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Generalized reciprocity principle for discrete symplectic systems

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Abstract. This paper studies transformations for conjoined bases of symplectic difference systems $Y_{i+1} = \mathcal{S}_i Y_i$ with the symplectic coefficient matrices $\mathcal{S}_i$. For an arbitrary symplectic transformation matrix P_i we formulate most general sufficient conditions for $\mathcal{S}_i$, P_i which guarantee that P_i preserves oscillatory properties of conjoined bases Y_i. We present examples which show that our new results extend the applicability of the discrete transformation theory.

Keywords: symplectic difference systems, focal points, symplectic transformations, reciprocity principle, comparative index.

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1 Introduction

In this paper we investigate transformations of the symplectic difference systems [2]

$$Y_{i+1} = \mathcal{S}_i\, Y_i, \qquad \mathcal{S}_i = \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}, \qquad Y_i = \begin{bmatrix} X_i \\ U_i \end{bmatrix}, \qquad i = M, M+1, \dots, \qquad M \in \mathbb{Z}, \tag{1.1}$$

where $\mathcal{S}_i$, Y_i are real partitioned matrices with $n \times n$ blocks $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{D}_i$, X_i, U_i. The matrix $\mathcal{S}_i$ is assumed to be symplectic, i.e.

$$\mathcal{S}_i^T J \mathcal{S}_i = J, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

and I, 0 are the identity and zero matrices.

Together with system (1.1) we consider the transformed system

$$\tilde{Y}_{i+1} = \tilde{\mathcal{S}}_i \tilde{Y}_i, \qquad \tilde{Y}_i = P_i Y_i, \qquad \tilde{\mathcal{S}}_i = P_{i+1} \mathcal{S}_i P_i^{-1}, \tag{1.2}$$

where P_i is an arbitrary symplectic transformation matrix, i.e.

$$P_i^T J P_i = J, \qquad i = M, M+1, \dots$$

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For the special case $P_i = J$, (1.2) takes the form of the so called *reciprocal* system [3]

$$JY_{i+1} = \tilde{\mathcal{S}}_i J Y_i, \qquad \tilde{\mathcal{S}}_i = J\mathcal{S}_i J^T = \begin{bmatrix} \mathcal{D}_i & -\mathcal{C}_i \\ -\mathcal{B}_i & \mathcal{A}_i \end{bmatrix}. \tag{1.3}$$

The main aim of the paper is to formulate the most general sufficient conditions for P_i and $\mathcal{S}_i$ such that systems (1.1), (1.2) have the same oscillatory properties.

Recall now some results from the continuous case which we are going to extend to (1.1). Consider the continuous counterpart of (1.1) – the differential Hamiltonian system

$$x' = A(t)x + B(t)u, \qquad u' = -C(t)x - A^T(t)u, \; B(t) = B^T(t), \qquad C(t) = C^T(t). \tag{1.4}$$

Let $P(t)$ be a $2n \times 2n$ continuously differentiable matrix and suppose that the matrix $P(t)$ is symplectic, i.e. $P^T(t)JP(t) = J$. Then the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = P(t) \begin{pmatrix} x \\ u \end{pmatrix} \tag{1.5}$$

transforms (1.4) into another Hamiltonian system

$$y' = \bar{A}(t)y + \bar{B}(t)z, \qquad z' = -\bar{C}(t)y - \bar{A}^T(t)z, \tag{1.6}$$

where the matrices $\bar{A}(t), \bar{B}(t), \bar{C}(t)$ may be expressed via $A(t), B(t), C(t)$ and blocks of $P(t)$ (see [1]). The natural problem is to look for invariants of the above transformation, in particular, to ask when this transformation preserves oscillatory properties of transformed systems. If $P(t) = J$ in (1.5) and the matrices $B(t), C(t)$ are nonnegative definite it has been shown in [15] that (1.4) is nonoscillatory iff (1.6) is nonoscillatory. This statement is now commonly referred as *reciprocity principle* for Hamiltonian systems. It has been shown that the reciprocity-type statement extends under natural additional assumptions to general transformation (1.5) (see [5]).

Discrete analogs of these results based on the reciprocity principle for the discrete Hamiltonian systems [13]

$$\begin{aligned} \Delta x_i &:= x_{i+1} - x_i = A_i x_{i+1} + B_i u_i, \qquad B_i = B_i^T, \\ \Delta u_i &= \, - C_i x_{i+1} - A_i^T u_i, \qquad C_i = C_i^T, \qquad \det(I - A_i) \neq 0, \end{aligned} \tag{1.7}$$

were presented for the first time in [3, Theorem 3]. Later this principle was generalized for symplectic systems (1.1) in [7, 11, 12].

