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# Local solutions for a hyperbolic equation

**Aldo Trajano Lourêdo**✉[1], **Manuel Milla Miranda**[1],
**Marcondes Rodrigues Clark**[2] and **Haroldo Rodrigues Clark**[3]

[1]Universidade Estadual da Paraíba, Departamento de Matemática, Campina Grande, PB, Brazil
[2]Universidade Federal do Piauí, Departamento de Matemática, Teresina, PI, Brazil
[3]Universidade Federal Fluminense, Instituto de Matemática e Estatística, Niterói, RJ, Brazil

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**Abstract.** Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ with its boundary $\Gamma$ constituted of two disjoint parts $\Gamma_0$ and $\Gamma_1$ with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. This paper deals with the existence of local solutions to the nonlinear hyperbolic problem

$$\left| \begin{array}{ll} u'' - \triangle u + |u|^\rho = f & \text{in } \Omega \times (0, T_0), \\ u = 0 & \text{on } \Gamma_0 \times (0, T_0), \\ \dfrac{\partial u}{\partial \nu} + h(\cdot, u') = 0 & \text{on } \Gamma_1 \times (0, T_0), \end{array} \right. \qquad (*)$$

where $\rho > 1$ is a real number, $\nu(x)$ is the exterior unit normal at $x \in \Gamma_1$ and $h(x,s)$ (for $x \in \Gamma_1$ and $s \in \mathbb{R}$) is a continuous function and strongly monotone in $s$. We obtain existence results to problem $(*)$ by applying the Galerkin method with a special basis, Strauss' approximations of continuous functions and trace theorems for non-smooth functions. As usual, restrictions on $\rho$ are considered in order to have the continuous embedding of Sobolev spaces.

**Keywords:** Galerkin method, local solutions, hyperbolic equation.

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## 1 Introduction

Motivated by a nonlinear theory of mesons field introduced by L. I. Schiff [27], K. Jörgens in [5, 6] began a rigorous mathematical investigation, from a mathematical point of view, of equations of the type

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + F'(|u|^2)u = 0. \tag{1.1}$$

Specifically, K. Jörgens [6] proved the existence and uniqueness of solutions for the equation

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + \mu^2 u + \eta^2 |u|^2 u = 0 \quad \text{in } \Omega \times (0, \infty),$$

✉Corresponding author. Email: aldotl@cct.uepb.edu.br

where $\Omega$ is a bounded open set of $\mathbb{R}^n$ with boundary $\Gamma$. This equation is of the type (1.1) when

$$F(s) = \mu^2 s + \frac{1}{2}\eta^2 s^2.$$

Motivated by the works of K. Jörgens [5, 6], the authors J.-L. Lions and W. A. Strauss [28] initiated and developed a large field of research on nonlinear evolution equations that includes K. Jörgens' model. See also, F. E. Browder [1], J. A. Goldstein [3, 4], L. A. Medeiros [14], I. E. Segal [26], W. A. Strauss [28] and von Wahl [30].

Medeiros et al. [15] proved the existence and uniqueness of global solutions of the nonlinear hyperbolic problem

$$\left| \begin{array}{ll} u'' - \triangle u + |u|^\rho = f & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x) & \text{in } \Omega, \end{array} \right. \tag{1.2}$$

where $\rho > 1$ is a real number with restrictions given by the continuous embedding of Sobolev spaces and the initial data $u^0$ and $u^1$ do not have restrictions on their norms.

Considering the boundary $\Gamma$ of $\Omega$ constituted of two disjoint parts $\Gamma_0$ and $\Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and denoting by $\nu(x)$ the unit exterior normal vector at $x \in \Gamma_1$, Milla Miranda and Medeiros [20] studied the existence and uniqueness of solutions of the problem

$$\left| \begin{array}{ll} u'' - \mu(t)\triangle u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \mu(t)\dfrac{\partial u}{\partial \nu} + \delta(x)u' = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in } \Omega. \end{array} \right. \tag{1.3}$$

When $\mu > 0$ is constant, existence and uniqueness of global strong solutions for (1.3) has been proved by Komornik and Zuazua [7], Quinn and Russell [25] applying semigroup theory. This method does not work for (1.3) because the boundary condition $(1.3)_3$ depends on $\mu(t)$. For this reason Milla Miranda and Medeiros [20] constructed a special basis where lie approximations of the initial data, so the Galerkin method works well with this basis. Using this approach they proved the well-posedness for (1.3).

The existence of solutions of problem (1.3) with nonlinear boundary conditions has been obtained, by using the theory of monotone operators by Zuazua [7], Lasiecka and Tataru [18], and applying the Galerkin method by Lourêdo and Milla Miranda [12].

Motivated by (1.2) and (1.3) we consider in this paper the following problem:

$$\left| \begin{array}{ll} u'' - \triangle u + |u|^\rho = f & \text{in } \Omega \times (0,T_0), \\ u = 0 & \text{on } \Gamma_0 \times (0,T_0), \\ \dfrac{\partial u}{\partial \nu} + h(\cdot,u') = 0 & \text{on } \Gamma_1 \times (0,T_0), \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x) & \text{in } \Omega. \end{array} \right. \tag{1.4}$$

With restrictions on the real number $\rho > 1$ due to the continuous embedding of Sobolev spaces, we obtain the existence of local solutions to problem (1.4) in two cases: first, $h(x,s) = \delta(x)p(s)$ with $p$ Lipschitzian and strongly monotone. In the second case $h(x,s)$ is only continuous in $s$ and strongly monotone in $s$, and the initial data belong to a class more regular than in the first case. In our approach, we apply the Galerkin method with a special basis, the Strauss' approximations of continuous functions and trace theorems for non-smooth functions.

It is worth noting that the term $\int_\Omega |u|^\rho u'\,dx$ does not have a definite sign. This fact brings serious difficulties to obtain global solutions to problem (1.4) without considering restrictions on the norms of the initial data.

Hereafter, this paper is organized in three sections, namely, Section 2 is devoted to the notations and statements of the two main results. In Section 3 we present the proof of Theorem 2.1 in which the case $h = \delta p$ is considered, where $p$ is a Lipschitz continuous function. In Section 4, Theorem 2.2 is proved which contains the case where $s \mapsto h(\cdot, s)$ is only a continuous function in $s$.

## 2 Notations and main results

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ with a $C^2$ boundary $\Gamma$, which has two disjoint parts $\Gamma_0$ and $\Gamma_1$ such that $\operatorname{meas}\Gamma_0 > 0$; $\operatorname{meas}\Gamma_1 > 0$; and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Let $\nu(x)$ be the unit normal vector at $x \in \Gamma_1$.

