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Nonradial solutions for semilinear Schrödinger equations with sign-changing potential

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Abstract. In this paper, we investigate the existence of infinite nonradial solutions for the Schrödinger equations

$$\begin{cases} -\triangle u + b(|x|)u = f(|x|, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where b is allowed to be sign-changing. Under some assumptions on $b \in C([0,\infty), \mathbb{R})$ and $f \in C([0,\infty) \times \mathbb{R}^N, \mathbb{R})$, we obtain that the above system possesses infinitely many nonradial solutions. The method of proof relies on critical point theorem.

Keywords: Schrödinger equations, variational methods, critical point, Sign-changing potential, nonradial solution.

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1 Introduction and statement of the main result

In this paper, we study the existence of infinitely many nonradial solutions for the following semilinear Schrödinger equation

$$\begin{cases} -\triangle u + b(|x|)u = f(|x|, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

We suppose that $b\colon [0,\infty) \to \mathbb{R}$ and $f\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following assumptions:

(B_1) $b \in C([0,\infty), \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} b(|x|) > -\infty$;

(B_2) there exists a constant $a > 0$ such that

$$\lim_{|y|\to+\infty} \text{meas}\left\{x \in \mathbb{R}^N : |x-y| \le a,\ b(|x|) \le M\right\} = 0, \qquad \forall\, M > 0,$$

where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;

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(B_3) $f \in C([0,\infty) \times \mathbb{R}^N, \mathbb{R})$ and there exist constants a_1, $a_2 > 0$ and $p \in (1, \frac{N+2}{N-2})$ such that

$$|f(x,u)| \le a_1|u| + a_2|u|^p, \qquad \forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}; \tag{1.2}$$

(B_4) there exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(r,u) := \mu \int_0^u f(r,v)\,dv \le uf(r,u), \quad \text{for any } r \ge 0 \text{ and } |u| \ge R; \tag{1.3}$$

(B_5) $f(|x|, -u) = -f(|x|, u)$, $\forall\ (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

We say that a solution $u\colon \mathbb{R}^N \to \mathbb{R}$ is a *radial solution* (see for instance in [4,7–9]) if $u(x) = u(|x|)$, that is, solution u has spherical symmetry. In the present paper, we consider the solutions of (1.1) which are different from the radial ones.

The following theorems are the main results of the paper.

Theorem 1.1. *Under assumptions (B_1)–(B_5), if $N = 4$ or $N \ge 6$, then system (1.1) possesses an unbounded sequence of solutions $\pm u_k$, $k \in \mathbb{N}$, which are not radial. The solutions are classical if f is locally Lipschitz with respect to u.*

Recently, by using variational methods and critical point theory, many authors have studied the existence of solution for system (1.1) or the following general type:

$$-\triangle u + b(x)u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

The interest in equation (1.1) or (1.4) originates from various problems in physics and mathematical physics. In cosmology and constructive field theory, system (1.1) or (1.4) is also called *nonlinear Euclidean scalar field equation* (see [8,9]). As it was mentioned in [4], a solution of (1.1) can also be interpreted as a *stationary state* (see [8,9]) of the reaction diffusion:

$$u_t = -\triangle u - b(|x|)u + f(|x|, u),$$

for more physics background of (1.1), we refer the readers to [8,9] and the references therein.

In [28], professor W. A. Strauss did pioneering work for the autonomous case of (1.1), that is:

$$-\triangle u = g(u), \qquad x \in \mathbb{R}^N, \tag{1.5}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and odd in u. In [8,9], Berestycki and Lions obtained the existence of infinitely many radial solutions of (1.5) under almost necessary growth conditions on g. The solutions they obtained have exponential decay at infinity. When $N = 1$, they obtained a necessary and sufficient condition for the existence of a solution of problem (1.5). Some open problems are also mentioned in [8,9]. For more results of radial solutions of (1.1) or (1.2) , we refer the readers to [5,7,32]. For more applications of critical point theory to PDE, we refer the readers to the work of Michel Willem [32], Strauss [30], Rabinowitz [26], Zou [35] and T. Bartsch, Z. Q. Wang, M. Willem [7].