In this paper we formulate the most general reciprocity-type statements for symplectic system (1.1) (see Theorem 3.3). Previous versions of reciprocity-type statements in [3, 7, 11] are based on the assumptions that some symmetric matrices associated with $\mathcal{S}_i$ and P_i are nonnegative definite. For example, it was proved in [11, Corollary 3.6] that system (1.1) and the reciprocal system (1.3) oscillate and do not oscillate simultaneously under the assumption

$$\mathcal{A}_i \mathcal{B}_i^T \geq 0, \qquad \mathcal{A}_i^T \mathcal{C}_i \leq 0, \tag{1.8}$$

where the inequality $A \geq 0 \, (A \leq 0)$ means that $A = A^T$ is nonnegative (nonpositive) definite. However, conditions of the given type impose serious restrictions on the applicability of the discrete transformation theory. For example, for the Fibonacci sequence $y_{i+2} = y_{i+1} + y_i$ rewritten in form (1.1), assumption (1.8) does not hold (see Section 4). Condition (1.8) was generalized to the case $\operatorname{ind}(\mathcal{A}_i \mathcal{B}_i^T) = \operatorname{ind}(\mathcal{A}_i^T \mathcal{C}_i)$ (see Theorem 3.2 and formula (3.11) in [12]), where $\operatorname{ind} A$ is the number of negative eigenvalues of $A = A^T$. However, [12, Theorem 3.2] deals with the constant transformation matrix $P_i = P$ in (1.2). The main theorem of this paper covers and explains all these special cases.

The paper is organized as follows. In the next section we recall basic facts concerning oscillatory properties of (1.1) (see [3,14]). We also recall relatively new results of the comparative index theory for (1.1) (see [9–12]) and complete their by new relations between the number of focal points for solutions of (1.1) and (1.2). In Section 3 we prove the main result of the paper (see Theorem 3.3) and its corollaries. In Section 4 we provide several examples illustrating the results of Section 3.

2 The comparative index in the transformation theory

We will use the following notation. For a matrix A, we denote by A^T, A^{-1}, A^{-T}, $A^\dagger$, $\operatorname{rank} A$, $\operatorname{ind} A$, $A \geq 0$, $A \leq 0$, respectively, its transpose, inverse, transpose and inverse, Moore–Penrose pseudoinverse, rank (i.e., the dimension of its image), index (i.e., the number of its negative eigenvalues), positive semidefiniteness, negative semidefiniteness. We also use the notation ΔA_k for the forward difference operator $A_{k+1} - A_k$ and the notation $A_i|_M^N$ for the difference $A_N - A_M$. By I and 0 we denote the identity and zero matrices of appropriate dimensions.

Oscillatory properties of discrete symplectic systems are defined using the concept of focal points of conjoined bases of (1.1). A $2n \times n$ matrix solution $Y = \binom{X}{U}$ of (1.1) is said to be a *conjoined basis* of this system if

$$X_i^T U_i = U_i^T X_i \quad \text{and} \quad \operatorname{rank} \begin{pmatrix} X_i \\ U_i \end{pmatrix} = n. \tag{2.1}$$

Note that if (2.1) holds for a fixed $i = i_0$, then it holds for any $i \in \mathbb{Z}$. The concept of the *multiplicity* of a *focal point* of a conjoined basis was introduced by W. Kratz [14] as follows. Given a conjoined basis, introduce the matrices

$$M_i = (I - X_{i+1}X_{i+1}^\dagger)\mathcal{B}_k, \qquad T_i = I - M_i^\dagger M_i, \qquad P_i = T_i^T X_i X_{i+1}^\dagger \mathcal{B}_i T_i.$$

Then obviously $M_i T_i = 0$ and it can be shown (see [14]) that the matrix P_i is symmetric. The *multiplicity* of a *forward focal point* of a conjoined basis $Y = \binom{X}{U}$ in the interval $(i, i+1]$ is defined as the number

$$m(i) := \operatorname{rank} M_i + \operatorname{ind} P_i.$$

The number of focal points $q(i)$ of a conjoined basis of (1.2) can be defined similarly. Recall (see [3]) that the conjoined basis $Y_i^{(M)}$ of (1.1) given by the initial conditions $X_M^{(M)} = 0$, $U_M^{(M)} = I$ is said to be the *principal solution* of (1.1) at M.

System (1.1) is said to be *nonoscillatory* (see [3]), if there exists $M \in \mathbb{N}$ such that the principal solution at M of (1.1) has no focal points in the discrete interval (M, ∞), i.e., $m(i) = 0$ for $i \in (M, \infty)$. In the opposite case (1.1) is said to be *oscillatory*.

Define the numbers of focal points in $(M, N+1]$

$$l(Y_i, M, N) = \sum_{i=M}^{N} m(i), \qquad l(\tilde{Y}_i, M, N) = \sum_{i=M}^{N} q(i) \tag{2.2}$$

for conjoined bases Y_i and $\tilde{Y}_i = P_i Y_i$.