The scalar product and norm of the space $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. We represent by $V$ the Hilbert space $V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_0\}$, equipped with the scalar product and norm

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\,dx \quad \text{and} \quad \|u\|^2 = ((u,u)),$$

respectively. All scalar functions considered in this paper are real-valued.

In what follows, we introduce necessary hypotheses on some objects of problem (1.4) in order to state our first result.

Let $p\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying:

$$\begin{aligned}&p \text{ is Lipschitz-continuous and strongly monotone in the second variable, i.e.}\\ &(p(s)-p(r))(s-r) \ge b_0(s-r)^2,\ \forall s,\ r \in \mathbb{R},\ (b_0 \text{ is a positive constant}).\end{aligned} \tag{2.1}$$

The function $\delta\colon \Gamma_1 \to \mathbb{R}$ is such that

$$\delta \in W^{1,\infty}(\Gamma_1) \text{ and } \delta(x) \ge \delta_0,\ \forall x \in \Gamma_1\ (\delta_0 \text{ is a positive constant}). \tag{2.2}$$

The real number $\rho$ is chosen according to the spatial dimension $n$.

$$\rho > 1 \quad \text{if } n = 1,2 \quad \text{and} \quad \frac{n+1}{n} \le \rho \le \frac{n}{n-2} \quad \text{if } n \ge 3. \tag{2.3}$$

**Theorem 2.1.** *Suppose* (2.1)–(2.3) *hold,* $f \in H^1(0,T;L^2(\Omega))$, $\{u^0,u^1\} \in (V \cap H^2(\Omega)) \times V$ *and satisfies the compatibility condition*

$$\frac{\partial u^0}{\partial \nu} + \delta(\cdot)p(u^1) = 0 \quad \text{on } \Gamma_1. \tag{2.4}$$

*Then there exist a real number* $T_0$ *with* $0 < T_0 \le T$ *and a unique function* $u$ *in the class*

$$\begin{aligned} &u \in L^\infty(0,T_0;V \cap H^2(\Omega)),\\ &u' \in L^\infty(0,T_0;V),\\ &u'' \in L^\infty(0,T_0;L^2(\Omega)) \cap L^2(0,T_0;L^2(\Gamma_1)),\end{aligned} \tag{2.5}$$

*satisfying the equations*

$$u'' - \triangle u + |u|^\rho = f \quad \text{in } L^\infty(0,T_0;L^2(\Omega)), \tag{2.6}$$

$$\begin{aligned} &\frac{\partial u}{\partial \nu} + \delta p(u') = 0 \quad \text{in } L^\infty(0,T_0;H^{1/2}(\Gamma_1)), \\ &\frac{\partial u'}{\partial \nu} + \delta p'(u')u'' = 0 \quad \text{in } L^\infty(0,T_0;L^2(\Gamma_1)), \end{aligned} \tag{2.7}$$

*and the initial conditions*

$$u(0) = u^0, \qquad u'(0) = u^1. \tag{2.8}$$

*Moreover, $T_0$ is explicitly given by*

$$T_0 = \min\left\{ \frac{1}{2L}\left(\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right)^{(1-\rho)/2},\ T \right\}, \tag{2.9}$$

*where*

$$L = \frac{(\rho-1)}{2}\left[2^{1/2}\|f\|_{L^\infty(0,T;L^2(\Omega))} + 2^{(\rho+1)/2}k_1^\rho\right] \tag{2.10}$$

*and $k_1 > 0$ is the constant of the continuous embedding of $V$ in $L^2(\Omega)$, defined in inequality (3.2).*

To state our second result we make the following considerations: let $A = -\triangle$ be the self-adjoint operator of $L^2(\Omega)$ defined by the triplet $\{V, L^2(\Omega); ((\cdot,\cdot))\}$. Then the domain of $-\triangle$ is given by

$$D(-\triangle) = \left\{ u \in V \cap H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\} \tag{2.11}$$

and it is known that $D(-\triangle)$ is dense in $V$, see this statement for instance, in Lions [10].

We suppose the function $h\colon \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned} &r \mapsto h(\cdot,r) \in C^0(\mathbb{R};L^\infty(\Gamma_1)),\ h(x,0) = 0, \text{ for almost all } x \in \Gamma_1, \\ &h \text{ is strongly monotone in the second variable, i.e.} \\ &[h(x,r) - h(x,s)]\,(r-s) \geq d_0(r-s)^2,\ \forall r,s \in \mathbb{R}, \text{ for almost all } x \in \Gamma_1, \\ &(d_0 \text{ is a positive constant}). \end{aligned} \tag{2.12}$$

**Theorem 2.2.** *Assume that hypotheses (2.3)–(2.12) are satisfied, $f \in H^1(0,T;L^2(\Omega))$ and $\{u^0,u^1\} \in D(-\triangle) \times H_0^1(\Omega)$. Then there exist a real number $T_0 > 0$ (the same $T_0$ given in Theorem 2.1) and at least one function $u$ in the class*

$$\begin{aligned} &u \in L^\infty(0,T_0;V),\ \triangle u \in L^\infty(0,T_0;L^2(\Omega)), \\ &u' \in L^\infty(0,T_0;V), \\ &u'' \in L^\infty(0,T_0;L^2(\Omega)) \cap L^2(0,T_0;L^2(\Gamma_1)), \end{aligned} \tag{2.13}$$

*satisfying*

$$u'' - \triangle u + |u|^\rho = f \quad \text{in } L^\infty(0,T_0;L^2(\Omega)), \tag{2.14}$$

$$\frac{\partial u}{\partial \nu} + h(\cdot,u') = 0 \quad \text{in } L^1(0,T_0;L^1(\Gamma_1)), \tag{2.15}$$

*and*

$$u(0) = u^0, \qquad u'(0) = u^1. \tag{2.16}$$

**Remark 2.3.** Note that in Theorem 2.2 the set of initial data satisfies $\frac{\partial u^0}{\partial \nu} + h(\cdot,u^1) = 0$.

**Remark 2.4.** As $h(x,s)$ is only continuous in $s$, the uniqueness of solutions of Theorem 2.2 is an open problem.

## 3 Case $p$ Lipschitz

We begin by making some considerations. Since $p\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function then $p(v) \in H^{1/2}(\Gamma_1)$ for $v \in H^{1/2}(\Gamma_1)$ and the mapping $p\colon H^{1/2}(\Gamma_1) \to H^{1/2}(\Gamma_1),\ \ v \mapsto p(v)$ is continuous, for this result we refer to Marcus and Mizel [13].