We are motivated by [4] written by T. Bartsch and Michel Willem. They make the following assumptions.

(A_1) $b \in C([0,\infty), \mathbb{R})$ is bounded from below by a positive constant a_0.

(A_2) $f \in C([0,\infty) \times \mathbb{R}, \mathbb{R})$ and there are positive constants a_1, R and a constant $1 < q < \frac{N+2}{N-2}$ such that

$$|f(r,u)| \le a_1|u|^q \quad \text{for any } r \ge 0,\ |u| \ge R.$$

(A_3) There exists $\mu > 2$ such that

$$\mu F(r,u) := \mu \int_0^u f(r,v)\,dv \le u f(r,u), \quad \text{for any } r \ge 0,\ u \in \mathbb{R}. \tag{1.6}$$

(A_4) There exists $K > 0$ such that $\inf_{r>0,|u|=K} F(r,u) > 0$.

(A_5) $f(r,u) = o(|u|)$ for $u \to 0$ uniformly in $r \ge 0$.

(A_6) f is odd in u: $f(r,-u) = -f(r,u)$ for any $r \ge 0$, $u \in \mathbb{R}$.

They state the following result.

Theorem 1.2. *Suppose $N = 4$ or $N \ge 6$. If the assumptions (A_1)–(A_6) hold, then there exists an unbounded sequence of solutions $\pm u_k$, $k \in \mathbb{N}$, of (1.1) which are not radial. The solutions are classical if f is locally Lipschitz with respect to u.*

Remark 1.3. (1) As it is mentioned in [4] that solutions of (1.1) always occur in pairs because of the oddness of f.

(2) Compared with Theorem 1.2, our result allows b to be sign-changing.

(3) Assumption (1.6) is known as *global A–R condition* which was introduced by A. Ambrosetti and R. H. Rabinowitz (see for instance in [26]). It is obvious that the second part of assumption (1.3) is weaker than (1.6).

(4) In our result, assumption (A_5) is not necessary.

In [4], (A_5) together with (A_3) plays a key role while discussing the functional φ (see later) corresponding to the system (1.1) satisfying the *(P.S.)-condition* (see [26,30,32,35]). If a function $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies (A_3) and (A_5), then for any $\varepsilon > 0$ (in application, we only concern about sufficiently small positive ε, that is $0 < \varepsilon \ll 1$), there exists a finite $C_\varepsilon = C(\varepsilon) > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon |u|^q. \tag{1.7}$$

Though in (1.7), C_ε may change for different $\varepsilon > 0$, but by (A_3) and (A_5) one can easily show that we can always assume that

$$C_\varepsilon < \infty, \quad \text{uniformly for any } \varepsilon > 0,\ x \in \mathbb{R}^N \text{ and } u \in \mathbb{R}. \tag{1.8}$$

That is, C_ε is independent of $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$. But under our assumptions in Theorem 1.1, (1.7) does not work any more, for example, let $f(r,u) = f(u) = u + |u|^{p-1}u$ $\left(1 < p < \frac{N+2}{N-2}\right)$, then one can easily check that f satisfies the conditions (B_3) and (B_5) in our result. Now we are going to prove that f also satisfies condition (B_4): firstly, we have $F(r,u) = \frac{u^2}{2} + \frac{|u|^{p+1}}{p+1}$ and $uf(r,u) = u^2 + |u|^{p+1}$, choose some $\mu \in (2, p+1)$.

