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A note on the existence of positive solutions of singular initial-value problem for second order differential equations

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Abstract. We are interested in the existence of positive solutions to initial-value problems for second-order nonlinear singular differential equations. Existence of solutions is proven under conditions which are directly applicable and considerably weaker than previously known conditions.

Keywords: second order equations, existence, Emden–Fowler equation.

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1 Introduction

In recent years, the studies of singular initial value problems for second order differential equation have attracted the attention of many mathematicians and physicists (see for example, [1–22])

Agarwal and O'Regan [1] established the existence theorems for the positive solution of the problems

$$\begin{aligned} (py')' + pqg(y) &= 0, \qquad t \in [0, T) \\ y(0) &= a > 0, \\ \lim_{t \to 0+} p(t)y'(t) &= 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} (py')' + pqg(y) &= 0, \qquad t \in [0, T) \\ y(0) &= a > 0, \\ y'(0) &= 0, \end{aligned} \tag{1.2}$$

where $0 < T \leq \infty$, $p \geq 0$, $q \geq 0$ and $g : [0, \infty) \to [0, \infty)$.

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Theorem 1.1 ([1]). *Suppose the following conditions are satisfied*

$$p \in C[0,T) \cap C^1(0,T) \quad \text{with } p > 0 \text{ on } (0,T) \tag{1.3}$$

$$q \in L^1_p[0,t^*] \quad \text{for any } t^* \in (0,T) \text{ with } q > 0 \text{ on } (0,T), \tag{1.4}$$

where $L^1_r[0,a]$ *is the space of functions* $u(t)$ *with* $\int_0^a |u(t)|\, r(t) dt < \infty$,

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \quad \text{for any } t^* \in (0,T) \tag{1.5}$$

and

$$g : [0,\infty) \to [0,\infty) \text{ is continuous, nondecreasing on } [0,\infty) \text{ and } g(u) > 0 \text{ for } u > 0. \tag{1.6}$$

Let

$$H(z) = \int_z^a \frac{dx}{g(x)} \quad \text{for } 0 < z \le a$$

and assume

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)dxds < a \ \text{ for any } t^* \in (0,T), \tag{1.7}$$

here

$$\tau(x) = g\left(H^{-1}\left(\int_0^x \frac{1}{p(w)} \int_0^w p(z)q(z)dzdw\right)\right).$$

Then (1.1) *has a solution* $y \in C[0,T)$ *with* $py' \in C[0,T)$, $(py')' \in L^1_{pq}(0,T)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T)$. *In addition if either*

$$p(0) \neq 0 \tag{1.8}$$

or

$$p(0) = 0 \quad \text{and} \quad \lim_{t \to 0+} \frac{p(t)q(t)}{p'(t)} = 0 \tag{1.9}$$

holds, then y *is a solution of* (1.2).

The condition (1.7) makes this theorem difficult for application. We try to establish a more general and applicable condition instead of (1.6) and (1.7).

2 Main results

Theorem 2.1. *Suppose* (1.3)–(1.5) *hold. In addition we assume*

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dxds < a \tag{2.1}$$

for any $t^* \in (0,T_0)$. *Then*

a) (1.1) *has a solution* $y \in C[0,T_0)$ *with* $py' \in C[0,T_0)$, $(py')' \in L^1_{pq}(0,T_0)$ *and* $0 < y(t) \le a$ *for* $t \in [0,T_0)$.

b) If $\int_{T_0}^{T_1} \frac{1}{p(s)} \int_0^s p(x)q(x)g(T_0)ds < y(T_0)$, *and conditions* (1.3)–(1.6) *satisfied then the solution can be extended to the interval* $[0,T_1)$.

By monotonicity of $g(x)$, it follows from (1.7) that

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)\tau(x)dxds \le \int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dxds,$$

and therefore the condition (2.1) is stronger than condition (1.7) in the Theorem 1.1. But the statement b) of the Theorem 2.1 allows to extend the condition to the new intervals $[T_0, T_1)$, $[T_1, T_2), \dots$ and therefore this theorem can be considered as a generalization of the Theorem 1.1.