Another important notion we use is the concept of the *comparative index* as introduced and treated in [9–12]. We define the comparative index for $2n \times n$ matrices $Y = \binom{X}{U}$, $\hat{Y} = \binom{\hat{X}}{\hat{U}}$ with condition (2.1) using the notation

$$\begin{cases} \mathcal{M} = (I - XX^{\dagger})\hat{X}, \\ \mathcal{T} = I - \mathcal{M}^{\dagger}\mathcal{M}, \\ \mathcal{D} = \mathcal{D}^T = \mathcal{T} w^T(Y, \hat{Y}) X^{\dagger} \hat{X} \mathcal{T}, \end{cases}$$

where $w(Y, \hat{Y})$ is the Wronskian given by

$$w(Y, \hat{Y}) = Y^T J \hat{Y}. \tag{2.3}$$

The comparative index is defined by

$$\mu(Y, \hat{Y}) = \mu_1(Y, \hat{Y}) + \mu_2(Y, \hat{Y}), \qquad \mu_1(Y, \hat{Y}) = \operatorname{rank} \mathcal{M}, \qquad \mu_2(Y, \hat{Y}) = \operatorname{ind} \mathcal{D}.$$

The *dual comparative index* is introduced as $\mu^*(Y, \hat{Y}) = \mu_1(Y, \hat{Y}) + \mu_2^*(Y, \hat{Y})$, where $\mu_2^*(Y, \hat{Y}) = \operatorname{ind}(-\mathcal{D})$. For the comparative indices $\mu(Y, \hat{Y})$, $\mu^*(Y, \hat{Y})$ we have the estimates (see Property 7 in [10, p. 449]):

$$\mu(Y, \hat{Y}) \le \operatorname{rank} w(Y, \hat{Y}) \le n, \qquad \mu^*(Y, \hat{Y}) \le \operatorname{rank} w(Y, \hat{Y}) \le n. \tag{2.4}$$

For the special case $Y := Y_{k+1}$, $\hat{Y} := \mathcal{S}_k [0\, I]^T$ the numbers μ_1 and μ_2 are actually equal to the quantities $\operatorname{rank} M_k$ and $\operatorname{ind} P_k$ from the definition of the multiplicity of a forward focal point (see [10, Lemma 3.1]). Based on this connection and properties of the comparative index [10] we prove the following result of the transformation theory of (1.1).

Lemma 2.1. *Let Y_i, $\tilde{Y}_i = P_i Y_i$ be conjoined bases of (1.1) and (1.2), then*

$$q(i) - m(i) - \Delta\mu(\tilde{Y}_i, P_i [0\, I]^T) = u_i, \tag{2.5}$$

$$\begin{aligned} u_i &= \mu(P_{i+1}[0\, I]^T, \tilde{\mathcal{S}}_i [0\, I]^T) - \mu^*(P_i^{-1}[0\, I]^T, \mathcal{S}_i^{-1}[0\, I]^T) \\ &= \mu^*(\mathcal{S}_i^{-1}[0\, I]^T, P_i^{-1}[0\, I]^T) - \mu(\tilde{\mathcal{S}}_i [0\, I]^T, P_{i+1}[0\, I]^T), \end{aligned} \tag{2.6}$$

where $m(i)$ and $q(i)$ are the numbers of focal points in $(i, i+1]$ for Y_i and $\tilde{Y}_i = P_i Y_i$, respectively.

Proof. The proof of the first representation of the sequence u_i in (2.6) is given in [11, Lemma 3.1]. Consider the proof of the second one. By Property 5 in [10, p. 448]

$$\mu(P_{i+1}[0\, I]^T, \tilde{\mathcal{S}}_i[0\, I]^T) + \mu(\tilde{\mathcal{S}}_i[0\, I]^T, P_{i+1}[0\, I]^T) = \operatorname{rank} w(P_{i+1}[0\, I]^T, \tilde{\mathcal{S}}_i[0\, I]^T),$$

analogously

$$\mu^*(P_i^{-1}[0\, I]^T, \mathcal{S}_i^{-1}[0\, I]^T) + \mu^*(\mathcal{S}_i^{-1}[0\, I]^T, P_i^{-1}[0\, I]^T) = \operatorname{rank} w(P_i^{-1}[0\, I]^T, \mathcal{S}_i^{-1}[0\, I]^T),$$

where the Wronskians are evaluated according to (2.3). It is easy to verify that

$$\begin{aligned} w(P_{i+1}[0\, I]^T, \tilde{\mathcal{S}}_i[0\, I]^T) &= [0\, I] P_{i+1}^T J \tilde{\mathcal{S}}_i [0\, I]^T = [0\, I] J P_{i+1}^{-1} \tilde{\mathcal{S}}_i [0\, I]^T \\ &= [0\, I] J \mathcal{S}_i P_i^{-1} [0\, I]^T = [0\, I] \mathcal{S}_i^{-T} J P_i^{-1} [0\, I]^T = -w^T(P_i^{-1}[0\, I]^T, \mathcal{S}_i^{-1}[0\, I]^T)). \end{aligned}$$