**Remark 3.1.** The regularity of the trace mapping of order zero, $\gamma_0\colon V \to H^{1/2}(\Gamma_1)$ ensures that the mapping $\widetilde{p} = p \circ \gamma_0$ with $\widetilde{p}\colon V \to H^{1/2}(\Gamma_1)$ is continuous.

**Remark 3.2.** Throughout this section, in order to facilitate the notation, the mapping $\widetilde{p}(v)$ for $v \in V$ will be denoted just by $p(v)$.

**Remark 3.3.** Since $\delta \in W^{1,\infty}(\Gamma_1)$ then $\delta v \in H^{1/2}(\Gamma_1)$ for all $v \in H^{1/2}(\Gamma_1)$, and the linear operator $\delta\colon H^{1/2}(\Gamma_1) \to H^{1/2}(\Gamma_1),\ \ v \mapsto \delta v$ is continuous. In fact, using the theory of interpolation for Hilbert spaces (see for instance the reference [11]) it can be shown that the linear operators $\delta\colon H^1(\Gamma_1) \to H^1(\Gamma_1),\ v \mapsto \delta v$ and $\delta\colon L^2(\Gamma_1) \to L^2(\Gamma_1),\ v \mapsto \delta v$ are continuous.

**Remark 3.4.** As a consequence of (2.11), the intersection $V \cap H^2(\Omega)$ is dense in $V$.

**Proposition 3.5.** *In $V \cap H^2(\Omega)$ the norm of $H^2(\Omega)$ and the norm*

$$u \mapsto \left[ |\triangle u|^2 + \left\| \frac{\partial u}{\partial \nu} \right\|^2_{H^{1/2}(\Gamma_1)} \right]^{1/2}$$

*are equivalent.*

**Proposition 3.6.** *Let $\delta \in W^{1,\infty}(\Gamma_1),\ \ p\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with $p(0) = 0$, $u^0 \in V \cap H^2(\Omega),\ \ u^1 \in V$, and*

$$\frac{\partial u^0}{\partial \nu} + \delta(\cdot)p(u^1) = 0 \quad \text{on } \Gamma_1.$$

*Then, for each $\varepsilon > 0$ there exist $w$ and $z$ in $V \cap H^2(\Omega)$ such that*

$$\|w - u^0\|_{V \cap H^2(\Omega)} < \varepsilon, \qquad \|z - u^1\| < \varepsilon \quad \text{and} \quad \frac{\partial w}{\partial \nu} + \delta(\cdot)p(z) = 0 \quad \text{on } \Gamma_1.$$

The proof of the preceding propositions can be found in Milla Miranda and Medeiros [20] and Milla Miranda and Lourêdo [19].

Under the restrictions (2.3) on $\rho$, we have $(\rho - 1)n \leq 2\rho \leq \frac{2n}{n-2} = q$ for $n \geq 3$, and this implies

$$V \hookrightarrow L^q(\Omega) \hookrightarrow L^{2\rho}(\Omega) \hookrightarrow L^{(\rho-1)n}(\Omega), \qquad n \geq 3. \tag{3.1}$$

In (3.1) we mean by $X \hookrightarrow Y$ that the spaces $X, Y$ satisfy $X \subset Y$ and the injection of $X$ in $Y$ is continuous. We denote by $k_0, k_1$ and $k_2$ the constants immersion that satisfying

$$\|u\|_{L^q(\Omega)} \leq k_0\|u\|, \qquad \|u\|_{L^{2\rho}(\Omega)} \leq k_1\|u\|, \qquad \|u\|_{L^{(\rho-1)n}(\Omega)} \leq k_2\|u\| \qquad \forall u \in V. \tag{3.2}$$

We now can proceed to the proof of our first result.

**Proof of Theorem 2.1.** Proposition 3.6 provides us sequences $(u_l^0)$ and $(u_l^1)$ of vectors of $V \cap H^2(\Omega)$ such that

$$\left| \begin{array}{ll} \lim_{l\to\infty} u_l^0 = u^0 & \text{in } V \cap H^2(\Omega), \\ \lim_{l\to\infty} u_l^1 = u^1 & \text{in } V, \\ \dfrac{\partial u_l^0}{\partial \nu} + \delta p(u_l^1) = 0 & \text{on } \Gamma_1 \text{ for } l \in \mathbb{N}. \end{array} \right. \tag{3.3}$$

We now construct a special basis of $V \cap H^2(\Omega)$ in the following way: for $l \in \mathbb{N}$ we consider the basis $\{w_1^l, w_2^l, \ldots, w_j^l, \ldots\}$ of $V \cap H^2(\Omega)$ satisfying $u_l^0, u_l^1 \in [w_1^l, w_2^l]$, where $[w_1^l, w_2^l]$ denotes the subspace generated by $w_1^l$, $w_2^l$. According to this basis we determine approximate solutions $u_{lm}(t)$ of problem (3.4) with $h = \delta p$, that is, $u_{lm}(t) = \sum_{j=1}^m g_{jlm}(t) w_j^l$, where $g_{jlm}(t)$ is defined as the solutions of the approximate problem

$$\left| \begin{array}{l} (u''_{lm}(t), v) + ((u_{lm}(t), v)) + (|u_{lm}(t)|^\rho, v) + \displaystyle\int_{\Gamma_1} \delta p(u'_{lm}(t)) v \, d\Gamma = (f(t), v), \quad \forall v \in V_m^l, \\ u_{lm}(0) = u_l^0, \; u'_{lm}(0) = u_l^1, \end{array} \right. \tag{3.4}$$

where $V_m^l$ is the subspace generated by $w_1^l, w_2^l, \ldots, w_m^l$. The solution $u_{lm}$ of (3.4) is defined on $[0, t_{lm})$ with $0 < t_{lm} \le T_0$. The next estimate enables us to extend $u_{lm}$ to the whole interval $[0, T_0]$.

