,\infty)$ and (2.20) holds. We will prove analogically that (2.27) holds. In the following we continue similarly to the case of the super-linear system, which completes the proof. $\square$

Theorem 2.1 is a generalization of Theorem 2.1 in [8].

Corollary 2.2. *If the assumptions of Theorem 2.1 are fulfilled, $y(t) \in W$ is a solution and $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} y_i(t) = 0,\ i = 2,3$, then*

$$
\lim_{t\to\infty}\frac{y_2(t)}{P_2(t)} = \lim_{t\to\infty}\frac{z_1(t)}{P_2(t)} = \lim_{t\to\infty}\frac{y_1(t)}{P_2(t)} = 0.
$$

Example 2.3. We consider (1.1) as follows

$$
\begin{aligned}
\left[y_1(t) + \frac{1}{6}y_1\left(\frac{3t}{2}\right)\right]' &= \mathrm{e}^{-t}y_2(t)\\
y_2'(t) &= \mathrm{e}^{-t}y_3\left(\frac{t}{2}\right)\\
y_3'(t) &= -\left(48\mathrm{e}^{-2t} + 40\mathrm{e}^{-6t}\right)y_1\left(\frac{t}{2}\right), \qquad t \ge 0,
\end{aligned}
\tag{2.37}
$$

where $p_1(t) = p_2(t) = \mathrm{e}^{-t}$, $f_2(t) = t$, $f_3(t,v) = -(48\mathrm{e}^{-2t} + 40\mathrm{e}^{-6t})\cdot v$, $a(t) = \frac{1}{6}$, $h_1(t) = \frac{t}{2}$, $h_3(t) = \frac{t}{2}$, $g(t) = \frac{3t}{2}$, $\omega(t,v) = (48\mathrm{e}^{-2t} + 40\mathrm{e}^{-6t})\cdot v$.

The system (2.37) is super-linear as well as sub-linear and for $t \ge 0$ has a non-oscillatory solution with components

$$
y_1(t) = \mathrm{e}^{-4t}, \qquad y_2(t) = -4\mathrm{e}^{-3t} - \mathrm{e}^{-5t}, \qquad y_3(t) = 12\mathrm{e}^{-4t} + 5\mathrm{e}^{-8t}.
$$

All assumptions of Theorem 2.1 are satisfied, moreover, $P_1(t) = \mathrm{e}^{-t}$, $P_2(t) = \mathrm{e}^{-t}$ and $Q(t) = \frac{\mathrm{e}^{-2t}}{2}$.

Thus

$$\lim_{t\to\infty} y_3(t) = 0, \qquad \lim_{t\to\infty} \frac{y_2(t)}{P_2(t)} = 0, \qquad \lim_{t\to\infty} \frac{z_1(t)}{P_2(t)} = 0,$$

meaning that the case (IV) stands.

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