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Nonexistence of solutions for singular nonlinear ordinary inequalities

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Abstract. In this paper we prove nonexistence theorems of nonnegative nontrivial solutions for a singular nonlinear ordinary inequality in bounded domains with singular points on the boundary. The proofs are based on the test function method developed by Mitidieri and Pohozaev. We also give the examples demonstrating that the conditions obtained are sharp in the case of the problem under consideration.

Keywords: nonlinear differential inequalities, nonexistence theorem, test function method.

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1 Introduction

In this paper, we shall consider nonexistence of nontrivial weak solutions of the singular nonlinear differential inequality

$$\begin{cases} (|u'|^{p-2}u')' \geq a(x)u^q & \text{for } x \in (0, x_0], \\ u(x) \geq 0 & \text{for } x \in (0, x_0], \\ u'(x_0) < 0, \end{cases} \tag{1.1}$$

where $x_0 > 0$, $p > 1$, $q > p-1$, and the function $a \in C((0, x_0])$ satisfies the estimate

$$a(x) \geq cx^{-\alpha} \tag{1.2}$$

for some constants $\alpha \in \mathbb{R}$ and $c > 0$.

There have been many results on the nonexistence of nonnegative nontrivial solutions for nonlinear differential inequalities (systems), see [1–32] and references therein. Tools based on different forms of the maximum principle like the moving planes method or moving spheres method, nonlinear capacitary estimates and Pohozaev type identities, energy methods and

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Harnack inequality type argument, have been proved to be very successful for solving interesting problems related to applications and to the general theory of partial differential equations.

Mitidieri and Pohozaev (see [22]) have developed a new effective approach to these problems on the basis of a special choice of test functions. By integration technique which uses suitable test functions, they have established a priori estimates of solutions and obtained the nonexistence results. This approach not only provides simple, accurate, and more general results but also is essentially different from the comparison method. Moreover, it can be applied to a wide class of nonlinear differential inequalities (see [5–7, 16–23, 25–29]) and systems (see [12, 14, 15, 24]). In particular, it was shown in [22] that the inequalities

$$\pm \Delta_p u \geq |x|^{-\alpha} u^q \quad \text{in } \mathbb{R}^N \tag{1.3}$$

have no weak positive solutions, and then, G. Caristi [3] perfected the results, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. In fact this method was applied to more general operators, including the generalized mean curvature operator (see [2, 16], [22–32]) and a wide class of anisotropic quasilinear operator (see [5, 6]). Later, by using refined techniques, Filippucci, Pucci and Rigoli (see [8–11]) proved very significant existence and nonexistence results for the coercive case.

In the present paper, by modifying the method developed by Mitidieri and Pohozaev in [22] and Galakhov in [16] , we will show nonexistence theorems for the nonlinear differential inequality (1.1) with singular points on the boundary.

We understand solutions to problem (1.1) in the sense of distributions and define the class of admissible solutions to problem (1.1) as

$$X((0, x_0]) := \{u : (0, x_0] \to \mathbb{R}_+, \ \ a(x)u^q, \ \ |u'|^p \in L^1_{loc}((0, x_0])\}.$$

We prove the following theorems.

Theorem 1.1. *Suppose that the function $a \in C((0, x_0])$ is nonnegative and satisfies inequality (1.2), and $q > p - 1$. If $\alpha > p$, then the problem (1.1) has no nontrivial nonnegative solutions in $X((0, x_0])$.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, the problem (1.1) with $\alpha = p$ has no nontrivial nonnegative solutions in $X((0, x_0]) \cap C((0, x_0])$.*

Remark 1.3. For $\alpha < p$ and $q > p - 1$, a solution of problem (1.1) with $a(x) = x^{-\alpha}$ can be written down explicitly as $u(x) = Cx^{\frac{\alpha - p}{q - p + 1}}$ with an appropriate constant $C > 0$. Thus, the assumption $\alpha \geq p$ is essential to deal with nonexistence results.

2 Proofs of Theorems 1.1 and 1.2

In this section, we will prove the two theorems. In doing so we will follow the argument of Theorem 2.1 in [22] and Theorem 3.4 in [16].

To establish a priori estimates of the solutions, we need to define some test functions that will be widely used in the sequel. We consider the test function $\xi \in C^1([0, x_0]; [0, 1])$ that satisfies

$$\xi(x) = \begin{cases} 1, & \eta < x < x_0, \\ 0, & 0 < x < \eta/2, \end{cases} \tag{2.1}$$

and

$$|\xi'(x)| \leq c\,\eta^{-1}, \qquad \forall\, x \in (0, x_0), \tag{2.2}$$

where $\eta \in (0, x_0)$ is a parameter and $c > 0$ is a constant. Set

$$\chi(x) = \xi^{\lambda}(x), \tag{2.3}$$

where $\lambda > 0$ is a parameter to be chosen later according to the nature of the problem.