**First estimate:** Setting $v = u'_{lm}(t)$ in (3.4), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u'_{lm}(t)|^2 + \frac{1}{2}\frac{d}{dt}\|u_{lm}(t)\|^2 + \int_{\Gamma_1} \delta p(u'_{lm}(t)) u'_{lm}(t) \, d\Gamma \\ &\quad = (f(t), u'_{lm}(t)) - (|u_{lm}(t)|^\rho, u'_{lm}(t)). \end{aligned} \tag{3.5}$$

By usual inequalities and (3.2) we get

$$|(|u_{lm}(t)|^\rho, u'_{lm}(t))| \le \|u_{lm}(t)\|^\rho_{L^{2\rho}(\Omega)} |u'_{lm}(t)| \le k_1^\rho \|u_{lm}(t)\|^\rho |u'_{lm}(t)|. \tag{3.6}$$

Taking

$$\varphi_{lm}(t) = \frac{1}{2}|u'_{lm}(t)|^2 + \frac{1}{2}\|u_{lm}(t)\|^2 + 1 \tag{3.7}$$

and combining (3.6), (3.7), (3.5), and using hypotheses (2.1) and (2.2) on $p$ and $\delta$, we get

$$\frac{d}{dt}\varphi_{lm}(t) + \delta_0 b_0 \int_{\Gamma_1} u'^2_{lm}(t) \, d\Gamma \le |f(t)||u'_{lm}(t)| + k_1^\rho \|u_{lm}(t)\|^\rho |u'_{lm}(t)|. \tag{3.8}$$

Observing that

$$\|u_{lm}(t)\|^\rho \le 2^{\frac{\rho}{2}} \varphi_{lm}^{\frac{\rho}{2}}(t) \quad \text{and} \quad |u'_{lm}(t)| \le 2^{\frac{1}{2}} \varphi_{lm}^{\frac{1}{2}}(t),$$

and together with $\varphi_{lm}(t) \ge 1$, we find

$$|f(t)||u'_{lm}(t)| + k_1^\rho \|u_{lm}(t)\|^\rho |u'_{lm}(t)| \le \left[ 2^{\frac{1}{2}} \|f\|_{L^\infty(0,T;L^2(\Omega))} + 2^{\frac{\rho+1}{2}} k_1^\rho \right] \varphi_{lm}^{\frac{\rho+1}{2}}(t).$$

Combining this inequality with (3.8), we derive

$$\frac{d}{dt}\varphi_{lm}(t) + \delta_0 b_0 \int_{\Gamma_1} u'^2_{lm}(t) \, d\Gamma \le M \varphi_{lm}^{\frac{\rho+1}{2}}(t),$$

where

$$M = 2^{\frac{1}{2}} \|f\|_{L^\infty(0,T;L^2(\Omega))} + 2^{\frac{\rho+1}{2}} k_1^{\rho}.$$

This means that

$$\varphi_{lm}^{-\frac{\rho+1}{2}}(t) \frac{d}{dt} \varphi_{lm}(t) \leq M, \tag{3.9}$$

and recalling the identity

$$\frac{d}{dt}\left[\frac{2}{1-\rho}\varphi_{lm}^{\frac{1-\rho}{2}}(t)\right] = \varphi_{lm}^{-\frac{\rho+1}{2}}(t)\frac{d}{dt}\varphi_{lm}(t).$$

We obtain by (3.9) that

$$\frac{d}{dt}\varphi_{lm}^{\frac{1-\rho}{2}}(t) \geq -\frac{\rho-1}{2}M,$$

which implies

$$\varphi_{lm}^{\frac{1-\rho}{2}}(t) \geq \varphi_{lm}^{\frac{1-\rho}{2}}(0) - Lt. \tag{3.10}$$

In the above expression $L$ was defined in (2.10). By the convergences in (3.3) we find

$$\varphi_{lm}^{\frac{\rho-1}{2}}(0) < \left[\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right]^{\frac{\rho-1}{2}}, \qquad \forall l \geq l_0,\ \forall m,$$

which is equivalent to

$$\varphi_{lm}^{\frac{1-\rho}{2}}(0) > \left[\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right]^{\frac{1-\rho}{2}}, \qquad \forall l \geq l_0,\ \forall m.$$

Thus, from (3.10) it follows that

$$\varphi_{lm}^{\frac{1-\rho}{2}}(t) \geq \left[\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right]^{\frac{1-\rho}{2}} - Lt. \tag{3.11}$$

By hypothesis (2.9), we obtain

$$Lt \leq \frac{1}{2}\left[\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right]^{\frac{1-\rho}{2}}, \qquad \forall t \in [0, T_0].$$

Then (3.11) provides

$$\varphi_{lm}^{\frac{1-\rho}{2}}(t) \geq \frac{1}{2}\left[\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right]^{\frac{1-\rho}{2}}, \qquad \forall t \in [0, T_0].$$

Thus

$$\varphi_{lm}(t) \leq 2^{\frac{2}{\rho-1}}\left[\frac{1}{2}|u^1|^2 + \frac{1}{2}\|u^0\|^2 + 2\right] = N, \qquad \forall t \in [0, T_0],\ \forall l \geq l_0,\ \forall m. \tag{3.12}$$

With this limitation and taking into account (3.7), we find

$$\begin{aligned} &(u_{lm}) \text{ is bounded in } L^\infty(0, T_0; V), \\ &(u'_{lm}) \text{ is bounded in } L^\infty(0, T_0; L^2(\Omega)), \\ &(u'_{lm}) \text{ is bounded in } L^2(0, T_0; L^2(\Gamma_1)), \end{aligned} \tag{3.13}$$

where these limitations are independent of $l \geq l_0$ and $m$.

**Second estimate:** Differentiating the approximate equation in (3.4)$_1$ with respect to t and taking $v = u''_{lm}(t)$, we find

$$\frac{1}{2}\frac{d}{dt}[|u''_{lm}(t)|^2 + \|u'_{lm}(t)\|^2] + (\rho|u_{lm}(t)|^{\rho-2}u_{lm}(t)u'_{lm}(t), u''_{lm}(t)) + \int_{\Gamma_1}\delta p'(u'_{lm}(t))[u''_{lm}(t)]^2 d\Gamma = (f'(t), u''_{lm}(t)) \tag{3.14}$$

By Hölder's inequality with $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$, we obtain

$$|(|u_{lm}(t)|^{\rho-2}u'_{lm}(t), u''_{lm}(t))| \le \|u_{lm}(t)\|^{\rho-1}_{L^{(\rho-1)n}(\Omega)}\|u'_{lm}(t)\|_{L^q(\Omega)}|u''_{lm}(t)|$$