From

$$0 < \mu F(r,u) \le u f(r,u),$$

that is,

$$\begin{aligned} &\frac{\mu u^2}{2} + \frac{\mu |u|^{p+1}}{p+1} \le u^2 + |u|^{p+1}, \\ \Leftrightarrow\ &\frac{\mu-2}{2}u^2 \le \frac{p+1-\mu}{p+1}|u|^{p+1}, \end{aligned}$$

$$\Leftrightarrow \frac{\mu-2}{2} \le \frac{p+1-\mu}{p+1}|u|^{p-1},$$
$$\Leftrightarrow \frac{(\mu-2)(p+1)}{2(p+1-\mu)} \le |u|^{p-1},$$
$$\Leftrightarrow \left[\frac{(\mu-2)(p+1)}{2(p+1-\mu)}\right]^{\frac{1}{p-1}} \le |u|.$$

Let $R = \left[\frac{(\mu-2)(p+1)}{2(p+1-\mu)}\right]^{\frac{1}{p-1}}$, then we know that f satisfies condition (B_4). But for any given $\varepsilon_0 > 0$, there does not exist a finite $C_{\varepsilon_0} > 0$ such that

$$|f(|x|,u)| \le \varepsilon_0|u| + C_{\varepsilon_0}|u|^p, \quad \text{uniformly for any } \varepsilon > 0,\ x \in \mathbb{R}^N \text{ and } u \in \mathbb{R}. \tag{1.9}$$

If not, we assume that for some $\varepsilon_0 > 0$ (without loss of generality, we suppose that $0 < \varepsilon_0 < 1$), there exists some finite $C = C_{\varepsilon_0} > 0$ such that

$$|f(|x|,u)| = |u| + |u|^p \le \varepsilon_0|u| + C_{\varepsilon_0}|u|^p. \tag{1.10}$$

By (1.10) we have $C_{\varepsilon_0} > 1$ and

$$(1-\varepsilon_0)|u| \le (C_{\varepsilon_0} - 1)|u|^p,$$

this implies that

$$C_{\varepsilon_0} \ge \frac{1-\varepsilon_0}{|u|^{p-1}} + 1 \to +\infty \quad \text{as } |u| \to 0.$$

This is obviously a contradiction. That is, in this case C_{ε_0} in (1.10) depends on u. Also, one can easily show that f does not satisfy (A_3) and (A_5). As far as we know, while using the *fountain theorem* [32,34] to discuss the existence of solutions of second order elliptic partial differential equations , many authors always assume that (A_5), or similar type: $f(x,u) = o(|u|)$ for $u \to 0$ uniformly in $x \in \mathbb{R}^N$ holds (see for instance in [12,25,32]).

Finally, we recall an abstract critical point lemma which we shall use later. Let X be a Banach space. We say that $I \in C^1(X,\mathbb{R})$ satisfies $(C)_c$-*condition* (or weak-(*P.S.*)-condition [35]) if any sequence $\{u_n\}$ such that

$$I(u_n) \to c, \qquad \|I'(u_n)\|(1+\|u_n\|) \to 0 \tag{1.11}$$

has a convergent subsequence.

Lemma 1.4 ([3,26]). *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X,\mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

- (I_1) *$I(0) = 0$, $I(-u) = I(u)$ for all $u \in X$;*
- (I_2) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap Z} \ge \alpha$;*
- (I_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $I(u) \le 0$ on $\tilde{X} \setminus B_R$;*

then I possesses an unbounded sequence of critical values.

2 Variational setting and proof of Theorem 1.1

Our proof is divided into a sequence of lemmas. Throughout this section, we make the following assumption instead of (B_1).
(B_1') $b \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} b(|x|) > 0$.

We work in the Hilbert space

$$X = \left\{ u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + b(|x|)u^2 \right) dx < +\infty \right\}$$

equipped with the inner product

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + b(|x|)uv)\, dx, \qquad u, v \in X,$$

the associated norm

$$\|u\| = \left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + b(|x|)u^2 \right) dx \right\}^{1/2}, \qquad u \in X.$$

Evidently, $C_0^\infty(\mathbb{R}^N, \mathbb{R}) \subset X$ and X is continuously embedded into $H^1(\mathbb{R}^N)$ and hence continuously embedded into $L^r(\mathbb{R}^N)$ for $2 \le r \le 2^*$, (where $2^* = \frac{2N}{N-2}$ for $N \ge 3$ and $2^* = \infty$ for $N = 1, 2$), i.e., there exists $S_r > 0$ such that

$$\|u\|_r \le S_r \|u\|, \qquad \forall\, u \in X, \tag{2.1}$$

where $\| \cdot \|_r$ denotes the usual norm in $L^r(\mathbb{R}^N)$ for all $2 \le r \le 2^*$. In fact we further have the following lemma due to [7].

