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Non-conjugate boundary value problem of a third order differential equation

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Abstract. This paper is devoted to prove the existence of the optimal interval where the Green's function is negative definite. The left and right endpoints of the interval are found. Then, a new principle of comparison of a third-order differential equation is established. As an application of our results, the solvability of a non-conjugate boundary value problem is discussed.

Keywords: third-order differential equation, non-conjugate boundary condition, Green's function, principle of comparison, upper and lower solutions.

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1 Introduction

The third-order differential equation has attracted considerable attention because of its wide applications in the deflection of a curved beam having a constant or varying cross section, in the three layer beam, in electromagnetic waves and gravity-driven flows [4]. Many authors have used a great number of theories and methods to deal with the third-order differential equation, such as the approaches based on differential inequality [5, 6] disconjugacy theory [1, 8], fixed point method [9], variational method [10], topological degree theory [7], the upper and lower solutions method [2]. These techniques can be interconnected and have proved to be very strong and fruitful.

The Green's function plays an important role in the solvability of differential equations. We can transform the differential equation into integral equations by utilizing Green's function.

Using of the spectral theory for self-adjoint operators, P. J. Torres [9] studied the second-order differential equations with periodic boundary value problem and obtained the condition under which the Green's function had constant sign. R. Ma [8] discussed a class of the third-order differential equations (**Non-self-adjoint operator**) with **conjugate boundary value condition** and got the symmetric optimal interval in the neighborhood of origin 0 by utilizing

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the disconjugacy theory [1]. But when it comes to non-conjugate boundary value condition, the disconjugacy theory does not work.

In this paper, we investigate the third-order differential equation (**Non-self-adjoint operator**) with the **non-conjugate boundary value condition**. Firstly, we prove there exists an optimal interval. And then, by constructing a right prism, we show the Green's function has constant sign when the parameter is on the interval. Finally, we verify a new principle of comparison which extends the result in [11].

This paper is organized as follows: in Section 2, some preliminaries are introduced; in Section 3, the existence of the optimal interval on which the Green's function will be negative definite is verified; the main result is given in Section 4 and Section 5. In Section 6, by using the upper and lower solutions method, we prove the existence of solution to a third order differential equation.

2 Preliminaries

Definition 2.1 ([1]). Let $p_k \in C[a,b]$ for $k = 1, \ldots, n$. A linear differential equation of order n

$$Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{2.1}$$

is said to be non-conjugate on an interval $[a,b]$ if every nontrivial solution has less than n zeros on $[a,b]$, multiple zeros being accounted according to their multiplicity.

Definition 2.2 ([8]). There are two cases which are called $(k, 3-k)$, $(k = 1, 2)$ conjugate boundary value problems for the third-order differential equation: $L_1 u = u'''(t) + Mu(t)$.

$$(1) \quad u \in \left\{ v \in C^3[0,1] \,\middle|\, v(0) = v'(0) = v(1) = 0 \right\};$$
$$(2) \quad u \in \left\{ v \in C^3[0,1] \,\middle|\, v(0) = v(1) = v'(1) = 0 \right\}.$$

We will discuss the following non-conjugate boundary value problem:

$$\begin{cases} u'''(t) + Mu(t) = f(t), \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{2.2}$$

Lemma 2.3. *Let $M = m^3$, the Green's function of (2.2) can be explicitly given by the expression*

(1) in the case $m \neq 0$,

$$G(t,s) = \begin{cases} K_1(m,t,s), & 0 \le s \le t \le 1; \\ K_2(m,t,s), & 0 \le t \le s \le 1, \end{cases} \tag{2.3}$$

where

$$\begin{aligned}
K_1(m,t,s) &= \frac{e^{\frac{m(t-s)}{2}} F_{t,s}(m)}{3m^2 g(m)}, \\
K_2(m,t,s) &= \frac{-e^{\frac{m(t-s)}{2}} h(mt) g(m(1-s))}{3m^2 g(m)}, \\
F_{t,s}(m) &\overset{\text{def}}{=} -h(mt)g(m(1-s)) + g(m)h(m(t-s)), \\
g(m) &\overset{\text{def}}{=} \cos\frac{\sqrt{3}m}{2} + \sqrt{3}\sin\frac{\sqrt{3}m}{2} - e^{-\frac{3m}{2}}, \\
h(m) &\overset{\text{def}}{=} e^{-\frac{3m}{2}} - \cos\frac{\sqrt{3}m}{2} + \sqrt{3}\sin\frac{\sqrt{3}m}{2}.
\end{aligned}$$

(2) in the case $m = 0$,

$$G(t,s) = \begin{cases} \frac{1}{2}s\left(s - 2t + t^2\right), & 0 \leq s \leq t \leq 1; \\ \\ \frac{1}{2}t^2(s-1), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4}$$

Proof. (1) In the case $m \neq 0$, the corresponding fundamental system of solutions to the homogeneous differential equation are

$$u_1(t) = e^{-mt}, \qquad u_2(t) = e^{\frac{mt}{2}} \cos\frac{\sqrt{3}mt}{2}, \qquad u_3(t) = e^{\frac{mt}{2}} \sin\frac{\sqrt{3}mt}{2}.$$

To get a particular solution, let

$$v(t) = c_1(t)e^{-mt} + c_2(t)e^{\frac{mt}{2}} \cos\frac{\sqrt{3}mt}{2} + c_3(t)e^{\frac{mt}{2}} \sin\frac{\sqrt{3}mt}{2},$$

then we have the system of linear inhomogeneous equations

$$\begin{cases} u_1(t){c'}_1(t) + u_2(t){c'}_2(t) + u_3(t){c'}_3(t) = 0, \\ {u'}_1(t){c'}_1(t) + {u'}_2(t){c'}_2(t) + {u'}_3(t){c'}_3(t) = 0, \\ {u''}_1(t){c'}_1(t) + {u''}_2(t){c'}_2(t) + {u''}_3(t){c'}_3(t) = f(t). \end{cases}$$

The solutions are

$$\begin{aligned} {c'}_1(t) &= \frac{e^{mt}}{3m^2} f(t), \\ {c'}_2(t) &= -\frac{e^{-\frac{mt}{2}}\left(\cos\frac{\sqrt{3}mt}{2} + \sqrt{3}\sin\frac{\sqrt{3}mt}{2}\right)}{3m^2} f(t), \\ {c'}_3(t) &= -\frac{e^{-\frac{mt}{2}}\left(\sin\frac{\sqrt{3}mt}{2} - \sqrt{3}\cos\frac{\sqrt{3}mt}{2}\right)}{3m^2} f(t). \end{aligned}$$