From this, notations (3.2) and using the constant $N$ introduced in (3.12) that bounds $\varphi_{lm}(t)$, we get

$$\begin{aligned}\rho|(|u_{lm}(t)|^{\rho-2}u'_{lm}(t), u''_{lm}(t))| &\le \rho k_2^{\rho-1}k_0\|u'_{lm}(t)\|^{\rho-1}\|u'_{lm}(t)\||u''_{lm}(t)| \\ &\le \rho k_2^{\rho-1}k_0 2^{\frac{\rho-1}{2}}N^{\frac{\rho-1}{2}}\|u'_{lm}(t)\||u''_{lm}(t))|,\end{aligned}$$

that is

$$\rho|(|u_{lm}(t)|^{\rho-2}u'_{lm}(t), u''_{lm}(t))| \le R\|u'_{lm}(t)\||u''_{lm}(t))|$$

where

$$R = \rho k_2^{\rho-1}k_0 2^{\frac{\rho-1}{2}}N^{\frac{\rho-1}{2}}.$$

Combining the last inequality with (3.14) and considering hypothesis (2.1)$_3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[|u''_{lm}(t)|^2 + \|u'_{lm}(t)\|^2] + \delta_0 b_0\int_{\Gamma_1}[u''_{lm}(t)]^2 d\Gamma \\ &\quad\le \frac{1}{2}|f'(t)|^2 + \frac{R^2}{2}\|u'_{lm}(t)\|^2 + \frac{1}{2}|u''_{lm}(t)|, \qquad \forall t \in [0, T_0].\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}\left[|u''_{lm}(t)|^2 + \|u'_{lm}(t)\|^2\right] + \delta_0 b_0\int_0^t\int_{\Gamma_1}[u''_{lm}(t)]^2\, d\Gamma \\ &\quad\le \frac{1}{2}|u''_{lm}(0)|^2 + \frac{1}{2}\|u_{lm}(0)\|^2 + \frac{1}{2}\int_0^T|f'(t)|^2\, dt \\ &\quad\quad + \int_0^t\left[\frac{R^2}{2}\|u'_{lm}(s)\|^2 + \frac{1}{2}|u''_{lm}(s)|\right] ds, \qquad \forall t \in [0, T_0].\end{aligned} \tag{3.15}$$

**Remark 3.7.** To apply Gronwall's lemma in inequality (3.15) we need to derive an upper bound for $(u''_{lm}(0))$. This is the key point of the proof of Theorem 2.1. We get this limitation thanks to the choice of the special basis of $V \cap H^2(\Omega)$, previously built in this section.

We make $t = 0$ in the approximate system (3.4)$_1$, take $v = u''_{lm}(0)$, and after that apply Gauss' theorem, to obtain

$$|u''_{lm}(0)|^2 + (-\triangle u_l^0, u'_{lm}(0)) + \int_{\Gamma_1}\left[\frac{\partial u_l^0}{\partial \nu} + \delta p(u_l^1)\right]u''_{lm}(0)\, d\Gamma + (|u_l^0|^\rho, u''_{lm}(0)) = (f(0), u''_{lm}(0)).$$

Using (3.3)$_3$, we find

$$|u''_{lm}(0)|^2 \le |\triangle u_l^0||u''_{lm}(0)| + \|u_l^0\|^\rho_{L^{2\rho}(\Omega)}|u''_{lm}(0)| + |f(0)||u''_{lm}(0)|.$$

From this and convergence (3.3)$_1$ and (3.3)$_2$, and notations (3.2) we get

$$|u''_{lm}(0)|^2 \le \left[|\triangle u_l^0| + k_1\|u^0\| + |f(0)| + 1 + k_1^\rho\right]|u''_{lm}(0)|, \qquad \forall l \ge l_0,\ \forall m.$$

Thus,

$$|u''_{lm}(0)| \le |\triangle u_l^0| + k_1\|u^0\| + |f(0)| + 1 + k_1^\rho = S, \qquad \forall l \ge l_0,\ \forall m. \tag{3.16}$$

Inequality (3.15), (3.16), convergences $(3.3)_1$, $(3.3)_2$ and Gronwall's lemma provide

$$\frac{1}{2}|u''_{lm}(t)| + \frac{1}{2}\|u'_{lm}(t)\|^2 + \delta_0 b_0 \int_0^t \|u''_{lm}(s)\|^2_{L^2(\Gamma_1)}\, d\Gamma \le P, \tag{3.17}$$

where the constant $P$ is independent of $l \ge l_0$, $m$ and $t \in [0, T_0]$.

With the above estimate, we obtain

$$\begin{aligned} &(u'_{lm}) \text{ is bounded in } L^\infty(0, T_0; V),\\ &(u''_{lm}) \text{ is bounded in } L^\infty(0, T_0; L^2(\Omega)),\\ &(u''_{lm}) \text{ is bounded in } L^2(0, T_0; L^2(\Gamma_1)). \end{aligned} \tag{3.18}$$

**Passage to the limit in *m*.** The constants $N$ and $P$ in (3.12) and (3.17) are independent of $l \ge l_0$, $m$ and $t \in [0, T_0]$. Thus, the estimates (3.13) and (3.18), allow us to find a subsequence of $(u_{lm})$, which still will be denoted by $(u_{lm})$, and a function $u_l$ such that

$$\begin{aligned} &u_{lm} \to u_l \text{ weak star in } L^\infty(0, T_0; V),\\ &u'_{lm} \to u'_l \text{ weak star in } L^\infty(0, T_0; V),\\ &u''_{lm} \to u''_l \text{ weak star in } L^\infty(0, T_0; L^2(\Omega)),\\ &u'_{lm} \to u'_l \text{ weak in } L^2(0, T_0; L^2(\Gamma_1)),\\ &u''_{lm} \to u''_l \text{ weak in } L^2(0, T_0; L^2(\Gamma_1)). \end{aligned} \tag{3.19}$$

The convergence $(3.19)_1$, $(3.19)_2$ and the Aubin–Lions theorem provide the convergence $u_{lm} \to u_l$ in $L^2(0, T_0; L^2(\Omega))$. Therefore,

$$|u_{lm}(x,t)|^\rho \to |u_l(x,t)|^\rho \quad \text{a.e. in } \Omega \times (0, T_0) = Q_0. \tag{3.20}$$

We also have

$$\int_\Omega [|u_{lm}(t)|^\rho]^2\, dx \le k_1 \|u_{lm}(t)\|^{2\rho} \le k_1^{2\rho} 2^{2\rho} N^\rho,$$

where $k_1$ and $N$ are introduced in (3.2) and (3.12), respectively. From this

$$\||u_{lm}|^\rho\|_{L^\infty(0,T_0;L^2(\Omega))} \le C, \qquad \forall l \ge l_0 \quad \text{and} \quad \forall m. \tag{3.21}$$

Applying Lemma 3.1 of Lions [8] and using (3.20) and (3.21),

$$|u_{lm}|^\rho \to |u_l|^\rho \text{ weak star in } L^\infty(0, T_0; L^2(\Omega)). \tag{3.22}$$