Lemma 2.1 ([7, Lemma 3.1])**.** *Under assumptions (B_1') and (B_2), the embedding from X into $L^s(\mathbb{R}^N)$ is compact for $2 \le s < 2^*$.*

Now we define a functional Φ on X by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + b(|x|)u^2 \right) dx - \int_{\mathbb{R}^N} F(|x|, u)\, dx \tag{2.2}$$

for all $u \in X$. Then it is well known that $u \in X$ is a solution of (1.1) if and only if u is a critical point of Φ in X. By assumption (B_3), we have

$$|F(x, u)| \le \frac{a_1}{2}|u|^2 + \frac{a_2}{p+1}|u|^{p+1}, \qquad \forall\, (x, u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.3}$$

Consequently, under assumptions (B_1'), (B_2) and (B_3), the functional Φ is of class $C^1(X, \mathbb{R})$. Moreover, we have

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(|x|, u)\, dx, \qquad \forall\, u \in X, \tag{2.4}$$

$$\langle \Phi'(u), v \rangle = (u, v) - \int_{\mathbb{R}^N} f(|x|, u) v\, dx, \qquad \forall\, u, v \in X. \tag{2.5}$$

By (2.3), for $|u| < R$ (R is the same as in (B_4)), we have

$$|f(|x|, u)u| + \mu |F(|x|, u)| \le d|u|^2, \tag{2.6}$$

where $d = \frac{2+\mu}{2}a_1 + \frac{p+1+\mu}{p+1}a_2R^{p-1}$.

Now, we shall show that Φ defined as (2.4) in X satisfies all the conditions in Lemma 1.4. By (B_5), it is obvious that $\Phi(0) = 0$ and $\Phi(-u) = \Phi(u)$ for all $u \in X$. That is, (I_1) is satisfied. In order to prove that Φ satisfies the $(C)_c$-condition, we firstly introduce an inequality (see for instance in [1]) which we will use later: if $1 \le p < \infty$ and $a, b \ge 0$, then

$$(a+b)^p \le 2^{p-1}(a^p + b^p). \tag{2.7}$$

Lemma 2.2. *Under assumptions (B_1') and (B_2)–(B_4), any sequence $\{u_n\} \subset X$ satisfying*

$$\Phi(u_n) \to c > 0, \qquad \langle \Phi'(u_n), u_n \rangle \to 0 \tag{2.8}$$

is bounded in X. Moreover, $\{u_n\}$ contains a convergent subsequence.

Proof. Let $\{u_n\} \subset X$ be a sequence satisfying (2.8), for the sake of discussion below, we introduce an auxiliary function $\mathcal{F}(|x|, u) = f(|x|, u)u - \mu F(|x|, u)$ and $\Omega_n = \{x \in \mathbb{R}^N : |u_n(x)| < R\}$ where R is the same as in (B_4). By (B_4) and (2.6), without loss of generality, we may assume that for all $n \in \mathbb{N}$, we have:

$$\begin{aligned}
c+1 &\ge \Phi(u_n) - \frac{1}{\mu}\langle \Phi'(u_n), u_n \rangle \\
&= \frac{\mu-2}{2\mu}\|u_n\|^2 + \frac{1}{\mu}\int_{\mathbb{R}^N} [f(|x|, u_n)u_n - \mu F(|x|, u_n)]\, dx \\
&= \frac{\mu-2}{2\mu}\|u_n\|^2 + \frac{1}{\mu}\int_{\Omega_n} \mathcal{F}(|x|, u_n)\, dx + \frac{1}{\mu}\int_{\mathbb{R}^N\setminus\Omega_n} \mathcal{F}(|x|, u_n)\, dx \\
&\ge \frac{\mu-2}{2\mu}\|u_n\|^2 + \frac{1}{\mu}\int_{\Omega_n} \mathcal{F}(|x|, u_n)\, dx \\
&\ge \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{1}{\mu}\int_{\Omega_n} [|f(|x|, u_n)u_n| + \mu|F(|x|, u_n)|]\, dx \\
&\ge \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{d}{\mu}\int_{\mathbb{R}^N} u_n^2\, dx \\
&= \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{d}{\mu}\|u_n\|_2^2.
\end{aligned} \tag{2.9}$$

By (2.9), we have

$$\frac{\|u_n\|_2^2}{\|u_n\|^2} \ge \frac{\mu-2}{2d} - \frac{\mu(c+1)}{d\|u_n\|^2}.$$

So for sufficiently large $\|u_n\|^2$ (actually we only require $\|u_n\|^2 \ge \frac{4(c+1)\mu}{(\mu-2)}$),

$$\frac{\|u_n\|_2^2}{\|u_n\|^2} \ge \frac{\mu-2}{4d} > 0. \tag{2.10}$$

If $\{u_n\} \subset X$ is an unbounded sequence in X, passing to a subsequence if necessary, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then (2.10) implies that

$$\|v_n\|_2^2 > 0. \tag{2.11}$$

Let $A_n = \{x \in \mathbb{R}^N : v_n \ne 0\}$, then $\text{meas}(A_n) > 0$. Furthermore, under the assumption that $\|u_n\| \to \infty$ as $n \to \infty$, we obtain

$$|u_n(x)| \to \infty \quad \text{as } n \to \infty \text{ for } x \in A_n. \tag{2.12}$$

Hence $A_n \subseteq \mathbb{R}^N \setminus \Omega_n$ for sufficiently large $n \in \mathbb{N}$. By (B_4), there exists some $d_1 > 0$ such that

$$F(|x|, u) \geq d_1 |u|^\mu \quad \text{for } x \in \mathbb{R}^N \text{ and } |u| \geq R.$$

Hence by $\mu > 2$, we obtain

$$\lim_{|u| \to \infty} \frac{F(|x|, u)}{|u|^2} = +\infty. \tag{2.13}$$

By (2.1), (2.3), (2.4), (2.11), (2.13) and Fatou's lemma [21], for sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} \\
&= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(|x|, u_n)}{u_n^2} v_n^2 \, dx \\
&= \frac{1}{2} - \int_{\Omega_n} \frac{F(|x|, u_n)}{u_n^2} v_n^2 \, dx - \int_{\mathbb{R}^N \setminus \Omega_n} \frac{F(|x|, u_n)}{u_n^2} v_n^2 \, dx \\
&\leq \frac{1}{2} + \left(\frac{a_1}{2} + \frac{a_2}{p+1} R^{p-1} \right) S_2 - \liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{F(|x|, u_n)}{u_n^2} v_n^2 \, dx \\
&\leq \frac{1}{2} + \left(\frac{a_1}{2} + \frac{a_2}{p+1} R^{p-1} \right) S_2 - \int_{\mathbb{R}^N \setminus \Omega_n} \liminf_{n\to\infty} \frac{F(|x|, u_n)}{u_n^2} v_n^2 \, dx \\
&\leq \frac{1}{2} + \left(\frac{a_1}{2} + \frac{a_2}{p+1} R^{p-1} \right) S_2 - \int_{A_n} \liminf_{n\to\infty} \frac{F(|x|, u_n)}{u_n^2} [\chi_{A_n}(x)] v_n^2 \, dx \\
&\to -\infty, \quad \text{as } n \to \infty.
\end{aligned} \tag{2.14}$$

This is an obvious contradiction. Hence $\{u_n\} \subset X$ is bounded.

