



## Bounded solutions of $k$ -dimensional system of nonlinear difference equations of neutral type

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**Abstract.** The  $k$ -dimensional system of neutral type nonlinear difference equations with delays in the following form

$$\begin{cases} \Delta(x_i(n) + p_i(n)x_i(n - \tau_i)) = a_i(n)f_i(x_{i+1}(n - \sigma_i)) + g_i(n), \\ \Delta(x_k(n) + p_k(n)x_k(n - \tau_k)) = a_k(n)f_k(x_1(n - \sigma_k)) + g_k(n), \end{cases}$$

where  $i = 1, \dots, k - 1$ , is considered. The aim of this paper is to present sufficient conditions for the existence of nonoscillatory bounded solutions of the above system with various  $(p_i(n))$ ,  $i = 1, \dots, k$ ,  $k \geq 2$ .

**Keywords:** system of difference equation,  $k$ -dimensional, neutral type, nonoscillatory solutions, boundedness, existence.

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### 1 Introduction

In this paper we consider a nonlinear difference system of  $k$  ( $k \geq 2$ ) equations of the form

$$\begin{cases} \Delta(x_i(n) + p_i(n)x_i(n - \tau_i)) = a_i(n)f_i(x_{i+1}(n - \sigma_i)) + g_i(n), \\ \Delta(x_k(n) + p_k(n)x_k(n - \tau_k)) = a_k(n)f_k(x_1(n - \sigma_k)) + g_k(n), \end{cases} \quad (1.1)$$

where  $n \in \mathbb{N}_0$ ,  $i = 1, \dots, k - 1$ ,  $\Delta$  is the forward difference operator defined by  $\Delta u(n) = u(n + 1) - u(n)$ . Here  $\mathbb{R}$  is a set of real numbers,  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\sigma_i, \tau_i \in \mathbb{N}$  for  $i = 1, \dots, k$ . By  $n_0$  we denote  $\max\{\tau_1, \dots, \tau_k, \sigma_1, \dots, \sigma_k\}$ , and  $\mathbb{N}_0 = \{n_0, n_0 + 1, \dots\}$ . Moreover  $a_i = (a_i(n))$ ,  $g_i = (g_i(n))$ ,  $p_i = (p_i(n))$  for  $i = 1, \dots, k$  are given sequences of real numbers,  $x_i = (x_i(n))$  for  $i = 1, \dots, k$  are unknown real sequences and functions  $f_i: \mathbb{R} \rightarrow \mathbb{R}$ . Throughout this paper  $X$  denotes an unknown vector  $(x_1, \dots, x_k)$  and  $X(n)$  denotes  $(x_1(n), \dots, x_k(n)) \in \mathbb{R}^k$ . For the elements of  $\mathbb{R}^k$  the symbol  $|\cdot|$  stands for the maximum norm.

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By  $\mathcal{B}$  we denote the Banach space of all bounded sequences in  $\mathbb{R}^k$  with the supremum norm, i.e.

$$\mathcal{B} = \left\{ X: \mathbb{N} \rightarrow \mathbb{R}^k : \|X\| = \sup_{n \in \mathbb{N}} |X(n)| < \infty \right\},$$

and by  $B$  the following subset of  $\mathcal{B}$

$$B = \{X = (x_1, \dots, x_k) \in \mathcal{B} : x_i \text{ is nonnegative or nonpositive for } i = 1, \dots, k\}.$$

A sequence of real numbers is said to be nonoscillatory if it is either eventually positive or eventually negative. By a solution of system (1.1) we mean a vector  $X$  such that its components, i.e.  $x_1, \dots, x_k$ , satisfy the system (1.1) for sufficiently large  $n$ . The solution  $X$  of system (1.1) is called nonoscillatory if all its components are nonoscillatory. The solution  $X$  of system (1.1) is called bounded if all its components are bounded.

Any higher-order nonlinear neutral difference equation could be rewritten as  $k$ -dimensional system of difference equations with one equation of neutral type but not vice-versa. Higher-order nonlinear neutral difference equations have been studied by many authors, see for example [2–4, 8–10, 13–23], and the references cited therein. The theorems presented here generalize and improve the results obtained for three dimensional system in [13].

The following definition and theorems will be used in the sequel.

**Definition 1.1** (Uniformly Cauchy subset, [6]). A set  $\Omega$  of sequences in  $l^\infty$  is uniformly Cauchy if for every  $\varepsilon > 0$ , there exists an integer  $n$  such that  $|X(i) - X(j)| < \varepsilon$  whenever  $i, j > n$  for any  $X \in \Omega$ .

**Lemma 1.2** (Arzelà–Ascoli theorem, [1]). A bounded and uniformly Cauchy subset of  $l^\infty$  is relatively compact.

**Theorem 1.3** (Krasnoselskii’s fixed point theorem, [7]). Let  $\Omega$  be a bounded closed convex subset of a Banach space and let  $F, T$  be maps such that  $Fx + Ty \in \Omega$  for every pair  $x, y \in \Omega$ . If  $F$  is a contraction and  $T$  is completely continuous, then the equation  $Fx + Tx = x$  has a solution in  $\Omega$ .

**Theorem 1.4** (Schauder’s fixed point theorem, [5]). Let  $\Omega$  be a nonempty, compact and convex subset of a Banach space and let  $T: \Omega \rightarrow \Omega$  be continuous. Then  $T$  has a fixed point in  $M$ .

## 2 Main results

In this section, using the Krasnoselskii’s fixed point theorem and Schauder’s fixed point theorem, we establish sufficient conditions for the existence of nonoscillatory bounded solutions of system (1.1).

**Theorem 2.1.** Assume that for  $i = 1, \dots, k$

$$\sum_{n=1}^{\infty} |a_i(n)| < \infty, \quad (2.1)$$

$$\sum_{n=1}^{\infty} |g_i(n)| < \infty, \quad (2.2)$$

$$f_i: \mathbb{R} \rightarrow \mathbb{R} \text{ is a continuous function} \quad (2.3)$$

and for any closed subset  $J \subset \mathbb{R}$

$$\max_{i=1,\dots,k} \sup_{t \in J} \{|f_i(t)|\} > 0. \quad (2.4)$$

Assume also that for each  $i = 1, \dots, k$  the terms of sequence  $p_i$  are of the same sign for  $n \in \mathbb{N}_0$ . If for each  $i = 1, \dots, k$  there exists a positive real constant  $c_{p_i}$  such that

$$0 \leq p_i(n) \leq c_{p_i} < 1, \quad n \in \mathbb{N}_0, \quad (2.5)$$

or

$$-1 < -c_{p_i} \leq p_i(n) \leq 0, \quad n \in \mathbb{N}_0, \quad (2.6)$$

then system (1.1) has a bounded nonoscillatory solution.