On the other hand, the convergence $(3.19)_2$ implies $u'_{lm} \to u'_l$ weak in $L^2(0, T_0; H^{1/2}(\Gamma_1))$. This, convergence $(3.19)_5$ and the Aubin–Lions theorem provide $u'_{lm} \to u'_l$ in $L^2(0, T_0; L^2(\Gamma_1))$. As $p$ is Lipschitzian, $\delta p(u'_{lm}) \to \delta p(u'_l)$ in $L^2(0, T_0; L^2(\Gamma_1))$. Note that $|\delta p(u'_{lm})|_{L^2(\Gamma_1)} \le C$ for all $l \ge l_0$, for all $m$ and $t \in [0, T_0]$. This and the preceding convergence provide

$$\delta p(u'_{lm}) \to \delta p(u'_l) \quad \text{in } L^\infty(0, T_0; L^2(\Gamma_1)). \tag{3.23}$$

Now taking $\theta \in L^2(0, T_0)$ and $v \in V$ then the convergence (3.19), (3.22), and (3.23) allow us to pass to the limit in the approximate equation (3.4). Moreover, observing that $V \cap H^2(\Omega)$ is dense in $V$, we obtain

$$\begin{aligned} &\int_0^{T_0} (u''_l(t), v)\theta\, dt + \int_0^{T_0} ((u_l(t), v))\theta\, dt + \int_0^{T_0} (|u_l(t)|^\rho, v)\theta\, dt + \int_0^{T_0}\int_{\Gamma_1} \delta p(u'_l(t)) v\theta\, d\Gamma\, dt\\ &\quad = \int_0^{T_0} (f(t), v)\theta\, dt. \end{aligned} \tag{3.24}$$

Taking, in particular, $v \in \mathcal{D}(\Omega)$ and $\theta \in \mathcal{D}(0, T_0)$, in the preceding equation yields

$$u_l'' - \triangle u_l + |u_l|^\rho = f \quad \text{in } \mathcal{D}'(Q_0), \quad \text{where } Q_0 = \Omega \times (0, T_0).$$

The regularities of $u_l$ and $f$ permit us to write

$$u_l'' - \triangle u_l + |u_l|^\rho = f \quad \text{in } L^\infty(0, T_0; L^2(\Omega)). \tag{3.25}$$

Since $u_l \in L^\infty(0, T_0; V)$ and $\triangle u_l \in L^\infty(0, T_0; L^2(\Omega))$, then $\frac{\partial u_l}{\partial \nu} \in L^\infty(0, T_0; H^{-1/2}(\Gamma_1))$, as shown in Milla Miranda [21]. Multiplying both sides of (3.25) by $v\theta$, with $v \in V$ and $\theta \in L^2(0, T_0)$, and integrating on $Q_0$, we obtain

$$\int_0^{T_0} (u_l''(t), v)\theta \, dt + \int_0^{T_0} ((u_l(t), v))\theta \, dt + \int_0^{T_0} (|u_l(t)|^\rho, v)\theta \, dt + \int_0^{T_0} \left\langle \frac{\partial u_l(t)}{\partial \nu}, v \right\rangle \theta \, dt$$
$$= \int_0^{T_0} (f(t), v)\theta \, dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$. This equality and (3.24) imply

$$\frac{\partial u_l}{\partial \nu} + \delta p(u_l') = 0 \quad \text{in } L^2(0, T_0; H^{-1/2}(\Gamma_1)).$$

Since $u_l' \in L^\infty(0, T_0; H^{1/2}(\Gamma_1))$, then $\delta p(u_l') \in L^\infty(0, T_0; H^{1/2}(\Gamma_1))$, and thus

$$\frac{\partial u_l}{\partial \nu} + \delta p(u_l') = 0 \quad \text{in } L^\infty(0, T_0; H^{1/2}(\Gamma_1)). \tag{3.26}$$

By the facts $u_l, \triangle u_l \in L^\infty(0, T_0; L^2(\Omega))$ and $\frac{\partial u_l}{\partial \nu} \in L^\infty(0, T_0; H^{\frac{1}{2}}(\Gamma_1))$, and by Proposition 3.5, we conclude

$$u_l \in L^\infty(0, T_0; V \cap H^2(\Omega)). \tag{3.27}$$

Differentiating with respect to $t$ the equality (3.26) and noting that $u_l'' \in L^2(0, T_0; L^2(\Gamma_1))$, we obtain the regularities in (2.7). As estimates (3.13) and (3.18) are independent of $l \geq l_0$, we obtain in a similar way a function $u$ in class (2.5), $u$ satisfying (2.6)-(2.8).

The verification of the initial data (2.8) follows from estimatee $(3.19)_1$– $(3.19)_3$.

**Uniqueness of solutions.** Let $u$ and $v$ be two functions in class (2.5) which satisfy equations (2.6), (2.7) and initial conditions (2.8). Considering the difference $w = u - v$, we have

$$\begin{aligned} & w'' - \triangle w + |u|^\rho - |v|^\rho = \quad \text{in } L^\infty(0, T_0; L^2(\Omega)), \\ & \frac{\partial w}{\partial \nu} + \delta\left[p(u') - p(v')\right] = 0 \quad \text{in } L^\infty(0, T_0; H^{\frac{1}{2}}(\Gamma_1)), \\ & w(0) = 0, \qquad w'(0) = 0. \end{aligned} \tag{3.28}$$

Multiplying both sides of $(3.28)_1$ by $w'$, integrating on $\Omega$ and using Gauss' theorem, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|w'(t)|^2 + \|w(t)\|^2\right] + \int_{\Gamma_1} \delta[p(u'(t)) - p(v'(t))] \, d\Gamma = -\left(|u(t)|^\rho - |v(t)|^\rho, w'(t)\right). \tag{3.29}$$

By the mean value theorem, $|u(x,t)|^\rho - |v(x,t)|^\rho = \rho|\xi|^{\rho-2}\xi w(x,t)$, where $\xi$ is between $u(x,t)$ and $v(x,t)$, and thus $\left||u(x,t)|^\rho - |v(x,t)|^\rho\right| \leq |g(x,t)|^{\rho-2}|w(x,t)|$, where $g(x,t) = |u(x,t)| +$

$|v(x,t)|$. Therefore

$$\begin{aligned}|(|u(t)|^\rho - |v(t)|^\rho)w'(t)| &\leq \rho \int_\Omega g^{\rho-1}|w(t)||w'(t)|\,dx \\ &\leq \rho\|g\|^{\rho-1}_{L^{(\rho-1)n}(\Omega)}\|w(t)\|_{L^q(\Omega)}|w'(t)| \\ &\leq \rho\left[\|u(t)\|_{L^{(\rho-1)n}(\Omega)} + \|v(t)\|_{L^{(\rho-1)n}(\Omega)}\right]^{\rho-1}\|w(t)\|_{L^q(\Omega)}|w'(t)|.\end{aligned} \tag{3.30}$$