Thus the particular solution is given by

$$v(t) = \int_0^t \frac{e^{m(s-t)} + e^{\frac{m(t-s)}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}m}{2}(t-s) - \cos\frac{\sqrt{3}m}{2}(t-s)\right]}{3m^2} f(s)\,\mathrm{d}s,$$

the corresponding general solution of the differential equation is

$$\begin{aligned} u(t) &= c_1 e^{-mt} + c_2 e^{\frac{mt}{2}} \cos\frac{\sqrt{3}mt}{2} + c_3 e^{\frac{mt}{2}} \sin\frac{\sqrt{3}mt}{2} \\ &\quad + \int_0^t \frac{e^{m(s-t)} + e^{\frac{m(t-s)}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}m}{2}(t-s) - \cos\frac{\sqrt{3}m}{2}(t-s)\right]}{3m^2} f(s)\,\mathrm{d}s. \end{aligned} \tag{2.5}$$

By the boundary condition $u(0) = u'(0) = u'(1) = 0$, we have

$$c_1 = \frac{-\int_0^1 e^{\frac{-m(s-1)}{2}}\left[-e^{\frac{3m(s-1)}{2}} + \cos\frac{\sqrt{3}m(1-s)}{2} + \sqrt{3}\sin\frac{\sqrt{3}m(1-s)}{2}\right] f(s)\,\mathrm{d}s}{3m^2 e^{\frac{m}{2}}(\cos\frac{\sqrt{3}m}{2} + \sqrt{3}\sin\frac{\sqrt{3}m}{2} - e^{-\frac{3m}{2}})},$$

$$c_2 = \frac{\int_0^1 e^{\frac{-m(s-1)}{2}}\left[-e^{\frac{3m(s-1)}{2}} + \cos\frac{\sqrt{3}m(1-s)}{2} + \sqrt{3}\sin\frac{\sqrt{3}m(1-s)}{2}\right] f(s)\,\mathrm{d}s}{3m^2 e^{\frac{m}{2}}(\cos\frac{\sqrt{3}m}{2} + \sqrt{3}\sin\frac{\sqrt{3}m}{2} - e^{-\frac{3m}{2}})},$$

$$c_3 = \frac{-\sqrt{3}\int_0^1 e^{\frac{-m(s-1)}{2}}\left[-e^{\frac{3m(s-1)}{2}} + \cos\frac{\sqrt{3}m(1-s)}{2} + \sqrt{3}\sin\frac{\sqrt{3}m(1-s)}{2}\right] f(s)\,\mathrm{d}s}{3m^2 e^{\frac{m}{2}}(\cos\frac{\sqrt{3}m}{2} + \sqrt{3}\sin\frac{\sqrt{3}m}{2} - e^{-\frac{3m}{2}})}.$$

Hence

$$G(t,s) = \begin{cases} \dfrac{e^{\frac{m(t-s)}{2}} F_{t,s}(m)}{3m^2 g(m)}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{-e^{\frac{m(t-s)}{2}} h(mt) g(m(1-s))}{3m^2 g(m)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.6}$$

(2) In the case $m = 0$, $0 \le s \le t \le 1$, since

$$\lim_{m\to 0} e^{\frac{m(t-s)}{2}} = 1,$$

then

$$\lim_{m\to 0} K_1(m,t,s) = \lim_{m\to 0} \frac{F_{t,s}(m)}{3m^2 g(m)}.$$

By Taylor's series expansion at the origin

$$F_{t,s}(m) = \frac{9}{2}m^3(s^2 - 2st + st^2) + o(m^4) = \frac{9}{2}m^3[s^2 - 2st + st^2 + o(1)m],$$

$$g(m) = 3m + o(m^2) = m[3 + o(1)m],$$

we have

$$\lim_{m\to 0} K_1(m,t,s) = \lim_{m\to 0} \frac{F_{t,s}(m)}{3m^2 g(m)} = \frac{1}{2}s\left(s - 2t + t^2\right).$$

For $0 \le t \le s \le 1$, the result can be proved by the same method as employed above. The Green's function is

$$G(t,s) = \begin{cases} \dfrac{1}{2}s\left(s - 2t + t^2\right), & 0 \le s \le t \le 1, \\[2ex] \dfrac{1}{2}t^2(s-1), & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

□

Remark 2.4. (1) This result just coincides with [3] in the case $m = 0$.

(2) In the case $0 \le s \le t \le 1$, $s = 0$ or $t = s = 1$, implies $G(t,s) = 0$.

(3) When $0 \le t \le s \le 1$, $t = 0$ or $t = s = 1$ implies $G(t,s) = 0$. In the following discussion about the sign of the Green's function, we make an appointment for $t, s \in (0,1)$ from Lemma 3.1 to Theorem 3.7.

3 Existence of the optimal interval

In this section, we prove the existence of the optimal interval on which the Green's function will have constant sign. Let $g(m)$, $h(m)$ be as in Lemma 2.3, $M = m^3$ for parameter M.

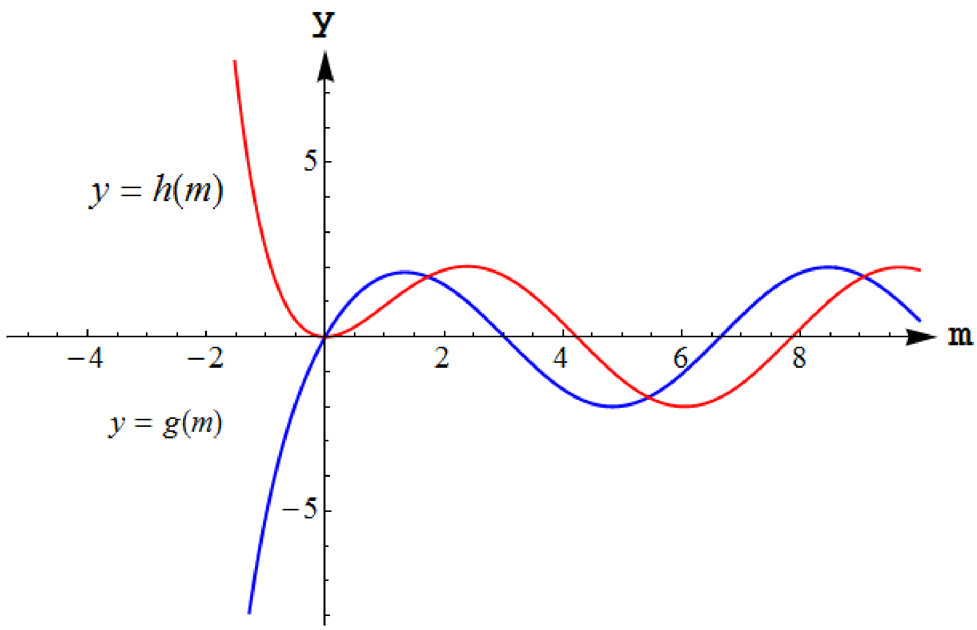

Figure 3.1

Lemma 3.1. *If $m < 0$, then $g(m) < 0$, and $h(m) > 0$.*

By Lemma 3.1 we can easily get the following.

