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Hyers–Ulam stability and exponential dichotomy of linear differential periodic systems are equivalent

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Abstract. Let m be a positive integer and q be a positive real number. We prove that the m-dimensional and q-periodic system

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{R}_+, \qquad x(t) \in \mathbb{C}^m \tag{$*$}$$

is Hyers–Ulam stable if and only if the monodromy matrix associated to the family $\{A(t)\}_{t\geq 0}$ possesses a discrete dichotomy, i.e. its spectrum does not intersect the unit circle.

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1 Introduction

The notion of exponential dichotomy comes from a paper published in 1930 by Oscar Perron [25]. Over the years this concept has proven to be very useful in investigating properties of the solutions of ordinary and functional differential equations. In particular, the existence of bounded and periodic solutions of several families of semi-linear systems has been studied using the Green matrix $G(t,s)$ of the system ($*$) and concluding that for any bounded f, the convolution $G * f$ is a bounded solution of the non-homogeneous linear system

$$\dot{x}(t) = A(t)x(t) + f(t). \tag{1.1}$$

In 1940 S. M. Ulam has tackled some open problems (see [30] and [31]), one of those problems concerns the stability of a certain functional equation. The first answer to that problem was provided by D. H. Hyers in 1941, see [15]. Later on, this was coined as the Hyers–Ulam problem and its study became an extensive object for many mathematicians. See for example [1, 3–7, 12, 13, 16–24, 27–29, 32] and the references therein.

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The set of all $m \times m$ matrices having complex entries will be denoted by $\mathbb{C}^{m\times m}$. Denote by I_m the identity matrix in $\mathbb{C}^{m\times m}$. Assume that the map $t \mapsto A(t)\colon \mathbb{R} \mapsto \mathbb{C}^{m\times m}$ is continuous and then the Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t), \quad t \in \mathbb{R}, \quad X(t) \in \mathbb{C}^{m\times m} \\ X(0) = I_m, \end{cases} \tag{1.2}$$

has a unique solution denoted by $\Phi_{\mathcal{A}}(t)$. It is well known that $\Phi_{\mathcal{A}}(t)$ is an invertible matrix and that its inverse is the unique solution of the Cauchy problem

$$\begin{cases} \dot{X}(t) = -X(t)A(t), \quad t \in \mathbb{R} \\ X(0) = I_m. \end{cases}$$

The evolution family $\mathcal{U}_{\mathcal{A}} = \{U_{\mathcal{A}}(t,s) : t,s \in \mathbb{R}\}$, where

$$U_{\mathcal{A}}(t,s) := \Phi_{\mathcal{A}}(t)\Phi_{\mathcal{A}}^{-1}(s),$$

has the following properties:

(i) $U_{\mathcal{A}}(t,t) = I_m$, for all $t \in \mathbb{R}$;

(ii) $U_{\mathcal{A}}(t,s) = U_{\mathcal{A}}(t,r)U_{\mathcal{A}}(r,s)$ for all $t,s,r \in \mathbb{R}$;

(iii) $\frac{\partial}{\partial t}U_{\mathcal{A}}(t,s) = A(t)U_{\mathcal{A}}(t,s)$ for all $t,s \in \mathbb{R}$;

(iv) $\frac{\partial}{\partial s}U_{\mathcal{A}}(t,s) = -U_{\mathcal{A}}(t,s)A(s)$ for all $t,s \in \mathbb{R}$;

(v) the map $(t,s) \mapsto U_{\mathcal{A}}(t,s) : \mathbb{R}^2 \to \mathbb{C}^{m\times m}$ is continuous.

If, in addition, the map $A(\cdot)$ is q- periodic, for some positive number q, then:

(vi) $U_{\mathcal{A}}(t+q,s+q) = U_{\mathcal{A}}(t,s)$ for all $t,s \in \mathbb{R}$;

(vii) there exist $\omega > 0$ and $M_\omega \geq 1$ such that

$$\|U_{\mathcal{A}}(t,s)\| \leq M_\omega e^{\omega(t-s)}, \qquad t \geq s;$$

(viii) $\Phi_{\mathcal{A}}(t+q) = \Phi_{\mathcal{A}}(t) \cdot \Phi_{\mathcal{A}}(q)$ for all $t \in \mathbb{R}$.