From the embedding (3.1) we find $\|u(t)\|_{L^{(\rho-1)n}(\Omega)} \leq k_2\|u(t)\| \leq C$, $\forall t \in [0, T_0]$ and similarly, $\|v(t)\|_{L^{(\rho-1)n}(\Omega)} \leq k_2\|u(t)\| \leq C$, $\forall t \in [0, T_0]$. Combining (3.30) with the two preceding inequalities, we get

$$|(|u(t)|^\rho - |v(t)|^\rho)| \leq C_1\|w(t)\||w'(t)| \leq \frac{C_1^2}{2}\|w(t)\|^2 + \frac{1}{2}|w'(t)|^2.$$

This inequality, (3.29) and properties (2.1) of $p$, imply

$$\frac{1}{2}\frac{d}{dt}\left[|w'(t)|^2 + \|w(t)\|^2\right] + \delta_0 b_0 \int_{\Gamma_1} w'(t)^2\,d\Gamma \leq \frac{1}{2}|w'(t)| + \frac{C_1^2}{2}\|w(t)\|^2.$$

Then the Gronwall inequality provides $w'(t) = 0$ and $w(t) = 0$ a.e. in $[0, T_0]$. This concludes the proof of Theorem 2.1. □

## 4 Case $h(x,s)$ continuous in $s$

Initially note that, since $h$ is a continuous function, the following Strauss' approximations were shown by Louredo and Milla Miranda [12] .

**Proposition 4.1.** *Assume that $h$ satisfies hypotheses (2.11). Then there exists a sequence $(h_l)$ of functions of $C^0(\mathbb{R}; L^\infty(\Gamma_1))$ satisfying the following conditions:*

*(i)* $h_l(x, 0) = 0$ *for almost all $x$ in $\Gamma_1$;*

*(ii)* $[h_l(x,s) - h_l(x,r)]\,(s-r) \geq d_0(s-r)^2$, $\forall s, r \in \mathbb{R}$ *and for almost all $x$ in $\Gamma_1$;*

*(iii) there exists a function $c_l \in L^\infty(\Gamma_1)$ such that*

$$|h_l(x,s) - h_l(x,r)| \leq c_l(x)|s-r|, \qquad \forall s, r \in \mathbb{R} \quad \textit{for almost all } x \textit{ in } \Gamma_1;$$

*(iv) $(h_l)$ converges to $h$ uniformly on bounded sets of $\mathbb{R}$ for almost all $x$ in $\Gamma_1$.*

**Proof of Theorem 2.1.** We proceed as in Theorem 2.1, changing the function $\delta(x)p(s)$ into $h_l(x,s)$. Let $(u_l^1)$ be a sequence of functions of $\mathcal{D}(\Omega)$ such that

$$u_l^1 \to u^1 \quad \text{in } H_0^1(\Omega). \tag{4.1}$$

Note that

$$\frac{\partial u^0}{\partial \nu} + h_l(\cdot, u_l^1) = 0, \qquad \forall l. \tag{4.2}$$

For fixed $l \in \mathbb{N}$ we construct the special basis $\{w_1^l, w_2^l, \ldots\}$ of $V \cap H^2(\Omega)$ such that $u_l^0$, $u_l^1$ belong to $[w_1^l, w_2^l]$. With this basis we determine the approximate solutions $u_{lm}(t) = \sum_{j=1}^m g_{jlm}(t) w_j^l$ of (4.3), where $g_{jlm}(t)$ is defined by the approximate problem

$$\left| \begin{array}{l} (u''_{lm}(t), v) + ((u_{lm}(t), v)) + (|u_{lm}(t)|^\rho, v) + \displaystyle\int_{\Gamma_1} h_l(\cdot, u'_{lm}(t)) v \, d\Gamma = (f(t), v), \quad \forall v \in V_m^l, \\ u_{lm}(0) = u^0, \; u'_{lm}(0) = u_l^1. \end{array} \right. \tag{4.3}$$

In a similar way as we made to obtain the estimates (3.12), (3.16) and (3.17) of Section 3, we find

$$\begin{gathered} \frac{1}{2}|u'_{lm}(t)|^2 + \frac{1}{2}\|u_{lm}(t)\|^2 \le N, \qquad d_0 \int_0^t \int_{\Gamma_1} [u'_{lm}(x,s)]^2 d\Gamma \, ds \le D, \\ \frac{1}{2}|u''_{lm}(t)|^2 + \frac{1}{2}\|u'_{lm}(t)\|^2 + d_0 \int_0^t \int_{\Gamma_1} [u''_{lm}(x,s)]^2 \, d\Gamma \, ds \le P, \end{gathered} \tag{4.4}$$

where the constants $N, D$ and $P$ are independent of $l \ge l_0$, $m$ and $t \in [0, T_0]$. The estimates (4.4) provide a subsequence of $(u_{lm})$, which still will be denoted by $(u_{lm})$, such that

$$\begin{aligned} & u_{lm} \to u_l \text{ weak star in } L^\infty(0, T_0; V), \\ & u'_{lm} \to u'_l \text{ weak star in } L^\infty(0, T_0; V), \\ & u''_{lm} \to u''_l \text{ weak star in } L^\infty(0, T_0; L^2(\Omega)), \\ & u'_{lm} \to u'_l \text{ weak in } L^2(0, T_0; L^2(\Gamma_1)), \\ & u''_{lm} \to u''_l \text{ weak in } L^2(0, T_0; L^2(\Gamma_1)). \end{aligned} \tag{4.5}$$

In a similar way as in the convergence (3.22) and (3.23), we get

$$\begin{aligned} & |u_{lm}|^\rho \to |u_l|^\rho \text{ weak star in } L^\infty(0, T_0; L^2(\Omega)), \\ & h_l(\cdot, u'_{lm}) \to h_l(\cdot, u'_l) \text{ weak star in } L^\infty(0, T_0; L^2(\Gamma_1)). \end{aligned} \tag{4.6}$$

Convergence (4.5) and (4.6) allow us to pass to limit in $m$ in the approximate equations of (4.3). Therefore,

$$\begin{aligned} & \int_0^{T_0} (u''_l(t), v)\theta \, dt + \int_0^{T_0} ((u_l(t), v))\theta \, dt + \int_0^{T_0} (|u_l(t)|^\rho, v)\theta \, dt + \int_0^{T_0} \int_{\Gamma_1} h_l(\cdot, u'_l) v\theta \, d\Gamma \, dt \\ & \quad = \int_0^{T_0} (f(t), v)\theta \, dt, \end{aligned}$$

for $v \in V$ and $\theta \in L^2(0, T_0)$. By analogous arguments used to obtain (3.25) and (3.26), we find

$$u''_l - \triangle u_l + |u_l|^\rho = f \quad \text{in } L^\infty(0, T_0; L^2(\Omega)), \tag{4.7}$$

$$\frac{\partial u_l}{\partial \nu} + h_l(\cdot, u'_l) = 0 \quad \text{in } L^\infty(0, T_0; H^{1/2}(\Gamma_1)). \tag{4.8}$$