Corollary 3.2. *If $m < 0$ and $0 < t \leq s < 1$, then $K_2(t,s) < 0$.*

Lemma 3.3. *There exist $\alpha < 0$, $\beta > 0$, such that*

(1) if $m \in (\alpha, 0)$, $0 < s \leq t < 1$, then $F_{t,s}(m) > 0$;

(2) if $m \in (0, \beta)$, $0 < s \leq t < 1$, then $F_{t,s}(m) < 0$.

Proof. (1) In the case $m < 0$, $0 < s \leq t < 1$
since

$$F_{t,s}(m) = F_{t,s}(0) + F'_{t,s}(0)m + \frac{F''{}_{t,s}(0)}{2!}m^2 + \cdots + \frac{F^{(5)}_{t,s}(0)}{5!}m^5 + \cdots,$$

and

$$\begin{aligned}
&F_{t,s}(0) = F'_{t,s}(0) = F''_{t,s}(0) = 0,\\
&F'''_{t,s}(0) = 27(s-t)^2 + 27t^2(s-1) = 27s(t^2 - 2t + s),\\
&F^{(4)}_{t,s}(0) = 54s(s^2 + st^2 - 3st - s - t^3 + t^2 + 2t) = 54s(s-t-1)(s+t^2-2t),\\
&F^{(5)}_{t,s}(0) = \frac{135s(t^2 - 2t + s)(t-s+1)^2}{2},
\end{aligned}$$

then

$$F_{t,s}(m) = \frac{9s(t^2-2t+s)}{2}m^3 + \frac{9s(s-t-1)(t^2-2t+s)}{4}m^4 + \frac{9s(s-t-1)^2(t^2-2t+s)}{16}m^5 + \cdots .$$

In view of

$$t^2 - 2t + s < 0 \qquad (\forall 0 < s \le t < 1),$$

then $\frac{F'''_{t,s}(0)}{3!}m^3 > 0$ for $m < 0$.

As a result, there exists $\alpha < 0$ such that when $m \in (\alpha, 0)$, $0 < s \le t < 1$,

$$F_{t,s}(m) > 0.$$

(2) In the case $m > 0$, $\frac{F'''_{t,s}(0)}{3!}m^3 < 0$. Consequently, there exists $\beta > 0$, such that when $m \in (0, \beta)$ and $0 < s \le t < 1$, $F_{t,s}(m) < 0$. □

Remark 3.4. When $m = 0$, the Green's function is

$$G(t,s) = \begin{cases} \frac{1}{2}s\left(s - 2t + t^2\right), & 0 < s \le t < 1, \\ \frac{1}{2}t^2(s-1), & 0 < t \le s < 1. \end{cases}$$

If $0 < s \le t < 1$, then $s - 2t + t^2 < 0$, so $G(t,s) < 0$.

Let us define

$$T_0 = \{\alpha < 0 \mid \forall m \in (\alpha, 0),\ \forall 0 < s \le t < 1,\ F_{t,s}(m) > 0\};$$
$$T_1 = \{\beta > 0 \mid \forall m \in (0, \beta),\ \forall 0 < s \le t < 1,\ F_{t,s}(m) < 0\}.$$

By Lemma 3.3, we know that $T_0 \neq \phi$, $T_1 \neq \phi$.

Lemma 3.5. *T_0, T_1 are bounded.*

Proof. In fact, let $m_1 = -4.3,\ s_1 = 0.004,\ t_1 = 0.946$, we get

$$-h(m_1 t_1)g(m_1(1-s_1)) + g(m_1)h(m_1(t_1 - s_1)) < 0.$$

So $m_1 \notin T_0$, this yields that m_1 is a lower bound of the set T_0.

If $m_2 = 3.1,\ s_2 = 10^{-3},\ t_2 = 10^{-3}$, then

$$-h(m_2 t_2)g(m_2(1-s_2)) + g(m_2)h(m_2(t_2 - s_2)) > 0.$$

Consequently $m_2 \notin T_1$, which implies m_2 is the upper bound of the set T_1. □

Let

$$\tau_0 = \sup\{\delta > 0 \mid -\delta \in T_0\}, \qquad \tau_1 = \sup\{\beta > 0 | \beta \in T_1\},$$
$$T_2 = \{\gamma > 0 \mid \forall m \in (0, \gamma),\ \forall\, 0 < t \le s < 1,\ G(t,s) < 0\};$$

It is easy to verify that $T_2 \neq \phi$,

$$\tau_2 = \sup\{\gamma > 0 \mid \forall m \in (0, \gamma),\ \forall 0 < t \le s < 1,\ G(t,s) < 0\};$$
$$\tau = \sup\{\omega > 0 \mid \forall m \in (0, \omega),\ G(t,s) < 0\}.$$

Lemma 3.6. *Let $m > 0$, λ ($\lambda \approx 3.01674$) be the smallest positive zero of the equation $g(m) = 0$ and μ ($\mu \approx 4.23321$) be the smallest positive zero of the equation $h(m) = 0$, then*

(1) for $\forall\, 0 < t \le s < 1$, $G(t,s) < 0$ implies $\tau_2 = \lambda$;

(2) for $\forall\, 0 < s \le t < 1$, $F_{t,s}(m) < 0$ implies $\tau_1 \le \lambda$.

Proof. (1) The proof will be given in two steps.
Step 1. $\tau_2 \le \lambda$.