To prove the latter statement, we remark that the map $t \mapsto \Phi_{\mathcal{A}}(t+q)(\Phi_{\mathcal{A}}(q))^{-1}$ is a solution of (1.2). Now, by using the uniqueness it must be $\Phi_{\mathcal{A}}(\cdot)$. The matrix $T_q := U_{\mathcal{A}}(q,0)$ is the matrix of monodromy associated with the family $\mathcal{A}$. Having in mind that T_q is invertible there exists a matrix $B \in \mathbb{C}^{m\times m}$ such that $T_q = e^{qB}$. Thus there is a periodic (period q) matrix function $t \mapsto R(t)$ such that $\Phi_{\mathcal{A}}(t) = R(t)e^{tB}$ for all $t \in \mathbb{R}$. This will be used to show that certain family of projections described below is periodic.

The complex unit circle is denoted by $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$. Recall that the matrix A is said to be dichotomic (or that it possesses a discrete dichotomy) if its spectrum does not intersect the unit circle, i.e. $\sigma(A) \cap \Gamma = \emptyset$. An $m \times m$ complex matrix P, verifying $P^2 = P$ is called projection. The circle and closed disk centered in the eigenvalue $\lambda_j \in \sigma(A)$ are respectively denoted by

$$C_r(\lambda_j) = \{z \in \mathbb{C} : |z - \lambda_j| = r\}$$

and

$$\overline{D}_r(\lambda_j) = \{z \in \mathbb{C} : |z - \lambda_j| \le r\}.$$

Here r is any positive real number, small enough such that $\sigma(A) \cap \overline{D}_r(\lambda_j) = \{\lambda_j\}$, for every $1 \le j \le k$. The projection $E_{\lambda_j}(A) := E_j(A) \colon \mathbb{C}^m \to \mathbb{C}^m$, defined by

$$E_j(A) = \frac{1}{2\pi i} \oint_{C_r(\lambda_j)} (zI_m - A)^{-1}\, dz,$$

is called spectral projection associated to the eigenvalue λ_j, [10, Chap. 7]. Obviously, $I_m = E_{\lambda_1}(A) + E_{\lambda_2}(A) + \cdots + E_{\lambda_k}(A)$. The stable spectral projection of A is given by

$$\Pi_-(A) := \frac{1}{2\pi i} \oint_{C_r(0)} (zI_m - A)^{-1}\, dz,$$

where $0 < r < 1$ is large enough such that

$$\{\lambda \in \sigma(A) : |\lambda| < 1\} \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}.$$

Clearly, $\Pi_-(A)$ commutes with any natural power of A.

Coming back to the non-autonomous case let $\Pi_- := \Pi_-(T_q)$ and let

$$\Pi_-(t) := \Phi_{\mathcal{A}}(t)\Pi_-(\Phi_{\mathcal{A}}(t))^{-1} \quad \text{and} \quad \Pi_+(t) := I_m - \Pi_-(t)$$

for $t \in \mathbb{R}$. Next, we list the main properties of this family of projections.

(i) $\Pi_-^2(t) = \Pi_-(t)$ and $\Pi_+^2(t) = \Pi_+(t)$ for all $t \in \mathbb{R}$.

(ii) $\Pi_\pm(t)U(t,s) = U(t,s)\Pi_\pm(s)$ for all $t, s \in \mathbb{R}$, (the signs correspond).

(iii) The maps $t \mapsto \Pi_\pm(t)$ are continuous on $\mathbb{R}$ and q-periodic.

(iv) $\Pi_-(t) + \Pi_+(t) = I_m$ and $\Pi_-(t) \cdot \Pi_+(t) = 0$ for all $t \in \mathbb{R}$.

(v) For each $t, s \in \mathbb{R}$, $U(t,s)$ is an isomorphism from $\ker(\Pi_-(s))$ to $\ker(\Pi_-(t))$.