*Proof.* For the fixed positive real number  $r$  we define a set

$$\Omega_1 = \left\{ X \in B : \frac{1}{8}(1 - c_{p_i})r \leq |x_i(n)| \leq r, \quad i = 1, \dots, k, \quad n \in \mathbb{N} \right\}.$$

Clearly  $\Omega_1$  is a bounded closed convex subset of the Banach space  $B$ . Since condition (2.3) is satisfied, we can take

$$M_f = \max_{i=1,\dots,k} \left\{ |f_i(t)| : |t| \in \left[ \frac{1}{8}(1 - c_{p_i})r, r \right] \right\}.$$

From (2.1) and (2.2), there exists such  $n_1 \in \mathbb{N}_0$  that

$$\sum_{n=n_1}^{\infty} |a_i(n)| \leq \frac{(1 - c_{p_i})r}{8M_f}, \quad \sum_{n=n_1}^{\infty} |g_i(n)| \leq \frac{(1 - c_{p_i})r}{4}.$$

Let  $I_1, I_2, I_3, I_4$  be subsets of the set  $\{1, \dots, k\}$  and moreover,  $I_i \cap I_j = \emptyset$  for  $i \neq j$ ,  $i, j = 1, 2, 3, 4$  and  $I_1 \cup I_2 \cup I_3 \cup I_4 = \{1, \dots, k\}$ .

We consider four cases

(i)

$$\begin{cases} 0 \leq p_i(n) \leq c_{p_i} < 1, \\ x_i(n) > 0, \quad \text{for } i \in I_1, \quad n \geq n_1, \end{cases}$$

(ii)

$$\begin{cases} -1 < -c_{p_i} \leq p_i(n) \leq 0, \\ x_i(n) < 0, \quad \text{for } i \in I_2, \quad n \geq n_1, \end{cases}$$

(iii)

$$\begin{cases} 0 \leq p_i(n) \leq c_{p_i} < 1, \\ x_i(n) < 0, \quad \text{for } i \in I_3, \quad n \geq n_1, \end{cases}$$

(iv)

$$\begin{cases} -1 < -c_{p_i} \leq p_i(n) \leq 0, \\ x_i(n) > 0, \quad \text{for } i \in I_4, \quad n \geq n_1. \end{cases}$$

Next, we define the maps  $F, T: \Omega_1 \rightarrow B$  where

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}, \quad T = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix},$$

$$(F_i X)(n) = \begin{cases} (F_i X)(n_1) & \text{for } i = 1, \dots, k, 0 \leq n < n_1, \\ -p_i(n)x_i(n - \tau_i) + \frac{(1+c_{p_i})r}{2} & \text{for } i \in I_1 \cup I_2, n \geq n_1, \\ -p_i(n)x_i(n - \tau_i) + \frac{(1-c_{p_i})r}{2} & \text{for } i \in I_3 \cup I_4, n \geq n_1, \end{cases} \quad (2.7)$$

and for  $i = 1, \dots, k-1$

$$(T_i X)(n) = \begin{cases} (T_i X)(n_1) & \text{for } 0 \leq n < n_1, \\ -\sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s) & \text{for } n \geq n_1, \end{cases} \quad (2.8)$$

and

$$(T_k X)(n) = \begin{cases} (T_k X)(n_1) & \text{for } 0 \leq n < n_1, \\ -\sum_{s=n}^{\infty} a_k(s) f_k(x_1(s - \sigma_k)) - \sum_{s=n}^{\infty} g_k(s) & \text{for } n \geq n_1. \end{cases} \quad (2.9)$$

We will show that  $F$  and  $T$  satisfy the assumptions of Theorem 1.3. First we prove that if  $X, \bar{X} \in \Omega_1$ , then  $FX + T\bar{X} \in \Omega_1$ .

For  $n \geq n_1, i \in I_1 \cup I_2$  and  $i \neq k$  we have

$$\begin{aligned} (F_i X)(n) + (T_i \bar{X})(n) &= -p_i(n)x_i(n - \tau_i) + \frac{(1+c_{p_i})r}{2} \\ &\quad - \sum_{s=n}^{\infty} a_i(s) f_i(\bar{x}_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\ &\leq \frac{(1+c_{p_i})r}{2} + \sum_{s=n}^{\infty} |a_i(s)| |f_i(\bar{x}_{i+1}(s - \sigma_i))| + \sum_{s=n}^{\infty} |g_i(s)| \\ &\leq \frac{1}{2}r + \frac{1}{2}c_{p_i}r + M_f \cdot \frac{(1-c_{p_i})r}{8M_f} + \frac{(1-c_{p_i})r}{4} \\ &= \frac{7}{8}r + \frac{1}{8}c_{p_i}r \leq r. \end{aligned}$$

Moreover,

$$\begin{aligned} (F_i X)(n) + (T_i \bar{X})(n) &= -p_i(n)x_i(n - \tau_i) + \frac{(1+c_{p_i})r}{2} \\ &\quad - \sum_{s=n}^{\infty} a_i(s) f_i(\bar{x}_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\ &\geq -|p_i(n)||x_i(n - \tau_i)| + \frac{(1+c_{p_i})r}{2} \\ &\quad - \sum_{s=n}^{\infty} |a_i(s)| |f_i(\bar{x}_{i+1}(s - \sigma_i))| - \sum_{s=n}^{\infty} |g_i(s)| \\ &\geq -c_{p_i}r + \frac{1}{2}r + \frac{1}{2}c_{p_i}r - M_f \cdot \frac{(1-c_{p_i})r}{8M_f} - \frac{(1-c_{p_i})r}{4} \\ &= \frac{1}{8}(1-c_{p_i})r. \end{aligned}$$

