Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **61**, 1–17; doi: 10.14232/ejqtde.2015.1.61 http://www.math.u-szeged.hu/ejqtde/

Multiple solutions to elliptic equations on $\mathbb{R}^N$ with combined nonlinearities

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Received 10 October 2014, appeared 7 October 2015
Communicated by Petru Jebelean

Abstract. In this paper, we are concerned with the multiplicity of nontrivial radial solutions for the following elliptic equation

$$\begin{cases} -\Delta u + V(x)u = -\lambda Q(x)|u|^{q-2}u + Q(x)f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to +\infty, \end{cases} \tag{P$_\lambda$}$$

where $1 < q < 2$, $\lambda \in \mathbb{R}^+$, $N \geq 3$, V and Q are radial positive functions, which can be vanishing or coercive at infinity, f is asymptotically linear or superlinear at infinity.

Keywords: weighted Sobolev embedding, sublinear, asymptotically linear, superlinear, critical point theory, variation methods.

2010 Mathematics Subject Classification: 35J10, 35J65, 58E05.

1 Introduction and main results

In this paper, we deal with the multiplicity of nontrivial radial solutions for the following elliptic equation

$$\begin{cases} -\Delta u + V(x)u = -\lambda Q(x)|u|^{q-2}u + Q(x)f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to +\infty, \end{cases} \tag{P$_\lambda$}$$

where $1 < q < 2$, $\lambda \in \mathbb{R}^+$, $N \geq 3$, V and Q are radial positive functions, which can be vanishing or coercive at infinity.

When Ω is a smooth bounded domain in $\mathbb{R}^N$, the problem

$$\begin{cases} -\Delta u = \pm\lambda |u|^{q-2}u + f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P$'_\pm$)$_\lambda$}$$

where $1 < q < 2$, $\lambda > 0$, $N \geq 3$, has been widely studied in the literature. It plays a central role in modern mathematical sciences, in the theory of heat conduction in electrically

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conductive materials, in the study of non-Newtonian fluids. However, it is not possible to give here a complete bibliography. Here we just list some representative results. In the case f is superlinear in u near infinity, problem $(P'_+)_\lambda$ is the famous concave-convex problem, after the celebrated works [2,6], this kind of problem has been drawn much attention. In the case f is linear in u, in [12], at least two nonnegative solutions have been got for a more general question

$$\begin{cases} -\Delta u = h(x)u^q + f(x,u), & x \in \Omega, \\ 0 \le u \in H_0^1(\Omega), & 0 < q < 1, \end{cases}$$

where the function $h \in L^\infty(\Omega)$ satisfies some additional conditions. For problem $(P'_-)_\lambda$, in the special case: $f(u) = au + |u|^p$, where $2 < p < \frac{2N}{N-2}$, one nonnegative solution for any $a \in \mathbb{R}$ and $\lambda > 0$ was found in [15] via the mountain pass theorem. Several papers have also been devoted to the study of nonlinearities with indefinite sign, for example [9,20] and the references therein.

When $\Omega = \mathbb{R}^N$, there are a large number of papers devoted to the following equation,

$$-\Delta u + V(x)u = f(x,u), \quad \text{with } u \in W^{1,2}(\mathbb{R}^N). \tag{Q}$$

So far, in almost all the results concerning equation (Q), the nonlinear function f is assumed to be globally superlinear, that is to say, $\lim_{|u|\to 0} \frac{f(x,u)}{u} = 0$ and there exists $\theta > 2$ such that $0 < \theta F(x,u) \le uf(x,u)$ for all $(x,u) \in \mathbb{R}^N \times (\mathbb{R}\setminus\{0\})$, where $F(x,u) = \int_0^u f(x,t)dt$. The case in which $V(x) \to +\infty$, $|x| \to \infty$ and f is globally superlinear was firstly studied by Rabinowitz in [16]. The assumptions in [16] ensure that the associated functional of the equation satisfies the Palais–Smale condition, this fact was observed in [4, 5] where the results in [16] were generalized. For a radially symmetric Schrödinger equation with an asymptotically linear term, one radial solution has been obtained in [17,25] by Stuart and Zhou and their results were generalized to more general situations in [8,10,11,13].

Since the class Sobolev embedding is $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $p \in (2, \frac{2N}{N-2})$, we cannot study the sublinear problems in $W^{1,2}(\mathbb{R}^N)$ via variational methods directly. In order to overcome this obstacle, a regular way is to add some restrictions on potentials V and Q. For example in [14], the authors obtained the existence of infinitely many nodal solutions for problem (Q), where $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \ge 1$, $\int_{\mathbb{R}^N} \frac{1}{V(x)} dx < +\infty$, the nonlinearity f is symmetric in the sense of being odd in u, and may involve a combination of concave and convex terms. There are also some other results about the concave and convex problem on $\mathbb{R}^N$, such as [7,21,23,24] and the references therein. However, to the best of our knowledge there is few result about problem $(P)_\lambda$ with both sublinear terms and asymptotically linear terms.

Recently, in [18], the authors established a weighted Sobolev type embedding of radially symmetric functions which provides a basic tool to study quasilinear elliptic equations with sublinear nonlinearities. Motivated by the works of [18], we consider $(P)_\lambda$ with more general potentials and nonlinearities. In this paper, we assume that

(V) $V \in C\big(\mathbb{R}^N, (0,+\infty)\big)$ is radially symmetric and there exists $a_1 \in \mathbb{R}$ such that

$$\liminf_{|x|\to+\infty} \frac{V(x)}{|x|^{a_1}} > 0;$$

(Q) $Q \in C\big(\mathbb{R}^N, (0,+\infty)\big)$ is radially symmetric and there exists $a_2 \in \mathbb{R}$ such that

$$\limsup_{|x|\to+\infty} \frac{Q(x)}{|x|^{a_2}} < \infty.$$

It is clear that the indexes a_1 and a_2 describe the behaviors of V and Q near infinity. On a_1, a_2, we assume that

(A_1) $a_2 \geq \dfrac{2(N-1)+a_1}{2} - N, \quad \dfrac{N-2}{2} - N \leq a_2 \leq -2;$

(A_2) $a_2 < \dfrac{2(N-1)+a_1}{4} - N, \quad \dfrac{N-2}{2} - N \leq a_2 \leq -2\,;$

(A_3) $a_1 \leq -2, \quad \dfrac{N-2}{2} - N < a_2 < \dfrac{2(N-1)+a_1}{2} - N;$

(A_4) $a_2 \leq \dfrac{N-2}{2} - N, \quad \dfrac{2(N-1)+a_1}{4} - N \leq a_2 < \dfrac{2(N-1)+a_1}{2} - N;$

(A_5) $a_1 \geq -2, \quad \dfrac{2(N-1)+a_1}{4} - N \leq a_2 < \dfrac{2(N-1)+a_1}{2} - N.$