For the existence of the inverse of the function $H(z)$, the Theorem 1.1 proposes the condition $q > 0$ on $(0, T)$ and $g(u) > 0$ for $u > 0$. Since we shall not deal with the function $H(z)$, we shall prove more general theorem.

Theorem 2.2. *Suppose the following conditions are satisfied*

$$p \in C[0, T_0) \cap C^1(0, T_0) \quad \text{with } p > 0 \text{ on } (0, T_0) \tag{2.2}$$

$$q \in L^1_p[0, t^*] \quad \text{for any } t^* \in (0, T_0) \quad \text{with } q \ge 0 \text{ on } (0, T_0), \tag{2.3}$$

where $L^1_r[0, a]$ *is the space of functions* $u(t)$ *with* $\int_0^a |u(t)|\, r(t)dt < \infty$,

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)dxds < \infty \quad \text{for any } t^* \in (0, T_0), \tag{2.4}$$

$$g : [0, \infty) \to [0, \infty) \text{ is nondecreasing on } [0, \infty) \text{ with } g(u) \ge 0 \text{ for } u > 0 \tag{2.5}$$

and

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dx < a$$

for any $t^* \in (0, T_0)$. *Then*

a) (1.1) *has a solution* $y \in C[0, T_0)$ *with* $py' \in C[0, T_0)$, $(py')' \in L^1_{pq}(0, T_0)$ *and* $0 < y(t) \le a$ *for* $t \in [0, T_0)$.

b) *If* $\int_{T_0}^{T_1} \frac{1}{p(s)} \int_0^s p(x)q(x)g(T_0)ds < y(T_0)$, *and conditions* (2.2)–(2.5) *satisfied then solution can be extended to the interval* $[0, T_1)$.

In addition if either (1.8) *and* (1.9) *holds, then* y *is a solution of* (1.2).

Proof of Theorem 2.2. Let us take $y_0(t) \equiv a$, and define $y_1(t)$, $y_2(t), \dots$ from the recurrence relations

$$y_n(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{n-1}(x))dxds, \qquad n = 1, 2, \dots \tag{2.6}$$

For the sequence $\{y_n(t)\}$ we obtain

$$y_1(t) \le y_0(t) = a \tag{2.7}$$

$$y_2(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_1(x))dxds \ge a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_0(x))dxds = y_1(t),$$

$$y_3(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_2(x))dxds \le a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_1(x))dxds = y_2(t),$$

$$y_3(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_2(x))dxds \ge a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dxds = y_1(t),$$

$$y_4(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_3(x))dxds \geq a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_2(x))dxds = y_3(t),$$
$$y_4(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_3(x))dxds \leq a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_1(x))dxds = y_2(t),$$
$$\vdots$$

That is, $\{y_{2n}(t)\}$ and $\{y_{2n+1}(t)\}$ are monotonically nonincreasing and nondecresing sequences, consecutively. Let us show that these sequences are equicontinuous. Indeed we have

$$|y_n(t) - y_n(r)| = \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y_{n-1}(x))dxds \leq M \int_r^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds,$$

where

$$M = \max\{g(u) : 0 \leq u \leq a\}$$

and it follows from (2.4) that the right-hand side can be taken $< \varepsilon$ for $|t - r| < \delta$, regardless of the choice of t and r: the function $\varphi(t) = \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)dxds$ is (uniformly) continuous on $[0, t^*]$ for any $t^* < T_0$. That is, the bounded and equicontinuous sequences $\{y_{2n}(t)\}$ and $\{y_{2n+1}(t)\}$ both have a limit. Denote by

$$\lim_{n\to\infty} y_{2n}(t) \equiv u(t),$$
$$\lim_{n\to\infty} y_{2n+1}(t) \equiv v(t).$$