For  $n \geq n_1$  and  $i \in I_3 \cup I_4$ , and  $i \neq k$  we have

$$\begin{aligned}
 (F_i X)(n) + (T_i \bar{X})(n) &= -p_i(n)x_i(n - \tau_i) + \frac{(1 - c_{p_i})r}{2} \\
 &\quad - \sum_{s=n}^{\infty} a_i(s) f_i(\bar{x}_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\
 &\leq |p_i(n)| |x_i(n - \tau_i)| + \frac{(1 - c_{p_i})r}{2} \\
 &\quad + \sum_{s=n}^{\infty} |a_i(s)| |f_i(\bar{x}_{i+1}(s - \sigma_i))| + \sum_{s=n}^{\infty} |g_i(s)| \\
 &\leq c_{p_i}r + \frac{1}{2}r - \frac{1}{2}c_{p_i}r + M_f \cdot \frac{(1 - c_{p_i})r}{8M_f} + \frac{(1 - c_{p_i})r}{4} \\
 &= \frac{1}{8}c_{p_i}r + \frac{7}{8}r \leq r.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (F_i X)(n) + (T_i \bar{X})(n) &= -p_i(n)x_i(n - \tau_i) + \frac{(1 - c_{p_i})r}{2} \\
 &\quad - \sum_{s=n}^{\infty} a_i(s) f_i(\bar{x}_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\
 &\geq \frac{(1 - c_{p_i})r}{2} - \sum_{s=n}^{\infty} |a_i(s)| |f_i(\bar{x}_{i+1}(s - \sigma_i))| - \sum_{s=n}^{\infty} |g_i(s)| \\
 &\geq \frac{1}{2}r - \frac{1}{2}c_{p_i}r - M_f \cdot \frac{(1 - c_{p_i})r}{8M_f} - \frac{(1 - c_{p_i})r}{4} \\
 &= \frac{1}{8}(1 - c_{p_i})r.
 \end{aligned}$$

For  $i = k$  there is a different definition of the mapping  $T_k$ , but all estimations are analogous, and hence omitted.

The task is now to prove that  $F$  is a contraction mapping. It is easy to see that

$$\begin{aligned}
 |(F_i X)(n) - (F_i \bar{X})(n)| &\leq |p_i(n)| |x_i(n - \tau_i) - \bar{x}_i(n - \tau_i)| \\
 &\leq c_{p_i} |x_i(n - \tau_i) - \bar{x}_i(n - \tau_i)|,
 \end{aligned}$$

for any  $X, \bar{X} \in \Omega_1$ ,  $i = 1, \dots, k$  and  $n \geq n_1$ . Hence

$$\|FX - F\bar{X}\| \leq \max_{i=1, \dots, k} \{c_{p_i}\} \cdot \|X - \bar{X}\|,$$

where, by (2.5) and (2.6), there is  $0 < \max_{i=1, \dots, k} \{c_{p_i}\} < 1$ .

The next step is to show continuity of  $T$ . Let  $X_j = (x_{1j}, \dots, x_{kj}) \in \Omega_1$  for  $j \in \mathbb{N}$  and for  $i = 1, \dots, k$  there is  $x_{ij}(n) \rightarrow x_i(n)$  as  $j \rightarrow \infty$ . Since  $\Omega_1$  is closed, we have  $X = (x_1, \dots, x_k) \in \Omega_1$ . By (2.1), (2.3), (2.8) and Lebesgue's dominated convergence theorem we obtain for  $i = 1, \dots, k - 1$

$$|(T_i X_j)(n) - (T_i X)(n)| \leq \sum_{s=n}^{\infty} |a_i(s)| |f_i(x_{i+1j}(s - \sigma_i)) - f_i(x_{i+1}(s - \sigma_i))| \rightarrow 0 \quad \text{if } j \rightarrow \infty,$$

where  $n \in \mathbb{N}$ . Analogously we conclude for  $i = k$ . Therefore

$$\|(TX_j) - (TX)\| \rightarrow 0 \quad \text{if } j \rightarrow \infty,$$

and we see that  $T$  is a continuous mapping.

In order to prove that  $T$  is completely continuous we can use Lemma 1.2. Hence we have to show that  $T\Omega_1$  is uniformly Cauchy (see Definition 1.1). We show transformations for any  $T_i, i = 1, \dots, k-1$ . Similar arguments apply to  $T_k$ .

Let  $X \in \Omega_1$ . We conclude from the assumptions (2.1), (2.2) and (2.3) that for any given  $\varepsilon > 0$  there exists an integer  $n_2 > n_1$  such that for  $n \geq n_2$  we have

$$\sum_{s=n}^{\infty} |a_i(s)| |f_i(x_{i+1}(s - \sigma_i))| + \sum_{s=n}^{\infty} |g_i(s)| < \frac{\varepsilon}{2}.$$

Hence, for  $n_4 > n_3 \geq n_2$ , we obtain

$$\begin{aligned} |(T_i X)(n_4) - (T_i X)(n_3)| &= \left| \sum_{s=n_4}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) + \sum_{s=n_4}^{\infty} g_i(s) \right. \\ &\quad \left. - \sum_{s=n_3}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{s=n_3}^{\infty} g_i(s) \right| < \varepsilon. \end{aligned}$$

Therefore  $T\Omega_1$  is uniformly Cauchy.

By Theorem 1.3, there exists  $X$  such that  $(FX)(n) + (TX)(n) = X(n)$ .

Finally, we verify that  $X$  satisfies system (1.1) for  $n \geq n_1$ . As  $(F_i X)(n) + (T_i X)(n) = x_i(n)$ ,  $i = 1, \dots, k$ , we have for  $i \in I_1 \cup I_2$  and  $i \neq k$

$$\begin{aligned} -p_i(n)x_i(n - \tau_i) + \frac{(1 + c_{p_i})r}{2} - \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s) &= x_i(n), \\ \Delta(x_i(n) + p_i(n)x_i(n - \tau_i)) &= -\Delta \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \Delta \sum_{s=n}^{\infty} g_i(s), \\ \Delta(x_i(n) + p_i(n)x_i(n - \tau_i)) &= a_i(n)f_i(x_{i+1}(n - \sigma_i)) + g_i(n). \end{aligned} \tag{2.10}$$

Similarly, we get (2.10) for  $i \in I_3 \cup I_4$  and  $i \neq k$ . In all cases, for  $i = k$ , the reasoning is also the same as above. The proof is complete.  $\square$

Note that for  $p_i(n) \equiv 0, i = 1, \dots, k$ , system (1.1) is not of the neutral type, but Theorem 2.1 is still true.