To prove the main results of this section, we need the following lemma.

Lemma 2.1. *Assume that $a \in C((0, x_0])$ is a nonnegative function. Let χ be defined as (2.3) and $q > p - 1$. Then each nontrivial nonnegative solution to problem (1.1) in $X((0, x_0])$ satisfies a priori estimate*

$$\int_0^{x_0} a(x) u^{q+\gamma} \chi dx \leq C \int_{\eta/2}^{\eta} a(x)^{-\frac{p-1+\gamma}{q-p+1}} |\xi'|^{\frac{p(q+\gamma)}{q-p+1}} dx \tag{2.4}$$

for $\gamma > 0$, with a constant $C > 0$ independent of u.

Proof. Without loss of generality, we suppose $u > 0$. If u is allowed to vanish at some points, we consider $u_\delta = u + \delta$ with arbitrary $\delta > 0$ and then pass to the limit as $\delta \to 0^+$. Let $\gamma \in \mathbb{R}$ be a parameter to be chosen later. Multiplying (1.1) by $u^\gamma \chi$ and integrating by parts, we get

$$\int_0^{x_0} a(x) u^{q+\gamma} \chi dx + \gamma \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \int_0^{x_0} |u'|^{p-1} u^\gamma |\chi'| dx. \tag{2.5}$$

Applying Young's inequality with exponents $l = \frac{p}{p-1}, l' = p > 1, \varepsilon > 0$ to the second integral on the right-hand side of (2.5), we obtain

$$\begin{aligned} &\int_0^{x_0} a(x) u^{q+\gamma} \chi dx + \gamma \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx \\ &\quad \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \varepsilon \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx + \varepsilon^{1-p} \int_0^{x_0} u^{p+\gamma-1} \frac{|\chi'|^p}{\chi^{p-1}} dx. \end{aligned} \tag{2.6}$$

Taking $\varepsilon = \gamma/2$, we have

$$\int_0^{x_0} a(x) u^{q+\gamma} \chi dx + \frac{\gamma}{2} \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \left(\frac{\gamma}{2}\right)^{1-p} \int_0^{x_0} u^{p+\gamma-1} \frac{|\chi'|^p}{\chi^{p-1}} dx. \tag{2.7}$$

By Hölder's inequality with exponents $m = \frac{q+\gamma}{p-1+\gamma} > 1$, $m' = \frac{q+\gamma}{q-p+1} > 1$ for every $\gamma > 0$ to the second integral on the right-hand side of (2.7) (since, by assumption, $q > p - 1$), we get

$$\begin{aligned} &\int_0^{x_0} a(x) u^{q+\gamma} \chi dx \\ &\quad \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \left(\frac{\gamma}{2}\right)^{1-p} \left(\int_0^{x_0} a(x) u^{q+\gamma} \chi dx\right)^{\frac{1}{m}} \left(\int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx\right)^{\frac{1}{m'}}, \end{aligned} \tag{2.8}$$

i.e.,

$$\begin{aligned} &\int_0^{x_0} a(x) u^{q+\gamma} \chi dx \\ &\quad \leq |u'(x_0)|^{p-2} u'(x_0) u^\gamma(x_0) \\ &\qquad + \left(\frac{\gamma}{2}\right)^{1-p} \left(\int_0^{x_0} a(x) u^{q+\gamma} \chi dx\right)^{\frac{1}{m}} \left(\int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx\right)^{\frac{1}{m'}}. \end{aligned} \tag{2.9}$$

Since $u'(x_0) < 0$, we get

$$\int_0^{x_0} a(x)u^{q+\gamma}\chi dx \leq \left(\frac{\gamma}{2}\right)^{1-p} \left(\int_0^{x_0} a(x)u^{q+\gamma}\chi dx\right)^{\frac{1}{m}} \left(\int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx\right)^{\frac{1}{m'}}. \tag{2.10}$$

Consequently, the above inequality yields

$$\int_0^{x_0} a(x)u^{q+\gamma}\chi dx \leq \left(\frac{\gamma}{2}\right)^{1-p} \int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx. \tag{2.11}$$