So we have $\operatorname{rank} w(P_{i+1}[0\,I]^T, \tilde{\mathcal{S}}_i[0\,I]^T) = \operatorname{rank} w(P_i^{-1}[0\,I]^T, \mathcal{S}_i^{-1}[0\,I]^T))$ and then the second representation of u_i in (2.6) follows from the identity

$$\begin{aligned}&\mu(P_{i+1}[0\,I]^T, \tilde{\mathcal{S}}_i[0\,I]^T) + \mu(\tilde{\mathcal{S}}_i[0\,I]^T, P_{i+1}[0\,I]^T)\\ &\qquad = \mu^*(P_i^{-1}[0\,I]^T, \mathcal{S}_i^{-1}[0\,I]^T) + \mu^*(\mathcal{S}_i^{-1}[0\,I]^T, P_i^{-1}[0\,I]^T).\end{aligned}$$

The proof is completed. □

Note that we can interchange the roles of systems (1.1) and (1.2) in Lemma 2.1. In this case we have to replace P_i, Y_i, $\mathcal{S}_i$ by $P_i^{-1}, \tilde{Y}_i, \tilde{\mathcal{S}}_i$, respectively. This approach makes it possible to derive new formulas presenting the difference $q(i) - m(i)$.

Lemma 2.2. *Under the notation of Lemma 2.1 we have*

$$q(i) - m(i) + \Delta\mu(Y_i, P_i^{-1}[0\,I]^T) = \tilde{u}_i, \tag{2.7}$$

$$\begin{aligned}\tilde{u}_i &= \mu^*(P_i[0\,I]^T, \tilde{\mathcal{S}}_i^{-1}[0\,I]^T) - \mu(P_{i+1}^{-1}[0\,I]^T, \mathcal{S}_i[0\,I]^T)\\ &= \mu(\mathcal{S}_i[0\,I]^T, P_{i+1}^{-1}[0\,I]^T) - \mu^*(\tilde{\mathcal{S}}_i^{-1}[0\,I]^T, P_i[0\,I]^T),\end{aligned} \tag{2.8}$$

where

$$\tilde{u}_i - u_i = \Delta \operatorname{rank}([I\,0]P_i[0\,I]^T) \tag{2.9}$$

for u_i given by (2.6).

Proof. As it was mentioned above, we derive (2.7), (2.8) just replacing the roles of (1.1) and (1.2) in (2.5). Then, by (2.7), (2.5) $\tilde{u}_i - u_i = \Delta\{\mu(Y_i, P_i^{-1}[0\,I]^T) + \mu(\tilde{Y}_i, P_i[0\,I]^T)\}$. By Property 9 in [10, p. 449] we have

$$\begin{aligned}\mu(\tilde{Y}_i, P_i[0\,I]^T) - \mu([0\,I]^T, P_i[0\,I]^T) &= \mu(P_i^{-1}[0\,I]^T, P_i^{-1}[0\,I]^T) - \mu(Y_i, P_i^{-1}[0\,I]^T)\\ &= -\mu(Y_i, P_i^{-1}[0\,I]^T),\end{aligned}$$

then $\mu(Y_i, P_i^{-1}[0\,I]^T) + \mu(\tilde{Y}_i, P_i[0\,I]^T) = \mu([0\,I], P_i[0\,I]^T) = \operatorname{rank}([I\,0]P_i[0\,I]^T)$ and the proof of (2.9) is completed. □

For the most important special case $P_i = J$ we have the following corollary to Lemmas 2.1, 2.2.

Corollary 2.3. *For the case $P_i = J$ the sequences u_i, $\tilde{u}_i$ in Lemmas 2.1, 2.2 are defined by the formulas*

$$u_i = \operatorname{ind}(-\mathcal{A}_i^T\mathcal{C}_i) - \operatorname{ind}(\mathcal{A}_i\mathcal{B}_i^T), \tag{2.10}$$

$$\tilde{u}_i = \operatorname{ind}(-\mathcal{C}_i\mathcal{D}_i^T) - \operatorname{ind}(\mathcal{B}_i^T\mathcal{D}_i), \tag{2.11}$$

$$u_i = \tilde{u}_i,$$

where $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{D}_i$ are the blocks of $\mathcal{S}_i$ in (1.1)*. Similarly, for the comparative indexes $\mu(\tilde{Y}_i, P_i[0\,I]^T)$, $\mu(Y_i, P_i^{-1}[0\,I]^T)$ in the left hand sides of* (2.5), (2.7) *for the case $P_i = J$ we have the representations*

$$\mu(JY_i, [I\,0]^T) = \operatorname{rank}(I - U_iU_i^\dagger) + \operatorname{ind}(X_i^TU_i), \tag{2.12}$$

$$\mu(Y_i, [-I\,0]^T) = \operatorname{rank}(I - X_iX_i^\dagger) + \operatorname{ind}(-X_i^TU_i) \tag{2.13}$$

Proof. Formulas (2.10), (2.11), (2.12), (2.13) are verified by direct computations according to the definition of the comparative index. Note that for the special case $P_i = J$ we have $\operatorname{rank}([I\,0]P_i[0\,I]^T) = n$, then $u_i = \tilde{u}_i$ according to (2.9). □

3 Generalized reciprocity principle

Based on Lemmas 2.1, 2.2 we can derive connections between total numbers of focal points (2.2) of conjoined bases of (1.1), (1.2).