**Example 2.2.** Consider a difference system

$$\begin{cases} \Delta(x_1(n) + \frac{1}{2n}x_1(n-1)) = \frac{5n^4 - 21n^3 + 22n^2 + 4n - 8}{4n^6 - 16n^5 + 10n^4 + 16n^3 - 14n^2} x_2(n-2) + \frac{1}{n^2}, \\ \Delta(x_2(n) - \frac{1}{2n}x_2(n-2)) = \frac{2n^5 - 17n^4 + 43n^3 - 48n^2 + 27n - 7}{2n^8 - 4n^7 - 6n^6 + 8n^5 + 8n^4} x_3^3(n-1) - \frac{1}{n^2}, \\ \Delta(x_3(n) + \frac{1}{2n}x_3(n-1)) = \frac{3n^3 - 3n^2 + 2}{4n^5 - 2n^4 - 6n^3} x_4(n-1) + \frac{1}{n^3}, \\ \Delta(x_4(n) - \frac{1}{2n}x_4(n-1)) = \frac{2n^2 - 5n + 3}{n^4 + n^3} x_1^2(n-1). \end{cases}$$

All assumptions of Theorem 2.1 are satisfied. The system above has the bounded (but not unique) solution  $X = ((1 + \frac{1}{n}), (-2 + \frac{1}{n^2}), (-1 - \frac{1}{n}), (2 - \frac{1}{n}))$  for  $n \geq 3$ .

**Theorem 2.3.** Assume that conditions (2.1), (2.2), (2.3) and (2.4) are satisfied. If there exist positive real numbers  $\tilde{c}_{p_i}$ ,  $i = 1, \dots, k$  that

$$1 < \tilde{c}_{p_i} \leq p_i(n), \quad n \in \mathbb{N}_0, \quad (2.11)$$

or

$$p_i(n) \leq -\tilde{c}_{p_i} < -1, \quad n \in \mathbb{N}_0, \quad (2.12)$$

then system (1.1) has a bounded nonoscillatory solution.

*Proof.* We define a subset  $\Omega_2$  of  $B$  in the following way

$$\Omega_2 = \left\{ X \in B : \frac{1}{8}(\tilde{c}_{p_i} - 1)r \leq |x_i(n)| \leq \tilde{c}_{p_i}r, \quad i = 1, \dots, k, \quad n \in \mathbb{N} \right\}.$$

where  $r$  is a fixed positive real number. Obviously  $\Omega_2$  is a bounded, closed and convex subset of  $B$ . Let us set

$$\tilde{M}_f = \max_{i=1, \dots, k} \left\{ |f_i(t)| : |t| \in \left[ \frac{1}{8}(\tilde{c}_{p_i} - 1)r, \tilde{c}_{p_i}r \right] \right\}.$$

From assumptions (2.1) and (2.2), we conclude that there exists  $n_5 \in \mathbb{N}_0$  that

$$\sum_{n=n_5}^{\infty} |a_i(n)| \leq \frac{(\tilde{c}_{p_i} - 1)r}{8\tilde{M}_f}, \quad \sum_{n=n_5}^{\infty} |g_i(n)| \leq \frac{(\tilde{c}_{p_i} - 1)r}{4}.$$

Let  $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4$  be such subsets of the set  $\{1, \dots, k\}$  that  $\tilde{I}_i \cap \tilde{I}_j = \emptyset$  for  $i \neq j$ ,  $i, j = 1, 2, 3, 4$  and  $\tilde{I}_1 \cup \tilde{I}_2 \cup \tilde{I}_3 \cup \tilde{I}_4 = \{1, \dots, k\}$ .

Since we seek for the nonoscillatory solution, we consider the following cases

(i)

$$\begin{cases} 1 < \tilde{c}_{p_i} \leq p_i(n), \\ x_i(n) > 0, \quad \text{for } i \in \tilde{I}_1, \quad n \geq n_5, \end{cases}$$

(ii)

$$\begin{cases} p_i(n) \leq -\tilde{c}_{p_i} < -1, \\ x_i(n) < 0, \quad \text{for } i \in \tilde{I}_2, \quad n \geq n_5, \end{cases}$$

(iii)

$$\begin{cases} 1 < \tilde{c}_{p_i} \leq p_i(n), \\ x_i(n) < 0, \quad \text{for } i \in \tilde{I}_3, \quad n \geq n_5, \end{cases}$$

(iv)

$$\begin{cases} p_i(n) \leq -\tilde{c}_{p_i} < -1, \\ x_i(n) > 0, \quad \text{for } i \in \tilde{I}_4, \quad n \geq n_5. \end{cases}$$

We define the maps  $F, T: \Omega_2 \rightarrow B$  in the following way

$$(F_i X)(n) = \begin{cases} (F_i X)(n_5) & \text{for } i = 1, \dots, k, \quad 0 \leq n < n_5, \\ -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} & \text{for } i \in \tilde{I}_1 \cup \tilde{I}_2, \quad n \geq n_5, \\ -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(\tilde{c}_{p_i}-1)r}{2} & \text{for } i \in \tilde{I}_3 \cup \tilde{I}_4, \quad n \geq n_5, \end{cases} \quad (2.13)$$

and for  $i = 1, \dots, k-1$

$$(T_i X)(n) = \begin{cases} (T_i X)(n_5) & \text{for } 0 \leq n < n_5, \\ -\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s-\sigma_i)) - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) & \text{for } n \geq n_5, \end{cases} \quad (2.14)$$

and

$$(T_k X)(n) = \begin{cases} (T_k X)(n_5) & \text{for } 0 \leq n < n_5, \\ -\frac{1}{p_k(n+\tau_k)} \sum_{s=n+\tau_k}^{\infty} a_k(s) f_k(x_1(s-\sigma_k)) - \frac{1}{p_k(n+\tau_k)} \sum_{s=n+\tau_k}^{\infty} g_k(s) & \text{for } n \geq n_5. \end{cases} \quad (2.15)$$

Let  $X, \bar{X} \in \Omega_2$ ,  $n \geq n_5$ . Then also  $FX + T\bar{X} \in \Omega_2$ . We will present all transformations for the  $i$ -th components of  $F$  and  $T$ , where  $i = 1, \dots, k-1$ . We have for  $i \in \tilde{I}_1 \cup \tilde{I}_2$

$$\begin{aligned} (F_i X)(n) + (T_i \bar{X})(n) &= -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} \\ &\quad - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s-\sigma_i)) \\ &\quad - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\ &\leq \frac{(1+\tilde{c}_{p_i})r}{2} + \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)| |f_i(x_{i+1}(s-\sigma_i))| \\ &\quad + \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\ &\leq \frac{1}{2} \tilde{c}_{p_i} r + \frac{1}{2} r + \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f} + \frac{(\tilde{c}_{p_i}-1)r}{4} \\ &= \frac{7}{8} \tilde{c}_{p_i} r + \frac{1}{8} r \leq \tilde{c}_{p_i} r. \end{aligned}$$