According to the indexes a_1, a_2, we define the bottom index 2_*,

$$2_* = \begin{cases} \dfrac{2(a_2+N)}{N-2}, & \text{if } (a_1,a_2) \in A_i,\ i = 1,2,3; \\ \dfrac{4(a_2+N)}{2(N-1)+a_1}, & \text{if } (a_1,a_2) \in A_i,\ i = 4,5. \end{cases}$$

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) := \left\{ u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is radial} \right\}.$$

Denote by $D_r^{1,2}(\mathbb{R}^N)$ the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Define

$$W_r^{1,2}(\mathbb{R}^N; V) := \left\{ u \in D_r^{1,2}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx < \infty \right\},$$

which is a Hilbert space [1,19] equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)|u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Let

$$L^p(\mathbb{R}^N; Q) := \left\{ u : \mathbb{R}^N \mapsto \mathbb{R} \;\middle|\; u \text{ is Lebesgue measurabe}, \int_{\mathbb{R}^N} Q(x)|u(x)|^p dx < \infty \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{L^p(\mathbb{R}^N;Q)} = \left(\int_{\mathbb{R}^N} Q(x)|u(x)|^p dx \right)^{\frac{1}{p}}.$$

Following [18, Theorem 1.2], under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, it holds that the embedding $W_r^{1,2}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; Q)$ is compact for $p \in (2_*, \frac{2N}{N-2})$. We remark that the index $2_* < 2$ by (A_i), $i = 1, \dots, 5$, so it is possible to study $(\mathrm{P})_\lambda$ with sublinear nonlinearities. We make the following assumptions on f:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) there exists a positive constant C, such that $|f(u)| \leq C(1 + |u|^{p-1})$, $2 < p < 2^* := \frac{2N}{N-2}$;

(f_3) there exist one small positive constant r_0 and another positive constant C', such that $|f(u)| \leq C'|u|$, for $|u| \leq r_0$;

(f_4) $\lim\limits_{|u| \to \infty} \dfrac{2F(u)}{|u|^2} = b$.

Since under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, it holds that the embedding $W_r^{1,2}(\mathbb{R}^N; V) \hookrightarrow L^2(\mathbb{R}^N; Q)$ is compact, the eigenvalue problem

$$\begin{cases} -\Delta u + V(x)u = \mu Q(x)u, & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{P$_\mu$}$$

has the eigenvalue sequence

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \to +\infty.$$

Similar to the eigenvalue problem in bounded domain, $\mu_1 > 0$ is simple, isolated and has an associated eigenfunction ϕ_1 which is positive in $\mathbb{R}^N$.

Our main results are the following.

Theorem 1.1. *Under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, if f satisfies (f_1), (f_3) and (f_4) with $\mu_1 < b$, $(\mathrm{P})_\lambda$ has at least two nontrivial solutions.*

Theorem 1.2. *Let us assume that conditions (V), (Q) and (A_i), $i = 1, \dots, 5$ hold, and that f satisfies (f_1) and (f_4) with $\mu_{k+1} < b < +\infty$ for some $k \in \mathbb{N}$, moreover,*

$(f_3)'$ $F(u) \geq \dfrac{\mu_m}{2} u^2$, $u \in \mathbb{R}$, $\limsup\limits_{u \to 0} \dfrac{2F(u)}{u^2} < \mu_{m+1}$, *for some* $m \in \mathbb{N}$, $m \leq k$;

(f_5) $\lim\limits_{|u| \to \infty} H(u) = +\infty$, *where* $H(u) = \dfrac{1}{2} f(u)u - F(u) \geq 0$, $u \in \mathbb{R}$, $F(u) = \displaystyle\int_0^u f(s)ds$.

Then, there exists $\lambda^* > 0$, such that for $\lambda \in (0, \lambda^*)$, $(\mathrm{P})_\lambda$ has at least three nontrivial solutions.

In the case $b = +\infty$, we establish the following version of Theorem 1.2.

Theorem 1.3. *Under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, if f satisfies (f_1), (f_2) and*

$(f_4)'$ $\lim\limits_{|u| \to +\infty} \dfrac{F(u)}{|u|^2} = +\infty$;

(f_6) *there exists* $\theta \geq 1$, *such that* $\theta H(u) \geq H(su)$, $(s, u) \in [0, 1] \times \mathbb{R}$. *Moreover,* $F(u) \geq \dfrac{\mu_m}{2} u^2$, $u \in \mathbb{R}$, $\limsup\limits_{u \to 0} \dfrac{F(u)}{u^2} < \dfrac{1}{2}\mu_{m+1}$, *for some* $m \in \mathbb{N}$.

Then, there exists $\lambda^{**} > 0$, such that for $\lambda \in (0, \lambda^{**})$, $(\mathrm{P})_\lambda$ has at least three nontrivial solutions.

Remark 1.4. In Theorem 1.1, f may be superlinear or asymptotically linear near zero, we can get two nontrivial solutions by the mountain pass theorem and the truncation technique. In Theorem 1.2 and 1.3, in order to guarantee the functional associated to problem $(\mathrm{P})_\lambda$ enjoys linking structure, f has to satisfy some stricter growth condition near zero.

Remark 1.5. In Theorems 1.2 and 1.3, we can get three nontrivial solutions: two mountain pass solutions and one linking solution. We can distinguish them by choosing special "paths".

Remark 1.6. As we have known, there are few results about problems on $\mathbb{R}^N$ with both sublinear and asymptotically linear nonlinearities at the same time.

The paper is organized as follows. In Section 2, we give some preliminary results. The proof of our main results will be given in Section 3.

2 Preliminary

In this section we give some preliminaries that will be used to prove the main results of this paper. We begin with a special case of results on Sobolev embedding which is due to [18].

Lemma 2.1 ([18])**.** *Let (V), (Q), (A_i), $i = 1, \dots, 5$ be satisfied, the Sobolev space $W_r^{1,2}(\mathbb{R}^N; V)$ is compactly embedded in $L^p(\mathbb{R}^N; Q)$, for any p such that $2_* < p < \frac{2N}{N-2}$.*

For $u \in W_r^{1,2}(\mathbb{R}^N; V)$, we denote

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u(x)|^q\,dx - \int_{\mathbb{R}^N} Q(x)F(u(x))\,dx, \tag{2.1}$$

$$I_\lambda^\pm(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u^\pm(x)|^q\,dx - \int_{\mathbb{R}^N} Q(x)F(u^\pm(x))\,dx, \tag{2.2}$$

where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, then under the conditions (f_1)–(f_3), by (2.1) and (2.2), I_λ and $I_\lambda^\pm \in C^1(W_r^{1,2}(\mathbb{R}^N; V), \mathbb{R})$.

