Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **20**, 1–12; http://www.math.u-szeged.hu/ejqtde/

Global existence and asymptotic behavior of solutions for a system of higher-order Kirchhoff-type equations

Yaojun Ye ✉

Department of Mathematics and Information Science, Zhejiang University of Science and Technology, Hangzhou 310023, P.R. China

Received 13 September 2014, appeared 25 March 2015
Communicated by Patrizia Pucci

Abstract. This paper deals with the global existence and energy decay of solutions for some coupled system of higher-order Kirchhoff-type equations with nonlinear dissipative and source terms in a bounded domain. We prove the existence of global solutions for this problem by constructing a stable set in $H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)$ and give the decay estimate of global solutions by applying a lemma of V. Komornik.

Keywords: system of higher-order Kirchhoff-type equations, initial-boundary value problem, global solutions, asymptotic behavior.

2010 Mathematics Subject Classification: 35B40, 35G20, 35L55, 35A05, 35L70.

1 Introduction

In this paper we investigate the following system of nonlinear higher-order Kirchhoff-type equations

$$u_{tt} + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)(-\Delta)^{m_1}u + a|u_t|^{q-2}u_t = f_1(u,v), \qquad x \in \Omega,\ t > 0, \tag{1.1}$$
$$v_{tt} + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)(-\Delta)^{m_2}v + a|v_t|^{q-2}v_t = f_2(u,v), \qquad x \in \Omega,\ t > 0, \tag{1.2}$$

with initial data

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{1.3}$$
$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega, \tag{1.4}$$

and boundary value

$$\frac{\partial^i u}{\partial \nu^i} = 0, \qquad i = 0,1,2,\dots,m_1 - 1, \quad x \in \partial\Omega,\ t \ge 0, \tag{1.5}$$
$$\frac{\partial^j v}{\partial \nu^j} = 0, \qquad j = 0,1,2,\dots,m_2 - 1, \quad x \in \partial\Omega,\ t \ge 0, \tag{1.6}$$

✉ Email: yjye2013@163.com

where $a > 0$ and $q \geq 2$ are real numbers and $m_i \geq 1$ $(i = 1,2)$ are positive integers. $\Phi(s)$ is a positive locally Lipschitz function like $\Phi(s) = \alpha + \beta s^\gamma$ with the constants $\alpha > 0$, $\beta \geq 0$, $\gamma \geq 1$ and $s \geq 0$. Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied, ν denotes the unit outward normal vector on $\partial\Omega$, and $\frac{\partial^i}{\partial\nu^i}$ denotes the ith order normal derivation. D denotes the gradient operator, that is $Du = \nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$. Moreover, $D^m u = \Delta^k u$ if $m = 2k$ and $D^m u = D\Delta^k u$ if $m = 2k+1$. $f_i(\cdot,\cdot)\colon \mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$ are given functions to be determined later.

When $m_1 = m_2 = 1$, (1.1)–(1.6) becomes the following initial-boundary value problem for the system of nonlinear wave equations of Kirchhoff-type:

$$u_{tt} - \Phi(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + a|u_t|^{q-2}u_t = f_1(u,v), \qquad x \in \Omega,\ t > 0, \tag{1.7}$$

$$v_{tt} - \Phi(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + a|v_t|^{q-2}v_t = f_2(u,v), \qquad x \in \Omega,\ t > 0, \tag{1.8}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{1.9}$$

$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega, \tag{1.10}$$

$$u(x,t) = v(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0. \tag{1.11}$$

The equation (1.7)–(1.8) has its origin in the nonlinear vibrations of an elastic string [19]. Many authors have investigated the global existence and uniqueness of solutions to the problem related to the system (1.7)–(1.8) through various approaches and assumptive conditions. L. Liu and M. Wang [15] have dealt with the global existence for regular and weak solutions for the problem (1.7)–(1.11) by using Galerkin method. When the initial energy $E(0)$ is non-positive or positive, applying the concavity method [12, 13] and the potential well method [3, 26, 28], they proved the blow-up of solutions in finite time, and give some estimates for the lifespan of solutions. When $\Phi(s) = s^\gamma$, $\gamma \geq 1$, J. Y. Park and J. J. Bae [22] studied the existence and uniform decay of strong solutions of the problem (1.7)–(1.11). In [23, 24], they showed the global existence and asymptotic behavior of solutions of the problem (1.7)–(1.11) under some restrictions on the initial energy. S. T. Wu and L. Y. Tsai [30] considered the system (1.7)–(1.11) with $\Phi(\|\nabla u\|^2 + \|\nabla v\|^2) = \Phi(\|\nabla u\|^2)$ in (1.7) and $\Phi(\|\nabla u\|^2 + \|\nabla v\|^2) = \Phi(\|\nabla v\|^2)$ in (1.8), respectively. They obtain the existence of local and global solutions and give the blow-up result for small positive initial energy. When nonlinear dissipative terms in (1.7) and (1.8) become the strong dissipative terms, S. T. Wu [31] discusses the existence, asymptotic behavior and blow-up of solutions of the problem (1.7)–(1.11) under some conditions. Moreover, he gives the decay estimates of the energy function and the estimates for the lifespan of solutions.

