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Existence of solutions for fourth order three-point boundary value problems on a half-line

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Abstract. In this paper, we apply Schauder's fixed point theorem, the upper and lower solution method, and topological degree theory to establish the existence of unbounded solutions for the following fourth order three-point boundary value problem on a half-line

$$\begin{aligned} &x''''(t) + q(t)f(t, x(t), x'(t), x''(t), x'''(t)) = 0, \qquad t \in (0, +\infty), \\ &x''(0) = A, \qquad x(\eta) = B_1, \qquad x'(\eta) = B_2, \qquad x'''(+\infty) = C, \end{aligned}$$

where $\eta \in (0, +\infty)$, but fixed, and $f\colon [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ satisfies Nagumo's condition. We present easily verifiable sufficient conditions for the existence of at least one solution, and at least three solutions of this problem. We also give two examples to illustrate the importance of our results.

Keywords: three-point boundary value problem, lower and upper solutions, half-line, Schauder's fixed point theorem, topological degree theory.

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1 Introduction

In this paper, we develop an existence theory for fourth order ordinary differential equations together with boundary conditions on a half-line

$$\begin{aligned} &x''''(t) + q(t)f(t, x(t), x'(t), x''(t), x'''(t)) = 0, \qquad t \in (0, +\infty), \\ &x''(0) = A, \qquad x(\eta) = B_1, \qquad x'(\eta) = B_2, \qquad \lim_{t\to+\infty} x'''(t) = x'''(+\infty) = C, \end{aligned} \tag{1.1}$$

where $\eta \in (0, +\infty)$, but fixed, $q\colon (0, +\infty) \to (0, +\infty)$, $f\colon [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ are continuous, and $A, B_1, B_2 \in \mathbb{R}$, $C \geq 0$. By using the upper and lower solution method, we present easily verifiable sufficient conditions for the existence of unbounded solutions of (1.1).

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The upper and lower solution method has been successfully used to provide existence results for two-point and multi-point boundary value problems (in short BVPs) for second-order and higher-order ordinary differential equations, see [6, 9, 10, 14, 19, 20, 26, 27] and references therein. All of these works deal with problems on finite intervals. In recent years the study of BVPs on $[0, +\infty)$ has attracted several researchers, for instance see [3, 5, 11, 12, 15, 16, 24, 25, 28] and references therein. In these works authors have applied either some fixed point theorem or a monotone iterative technique to establish the existence of bounded or unbounded solutions.

Fourth-order differential equations appear in mathematical modeling of physical, biological, and chemical phenomena such as viscoelastic and inelastic flows, deformation of beams and plate deflection problems [2, 7, 8, 13, 18, 29]. On a finite interval fourth-order differential equations together with two-point boundary conditions have been studied in [1, 17, 21, 22], and with multi-point conditions in [6, 19, 23, 27]. It seems that [12] is the only paper which considers a particular fourth-order differential equation on $[0, +\infty)$ (with entirely different technique and boundary conditions than ours). Thus, to fill a gap in this paper we present an existence theory of unbounded solutions for the BVP (1.1).

The plan of our paper is as follows: in Section 2, we give some definitions and lemmas which we need to prove the main results. This includes the construction of Green's function for a related fourth order boundary value problem with nonhomogeneous three-point boundary conditions, definitions of upper and lower solutions of (1.1), and Nagumo's condition. In Section 3, we present two main results. In our first result we use Schauder's fixed point theorem to establish the existence of at least one solution of (1.1) which lies between the assumed pair of upper and lower solutions. In our second result we assume the existence of two pairs of upper and lower solutions and employ the degree theory to prove the existence of at least three solutions of (1.1). We demonstrate the importance of our results through two illustrative examples.

2 Preliminaries

We begin with constructing Green's function for the linear boundary value problem

$$\begin{aligned} &x''''(t) + v(t) = 0, \qquad t \in (0, +\infty), \\ &x''(0) = A, \qquad x(\eta) = B_1, \qquad x'(\eta) = B_2, \qquad x'''(+\infty) = C. \end{aligned} \tag{2.1}$$

Lemma 2.1. *Let $v \in \mathcal{C}[0, +\infty)$ and $\int_0^\infty v(t)\,dt < +\infty$. Then the solution $x \in \mathcal{C}^3[0, +\infty) \cap \mathcal{C}^4(0, +\infty)$ of the problem (2.1) can be expressed as*

$$x(t) = B_1 + \left(B_2 - A\eta - \frac{C\eta^2}{2}\right)(t - \eta) + \frac{A}{2}(t^2 - \eta^2) + \frac{C}{6}(t^3 - \eta^3) + \int_0^\infty G(t, s)v(s)\,ds,$$

where

$$G(t, s) = \begin{cases} s\left(\dfrac{t^2}{2} - \eta t + \dfrac{\eta^2}{2}\right), & s \le \min\{\eta, t\}; \\ \dfrac{t^3}{6} + \dfrac{s^2 t}{2} - \eta t s + \dfrac{\eta^2 s}{2} - \dfrac{s^3}{6}, & t \le s \le \eta; \\ \dfrac{st^2}{2} - \dfrac{s^2 t}{2} - \dfrac{\eta^2 t}{2} + \dfrac{s^3}{6} + \dfrac{\eta^3}{3}, & \eta \le s \le t; \\ \dfrac{t^3}{6} - \dfrac{\eta^2 t}{2} + \dfrac{\eta^3}{3}, & \max\{\eta, t\} \le s. \end{cases} \tag{2.2}$$

Proof. Since $v \in \mathcal{C}[0,+\infty)$ and $\int_0^\infty v(t)\,dt < +\infty$, we can integrate (2.1) from t to $+\infty$, and use $x'''(+\infty) = C$, to get

$$x'''(t) = C + \int_t^\infty v(s)\,ds.$$

Integrating the above equation on $[0,t]$, applying Fubini's theorem, and using $x''(0) = A$, we obtain

$$x''(t) = A + Ct + \int_0^t sv(s)\,ds + \int_t^\infty tv(s)\,ds.$$

Again integrating the above equation on $[0,t]$, we find

$$x'(t) = x'(0) + At + \frac{C}{2}t^2 + \int_0^t \left(st - \frac{s^2}{2}\right) v(s)\,ds + \int_t^\infty \frac{t^2}{2} v(s)\,ds. \tag{2.3}$$

Since $x'(\eta) = B_2$, it follows that

$$x'(0) = \left(B_2 - A\eta - \frac{C\eta^2}{2}\right) - \int_0^\eta \left(s\eta - \frac{s^2}{2}\right) v(s)\,ds - \int_\eta^\infty \frac{\eta^2}{2} v(s)\,ds.$$

Hence from (2.3), we have

$$\begin{aligned} x'(t) = &\left(B_2 - A\eta - \frac{C\eta^2}{2}\right) + At + \frac{C}{2}t^2 \\ &+ \int_0^t s(t-\eta)v(s)\,ds + \int_t^\eta \left(\frac{t^2}{2} + \frac{s^2}{2} - s\eta\right) v(s)\,ds \\ &+ \int_\eta^\infty \frac{1}{2}(t^2 - \eta^2) v(s)\,ds, \quad \text{if } t \le \eta \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} x'(t) = &\left(B_2 - A\eta - \frac{C\eta^2}{2}\right) + At + \frac{C}{2}t^2 \\ &+ \int_0^\eta s(t-\eta)v(s)ds + \int_\eta^t \left(st - \frac{s^2}{2} - \frac{\eta^2}{2}\right) v(s)ds \\ &+ \int_t^\infty \frac{1}{2}(t^2 - \eta^2) v(s)\,ds, \quad \text{if } \eta \le t. \end{aligned} \tag{2.5}$$

When $t \le \eta$ we integrate (2.4) from t to η, and when $\eta \le t$ we integrate (2.5) from η to t, to obtain

$$\begin{aligned} x(t) = &B_1 + \left(B_2 - A\eta - \frac{C\eta^2}{2}\right)(t-\eta) + \frac{A}{2}(t^2 - \eta^2) + \frac{C}{6}(t^3 - \eta^3) \\ &+ \begin{cases} \displaystyle\int_0^t s\left(\frac{t^2}{2} - \eta t + \frac{\eta^2}{2}\right) v(s)\,ds + \int_t^\eta \left(\frac{t^3}{6} + \frac{s^2 t}{2} - \eta ts + \frac{\eta^2 s}{2} - \frac{s^3}{6}\right) v(s)\,ds \\ \qquad\qquad\qquad \displaystyle + \int_\eta^\infty \left(\frac{t^3}{6} - \frac{\eta^2 t}{2} + \frac{\eta^3}{3}\right) v(s)\,ds, & t \le \eta; \\ \displaystyle\int_0^\eta s\left(\frac{t^2}{2} - \eta t + \frac{\eta^2}{2}\right) v(s)\,ds + \int_\eta^t \left(\frac{st^2}{2} - \frac{s^2 t}{2} - \frac{\eta^2 t}{2} + \frac{s^3}{6} + \frac{\eta^3}{3}\right) v(s)\,ds \\ \qquad\qquad\qquad \displaystyle + \int_t^\infty \left(\frac{t^3}{6} - \frac{\eta^2 t}{2} + \frac{\eta^3}{3}\right) v(s)\,ds, & \eta \le t; \end{cases} \end{aligned}$$

which is the same as

$$x(t) = B_1 + \left(B_2 - A\eta - \frac{C\eta^2}{2}\right)(t-\eta) + \frac{A}{2}(t^2-\eta^2) + \frac{C}{6}(t^3-\eta^3)$$
$$+ \int_0^\infty G(t,s)v(s)\,ds, \qquad \forall t \in [0,+\infty).$$

This completes the proof of the lemma. □

Let

$$X = \left\{ x \in C^3[0,+\infty) : \lim_{t\to+\infty} \frac{x(t)}{1+t^3}, \lim_{t\to+\infty} \frac{x'(t)}{1+t^2}, \lim_{t\to+\infty} \frac{x''(t)}{1+t} \text{ and } \lim_{t\to+\infty} x'''(t) \text{ exist} \right\}$$

with the norm $\|x\| = \max\{\|x\|_1, \|x\|_2, \|x\|_3, \|x\|_4\}$, where

$$\|x\|_1 = \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1+t^3}, \qquad \|x\|_2 = \sup_{t\in[0,+\infty)} \frac{|x'(t)|}{1+t^2},$$
$$\|x\|_3 = \sup_{t\in[0,+\infty)} \frac{|x''(t)|}{1+t}, \qquad \|x\|_4 = \sup_{t\in[0,+\infty)} |x'''(t)|.$$

Then by standard arguments, it follows that $(X, \|.\|)$ is a Banach space. In what follows, we shall need the following modified version of the Arzelà–Ascoli lemma [4,24].