When $0 < t \le s < 1$, by contradiction, we assume $\lambda < \tau_2 < \mu (0 < \frac{\lambda}{\tau_2} < 1)$. It is easy to see that

$$3\tau_2^2 \left(\cos\frac{\sqrt{3}\tau_2}{2} + \sqrt{3}\sin\frac{\sqrt{3}\tau_2}{2} - e^{-\frac{3\tau_2}{2}} \right) < 0$$

and

$$-h(\tau_2 t) = \cos\frac{\sqrt{3}\tau_2 t}{2} - \sqrt{3}\sin\frac{\sqrt{3}\tau_2 t}{2} - e^{-\frac{3\tau_2 t}{2}} < 0.$$

But there exist $s_0 = 1 + (\theta_0 - 1)\frac{\lambda}{\tau_2}$, $\theta_0 \in (0,1)$, such that

$$g(\tau_2(1-s_0)) = \cos\frac{\sqrt{3}\tau_2(1-s_0)}{2} + \sqrt{3}\sin\frac{\sqrt{3}\tau_2(1-s_0)}{2} - e^{-\frac{3\tau_2(1-s_0)}{2}} > 0.$$

In fact, when $\tau_2(1-s_0) < \lambda$, namely $1 - \frac{\lambda}{\tau_2} < s_0 < 1$.

Let $s_0 = 1 + (\theta_0 - 1)\frac{\lambda}{\tau_2}, \theta_0 \in (0,1)$, we can get

$$g(\tau_2(1-s_0)) = \cos\frac{\sqrt{3}\tau_2(1-s_0)}{2} + \sqrt{3}\sin\frac{\sqrt{3}\tau_2(1-s_0)}{2} - e^{-\frac{3\tau_2(1-s_0)}{2}} > 0, \text{ i.e. } G(t_0,s_0) > 0.$$

This contradicts the assumption that $\forall\, 0 < t \le s < 1,\ G(t,s) < 0$, therefore $\tau_2 \le \lambda < +\infty$.
Step 2. $\tau_2 = \lambda$.

Noting that when $m \in (0,\lambda)$, $G(t,s) < 0$, then $\tau_2 = \lambda$.
(2) In the case $0 < s \le t < 1$, contrarily, suppose that $\lambda < \tau_1 < \mu$ $\left(0 < \frac{\lambda}{\tau_1} < 1\right)$.

As a consequence

$$-h(\tau_1 t) = \cos\frac{\sqrt{3}\tau_1 t}{2} - \sqrt{3}\sin\frac{\sqrt{3}\tau_1 t}{2} - e^{-\frac{3\tau_1 t}{2}} < 0.$$

But there exists $t_1 = s_1 = \left(1 - \frac{\lambda}{\tau_1}\right)/10$, such that

$$F_{t_1,s_1}(\tau_1) = -h(\tau_1 t_1) g(\tau_1(1-s_1)) > 0.$$

In fact, as long as $\tau_1(1-s_1) > \lambda$, namely $0 < s_1 < 1 - \frac{\lambda}{\tau_1}$ $\left(0 < \frac{\lambda}{\tau_1} < 1\right)$, let $s_1 = \left(1 - \frac{\lambda}{\tau_1}\right)/10$, we can get

$$g(\tau_1(1-s_1)) = \cos\frac{\sqrt{3}\tau_1(1-s_1)}{2} + \sqrt{3}\sin\frac{\sqrt{3}\tau_1(1-s_1)}{2} - e^{-\frac{3\tau_1(1-s_1)}{2}} < 0,$$

which means that $F_{t_1,s_1}(\tau_1) = -h(\tau_1 t_1) g(\tau_1(1-s_1)) > 0$. This contradicts the assumption that $F_{t,s}(m) < 0,\ 0 < s \le t < 1$, so $\tau_1 \le \lambda$ and $\tau_2 \ge \tau_1$.

Thus from (1) and (2), we arrive at the conclusion that $\lambda \ge \tau = \min\{\tau_1, \tau_2\} = \tau_1$. □

Theorem 3.7. *There exists an optimal interval around 0 such that for $\forall m \in (-\tau_0, \tau_1)$, $G(t,s) < 0$.*

Proof. It is easy to verify the assertion by Lemma 3.5 and Lemma 3.6. □

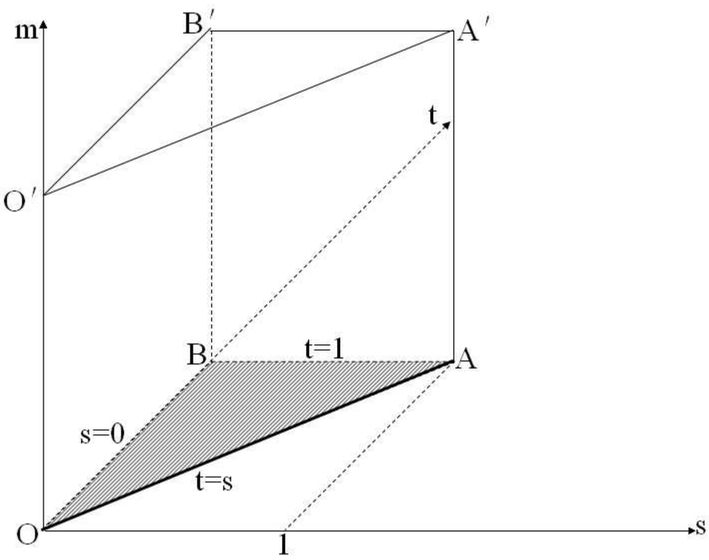

Figure 4.1

4 Right and left endpoints of the optimal interval

In this section, we set out to find the right and left endpoints of the optimal interval.

Lemma 4.1. *If* $m \in [0,\lambda],\ 0 \le s \le t \le 1$, *then* $F_{t,s}(m) \le 0$, *therefore* $G(t,s) \le 0$.

Note: In fact, we just need to inspect whether the maximum value of $F_{t,s}(m)$ is 0, namely to calculate every place of the three-dimensional range where the ternary function $F_{t,s}(m)$ will possibly get maximum value. So our task is to check 9 edges and the stagnation points on 5 faces and the interior of the right prism region.

Proof. (See Figure 4.1.) For the edges

$$AB = \{(m,t,s) \mid m = 0,\ t = 1,\ s \in [0,1]\}\,,$$
$$OB = \{(m,t,s) \mid m = 0,\ s = 0,\ t \in [0,1]\}\,,$$
$$AO = \{(m,t,s) \mid m = 0,\ t = s \in [0,1]\}\,;$$

and the face

$$ABO = \{(m,t,s) \mid m = 0,\ 0 \le s \le t \le 1\}\,,$$

since $m = 0$, $F_{t,s}(m) \equiv 0$.