For the initial boundary value problem of a single nonlinear higher-order wave equation of Kirchhoff-type

$$u_{tt} + \Phi(\|D^m u\|^2)(-\Delta)^m u + a|u_t|^{q-2}u_t = b|u|^{p-2}u, \qquad x \in \Omega,\ t > 0, \tag{1.12}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{1.13}$$

$$\frac{\partial^i u}{\partial \nu^i} = 0, \qquad i = 0,1,2,\ldots,m-1, \qquad x \in \partial\Omega,\ t \geq 0. \tag{1.14}$$

The original physical models governed by (1.12) are vibrating beams of the Woinowsky–Krieger type with a nonlinear damping $a|u_t|^{q-2}u_t$ effective in Ω, but without internal material damping term of the Kelvin–Voigt type [4, 10, 25]. G. Autuori et al. [4] studied the asymptotic stability for solutions of the equation (1.12)–(1.14). Q. Gao et al. [7] proved the local existence and the blow-up property of solution for the problem (1.12)–(1.14).

When $\Phi(s) = \beta s^\gamma$ in (1.12), F. C. Li [14] investigated the problem (1.12)–(1.14) and obtained that the solution exists globally if $p \leq q$, while if $p > \max\{q, 2\gamma\}$, then for any initial data with negative initial energy, the solution blows up at finite time in $L^{\gamma+2}$ norms. Later, S. A. Messaoudi and B. Said-Houari [18] improved the results in [14] by modification of the proof and showed the same result when the initial energy has an upper bound. Meanwhile, V. A. Galaktionov and S. I. Pohozaev [6] proved the global existence and nonexistence results of solutions for the Cauchy problem of equation (1.12) without the dissipation (i.e., (1.12) without the term $a|u_t|^{q-2}u_t$) in the whole space $\mathbb{R}^n$. However, their approach can not be applied to the problem (1.12)–(1.14).

Motivated by the above researches, in this paper, we prove the global existence for the problem (1.1)–(1.6) by constructing a stable set in $H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)$ and give the energy decay of global solutions by applying a lemma of V. Komornik [11].

We adopt the usual notations and convention. Let $H^m(\Omega)$ denote the Sobolev space with the usual scalar products and norm. Meanwhile, $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|\cdot\|_r$ the Lebesgue space $L^r(\Omega)$ norm and $\|\cdot\|$ denotes $L^2(\Omega)$ norm, we write equivalent norm $\|D^m\cdot\|$ instead of $H_0^m(\Omega)$ norm $\|\cdot\|_{H_0^m(\Omega)}$ (see [2,5,8]). Moreover, C_i $(i = 0, 1, 2, 3, \dots)$ denotes various positive constants which depend on the known constants and may be different at each appearance.

This paper is organized as follows: in the next section, we give some preliminaries. In Section 3, we prove the existence of global solutions for problem (1.1)–(1.6). The Section 4 is devoted to the study of the energy decay of global solutions.

2 Preliminaries

To state and prove our main results, we make the following assumptions:

(A1) $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1-class locally Lipschitz function satisfying

$$\Phi(s) \geq \alpha, \qquad s\Phi(s) \geq \int_0^s \Phi(\theta)\,d\theta.$$

(A2) p satisfies

$$\begin{aligned} &1 < p < +\infty, && n \leq 2\min(m_1, m_2),\\ &1 < p \leq \min\left(\frac{n}{n-2m_1}, \frac{n}{n-2m_2}\right), && n > 2\max(m_1, m_2). \end{aligned}$$

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we assume that

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(p-1)}(u+v) + b_2|u|^{p-2}u|v|^p,\\ f_2(u,v) &= b_1|u+v|^{2(p-1)}(u+v) + b_2|v|^{p-2}v|u|^p, \end{aligned} \tag{2.1}$$

where $b_1, b_2 > 0$ and $p > 1$ are constants.

It is easy to see that

$$uf_1(u,v) + vf_2(u,v) = 2pF(u,v), \quad \forall (u,v) \in \mathbb{R}^2, \tag{2.2}$$

where

$$F(u,v) = \frac{b_1}{2p}|u+v|^{2p} + \frac{b_2}{p}|uv|^p. \tag{2.3}$$

Moreover, a quick computation will show that there exist two positive constants C_0 and C_1 such that the following inequality holds (see [17])

$$\frac{C_0}{2p}(|u|^{2p}+|v|^{2p}) \le F(u,v) \le \frac{C_1}{2p}(|u|^{2p}+|v|^{2p}). \tag{2.4}$$

Now, we define the following functionals:

$$J([u,v]) = \frac{1}{2}\int_0^{\|D^{m_1}u\|^2+\|D^{m_2}v\|^2} \Phi(s)\,ds - \int_\Omega F(u,v)\,dx, \tag{2.5}$$

$$K([u,v]) = \int_0^{\|D^{m_1}u\|^2+\|D^{m_2}v\|^2} \Phi(s)\,ds - 2p\int_\Omega F(u,v)\,dx, \tag{2.6}$$

for $[u,v] \in H_0^{m_1}(\Omega)\times H_0^{m_2}(\Omega)$.