Lemma 2.2. *Let $M \subset X$. Then M is relatively compact if the following conditions hold:*

- ***(i)*** *M is bounded in X;*
- ***(ii)*** *functions in $\{y : y = \frac{x}{1+t^3},\ x \in M\}$, $\{z : z = \frac{x'}{1+t^2},\ x \in M\}$, $\{u : u = \frac{x''}{1+t},\ x \in M\}$ and $\{w : w = x'''(t),\ x \in M\}$ are locally equi-continuous on $[0,+\infty)$;*
- ***(iii)*** *functions in $\{y : y = \frac{x}{1+t^3},\ x \in M\}$, $\{z : z = \frac{x'}{1+t^2},\ x \in M\}$, $\{u : u = \frac{x''}{1+t},\ x \in M\}$ and $\{w : w = x'''(t),\ x \in M\}$ are equi-convergent at $+\infty$.*

Definition 2.3. A function $\alpha \in X \cap C^4(0,+\infty)$ is called a lower solution of (1.1) if

$$\alpha''''(t) + q(t)f(t,\alpha(t),\alpha'(t),\alpha''(t),\alpha'''(t)) \geq 0, \qquad t \in (0,+\infty), \tag{2.6}$$
$$\alpha''(0) \leq A, \qquad \alpha(\eta) \leq B_1, \qquad \alpha'(\eta) = B_2, \qquad \alpha'''(+\infty) \leq C. \tag{2.7}$$

Similarly, a function $\beta \in X \cap C^3(0,+\infty)$ is called an upper solution of (1.1) if

$$\beta''''(t) + q(t)f(t,\beta(t),\beta'(t),\beta''(t),\beta'''(t)) \leq 0, \qquad t \in (0,+\infty), \tag{2.8}$$
$$\beta''(0) \geq A, \qquad \beta(\eta) \geq B_1, \qquad \beta'(\eta) = B_2, \qquad \beta'''(+\infty) \geq C. \tag{2.9}$$

Also, we say $\alpha(\beta)$ is a strict lower solution (strict upper solution) for problem (1.1) if the above inequalities are strict.

Remark 2.4. If

$$\alpha''(t) \leq \beta''(t) \quad \text{for all } t \in (0,+\infty), \tag{2.10}$$

then by integrating (2.10) and using the continuity of $\alpha(t)$ and $\beta(t)$, and the fact that $\alpha'(\eta) = B_2 = \beta'(\eta)$, it follows that $\beta'(t) \leq \alpha'(t)$ for all $t \in [0,\eta)$ and $\alpha'(t) \leq \beta'(t)$ for all $t \in [\eta,+\infty)$. A further integration then yields $\alpha(t) \leq \beta(t)$ for all $t \in [0,+\infty)$.

Definition 2.5. Let $\alpha, \beta \in X \cap \mathcal{C}^4(0,+\infty)$ be a pair of lower and upper solutions of (1.1) satisfying $\alpha''(t) \leq \beta''(t),\ t \in [0,+\infty)$. A continuous function $f\colon [0,+\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is said to satisfy Nagumo's condition with respect to the pair of functions α, β, if there exist a nonnegative function $\phi \in \mathcal{C}[0,+\infty)$ and a positive function $h \in \mathcal{C}[0,+\infty)$ such that

$$|f(t,y,z,u,w)| \leq \phi(t)h(|w|) \tag{2.11}$$

for all $(t,y,z,u,w) \in [0,\eta) \times [\alpha(t),\beta(t)] \times [\beta'(t),\alpha'(t)] \times [\alpha''(t),\beta''(t)] \times \mathbb{R}$ and $(t,y,z,u,w) \in [\eta,+\infty) \times [\alpha(t),\beta(t)] \times [\alpha'(t),\beta'(t)] \times [\alpha''(t),\beta''(t)] \times \mathbb{R}$, and

$$\int_0^\infty \frac{s}{h(s)}\,ds = +\infty. \tag{2.12}$$

3 Main results

The following result provides sufficient conditions for the existence of at least one solution of the problem (1.1).

Theorem 3.1. *Assume that α, β are lower and upper solutions of (1.1) satisfying $\alpha''(t) \leq \beta''(t)$, $t \in [0,+\infty)$, and suppose that $f\colon [0,+\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous satisfying Nagumo's condition with respect to the pair of functions α, β. Further, assume that*

$$f(t,\alpha(t),z,u,w) \leq f(t,y,z,u,w) \leq f(t,\beta(t),z,u,w) \tag{3.1}$$

and

$$f(t,y,\alpha'(t),u,w) \leq f(t,y,z,u,w) \leq f(t,y,\beta'(t),u,w) \tag{3.2}$$

for $(t,y,z,u,w) \in [0,\eta) \times [\alpha(t),\beta(t)] \times [\beta'(t),\alpha'(t)] \times [\alpha''(t),\beta''(t)] \times \mathbb{R}$ and $(t,y,z,u,w) \in [\eta,+\infty) \times [\alpha(t),\beta(t)] \times [\alpha'(t),\beta'(t)] \times [\alpha''(t),\beta''(t)] \times \mathbb{R}$. If

$$\int_0^\infty \max\{s,1\}q(s)\,ds < +\infty, \qquad \int_0^\infty \max\{s,1\}\phi(s)q(s)\,ds < +\infty \tag{3.3}$$

and there exists a constant $\gamma > 1$ such that

$$m = \sup_{t\in[0,+\infty)} (1+t)^\gamma q(t)\phi(t) < +\infty, \tag{3.4}$$

where $\phi(t)$ is the function in Nagumo's condition of f, then (1.1) has at least one solution $x \in X \cap \mathcal{C}^4(0,+\infty)$ satisfying

$$\begin{aligned}
&\alpha(t) \leq x(t) \leq \beta(t), \qquad t \in [0,+\infty),\\
&\beta'(t) \leq x'(t) \leq \alpha'(t), \qquad t \in [0,\eta), \qquad \alpha'(t) \leq x'(t) \leq \beta'(t), \qquad t \in [\eta,+\infty),\\
&\alpha''(t) \leq x''(t) \leq \beta''(t), \qquad |x'''(t)| < N, \qquad t \in [0,+\infty);
\end{aligned}$$

here, N is a constant depending on α, β, h and C.

Proof. We can choose an

$$r \geq \max\left\{\sup_{t\in[0,+\infty)} |\alpha'''(t)|, \sup_{t\in[0,+\infty)} |\beta'''(t)|, C\right\} \tag{3.5}$$

and an $N > r$ such that

$$\int_r^N \frac{s}{h(s)}\, ds > m \left(\sup_{t\in[0,+\infty)} \frac{\beta''(t)}{(1+t)^\gamma} - \inf_{t\in[0,+\infty)} \frac{\alpha''(t)}{(1+t)^\gamma} + \frac{\gamma}{\gamma-1} \max\{\|\beta\|_3, \|\alpha\|_3\} \right). \tag{3.6}$$

We define the following auxiliary functions

$$f_0(t,y,z,u,w) = \begin{cases} f(t,\beta,z,u,w), & y > \beta(t); \\ f(t,y,z,u,w), & \alpha(t) \le y \le \beta(t); \\ f(t,\alpha,z,u,w), & y < \alpha(t), \end{cases}$$

$$f_1(t,y,z,u,w) = \begin{cases} t \in [0,\eta), & \begin{cases} f_0(t,y,\beta',u,w), & z < \beta'(t); \\ f_0(t,y,z,u,w), & \beta'(t) \le z \le \alpha'(t); \\ f_0(t,y,\alpha',u,w), & z > \alpha'(t), \end{cases} \\ t \in [\eta,+\infty), & \begin{cases} f_0(t,y,\beta',u,w), & z > \beta'(t); \\ f_0(t,y,z,u,w), & \alpha'(t) \le z \le \beta'(t); \\ f_0(t,y,\alpha',u,w), & z < \alpha'(t), \end{cases} \end{cases}$$

and

$$f^*(t,y,z,u,w) = \begin{cases} f_1(t,y,z,\beta'',w^*) + \frac{\beta''(t)-u}{1+|\beta''(t)-u|}, & u > \beta''(t); \\ f_1(t,y,z,u,w^*), & \alpha''(t) \le u \le \beta''(t); \\ f_1(t,y,z,\alpha'',w^*) + \frac{\alpha''(t)-u}{1+|\alpha''(t)-u|}, & u < \alpha''(t), \end{cases} \tag{3.7}$$

where

$$w^* = \begin{cases} N, & w > N; \\ w, & -N \le w \le N; \\ -N, & w < -N. \end{cases}$$

Now we consider the modified problem

$$\begin{aligned} &x''''(t) + q(t) f^*(t, x(t), x'(t), x''(t), x'''(t)) = 0, \qquad t \in (0,+\infty), \\ &x''(0) = A, \qquad x(\eta) = B_1, \qquad x'(\eta) = B_2, \qquad x'''(+\infty) = C. \end{aligned} \tag{3.8}$$

As an application of Schauder's fixed point theorem first we will prove that (3.8) has at least one solution x. To show this, for $x \in X$, we define two operators as follows

$$(T_1x)(t) = \int_0^\infty G(t,s)q(s)f^*(s,x(s),x'(s),x''(s),x'''(s))\, ds, \qquad t \in [0,+\infty)$$

and

$$\begin{aligned} (Tx)(t) = B_1 + \left(B_2 - A\eta - \frac{C\eta^2}{2}\right)(t-\eta) + \frac{A}{2}(t^2-\eta^2) \\ + \frac{C}{6}(t^3 - \eta^3) + (T_1x)(t), \qquad t \in [0,+\infty). \end{aligned} \tag{3.9}$$

Now we shall prove that $T\colon X \to X$ is completely continuous. We divide the proof in the following three parts.