The edge

$$OO' = \{(m,t,s) \mid m \in [0,\lambda]\,,\ t = s = 0\}\,,$$

so $F_{t,s}(m) \equiv 0$ on OO'.

The edge

$$AA' = \{(m,t,s) \mid m \in [0,\lambda]\,,\ t = s = 1\}\,,$$

so $F_{t,s}(m) \equiv 0$ on AA'.

The edge

$$O'A' = \{(m,t,s) \mid m = \lambda,\ t = s \in [0,1]\}\,.$$

By the properties of functions $h(m)$ and $g(m)$, we know that

$$F_{t,s}(\lambda) = -h(\lambda t)g(\lambda(1-t)) \leq 0 \quad \text{on } O'A'.$$

The face

$$OO'A'A = \{(m,t,s) \mid m \in [0,\lambda]\,,\ t = s \in [0,1]\}\,.$$

Using the properties of functions $h(m)$ and $g(m)$, we have

$$F_{t,s}(m) = -h(mt)g(m(1-t)) \leq 0 \quad \text{on } OO'A'A.$$

For the edges

$$\begin{aligned} O'B' &= \{(m,t,s) \mid m = \lambda,\ s = 0,\ t \in [0,1]\}\,, \\ BB' &= \{(m,t,s) \mid t = 1,\ s = 0,\ m \in [0,\lambda]\}\,, \end{aligned}$$

and the face

$$O'B'BO = \{(m,t,s) \mid t \in [0,1]\,,\ s = 0,\ m \in [0,\lambda]\}\,,$$

because $s = 0$, $F_{t,s}(m) \equiv 0$.

For the edge

$$\begin{aligned} A'B' &= \{(m,t,s) \mid t = 1,\ s \in [0,1]\,,\ m = \lambda\}\,, \\ F_{t,s}(\lambda) &= -h(\lambda)g(\lambda(1-s)) + g(\lambda)h(\lambda(1-s)). \end{aligned}$$

Since λ is the smallest positive solution of the function $g(m)$, we have

$$F_{t,s}(\lambda) = -h(\lambda)g(\lambda(1-s)) \leq 0 \quad \text{on } A'B'.$$

The face

$$O'A'B' = \{(m,t,s) \mid m = \lambda, 0 \leq s \leq t \leq 1\}\,,$$

$F_{t,s}(\lambda) = -h(\lambda t)g(\lambda(1-s)) + g(\lambda)h(\lambda(t-s)) = -h(\lambda t)g(\lambda(1-s)) \leq 0$ on $O'A'B'$.

The face

$$ABB'A' = \{(m,t,s) \mid t = 1, m \in [0,\lambda]\,, s \in [0,1]\}\,,$$

let $F_{t,s}(m)\,|_{t=1} \stackrel{\text{def}}{=} J(m,s)$, then $J(m,s) = -h(m)g(m(1-s)) + g(m)h(m(1-s))$.

We have verified 4 edges for the 3-dimensional area, and it remains to calculate the stagnation points inside the face. Let

$$\frac{\partial J(m,s)}{\partial s} = (-m)\left[g(m)h'(m(1-s)) - h(m)g'(m(1-s))\right] = 0; \tag{4.1}$$

$$\begin{aligned} \frac{\partial J(m,s)}{\partial m} = &-\left[h'(m)g(m(1-s)) + h(m)g'(m(1-s))(1-s)\right] \\ &+ g'(m)h(m(1-s)) + g(m)h'(m(1-s))(1-s) = 0. \end{aligned} \tag{4.2}$$

Obviously, $\{(m,s) \mid m = 0, s \in [0,1]\}$ solve the system of nonlinear equations. While there may exist other solutions. Combining (4.1) with (4.2), we can get corresponding curves of the system in Figure 4.2, which clearly shows there exists no other solution. Hence, all the solutions of the system are $\{(m,s) \mid m = 0, s \in [0,1]\}$. When $m = 0$, $F_{t,s}(0) \equiv 0$; thus, on the face $ABB'A'$, $F_{t,s}(m) \leq 0$.

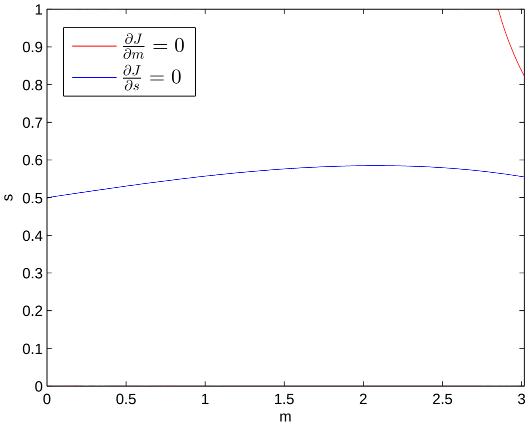

Figure 4.2

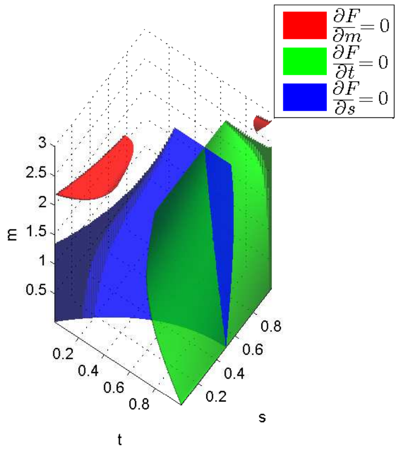

Figure 4.3

The interior of the right prism is $m \in [0, \lambda]$, $0 \le s \le t \le 1$. Let

$$\frac{\partial F_{t,s}(m)}{\partial t} = m\left[g(m)h'(m(t-s)) - g(m(1-s))h'(mt)\right] = 0 \tag{4.3}$$

$$\frac{\partial F_{t,s}(m)}{\partial s} = (-m)\left[h'(m(t-s))g(m) - h(mt)g'(m(1-s))\right] = 0 \tag{4.4}$$

$$\begin{aligned}\frac{\partial F_{t,s}(m)}{\partial m} =& -\left[th'(mt)g(m(1-s)) + (1-s)h(mt)g'(m(1-s))\right] \\ &+ g'(m)h(m(t-s)) + g(m)h'(m(t-s))(t-s) = 0.\end{aligned} \tag{4.5}$$

Combining (4.3), (4.4), with (4.5), we know that $\{(m,t,s) \mid m = 0,\ \forall 0 \le s \le t \le 1\}$ or $\{(m,t,s) \mid \forall m \in [0,\lambda],\ t = s = 0\}$ are the solutions of the system. While there may be other solutions. But in Figure 4.3 it is obvious that there exists no other solution for the system. So we obtain that all the solutions of the system are $\{(m,t,s) \mid m = 0,\ \forall 0 \le s \le t \le 1\}$ or $\{(m,t,s) \mid \forall\, m \in [0,\lambda],\ t = s = 0\}$. When $m = 0$ or $t = s = 0$, we can get $F_{t,s}(m) \equiv 0$.