Estimates (4.4) imply in the existence of a subsequence of $(u_l)$, which still will be denoted by $(u_l)$, and a function $u$ such that

$$\begin{aligned} & u_l \to u \text{ weak star in } L^\infty(0, T_0; V), \\ & u'_l \to u' \text{ weak star in } L^\infty(0, T_0; V), \\ & u''_l \to u'' \text{ weak star in } L^\infty(0, T_0; L^2(\Omega)), \\ & u'_l \to u' \text{ weak in } L^2(0, T_0; L^2(\Gamma_1)), \\ & u''_l \to u'' \text{ weak in } L^2(0, T_0; L^2(\Gamma_1)). \end{aligned} \tag{4.9}$$

As in $(4.6)_1$, we derive

$$|u_l|^\rho \to |u|^\rho \text{ weak star in } L^\infty(0,T_0;L^2(\Omega)).$$

By using this, convergence $(4.9)_1$ and $(4.9)_3$ in (4.7) yield

$$u'' - \triangle u + |u|^\rho = f \quad \text{in } L^\infty(0,T_0;L^2(\Omega)).$$

Using the convergence $(4.9)_2$, $(4.9)_5$, and the Aubin–Lions theorem, we obtain $u_l' \to u'$ strong in $L^2(0,T_0;L^2(\Gamma_1))$, which implies

$$u_l' \to u' \quad \text{a.e. in } \Sigma_0. \tag{4.10}$$

Let $(x,t) \in \Sigma_0 = \Gamma_1 \times (0,T_0)$ be fixed, then the convergence (4.10) ensures that the set $\{u_l'(x,t) : l \in \mathbb{N}\}$ is bounded. The item *(iv)* of Proposition 4.1 guarantees that $h_l$ converges to $h$ uniformly on bounded sets of $\mathbb{R}$ and for almost all in $\Gamma_1$. These two results and the convergence (4.10) provide

$$h_l(\cdot,u_l') \to h(\cdot,u') \quad \text{a.e on } \Sigma_0. \tag{4.11}$$

On the other hand, by equations (4.7) and (4.8), we obtain

$$\int_{\Gamma_1} h_l(\cdot,u_l'(t))u_l'(t)\,d\Gamma = -\frac{1}{2}\frac{d}{dt}|u_l'(t)|^2 - \frac{1}{2}\frac{d}{dt}\|u_l(t)\|^2 - (|u_l(t)|^\rho,u_l'(t)) + (f(t),u_l'(t)). \tag{4.12}$$

By (3.2), hypothesis on $f$ and estimates (4.4), we obtain

$$|(|u_l(t)|^\rho,u_l'(t))| \le C \quad \text{and} \quad |(f(t),u_l'(t))| \le C,$$

where the constant $C > 0$ is independent of $l \ge l_0$ and $t \in [0,T_0]$. By (4.9) we find that $u_l \in C^0([0,T_0];V)$ and $u_l' \in C^0([0,T_0];L^2(\Omega))$, and by (4.4), the sequences $(u_l(T_0))$ and $(u_l'(T_0))$ are bounded in $V$ and $L^2(\Omega)$, respectively. Thus by (4.12), we obtain

$$\int_0^{T_0}\int_{\Gamma_1} h_l(\cdot,u_l'(t))u_l'(t)\,d\Gamma\,dt = -\frac{1}{2}|u_l'(T_0)|^2 + \frac{1}{2}|u_0^1|^2 - \frac{1}{2}\|u_l(T_0)\|^2 + \frac{1}{2}\|u^0\|^2 \le C,$$

para todo $l \ge l_0$ and $t \in [0,T_0]$. Since $h_l(x,s)s \ge 0$, we derive

$$\int_0^{T_0}\int_{\Gamma_1} h_l(\cdot,u_l'(t))u_l'(t)\,d\Gamma\,dt \le C, \qquad \forall l \ge l_0 \text{ and } t \in [0,T_0]. \tag{4.13}$$

By (4.11), (4.13) and Strauss' theorem in [28], we find

$$h_l(\cdot,u_l') \to h(\cdot,u') \quad \text{in } L^1(\Gamma_1 \times (0,T_0)). \tag{4.14}$$

On the other hand, by convergence $(4.9)_1$–$(4.9)_3$ and equation (4.7), we find $u_l \to u$ weak in $L^2(0,T_0,V)$ and $\triangle u_l \to \triangle u$ weak in $L^2(0,T_0,L^2(\Omega))$. From this, as shown in Milla Miranda [21], we conclude

$$\frac{\partial u_l}{\partial \nu} \to \frac{\partial u}{\partial \nu} \text{ weak in } L^2(0,T_0,H^{-1/2}(\Gamma_1)).$$

From convergence (4.14) and equation (4.8), we obtain $\frac{\partial u_l}{\partial \nu} \to -h(\cdot,u')$ in $L^1(\Gamma_1 \times (0,T_0))$. These last two convergence assure that

$$\frac{\partial u}{\partial \nu} + h(\cdot,u') = 0 \quad \text{in } L^1(0,T_0;L^1(\Gamma_1)).$$

The verification of the initial data follow from the convergence (4.9). This conclude the proof of Theorem 2.2 for the case $n \geq 3$.

When $n = 1, 2$ we consider $\rho > 1$, and in this case $V \hookrightarrow L^r(\Omega)$ for all $r \geq 1$, $r \in \mathbb{R}$. Thus we obtain inequality (3.6) since $V \hookrightarrow L^{2\rho}(\Omega)$. For second estimate we apply the Hölder inequality in

$$|(|u_{lm}(t)|^{\rho-2}u_{lm}(t)u'_{lm}(t), u''_{lm}(t))|$$

considering

$$\frac{1}{(\rho-1)k} + \frac{1}{s} + \frac{1}{2} = 1,$$

where $k$ is a natural number such that $(\rho - 1)k > 2$ and $s = \frac{2(\rho-1)k}{(\rho-1)k-2}$. With these considerations and applying similar arguments to those used in Sections 3 and 4 we are able to conclude the proofs of Theorems 2.1 and 2.2 for the case $n = 1, 2$. □

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