Proposition 1.1. *The following two statements, concerning an invertible $m \times m$ matrix A, are equivalent.*

(1) *A possesses a discrete dichotomy.*

(2) *There exist four positive constants $N_1 = N_1(A)$, $N_2 = N_2(A)$, $\nu_1 = \nu_1(A)$, $\nu_2 = \nu_2(A)$ such that*

(i) $\|A^n\Pi_-(A)x\| \le N_1 e^{-\nu_1 n}\|\Pi_-(A)x\|$, *for all $x \in \mathbb{C}^m$ and all $n \in \mathbb{Z}_+$.*

(ii) $\|A^n\Pi_+(A)x\| \le N_2 e^{\nu_2 n}\|\Pi_+(A)x\|$, *for all $x \in \mathbb{C}^m$ and all $n \in \mathbb{Z}_- := \{0, -1, -2, \dots\}$.*

The argument is standard and the details are omitted. Mention that the above result can be stated in a more general form with any projection P, commuting with A, instead of $\Pi_-(A)$. Moreover, the assumption of invertibility can be removed. See, for example, Proposition 2.1 from [2]. For further details about the concept of dichotomy see for example [8,26].

Let $t \mapsto f(t)$ be a $\mathbb{C}^m$-valued locally Riemann integrable function on $\mathbb{R}_+$ and let $x \in \mathbb{C}^m$ be a given vector. Consider the Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t), & t \geq 0 \\ x(0) = x. \end{cases} \tag{1.3}$$

The solution of (1.3) is given by

$$\phi_{f,x}(t) = U_{\mathcal{A}}(t,0)x + \int_0^t U_{\mathcal{A}}(t,s)f(s)\,ds.$$

In order to prove Theorem 1.3 below, we need the following proposition, which contains equivalent characterizations for exponential dichotomy.

Proposition 1.2. *The following three statements concerning the matrix family $\mathcal{A}$ are equivalent.*

(1) *T_q is dichotomic.*

(2) *There exist the positive constants N_1', N_2', ν_1', ν_2' such that*

- **(i)** $\|U_{\mathcal{A}}(t,s)\Pi_-(s)\| \leq N_1' e^{-\nu_1'(t-s)}$, *for all $t \geq s \geq 0$, and*
- **(ii)** $\|U_{\mathcal{A}}(t,s)\Pi_+(s)\| \leq N_2' e^{\nu_2'(t-s)}$, *for all $0 \leq t \leq s$.*

(3) *For each locally Riemann integrable and bounded function $f\colon \mathbb{R}_+ \to \mathbb{C}^m$ there exists a unique $x \in \ker(\Pi_-)$, such that $\phi_{f,x}(\cdot)$ is bounded on $\mathbb{R}_+$.*

Proof. **(1)** $\Rightarrow$ **(2)** Let $t \geq s \in \mathbb{R}_+$ and let n and k be the integer parts of $\frac{t}{q}$ and $\frac{s}{q}$ respectively, i.e., $n = [\frac{t}{q}]$ and $k = [\frac{s}{q}]$. Therefore $t = nq + \mu$ and $s = kq + \rho$, with $n, k \in \mathbb{Z}_+$ and $\mu, \rho \in [0, q)$. We analyze the following cases.

Case 1. When $n > k$, then

$$\begin{aligned} U_{\mathcal{A}}(t,s)\Pi_-(s) &= U_{\mathcal{A}}(nq+\mu, nq)U_{\mathcal{A}}(nq, (k+1)q)U_{\mathcal{A}}((k+1)q, kq+\rho)\Pi_-(kq+\rho) \\ &= U_{\mathcal{A}}(\mu, 0)U_{\mathcal{A}}((n-k-1)q, 0)U_{\mathcal{A}}(q, \rho)\Pi_-(\rho) \\ &= U_{\mathcal{A}}(\mu, 0)T_q^{n-k-1}\Pi_- U_{\mathcal{A}}(q, \rho). \end{aligned}$$

In the view of Proposition 1.1 and taking into account that $\Pi_-(0) = \Pi_-(q) = \Pi_-$, we get

$$\begin{aligned} \|U_{\mathcal{A}}(t,s)\Pi_-(s)\| &\leq Me^{\omega q} N_1 e^{-\nu_1(n-k-1)} Me^{\omega q}\|\Pi_-\| \\ &\leq N_1' e^{-\nu_1'(t-s)}, \end{aligned}$$

where $N_1' = N_1 M^2 e^{2\omega q} e^{2\nu_1}\|\Pi_-\|$ and $\nu_1' = \frac{\nu_1}{q}$.