Recall that a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence for a functional I, if

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty.$$

a sequence $\{u_n\}$ is a $(\mathrm{C})_c$ sequence for a functional I, if

$$I(u_n) \to c, \quad (1 + \|u_n\|)I'(u_n) \to 0, \quad \text{as } n \to \infty.$$

Definition 2.2. Assume X is a Banach space, $I \in C^1(X, \mathbb{R})$, we say that I satisfies the $(\mathrm{PS})_c$ condition, if every $(\mathrm{PS})_c$ sequence $\{u_n\}$ has a convergent subsequence. I satisfies (PS) condition if I satisfies $(\mathrm{PS})_c$ at any $c \in \mathbb{R}$.

Definition 2.3. Assume X is a Banach space, $I \in C^1(X, \mathbb{R})$, we say that I satisfies the $(\mathrm{C})_c$ condition, if every $(\mathrm{C})_c$ sequence $\{u_n\}$ has a convergent subsequence. I satisfies (C) condition if I satisfies $(\mathrm{C})_c$ at any $c \in \mathbb{R}$.

Lemma 2.4 (Mountain pass theorem, Ambrosetti–Rabinowitz, 1973, [22])**.** *Let X be a Banach space, $I \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} I(u) > I(0) \geq I(e).$$

If I satisfies the $(PS)_c$ condition with

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0,\ \gamma(1) = e\}.$$

Then c is a critical value of I.

Lemma 2.5 (Linking theorem, Rabinowitz, 1978, [22]). *Let $X = Y \oplus Z$ be a Banach space with* $\dim Y < \infty$. *Let $R > r > 0$ and $z \in Z$ be such that $\|z\| = r$. Define*

$$\begin{aligned} \mathcal{M} &:= \{u = y + tz \mid \|u\| \le R,\ t \ge 0,\ y \in Y\}, \\ \mathcal{M}_0 &:= \{u = y + tz \mid y \in Y,\ \|u\| = R \text{ and } t \ge 0 \text{ or } \|u\| \le R \text{ and } t = 0\}, \\ \mathcal{N} &:= \{u \in Z \mid \|u\| = r\}. \end{aligned}$$

Let $I \in C^1(X, \mathbb{R})$ be such that

$$d := \inf_{\mathcal{N}} I > a := \max_{\mathcal{M}_0} I.$$

If I satisfies the (PS)$_c$ condition with

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}} I(\gamma(u)),$$

$$\Gamma := \{\gamma \in C(\mathcal{M}, X) \mid \gamma|_{\mathcal{M}_0} = id\}.$$

Then c is a critical value of I.

It is well known that the above two minimax theorems are still valid under (C)$_c$ condition. In our paper, we denote $X := W_r^{1,2}(\mathbb{R}^N; V)$, C denotes various positive constants.

Lemma 2.6. *Under the assumptions (V), (Q) and (A_i), $i = 1, \ldots, 5$, if f satisfies (f_1)–(f_3), then for given $\lambda > 0$, there exist ρ_1, $\beta_1 > 0$, such that*

$$\inf_{u \in X,\ \|u\| = \rho_1} I_\lambda^+(u) \ge \beta_1 > 0. \tag{2.3}$$

Proof. By (f_1)–(f_3), there exists $c > 0$, such that $|F(u)| \le c(|u|^2 + |u|^p)$. Then,

$$\begin{aligned} I_\lambda^+(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{q} \int_{\mathbb{R}^N} Q(x)|u^+(x)|^q \, dx - \int_{\mathbb{R}^N} Q(x) F(u^+(x)) \, dx \\ &\ge \frac{1}{2}\|u\|^2 - c \int_{\mathbb{R}^N} Q(x)|u^+(x)|^2 \, dx - c \int_{\mathbb{R}^N} Q(x)|u(x)|^p \, dx + \frac{\lambda}{q} \int_{\mathbb{R}^N} Q(x)|u^+(x)|^q \, dx \\ &\ge \frac{1}{2}\|u\|^2 - c \int_{\mathbb{R}^N} Q(x)|u^+(x)|^2 \, dx - c'\|u\|^p + \frac{\lambda}{q} \int_{\mathbb{R}^N} Q(x)|u^+(x)|^q \, dx. \end{aligned}$$

Hence, for $\|u\|$ small enough

$$I_\lambda^+(u) \ge \frac{1}{3}\|u\|^2 - c \int_{\mathbb{R}^N} Q(x)|u^+(x)|^2 \, dx + \frac{\lambda}{q} \int_{\mathbb{R}^N} Q(x)|u^+(x)|^q \, dx. \tag{2.4}$$

Then, we can choose $i \in \mathbb{N}$, such that $c \in \left(\frac{\mu_i}{4}, \frac{\mu_{i+1}}{4}\right)$. Let $X_j := \operatorname{span}\{\phi_j\}$, $j \in \mathbb{N}$, and $G_i := X_1 \oplus X_2 \oplus \cdots \oplus X_i$, where ϕ_j is the eigenfunction associated to the eigenvalue μ_j and $\oplus$ means the orthogonal sum of the subspace. We note that $X := G_i \oplus G_i^\perp$, thus, u^+ can be decomposed as $u^+ = v + w$, where $v \in G_i$, $w \in G_i^\perp$. Observe that for $v \in G_i$, there holds

$$\|v\|^2 \ge \mu_1 \int_{\mathbb{R}^N} Q(x) v^2 \, dx,$$

and for $w \in G_i^\perp$, there holds

$$\|w\|^2 \ge \mu_{i+1} \int_{\mathbb{R}^N} Q(x) w^2 \, dx.$$

Therefore, (2.4) implies that

$$\begin{aligned} I_\lambda^+(u) &\geq \frac{1}{12}\|u\|^2 + \frac{1}{4}\left(1-\frac{4c}{\mu_{i+1}}\right)\|w\|^2 - \frac{1}{4}\left(\frac{4c}{\mu_1}-1\right)\|v\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u^+(x)|^q\,dx \\ &=: \frac{1}{12}\|u\|^2 + \xi\|w\|^2 - \eta\|v\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u^+(x)|^q\,dx. \end{aligned} \tag{2.5}$$

It suffices to show that there exists $\rho_1 > 0$ small enough, such that for $0 < \|u\| \leq \rho_1$,

$$I_{\lambda 1}^+(u) := \xi\|w\|^2 - \eta\|v\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u^+(x)|^q\,dx \geq 0. \tag{2.6}$$

Seeking a contradiction, we suppose that there exist $u_n \neq 0$ satisfying $\|u_n\| \to 0$ as $n \to \infty$, and $I_{\lambda 1}^+(u_n) \leq 0$. Decompose $u_n^+ = v_n + w_n$, where $v_n \in G_i$, $w_n \in G_i^\perp$. We have

$$I_{\lambda 1}^+(u_n) = \xi\|w_n\|^2 - \eta\|v_n\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u_n^+(x)|^q\,dx < 0. \tag{2.7}$$