Then we can define the stable set W of the problem (1.1)–(1.6) as follows

$$W = \{[u,v] \in H_0^{m_1}(\Omega)\times H_0^{m_2}(\Omega) : K([u,v]) > 0\} \cup \{[0,0]\}.$$

We denote the total energy related to the equations (1.1) and (1.2) by

$$\begin{aligned} E(t) &= \frac{1}{2}(\|u_t\|^2+\|v_t\|^2) + \frac{1}{2}\int_0^{\|D^{m_1}u\|^2+\|D^{m_2}v\|^2} \Phi(s)\,ds - \int_\Omega F(u,v)\,dx \\ &= \frac{1}{2}(\|u_t\|^2+\|v_t\|^2) + J([u,v]) \end{aligned} \tag{2.7}$$

for $[u,v] \in H_0^{m_1}(\Omega)\times H_0^{m_2}(\Omega),\ t \ge 0$ and

$$E(0) = \frac{1}{2}(\|u_1\|^2+\|v_1\|^2) + \frac{1}{2}\int_0^{\|D^{m_1}u_0\|^2+\|D^{m_2}v_0\|^2} \Phi(s)\,ds - \int_\Omega F(u_0,v_0)\,dx \tag{2.8}$$

is the initial total energy.

We state some known lemmas which will be needed later.

Lemma 2.1. *Let r be a number with $2 \le r < +\infty$ if $n \le 2m$ and $2 \le r \le \frac{2n}{n-2m}$ if $n > 2m$. Then there is a constant C depending on Ω and r such that*

$$\|u\|_r \le C\left\|(-\Delta)^{\frac{m}{2}}u\right\| = C\|D^m u\|, \qquad \forall u \in H_0^m(\Omega).$$

Lemma 2.2 (Young's inequality)**.** *Let X, Y and ε be positive constants and $\varsigma,\ \sigma \ge 1$, $\frac{1}{\varsigma}+\frac{1}{\sigma}=1$. Then one has the inequality*

$$XY \le \frac{\varepsilon^\varsigma X^\varsigma}{\varsigma} + \frac{Y^\sigma}{\sigma\varepsilon^\sigma}.$$

Lemma 2.3. *Let $[u,v]$ be a solution of the problem (1.1)–(1.6), then $E(t)$ is a non-increasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -a(\|u_t\|_q^q + \|v_t\|_q^q) \le 0. \tag{2.9}$$

Proof. Multiplying equation (1.1) by u_t and (1.2) by v_t, and integrating over $\Omega\times[0,t]$, then, adding them together, and integrating by parts, we get

$$E(t) - E(0) = -a\int_0^t \left(\|u_t(s)\|_q^q + \|v_t(s)\|_q^q\right) ds \tag{2.10}$$

for $t \ge 0$.

Being the primitive of an integrable function, $E(t)$ is absolutely continuous and equality (2.9) is satisfied. □

The local existence and uniqueness of solutions for the problem (1.1)–(1.6) can be obtained by a similar way as done in [1,7,16,20,21,27,32]. The result reads as follows.

Theorem 2.4 (Local existence)**.** *Suppose that the assumptions* **(A1)** *and* **(A2)** *hold. If* $[u_0, v_0] \in (H_0^{m_1}(\Omega) \cap H^{2m_1}(\Omega)) \times (H_0^{m_2}(\Omega) \cap H^{2m_2}(\Omega))$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$, *then there exists* $T > 0$ *such that the problem* (1.1)–(1.6) *has a unique local solution* $[u, v]$ *which satisfies*

$$\begin{aligned}
[u, v] &\in C([0, T); \quad H_0^{m_1}(\Omega) \times H_0^{m_2}(\Omega)),\\
u_t &\in C([0, T); \quad L^2(\Omega)) \cap L^q(\Omega \times [0, T)),\\
v_t &\in C([0, T); \quad L^2(\Omega)) \cap L^q(\Omega \times [0, T)).
\end{aligned}$$

Moreover, at least one of the following statements holds true:

(1) $\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2 \to \infty$ *as* $t \to T^-$;

(2) $T = +\infty$.

3 Global existence of solutions

The following lemmas play an important role in the proof of global existence of solutions.