Case 2. When $n = k$, then $\mu \geq \rho$ and

$$\|U_{\mathcal{A}}(s,t)\Pi_-(s)\| = \|U_{\mathcal{A}}(\mu, \rho)\Pi_-(\rho)\|.$$

By using $\sup_{\rho \in [0,q]} \|\Pi_-(\rho)\| \leq c < \infty$ and letting ν be an arbitrary positive number, we may choose $N \in \mathbb{R}_+$ large enough, such that

$$\|U_{\mathcal{A}}(t,s)\Pi_-(s)\| \leq Me^{\omega(\mu-\rho)}\|\Pi_-(\rho)\| \leq cNe^{-\nu(\mu-\rho)} = N_1' e^{-\nu_1'(t-s)}.$$

Similar estimations can be obtained in order to prove **(2) (ii)**. We omit the details.

(2) $\Rightarrow$ **(1)**. Put $s = 0$ and $t = nq$ in **(2) (i), (ii)** and apply Proposition 1.1 with T_q instead of A.

(2) $\Rightarrow$ **(3)**. The map

$$t \mapsto y(t) := \int_0^t U_{\mathcal{A}}(t,s)\Pi_-(s)f(s)\,ds - \int_t^\infty U_{\mathcal{A}}(t,s)\Pi_+(s)f(s)\,ds$$

is a solution of (1.1), [8, Chap. 3]. Indeed, the second integral is well defined because, from **(2) (ii)**, have that

$$\begin{aligned}\int_t^\infty \|U_{\mathcal{A}}(t,s)\Pi_+(s)f(s)\|\,ds &\le \int_t^\infty N_2' e^{\nu_2'(t-s)}\|f\|_\infty\,ds\\ &= \frac{N_2'}{\nu_2'}\|f\|_\infty.\end{aligned}$$

Also from **(2)**, the solution is bounded, and

$$\sup_{t\ge 0}|y(t)| \le \left(\frac{N_1'}{\nu_1'} + \frac{N_2'}{\nu_2'}\right)\sup_{t\ge 0}|f(t)|.$$

Moreover, since $\ker(\Pi_-)$ is a closed subspace, the initial value

$$y(0) = -\int_0^\infty U_{\mathcal{A}}(0,s)\Pi_+(s)f(s)\,ds \in \ker(\Pi_-).$$

Let us suppose that there exist two bounded solutions of the differential equation $\dot{x}(t) = A(t)x(t) + f(t),\ t \ge 0$ having their start in $\ker(\Pi_-)$. Denote them by $y_1(\cdot)$ and $y_2(\cdot)$. Then

$$y_1(t) = U_{\mathcal{A}}(t,0)x_1 + \int_0^t U_{\mathcal{A}}(t,s)f(s)\,ds, \qquad x_1 \in \ker(\Pi_-)$$

and

$$y_2(t) = U_{\mathcal{A}}(t,0)x_2 + \int_0^t U_{\mathcal{A}}(t,s)f(s)\,ds, \qquad x_2 \in \ker(\Pi_-).$$

Their difference is bounded and $y_1(t) - y_2(t) = U_{\mathcal{A}}(t,0)(x_1 - x_2)$. Since the map $y_1(\cdot) - y_2(\cdot)$ is bounded on $\mathbb{R}_+$, and because T_q is dichotomic it follows that $x_1 - x_2 \in \mathrm{Range}(\Pi_-)$. On the other hand, $x_1, x_2 \in \ker(\Pi_-)$ yields $x_1 - x_2 \in \ker(\Pi_-)$ and therefore $x_1 = x_2$.

(3) $\Rightarrow$ **(1)**. Suppose that there exists $\lambda \in \sigma(T)$, with $|\lambda| = 1$. Then, there exists $x_0 \neq 0$ such that $T_q x_0 = \lambda x_0$, and therefore $U_{\mathcal{A}}(nq,0) = \lambda^n x_0$, for all $n \in \mathbb{Z}_+$.

Set

$$f(t) := \begin{cases} U_{\mathcal{A}}(s,0)x_0, & \text{if } s \in [0,q) \\ x_0, & \text{if } s = q, \end{cases}$$

and let us denote also by f its continuation by periodicity on $\mathbb{R}_+$. By assumption there exists a unique $y_0 \in \ker(\Pi_-)$ such that the map

$$t \mapsto \psi(t) := U_{\mathcal{A}}(t,0)y_0 + \int_0^t U_{\mathcal{A}}(t,s)f(s)\,ds$$

is bounded on $\mathbb{R}_+$. Next we analyze two cases.