Theorem 3.1. *Let Y_i, $\tilde{Y}_i = P_i Y_i$ be conjoined bases of (1.1) and (1.2) then*

$$\begin{aligned} &l(\tilde{Y}, M, N) - l(Y, M, N) - \mu(\tilde{Y}_i, P_i[0\, I]^T)|_M^{N+1} = S(M,N), \\ &S(M,N) = \tilde{S}(M,N) - \operatorname{rank}([I\, 0] P_i [0\, I]^T)|_M^{N+1}, \\ &S(M,N) = \sum_{i=M}^{N} u_i, \ \tilde{S}(M,N) = \sum_{i=M}^{N} \tilde{u}_i, \end{aligned} \tag{3.1}$$

where the sequences $u_i, \tilde{u}_i$ are defined by (2.6), (2.8), respectively and $l(Y_i, M, N)$, $\tilde{l}(\tilde{Y}_i, M, N)$ given by (2.2) are the numbers of focal points for Y_i, $\tilde{Y}_i$ in $(M,\ N+1]$.

Proof. Summing (2.5), (2.7) from $i = M$ to $i = N$ and using (2.9) we derive (3.1). □

Remark 3.2. Note that by (2.4) and (2.9) for the partial sums $S(M,N)$, $\tilde{S}(M,N)$ in (3.1) we have the estimate

$$|S(M,N) - \tilde{S}(M,N)| \le \max(\operatorname{rank}([I\, 0]^T P_{N+1} [0\, I]^T), \operatorname{rank}([I\, 0]^T P_M [0\, I]^T)) \le n. \tag{3.2}$$

It follows from (3.2) that either the partial sums $S(M,N)$ and $\tilde{S}(M,N)$ are simultaneously bounded for a fixed $M \in \mathbb{Z}$ as $N \to \infty$, i.e. the inequalities

$$\begin{aligned} |S(M,N)| &\le C(M), \qquad \forall N \ge M, \\ |\tilde{S}(M,N)| &\le \tilde{C}(M), \qquad \forall N \ge M. \end{aligned} \tag{3.3}$$

hold for some $C(M) > 0$, $\tilde{C}(M) > 0$ or these sums are simultaneously unbounded.

The main result of this paper is the following.

Theorem 3.3 (Generalized reciprocity principle)**.**

(i) If the sequences $S(M,N)$, $\tilde{S}(M,N)$ defined in Theorem 3.1 are bounded as $N \to \infty$, i.e. there exists $M \in \mathbb{Z}$ such that (3.3) hold, then systems (1.1) and (1.2) oscillate or do not oscillate simultaneously.

(ii) If systems (1.1), (1.2) are nonoscillatory, then the sequences $S(M,N)$, $\tilde{S}(M,N)$ are bounded, i.e. (3.3) hold;

(iii) If the sequences $S(M,N)$, $\tilde{S}(M,N)$ defined in Theorem 3.1 are unbounded, then at least one of systems (1.1), (1.2) is oscillatory.

Proof.

1. Consider the proof of (i). Note first that in the definition of nonoscillation of (1.1) (see Section 2) it is possible to replace the principal solution $Y_i^{(M)}$ by any conjoined basis of (1.1) according to the inequality

$$|l(Y, M, N) - l(Y^{(M)}, M, N)| \le n \tag{3.4}$$

proved in [8]. Our purpose is to show that under assumption (3.3) the similar inequality holds for the numbers of focal points $l(Y, M, N)$, $l(\tilde{Y}, M, N)$ of conjoined bases of (1.1), (1.2). Indeed, by (3.3), (3.1), and (2.4) we have

$$\begin{aligned}-C(M) &\le l(\tilde{Y}, M, N) - l(Y, M, N) - \mu(\tilde{Y}_{N+1}, P_{N+1}) + \mu(\tilde{Y}_M, P_M) \\ &= S(M, N) \le C(M), \\ -C(M) - n \le -C(M) - \mu(\tilde{Y}_M, P_M) &\le l(\tilde{Y}, M, N) - l(Y, M, N) \\ &\le C(M) + \mu(\tilde{Y}_{N+1}, P_{N+1}) \le C(M) + n,\end{aligned}$$

then,

$$|l(\tilde{Y}, M, N) - l(Y, M, N)| \le C(M) + n, \qquad \forall\, N \ge M. \tag{3.5}$$

So we have proved that (3.3) implies (3.5). Since $l(Y, M, N)$, $l(\tilde{Y}, M, N)$ are the partial sums of the series with natural or zero members, then, by (3.5), $l(Y, M_1, N) = 0$ for for some M_1 and for all $N \ge M_1$ iff $l(\tilde{Y}, M_2, N) = 0$ for some M_2 and for all $N > M_2$. So we see that (1.1) is nonoscillatory if and only if so is (1.2).