Lemma 3.1. *If* $[u, v] \in W$, *then*

$$\frac{p-1}{2p} \int_0^{\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2} \Phi(s)\, ds < J([u, v]). \tag{3.1}$$

Proof. By (2.5) and (2.6), we have the following equality

$$J([u, v]) = \frac{p-1}{2p} \int_0^{\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2} \Phi(s)\, ds + \frac{1}{2p} K([u, v]). \tag{3.2}$$

Since $[u, v] \in W$, so we get $K([u, v]) > 0$. Therefore, by (3.2), we find that (3.1) is valid. □

Lemma 3.2. *Let* **(A1)** *and* **(A2)** *hold. If* $[u_0, v_0] \in W$ *and* $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$ *such that*

$$\eta = \frac{C_2 B^{2p}}{\alpha} \left[\frac{2p}{(p-1)\alpha} E(0) \right]^{p-1} < 1, \tag{3.3}$$

where C_2 *is given by* (3.10), *then* $[u, v] \in W$, *for each* $t \in [0, T)$.

Proof. Since $[u_0, v_0] \in W$, so $K([u_0, v_0]) > 0$. Then it follows from the continuity of $[u, v]$ on t that

$$K([u, v]) \geq 0, \tag{3.4}$$

for some interval near $t = 0$. Let $\tau > 0$ be a maximal time (possibly $\tau = T$), when (3.4) holds on $[0, \tau)$.

We have from **(A1)** and (3.1) that

$$\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2 \leq \frac{2p}{(p-1)\alpha} J([u, v]). \tag{3.5}$$

It follows from (2.7), (3.5) and (2.9) in Lemma 2.3 that

$$\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2 \leq \frac{2p}{(p-1)\alpha} E(t) \leq \frac{2p}{(p-1)\alpha} E(0), \tag{3.6}$$

for $\forall t \in [0,\tau)$.

By Minkowski's inequality and Lemma 2.1, we get that

$$\|u+v\|_{2p}^2 \le 2(\|u\|_{2p}^2 + \|v\|_{2p}^2) \le 2B^2(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2), \tag{3.7}$$

where $B = \max(B_1, B_2)$ and B_i $(i = 1,2)$ is the optimal Sobolev's constant from $H_0^{m_i}(\Omega)$ $(i = 1,2)$ to $L^{2p}(\Omega)$.

Also, we have from Hölder's inequality, Lemma 2.1 and Lemma 2.2 that

$$\|uv\|_p \le \|u\|_{2p} \cdot \|v\|_{2p} \le \frac{1}{2}(\|u\|_{2p}^2 + \|v\|_{2p}^2) \le \frac{B^2}{2}(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2). \tag{3.8}$$

We get from **(A1)**, (2.3), (3.3) in Lemma 3.2, (3.6)–(3.8) that

$$\begin{aligned} 2p\int_\Omega F(u,v)\,dx &\le C_2B^{2p}(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)^p \\ &\le C_2B^{2p}\left[\frac{2p}{(p-1)\alpha}E(0)\right]^{p-1}(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) \\ &\le \frac{C_2B^{2p}}{\alpha}\left[\frac{2p}{(p-1)\alpha}E(0)\right]^{p-1}\alpha(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) \\ &< \alpha(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) \le \int_0^{\|D^{m_1}u\|^2+\|D^{m_2}v\|^2}\Phi(s)\,ds, \end{aligned} \tag{3.9}$$

for all $t \in [0,\tau)$. Here

$$C_2 = 2^p b_1 + \frac{b_2}{2^{p-1}}. \tag{3.10}$$

Therefore,

$$\int_0^{\|D^{m_1}u\|^2+\|D^{m_2}v\|^2}\Phi(s)\,ds - 2p\int_\Omega F(u,v)\,dx > 0, \qquad \forall t \in [0,\tau), \tag{3.11}$$

which implies that $[u,v] \in W$ for $\forall t \in [0,\tau)$. By repeating this procedure (3.5)–(3.11), and using the fact that

$$\lim_{t\to\tau}\frac{C_2B^{2p}}{\alpha}\left[\frac{2p}{(p-1)\alpha}E(t)\right]^{p-1} < 1,$$

τ is extended to T. Thus, we conclude that $[u,v] \in W$ on $[0,T)$. □

The main result in this section reads as follows.

Theorem 3.3 (Global solutions)**.** *Suppose that* (3.3), **(A1)** *and* **(A2)** *hold, and* $[u,v]$ *is a local solution of problem* (1.1)–(1.6) *on* $[0,T)$. *If* $[u_0,v_0] \in W$, $[u_1,v_1] \in L^2(\Omega)\times L^2(\Omega)$, *then* $[u,v]$ *is a global solution of the problem* (1.1)–(1.6).