Clearly we have $u(t) \geq v(t)$. Now Lebesgue's dominated theorem guarantees that

$$\begin{aligned} u(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(v(x))dxds \qquad \text{and} \\ v(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds. \end{aligned} \tag{2.8}$$

If $u(t) = v(t)$ we have that the function $u(t)$ is the solution of the problem (1.1), indeed it follows from

$$u(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(u(x))dxds$$

that

$$u'(t) = -\frac{1}{p(t)} \int_0^t p(x)q(x)g(u(x))dx,$$
$$pu' = -\int_0^t p(x)q(x)g(u(x))dx,$$
$$(pu')' = -pqg(u).$$

So we suppose $u(t) \neq v(t)$ and consider the operator $N : C[0, T_0) \to C[0, T_0)$ defined by

$$Ny(t) = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y(x))dxds. \tag{2.9}$$

Next let

$$K = \{y \in C[0, T_0) : v(t) \leq y(t) \leq u(t) \text{ for } t \in [0, T_0)\}.$$

Clearly K is closed, convex, bounded subset of $C[0,T_0)$ and $N : K \to K$. Let us show that $N : K \to K$ is continuous and compact. Continuity follows from Lebesgue's dominated convergence theorem: if $y_n(t) \to y(t)$, then $Ny_n(t) \to Ny(t)$. To show that N is completely continuous let $y(t) \in K$, $t^* < T_0$, then

$$|Ny(t) - Ny(r)| \le M \left| \int_r^t \frac{1}{p(x)} \int_0^x p(z)q(z)dzds \right| \quad \text{for } t,r \in [0,t^*],$$

that is N completely continuous on $[0,T_0)$.

The Schauder–Tychonoff theorem guarantees that N has a fixed point $w \in K$, i.e. w is a solution of (1.1).

Now if $w(T_0) > 0$, and $\int_{T_0}^{T_1} \frac{1}{p(s)} \int_0^s p(x)q(x)g(T_0)ds < w(T_0) = b$ we take

$$\begin{aligned} \overline{y}_0(t) &= \begin{cases} w, & \text{if } 0 \le t \le T_0 \\ b, & \text{if } T_0 \le t < T_1 \end{cases} \\ \overline{y}_n(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(\overline{y}_{n-1}(x))dxds, \qquad n = 1,2,\dots \end{aligned} \tag{2.10}$$

and in like manner we obtain the solution $\overline{w}$ of the problem (1.2) on the interval $[0,T_1)$. Clearly we obtain for this solution

$$\overline{w}(T_0) = b = w(T_0) = a - \int_0^{T_0} \frac{1}{p(s)} \int_0^s p(x)q(x)g(w(x))dxds$$

and

$$\begin{aligned} \overline{w}'(T_0+) &= \lim_{t\to 0+} \frac{\overline{w}(T_0+t) - \overline{w}(T_0)}{t} \\ &= -\lim_{t\to 0+} \frac{\int_{T_0}^{T_0+t} \frac{1}{p(s)} \int_0^s p(x)q(x)g(\overline{w}(x))dxds}{t}. \end{aligned}$$

Using L'Hôpital's rule we obtain

$$\begin{aligned} \overline{w}'(T_0+) &= -\lim_{t\to 0+} \frac{1}{p(T_0+t)} \int_0^{T_0+t} p(x)q(x)g(\overline{w}(x))dx \\ &= -\frac{1}{p(T_0)} \int_0^{T_0} p(x)q(x)g(\overline{w}(x))dx = w'(T_0-) \end{aligned}$$

and therefore $\overline{w}' \in C[0,T_1)$. It is also clear that $p\overline{w}'$ is differentiable and

$$(p\overline{w}')' = -pqg(\overline{w})$$

for all $t \in [0,T_1)$.