(1) $T\colon X \to X$ is well defined. For each $x \in X$, in view of (2.11), (3.3) and (3.7) as in [5], we have

$$\begin{aligned}\left|\int_0^\infty q(s)f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right| &\le \int_0^\infty q(s)(H_0\phi(s)+1)\,ds \\ &\le \int_0^\infty \max\{1,s\}q(s)(H_0\phi(s)+1)\,ds < +\infty,\end{aligned} \tag{3.10}$$

where $H_0 = \max_{0\le t\le \|x\|_4} h(t)$. For $x \in X$, we find from (3.10) that

$$\int_1^\infty sq(s)(H_0\phi(s)+1)\,ds \le \int_0^\infty \max\{s,1\}q(s)(H_0\phi(s)+1))\,ds < +\infty, \tag{3.11}$$

which implies

$$\lim_{t\to+\infty} tq(t)(H_0\phi(t)+1) = 0. \tag{3.12}$$

Since

$$\int_t^\infty q(s)(H_0\phi(s)+1)\,ds \le \int_t^\infty sq(s)(H_0\phi(s)+1))\,ds < +\infty, \qquad t\ge 1, \tag{3.13}$$

it also follows that

$$\lim_{t\to+\infty}\int_t^\infty q(s)(H_0\phi(s)+1)\,ds = 0. \tag{3.14}$$

Thus by Lebesgue's dominated convergence theorem, l'Hôpital's rule, (3.12) and (3.14), we find

$$\begin{aligned}&\left|\lim_{t\to+\infty}\frac{(T_1x)(t)}{1+t^3}\right| \\ &\quad\le \lim_{t\to+\infty}\int_0^\infty \frac{|G(t,s)|}{1+t^3}q(s)(H_0\phi(s)+1)\,ds \\ &\quad= \lim_{t\to+\infty}\Bigg[\int_0^\eta s\frac{(\frac{t^2}{2}-\eta t+\frac{\eta^2}{2})}{1+t^3}q(s)(H_0\phi(s)+1)\,ds \\ &\qquad\qquad + \int_\eta^t \frac{(\frac{st^2}{2}-\frac{s^2t}{2}-\frac{\eta^2 t}{2}+\frac{s^3}{6}+\frac{\eta^3}{3})}{1+t^3}q(s)(H_0\phi(s)+1)\,ds \\ &\qquad\qquad + \int_t^\infty \frac{(\frac{t^3}{6}-\frac{\eta^2 t}{2}+\frac{\eta^3}{3})}{1+t^3}q(s)(H_0\phi(s)+1)\,ds\Bigg] \\ &\quad= \lim_{t\to+\infty}\int_\eta^t \frac{(st-\frac{s^2}{2}-\frac{\eta^2}{2})}{3t^2}q(s)(H_0\phi(s)+1)\,ds + \lim_{t\to+\infty}\frac{(-\frac{\eta^2 t}{2}+\frac{t^3}{6}+\frac{\eta^3}{3})}{3t^2}q(t)(H_0\phi(t)+1) \\ &\qquad + \lim_{t\to+\infty}\int_t^\infty \frac{(\frac{t^2}{2}-\frac{\eta^2}{2})}{3t^2}q(s)(H_0\phi(s)+1)\,ds - \lim_{t\to+\infty}\frac{(\frac{t^3}{6}-\frac{\eta^2 t}{2}+\frac{\eta^3}{3})}{3t^2}q(t)(H_0\phi(t)+1) \\ &\quad= \lim_{t\to+\infty}\int_\eta^t \frac{s}{6t}q(s)(H_0\phi(s)+1)ds + \lim_{t\to+\infty}\frac{(t^2-\frac{t^2}{2}-\frac{\eta^2}{2})}{6t}q(t)(H_0\phi(t)+1) \\ &\qquad + \lim_{t\to+\infty}\int_t^\infty \frac{t}{6t}q(s)(H_0\phi(s)+1)ds - \lim_{t\to+\infty}\frac{(\frac{t^2}{2}-\frac{\eta^2}{2})}{6t}q(t)(H_0\phi(t)+1) \\ &\quad= \frac{1}{6}\lim_{t\to+\infty}\int_t^\infty q(s)(H_0\phi(s)+1)ds = 0,\end{aligned}$$

which implies that $\lim_{t\to+\infty}\frac{(T_1x)(t)}{1+t^3}=0$. Therefore, it follows that

$$\begin{aligned}\lim_{t\to+\infty}\frac{(Tx)(t)}{1+t^3} &= \lim_{t\to+\infty}\frac{B_1+(B_2-A\eta-\frac{C\eta^2}{2})(t-\eta)+\frac{A}{2}(t^2-\eta^2)+\frac{C}{6}(t^3-\eta^3)}{1+t^3}\\ &\quad+\lim_{t\to+\infty}\frac{(T_1x)(t)}{1+t^3}\\ &=\frac{C}{6}.\end{aligned}$$

By Lebesgue's dominated convergence theorem, l'Hôpital's rule, (3.12) and (3.14), we have

$$\begin{aligned}\left|\lim_{t\to+\infty}\frac{(T_1x)'(t)}{1+t^2}\right| &\le \lim_{t\to+\infty}\left[\int_0^\eta \frac{s(t-\eta)}{1+t^2}q(s)(H_0\phi(s)+1)\,ds\right.\\ &\qquad+\int_\eta^t \frac{(st-\frac{s^2}{2}-\frac{\eta^2}{2})}{1+t^2}q(s)(H_0\phi(s)+1)\,ds\\ &\qquad\left.+\int_t^\infty \frac{\frac{1}{2}(t^2-\eta^2)}{1+t^2}q(s)(H_0\phi(s)+1)\,ds\right]\\ &=\lim_{t\to+\infty}\frac{\int_\eta^t sq(s)(H_0\phi(s)+1)\,ds+(t^2-\frac{t^2}{2}-\frac{\eta^2}{2})q(t)(H_0\phi(t)+1)}{2t}\\ &\quad+\lim_{t\to+\infty}\frac{\int_t^\infty tq(s)(H_0\phi(s)+1)\,ds-\frac{1}{2}(t^2-\eta^2)q(t)(H_0\phi(t)+1)}{2t}\\ &=\frac{1}{2}\lim_{t\to+\infty}\int_t^\infty q(s)(H_0\phi(s)+1)\,ds=0,\end{aligned}$$

which implies that $\lim_{t\to+\infty}\frac{(T_1x)'(t)}{1+t^2}=0$. Therefore, it follows that

$$\lim_{t\to+\infty}\frac{(Tx)'(t)}{1+t^2}=\lim_{t\to+\infty}\frac{(B_2-A\eta-\frac{C\eta^2}{2})+At+\frac{C}{2}t^2}{1+t^2}+\lim_{t\to+\infty}\frac{(T_1x)'(t)}{1+t^2}=\frac{C}{2}.$$

Again, using Lebesgue's dominated convergence theorem, l'Hôpital's rule, (3.12) and (3.14), we obtain

$$\begin{aligned}\left|\lim_{t\to+\infty}\frac{(T_1x)''(t)}{1+t}\right| &= \lim_{t\to+\infty}\left|\frac{1}{1+t}\left(\int_0^t sq(s)f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right.\right.\\ &\qquad\left.\left.+\int_t^\infty tq(s)f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right)\right|\\ &\le \lim_{t\to+\infty}\frac{1}{1+t}\left(\int_0^t sq(s)(H_0\phi(s)+1)\,ds+\int_t^\infty tq(s)q(s)(H_0\phi(s)+1)\,ds\right)\\ &=\lim_{t\to+\infty}\int_t^\infty q(s)(H_0\phi(s)+1)\,ds=0,\end{aligned}$$

which implies that $\lim_{t\to+\infty}\frac{(T_1x)''(t)}{1+t}=0$. Hence, we find

$$\lim_{t\to+\infty}\frac{(Tx)''(t)}{1+t}=\lim_{t\to+\infty}\frac{A+Ct}{1+t}+\lim_{t\to+\infty}\frac{(T_1x)''(t)}{1+t}=C.$$

Now from (3.12), we have

$$\left|\lim_{t\to+\infty}\int_t^\infty q(s)f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right|\le\lim_{t\to+\infty}\int_t^\infty q(s)(H_0\phi(s)+1)\,ds=0,$$

and hence,

$$\lim_{t\to+\infty}(Tx)'''(t) = \lim_{t\to+\infty} C + \int_t^{\infty} q(s)f^*(s, x(s), x'(s), x''(s), x'''(s))\, ds = C.$$