Proof. It suffices to show that $\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2$ is bounded independently of t. Under the hypotheses in Theorem 3.3, we get from Lemma 3.2 that $[u,v] \in W$ on $[0,T)$. So the formula (3.5) holds on $[0,T)$. Thus, we have from (3.5) that

$$\begin{aligned} &\frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{(p-1)\alpha}{2p}(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) \\ &\le \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + J([u,v]) = E(t) \le E(0). \end{aligned} \tag{3.12}$$

Therefore, we get

$$\|u_t\|^2 + \|v_t\|^2 + \|D^{m_1}u\|^2 + \|D^{m_2}v\|^2 \le \max\left(2, \frac{2p}{(p-1)\alpha}\right)E(0) < +\infty.$$

The above inequality and the standard continuation principle [9, 29] lead to the global existence of the solution, that is, $T = +\infty$. Hence, the solution $[u, v]$ is a global solution of the problem (1.1)–(1.6). □

4 Energy decay of global solution

In order to study the decay estimate of global solutions for the problem (1.1)–(1.6), we need the following lemma.

Lemma 4.1 ([6]). *Let $Y(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and assume that there are two constants $\eta \ge 1$ and $M > 0$ such that*

$$\int_\tau^{+\infty} Y(t)^{\frac{\eta+1}{2}}\,dt \le MY(\tau), \qquad 0 \le \tau < +\infty,$$

then $Y(t) \le CY(0)(1+t)^{-\frac{2}{\eta-1}}$, $\forall t \ge 0$, if $\eta > 1$ and $Y(t) \le CY(0)e^{-\omega t}$, $\forall t \ge 0$ if $\eta = 1$, where C and ω are positive constants independent of $Y(0)$.

The following result is concerned with the energy decay estimate of global solutions for the problem (1.1)–(1.6). The theorem reads as follows.

Theorem 4.2. *Under the assumptions of Theorem 3.3, we further supposed that q satisfies*

$$\begin{aligned} &2 < q < +\infty, && n \le 2\min(m_1, m_2), \\ &2 < q \le \min\left(\frac{2n}{n-2m_1}, \frac{2n}{n-2m_2}\right), && n > 2\max(m_1, m_2). \end{aligned} \tag{4.1}$$

If $[u_0, v_0] \in W$ and $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$ satisfy (3.3), then the global solution $[u, v]$ of the problem (1.1)–(1.6) have the following decay properties:

$$E(t) \le K(1+t)^{-\frac{2}{q-2}},$$

where $K > 0$ is a constant depending on initial energy $E(0)$.

Proof. Multiplying the equation (1.1) by $E(t)^{\frac{q-2}{2}}u$ and integrating over $\Omega \times [S, T]$, we obtain that

$$0 = \int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega u\Big[u_{tt} + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)(-\Delta)^{m_1}u + a|u_t|^{q-2}u_t - f_1(u, v)\Big]\,dx\,dt, \tag{4.2}$$

where $0 \le S < T < +\infty$.

Since

$$\begin{aligned} &\int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega uu_{tt}\,dx\,dt \\ &\quad = \left[E(t)^{\frac{q-2}{2}} \int_\Omega uu_t\,dx\right]_S^T - \int_S^T \left[\frac{q-2}{2}E(t)^{\frac{q-4}{2}}E'(t) + E(t)^{\frac{q-2}{2}}\right] \int_\Omega uu_t\,dx\,dt \\ &\qquad - \int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega |u_t|^2\,dx\,dt. \end{aligned} \tag{4.3}$$

So, substituting the formula (4.3) into the right-hand side of (4.2), we get that

$$\begin{aligned} 0 = &\int_S^T E(t)^{\frac{q-2}{2}} \left[\|u_t\|^2 + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)\|D^{m_1}u\|^2 \right] dt \\ &- \int_S^T \int_\Omega E(t)^{\frac{q-2}{2}} \left[2|u_t|^2 - a|u_t|^{q-2}u_t u \right] dx\, dt \\ &- \int_S^T \left[\frac{q-2}{2} E(t)^{\frac{q-4}{2}} E'(t) + E(t)^{\frac{q-2}{2}} \right] \int_\Omega u u_t \, dx\, dt \\ &+ \left[E(t)^{\frac{q-2}{2}} \int_\Omega u u_t \, dx \right]_S^T - \int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega u f_1(u,v)\, dx\, dt. \end{aligned} \tag{4.4}$$

Similarly, multiplying (1.2) by $E(t)^{\frac{q-2}{2}} v$ and integrating over $\Omega \times [S,T]$, we have

$$\begin{aligned} 0 = &\int_S^T E(t)^{\frac{q-2}{2}} \left[\|v_t\|^2 + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)\|D^{m_2}v\|^2 \right] dt \\ &- \int_S^T \int_\Omega E(t)^{\frac{q-2}{2}} \left[2|v_t|^2 - a|v_t|^{q-2}v_t v \right] dx\, dt \\ &- \int_S^T \left[\frac{q-2}{2} E(t)^{\frac{q-4}{2}} E'(t) + E(t)^{\frac{q-2}{2}} \right] \int_\Omega v v_t \, dx\, dt \\ &+ \left[E(t)^{\frac{q-2}{2}} \int_\Omega v v_t dx \right]_S^T - \int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega v f_2(u,v)\, dx\, dt. \end{aligned} \tag{4.5}$$