Recalling the definition of the function χ in (2.2), we get

$$\frac{|\chi'|^{pm'}}{\chi^{pm'-1}} = \lambda^{pm'} \xi^{\lambda - pm'} |\xi'|^{pm'}, \tag{2.12}$$

which leads to

$$\int_0^{x_0} a(x)u^{q+\gamma}\chi dx \leq \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{pm'} \int_0^{x_0} a(x)^{-\frac{m'}{m}} \xi^{\lambda - pm'} |\xi'|^{pm'} dx. \tag{2.13}$$

Since $\xi \in C^1([0, x_0]; [0, 1])$ satisfy (2.1), then

$$\begin{aligned} \int_0^{x_0} a(x)u^{q+\gamma}\chi dx &\leq \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{pm'} \int_{\eta/2}^{\eta} a(x)^{-\frac{m'}{m}} \xi^{\lambda - pm'} |\xi'|^{pm'} dx \\ &\leq \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{\frac{p(q+\gamma)}{q-p+1}} \int_{\eta/2}^{\eta} a(x)^{-\frac{p-1+\gamma}{q-p+1}} |\xi'|^{\frac{p(q+\gamma)}{q-p+1}} dx, \end{aligned} \tag{2.14}$$

by choosing λ large enough. Hence (2.4) holds with a constant $C = (\frac{\gamma}{2})^{1-p} \lambda^{\frac{p(q+\gamma)}{q-p+1}}$. The lemma is proved. □

Proof of Theorem 1.1. Now let $\xi \in C^1([0, x_0]; [0, 1])$ satisfy (2.1) and (2.2). Then (2.4) takes the form

$$\begin{aligned} \int_\eta^{x_0} a(x)u^{q+\gamma} dx &\leq \int_0^{x_0} a(x)u^{q+\gamma}\chi dx \\ &\leq C\lambda^{\frac{p(q+\gamma)}{q-p+1}} \int_{\eta/2}^{\eta} x^{\frac{\alpha(p-1+\gamma)}{q-p+1}} \eta^{-\frac{p(q+\gamma)}{q-p+1}} dx \\ &\leq C\lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{-\frac{p(q+\gamma)}{q-p+1}} \int_{\eta/2}^{\eta} x^{\frac{\alpha(p-1+\gamma)}{q-p+1}} dx \\ &\leq C'\lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{\sigma}, \end{aligned} \tag{2.15}$$

where

$$\sigma = \frac{q - p + 1 - pq + \alpha(p-1) + \gamma(\alpha - p)}{q - p + 1}.$$

If we choose γ large enough, then the assumption $\alpha > p$ implies $\sigma > 0$. Hence

$$0 \leq \int_\eta^{x_0} a(x)u^{q+\gamma} dx \leq C'\lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{\sigma}. \tag{2.16}$$

Letting $\eta \to 0$ in (2.16), we get

$$\int_0^{x_0} a(x)u^{q+\gamma} dx = 0. \tag{2.17}$$

Thus $u \equiv 0$. This completes the proof. □

Proof of Theorem 1.2. If $\alpha = p$, one has $\sigma = 1 - p$ for every $\gamma > 0$ in (2.15). Now fix a number $b > 0$. We may choose the parameters γ and λ in (2.13) so that

$$pm' < \lambda < b^{\frac{q-p+1}{p}}. \tag{2.18}$$

For $u \in C((0, x_0])$, we can consider the set

$$M_{\eta,b} = \{x \in (\eta, x_0) : u(x) \geq b\}. \tag{2.19}$$

Due to (2.15) with $\alpha = p$, we get

$$c{x_0}^{-p} b^{q+\gamma} \mu(M_{\eta,b}) \leq \int_{M_{\eta,b}} cx^{-p} u^{q+\gamma} dx \leq \int_{M_{\eta,b}} a(x) u^{q+\gamma} dx \leq \int_{\eta}^{x_0} a(x) u^{q+\gamma} dx \tag{2.20}$$

and by (2.16)

$$c{x_0}^{-p} b^{q+\gamma} \mu(M_{\eta,b}) \leq \ C' \lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{1-p}, \tag{2.21}$$

which leads to

$$\mu(M_{\eta,b}) \leq c^{-1} C' {x_0}^p \eta^{1-p} \left(\frac{\lambda^{\frac{p}{q-p+1}}}{b} \right)^{q+\gamma} \to 0 \tag{2.22}$$

for each b, η fixed and $\gamma \to \infty$, since the fraction in parentheses is less than 1 by (2.18). For each b and η, one obtains

$$\mu(M_{\eta,b}) = 0,$$

which means $u \equiv 0$. Thus we obtain the conclusion. □

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