Consequently, it follows that $Tx \in X$.
(2) $T\colon X \to X$ is continuous. For any convergent sequence $x_n \to x$ in X, we have

$$x_n(t) \to x(t),\quad x_n'(t) \to x'(t),\quad x_n''(t) \to x''(t),\quad x_n'''(t) \to x'''(t),\quad n \to +\infty,\ t \in [0,+\infty).$$

Thus the continuity of f^* implies that

$$|f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))| \to 0,\quad n \to +\infty,\ s \in [0,+\infty).$$

Since $x_n'''(t) \to x'''(t)$, we have $\sup\limits_{n\in\mathbb{N}} \|x_n\|_4 < +\infty$. Let

$$H_1 = \max_{0\le t\le \max\{\|x\|_4, \sup_{n\in\mathbb{N}}\|x_n\|_4\}} h(t).$$

Then we obtain

$$\begin{aligned}&\int_0^{\infty} sq(s)|f^*(s, x_n(s), x_n'(s), x_n''(s), x'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds\\ &\qquad \le 2\int_0^{\infty} sq(s)(H_1\phi(s)+1)\, ds < +\infty.\end{aligned} \tag{3.15}$$

Therefore from Lebesgue's dominated convergence theorem and (3.15) for $\eta \le t$ it follows that

$$\begin{aligned}
\frac{|Tx_n(t) - Tx(t)|}{1+t^3} &= \frac{|T_1x_n(t) - T_1x(t)|}{1+t^3}\\
&= \left|\int_0^{\infty} \frac{G(t,s)}{1+t^3} q(s)(f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s)))\, ds\right|\\
&\le \int_0^{\eta} \frac{s(t-\eta)^2}{2(1+t^3)} q(s)|f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds\\
&\quad + \int_{\eta}^{t} \frac{s(\frac{t^2}{2} - \frac{st}{2} + \frac{\eta t}{2} + \frac{s^2}{6} + \frac{\eta^2}{3})}{1+t^3} q(s)\\
&\qquad \times |(f^*(s, x_n(s), x_n'(s), x_n''(s), x'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds\\
&\quad + \int_t^{\infty} \frac{s(\frac{t^2}{6} + \frac{\eta t}{2} + \frac{\eta^2}{3})}{1+t^3} q(s)\\
&\qquad \times |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds\\
&\le \int_0^{\infty} s\frac{t^2}{1+t^3} q(s)|f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds\\
&\le \int_0^{\infty} sq(s)|f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds
\end{aligned}$$

and for $t \le \eta$

$$
\begin{aligned}
\frac{|Tx_n(t) - Tx(t)|}{1+t^3} &= \frac{|T_1x_n(t) - T_1x(t)|}{1+t^3} \\
&= \left| \int_0^\infty \frac{G(t,s)}{1+t^3} q(s)(f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s)))\, ds \right| \\
&\le \int_0^t \frac{s(t-\eta)^2}{2(1+t^3)} q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\quad + \int_t^\eta \frac{s(\frac{t^2}{6} + \frac{st}{2} + \eta t + \frac{\eta^2}{2} + \frac{s^2}{6})}{1+t^3} q(s) \\
&\qquad \times |(f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\quad + \int_\eta^\infty \frac{s(\frac{t^2}{6} + \frac{\eta^2}{2} + \frac{\eta^2}{3})}{1+t^3} q(s) \\
&\qquad \times |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\le \int_0^\infty s \frac{\frac{7\eta^2}{3}}{1+t^3} q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\le \frac{7\eta^2}{3} \int_0^\infty s q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds.
\end{aligned}
$$

Combining the above inequalities, we obtain

$$
\begin{aligned}
&\|Tx_n - Tx\|_1 \\
&\quad \le \max\left\{1, \tfrac{7\eta^2}{3}\right\} \int_0^\infty s q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds,
\end{aligned}
$$

which approaches zero as $n \to \infty$. Similarly, from Lebesgue's dominated convergence theorem and (3.15) for $\eta \le t$, we find

$$
\begin{aligned}
\frac{|(Tx_n)'(t) - (Tx)'(t)|}{1+t^2} &= \frac{|(T_1x_n)'(t) - (T_1x)'(t)|}{1+t^2} \\
&\le \int_0^\eta \frac{s(t-\eta)}{1+t^2} q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\quad + \int_\eta^t \frac{(st - \frac{s^2}{2} - \frac{\eta^2}{2})}{1+t^2} q(s) \\
&\qquad \times |(f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\quad + \int_t^\infty \frac{\frac{1}{2}(t^2 - \eta^2)}{1+t^2} q(s) \\
&\qquad \times |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\le \int_0^\infty \frac{st}{1+t^2} q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\le \frac{1}{2} \int_0^\infty s q(s) |f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds
\end{aligned}
$$

and for $t \le \eta$

$$\begin{aligned}
\frac{|(Tx_n)'(t)-(Tx)'(t)|}{1+t^2} &= \frac{|(T_1x_n)'(t)-(T_1x)'(t)|}{1+t^2} \\
&= \left| \int_0^\infty \frac{G(t,s)}{1+t^2} q(s)(f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s)))\,ds \right| \\
&\le \int_0^t \frac{s(\eta-t)}{1+t^2} q(s)|f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds \\
&\quad + \int_t^\eta \frac{(s\eta - \frac{t\eta}{2} - \frac{s^2}{2})}{1+t^2} q(s) \\
&\qquad \times |(f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds \\
&\quad + \int_\eta^\infty \frac{\frac{1}{2}(\eta^2-t^2)}{1+t^2} q(s) \\
&\qquad \times |f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds \\
&\le \int_0^\infty \frac{s\eta}{1+t^2} q(s)|f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds \\
&\le \eta \int_0^\infty sq(s)|f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds.
\end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned}
&\|Tx_n - Tx\|_2 \\
&\quad \le \max\left\{\tfrac{1}{2},\eta\right\} \int_0^\infty sq(s)|f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds,
\end{aligned}$$

which approaches zero as $n \to \infty$. We also have

$$\begin{aligned}
&\|Tx_n - Tx\|_3 \\
&\quad = \sup_{t\in[0,+\infty)} \frac{|(Tx_n)''(t)-(Tx)''(t)|}{1+t} = \sup_{t\in[0,+\infty)} \frac{|(T_1x_n)'(t)-(T_1x)'(t)|}{1+t} \\
&\quad \le \sup_{t\in[0,+\infty)} \Bigg[\int_0^t \frac{s}{1+t} q(s) \\
&\qquad\qquad \times |f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds \\
&\qquad\quad + \int_t^\infty \frac{t}{1+t} q(s) \\
&\qquad\qquad \times |f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds \Bigg] \\
&\quad \le \int_0^\infty sq(s)|f^*(s,x_n(s),x_n'(s),x_n''(s),x_n'''(s)) - f^*(s,x(s),x'(s),x''(s),x'''(s))|\,ds,
\end{aligned}$$

which approaches zero as $n \to \infty$. Finally, from (3.15) we have

$$\begin{aligned}
&\|(Tx_n) - (Tx)\|_4 \\
&\quad = \sup_{t\in[0,+\infty)} |(Tx_n)'''(t) - (Tx)'''(t)| = \sup_{t\in[0,+\infty)} |(T_1x_n)'''(t) - (T_1x)'''(t)| \\
&\quad = \sup_{t\in[0,+\infty)} \left| \int_t^\infty q(s)(f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s)))\, ds \right| \\
&\quad \le \int_0^\infty q(s)|f^*(s, x_n(s), x_n'(s), x_n''(s), x_n'''(s)) - f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds,
\end{aligned}$$

which approaches zero as $n \to \infty$. As a result, we conclude that $\|Tx_n - Tx\| \to 0$, as $n \to +\infty$; and therefore $T\colon X \to X$ is continuous.
(3) $T\colon X \to X$ is compact. For this it is suffices to show that T maps bounded subsets of X into relatively compact sets. We assume that M_1 is any bounded subset of X, then for $x \in M_1$, we let $H_2 = \max_{0\le t\le\|x\|_4, x\in M_1} h(t) < +\infty$. Now following as above, we get

$$\begin{aligned}
\|Tx\|_1 &= \sup_{t\in[0,+\infty)} \frac{|Tx(t)|}{1+t^3} \\
&= \sup_{t\in[0,+\infty)} \left| \frac{B_1 + (B_2 - A\eta - \frac{C\eta^2}{2})(t-\eta) + \frac{A}{2}(t^2-\eta^2) + \frac{C}{6}(t^3-\eta^3)}{1+t^3} \right. \\
&\qquad\qquad \left. + \int_0^\infty \frac{G(t,s)}{1+t^3} q(s) f^*(s, x(s), x'(s), x''(s), x'''(s))\, ds \right| \\
&\le |B_1| + \left| B_2 - A\eta - \frac{C\eta^2}{2} \right| + \frac{|A|}{2} + \frac{C}{6} \\
&\quad + \max\left\{1, \frac{7\eta^2}{3}\right\} \int_0^\infty sq(s)|f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\le |B_1| + \left| B_2 - A\eta - \frac{C\eta^2}{2} \right| + \frac{|A|}{2} + \frac{C}{6} + \max\left\{1, \frac{7\eta^2}{3}\right\} \int_0^\infty sq(s)(H_2\phi(s)+1)\, ds \\
&< +\infty,
\end{aligned}$$