Then $u_n^+ \neq 0$, in X. Let $z_n = \frac{u_n^+}{\|u_n^+\|}$, up to a subsequence, we get that

$$\begin{aligned} &z_n \rightharpoonup z, \quad \text{as } n \to \infty, \text{ in } X, \\ &z_n \to z, \quad \text{as } n \to \infty, \text{ in } L^s(\mathbb{R}^N; Q),\ 2_* < s < \frac{2N}{N-2}, \\ &z_n(x) \to z(x), \quad \text{as } n \to \infty,\ a.e.\ x \in \mathbb{R}^N. \end{aligned}$$

Dividing both sides of (2.7) by $\|u_n^+\|^q$,

$$\frac{\xi\|w_n\|^2 - \eta\|v_n\|^2}{\|u_n^+\|^2}\|u_n^+\|^{2-q} + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|z_n(x)|^q\,dx \leq 0.$$

Let $n \to \infty$, there holds $\int_{\mathbb{R}^N} Q(x)|z(x)|^q\,dx \leq 0$, in view of $\|u_n^+\| \to 0$, as $n \to \infty$. Thus, $z(x) = 0$, $a.e.\ x \in \mathbb{R}^N$. Then, we have

$$\frac{\|v_n\|^2}{\|u_n^+\|^2} \leq \frac{C\|v_n\|^2_{L^2(\mathbb{R}^N;Q)}}{\|u_n^+\|^2} \leq C\|z_n\|^2_{L^2(\mathbb{R}^N;Q)} \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Using the equivalence of all norms on the finite dimensional space, choosing n sufficient large, we obtain that

$$\begin{aligned} I_{\lambda 1}^+(u_n) &= \xi\|w_n\|^2 - \eta\|v_n\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u_n^+(x)|^q\,dx \\ &\geq \xi\|u_n^+\|^2 - (\xi+\eta)\|v_n\|^2 \\ &= \left[\xi - (\xi+\eta)\frac{\|v_n\|^2}{\|u_n^+\|^2}\right]\|u_n^+\|^2, \end{aligned} \tag{2.9}$$

in view of (2.8), we get a contradiction from (2.9). Therefore, from (2.5) and (2.6) we can choose $\rho_1 > 0$ small enough such that for $\|u\| = \rho_1$,

$$I_\lambda^+(u) \geq \frac{1}{12}\|u\|^2 + I_{\lambda 1}^+(u) \geq \frac{1}{12}\rho_1^2 =: \beta_1 > 0.$$

Therefore, (2.3) is true. □

Using a similar argument as in Lemma 2.6, we have the following result.

Lemma 2.7. *Under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, if f satisfies (f_1)–(f_3), then for given $\lambda > 0$, there exist ρ_2, $\beta_2 > 0$, such that*

$$\inf_{u \in X,\ \|u\| = \rho_2} I_\lambda^-(u) \geq \beta_2 > 0.$$

Remark 2.8. In fact, for any $\lambda > 0$, 0 is a local minimizer of $I_\lambda^\pm$. In the bounded domain, it is easy to obtain this result by the fact that the local $H_0^1(\Omega)$-minimizer is also the local $C_0^1(\overline{\Omega})$-minimizer (see [3]).

3 Proof of main results

Proof of Theorem 1.1. It is easy to see that $I_\lambda^+(0) = 0$. We note that (f_1) and (f_4) with $\mu_1 < b < +\infty$ imply (f_2). Then, from Lemma 2.6, given $\lambda > 0$, there exist $\rho_1 > 0$, $\beta_1 > 0$, such that

$$\inf_{u \in X,\ \|u\| = \rho_1} I_\lambda^+(u) \geq \beta_1 > 0.$$

On the other hand, (f_1) and (f_4) imply that

$$\lim_{|u| \to +\infty} \frac{2F(u)}{|u|^2} = b > \mu_1. \tag{3.1}$$

Thus, choosing $u = \phi_1$,

$$\begin{aligned} I_\lambda^+(t\phi_1) &= \frac{1}{2} t^2 \|\phi_1\|^2 + \frac{\lambda t^q}{q} \int_{\mathbb{R}^N} Q(x) |\phi_1|^q \, dx - \int_{\mathbb{R}^N} Q(x) F(t\phi_1) \, dx \\ &= t^2 \left(\frac{1}{2} \|\phi_1\|^2 + \frac{\lambda t^{q-2}}{q} \int_{\mathbb{R}^N} Q(x) |\phi_1|^q \, dx - \int_{\mathbb{R}^N} Q(x) \frac{F(t\phi_1)}{t^2 \phi_1^2} \varphi_1^2 \, dx \right). \end{aligned}$$

By (3.1) and $1 < q < 2$, we have $I_\lambda^+(t\phi_1) \to -\infty$, as $t \to +\infty$. Thus, for large enough t_1, we have $\|t_1 \phi_1\| > \rho_1$, and $I_\lambda^+(t_1 \phi_1) < 0$. Define

$$c^+ = \inf_{\gamma \in \Gamma^+} \max_{0 \leq t \leq 1} I_\lambda^+(\gamma(t))$$

where $\Gamma^+ := \{\gamma \in C([0,1],\ X) \mid \gamma(0) = 0,\ \gamma(1) = t_1 \phi_1\}$.

Now, in order to apply Lemma 2.4 to prove Theorem 1.1, it is sufficient to verify that I_λ^+ satisfies the $(C)_{c^+}$ condition.

Lemma 3.1. *Under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, if f satisfies (f_1), (f_3) and (f_4) with $\mu_1 < b$, then for any fixed $\lambda > 0$, the functional I_λ^+ satisfies the $(C)_{c^+}$ condition.*

Proof. For every $(C)_{c^+}$ sequence $\{u_n\}$

$$I_\lambda^+(u_n) \to c^+, \quad \text{as } n \to \infty, \tag{3.2}$$

$$(1 + \|u_n\|) I_\lambda^{+'}(u_n) \to 0, \quad \text{as } n \to \infty, \tag{3.3}$$

we claim that the sequence $\{u_n\}$ is bounded in X. Seeking a contradiction, we suppose that $\|u_n\| \to \infty$. Let $z_n = \frac{u_n}{\|u_n\|}$, up to a subsequence, we get that

$$\begin{aligned} &z_n \rightharpoonup z, \quad \text{as } n \to \infty, \text{ in } X, \\ &z_n \to z, \quad \text{as } n \to \infty, \text{ in } L^s(\mathbb{R}^N; Q),\ 2_* < s < \frac{2N}{N-2}, \\ &z_n(x) \to z(x), \quad \text{as } n \to \infty,\ a.e.\ x \in \mathbb{R}^N. \end{aligned}$$