Taking the sum of (4.4) and (4.5), we obtain that

$$\begin{aligned} &\int_S^T E(t)^{\frac{q-2}{2}} \Big[\|u_t\|^2 + \|v_t\|^2 + \Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2) \\ &\qquad\qquad - 2\int_\Omega F(u,v)\, dx \Big] dt \\ &= -\left[E(t)^{\frac{q-2}{2}} \int_\Omega (u u_t + v v_t)\, dx \right]_S^T + \int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega 2(|u_t|^2 + |v_t|^2)\, dx\, dt \\ &\quad - \int_S^T E(t)^{\frac{q-2}{2}} \int_\Omega a(|u_t|^{q-2}u_t u + |v_t|^{q-2}v_t v)\, dx\, dt \\ &\quad + \frac{q-2}{2} \int_S^T \int_\Omega E(t)^{\frac{q-4}{2}} E'(t)(u u_t + v v_t)\, dx\, dt \\ &\quad + 2(p-1) \int_S^T \int_\Omega E(t)^{\frac{q-2}{2}} F(u,v)\, dx\, dt. \end{aligned} \tag{4.6}$$

We obtain from (3.3), (3.6) and (3.9) that

$$2(p-1)\int_\Omega F(u,v)\, dx \le 2\eta E(t). \tag{4.7}$$

We derive from **(A1)** that

$$\int_0^{\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2} \Phi(s)\, ds \le (\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2)\Phi(\|D^{m_1}u\|^2 + \|D^{m_2}v\|^2). \tag{4.8}$$

It follows from (4.6)–(4.8) that

$$\begin{aligned}2(1-\eta)\int_S^T E(t)^{\frac{q}{2}}\,dt \le\; & -\left[E(t)^{\frac{q-2}{2}}\int_\Omega (uu_t+vv_t)\,dx\right]_S^T \\ & + \int_S^T E(t)^{\frac{q-2}{2}}\int_\Omega 2(|u_t|^2+|v_t|^2)\,dx\,dt \\ & - \int_S^T E(t)^{\frac{q-2}{2}}\int_\Omega a(|u_t|^{q-2}u_t u+|v_t|^{q-2}v_t v)\,dx\,dt \\ & + \frac{q-2}{2}\int_S^T E(t)^{\frac{q-4}{2}}E'(t)\int_\Omega (uu_t+vv_t)\,dx\,dt \\ =\; & I_1+I_2+I_3+I_4.\end{aligned} \tag{4.9}$$

In the following, we estimate these terms I_i $(i=1,2,3,4)$ respectively.

Since $E(t)$ is non-increasing, we find from the Cauchy–Schwarz inequality and Lemma 2.1 that

$$\begin{aligned}I_1 = E(S)^{\frac{q-2}{2}}&\int_\Omega [u(S)u_t(S)+v(S)v_t(S)]\,dx \\ &- E(T)^{\frac{q-2}{2}}\int_\Omega [u(T)u_t(T)+v(T)v_t(T)]\,dx \le C_3E(S)^{\frac{q}{2}}.\end{aligned} \tag{4.10}$$

We get from Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned}I_2 &\le \int_S^T\int_\Omega \left[(\varepsilon_1+\varepsilon_2)E(t)^{\frac{q}{2}}+(C_{\varepsilon_1}+C_{\varepsilon_2})(|u_t|^q+|v_t|^q)\right]dx\,dt \\ &\le C_4(\varepsilon_1+\varepsilon_2)\int_S^T E(t)^{\frac{q}{2}}\,dt+(C_{\varepsilon_1}+C_{\varepsilon_2})\int_S^T (\|u_t\|_q^q+\|v_t\|_q^q)\,dt \\ &\le C_4(\varepsilon_1+\varepsilon_2)\int_S^T E(t)^{\frac{q}{2}}\,dt+\frac{(C_{\varepsilon_1}+C_{\varepsilon_2})}{a}\int_S^T (-E'(t))\,dt \\ &= C_4(\varepsilon_1+\varepsilon_2)\int_S^T E(t)^{\frac{q}{2}}\,dt-\frac{(C_{\varepsilon_1}+C_{\varepsilon_2})}{a}(E(T)-E(S)) \\ &\le C_4(\varepsilon_1+\varepsilon_2)\int_S^T E(t)^{\frac{q}{2}}\,dt+C_5E(S).\end{aligned} \tag{4.11}$$