Case 1. When $\lambda = 1$. The sequence $(\psi(nq))_{n\in\mathbb{Z}_+}$ should be bounded. But,

$$\begin{aligned}
\psi(nq) &:= U_{\mathcal{A}}(nq,0)y_0 + \int_0^{nq} U_{\mathcal{A}}(nq,s)f(s)\,ds \\
&= U_{\mathcal{A}}(nq,0)y_0 + \sum_{k=0}^{n-1}\int_{kq}^{(k+1)q} U_{\mathcal{A}}(nq,s)f(s)\,ds \\
&= U_{\mathcal{A}}(nq,0)y_0 + \sum_{k=0}^{n-1} U_{\mathcal{A}}(nq,(k+1)q)\int_0^{q} U_{\mathcal{A}}(q,r)f(r)\,dr \\
&= U_{\mathcal{A}}(nq,0)y_0 + \sum_{k=0}^{n-1} U_{\mathcal{A}}(nq,kq)x_0 = U_{\mathcal{A}}(nq,0)y_0 + nx_0.
\end{aligned}$$

If $y_0 = 0$, obviously we arrive at a contradiction, since the map $n \mapsto nx_0$ is unbounded, and if $y_0 \neq 0$, let denote $y_0(n) := U_{\mathcal{A}}(nq,0)y_0$. Then, one has

$$\begin{aligned}
\|y_0\| &= \|U_{\mathcal{A}}(0,nq)y_0(n)\| \\
&= \|U_{\mathcal{A}}(0,nq)\Pi_+ y_0(n)\| \leq N_2' e^{-\nu_2' nq}\|U_{\mathcal{A}}(nq,0)y_0\|,
\end{aligned}$$

where the fact that

$$\Pi_+ y_0(n) = U_{\mathcal{A}}(nq,0)\Pi_+ y_0 = U_{\mathcal{A}}(nq,0)(y_0 - \Pi_- y_0) = y_0(n),$$

was used. This yields

$$\|U_{\mathcal{A}}(nq,0)y_0\| \geq \frac{1}{N_2'} e^{\nu_2' nq}\|y_0\|,$$

and a contradiction arises again.

Case 2. When $\lambda = e^{iuq} \neq 1, u \in \mathbb{R}, i^2 = -1$. Then $1 \in \sigma(e^{-iuq}T)$, $T_u(q) := e^{-iuq}T_q$ is the monodromy matrix of the evolution family

$$\{U_{\mathcal{A},u}(t,s) := e^{-iu(t-s)}U_{\mathcal{A}}(t,s) : t,s \in \mathbb{R}\}$$

and, as before, we obtain that the sequence

$$(e^{-iunq}\psi(nq))_{n\in\mathbb{Z}_+} = (U_{\mathcal{A},u}(nq,0)y_0 + qnx_0)_{n\in\mathbb{Z}_+},$$

is unbounded, which is a contradiction. □

In the present paper we assume that the matrix-valued map $t \mapsto A(t)$ is continuous and q-periodic for some positive q. Next we outline the Hyers–Ulam problem for a family of $m\times m$ matrices $\mathcal{A} = \{A(t)\}_{t\geq 0}$, m being a positive integer. Let $\mathbb{R}_+$ be the set of all nonnegative real numbers and let $\rho(\cdot)$ be a $\mathbb{C}^m$-valued function defined on $\mathbb{R}_+$. Consider the systems

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{R}, \qquad x(t) \in \mathbb{C}^m \tag{1.4}$$

and

$$\dot{x}(t) = A(t)x(t) + \rho(t), \qquad t \in \mathbb{R}_+, \qquad x(t) \in \mathbb{C}^m. \tag{1.5}$$

Let ε be a positive real number. A continuous $\mathbb{C}^m$-valued function $y(\cdot)$ defined on $\mathbb{R}_+ := [0,\infty)$ is called ε-approximate solution for (1.4) if it is continuously differentiable on $\mathbb{R}_+ \setminus (q\mathbb{Z}_+)$ and

$$\|\dot{y}(t) - A(t)y(t)\| \leq \varepsilon, \qquad \forall t \in \mathbb{R}_+ \setminus (q\mathbb{Z}_+). \tag{1.6}$$

The family $\mathcal{A}$ is said to be Hyers–Ulam stable if there exists a nonnegative constant L such that, for every ε-approximate solution $\phi(\cdot)$ of (1.4), there exists an exact solution $\theta(\cdot)$ of (1.4) such that

$$\sup_{t\in\mathbb{R}_+} \|\phi(t)-\theta(t)\| \le L\varepsilon. \tag{1.7}$$

The result of this paper reads as follows.