Now we shall prove $\{u_n\}$ contains a convergent subsequence. Without loss of generality, by the Eberlein–Shmulyan theorem (see for instance in [33]), passing to a subsequence if necessary, there exists a $u \in X$ such that $u_n \rightharpoonup u$ in X. Again by Lemma 2.1, $u_n \to u$ in $L^r(\mathbb{R}^N)$ for $2 \leq r < 2^*$, that is

$$\|u_n - u\|_r \to 0 \quad \text{as } n \to \infty, \tag{2.15}$$

and $u_n \to u$ a.e. $x \in \mathbb{R}^N$. Observe that

$$\|u_n - u\|^2 = \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle + \int_{\mathbb{R}^N} [f(|x|, u_n) - f(|x|, u)](u_n - u) \, dx. \tag{2.16}$$

It is clear that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \to 0, \qquad n \to \infty. \tag{2.17}$$

By (B_3), (2.7), (2.15), Hölder's inequality and the fact $2 < p + 1 < 2^*$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(f(|x|, u_n) - f(|x|, u))(u_n - u)| \, dx \\
&\quad \leq \int_{\mathbb{R}^N} (|f(|x|, u_n)| + |f(|x|, u)|)|u_n - u| \, dx \\
&\quad \leq \int_{\mathbb{R}^N} \left[a_1(|u_n| + |u|) + a_2(|u_n|^p + |u|^p) \right] |u_n - u| \, dx \\
&\quad \leq \int_{\mathbb{R}^N} \left[2a_1(|u_n - u| + |u|) + 2^p a_2(|u_n - u|^p + |u|^p) \right] |u_n - u| \, dx \\
&\quad = \int_{\mathbb{R}^N} \left[2a_1(|u_n - u|^2 + |u||u_n - u|) + 2^p a_2(|u_n - u|^{p+1} + |u|^p |u_n - u|) \right] dx \\
&\quad \leq 2a_1(\|u_n - u\|_2^2 + \|u\|_2 \|u_n - u\|_2) + 2^p a_2 \left(\|u_n - u\|_{p+1}^{p+1} + \|u\|_{p+1}^{\frac{p}{p+1}} \|u_n - u\|_{p+1} \right) \\
&\quad \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{2.18}$$

This together with (2.16), (2.17) implies $u_n \to u$ in X as $n \to \infty$. □

Let $\{e_j\}$ be an orthonormal basis of X and define $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^{k} X_j, \qquad Z_k = \oplus_{j=k+1}^{\infty} X_j, \qquad k \in \mathbb{Z}. \tag{2.19}$$

Lemma 2.3. *Under assumptions (B_1') and (B_2), for $2 \le r < 2^*$, we have*

$$\beta_k(s) := \sup_{u \in Z_k, \|u\|=1} \|u\|_s \to 0, \qquad k \to \infty. \tag{2.20}$$

Proof. By Lemma 2.1, $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact, then Lemma 2.3 can be proved by a similar way as Lemma 3.8 in [28] or Lemma 3.3 in [4]. □

By Lemma 2.3, we can choose an integer $m \ge 1$ such that

$$\|u\|_2^2 \le \frac{1}{2a_1}\|u\|^2, \qquad \|u\|_{p+1}^{p+1} \le \frac{p+1}{4a_2}\|u\|^{p+1}, \qquad \forall\, u \in Z_m. \tag{2.21}$$

Lemma 2.4. *Under the assumptions (B_1'), (B_2) and (B_3), there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z_m} \ge \alpha$.*

Proof. By (2.4) and (2.21), we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{a_1}{2}\|u\|_2^2 - \frac{a_2}{p+1}\|u\|_{p+1}^{p+1} \\ &\ge \frac{1}{4}(\|u\|^2 - \|u\|^{p+1}).\end{aligned} \tag{2.22}$$

Let $0 < \rho < 1$, then $\alpha = \frac{1}{4}(\rho^2 - \rho^{p+1}) > 0$ satisfies the conditions of the lemma. □