$$\begin{aligned}
\|Tx\|_2 &= \sup_{t\in[0,+\infty)} \frac{|(Tx)'(t)|}{1+t^2} \\
&= \sup_{t\in[0,+\infty)} \left| \frac{(B_2 - A\eta - \frac{C\eta^2}{2}) + At + \frac{C}{2}t^2}{1+t^2} \right. \\
&\qquad\qquad \left. + \int_0^\infty \frac{\frac{\partial G(t,s)}{\partial t}}{1+t^2} q(s) f^*(s, x(s), x'(s), x''(s), x'''(s))\, ds \right| \\
&\le \left| B_2 - A\eta - \frac{C\eta^2}{2} \right| + |A| + \frac{C}{2} \\
&\quad + \max\left\{\frac{1}{2}, \eta\right\} \int_0^\infty sq(s)|f^*(s, x(s), x'(s), x''(s), x'''(s))|\, ds \\
&\le \left| B_2 - A\eta - \frac{C\eta^2}{2} \right| + |A| + \frac{C}{2} + \max\left\{\frac{1}{2}, \eta\right\} \int_0^\infty sq(s)(H_2\phi(s)+1)\, ds \\
&< +\infty,
\end{aligned}$$

$$\begin{aligned}\|Tx\|_3 &= \sup_{t\in[0,+\infty)} \frac{|(Tx)''(t)|}{1+t}\\ &= \sup_{t\in[0,+\infty)} \left|\frac{A+Ct}{1+t} + \frac{1}{1+t}\int_0^\infty \frac{\partial^2 G(t,s)}{\partial t^2} q(s) f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right|\\ &\le |A| + C + \int_0^\infty s q(s)(H_2\phi(s)+1)\,ds < +\infty,\end{aligned}$$

and

$$\begin{aligned}\|(Tx)\|_4 &= \sup_{t\in[0,+\infty)} |(Tx)'''(t)|\\ &= \sup_{t\in[0,+\infty)} \left|C + \int_t^\infty q(s) f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right|\\ &\le C + \int_0^\infty q(s)(H_2\phi(s)+1)\,ds < +\infty,\end{aligned}$$

which implies that $\|Tx\| < +\infty$. Thus TM_1 is uniformly bounded. Furthermore, for any $k > 0$, for $t_1, t_2 \in [0,k]$, we have

$$\begin{aligned}\left|\frac{(Tx)(t_1)}{1+t_1^3} - \frac{(Tx)(t_2)}{1+t_2^3}\right| \le &\left|\frac{B_1 + (B_2 - A\eta - \frac{C\eta^2}{2})(t_1-\eta) + \frac{A}{2}(t_1^2-\eta^2) + \frac{C}{6}(t_1^3-\eta^3)}{1+t_1^3}\right.\\ &\left. - \frac{B_1 + (B_2 - A\eta - \frac{C\eta^2}{2})(t_2-\eta) + \frac{A}{2}(t_2^2-\eta^2) + \frac{C}{6}(t_2^3-\eta^3)}{1+t_2^3}\right|\\ &+ \int_0^\infty \left|\frac{G(t_1,s)}{1+t_1^3} - \frac{G(t_2,s)}{1+t_2^3}\right| q(s)(H_2\phi(s)+1)\,ds,\end{aligned}$$

$$\begin{aligned}\left|\frac{(Tx)'(t_1)}{1+t_1^2} - \frac{(Tx)'(t_2)}{1+t_2^2}\right| \le &\left|\frac{(B_2 - A\eta - \frac{C\eta^2}{2}) + At_1 + \frac{Ct_1^2}{2}}{1+t_1^2} - \frac{(B_2 - A\eta - \frac{C\eta^2}{2}) + At_2 + \frac{Ct_2^2}{2}}{1+t_2^2}\right|\\ &+ \int_0^\infty \left|\frac{\frac{\partial G(t_1,s)}{\partial t}}{1+t_1^2} - \frac{\frac{\partial G(t_2,s)}{\partial t}}{1+t_2^2}\right| q(s)(H_2\phi(s)+1)\,ds,\end{aligned}$$

$$\left|\frac{(Tx)''(t_1)}{1+t_1} - \frac{(Tx)''(t_2)}{1+t_2}\right| \le \left|\frac{A+Ct_1}{1+t_1} - \frac{A+Ct_2}{1+t_2}\right| + \int_0^\infty \left|\frac{\frac{\partial^2 G(t_1,s)}{\partial t^2}}{1+t_1^2} - \frac{\frac{\partial^2 G(t_2,s)}{\partial t^2}}{1+t_2^2}\right| q(s)(H_2\phi(s)+1)s$$

and

$$\begin{aligned}&|(Tx)'''(t_1) - (Tx)'''(t_2)|\\ &\quad= \left|\int_{t_1}^\infty q(s)(f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds - \int_{t_2}^\infty q(s)(f^*(s,x(s),x'(s),x''(s),x'''(s))\,ds\right|\\ &\quad\le \int_{t_1}^{t_2} q(s)(H_2\phi(s)+1)\,ds.\end{aligned}$$

Thus, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\frac{(Tx)(t_1)}{1+t_1^3} - \frac{(Tx)(t_2)}{1+t_2^3}\right| < \epsilon, \qquad \left|\frac{(Tx)'(t_1)}{1+t_1^2} - \frac{(Tx)'(t_2)}{1+t_2^2}\right| < \epsilon,$$
$$\left|\frac{(Tx)''(t_1)}{1+t_1} - \frac{(Tx)''(t_2)}{1+t_2}\right| < \epsilon, \qquad |(Tx)'''(t_1) - (Tx)'''(t_2)| < \epsilon$$

provided $|t_1 - t_2| < \delta,\ t_1, t_2 \in [0, k]$.

Since k is arbitrary, it follows that functions belonging to $\{\frac{TM_1}{1+t^3}\}$, $\{\frac{(TM_1)'}{1+t^2}\}$, $\{\frac{(TM_1)''}{1+t}\}$ and $\{(TM_1)'''\}$ are locally equicontinuous on $[0, +\infty)$. Now for $x \in M_1$, we have

$$\left|\frac{(Tx)(t)}{1+t^3} - \lim_{t\to+\infty}\frac{(Tx)(t)}{1+t^3}\right| = \left|\frac{(Tx)(t)}{1+t^3} - \frac{C}{6}\right| \to 0, \quad \text{as } t \to +\infty,$$
$$\left|\frac{(Tx)'(t)}{1+t^2} - \lim_{t\to+\infty}\frac{(Tx)'(t)}{1+t^2}\right| = \left|\frac{(Tx)'(t)}{1+t^2} - \frac{C}{2}\right| \to 0, \quad \text{as } t \to +\infty,$$
$$\left|\frac{(Tx)''(t)}{1+t} - \lim_{t\to+\infty}\frac{(Tx)''(t)}{1+t}\right| = \left|\frac{(Tx)''(t)}{1+t} - C\right| \to 0, \quad \text{as } t \to +\infty,$$

and

$$\begin{aligned}|(Tx)'''(t) - \lim_{t\to+\infty}(Tx)'''(t)| &= |(Tx)'''(t) - C| \\ &= \left|\int_t^\infty q(s)(f^*(s, x(s), x'(s), x''(s), x'''(s))\,ds\right| \to 0,\end{aligned}$$

as $t \to +\infty$, which show that the functions from $\{\frac{TM_1}{1+t^3}\}$, $\{\frac{(TM_1)'}{1+t^2}\}$, $\{\frac{(TM_1)''}{1+t}\}$ and $\{(TM_1)'''\}$ are equiconvergent at $+\infty$. Consequently, the conditions of Lemma 2.2 hold, and hence, TM_1 is relatively compact.

Therefore $T\colon X \to X$ is completely continuous, and Schauder's fixed point theorem guarantees that T has at least one fixed point $x \in X$, which is a solution of (3.8). Next, we shall show that this x satisfies

$$\alpha''(t) \le x''(t) \le \beta''(t), \qquad t \in [0, +\infty) \tag{3.16}$$

which in view of Remark 2.4 will imply that

$$\begin{aligned}&\beta'(t) \le x'(t) \le \alpha'(t), \quad t \in [0, \eta), \quad \alpha'(t) \le x'(t) \le \beta'(t), \quad t \in [\eta, +\infty), \\ &\alpha(t) \le x(t) \le \beta(t), \quad t \in [0, +\infty).\end{aligned} \tag{3.17}$$

For this, we shall show that $x''(t) \le \beta''(t)$ for all $t \in [0, +\infty)$. If this is not true then there exists a $t_0 \in [0, +\infty)$ such that

$$x''(t_0) - \beta''(t_0) = \sup_{t\in[0,\infty)}(x''(t) - \beta''(t)) > 0.$$

Now in view of $\lim_{t\to+\infty}(x'''(t) - \beta'''(t)) \le 0$, there are three cases to consider.

Case I. If $t_0 = 0$, then $x''(0) - \beta''(0) = \lim_{t\to 0^+} x''(t) - \beta''(t) = \sup_{t\in[0,+\infty)}(x''(t) - \beta''(t)) > 0$. But from the boundary condition (2.9), we have the contradiction $x''(0) - \beta''(0) \le 0$.