In summary, when $\forall m \in [0,\lambda]$, $\forall 0 \le s \le t \le 1$, the ternary function $F_{t,s}(m) \le 0$, namely $G(t,s) \le 0$ $(\forall 0 \le s \le t \le 1)$. Moreover, in the case $0 \le t \le s \le 1$, from the expression of the Green's function, we can get $G(t,s) \le 0$. □

Theorem 4.2. *The smallest positive solution of the equation:* $g(m) = 0$ *is just the right endpoint of the optimal interval where the Green's function will be negative definite.*

Proof. From Lemma 3.6, we know that the right endpoint value of the optimal interval couldn't be larger than λ. Combining it with Lemma 4.1, we have $G(t,s) \le 0$ for arbitrary $m \in [0,\lambda]$. This completes the proof. □

Lemma 4.3. *Assume* $m \le 0$, $0 \le s \le t \le 1$.

(1) $F_{t,s}(m) \ge 0$ *implies* $m \ge -\mu$;

(2) if $m \in [-\mu, 0]$ *then* $F_{t,s}(m) \ge 0$.

Proof. (See Figure 4.4) Following the same manner of Lemma 4.1, it is easy to prove that on the edges and the faces AB, OB, AO, ABO,OO', AA', $O'A'$, $OO'AA'$, $O'B'$, $O'B'BO$, BB', $F_{t,s}(m) \ge 0$ always holds.

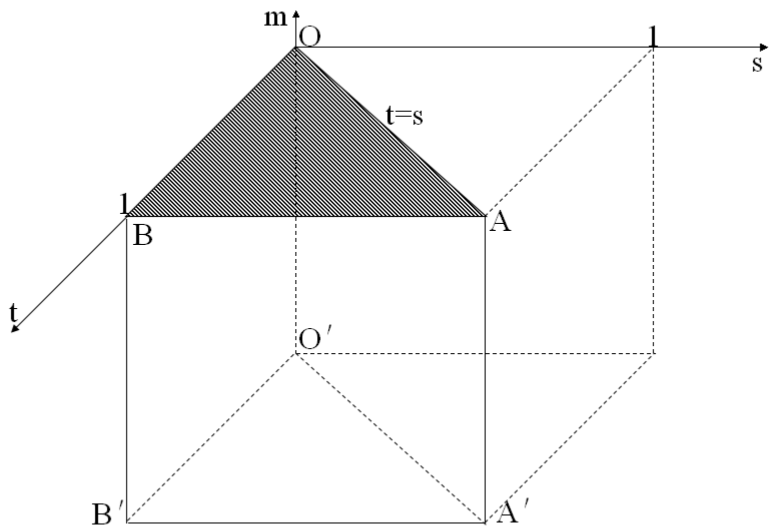

Figure 4.4

The face

$$ABB'A' = \{(m,t,s) \mid t = 1, m \in [-\mu, 0]\,,\ s \in [0,1]\}\,.$$

We get

$$\begin{aligned} J(m,s) &= -h(m)g(m(1-s)) + g(m)h(m(1-s))(m<0) \\ &= 2\sqrt{3}\left[e^{-\frac{3m}{2}}\left(e^{\frac{3ms}{2}}\sin\frac{\sqrt{3}m}{2} - \sin\frac{\sqrt{3}m(1-s)}{2}\right) - \sin\frac{\sqrt{3}ms}{2}\right]. \end{aligned}$$

By Taylor's expansion at $s = 0$ for the function $J(m,s)$, we could gain

$$J_m(s) = 2\sqrt{3}\left\{\tfrac{1}{2}m\left[-\sqrt{3} + e^{-\frac{3m}{2}}\left(\sqrt{3}\cos\tfrac{\sqrt{3}m}{2} + 3\sin\tfrac{\sqrt{3}m}{2}\right)\right]s + \tfrac{3}{2}m^2e^{-\frac{3m}{2}}\sin\tfrac{\sqrt{3}m}{2}s^2 + \cdots\right\}.$$

Set

$$\begin{aligned} \psi(m) &\stackrel{\text{def}}{=} -\sqrt{3} + e^{-\frac{3m}{2}}\left(\sqrt{3}\cos\frac{\sqrt{3}m}{2} + 3\sin\frac{\sqrt{3}m}{2}\right) \\ &= \sqrt{3}e^{-\frac{3m}{2}}\left(-e^{\frac{3m}{2}} + \cos\frac{\sqrt{3}m}{2} + \sqrt{3}\sin\frac{\sqrt{3}m}{2}\right), \end{aligned}$$

since $m \le 0$, let $m = -\mu$, then $h(\mu) = e^{-\frac{3\mu}{2}} - \cos\frac{\sqrt{3}\mu}{2} + \sqrt{3}\sin\frac{\sqrt{3}\mu}{2}$, which means $\psi(m) = \sqrt{3}e^{\frac{3\mu}{2}}(-e^{-\frac{3\mu}{2}} + \cos\frac{\sqrt{3}\mu}{2} - \sqrt{3}\sin\frac{\sqrt{3}\mu}{2}) = -\sqrt{3}e^{\frac{3\mu}{2}}h(\mu)$, thus the biggest negative solution of $\psi(m) = 0$ is just the smallest positive solution of $h(m) = 0$.

Because $\psi(0) = 0, \psi(-2) \doteq -66.793 < 0$, we can get $\psi(m) \le 0\,(m \in [-\mu, 0])$, and also we can get the first term of $J_m(s)$: $\frac{1}{2}m\psi(m) \ge 0$ (see Figure 4.5 and 4.6).

Contrary to (1), we assume that $m < -\mu$, so the first term is $\frac{1}{2}m\psi(m) < 0$.