On the other hand,

$$\begin{aligned} (F_i X)(n) + (T_i \bar{X})(n) &= -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} \\ &\quad - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s-\sigma_i)) \\ &\quad - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\ &\geq -\frac{|x_i(n+\tau_i)|}{|p_i(n+\tau_i)|} + \frac{(1+\tilde{c}_{p_i})r}{2} \\ &\quad - \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)| |f_i(x_{i+1}(s-\sigma_i))| \\ &\quad - \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\ &\geq -r + \frac{1}{2} \tilde{c}_{p_i} r + \frac{1}{2} r - \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f} - \frac{(\tilde{c}_{p_i}-1)r}{4} \\ &= \frac{1}{8} (\tilde{c}_{p_i}-1)r. \end{aligned}$$



Next we have for  $i \in \tilde{I}_3 \cup \tilde{I}_4$

$$\begin{aligned}
 (F_i X)(n) + (T_i \bar{X})(n) &= -\frac{x_i(n + \tau_i)}{p_i(n + \tau_i)} + \frac{(\tilde{c}_{p_i} - 1)r}{2} \\
 &\quad - \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(\bar{x}_{i+1}(s - \sigma_i)) \\
 &\quad - \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\
 &\leq \frac{|x_i(n + \tau_i)|}{|p_i(n + \tau_i)|} + \frac{(\tilde{c}_{p_i} - 1)r}{2} \\
 &\quad + \frac{1}{|p_i(n + \tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)| |f_i(\bar{x}_{i+1}(s - \sigma_i))| \\
 &\quad + \frac{1}{|p_i(n + \tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\
 &\leq r + \frac{1}{2} \tilde{c}_{p_i} r - \frac{1}{2} r + \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i} - 1)r}{8\tilde{M}_f} + \frac{(\tilde{c}_{p_i} - 1)r}{4} \\
 &= \frac{7}{8} \tilde{c}_{p_i} r + \frac{1}{8} r \leq \tilde{c}_{p_i} r.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 (F_i X)(n) + (T_i \bar{X})(n) &= -\frac{x_i(n + \tau_i)}{p_i(n + \tau_i)} + \frac{(\tilde{c}_{p_i} - 1)r}{2} \\
 &\quad - \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(\bar{x}_{i+1}(s - \sigma_i)) \\
 &\quad - \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\
 &\geq \frac{(\tilde{c}_{p_i} - 1)r}{2} \\
 &\quad - \frac{1}{|p_i(n + \tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)| |f_i(\bar{x}_{i+1}(s - \sigma_i))| \\
 &\quad - \frac{1}{|p_i(n + \tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\
 &\geq \frac{1}{2} \tilde{c}_{p_i} r - \frac{1}{2} r - \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i} - 1)r}{8\tilde{M}_f} - \frac{(\tilde{c}_{p_i} - 1)r}{4} \\
 &= \frac{1}{8} (\tilde{c}_{p_i} - 1)r.
 \end{aligned}$$

To see that  $F$  is a contraction mapping let us observe that for  $i = 1, \dots, k$

$$\begin{aligned}
 |(F_i X)(n) - (F_i \bar{X})(n)| &\leq \frac{1}{|p_i(n + \tau_i)|} |x_i(n + \tau_i) - \bar{x}_i(n + \tau_i)| \\
 &\leq \frac{1}{\tilde{c}_{p_i}} |x_i(n + \tau_i) - \bar{x}_i(n + \tau_i)|.
 \end{aligned}$$

Hence

$$\|FX - F\bar{X}\| \leq \frac{1}{\min_{i=1, \dots, k} \{\tilde{c}_{p_i}\}} \|X - \bar{X}\|,$$

but  $\frac{1}{\min_{i=1,\dots,k} \{\tilde{c}_{p_i}\}} < 1$  by (2.11) and (2.12).

The proof of the continuity of the mapping  $T$  can be performed exactly in the same way as previously.

By virtue of Theorem 1.3, there exists  $X$  that  $(FX)(n) + (TX)(n) = X(n)$ . Finally, we show that  $X$  satisfies system (1.1) for  $n \geq n_5$ . Let  $(F_i X)(n) + (T_i X)(n) = x_i(n)$  for  $i = 1, \dots, k$ . We show all transformations only for  $i \in \tilde{I}_1 \cup \tilde{I}_2$  and  $i \neq k$ , because for the other cases they are analogous. Since

$$\begin{aligned} x_i(n) &= -\frac{x_i(n + \tau_i)}{p_i(n + \tau_i)} + \frac{(1 + \tilde{c}_{p_i})r}{2} - \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \\ &\quad - \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s), \end{aligned}$$

then we have

$$\begin{aligned} \Delta \left( x_i(n) + \frac{x_i(n + \tau_i)}{p_i(n + \tau_i)} \right) &= -\Delta \left( \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \right) \\ &\quad - \Delta \left( \frac{1}{p_i(n + \tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \right). \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{p_i(n + \tau_i + 1)} \Delta \left( x_i(n + \tau_i) + p_i(n + \tau_i) x_i(n) \right) + \left( \Delta \frac{1}{p_i(n + \tau_i)} \right) \left( x_i(n + \tau_i) + p_i(n + \tau_i) x_i(n) \right) \\ &= -\frac{1}{p_i(n + \tau_i + 1)} \Delta \left( \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \right) - \frac{1}{p_i(n + \tau_i + 1)} \Delta \left( \sum_{s=n+\tau_i}^{\infty} g_i(s) \right) \\ &\quad - \left( \Delta \frac{1}{p_i(n + \tau_i)} \right) \left( \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \right) - \left( \Delta \frac{1}{p_i(n + \tau_i)} \right) \left( \sum_{s=n+\tau_i}^{\infty} g_i(s) \right). \end{aligned}$$

It is easy to notice that

$$-\Delta \left( \sum_{s=n+\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \right) = a_i(n + \tau_i) f_i(x_{i+1}(n + \tau_i - \sigma_i)),$$

and

$$-\Delta \left( \sum_{s=n+\tau_i}^{\infty} g_i(s) \right) = g_i(n + \tau_i).$$

Then

$$\Delta \left( x_i(n + \tau_i) + p_i(n + \tau_i) x_i(n) \right) = a_i(n + \tau_i) f_i(x_{i+1}(n + \tau_i - \sigma_i)) + g_i(n + \tau_i).$$

Now we can transform the last equation into

$$\Delta \left( x_i(n) + p_i(n) x_i(n - \tau_i) \right) = a_i(n) f_i(x_{i+1}(n - \sigma_i)) + g_i(n).$$