Case II. If $t_0 \in (0, +\infty)$, then we have $x'''(t_0) = \beta'''(t_0)$ and $x''''(t_0) \le \beta''''(t_0)$. But then from (3.7), (3.8) and $N > \sup_{t\in[0,+\infty)} |\beta'''(t)|$, we find

$$\begin{aligned}x''''(t_0) &= -q(t_0)f^*(t_0, x(t_0), x'(t_0), x''(t_0), x'''(t_0)) \\ &= -q(t_0)\left[f_1(t_0, x(t_0), x'(t_0), \beta''(t_0), \beta'''(t_0)) + \frac{\beta''(t_0) - x''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}\right].\end{aligned}$$

Now using the condition (3.2) and the definition of f_1, we obtain for $t_0 \in [0, \eta)$

$$\begin{aligned} x''''(t_0) = &-q(t_0) f(t_0, x(t_0), \beta' t_0), \beta''(t_0), \beta'''(t_0)) \\ &+ q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}, \quad \text{if } x'(t_0) \geq \beta'(t_0), \\ x''''(t_0) \geq &-q(t_0) f(t_0, x(t_0), \beta'(t_0), \beta''(t_0), \beta'''(t_0)) \\ &+ q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}, \quad \text{if } x'(t_0) < \beta'(t_0) \end{aligned}$$

and for $t_0 \in [\eta, +\infty)$

$$\begin{aligned} x''''(t_0) = &-q(t_0) f(t_0, x(t_0), \beta'(t_0), \beta''(t_0), \beta'''(t_0)) \\ &+ q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}, \quad \text{if } x'(t_0) > \beta'(t_0), \\ x''''(t_0) \geq &-q(t_0) f(t_0, x(t_0), \beta'(t_0), \beta''(t_0), \beta'''(t_0)) \\ &+ q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}, \quad \text{if } x'(t_0) \leq \beta'(t_0). \end{aligned}$$

Next using the condition (3.1) and the definition of f_0, we obtain

$$\begin{aligned} x''''(t_0) = &-q(t_0) f(t_0, \beta(t_0), \beta' t_0), \beta''(t_0), \beta'''(t_0)) \\ &+ q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}, \quad \text{if } x(t_0) \geq \beta(t_0), \\ x''''(t_0) \geq &-q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), \beta'''(t_0)) \\ &+ q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|}, \quad \text{if } x(t_0) < \beta(t_0). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} x''''(t_0) &\geq -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), \beta'''(t_0)) + q(t_0) \frac{x''(t_0) - \beta''(t_0)}{1 + |\beta''(t_0) - x''(t_0)|} \\ &> -q(t_0) f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), \beta'''(t_0)) \geq \beta''''(t_0), \end{aligned}$$

which is a contradiction.
Case III. If $t_0 = +\infty$ then

$$x''(+\infty) - \beta''(+\infty) = \lim_{t \to +\infty} x''(t) - \beta''(t) = \sup_{t \in [0, +\infty)} (x''(t) - \beta''(t)) > 0.$$

But then from the boundary condition (2.9), we have the contradiction $x''''(+\infty) - \beta''''(+\infty) \leq 0$.

Consequently, $x''(t) \leq \beta''(t)$ for $t \in [0, +\infty)$. The proof of $\alpha''(t) \leq x''(t)$ for $t \in [0, +\infty)$ is similar. Hence (3.16) holds, and consequently (3.17) follows.

Finally, we will show that $|x'''(t)| < N$ for all $t \in [0, +\infty)$. Suppose there exists a $t_0 \in [0, +\infty)$ such that $|x'''(t_0)| \geq N$. Since $\lim_{t \to +\infty} x'''(t) = C < N$, there exists a $T > 0$ such that

$$|x'''(t)| < N \quad \text{for } t \geq T.$$

Let $t_1 = \inf\{t \leq T : |x'''(s)| < N, \forall s \in [t, +\infty)\}$. Then $|x'''(t_1)| = N$ and $|x'''(t)| < N$ for all $t > t_1$, and there exists a $t_2 < t_1$ such that $|x'''(t)| \geq N$ for $t \in [t_2, t_1]$. We need to consider two cases $x'''(t_1) = N$ and $x'''(t) \geq N$ for $t \in [t_2, t_1]$, or $x'''(t_1) = -N$ and $x'''(t) \leq -N$ for $t \in [t_2, t_1]$. We assume that $x'''(t_1) = N$ and $x'''(t) \geq N$ for $t \in [t_2, t_1]$, then we have

$$\begin{aligned}
\int_r^N \frac{s}{h(s)}\,ds &\leq \int_C^N \frac{s}{h(s)}\,ds \\
&= -\int_{t_1}^{\infty} \frac{x'''(s)}{h(x'''(s))} x''''(s)\,ds \\
&= -\int_{t_1}^{\infty} \frac{-q(s)f(s,x(s),x'(s),x''(s),x'''(s))x''(s)}{h(x''(s))}\,ds \\
&\leq \int_{t_1}^{\infty} q(s)\phi(s)x'''(s)\,ds \\
&\leq m\int_{t_1}^{\infty} \frac{x'''(s)}{(1+s)^{\gamma}}\,ds \\
&= m\left(\int_{t_1}^{\infty}\left(\frac{x''(s)}{(1+s)^{\gamma}}\right)' ds - \int_{t_1}^{\infty} x''(s)\left(\frac{1}{(1+s)^{\gamma}}\right)' ds\right) \\
&\leq m\left(\sup_{t\in[0,+\infty)} \frac{\beta''(t)}{(1+t)^{\gamma}} - \inf_{t\in[0,+\infty)} \frac{\alpha''(t)}{(1+t)^{\gamma}} + \frac{\gamma}{\gamma-1}\max\{\|\beta\|_3, \|\alpha\|_3\}\right) \\
&< \int_r^N \frac{s}{h(s)}\,ds,
\end{aligned}$$

which is a contradiction. The case $x'''(t_1) = -N$ and $x'''(t) \leq -N$ for $t \in [t_2, t_1]$ leads to a similar contradiction. Hence, $|x'''(t)| < N$ for all $t \in [0, +\infty)$. Consequently, we have

$$\begin{aligned}
x''''(t) &= -q(t)f^*(t,x(t),x'(t),x''(t),x'''(t)) = -q(t)f_1(t,x(t),x'(t),x''(t),x'''(t)) \\
&= -q(t)f_0(t,x(t),x'(t),x''(t),x'''(t)) = -q(t)f(t,x(t),x'(t),x''(t),x'''(t)),
\end{aligned}$$

and hence, x is a solution of (1.1). □

Example 3.2. *Let us consider the fourth-order nonlinear boundary value problem on the half-line*

$$\begin{aligned}
&x''''(t) + \frac{(x'''(t)-6)^2}{(1+t)^{15}}\left[(6t + x''(t)) + (x'(t)-1)^2 + (t^3 + x(t))\right] = 0, \qquad t \in (0, +\infty) \\
&x''(0) = 1, \qquad x(1) = -1, \qquad x'(1) = -3, \qquad x'''(+\infty) = 0.
\end{aligned} \tag{3.18}$$

Clearly (3.18) is a particular case of (1.1) with

$$q(t) = \frac{1}{(1+t)^{11}}, \qquad f(t,y,z,u,w) = \frac{(w-6)^2}{(1+t)^4}\left[(6t+u) + (z-1)^2 + (t^3+y)\right]$$

and $A = 1$, $B_1 = -1$, $B_2 = -1$, $C = 0$. For (3.18) a direct substitution shows that

$$\beta(t) = t^3 + t^2 - 8t + 6, \qquad \alpha(t) = -t^3$$

are upper and lower solutions such that $\beta, \alpha \in X \cap C^4(0, +\infty)$. Further, for these functions we have

$$\alpha''(t) = -6t \leq \beta''(t) = 6t + 2, \qquad t \in [0, +\infty).$$

We also note that when

$$\begin{aligned}
\alpha(t) = -t^3 \le y \le \beta(t) = t^3 + t^2 - 8t + 6, \qquad & t \in [0, +\infty), \\
\beta'(t) = 3t^2 + 2t - 8 \le z \le \alpha'(t) = -3t^2, \qquad & t \in [0, 1), \\
\alpha'(t) = -3t^2 \le z \le \beta'(t) = 3t^2 + 2t - 8, \qquad & t \in [1, +\infty), \\
\alpha''(t) = -6t \le u \le \beta''(t) = 6t + 2, \qquad & t \in [0, +\infty),
\end{aligned}$$

the function f is continuous and satisfies Nagumo's condition with respect to α and β, that is,

$$\begin{aligned}
|f(t, y, z, u, w)| &= \left| (w-6)^2 \frac{(6t+u) + (z-1)^2 + (t^3 + y)}{(1+t)^4} \right| \\
&\le \left(\sup_{t \in [0, +\infty)} \frac{89 + 4t + 7t^2 + 14t^3 + 9t^4}{(1+t)^4} \right) (|w| + 6)^2 \\
&\le 89(|w| + 6)^2.
\end{aligned}$$

Hence we can take $\varphi(t) = 89$ and $h(w) = (w+1)^2$. Now if $1 < \gamma \le 11$, then

$$\sup_{t \in [0, +\infty)} (1+t)^{\gamma} \frac{89}{(1+t)^{11}} = \sup_{t \in [0, +\infty)} \frac{89}{(1+t)^{11-\gamma}} \le 89 < +\infty,$$

and

$$\int_0^{\infty} \frac{1}{(1+s)^{11}}\, ds < +\infty, \qquad \int_0^{\infty} \frac{s}{(1+s)^{11}}\, ds < +\infty, \qquad \int_0^{\infty} \frac{s}{h(s)}\, ds = \int_0^{\infty} \frac{s}{(s+6)^2}\, ds = +\infty$$

these imply that conditions (2.12), (3.3) and (3.4) are fulfilled. Clearly, f is increasing in y, decreasing in z on $[0, 1) \times [\alpha(t), \beta(t)] \times [\beta'(t), \alpha'(t)] \times [\alpha''(t), \beta''(t)] \times \mathbb{R}$ and increasing on $[1, +\infty) \times [\alpha(t), \beta(t)] \times [\beta'(t), \alpha'(t)] \times [\alpha''(t), \beta''(t)] \times \mathbb{R}$ in y, z. Thus f satisfies condition (3.1) and (3.2). Theorem 3.1 now ensures that the BVP (3.18) has at least one solution $x(t)$ such that

$$\begin{aligned}
-t^3 \le x(t) \le t^3 + t^2 - 8t + 6, \quad & \text{for all } t \in [0, +\infty), \quad \text{(See Figure 1)} \\
3t^2 + 2t - 8 \le x'(t) \le -3t^2, \quad & \text{for all } t \in [0, 1), \\
-3t^2 \le x'(t) \le 3t^2 + 2t - 8, \quad & \text{for all } t \in [1, +\infty), \quad \text{(See Figure 2)} \\
-6t \le x''(t) \le 6t + 2 \quad & \text{for all } t \in [0, +\infty) \quad \text{(See Figure 3)}.
\end{aligned}$$

Also, $\|x\|_4 < N$ (see Figure 4).