From the continuity of the function, we know that there exists a neighborhood such that $J_m(s) < 0$, but $J_m(s)$ is the special case of the function $F_{t,s}(m)$ when $t = 1$. If so, that will

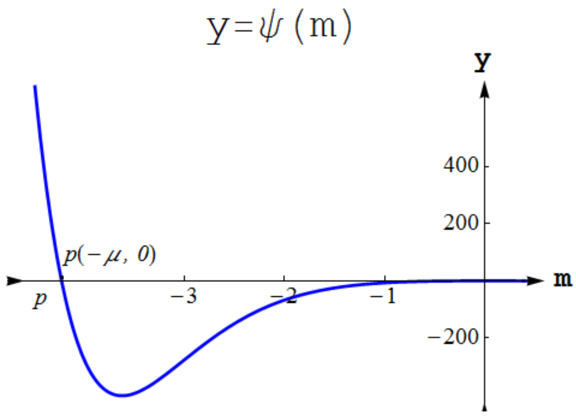

Figure 4.5

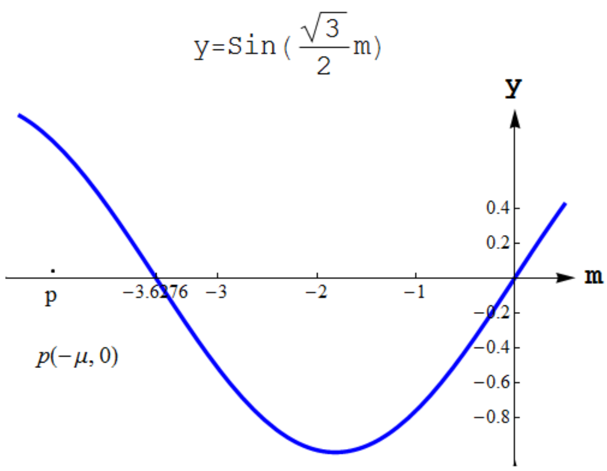

Figure 4.6

lead to $F_{t,s}(m)\,|_{t=1} < 0$. As a result, the Green's function will not be negative definite when $0 \le s \le t \le 1$. This completes the proof of assertion (1).

For the edge

$$A'B' = \{(m,t,s) \mid m = -\mu,\ t = 1,\ s \in [0,1]\},$$

$$J(-\mu,s) = 3\sqrt{3}\mu^2 e^{\frac{3\mu}{2}} \sin\frac{\sqrt{3}}{2}(-\mu)s^2 + \cdots .$$

The first term becomes $3\sqrt{3}\mu^2 e^{\frac{3\mu}{2}} \sin\frac{\sqrt{3}}{2}(-\mu)s^2 > 0$, and then it is a unary function with respect to s. Noting that $J_m(0) = J_m(1) \equiv 0$, we only need to compute the stagnation points with respect to $s \in [0,1]$. Additionally,

$$\frac{\partial J(m,s)}{\partial s} = 3me^{-\frac{3m}{2}}\left[\cos\frac{\sqrt{3}m(1-s)}{2} - e^{\frac{3m}{2}}\cos\frac{\sqrt{3}ms}{2} + \sqrt{3}e^{\frac{3ms}{2}}\sin\frac{\sqrt{3}m}{2}\right] \tag{4.6}$$

Namely, to seek out the stagnation points of the system (4.7) on $[0,1]$

$$\begin{cases} \cos\frac{\sqrt{3}m(1-s)}{2} - e^{\frac{3m}{2}}\cos\frac{\sqrt{3}ms}{2} + \sqrt{3}e^{\frac{3ms}{2}}\sin\frac{\sqrt{3}m}{2} = 0 \\ m = -\mu \end{cases} \tag{4.7}$$

By calculating, we get $s_* \doteq 0.564746$, and $J(-\mu, s_*) \doteq 2012.74 \gg 0$. So on the edge $A'B'$,we

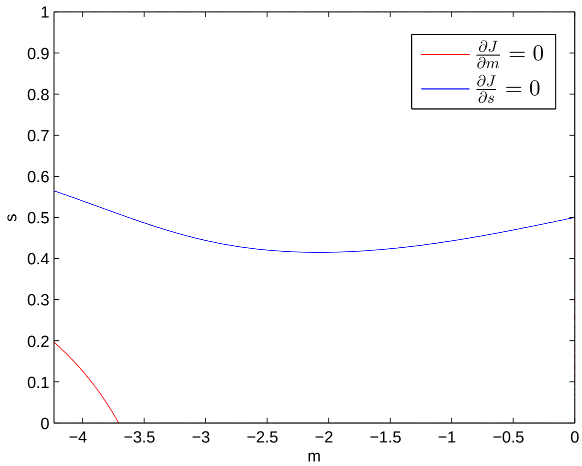

Figure 4.7

have $F_{t,s}(-\mu) \geq 0$. When $m = 0$, $F_{t,s}(0) = J(0,s) \equiv 0$, and we have verified 4 edges on the face $ABB'A'$. In the following step we will compute the stagnation points inside this face, since

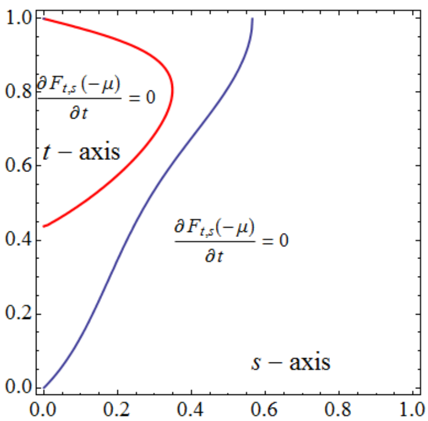

Figure 4.8

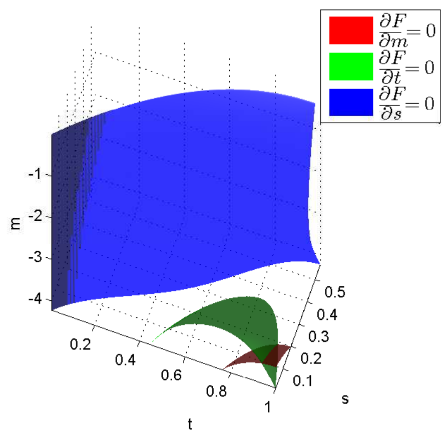

Figure 4.9

$$\frac{\partial J(m,s)}{\partial s} = 3e^{-\frac{3m}{2}}\Bigg\{ - se^{\frac{3m}{2}}\cos\frac{\sqrt{3}ms}{2} + (s-1)\cos\frac{\sqrt{3}m\,(s-1)}{2} + e^{\frac{3ms}{2}}\left[\cos\frac{\sqrt{3}m}{2} + \sqrt{3}\,(s-1)\sin\frac{\sqrt{3}m}{2}\right] + \sqrt{3}\sin\frac{\sqrt{3}m\,(1-s)}{2}\Bigg\}. \tag{4.8}$$