If (1.8) or (1.9) holds we easily have $\overline{w}'(0) = 0$ and therefore $\overline{w}$ is the solution of (1.2). $\square$

Now we will prove the stronger result which generalizes the Theorems 1.1, 2.1 and 2.2. Consider the problem

$$\begin{aligned} (py')' + p(t)h(t,y) &= 0, \qquad t \in [0,T) \\ y(0) &= a > 0, \\ \lim_{t\to 0+} p(t)y'(t) &= 0 \end{aligned} \tag{2.11}$$

or

$$(py')' + p(t)h(t,y) = 0, \qquad t \in [0,T)$$
$$y(0) = a > 0, \tag{2.12}$$
$$y'(0) = 0$$

and some preliminary problem

$$(pz')' + p(t)\varphi(t) = 0, \qquad t \in [0,T)$$
$$z(0) = a > 0, \tag{2.13}$$
$$\lim_{t\to 0+} p(t)z'(t) = 0$$

or

$$(pz')' + p(t)\varphi(t) = 0, \qquad t \in [0,T)$$
$$z(0) = a > 0, \tag{2.14}$$
$$z'(0) = 0.$$

Theorem 2.3. *Suppose that $p \in C[0,T_0) \cap C^1(0,T_0)$ with $p > 0$ on $(0,T_0)$ and p is integrable,*

$$k(t) \geq \varphi(t) - h(t,y) \geq 0,\ t \in (0,T_0) \tag{2.15}$$

where $p(t)k(t)$ and $p(t)\varphi(t)$ are integrable with

$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(t)k(x)dxds < \infty \quad \text{and}$$
$$\int_0^{t^*} \frac{1}{p(s)} \int_0^s p(t)\varphi(x)dxds < \infty \quad \text{for any } t^* \in (0,T_0), \tag{2.16}$$

$\varphi(t) \in C(0,T_0)$ and such that the problem (2.13) has nonnegative solution $z(t)$, for each $t \in (0,T_0)$, $h(t,\cdot)$ is continuous, for each fixed y, $h(\cdot,y)$ is measurable on $[0,T_0]$, then the problem (2.11) has nonnegative solution on $[0,T_0)$.

Note 2.4. Theorem 2.3 is the generalization of Theorem 2.2 in the following sense: since $g(y)$ in the Theorem 2.2 is nonincreasing we have that $g(a) \geq g(y)$ and therefore for the solution of the problem (1.1) we have

$$\begin{aligned} y(t) &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(y)dxds \\ &= a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\left[g(y) - g(a)\right]dxds - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)g(a)dxds, \end{aligned}$$

that is

$$y(t) = z(t) + \int_0^t \frac{1}{p(s)} \int_0^s \left[g(a) - g(y)\right] p(x)q(x)dxds,$$

where $z(t)$ is the solution of the problem (2.13) with $\varphi(t) = g(a)$. Thus Theorem 2.2 is a special case of Theorem 2.3 with $\varphi(t) = g(a)$ and $k(t) = g(a) - g(y)$.

Note 2.5. Theorem 2.3 shows that the nondecreasing condition of $g(y)$ in the statement of the Theorem 2.2 can be omitted and therefore the scope of problems can be seriously extended.

Proof of Theorem 2.3. It follows from the condition (2.16) that the problem (2.13) has nonnegative solution:

$$z = a - \int_0^t \frac{1}{p(s)} \int_0^s p(x)\varphi(x))dxds \tag{2.17}$$

on some interval $[0, T_0)$.