Theorem 3.3. *Assume that there exist strict lower and upper solutions α_2, β_1, and lower and upper solutions α_1, β_2 of BVP (1.1), satisfying*

$$\alpha_1''(t) \le \alpha_2''(t) \le \beta_2''(t), \quad \alpha_1''(t) \le \beta_1''(t) \le \beta_2'(t), \quad \alpha_2''(t) \nleq \beta_1''(t) \quad \textit{for all } t \in [0, +\infty). \tag{3.19}$$

Suppose further that $f\colon [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous satisfying Nagumo's condition with respect to the pair of functions α_1, β_2, and

$$f(t, \alpha_1(t), z, u, w) \le f(t, y, z, u, w) \le f(t, \beta_2(t), z, u, w) \tag{3.20}$$

and

$$f(t, y, \alpha_1'(t), u, w) \le f(t, y, z, u, w) \le f(t, y, \beta_2'(t), u, w) \tag{3.21}$$

for $(t,y,z,u,w) \in [0,\eta) \times [\alpha(t),\beta(t)] \times [\beta'(t),\alpha'(t)] \times [\alpha''(t),\beta''(t)] \times \mathbb{R}$ *and* $(t,y,z,u,w) \in [\eta,+\infty) \times [\alpha(t),\beta(t)] \times [\alpha'(t),\beta'(t)] \times [\alpha''(t),\beta''(t)] \times \mathbb{R}$. *If* (3.3) *and* (3.4) *hold then* (1.1) *has at least three solutions* $x_1, x_2, x_3 \in X \cap C^4(0,+\infty)$ *satisfying*

$$\begin{aligned}
&\alpha_1(t) \le x_1(t) \le \beta_1(t), \quad \alpha_2(t) \le x_2(t) \le \beta_2(t), \quad t \in [0,+\infty),\\
&\beta_i'(t) \le x_i'(t) \le \alpha_i'(t), \quad t \in [0,\eta), \quad \alpha_i'(t) \le x_i'(t) \le \beta_i'(t), \quad t \in [\eta,+\infty), \quad i = 1,2,\\
&\alpha_1''(t) \le x_1''(t) \le \beta_1''(t), \quad \alpha_2''(t) \le x_2''(t) \le \beta_2''(t), \quad t \in [0,+\infty),\\
&\alpha_1(t) \le x_3(t) \le \beta_2(t), \quad t \in [0,+\infty),\\
&\beta_2'(t) \le x_3'(t) \le \alpha_1'(t), \quad t \in [0,\eta), \quad \alpha_1'(t) \le x_3'(t) \le \beta_2'(t), \quad t \in [\eta,+\infty),\\
&\alpha_1''(t) \le x_3''(t) \le \beta_2''(t), \quad x_3''(t) \ge \beta_1''(t), \quad x_3''(t) \le \alpha_2''(t), \quad t \in [0,+\infty).
\end{aligned}$$

Proof. We define auxiliary functions $\tilde{f}_0$, $\tilde{f}_1$, $\tilde{f}^*$ same as f_0, f_1, f^* in Theorem 3.1 except α and β replaced by α_1 and β_2, respectively. We consider the modified problem

$$\begin{aligned}
&x''''(t) + q(t)\tilde{f}^*(t,x(t),x'(t),x''(t),x'''(t)) = 0, \quad t \in (0,+\infty),\\
&x''(0) = A, \quad x(\eta) = B_1, \quad x'(\eta) = B_2, \quad x'''(+\infty) = C.
\end{aligned} \tag{3.22}$$

We want to show that (3.22) has at least three solutions. For this we define two operators as

$$(\tilde{T}_1x)(t) = \int_0^\infty G(t,s)q(s)\tilde{f}^*(s,x(s),x'(s),x''(s),x'''(s))\,ds, \quad t \in [0,+\infty)$$

and

$$(\tilde{T}x)(t) = B_1 + \left(B_2 - A\eta - \frac{C\eta^2}{2}\right)(t-\eta) + \frac{A}{2}(t^2-\eta^2) + \frac{C}{6}(t^3-\eta^3) + (\tilde{T}_1x)(t),\ t \in [0,+\infty).$$

As for T in Theorem 3.1 we can show that $\tilde{T}: X \to X$ is completely continuous. Now by using the degree theory, we will show that $\tilde{T}$ has at least three fixed points which are solutions of (3.22). For $x \in X$, as in Theorem 3.1, we have

$$\begin{aligned}
&\|\tilde{T}x\|_1 \le |B_1| + \left|B_2 - A\eta - \frac{C\eta^2}{2}\right| + \frac{|A|}{2} + \frac{C}{6} + \max\left\{1, \tfrac{7\eta^2}{3}\right\} \int_0^\infty sq(s)(H_3\phi(s)+1)\,ds =: Q_1,\\
&\|\tilde{T}x\|_2 \le \left|B - A\eta - \frac{C\eta^2}{2}\right| + |A| + \frac{C}{2} + \max\left\{\tfrac{1}{2}, \eta\right\} \int_0^\infty sq(s)(H_3\phi(s)+1)\,ds =: Q_2,\\
&\|\tilde{T}x\|_3 \le |A| + C + \int_0^\infty sq(s)(H_3\phi(s)+1)\,ds =: Q_3,\\
&\|\tilde{T}x\|_4 \le C + \int_0^\infty q(s)(H_3\phi(s)+1)\,ds =: Q_4,
\end{aligned}$$

where

$$H_3 = \sup_{0 \le t \le \|x\|_4} h(t) < +\infty.$$

Let $\Delta = \{x \in X : \|x\| < K\}$ where $K > \max\{Q_1, Q_2, Q_3, Q_4\}$. Then we have $\|\tilde{T}x\| < K$, which implies that $\tilde{T}\overline{\Delta} \subset \Delta$. Thus, $\deg(I - \tilde{T}, \Delta, 0) = 1$. Next, we set

$$\Delta_{\alpha_2} = \{x \in \Delta : x''(t) > \alpha_2''(t),\ t \in [0,+\infty)\}, \qquad \Delta^{\beta_1} = \{x \in \Delta : x''(t) < \beta_1''(t),\ t \in [0,+\infty)\}.$$

Since $\alpha_1''(t) \le \alpha_2''(t) \le \beta_2''(t)$, $\alpha_1''(t) \le \beta_1''(t) \le \beta_2''(t)$, $\alpha_2''(t) \not\le \beta_1''(t)$ and $\alpha_2''(t) \not\le \beta_1''(t)$, $t \in [0,+\infty)$, we find $\Delta_{\alpha_2} \ne \emptyset \ne \Delta^{\beta_1}$ and $\overline{\Delta}_{\alpha_2} \cap \overline{\Delta^{\beta_1}} = \emptyset$ whereas the set $\Delta \setminus \overline{\Delta_{\alpha_2} \cup \Delta^{\beta_1}} \ne \emptyset$. Hence in view of the strict upper and lower solutions β_1 and α_2, $\widetilde{T}$ has no solution in $\partial\Delta_{\alpha_2} \cup \partial\Delta^{\beta_1}$. The additivity of degree implies that

$$\deg(I - \widetilde{T}, \Delta, 0) = \deg(I - \widetilde{T}, \Delta \setminus \overline{\Delta_{\alpha_2} \cup \Delta^{\beta_1}}, 0) + \deg(I - \widetilde{T}, \Delta_{\alpha_2}, 0) + \deg(I - \widetilde{T}, \Delta^{\beta_1}, 0). \quad (3.23)$$

Now we shall show that $\deg(I - \widetilde{T}, \Delta_{\alpha_1}, 0) = 1$. For this, we define a completely continuous operator $\widehat{T}_1, \widehat{T} \colon \overline{\Delta} \to \overline{\Delta}$ by

$$(\widehat{T}_1 x)(t) = \int_0^{\infty} G(t,s) q(s) \widetilde{f}^*(s, x(s), x'(s), x''(s), x'''(s))\, ds, \qquad t \in [0, +\infty)$$

and

$$\begin{aligned}(\widehat{T}x)(t) = B_1 &+ \left(B_2 - A\eta - \frac{C\eta^2}{2}\right)(t - \eta) \\ &+ \frac{A}{2}(t^2 - \eta^2) + \frac{C}{6}(t^3 - \eta^3) + (\widehat{T}_1 x)(t), \qquad t \in [0, +\infty).\end{aligned}$$

where the functions $\widehat{f}_0$, $\widehat{f}_1$, $\widehat{f}^*$ are the same as $\widetilde{f}_0$, $\widetilde{f}_1$, $\widetilde{f}^*$ except α_1 is replaced by α_2. Again as in Theorem 3.1 it is easy to show that $\widehat{T}$ has a fixed point x satisfying $\alpha_2''(t) \le x''(t) \le \beta_2''(t)$, $t \in [0, +\infty)$. Since the lower solution α_2 is strict, $x''(t) \ne \alpha_2''(t)$, $t \in [0, +\infty)$. Therefore, $x \in \Delta_{\alpha_2}$. Hence, it follows that

$$\deg(I - \widehat{T}, \Delta \setminus \overline{\Delta}_{\alpha_2}, 0) = 0.$$

Further, we can show that $\widehat{T}\overline{\Delta} \subset \Delta$. Then we have

$$\deg(I - \widehat{T}, \Delta, 0) = 1. \quad (3.24)$$

Since $\widehat{f}^* = f$ in the region Δ_{α_2}, we find

$$\begin{aligned}\deg(I - \widetilde{T}, \Delta_{\alpha_2}, 0) &= \deg(I - \widehat{T}, \Delta_{\alpha_2}, 0) \\ &= \deg(I - \widehat{T}, \Delta_{\alpha_2}, 0) + \deg(I - \widehat{T}, \Delta \setminus \overline{\Delta}_{\alpha_2}, 0) \\ &= \deg(I - \widehat{T}, \Delta, 0) = 1.\end{aligned}$$