The proof is complete.  $\square$

**Example 2.4.** Now, let us consider a difference system

$$\begin{cases} \Delta(x_1(n) + (2 + \frac{1}{2^n})x_1(n-2)) = -\frac{13 \cdot 8^{n-1} + 3 \cdot 4^{n-1}}{16^n - 4 \cdot 8^n + 4^{n+1}}x_2^2(n-2) + \frac{1}{2^n}, \\ \Delta(x_2(n) + (-1 - \frac{1}{2^n})x_2(n-2)) = \frac{-3 \cdot 4^n - 6 \cdot 2^n}{2 \cdot 8^n + 4 \cdot 4^n}x_3(n-1) - \frac{1}{2^n}, \\ \Delta(x_3(n) + (1 + \frac{1}{2^n})x_3(n-1)) = \frac{4 \cdot 4^n + 3 \cdot 2^n}{6 \cdot 8^n + 4 \cdot 4^n}x_4(n-1), \\ \Delta(x_4(n) + (-1 - \frac{1}{2^n})x_4(n-1)) = \frac{4 \cdot 8^n + 3 \cdot 4^n}{8 \cdot 16^n + 32 \cdot 8^n + 32 \cdot 4^n}x_1^2(n-2). \end{cases}$$

All assumptions of Theorem 2.3 are satisfied. The sequence

$$X = \left( \left(2 + \frac{1}{2^n}\right), \left(-2 + \frac{1}{2^n}\right), \left(-1 - \frac{1}{2^n}\right), \left(3 + \frac{1}{2^n}\right) \right) \quad \text{for } n \geq 2$$

is the bounded solution of the above system.

Now we can formulate the theorem that join both Theorem 2.1 and Theorem 2.3.

Let  $I_5, I_6, I_7, I_8$  be subsets of the set  $\{1, \dots, k\}$  such that  $I_i \cap I_j = \emptyset$  for  $i \neq j, i, j = 5, 6, 7, 8$  and  $I_5 \cup I_6 \cup I_7 \cup I_8 = \{1, \dots, k\}$ .

**Theorem 2.5.** Let assumptions (2.1), (2.2), (2.3) and (2.4) hold. If there exist positive real numbers  $c_{p_i}, i \in I_5 \cup I_6$  and  $\tilde{c}_{p_i}, i \in I_7 \cup I_8$  that satisfy the inequalities

$$\begin{aligned} 0 \leq p_i(n) \leq c_{p_i} < 1, & \quad \text{for } i \in I_5, n \in \mathbb{N}_0, \\ -1 < -c_{p_i} \leq p_i(n) \leq 0, & \quad \text{for } i \in I_6, n \in \mathbb{N}_0, \\ 1 < \tilde{c}_{p_i} \leq p_i(n), & \quad \text{for } i \in I_7, n \in \mathbb{N}_0, \\ p_i(n) \leq -\tilde{c}_{p_i} < -1, & \quad \text{for } i \in I_8, n \in \mathbb{N}_0, \end{aligned}$$

then system (1.1) has a bounded nonoscillatory solution.

*Proof.* For the fixed positive real number  $r$  we define the set

$$\Omega_3 = \left\{ X \in B : \frac{1}{8}(1 - c_{p_i})r \leq |x_i(n)| \leq r, i \in I_5 \cup I_6, \right. \\ \left. \frac{1}{8}(\tilde{c}_{p_i} - 1)r \leq |x_i(n)| \leq \tilde{c}_{p_i}r, i \in I_7 \cup I_8, n \in \mathbb{N} \right\}.$$

$\Omega_3$  is bounded closed convex subset of the Banach space  $B$ .

Let  $n_6 = \max\{c_1, c_5\}$ . From assumptions (2.1) and (2.2) we have

$$\begin{aligned} \sum_{n=n_6}^{\infty} |a_i(n)| &\leq \frac{(1 - c_{p_i})r}{8M_f}, & i \in I_5 \cup I_6, \\ \sum_{n=n_6}^{\infty} |g_i(n)| &\leq \frac{(1 - c_{p_i})r}{4}, & i \in I_5 \cup I_6, \\ \sum_{n=n_6}^{\infty} |a_i(n)| &\leq \frac{(\tilde{c}_{p_i} - 1)r}{8\tilde{M}_f}, & i \in I_7 \cup I_8, \\ \sum_{n=n_6}^{\infty} |g_i(n)| &\leq \frac{(\tilde{c}_{p_i} - 1)r}{4}, & i \in I_7 \cup I_8, \end{aligned}$$

where

$$M_f = \max_{i \in I_5 \cup I_6} \left\{ |f_i(t)| : |t| \in \left[ \frac{1}{8}(1 - c_{p_i})r, r \right] \right\},$$

$$\tilde{M}_f = \max_{i \in I_7 \cup I_8} \left\{ |f_i(t)| : |t| \in \left[ \frac{1}{8}(\tilde{c}_{p_i} - 1)r, \tilde{c}_{p_i}r \right] \right\}.$$

We can now proceed analogously as in the proof of Theorem 2.1 and Theorem 2.3. Repeating reasoning in these proofs we define for  $n \geq n_6$  the maps  $F, T: \Omega_3 \rightarrow B$  by formulas (2.7)–(2.8) for  $i \in I_5 \cup I_6$  and (2.13)–(2.15) for  $i \in I_7 \cup I_8$ . The rest of the proof also runs as in Theorem 2.1 and Theorem 2.3.  $\square$

In the next theorem we consider the case  $p_i(n) \equiv 1$ ,  $i = 1, \dots, k$  and get even better result than in the previous theorems.