Theorem 1.3. *The family $\mathcal{A}=\{A(t)\}_{t\ge 0}$ is Hyers–Ulam stable if and only if its monodromy matrix T_q possesses a discrete dichotomy.*

2 Hyers–Ulam stability and exponential dichotomy for linear differential systems

We can see an ε-approximate solution of (1.4) as an exact solution of (1.5) corresponding to a forced term $\rho(\cdot)$ which is bounded by ε.

Remark 2.1. Let ε be a given positive number. The following two statements are equivalent:

1. The matrix family $\mathcal{A}$ (or the system (1.4)) is Hyers–Ulam stable.
2. There exists a nonnegative constant L such that for every function $\rho(\cdot)$, continuous on $\mathbb{R}_+\setminus(q\mathbb{Z}_+)$, with $\sup_{t\ge 0}\|\rho(t)\|\le\varepsilon$, and every $x\in\mathbb{C}^m$ there exists $x_0\in\mathbb{C}^m$ and

$$\sup_{t\ge 0}\left\|U_{\mathcal{A}}(t,0)(x-x_0)+\int_0^t U_{\mathcal{A}}(t,s)\rho(s)\,ds\right\|\le L\varepsilon. \tag{2.1}$$

Proof. Let ε be a given positive number. Assume first that the system (1.4) is Hyers–Ulam stable and let L be a positive constant verifying (1.7). Let $\rho(\cdot)$ be as assumed in the second statement and $x\in\mathbb{C}^m$. Obviously, the solution $\phi(\cdot)$ of the Cauchy problem

$$\dot{x}(t)=A(t)x(t)+\rho(t),\qquad x(0)=x$$

is an ε-approximative solution for (1.4). Thus, by assumption, there exists an exact solution $\theta(\cdot)$ of (1.4) such that (1.7) holds true. Let $x_0:=\theta(0)$. Now, in view of (1.6) the inequality in (2.1) holds true as well.

Now assume that the second statement is true and let L be a positive constant verifying (2.1) and $\phi(\cdot)$ be an ε-approximative solution of (1.4). Set $\rho(t):=\dot{\phi}(t)-A(t)\phi(t)$ for $t\in\mathbb{R}_+\setminus(q\mathbb{Z}_+)$ and $\rho(t):=\varepsilon$ in the rest, and let $x:=\phi(0)$. Thus $\|\rho\|_\infty\le\varepsilon$ and, by assumption (2.1) holds true for a certain $x_0\in\mathbb{C}^m$. The required exact solution of (1.4), verifying (1.7), is defined by $\theta(t):=U(t,0)x_0$. □

Proof of Theorem 1.3.
Necessity. Suppose that T_q is not dichotomic. Then, there exist an integer j with $1\le j\le k$ and $\lambda_j=e^{i\mu_j q}\in\sigma(T_q)$, where μ_j is a certain real number. Let $\varepsilon>0$ be fixed and let

$$\rho(t):=\begin{cases}U_{\mathcal{A}}(s,0)u_0, & \text{if } s\in[0,q)\\ u_0, & \text{if } s=q,\end{cases}$$

where $u_0\in\mathbb{C}^m$ and $\|u_0\|\le(M_\omega e^{\omega q})^{-1}\varepsilon$. Let us denote also by ρ the continuation by periodicity of the previous function. Obviously, the function $\rho(\cdot)$ is locally Riemann integrable on