2. To prove (ii) we assume that (1.1), (1.2) are simultaneously nonoscillatory. Then there exists sufficiently large M such that $l(Y, M, N) = l(\tilde{Y}, M, N) = 0$. Then, according to (3.1), (2.4) we see that $|S(M, N)| \le n$ and by Remark 3.2 the sequence $\tilde{S}(M, N)$ is bounded as well.

3. It is easy to see that assertion (ii) is equivalent to assertion (iii).

The proof is completed. □

Note that Theorem 3.3 answers the question about the oscillation (nonoscillation) of one system ((1.1) or (1.2)) provided we posses information on oscillation (nonoscillation) of other one in all situations except the case when $S(M, N)$, $\tilde{S}(M, N)$ are unbounded (see Theorem 3.3 (iii)) and one of the systems ((1.1) or (1.2)) is oscillatory. This case demands additional information. For example, we can offer the following criterion.

Corollary 3.4. *Assume that*

$$\lim_{N\to\infty} S(M, N) = \infty\ (-\infty) \tag{3.6}$$

and system (1.1) ((1.2)) is oscillatory. Then system (1.2) ((1.1)) is oscillatory as well.

Proof. Assume the converse, i.e. that (1.2) is not oscillatory. Then there exists M such that $l(\tilde{Y}, M, N) = 0$ for all $N > M$. Applying (3.1) we have

$$S(M, N) + l(Y, M, N) = -\mu(\tilde{Y}_i, P_i[0\ I]^T)|_M^{N+1},$$

and then, by (2.4) the sum $S(M, N) + l(Y, M, N)$ is bounded as $N \to \infty$. This contradiction proves the first claim. The proof of the second claim (for the case $-\infty$) is similar. Certainly, by Remark 3.2 the sum $S(M, N)$ in (3.6) can be replaced by $\tilde{S}(M, N)$. □

The following theorem formulates the simplest sufficient conditions for the boundedness of $S(M, N)$, $\tilde{S}(M, N)$ in (3.1).

Theorem 3.5. *Systems* (1.1) *and* (1.2) *oscillate and do not oscillate simultaneously if at least one of the sequences u_i, $\tilde{u}_i$ given by* (2.6), (2.8) *tends to zero as $i \to \infty$, i.e. there exists $M > 0$ such that*

$$u_i = 0 \Leftrightarrow \mu(P_{i+1}[0\,I]^T, \tilde{\mathcal{S}}_i[0\,I]^T) = \mu^*(P_i^{-1}[0\,I]^T, \mathcal{S}_i^{-1}[0\,I]^T), \qquad i \geq M, \tag{3.7}$$

or

$$\tilde{u}_i = 0 \Leftrightarrow \mu^*(P_i[0\,I]^T, \tilde{\mathcal{S}}_i^{-1}[0\,I]^T) = \mu(P_{i+1}^{-1}[0\,I]^T, \mathcal{S}_i[0\,I]^T), \qquad i \geq M. \tag{3.8}$$

In particular, for $P_i = J$ we have the corollary to Theorem 3.5.

Corollary 3.6. *Systems* (1.1) *and* (1.3) *oscillate and do not oscillate simultaneously if there exists $M > 0$ such that*

$$\operatorname{ind}(-\mathcal{A}_i^T\mathcal{C}_i) = \operatorname{ind}(\mathcal{A}_i\mathcal{B}_i^T), \qquad i \geq M, \tag{3.9}$$

and (3.9) *is equivalent to*

$$\operatorname{ind}(-\mathcal{C}_i\mathcal{D}_i^T) = \operatorname{ind}(\mathcal{B}_i^T\mathcal{D}_i), \qquad i \geq M. \tag{3.10}$$

Remark 3.7.

(i) Note that for the case $\operatorname{rank}([I\,0]P_i[0\,I]^T) = \text{const}$, $i \geq M$ conditions (3.7) and (3.8) are equivalent according to (2.9). In particular, $\operatorname{rank}([I\,0]P_i[0\,I]^T) = n$ for the case $P_i = J$ (see Corollary 3.6).

(ii) Conditions (3.7), (3.8) will be satisfied if we assume

$$\mu(P_{i+1}[0\,I]^T, \tilde{\mathcal{S}}_i[0\,I]^T) = \mu^*(P_i^{-1}[0\,I]^T, \mathcal{S}_i^{-1}[0\,I]^T) = 0, \qquad i \geq M, \tag{3.11}$$

or

$$\mu^*(P_i[0\,I]^T, \tilde{\mathcal{S}}_i^{-1}[0\,I]^T) = \mu(P_{i+1}^{-1}[0\,I]^T, \mathcal{S}_i[0\,I]^T) = 0, \qquad i \geq M. \tag{3.12}$$

In particular, for the case $P = J$ from (3.11) we derive conditions (1.8) while (3.12) implies

$$\mathcal{C}_i\mathcal{D}_i^T \leq 0, \qquad \mathcal{B}_i^T\mathcal{D}_i \geq 0. \tag{3.13}$$

In the next section we give examples illustrating the applicability of Theorems 3.3, 3.5.