**Theorem 2.6.** *Assume that conditions (2.1), (2.2), (2.3) and (2.4) are satisfied. If  $p_i(n) \equiv 1$ ,  $i = 1, \dots, k$  then for any real constants  $d_1, \dots, d_k$  there exists a solution  $X$  of system (1.1) that  $\lim_{n \rightarrow \infty} X(n) = (d_1, \dots, d_k)$ .*

*Proof.* Let  $d_i \in \mathbb{R}$ ,  $i = 1, \dots, k$  and let  $\varepsilon$  be any positive real number. There exists a constant  $M > 0$  such that

$$|f_i(t)| \leq M \quad \text{for } t \in [d_i - \varepsilon, d_i + \varepsilon], \quad i = 1, \dots, k.$$

Let us denote

$$S_{a_i}(n) = \sum_{j=n}^{\infty} |a_i(j)|, \quad S_{g_i}(n) = \sum_{j=n}^{\infty} |g_i(j)|, \quad i = 1, \dots, k.$$

By (2.1) and (2.2) there exists such an index  $n_7 \geq n_0$  that for  $n \geq n_7$  we have

$$S_{a_i}(n) \leq \frac{\varepsilon}{2M}, \quad \text{and} \quad S_{g_i}(n) \leq \frac{\varepsilon}{2}, \quad i = 1, \dots, k.$$

We define a subset  $\Omega_5$  of  $\mathcal{B}$  by

$$\Omega_5 = \{X \in \mathcal{B} : X(0) = \dots = X(n_7 - 1) = D \text{ and } |X(n) - D| \leq M|S_A(n)| + |S_G(n)| \text{ for } n \geq n_7\},$$

where  $D = (d_1, \dots, d_k)$ ,  $S_A = (S_{a_1}, \dots, S_{a_k})$ ,  $S_G = (S_{g_1}, \dots, S_{g_k})$ . It is easy to check, that  $\Omega_5$  is the convex subset of  $\mathcal{B}$ . It can be also shown that  $\Omega_5$  is compact (see, for example, the proof of Theorem 1 in [12] or Lemma 4.7 in [11]). Now, for  $n \geq 0$ , we define a map

$$T: \Omega_5 \rightarrow \mathcal{B},$$

as follows, for  $i = 1, \dots, k - 1$

$$(T_i X)(n) = \begin{cases} d_i, & \text{for } n < n_7, \\ d_i - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} g_i(s), & \text{for } n \geq n_7 \text{ and } \tau_i > 0, \\ d_i - \frac{1}{2} \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \frac{1}{2} \sum_{s=n}^{\infty} g_i(s), & \text{for } n \geq n_7 \text{ and } \tau_i = 0, \end{cases}$$

and

$$(T_k X)(n) = \begin{cases} d_k, & \text{for } n < n_7, \\ d_k - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_k}^{n+2j\tau_k-1} a_k(s) f_k(x_1(s - \sigma_k)) - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_k}^{n+2j\tau_k-1} g_k(s), & \text{for } n \geq n_7 \text{ and } \tau_k > 0, \\ d_k - \frac{1}{2} \sum_{s=n}^{\infty} a_k(s) f_k(x_1(s - \sigma_k)) - \frac{1}{2} \sum_{s=n}^{\infty} g_k(s), & \text{for } n \geq n_7 \text{ and } \tau_k = 0. \end{cases}$$

We will show that  $T(\Omega_5) \subseteq \Omega_5$ . It is obvious that

$$\sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} |a_i(s)| \leq \sum_{s=n}^{\infty} |a_i(s)|, \quad i = 1, \dots, k, \quad (2.16)$$

$$\sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} |g_i(s)| \leq \sum_{s=n}^{\infty} |g_i(s)|, \quad i = 1, \dots, k. \quad (2.17)$$

Moreover, if  $X \in \Omega_5$ , then  $|x_i(n) - d_i| \leq h$  for all  $n \in \mathbb{N}$ ,  $i = 1, \dots, k$ . Hence  $|f_i(x_{i+1}(n))| \leq M$ ,  $i = 1, \dots, k-1$  and also  $|f_k(x_1(n))| \leq M$  for every  $X \in \Omega_5$ ,  $n \in \mathbb{N}$ . Therefore and by (2.16) and (2.17), for  $n \geq n_7$  and  $\tau_i > 0$ , we get

$$|(T_i X)(n) - d_i| \leq M \sum_{s=n}^{\infty} |a_i(s)| + \sum_{s=n}^{\infty} |g_i(s)| = MS_{a_i}(n) + S_{g_i}(n), \quad (2.18)$$

for  $i = 1, \dots, k-1$ . The same estimation holds for  $i = k$ .

For  $n \geq n_7$ ,  $\tau_i = 0$  we have

$$\begin{aligned} |(T_i X)(n) - d_i| &= \left| \frac{1}{2} \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) + \frac{1}{2} \sum_{s=n}^{\infty} g_i(s) \right| \\ &\leq MS_{a_i}(n) + S_{g_i}(n), \quad i = 1, \dots, k-1, \end{aligned}$$

and similarly for  $i = k$ . This gives  $T(X) \in \Omega_5$  for every  $X \in \Omega_5$  and  $T(\Omega_5) \subseteq \Omega_5$ . Similarly as in the proof of Theorem 2.1, it can be shown that  $T$  is continuous.

By Schauder's fixed point theorem there exists  $X \in \Omega_5$  such that  $T(X) = X$ , which is a solution of system (1.1). In fact, for  $n \geq n_7$ ,  $\tau_i > 0$  and  $i = 1, \dots, k-1$  we have

$$x_i(n) = d_i - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} g_i(s).$$

Hence

$$\begin{aligned} x_i(n) + x_i(n - \tau_i) &= 2d_i - \sum_{j=1}^{\infty} \sum_{s=n+2(j-1)\tau_i}^{n+2j\tau_i-1} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \\ &\quad - \sum_{j=1}^{\infty} \sum_{s=n+2(j-1)\tau_i}^{n+2j\tau_i-1} g_i(s) \\ &= 2d_i - \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s), \quad i = 1, \dots, k-1. \end{aligned}$$

Therefore

$$\begin{aligned}\Delta(x_i(n) + x_i(n - \tau_i)) &= - \sum_{s=n+1}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \\ &\quad + \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) \\ &\quad - \sum_{s=n+1}^{\infty} g_i(s) + \sum_{s=n}^{\infty} g_i(s), \quad i = 1, \dots, k-1,\end{aligned}$$

and finally

$$\Delta(x_i(n) + x_i(n - \tau_i)) = a_i(n) f_i(x_{i+1}(n - \sigma_i)) + g_i(n), \quad i = 1, \dots, k-1.$$

In the case  $\tau_i = 0$  we obtain

$$\begin{aligned}\Delta(x_i(n) + x_i(n)) &= 2\Delta x_i(n) \\ &= 2\Delta \left( d_i - \frac{1}{2} \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \frac{1}{2} \sum_{s=n}^{\infty} g_i(s) \right) \\ &= a_i(n) f_i(x_{i+1}(n - \sigma_i)) + g_i(n), \quad i = 1, \dots, k-1.\end{aligned}$$

The same reasoning applies to the case  $i = k$ . It is clear that  $X$  fulfills system (1.1) for  $n \geq n_7$ . By (2.1) and (2.2) sequences  $S_{a_i}$  and  $S_{g_i}$ ,  $i = 1, \dots, k$ , tend to zero. From (2.18) we get  $\lim_{n \rightarrow \infty} X(n) = D$ , that is our claim.  $\square$