It follows from the Cauchy–Schwarz inequality, Lemma 2.1 and (3.12) that

$$\begin{aligned}I_4 &\le \int_S^T \frac{q-2}{2}E(t)^{\frac{q-4}{2}}E'(t)\left|\int_\Omega (uu_t+vv_t)\,dx\right|dt \\ &\le \frac{q-2}{2}C_6\int_S^T E(t)^{\frac{q-4}{2}}(-E'(t))[\|D^{m_1}u\|\cdot\|u_t\|+\|D^{m_2}v\|\cdot\|v_t\|]\,dt \\ &\le \frac{q-2}{2}C_6\int_S^T E(t)^{\frac{q-2}{2}}(-E'(t))\,dt\le C_7E(S)^{\frac{q}{2}}.\end{aligned} \tag{4.12}$$

Now, we estimate the term I_3 in order to apply the results of Lemma 4.1. From Hölder's inequality, Lemma 2.1 and Lemma 2.2, we obtain that

$$\begin{aligned}I_3 &\le a\int_S^T E(t)^{\frac{q-2}{2}}\left[(\varepsilon_5+\varepsilon_6)(\|u\|_q^q+\|v\|_q^q)+(C_{\varepsilon_5}+C_{\varepsilon_6})(\|u_t\|_q^q+\|v_t\|_q^q)\right]dt \\ &\le aC^q(\varepsilon_5+\varepsilon_6)\int_S^T E(t)^{\frac{q-2}{2}}(\|D^{m_1}u\|^q+\|D^{m_2}v\|^q)\,dt \\ &\quad + a(C_{\varepsilon_5}+C_{\varepsilon_6})\int_S^T E(t)^{\frac{q-2}{2}}(\|u_t\|_q^q+\|v_t\|_q^q)\,dt \\ &= L_1+L_2.\end{aligned} \tag{4.13}$$

We have from (3.6) and Lemma 2.3 that

$$\begin{aligned} L_1 &\le aC^q \left[\frac{2pE(0)}{(p-1)\alpha}\right]^{\frac{q-2}{2}} (\varepsilon_5+\varepsilon_6) \int_S^T E(t)^{\frac{q}{2}}\,dt \\ &\le C_8(\varepsilon_5+\varepsilon_6) \int_S^T E(t)^{\frac{q}{2}}\,dt. \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} L_2 &\le a(C_{\varepsilon_5}+C_{\varepsilon_6}) \int_S^T E(t)^{\frac{q-2}{2}}(-E'(t))\,dt \\ &\le \frac{2a(C_{\varepsilon_5}+C_{\varepsilon_6})}{q}\left[E(S)^{\frac{q}{2}} - E(T)^{\frac{q}{2}}\right] \le C_9 E(S)^{\frac{q}{2}}. \end{aligned} \tag{4.15}$$

By combining (4.13)–(4.15), we get

$$I_3 \le C_8(\varepsilon_5+\varepsilon_6) \int_S^T E(t)^{\frac{q}{2}}\,dt + C_9 E(S)^{\frac{q}{2}}. \tag{4.16}$$

Therefore, it follows from (4.9)–(4.12) and (4.16) that

$$2(1-\eta)\int_S^T E(t)^{\frac{q}{2}}\,dt \le C_{10}E(S) + C_{11}E(S)^{\frac{q}{2}} + C_{12}\sum_{i=1}^{6}\varepsilon_i \int_S^T E(t)^{\frac{q}{2}}\,dt. \tag{4.17}$$

Choosing ε_i $(i = 1, 2, \dots, 6)$ small enough such that $\frac{1}{2}C_{12}\sum_{i=1}^{6}\varepsilon_i + \eta < 1$. Then we deduce from (4.17) that

$$\int_S^T E(t)^{\frac{q}{2}}\,dt \le C_{13}E(S) + C_{14}E(S)^{\frac{q}{2}} \le C_{15}\left[1 + E(0)^{\frac{q-2}{2}}\right]E(S),$$

Consequently, we have from Lemma 4.1 that

$$E(t) \le C_{16}(1+t)^{-\frac{2}{q-2}}, \qquad t \in [0, +\infty).$$

where C_{16} is a positive constants dependent of $E(0)$. Thus, we finish the proof of Theorem 4.2. □

Acknowledgements

This Research was supported by National Natural Science Foundation of China (No. 61273016), The Natural Science Foundation of Zhejiang Province (No. Y6100016), The Middle-aged and Young Leader in Zhejiang University of Science and Technology(2008-2012) and Interdisciplinary Pre-research Project of Zhejiang University of Science and Technology (2010–2012).