We will show that the problem (2.11) is equivalent to the (integral) equation:

$$y(t) = z(t) + \int_0^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y)\right] dxds. \tag{2.18}$$

Let us calculate the derivatives $y'(t)$ and $(py'(t))'$ from (2.18) by using the Leibniz rule:

$$\begin{aligned} y'(t) &= z'(t) + \frac{1}{p(t)} \int_0^t p(x)\left[\varphi(x) - h(x,y(x))\right] dx, \\ p(t)y'(t) &= p(t)z'(t) + \int_0^t p(x)\left[\varphi(x) - h(x,y(x))\right] dx, \\ (py'(t))' &= (pz'(t))' + p(t)\varphi(t) - p(t)h(t,y(t)), \end{aligned}$$

and since $(pz')' + p(t)\varphi(t) = 0$ we obtain $(py'(t))' + p(t)h(t,y(t)) = 0$. That is, the equation (2.18) is equivalent to the problem (2.11). Let us consider the recurrence relations

$$\begin{aligned} y_0(t) &= z(t), \\ y_1(t) &= z(t) + \int_0^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y_0)\right] dxds, \ldots \\ y_n(t) &= z(t) + \int_0^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y_{n-1})\right] dxds, \ldots \end{aligned} \tag{2.19}$$

We have

$$|y_n(t) - z(t)| \le \left| \int_0^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y_{n-1})\right] dxds \right| \le \left| \int_0^t \frac{1}{p(s)} \int_0^s p(x)k(x)dxds \right|$$

and

$$\begin{aligned} &|y_n(t) - z(t) - (y_n(r) - z(r))| \\ &\quad = \left| \int_r^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y_{n-1})\right] dxds \right| \\ &\quad \le \int_r^t \frac{1}{p(s)} \int_0^s p(x)k(x)dxds. \end{aligned} \tag{2.20}$$

Thus, the sequence $\{y_n(t) - z(t)\}$ is uniformly bounded and equicontinuous on $[0, t^*]$ for any $t^* < T_0$ and therefore by Ascoli–Arzelà lemma, there exists a continuous $w(t)$ such that $y_{n_k}(t) - z(t) \to w(t)$ uniformly on $[0, t^*]$. Without loss of generality, say $y_n(t) - z(t) \to w(t)$ or $y_n(t) \to z(t) + w(t) \equiv y(t)$. Then we obtain

$$\begin{aligned} y(t) &= z(t) + \lim_{n\to\infty} \int_0^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y_n)\right] dxds \\ &= z(t) + \int_0^t \frac{1}{p(s)} \int_0^s p(x)\left[\varphi(x) - h(x,y(x))\right] dxds \end{aligned} \tag{2.21}$$

using the Lebesgue dominated convergence theorem. Thus $y(t) \ge 0$ is the solution of the problem (2.11). □

Example 2.6. The problem

$$(t^{-1/3}y')' + t^{-1/3}\left(\frac{1}{2}\frac{t^{2,5}}{t^3+(y-1)^2}\right) = 0, \qquad t \in \left[0, \frac{1}{\sqrt[3]{4}}\right]$$
$$y(0) = 1,$$
$$\lim_{t\to 0+} p(t)y'(t) = 0$$

has a positive solution $y = -t^{3/2} + 1$. For $h(t,y)$ we have

$$h(t,y) = \frac{1}{2}\frac{t^{2,5}}{t^3+(y-1)^2} \le \frac{1}{2\sqrt{t}} \equiv \varphi(t),$$

and

$$z(t) = -2t^{3/2} + 1 \ge 0$$

is the solution of the problem

$$(t^{-1/3}z')' + t^{-1/3}\varphi(t) = 0,$$
$$z(0) = 1,$$
$$\lim_{t\to 0+} p(t)z'(t) = 0.$$

For the iteration sequence y_n in (2.19) we obtain (by using standard Latex tools)

$$\begin{aligned}
y_0(t) &= -2t^{3/2} + 1,\\
y_1(t) &= -2t^{3/2} + 1 + \int_0^t s^{1/3}\int_0^s x^{-1/3}\left(\frac{1}{2x^{1/2}} - \frac{1}{2}\frac{x^{2.5}}{x^3+4x^3}\right)dxds\\
&= -2t^{3/2} + 1 + \frac{8}{5}t^{3/2} = 1 - \frac{2}{5}t^{\frac{3}{2}},\\
y_2(t) &= 1 - \frac{50}{29}t^{\frac{3}{2}}, \qquad y_3(t) = 1 - \frac{1682}{3341}t^{\frac{3}{2}}, \qquad y_4(t) = 1 - 1.595\,6t^{\frac{3}{2}},\\
y_5(t) &= 1 - 0.564\,03t^{\frac{3}{2}}, \dots,\\
y_{17}(t) &= 1 - 0.711\,85t^{\frac{3}{2}}, \dots
\end{aligned}$$