**Example 2.7.** Let us consider the following system

$$\begin{cases} \Delta(x_1(n) + x_1(n-1)) = -\frac{11 \cdot 3^n}{12 \cdot 9^n - 108 \cdot 3^n + 243} x_2^2(n-2) + \frac{1}{3^n}, \\ \Delta(x_2(n) + x_2(n-2)) = \frac{20}{6 \cdot 3^n + 81} x_3(n-3), \\ \Delta(x_3(n) + x_3(n-2)) = \frac{23}{9 \cdot 3^n - 9} x_4(n-1) - \frac{1}{3^n}, \\ \Delta(x_4(n) + x_4(n-3)) = \frac{35 \cdot 3^n}{12 \cdot 9^n + 108 \cdot 3^n + 243} x_1^2(n-2) + \frac{7}{3^n}. \end{cases}$$

All assumptions of Theorem 2.6 are satisfied. It is easy to check that

$$X = \left( \left( 2 + \frac{1}{3^n} \right), \left( -2 + \frac{1}{3^n} \right), \left( -2 - \frac{1}{3^n} \right), \left( 3 - \frac{1}{3^n} \right) \right)$$

for  $n \geq 3$  is the solution of the above system having the property  $\lim_{n \rightarrow \infty} X(n) = (2, -2, -2, 3)$ .

In the theorem below we consider the case  $p_i(n) \equiv -1$ ,  $i = 1, \dots, k$ .

**Theorem 2.8.** Let conditions (2.3) and (2.4) be satisfied and assume

$$\sum_{n=1}^{\infty} n |a_i(n)| < \infty, \quad i = 1, \dots, k, \quad (2.19)$$

$$\sum_{n=1}^{\infty} n |g_i(n)| < \infty, \quad i = 1, \dots, k. \quad (2.20)$$

If  $p_i(n) \equiv -1$ ,  $i = 1, \dots, k$ , then for any real constants  $d_1, \dots, d_k$  there exists a solution  $X$  of system (1.1) that  $\lim_{n \rightarrow \infty} X(n) = (d_1, \dots, d_k)$ .

*Proof.* We can now proceed analogously to the proof of Theorem 2.6. Let  $d_i \in \mathbb{R}$ ,  $i = 1, \dots, k$  and let  $\varepsilon$  be any positive real number. There exists a constant  $M > 0$  such that

$$|f_i(t)| \leq M \quad \text{for } t \in [d_i - \varepsilon, d_i + \varepsilon], \quad i = 1, \dots, k.$$

Write

$$S_{a_i}(n) = \sum_{j=n}^{\infty} j |a_i(j)|, \quad S_{g_i}(n) = \sum_{j=n}^{\infty} j |g_i(j)|, \quad i = 1, \dots, k.$$

If the sequences  $a_1, \dots, a_k$  and  $g_1, \dots, g_k$  satisfy (2.19) and (2.20), then immediately satisfy (2.1) and (2.2) consequently. Hence, for  $n \geq n_7$ , we have

$$S_{a_i}(n) \leq \frac{\varepsilon}{2M}, \quad \text{and} \quad S_{g_i}(n) \leq \frac{\varepsilon}{2}, \quad i = 1, \dots, k.$$

We define a map

$$T : \Omega_5 \rightarrow \mathcal{B},$$

in the following way, for  $i = 1, \dots, k-1$

$$(T_i X)(n) = \begin{cases} d_i & \text{for } n < n_7 \\ d_i - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} g_i(s) & \text{for } n \geq n_7, \end{cases}$$

and

$$(T_k X)(n) = \begin{cases} d_k & \text{for } n < n_7, \\ d_k - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_k}^{\infty} a_k(s) f_k(x_1(s - \sigma_k)) - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_k}^{\infty} g_k(s) & \text{for } n \geq n_7. \end{cases}$$

We will prove that  $T(\Omega_5) \subseteq \Omega_5$ . It is easy to observe that

$$\sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} |a_i(s)| \leq \sum_{s=n}^{\infty} s |a_i(s)|, \quad i = 1, \dots, k, \quad (2.21)$$

$$\sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} |g_i(s)| \leq \sum_{s=n}^{\infty} s |g_i(s)|, \quad i = 1, \dots, k. \quad (2.22)$$

By (2.21) and (2.22) for  $n \geq n_7$  we get

$$\begin{aligned} |(T_i X)(n) - d_i| &= \left| \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) + \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} g_i(s) \right| \\ &\leq M \sum_{s=n}^{\infty} s |a_i(s)| + \sum_{s=n}^{\infty} s |g_i(s)| = M S_{a_i}(n) + S_{g_i}(n) \end{aligned} \quad (2.23)$$

for  $i = 1, \dots, k-1$ . Analogously we get this for  $i = k$ . Hence  $T(X) \in \Omega_5$  for any  $X \in \Omega_5$  and  $T(\Omega_5) \subseteq \Omega_5$ . Reasoning similarly as in the proof of Theorem 2.1, it can be shown that  $T$  is continuous.

By Schauder's fixed point theorem there exists  $X \in \Omega_5$  such that  $T(X) = X$  and it is a solution of system (1.1). For  $n \geq n_7$  we have

$$x_i(n) = d_i - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} g_i(s),$$

for  $i = 1, \dots, k - 1$ , and

$$x_i(n - \tau_i) = d_i - \sum_{j=1}^{\infty} \sum_{s=n+(j-1)\tau_i}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{j=1}^{\infty} \sum_{s=n+(j-1)\tau_i}^{\infty} g_i(s).$$

Since

$$x_i(n) - x_i(n - \tau_i) = - \sum_{s=n}^{\infty} a_i(s) f_i(x_{i+1}(s - \sigma_i)) - \sum_{s=n}^{\infty} g_i(s),$$

we have

$$\Delta(x_i(n) - x_i(n - \tau_i)) = a_i(n) f_i(x_{i+1}(n - \sigma_i)) + g_i(n), \quad i = 1, \dots, k - 1.$$

The same conclusion can be drawn for  $i = k$ .

Finally we see that  $X$  satisfies system (1.1) for  $n \geq n_7$ . By (2.19) and (2.20) sequences  $S_{a_i}$  and  $S_{g_i}$ ,  $i = 1, \dots, k$ , tend to zero. From (2.23) we get

$$\lim_{n \rightarrow \infty} X(n) = D. \quad \square$$

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