Lemma 2.5. *Under assumptions (B_1'), (B_2)–(B_4), for any finite dimensional subspace $\tilde{X} \subset X$, there holds*

$$\Phi(u) \to -\infty, \qquad \|u\| \to \infty,\ u \in \tilde{X}. \tag{2.23}$$

Proof. Arguing indirectly, assume that there exists a sequence $\{u_n\} \subset \tilde{X}$ with $\|u_n\| \to \infty$ and $M > 0$ such that $\Phi(u_n) \ge -M$ for all $n \in \mathbb{N}$. Set $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in X. Since $\tilde{X}$ is finite dimensional, then $v_n \to v \in \tilde{X}$ in X, $v_n \to v$ a.e. on $\mathbb{R}^N$, and so $\|v\| = 1$. Hence, we can conclude a contradiction by a similar fashion as (2.14). □

By (B_1), there exists a constant $b_0 > 0$ such that $\bar{b}(|x|) := b(|x|) + b_0 \ge 1$ for all $x \in \mathbb{R}^N$. Let $\bar{f}(|x|, u) = f(|x|, u) + b_0 u$. Then $\bar{b}$ and $\bar{f}$ satisfy (B_1'), (B_2)–(B_5) and it is also easy to verify the following lemma.

Lemma 2.6. *Problem (1.1) is equivalent to the following problem*

$$\begin{cases} -\triangle u + \bar{b}(|x|)u = \bar{f}(|x|, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{2.24}$$

At last, to complete the proof of Theorem 1.1, we need the following result (see [4]).

Lemma 2.7. *Let G be a group acting on X via orthogonal maps $\rho(g)\colon X \to X$ and such that the following hold.*

(i) $\Phi\colon X \to \mathbb{R}$ *is G-invariant.*

(ii) The inclusion $X^G \hookrightarrow L^s(\mathbb{R}^N)$ is compact for every $s \in (2, 2^*)$.

(iii) $\dim X^G = \infty$.

Here $X^G = \{u \in X : \rho(g)u = u \text{ for all } g \in G\}$ is the G-fixed point set. Then Φ has unbounded sequence of critical values with associated critical points lying in X^G.

Proof of Theorem 1.1. Firstly we shall find a group G and an action of G on X which satisfies the assumptions of Lemma 2.7. We should point out the main idea of the discussion below due to the work of T. Bartsch and M. Willem in [4]. $G \subset O(N)$ is defined as follows. Choose an integer $2 \le m \le \frac{N}{2}$ satisfying $2m \ne N-1$. This always holds for $N = 4$ or $N \ge 6$. The action of

$$H = O(m) \times O(m) \times O(N-2m)$$

on X is defined by

$$gu(x) = u(g^{-1}x).$$

Let τ be the involution defined on $\mathbb{R}^N = \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{N-2m}$ by

$$\tau(x_1, x_2, x_3) \doteq (x_2, x_1, x_3),$$

where $(x_1, x_2, x_3) \in \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{N-2m}$. Let $G = \langle H \cup \{\tau\} \rangle \subset O(N)$. Then elements of G can be represented uniquely as h or $h\tau$ with $h \in H$ and the action of G on X defined as

$$\begin{aligned} \rho(g)u(x) &:= hu(x) = u(h^{-1}x), && g = h \in H, \\ &:= -\tau u(x) = -u(\tau x), && g = h\tau. \end{aligned}$$

Then it is clear that 0 is the only radial function in X^G. By the work of T. Bartsh and M. Willem in [4] (also see in [32]) we know that G and the action $\rho(g)$ satisfy all the assumptions in Lemma 2.7. Thus we obtain an unbounded sequence of critical values c_k of $\Phi\colon X \to \mathbb{R}$. By Lemma 2.7, we know the associated critical points u_k lie in X^G, from discussion above we know that u_k are of nonradial solutions of (2.24). By Lemma 2.6, we know that u_k are also of nonradial solutions of (1.1). When f is locally Lipschitz with respect to u, by [14] we know that u_k are classical. □

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