$\mathbb{R}_+$ and bounded by ε. By assumption, the family matrix $\mathcal{A}$ is Hyers–Ulam stable. Hence, the solution

$$\phi(t) = U_{\mathcal{A}}(t,0)(x-x_0) + \int_0^t U_{\mathcal{A}}(t,s)\rho(s)\,ds,$$

of the Cauchy problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + \rho(t), & t \geq 0 \\ y(0) = x - x_0, \end{cases}$$

is bounded by $L\varepsilon$ for certain $x - x_0 \in \mathbb{C}^m$. Then, the sequence

$$n \mapsto E_j(T_q)\phi(nq) = E_j(T_q)\left[U_{\mathcal{A}}(nq,0)(x-x_0) + \int_0^{nq} U_{\mathcal{A}}(nq,s)\rho(s)\,ds\right]$$

should be also bounded by $L\varepsilon$. On the other hand, see for example [2, Lemma 4.5], [9,11,14], there exists an $m \times m$ matrix-valued polynomial $P_j = P_j(T_q)$ (in n) having the degree at most $m_j - 1$, such that

$$E_j(T_q)U_{\mathcal{A}}(nq,0) = e^{i\mu_j qn}P_j(n), \qquad \forall n \in \mathbb{Z}_+.$$

But,

$$\begin{aligned}
&E_j(T_q)\left[U_{\mathcal{A}}(nq,0)(x-x_0) + \int_0^{nq} U_{\mathcal{A}}(nq,s)\rho(s)\,ds\right] \\
&\quad = e^{i\mu_j nq}P_j(n)(x-x_0) + \int_0^{nq} E_j(T_q)U_{\mathcal{A}}(nq,s)\rho(s)\,ds \\
&\quad = e^{i\mu_j nq}P_j(n)(x-x_0) + \sum_{k=0}^{n-1}\int_{kq}^{(k+1)q} E_j(T_q)U_{\mathcal{A}}(nq,s)\rho(s)\,ds \\
&\quad = e^{i\mu_j nq}P_j(n)(x-x_0) + \sum_{k=0}^{n-1}\int_0^{q} E_j(T_q)U_{\mathcal{A}}(nq,(k+1)q)U_{\mathcal{A}}(q,s)\rho(s)\,ds \\
&\quad = e^{i\mu_j nq}P_j(n)(x-x_0) + \sum_{k=0}^{n-1}\lambda_j^{n-k}P_j(n-k)u_0.
\end{aligned}$$

Now, if $\lambda_j = 1$ then by choosing an appropriate $u_0 \neq 0$, we have that

$$\begin{aligned}
\deg[P_j(n)(x-x_0)] \leq \deg[P_j(n)] = \deg[P_j(n)u_0] &< 1 + \deg[P_j(n)] \\
&= \deg[q_j(n)],
\end{aligned}$$

where $q_j(n) := \sum_{k=0}^{n-1} P_j(n-k)u_0$ and the fact that the degree of the polynomial in n, $p(n) = 1^k + 2^k + \cdots + n^k$, is equal to $k+1$ was used. Therefore, the sequence $(P_j(n)(x-x_0) + q_j(n))_{n\in\mathbb{Z}_+}$, is unbounded and a contradiction arises.

When $\lambda_j \neq 1$, then $1 \in \sigma(T_{\mu_j}(q))$ and the map $t \mapsto e^{-i\mu_j t}\phi(t)$ should be bounded on $\mathbb{R}_+$. Then the sequence

$$n \mapsto e^{-i\mu_j nq}\phi(nq), \qquad n \in \mathbb{Z}_+,$$

is bounded as well. On the other hand

$$e^{-i\mu_j nq}E_j(T_{\mu_j}(q))\phi(nq) = E_j(T_{\mu_j}(q))\left[U_{\mathcal{A},\mu_j}(nq,0)(x-x_0) + \int_0^{nq} U_{\mathcal{A},\mu_j}(nq,s)e^{-i\mu_j s}\rho(s)\,ds\right].$$

Again, as above, there exists a matrix valued polynomial $Q_j(n) = Q_j(T_{\mu_j}(q))$ (in n) having the degree at most $m_j - 1$ such that

$$E_j(T_{\mu_j}(q))U_{\mathcal{A},\mu_j}(nq,0) = Q_j(n) \quad \text{for every } n \in \mathbb{Z}_+.$$

Thus after a standard calculation

$$e^{-i\mu_j nq}E_j(T_{\mu_j}(q))\phi(nq) = Q_j(n)(x-x_0) + \sum_{k=0}^{n-1} Q_j(n-k)u_0.$$

For an appropriate $u_0 \in \mathbb{C}^m$, the last expression is a vector valued polynomial of degree at least one and so it is unbounded and a contradiction is provided again.