References

[1] K. Agre, M. A. Rammaha, Systems of nonlinear wave equations with damping and source terms, *Differential Integral Equations* **19**(2006), 1235–1270. MR2278006

[2] G. Autuori, F. Colasuonno, P. Pucci, On the existence of stationary solutions for higher-order p-Kirchhoff problems, *Commun. Contemp. Math.* **16**(2014), 1–43. MR3253900

[3] G. Autuori, F. Colasuonno, P. Pucci, Lifespan estimates for solutions of polyharmonic Kirchhoff systems, *Math. Models Methods Appl. Sci.* **22**(2012), 1–36. MR2887665

[4] G. Autuori, P. Pucci, M. C. Salvatori, Asymptotic stability for nonlinear Kirchhoff systems, *Nonlinear Anal. Real World Appl.* **10**(2009), 889–909. MR2474268

[5] F. Colasuonno, P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, *Nonlinear Anal.* **74**(2011), 5962–5974. MR2833367

[6] V. A. Galaktionov, S. I. Pohozaev, Blow-up and critical exponents for nonlinear hyperbolic equations, *Nonlinear Anal.* **53**(2003), 453–466. MR1964337

[7] Q. Gao, F. Li, Y. Wang, Blow-up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation, *Cent. Eur. J. Math.* **9**(2011), 686–698. MR2784038

[8] F. Gazzola, H. C. Grunau, G. Sweers, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, Vol. 1991, Springer-Verlag, Berlin, 2010. MR2667016

[9] V. Georgiev, D. Todorova, Existence of solutions of the wave equations with nonlinear damping and source terms, *J. Differential Equations* **109**(1994), 295–308. MR1273304

[10] G. C. Gorain, Exponential energy decay estimates for the solutions of n-dimensional Kirchhoff type wave equation, *Appl. Math. Comput.* **177**(2006), 235–242. MR2234515

[11] V. Komornik, *Exact controllability and stabilization. The multiplier method*, RAM: Research in Applied Mathematics, Masson-John Wiley, Paris, 1994. MR1359765

[12] H. A. Levine, Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: the method of unbounded Fourier coefficient, *Math. Ann.* **214**(1975), 205–220. MR0385336

[13] H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + f(u)$, *Trans. Amer. Math. Soc.* **192**(1974), 1–21. MR0344697

[14] F. C. Li, Global existence and blow-up of solutions for a higher-order Kirchhoff-type equation with nonlinear dissipation, *Appl. Math. Lett.* **17**(2004), 1409–1414. MR2103466

[15] L. Liu, M. Wang, Global existence and blow-up of solutions for some hyperbolic systems with damping and source terms, *Nonlinear Anal.* **64**(2006), 69–91. MR2183830

[16] S. A. Messaoudi, Global existence and nonexistence in a system of Petrovsky, *J. Math. Anal. Appl.* **265**(2002), 296–308. MR1876141

[17] S. A. Messaoudi, B. Said-Houari, Global nonexistence of positive initial energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, *J. Math. Anal. Appl.* **365**(2010), 277–287. MR2585099

[18] S. A. Messaoudi, B. Said-Houari, A blow-up result for a higher-order nonlinear Kirchhoff-type hyperbolic equation, *Appl. Math. Lett.* **20**(2007), 866–871. MR2323123

[19] K. Narasimha, Nonlinear vibration of an elastic string, *J. Sound Vib.* **8**(1968), 134–146. url

[20] K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, *J. Math. Anal. Appl.* **216**(1997), 321–342. MR1487267

[21] K. Ono, On global existence, decay and blow-up of solutions for some mildly degenerate Kirchhoff strings, *J. Differential Equations* **137**(1997), 273–301.

[22] J. Y. Park, J. J. Bae, On the existence of solutions of the degenerate wave equations with nonlinear damping terms, *J. Korean Math. Soc.* **35**(1998), 465–489. url

[23] J. Y. Park, J. J. Bae, On the existence of solutions of nondegenerate wave equations with nonlinear damping terms, *Nihonkai Math. J.* **9**(1998), 27–46. url

[24] J. Y. Park, J. J. Bae, Variational inequality for quasilinear wave equations with nonlinear damping terms, *Nonlinear Anal.* **50**(2002), 1065–1083. MR1914228

[25] V. Pata, S. Zelik, Smooth attractors for strongly damped wave equations, *Nonlinearity* **19**(2006), 1495–1506. MR2229785

[26] L. E. Payne, D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.* **22**(1975), 273–303. MR0402291

[27] M. A. Rammaha, S. Sakuntasathien, Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms, *Nonlinear Anal.* **72**(2010), 2658–2683. MR2577827

[28] D. H. Sattinger, On global solutions for nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.* **30**(1968), 148–172.

[29] I. Segal, Nonlinear semi-groups, *Ann. of Math. (2)* **78**(1963), 339–364. MR0152908

[30] S. T. Wu, L. Y. Tsai, On a system of nonlinear wave equations of Kirchhoff type with a strong dissipation, *Tamkang J. Math.* **38**(2007), 1–20. MR2321028

[31] S. T. Wu, L. Y. Tsai, On coupled nonlinear wave equations of Kirchhoff type with damping and source terms, *Taiwanese J. Math.* **14**(2010), 585–610. MR2655788

[32] S. T. Wu, L. Y. Tsai, Blow up of positive-initial-energy solutions for an integro-differential equation with nonlinear damping, *Taiwanese J. Math.* **14**(2010), 2043–2058. MR2724148