We claim that $z \neq 0$. otherwise, $z = 0$, since by (3.3)

$$o(1) = \langle I_\lambda^{+\prime}(u_n),\ u_n\rangle = \|u_n\|^2 + \lambda \int_{\mathbb{R}^N} Q(x) u_n^+(x)^q\, dx - \int_{\mathbb{R}^N} Q(x) f(u_n^+(x)) u_n^+(x)\, dx. \tag{3.4}$$

Dividing both sides of (3.4) by $\|u_n^+\|^2$,

$$o(1) = 1 - \int_{\mathbb{R}^N} Q(x) \frac{f(u_n^+(x))u_n^+(x)}{\|u_n\|^2}\, dx. \tag{3.5}$$

Assumptions (f_1), (f_3) and (f_4) with $\mu_1 < b < +\infty$ imply that there exists $C > 0$, such that

$$|f(u_n^+(x))| \leq C u_n^+(x). \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\begin{aligned} 1 &= \int_{\mathbb{R}^N} Q(x) \frac{f(u_n^+(x))u_n^+(x)}{\|u_n\|^2}\, dx + o(1) \\ &\leq C \int_{\mathbb{R}^N} Q(x) |z_n^+(x)|^2\, dx + o(1). \end{aligned}$$

Letting $n \to \infty$, we get a contradiction. Thus, $z \neq 0$ in X.

Set

$$P_n(x) = \begin{cases} \dfrac{f(u_n(x))}{u_n(x)}, & \text{for } x \in \mathbb{R}^N,\ u_n(x) > 0, \\ 0, & \text{for } x \in \mathbb{R}^N,\ u_n(x) \leq 0. \end{cases}$$

From $I_\lambda^{+\prime}(u_n) = o(1)$, we can get that

$$\int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx + \int_{\mathbb{R}^N} V(x) u_n \phi\, dx + \lambda \int_{\mathbb{R}^N} Q(x) (u_n^+)^{q-1} \phi\, dx - \int_{\mathbb{R}^N} Q(x) f(u_n^+) \phi\, dx = o(1),$$

for all $\phi \in C_{0,r}^\infty(\mathbb{R}^N)$. Dividing $\|u_n\|$ in both sides of the above equality, there holds

$$\int_{\mathbb{R}^N} \nabla z_n \nabla \phi\, dx + \int_{\mathbb{R}^N} V(x) z_n \phi\, dx - \int_{\mathbb{R}^N} Q(x) P_n(x) z_n^+ \phi\, dx = o(1). \tag{3.7}$$

By (3.6), $|P_n(x)| \leq C$ for $x \in \mathbb{R}^N$. Then we have

$$\begin{aligned} &\left| \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\}} Q(x) P_n(x) z_n^+(x) \phi(x)\, dx \right| \\ &\quad \leq C \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\}} Q(x) z_n^+(x) |\phi(x)|\, dx \\ &\quad = o(1) + C \int_{\{x \in \mathbb{R}^N | z^+(x) = 0\}} Q(x) z^+(x) |\phi(x)|\, dx = o(1). \end{aligned} \tag{3.8}$$

On the other hand, since $z_n^+(x) \to z^+(x)$ for *a.e.* $x \in \mathbb{R}^N$, we have $\lim_{n\to\infty} u_n^+(x) = +\infty$ for *a.e.* $x \in \{x \in \mathbb{R}^N \mid z^+(x) > 0\}$, which implies that $\lim_{n\to\infty} P_n(x) = b$, for *a.e.* $x \in \{x \in \mathbb{R}^N \mid z^+(x) > 0\}$. Besides, $|P_n(x)| \le C$, for *a.e.* $x \in \mathbb{R}^N$. Using Lebesgue's dominated convergence theorem, we obtain that

$$\begin{aligned} &\left| \int_{\{x\in\mathbb{R}^N \mid\ z^+(x)>0\}} Q(x)(P_n(x)-b)z_n^+(x)\phi(x)\,dx \right| \\ &\quad \le \int_{\{x\in\mathbb{R}^N \mid\ z^+(x)>0\}} Q(x)|P_n(x)-b|z_n^+(x)|\phi(x)|\,dx \\ &\quad \le C\left(\int_{\{x\in\mathbb{R}^N \mid\ z^+(x)>0\}} Q(x)|P_n(x)-b|^2|\phi(x)|\,dx \right)^{\frac{1}{2}} \\ &\quad = o(1). \end{aligned} \tag{3.9}$$

By (3.8) and (3.9),

$$\begin{aligned} &\int_{\mathbb{R}^N} Q(x)P_n(x)z_n^+(x)\phi(x)\,dx \\ &\quad = \int_{\{x\in\mathbb{R}^N \mid z^+(x)=0\}} Q(x)P_n(x)z_n^+(x)\phi(x)\,dx \\ &\qquad + \int_{\{x\in\mathbb{R}^N \mid z^+(x)>0\}} Q(x)P_n(x)z_n^+(x)\phi(x)\,dx \\ &\quad = o(1) + \int_{\{x\in\mathbb{R}^N \mid z^+(x)>0\}} Q(x)P_n(x)z_n^+(x)\phi(x)\,dx \\ &\quad = o(1) + b\int_{\mathbb{R}^N} Q(x)z^+(x)\phi(x)\,dx. \end{aligned} \tag{3.10}$$

Combining (3.7) and (3.10), letting $n \to \infty$, there holds

$$\int_{\mathbb{R}^N} (\nabla z \nabla \phi + V(x)z\phi)\,dx = b\int_{\mathbb{R}^N} Q(x)z^+\phi\,dx. \tag{3.11}$$

We claim that $\text{meas}\{x \in \mathbb{R}^N \mid z^+(x) \neq 0\} > 0$. Otherwise $z^+ = 0$, taking $\phi = z$ in (3.11), we have $z = 0$, which is impossible. Taking $\phi = z^-$ in (3.11), we can get $z \ge 0$. Moreover by the Hopf lemma, we also can get $z > 0$ in $\mathbb{R}^N$. Taking $\phi = \phi_1$ in (3.11), we obtain

$$\int_{\mathbb{R}^N} (\nabla z \nabla \phi_1 + V(x)z\phi_1)\,dx = b\int_{\mathbb{R}^N} Q(x)z^+\phi_1\,dx.$$

Since $\phi_1 > 0$ is the eigenfunction associated to μ_1, and $z \ge 0$, we have

$$\int_{\mathbb{R}^N} (\nabla z \nabla \phi_1 + V(x)z\phi_1)\,dx = \mu_1\int_{\mathbb{R}^N} Q(x)z\phi_1\,dx.$$