4 Applications

The following example illustrates Theorem 3.3 (iii). According to Theorem 3.3 (iii), if (1.1) does not oscillate and the sum $S(M,N)$ is unbounded, then (1.2) is necessary oscillatory.

Example 4.1. Consider system (1.1) with the matrix $\mathcal{S}_i = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]$. It is easy to verify that for any conjoined basis the number of focal points $m(i) = 0$, i.e. this system is nonoscillatory. For the transformation matrix $P_i = J$ the assumptions of Theorem 3.3 (iii) are satisfied by virtue of $u_i = \operatorname{ind}(-\mathcal{A}_i^T\mathcal{C}_i) = 1$, $S(M,N) = \sum_{i=M}^{N}(1) = N - M + 1$, i.e. $S(M,N)$ is unbounded. The transformed system (1.3) with the matrix $\tilde{\mathcal{S}}_i = J^T\mathcal{S}_iJ = \left[\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix}\right]$ is oscillatory. Indeed, the conjoined basis $Y_i = [1\,0]^T$ of this system has $l(Y,M,N) = \sum_{i=M}^{N}\operatorname{ind}(-3) = N - M + 1$ focal points in $(M, N+1]$.

The following example presents the situation when conditions (1.8) do not hold, but (3.9) is true.

Example 4.2. Consider the second order difference equation

$$\Delta((-1)^i \Delta y_i) + (-1)^i y_{i+1} = 0$$

associated with Fibonacci sequence $y_{i+2} = y_{i+1} + y_i$. If we introduce $Y_i = [y_i \quad (-1)^i \Delta y_i]^T$, then symplectic system (1.1) has the matrix

$$\mathcal{S}_i = \begin{bmatrix} 1 & (-1)^i \\ (-1)^{i+1} & 0 \end{bmatrix}.$$

Since for the principal solution at 0 we have $y_0 = 0$, $y_1 = 1$, $y_{i+2} = y_i + y_{i+1} > 0$, then the number of focal points of this solution is defined as $m(i) = m_2(i) = \operatorname{ind}(-1)^i$, $i \geq 1$, i.e. system (1.1) is oscillatory. Note that for the given system condition (1.8) does not hold, but (3.9) is true for all i, then the transformed system (1.3) is also oscillatory. Point out that for the given example conditions (3.13) are trivially satisfied by $D_i = 0$.

Example 4.3. This example illustrates the situation when condition (3.9) does not hold, but (3.3) is true. Consider system (1.1) with the matrix

$$\begin{aligned} \mathcal{S}_i &= \begin{bmatrix} 1 & 0 \\ -(-2)^{i+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (-2)^i & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + (-2)^i & 1 \\ (-2)^i (3 - (-2)^{i+1}) & 1 - (-2)^{i+1} \end{bmatrix}. \end{aligned} \tag{4.1}$$

This system is nonoscillatory because it is derived using the low triangular transformation matrix

$$\begin{bmatrix} 1 & 0 \\ -(-2)^i & 1 \end{bmatrix}$$

applied to conjoined bases Y_i of the nonoscillatory symplectic system $Y_{i+1} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] Y_i$. Indeed, the number of focal points of the conjoined basis $Y_i = [1\,0]$ of the last system equals $m(i) = \operatorname{ind}(1) = 0$ and low triangular transformation matrices do not change the number of focal points (see [6, Corollary 2.2]). For the matrix $\mathcal{S}_i$ given by (4.1) we have

$$\operatorname{ind}(\mathcal{B}_i \mathcal{A}_i^T) = \operatorname{ind}(1 + (-2)^i) = \begin{cases} 0, & i = 2k; \\ 1, & i = 2k+1 \end{cases}$$

and

$$\operatorname{ind}(-\mathcal{A}_i^T \mathcal{C}_i) = \operatorname{ind}((1 + (-2)^i)(-2)^i(-3 + (-2)^{i+1})) = \begin{cases} 1, & i = 2k; \\ 0, & i = 2k+1. \end{cases}$$

We see that condition (3.9) is not satisfied, but the partial sum $S(M, N) = \sum_{i=M}^{N} (-1)^i$ is bounded, then reciprocal system (1.3) associated with (1.1) given by (4.1) is nonoscillatory by Theorem 3.3 (i).

The last example is devoted to the so-called trigonometric difference systems [4] and illustrates recent results of the transformation theory in [6, Lemma 3.2].