Similar to the proof of (3.25), we also have

$$\deg(I - \widetilde{T}, \Delta^{\beta_1}, 0) = 1. \quad (3.25)$$

Thus from (3.23), (3.24) and (3.25), we obtain

$$\deg(I - \widetilde{T}, \Delta \setminus \overline{\Delta_{\alpha_2} \cup \Delta^{\beta_1}}, 0) = -1.$$

Therefore, $\widetilde{T}$ has at least three fixed points $x_1 \in \Delta^{\beta_1}$, $x_2 \in \Delta_{\alpha_2}$, $x_3 \in \Delta \setminus \overline{\Delta_{\alpha_2} \cup \Delta^{\beta_1}}$ which are solutions of the problem (1.1). This completes the proof. □

Example 3.4. Consider the fourth-order three-point boundary value problem

$$\begin{aligned}&x''''(t) + \frac{1}{(1+t)^7} f(t, x(t), x'(t), x''(t), x'''(t)) = 0, \qquad t \in (0, +\infty) \\ &x''(0) = -1, \qquad x(1) = 16, \qquad x'(1) = 16, \qquad x''(+\infty) = 11,\end{aligned} \quad (3.26)$$

where

$$\begin{aligned}&f(t, x(t), x'(t), x''(t), x'''(t)) \\ &\quad = \left[(7 - x'''(t)) + \frac{(18^2 - x'''(t)^2)(x'''(t) - 12)}{x'''(t)^2 + 18^2}[(1 + x''(t)) + (x'(t) - 1)^2 + (x(t) + 1)]\right].\end{aligned}$$

Here, $q(t) = \frac{1}{(1+t)^7}$, $f(t,y,z,u,w) = (7-w) + \frac{(18^2-w^2)(w-12)}{w^2+18^2}[(1+u)+(z-1)^2+(y+1)]$, $\eta = 1$, $A = -1$, $B_1 = 16$, $B_2 = 16$, $C = 11$. It is easy to check that $\alpha_1(t) = -3t^3 - t^2 + 27t - 13$, $\beta_2(t) = 3t^3 + 6t^2 - 5t + 23$ are lower and upper solutions of (3.26) and $\alpha_2(t) = t^3 + 6t^2 + t + 7$, $\beta_1(t) = 2t^3 + 2t^2 + 6t + 5$ are strict lower and upper solutions of (3.26). Further, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in X$ and satisfy (3.19). Clearly, f is continuous. Moreover, with respect to $\alpha_1(t) = -3t^3 - t^2 + 27t - 13$ and $\beta_2(t) = 3t^3 + 6t^2 - 5t + 23$, when $0 \le t < +\infty$,

$$\begin{aligned} -3t^3 - t^2 + 27t - 13 = \alpha_1(t) \le y \le \beta_2(t) = 3t^3 + 6t^2 - 5t + 23, \qquad & t \in [0,+\infty), \\ \beta_2'(t) = 9t^2 + 12t - 5 \le z \le \alpha_1'(t) = -9t^2 - 2t + 27, \qquad & t \in [0,1), \\ -9t^2 - 2t + 27 = \alpha_1'(t) \le z \le \beta_2'(t) = 9t^2 + 12t - 5, \qquad & t \in [1,+\infty), \\ -18t - 2 = \alpha_1''(t) \le u \le \beta_2''(t) = 18t + 12 & \end{aligned}$$

and $w \in \mathbb{R}$, f satisfies

$$|f(t,y,z,u,w)| \le \phi(t)h(|w|),$$

where $\phi(t) = 713 + 131t + 42t^2 + 219t^3 + 81t^4$, $h(w) = 12 + w$, and hence

$$\int_0^\infty \frac{s}{h(s)}\,ds = \int_0^\infty \frac{s}{12+s}\,ds = +\infty.$$

We also have

$$\sup_{t\in[0,+\infty)} (1+t)^\gamma \frac{713 + 131t + 42t^2 + 219t^3 + 81t^4}{(1+t)^7} < +\infty,$$

if $1 < \gamma \le 2$ and

$$\int_0^\infty \max\{s,1\} q(s)\,ds = \int_0^1 \frac{1}{(1+s)^7} ds + \int_1^\infty \frac{s}{(1+s)^7}\,ds < +\infty,$$

$$\begin{aligned} \int_0^\infty \max\{s,1\}\phi(s)q(s)\,ds = & \int_0^1 \frac{713 + 131s + 42s^2 + 219s^3 + 81s^4}{(1+s)^7}\,ds \\ & + \int_1^\infty \frac{s(713 + 131s + 42s^2 + 219s^3 + 81s^4)}{(1+s)^7}\,ds < +\infty; \end{aligned}$$

that is, (3.3) and (3.4) are also satisfied. Clearly, f is increasing in y, decreasing in z on $[0,1) \times [\alpha_1(t), \beta_2(t)] \times [\beta_2'(t), \alpha_1'(t)] \times [\alpha_1''(t), \beta_2''(t)] \times \mathbb{R}$ and increasing on $[1,+\infty) \times [\alpha_1(t), \beta_2(t)] \times [\beta_1'(t), \alpha_2'(t)] \times [\alpha_1''(t), \beta_2''(t)] \times \mathbb{R}$ in y, z. Thus f satisfies condition (3.20) and (3.21). Therefore, Theorem 3.3 confirms that the boundary value problem (3.26) has at least three solutions.

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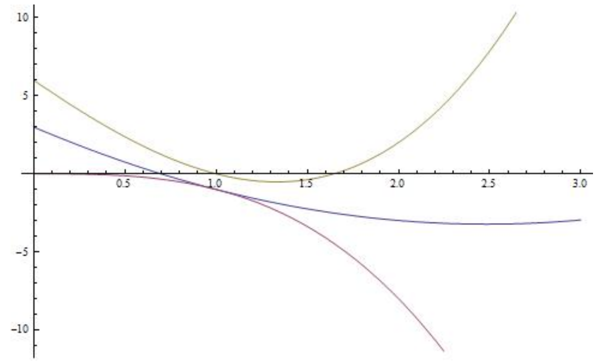

Figure 1: The solution $x(t)$ of the BVP (3.18) and $\alpha(t) = -t^3,\ \beta(t) = t^3 + t^2 - 8t + 6$

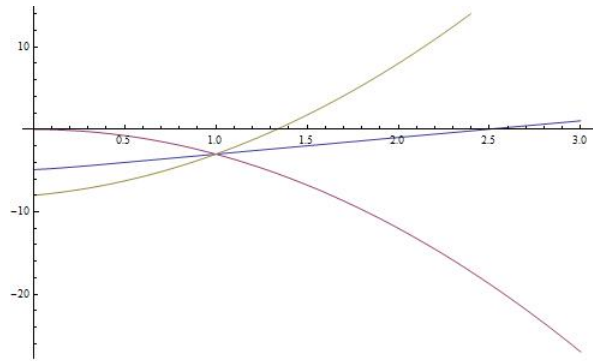

Figure 2: The solution derivative $x'(t)$ of the BVP (3.18) and $\alpha'(t) = -3t^2,\ \beta'(t) = 3t^2 + 2t - 8$

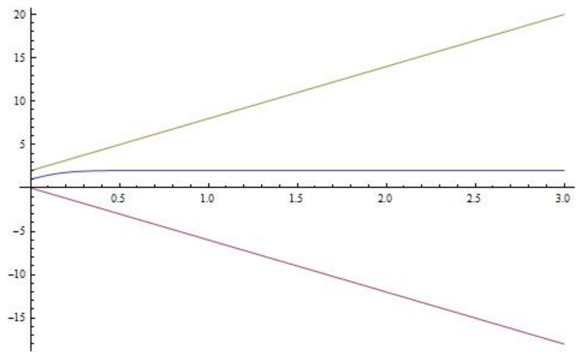

Figure 3: The solution second derivative $x''(t)$ of the BVP (3.18) and $\alpha''(t) = -6t,\ \beta''(t) = 6t + 2$

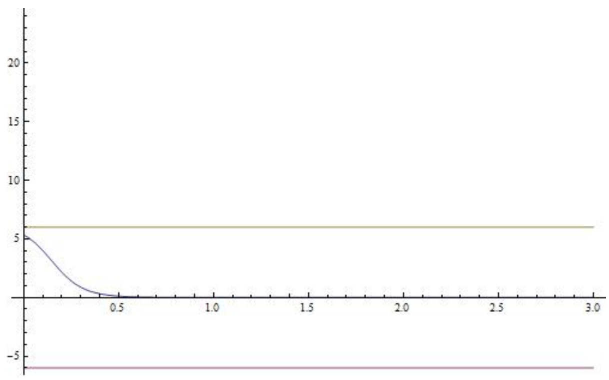

Figure 4: The solution third derivative of the BVP (3.18) satisfies $|x'''(t)| < N$