This is impossible, since $b > \mu_1$. Then $\{u_n\}$ is bounded in X. Since the embedding from X into $L^s(\mathbb{R}^N; Q)$, $s \in (2_*, \frac{2N}{N-2})$ is compact, there exists $u \in X$, such that $u_n \to u$ strongly in X. Thus from (3.2) and (3.3), we can get that

$$I_\lambda^+(u) = c^+ \ge \beta_1, \qquad {I_\lambda^+}'(u) = 0.$$

□

Finally, we are now ready to conclude the proof of Theorem 1.1. Since ${I_\lambda^+}'(u)u^- = 0$, then

$$\int_{\mathbb{R}^N} (\nabla u \nabla u^- + V(x)uu^-)\,dx = -\lambda \int_{\mathbb{R}^N} Q(x)(u^+)^{q-1}u^-\,dx + \int_{\mathbb{R}^N} Q(x)f(u^+)u^-\,dx = 0.$$

We have $u^- = 0$, *i.e.* $u \geq 0$. Thus, u is a nonnegative solution for problem $(\mathrm{P})_\lambda$. Similarly, for

$$I_\lambda^-(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u^-|^q\,dx - \int_{\mathbb{R}^N} Q(x)F(u^-)\,dx,$$

we can also get a nonpositive solution for problem $(\mathrm{P})_\lambda$. Thus, problem $(\mathrm{P})_\lambda$ has at least two nontrivial solutions. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. In order to prove Theorem 1.2, we firstly verify that the functional I_λ enjoys the linking structure.

Lemma 3.2. *Under the assumptions (V), (Q) and (A_i), $i = 1, \dots, 5$, if f satisfies the assumptions of Theorem 1.2, then there exist positive numbers r, d, R and $\eta = \eta(\lambda)$, such that*

(i) for all $u \in X_m := \overline{\bigoplus_{j=1}^m ker(-\Delta + V - \mu_j Q)}$, we have

$$I_\lambda(u) \leq \eta(\lambda), \qquad \lim_{\lambda\to 0^+} \eta(\lambda) = 0;$$

(ii) for all $u \in \mathcal{N} := \{u \in X_m^\perp \mid \|u\| = r\}$, we have

$$I_\lambda(u) \geq d > 0, \quad \textit{for all } \lambda > 0;$$

(iii) for all $u \in X_{m+1}$, and $\|u\| \geq R$, we have $I_\lambda(u) \leq 0$.

Proof. (i) Let $u \in X_m$, since $F(u) > \frac{1}{2}\mu_m u^2$, $u \in \mathbb{R}$, there exists $\varepsilon_1 > 0$, such that $F(u) \geq \frac{1}{2}(\mu_m + \varepsilon)u^2$, then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u|^q\,dx - \int_{\mathbb{R}^N} Q(x)F(u)\,dx \\ &\leq \left(\frac{1}{2} - \frac{\mu_m + \varepsilon_1}{2\mu_m}\right)\|u\|^2 + \frac{C\lambda}{q}\|u\|^q. \end{aligned} \tag{3.12}$$

Let $g(t) = \left(\frac{1}{2} - \frac{\mu_m+\varepsilon_1}{2\mu_m}\right)t^2 + \frac{C\lambda}{q}t^q$, we have

$$\max_{t>0} g(t) = g(t_0), \quad \text{where } t_0 = \left(\frac{\mu_m}{\varepsilon_1}C\lambda\right)^{\frac{1}{2-q}},$$

and

$$\begin{aligned} g(t_0) &= -\frac{\varepsilon_1}{2\mu_m}\left(\frac{\mu_m}{\varepsilon_1}C\right)^{\frac{2}{2-q}}\lambda^{\frac{2}{2-q}} + \frac{C\lambda}{q}\left(\frac{\mu_m}{\varepsilon_1}C\lambda\right)^{\frac{q}{2-q}} \\ &= \left(-\frac{\varepsilon_1}{2\mu_m}\left(\frac{\mu_m}{\varepsilon_1}\right)^{\frac{2}{2-q}} + \frac{1}{q}\left(\frac{\mu_m}{\varepsilon_1}\right)^{\frac{q}{2-q}}\right)(C\lambda)^{\frac{2}{2-q}}. \end{aligned}$$

Then from (3.12), we can get that

$$I_\lambda(u) \leq \left(-\frac{\varepsilon_1}{2\mu_m}\left(\frac{\mu_m}{\varepsilon_1}\right)^{\frac{2}{2-q}} + \frac{1}{q}\left(\frac{\mu_m}{\varepsilon_1}\right)^{\frac{q}{2-q}}\right)(C\lambda)^{\frac{2}{2-q}} =: \eta(\lambda) \to 0, \quad \text{as } \lambda \to 0.$$

(ii) Let $u \in X_m^{\perp}$, by (f_1) and (f_4) with $\mu_{k+1} < b < +\infty$, and $\limsup_{u\to 0} \frac{F(u)}{u^2} < \frac{1}{2}\mu_{m+1}$, we have that there exists $\varepsilon_2 > 0, C > 0, p > 2$, such that $F(u) \le \frac{1}{2}(\mu_{m+1} - \varepsilon_2)u^2 + C|u|^p$. Then

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u|^q\,dx - \int_{\mathbb{R}^N} Q(x)F(u)\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}(\mu_{m+1} - \varepsilon_2)\int_{\mathbb{R}^N} Q(x)|u|^2\,dx - C\int_{\mathbb{R}^N} Q(x)|u|^p\,dx \\ &\ge \frac{1}{2}\left(1 - \frac{\mu_{m+1} - \varepsilon_2}{\mu_{m+1}}\right)\|u\|^2 - C\|u\|^p, \end{aligned} \tag{3.13}$$

thus, choosing $r > 0$ small enough, (3.13) implies that

$$\inf_{u\in X_m^{\perp},\ \|u\|=r} I_\lambda(u) \ge d > 0, \quad \text{independent of } \lambda > 0.$$

(iii) For any $u \in X_{m+1}$, set $f(u) = bu + g(u)$, by (f_4), we have $\frac{G(u)}{u^2} \to 0$, as $|u| \to \infty$, where $G(u) = \int_0^u g(s)\,ds$. Then

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u|^q\,dx - \frac{b}{2}\int_{\mathbb{R}^N} Q(x)|u|^2\,dx - \int_{\mathbb{R}^N} Q(x)G(u)\,dx.$$

Since $b > \mu_{m+1}$, for every $z \in \operatorname{span}\{\phi_{m+1}\},\ t \in \mathbb{R}, w \in X_m$,

$$t^2\|z\|^2 + \|w\|^2 - b\int_{\mathbb{R}^N} Q(tz + w)^2\,dx < 0. \tag{3.14}$$

Arguing by contradiction, we find a sequence $\{u_n\}$, satisfying $\|u_n\| \to \infty$, $u_n = t_n z_0 + w_n$, where $z_0 \in \operatorname{span}\{\phi_{m+1}\},\ t_n \in \mathbb{R}, w_n \in X_m$, such that

$$I_\lambda(u_n) = \frac{1}{2}t_n^2\|z_0\|^2 + \frac{1}{2}\|w_n\|^2 + \frac{\lambda}{q}\int_{\mathbb{R}^N} Q(x)|u_n|^q\,dx - \int_{\mathbb{R}^N} Q(x)F(u_n)\,dx \ge 0. \tag{3.15}$$