Example 4.4. Consider the trigonometric difference system (1.1) for $M = 0$ with the orthogonal matrix

$$\mathcal{S}_i = \begin{bmatrix} \cos(\varphi_i^1) & 0 & \sin(\varphi_i^1) & 0 \\ 0 & \cos(\varphi_i^2) & 0 & \sin(\varphi_i^2) \\ -\sin(\varphi_i^1) & 0 & \cos(\varphi_i^1) & 0 \\ 0 & -\sin(\varphi_i^2) & 0 & \cos(\varphi_i^2) \end{bmatrix}, \qquad \varphi_i^1 = \frac{\pi i}{2}, \qquad \varphi_i^2 = \frac{\pi(i+1)}{2}. \tag{4.2}$$

The principal solution at 0 for this system has the upper blocks X_i given by

$$X_i = \begin{bmatrix} \sin\left(\frac{i(i-1)\pi}{4}\right) & 0 \\ 0 & \sin\left(\frac{i(i+1)\pi}{4}\right) \end{bmatrix},$$

then we can calculate the numbers of focal points of (1.1) which form the periodic sequence with the minimal period 4: $m(0) = m(1) = 0$, $m(2) = m(3) = 1$, $m(i) = m(i+4)$, $i \geq 0$. So we see that system (1.1) is oscillatory. Note that the block $\mathcal{B}_i$ in (1.1) associated with (4.2) is singular for all i and rank $B_i = 1$. Introduce the following orthogonal transformation matrix

$$P_i = \begin{bmatrix} \cos(\alpha_i)I & -\sin(\alpha_i)I \\ \sin(\alpha_i)I & \cos(\alpha_i)I \end{bmatrix}, \qquad \alpha_i = \frac{\pi}{4(i+1)}. \tag{4.3}$$

The matrix of the transformed system (1.2) takes the form (4.2) where the angles $\varphi_i^{1,2}$ have to be replaced by $\tilde{\varphi}_i^{1,2} = \varphi_i^{1,2} - \Delta\alpha_i$. The upper blocks of the transformed principal solution are

$$\tilde{X}_i = \begin{bmatrix} \sin\left(\frac{i(i-1)\pi}{4} - \alpha_i\right) & 0 \\ 0 & \sin\left(\frac{i(i+1)\pi}{4} - \alpha_i\right) \end{bmatrix},$$

then the transformation with (4.3) regularizes the system (1.1) in the following sense: transformed system (1.2) has the nonsingular block $\tilde{B}_i$ and, additionally, the transformed principal solution has the nonsingular upper block $\tilde{X}_i$. Moreover, the transformation with (4.3) preserves the oscillation properties of (1.1), i.e. system (1.2) is also oscillatory. Indeed, applying (2.5) we have

$$u_i = \mu(P_{i+1}[0\ I]^T, \tilde{\mathcal{S}}_i[0\ I]^T) - \mu^*(P_i^T[0\ I]^T, \mathcal{S}_i^T[0\ I]^T),$$

where μ is the comparative index and μ^* is the dual comparative index. As can be verified by a direct computation

$$\mu(P_{i+1}[0\ I]^T, \tilde{\mathcal{S}}_i[0\ I]^T) = \operatorname{ind}(\operatorname{diag}(\theta_i^1, \theta_i^2)), \qquad \theta_i^j = \frac{\sin(\varphi_i^j + \alpha_i)\sin(\varphi_i^j - \Delta\alpha_i)}{\sin(\alpha_{i+1})}, \tag{4.4}$$

$$\mu^*(P_i^T[0\ I]^T, \mathcal{S}_i^T[0\ I]^T) = \operatorname{ind}(\operatorname{diag}(\vartheta_i^1, \vartheta_i^2)), \qquad \vartheta_i^j = \frac{\sin(\varphi_i^j + \alpha_i)\sin(\varphi_i^j)}{\sin(\alpha_i)}. \tag{4.5}$$

Then

$$u_i = \sum_{j=1}^{2}(\operatorname{ind}(\theta_i^j) - \operatorname{ind}(\vartheta_i^j)).$$

Using (4.4), (4.5) it is possible to show that $u_i = 0$, $i \geq 0$. Assume first that for the fixed $j = 1, 2$ we have $\varphi_i^j = \pi k$, then $\vartheta_i^j = 0$ while $\theta_i^j > 0$ because of

$$\sin(\alpha_{i+1}) > 0,\ \sin(\varphi_i^j + \alpha_i)\sin(\varphi_i^j - \Delta\alpha_i) = \sin(\alpha_i)\sin(-\Delta\alpha_i) > 0.$$

Then, for the given case $\operatorname{ind}(\theta_i^j) = \operatorname{ind}(\vartheta_i^j) = 0$. For the opposite case $\sin(\varphi_i^j) \neq 0$ we have that the signs of $\sin(\varphi_i^j)$ and $\sin(\varphi_i^j - \Delta\alpha_i)$ are the same because of the definition of the angles in (4.2), (4.3). Then for this case $\operatorname{ind}(\theta_i^j) - \operatorname{ind}(\vartheta_i^j) = 0$. Applying Theorem 3.5 we see that system (1.2) is oscillatory. This fact can be verified by a direct computation. We have that $q(0) = q(1) = 1$, $q(2) = q(3) = 0$, $q(i+4) = q(i)$, $i \geq 0$, where $q(i)$ is the number of focal points of the transformed principal solution $\tilde{Y}_i$.

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