Conflict of interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] R. P. Agarwal, D. O'Regan, Second order initial value problems of Lane–Emden type, *Appl. Math. Lett.* **20**(2007), No. 12, 1198–1205. MR2384246; url

[2] A. Aslanov, A generalization of the Lane–Emden equation, *Int. J. Comput. Math.* **85**(2008), 1709–1725. MR2457778; url

[3] M. Beech, An approximate solution for the polytrope $n = 3$, *Astrophys. and Space Sci.* **132**(1987), 393–396. MR887764; url

[4] R. Bellman, *Stability theory of differential equations*, McGraw-Hill, New York, 1953. MR0061235

[5] D. C. Biles, M. P. Robinson, J. S. Spraker, A generalization of the Lane–Emden equation, *J. Math. Anal. Appl.* **273**(2002), 654–666. MR1932513; url

[6] S. Chandrasekhar, *Introduction to the study of stellar structure*, Dover Publications, New York, 1967. MR0092663

[7] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York–Toronto–London, 1955. MR0069338

[8] H. T. Davis, *Introduction to nonlinear differential and integral equations*, Dover Publications, New York, 1962. MR0181773

[9] P. Habets, F. Zanolin, Upper and lower solutions for a generalized Emden–Fowler equation. *J. Math. Anal. Appl.* **181**(1994), 684–700. MR1264540; url

[10] D. Hinton, An oscillation criterion for solutions of $(ry')^2 + qy^\gamma = 0$. *Michigan Math. J.* **16**(1969), 349–352. MR0249728

[11] G. P. Horedt, Exact solutions of the Lane–Emden equation in N-dimensional space, *Astron. Astrophys.* **172**(1987), 359–367.

[12] J. Janus, J. Myjak, A generalized Emden–Fowler equation with a negative exponent, *Nonlinear Anal.* **23**(1994), 953–970. MR1304238; url

[13] P. G. L. Leach, First integrals for the modified Emden equation $q'' + \alpha(t)q' + q^n = 0$, *J. Math. Phys.* **26**(1985), 2510–2514. MR803793; url

[14] S. Liao, A new analytic algorithm of Lane–Emden type equations. *Appl. Math. Comput.* **142**(2003), 1–16. MR1978243; url

[15] F. K. Liu, Polytropic gas spheres: an approximate analytic solution of the Lane–Emden equation, *Mon. Not. R. Astron. Soc.* **281**(1996), 1197–1205. url

[16] R. K. Miller, A. N. Michel, *Ordinary differential equations*, Academic Press, New York, 1982. MR660250

[17] W. Mydlarczyk, A singular initial-value problem for second and third order differential equations, *Colloq. Math.* **68**(1995), 249–257. MR1321047

[18] O. U. Richardson, *The emission of electricity from hot bodies*, Longman's Green and Company, London, 1921.

[19] P. Rosenau, A note on integration of the Emden–Fowler equation, *Internat. J. Non-Linear Mech.* **19**(1984), 303–308. MR755628; url

[20] Z. F. Seidov, R. K. Kyzakhmetov, Solutions of the Lane–Emden problem in series, *Soviet Astronomy* **21**(1977), 399–400.

[21] N. T. Shawagfeh, Nonperturbative approximate solution for Lane–Emden equations, *J. Math. Phys.* **34**(1993), 4364–4369. MR1233277; url

[22] A. M. Wazwaz, A new algorithm for solving differential equations of Lane–Emden type, *Appl. Math. Comput.* **118**(2001), 287–310. MR1812960; url