Dividing both sides of (3.15) by $\|u_n\|^2$, there holds

$$\frac{I_\lambda(u_n)}{\|u_n\|^2} = \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 + \frac{\lambda}{q\|u_n\|^2}\int_{\mathbb{R}^N} Q(x)|u_n|^q\,dx - \int_{\mathbb{R}^N} Q(x)\frac{F(u_n)}{\|u_n\|^2}\,dx \ge 0, \tag{3.16}$$

where $\tau_n := \frac{t_n}{\|u_n\|}$, $v_n := \frac{w_n}{\|u_n\|}$. Since $\tau_n^2\|z_0\|^2 + \|v_n\|^2 = 1$, after passing to a subsequence $\tau_n \to \tau$, in $\mathbb{R}$, $v_n \to v$ in X_m. Let $u' = \tau z_0 + v$, by (3.14), there exists a bounded domain $\Omega \subset \mathbb{R}^N$, such that

$$\tau^2\|z_0\|^2 + \|v\|^2 - b\int_\Omega Q(x)(\tau z_0 + v)^2\,dx < 0. \tag{3.17}$$

As $F(u) = \frac{1}{2}bu^2 + G(u)$, it follows from (3.16) that

$$\begin{aligned} 0 &\le \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 - \int_\Omega Q(x)\frac{F(u_n)}{\|u_n\|^2}\,dx + \frac{\lambda}{q\|u_n\|^2}\int_{\mathbb{R}^N} Q(x)|u_n|^q\,dx \\ &= \frac{1}{2}\tau_n^2\|z_0\|^2 + \frac{1}{2}\|v_n\|^2 - \frac{1}{2}b\int_\Omega Q(x)(\tau_n z_0 + v_n)^2\,dx \\ &\quad - \int_\Omega Q(x)\frac{G(u_n)}{\|u_n\|^2}\,dx + \frac{\lambda}{q\|u_n\|^2}\int_{\mathbb{R}^N} Q(x)|u_n|^q\,dx. \end{aligned}$$

Clearly, $|G(u)| \leq c_0 u^2$, for some $c_0 > 0$ and $\frac{G(u)}{u^2} \to 0$, as $|u| \to \infty$. Since $\tau_n \to \tau$, in $\mathbb{R}$, $v_n \to v$, in X_m, then $\tau_n z_0 + v_n \to u' = \tau z_0 + v$, in $L^2(\mathbb{R}^N; Q)$. It is easy to see from Lebesgue's dominated converge theorem that

$$\int_\Omega Q(x) \frac{G(u_n)}{\|u_n\|^2}\, dx = \int_\Omega Q(x) \frac{G(u_n)}{u_n^2} (\tau_n^2 + v_n^2)\, dx \to 0.$$

Hence, together with (3.17)

$$0 \leq \frac{1}{2}\tau^2 \|z_0\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{2} b \int_\Omega Q(x) |tz_0 + v|^2\, dx < 0,$$

this is impossible. □

Therefore, for $Y = X_m$, $Z = X_m^\perp$, $z \in \operatorname{span}\{\phi_{m+1}\}$ with $\|z\| = r$,

$$\mathcal{M} := \{u = y + tz \mid \|u\| \leq R,\ t \geq 0,\ y \in Y\},$$

$$\mathcal{M}_0 := \{u = y + tz \mid y \in Y,\ \|u\| = R \text{ and } t \geq 0 \text{ or } \|u\| \leq R \text{ and } t = 0\},$$

$$\mathcal{N} := \{u \in Z \mid \|u\| = r\}.$$

Lemma 3.2 implies that there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$,

$$\inf_{u \in \mathcal{N}} I_\lambda(u) > \sup_{u \in \mathcal{M}_0} I_\lambda(u).$$

Define

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}} I_\lambda(\gamma(u)),$$

$$\Gamma := \{\gamma \in C(\mathcal{M}, X) \mid \gamma|_{\mathcal{M}_0} = \mathrm{id}\}.$$

Next, we prove that the functional I_λ satisfies the $(C)_{c_\lambda}$ condition.

Lemma 3.3. *Under the assumptions (V), (Q) and (A_i), $i = 1, \ldots, 5$, if f satisfies the assumptions of Theorem 1.2, then for any given $\lambda > 0$, the functional I_λ satisfies the $(C)_{c_\lambda}$ condition.*

Proof. For every $(C)_{c_\lambda}$ sequence $\{u_n\}$,

$$I_\lambda(u_n) \to c_\lambda, \quad \text{as } n \to +\infty. \tag{3.18}$$

$$(1 + \|u_n\|) I'_\lambda(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{3.19}$$

We just prove that $\{u_n\}$ is bounded. Seeking a contradiction we suppose that $\|u_n\| \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, up to a subsequence, we get that

$$\begin{aligned}
&w_n \rightharpoonup w \quad \text{in } X,\\
&w_n \to w \quad \text{in } L^s(\mathbb{R}^N; Q),\ 2_* < s < \frac{2N}{N-2},\\
&w_n(x) \to w(x) \quad \textit{a.e. } x \in \mathbb{R}^N.
\end{aligned}$$

Now, we consider the two possible cases.

Case 1. $w = 0$ in X. From $o(1) = \langle I'_\lambda(u_n), u_n \rangle$, we have

$$o(1) = \|u_n\|^2 + \lambda \int_{\mathbb{R}^N} Q(x) |u_n|^q\, dx - \int_{\mathbb{R}^N} Q(x) f(u_n) u_n\, dx.$$

Dividing both sides of the above equality by $\|u_n\|^2$, we get that

$$o(1) = 1 - \int_{\mathbb{R}^N} Q(x)\frac{f(u_n)u_n}{\|u_n\|^2}\,dx. \tag{3.20}$$

Assumptions (f_1), $(f_3)'$ and (f_4) with $\mu_{k+1} < b < +\infty$ imply that there exists $C > 0$, such that

$$|f(u_n)u_n| \le C|u_n|^2. \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$1 = \int_{\mathbb{R}^N} Q(x)\frac{f(u_n)u_n}{\|u_n\|^2}\,dx + o(1) \le C\int_{\mathbb{R}^N} Q(x)|w_n|^2\,dx + o(1).$$

Letting $n \to \infty$, we get a contradiction.