Sufficiency. The absolute constant L will be settled later. Let $\rho\colon \mathbb{R}_+ \to \mathbb{C}^m$ be a bounded locally Riemann integrable function on $\mathbb{R}_+$, with $\|\rho\|_\infty \le \varepsilon$ and let $x \in \mathbb{C}^m$. By Proposition 1.1, there exists a unique bounded solution $y(\cdot)$ of the equation (1.5) starting from the subspace $\ker(\Pi_-)$. Let denote $u_0 := y(0)$. Then

$$\begin{aligned}
\|y(t)\| &= \left\| U_{\mathcal{A}}(t,0)u_0 + \int_0^t U_{\mathcal{A}}(t,s)\rho(s)\,ds \right\| \\
&= \left\| \int_0^t U_{\mathcal{A}}(t,s)\Pi_-(s)\rho(s)ds - \int_t^\infty U_{\mathcal{A}}(t,s)\Pi_+(s)\rho(s)\,ds \right\| \\
&\le \left(\frac{N_1'}{\nu_1'} + \frac{N_2'}{\nu_2'} \right)\varepsilon.
\end{aligned}$$

The desired assertion follows by choosing $L = \left(\frac{N_1'}{\nu_1'} + \frac{N_2'}{\nu_2'}\right)$ and setting $x_0 = x - u_0$. □

A more general result, described in the following, can be stated. Its proof is very similar to that given before and we omit the details.

Let X be a complex, finite dimensional Banach space and let $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}_+}$ and $\mathcal{P} = \{P(t)\}_{t\in\mathbb{R}_+}$ be two families of linear operators acting on X. Assume the following.

H1. $A(t+q) = A(t)$ and $P(t+q) = P(t)$, for all $t \in \mathbb{R}_+$ and some positive q.

H2. $P(t)^2 = P(t)$, for all $t \in \mathbb{R}_+$, i.e., $\mathcal{P}$ is a family of projections.

H3. $U_{\mathcal{A}}(t,s)P(s) = P(t)U_{\mathcal{A}}(t,s)$, for any $t \ge s \in \mathbb{R}_+$. In particular, this yields that $U_{\mathcal{A}}(t,s)x \in \ker(P(t))$ for each $x \in \ker(P(s))$.

H4. For each $t \ge s \in \mathbb{R}_+$, the map

$$x \mapsto U_{\mathcal{A}}(t,s)x : \ker(P(s)) \to \ker(P(t))$$

is invertible. Denote by $U_{\mathcal{A}|}(s,t)$ its inverse.

We say that the family $\mathcal{A}$ is $\mathcal{P}$-dichotomic if there exist four positive constants N_1, N_2, ν_1 and ν_2 such that

(i) $\|U_{\mathcal{A}}(t,s)P(s)\| \le N_1 e^{-\nu_1(t-s)}$ for all $t \ge s \ge 0$;

(ii) $\|U_{\mathcal{A}|}(t,s)(I-P(s))\| \le N_2 e^{\nu_2(t-s)}$ for all $0 \le t < s$.

Proposition 2.2. *Assume that the families $\mathcal{A}$ and $\mathcal{P}$ satisfy* **H1–H4** *above. Thus the following three statements are equivalent.*

(1) *T_q possesses a discrete dichotomy.*

(2) *The family $\mathcal{A}$ is $\mathcal{P}$-dichotomic.*

(3) *The family $\mathcal{A}$ is Hyers–Ulam stable.*

We conclude this note with the one-dimensional version of our result.

Corollary 2.3. *Let $t \mapsto a(t) \colon \mathbb{R}_+ \to \mathbb{C}$ be a given continuous and q-periodic function (for some positive q). The scalar differential equation*

$$\dot{x}(t) = a(t)x(t), \qquad t \in \mathbb{R}_+, \qquad x(t) \in \mathbb{C} \tag{2.2}$$

is Hyers–Ulam stable if and only if

$$\int_0^q \Re[a(r)]\, dr \neq 0.$$

Proof. Indeed, we have

$$T_q = e^{\int_0^q a(r)\, dr}, \qquad \sigma(T_q) = \{T_q\} \quad \text{and} \quad |T_q| = e^{\int_0^q \Re[a(r)]\, dr}.$$

From Theorem 1.3 follows that (2.2) is Hyers–Ulam stable if and only if $|T_q| \neq 1$ or equivalently if and only if $\int_0^q \Re[a(r)]\, dr \neq 0$. □

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