Case 2. $w \neq 0$ in X. (3.18) and (3.19) imply that

$$\begin{aligned} c_\lambda + o(1) &= I_\lambda(u_n) - \frac{1}{2}\langle I'_\lambda(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^N} Q(x)\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx + \lambda\left(\frac{1}{q} - \frac{1}{2}\right)\int_{\mathbb{R}^N} Q(x)|u_n|^q\,dx \\ &\ge \int_{\mathbb{R}^N} Q(x)\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx. \end{aligned}$$

Set $\Omega_1 := \{x \in \mathbb{R}^N \mid w(x) \neq 0\}$, thus, for $x \in \Omega_1$, $u_n(x) \to +\infty$, as $n \to +\infty$. By (f_5), we obtain that

$$c_\lambda + o(1) \ge \int_{\Omega_1} Q(x)H(u_n)\,dx. \tag{3.22}$$

Since $|\Omega_1| > 0$ and for *a.e.* $x \in \Omega_1$, $\lim_{n\to\infty} H(u_n) = +\infty$. Using Fatou's lemma,

$$\lim_{n\to\infty}\int_{\Omega_1} Q(x)H(u_n)\,dx = +\infty,$$

which contradicts to (3.22). Thus, every $(C)_{c_\lambda}$ sequence is bounded. □

Finally, following the above two lemmas, all the conditions of Lemma 2.5 are verified. Then, there exists $u_\lambda \in X$, such that $I_\lambda(u_\lambda) = c_\lambda > d$, $I'_\lambda(u_\lambda) = 0$. Thus, there exists $\lambda^* > 0$, such that for $0 < \lambda < \lambda^*$, u_λ is a nontrivial solution of problem $(P)_\lambda$. Furthermore, from the proof of Theorem 1.1, we know that for any given $\lambda \in (0, \lambda^*)$, problem $(P)_\lambda$ has already had at least two nontrivial solutions. We can estimate the mountain pass levels by considering paths $\gamma \in \Gamma^\pm$, whose images are contained in X_m. By Lemma 3.2 (i), the maxima on such paths are at most $\eta(\lambda) < d$, for $0 < \lambda < \lambda^*$, then $c^\pm < d$, this implies that the linking solution is different from the two mountain pass solutions. Thus, for $0 < \lambda < \lambda^*$, problem $(P)_\lambda$ has at least three nontrivial solutions, where one is nonnegative, one is nonpositive. this finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. In view of the proof of Theorem 1.1 and Theorem 1.2, under the assumptions of Theorem 1.3 we can get the functionals enjoy the mountain pass structure and linking structure easily. We only need to prove that for any $c \in \mathbb{R}$ all the $(C)_c$ sequences for I_λ and $I^\pm_\lambda$ are bounded under assumptions of Theorem 1.3.

For $\{u_n\}$ satisfying

$$\begin{aligned} c + o(1) &= I_\lambda(u_n), \\ o(1) &= (1 + \|u_n\|)I'_\lambda(u_n). \end{aligned} \tag{3.23}$$

We will prove that $\{u_n\}$ is bounded in X.

Since $\{I_\lambda(u_n)\}$ is bounded, dividing both sides of (3.23) by $\|u_n\|^2$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)\frac{F(u_n)}{\|u_n\|^2}\,dx = \frac{1}{2} < +\infty. \tag{3.24}$$

Set $v_n := \frac{u_n}{\|u_n\|}$, then there exists $v \in X$ such that $v_n \rightharpoonup v$ in X, denote $\Omega := \{x \in \mathbb{R}^N \mid v(x) \neq 0\}$, then $|u_n(x)| \to +\infty$ for *a.e.* $x \in \Omega$. If $\text{meas}(\Omega) > 0$, then by $(f_4)'$

$$\frac{F(u_n)}{\|u_n\|^2} = \frac{F(u_n)}{|u_n|^2}v_n^2 \to +\infty, \quad \text{as } n \to \infty.$$

Since $Q(x) > 0$, using Fatou's lemma, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)\frac{F(u_n)}{\|u_n\|^2}\,dx = +\infty,$$

which is a contradiction to (3.24). So $\text{meas}(\Omega) = 0$. Therefore, $v = 0$.

Next, we set

$$I_\lambda(t_n u_n) = \max_{t\in[0,1]} I_\lambda(t u_n).$$

For any $M > 0$, set $\overline{v_n} = \sqrt{2M}v_n$, by (f_1), (f_2) and (f_6), there exists $C > 0$ such that

$$|F(u)| \le C(|u|^2 + |u|^p), \quad \text{for all } u \in \mathbb{R}.$$

Then

$$\left|\int_{\mathbb{R}^N} Q(x)F(\overline{v_n})\,dx\right| \le C\int_{\mathbb{R}^N} Q(x)(|\overline{v_n}|^2 + |\overline{v_n}|^p)\,dx \to 0, \quad \text{as } n \to \infty.$$

Consequently, for large enough n, one has

$$I_\lambda(t_n u_n) \ge I_\lambda(\overline{v_n}) \ge \frac{1}{2}\|\overline{v_n}\|^2 - \int_{\mathbb{R}^N} Q(x)F(\overline{v_n})\,dx \ge \frac{M}{2}.$$

This means that $\lim_{n\to\infty} I_\lambda(t_n u_n) = +\infty$ and $t_n \in (0,1)$.

In view of the choice of t_n, we know that $\langle I'_\lambda(t_n u_n), t_n u_n\rangle = 0$. Hence

$$\begin{aligned}
+\infty \leftarrow\ & 2I_\lambda(t_n u_n) - \langle I'_\lambda(t_n u_n), t_n u_n\rangle \\
&= \left(\frac{2}{q} - 1\right)\lambda\int_{\mathbb{R}^N} Q(x)|t_n u_n|^q\,dx + \int_{\mathbb{R}^N} Q(x)\big(f(t_n u_n)t_n u_n - 2F(t_n u_n)\big)\,dx \\
&\le \left(\frac{2}{q} - 1\right)\lambda\int_{\mathbb{R}^N} Q(x)|u_n|^q\,dx + \theta\int_{\mathbb{R}^N} Q(x)\big(f(u_n)u_n - 2F(u_n)\big)\,dx \\
&\le \theta\big(2I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n\rangle\big).
\end{aligned}$$

This is a contradiction to (3.23), so $\{u_n\}$ is bounded in X.

Similarly, we can prove that $I^{\pm}_\lambda$ satisfies the $(C)_c$ condition, the details are omitted. □

Remark 3.4. The sublinear term $|u|^{q-2}u$ can be relaxed to more general type, and the function Q before the sublinear term and the asymptotically linear or superlinear term can also be different.

Remark 3.5. Generally, the sublinear term determines the geometry structure of the functional near zero. In this paper, 0 is a local minimizer. In our forthcoming work, we will discuss the other case

$$\begin{cases} -\Delta u + V(x)u = \lambda Q(x)|u|^{q-2}u + Q(x)f(u), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to +\infty. \end{cases} \tag{Q$_\lambda$}$$

Acknowledgments

The authors are grateful to the anonymous referees for their helpful comments and suggestions.

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