Combining (4.6) with (4.8), we could obtain

$$\begin{cases} \dfrac{\partial J(m,s)}{\partial s} = 0, \\ \dfrac{\partial J(m,s)}{\partial m} = 0, \end{cases} \quad \text{where } s \in [0,1], \ m \in [-\mu, 0]. \tag{4.9}$$

It is easy to know that $\{(m,s) \mid m = 0, s \in [0,1]\}$ solves the system (4.9), while there may be other solutions. It can be seen from the Figure 4.7 that there exists no other solution for the system (4.9). When $m = 0$, $F_{t,s}(0) \equiv 0$. So on the face $ABB'A'$ we have $F_{t,s}(m) \geq 0$.

The face

$$O'A'B' = \{(m,t,s) \mid m = -\mu, 0 \leq s \leq t \leq 1\},$$

$$F_{t,s}(-\mu) = -h(-\mu t)g(-\mu(1-s)) + g(-\mu)h(-\mu(t-s)).$$

If $t = 0$, then $s = 0$, so $F_{t,s}(-\mu) = 0$.

In the case $t = 1$, it is just the edge $A'B'$, hence $J(-\mu, s) \geq 0$.

When $s = 0$, $F_{t,s}(-\mu) \equiv 0$.

If $s = 1$, then $t = 1$, thus $F_{t,s}(-\mu) \equiv 0$. It only remains to verify the stagnation points inside the face $O'A'B'$, so let

$$\begin{cases} \dfrac{\partial F_{t,s}(-\mu)}{\partial t} = 0, \\ \dfrac{\partial F_{t,s}(-\mu)}{\partial t} = 0, \end{cases} \quad \text{where} \quad 0 \leq s \leq t \leq 1. \tag{4.10}$$

It is evident to know that from (4.3), (4.4) $\{(t,s) \mid t = s = 0\}$ solves the system (4.10), while there may be other solutions. It can be seen from the Figure 4.8 that there exists no other solution for the system (4.10), so all the solution of the system (4.10) is $\{(t,s) \mid t = s = 0\}$. When $t = s = 0$, $F(0,0,-\mu) \equiv 0$. Thus on the face $O'A'B'$ we can get $F_{t,s}(m) \geq 0$.

Finally, we will compute the stagnation points inside the 3-dimensional region

$$\{(m,t,s) \mid m \in [-\mu, 0], 0 \leq s \leq t \leq 1\}.$$

Combining (4.3), (4.4), with (4.5), we have

$$\begin{cases} \dfrac{\partial F(t,s,m)}{\partial t} = 0 \\ \dfrac{\partial F(t,s,m)}{\partial s} = 0 \\ \dfrac{\partial F(t,s,m)}{\partial m} = 0. \end{cases} \tag{4.11}$$

Obviously, $\{(m,t,s) \mid m = 0, 0 \leq s \leq t \leq 1\}$ or $\{(m,t,s) \mid m \in [-\mu, 0], t = s = 0\}$ solve the system of nonlinear equations, while there may be other solutions. It can be seen from Figure 4.9 that there exists no other solution for the system (4.11), and we will find $F_{t,s}(m) \equiv 0$ in both cases.

All in all, we know that the ternary function $F(t,s,m) \geq 0$ always holds in the 3-dimensional area $(\{(m,t,s) \mid m \in [-\mu, 0], \ 0 \leq s \leq t \leq 1\})$. □

Theorem 4.4. *The left endpoint of the optimal interval where the Green's function will be negative definite is just the biggest negative solution* $(m = -\mu)$ *of the equation:* $\psi(m) = 0$.

Proof. It is an immediate consequence of Corollary 3.2 and Lemma 4.3. □

5 A new principle of comparison

Theorem 5.1. *Let $f(t) \in C([0,1],(-\infty,0])$, and $b,c \geq 0$ be two constants. If $u(t) \in C^3[0,1]$ satisfies*

$$\begin{cases} u'''(t) + Mu(t) = f(t), & t \in [0,1] \\ u(0) = 0,\ u'(0) = b,\ u'(1) = c, \end{cases} \tag{5.1}$$

then $u(t) \geq 0$, provided $\sqrt[3]{M} \in [-\mu, \lambda)$.

Proof. It is easy to verify that such a $u(t)$ can be given by the expression

$$u(t) = \int_0^1 G(t,s)f(s)\,\mathrm{d}s + R(t), \qquad t \in [0,1]$$

where

$$m = \sqrt[3]{M}, \qquad R(t) = \frac{B+C}{\sqrt{3}me^{\frac{3m}{2}}g(m)};$$

$$B = be^{-mt}\Bigg\{ -e^{\frac{3m}{2}}\left(\sqrt{3}\cos\frac{\sqrt{3}m}{2} + \sin\frac{\sqrt{3}m}{2}\right) + e^{\frac{3m(t+1)}{2}}\left[\sqrt{3}\cos\frac{\sqrt{3}m(t-1)}{2} - \sin\frac{\sqrt{3}m(t-1)}{2}\right] - 2e^{\frac{3mt}{2}}\sin\frac{\sqrt{3}mt}{2}\Bigg\};$$

$$C = \sqrt{3}ce^{\frac{m(t+2)}{2}}h(mt).$$

By Remark 3.4, Theorem 4.2 and 4.4, we know that when $m \in [-\mu, \lambda)$:

$$G(t,s) \leq 0, \qquad \forall (t,s) \in [0,1]\times[0,1], \quad \text{so} \quad \int_0^1 G(t,s)f(s)\,\mathrm{d}s \geq 0.$$

(1) In the case $m = 0$, the denominator $R(t)$ is equal to zero, therefore,

$$\lim_{m\to 0}\frac{B}{\sqrt{3}me^{\frac{3m}{2}}g(m)} = \frac{b}{2}(2-t)t, \qquad \lim_{m\to 0}\frac{C}{\sqrt{3}me^{\frac{3m}{2}}g(m)} = \frac{c}{2}t^2;$$

hence $R(t) \geq 0$, namely $u(t) \geq 0$.

(2) In the case $m \neq 0$, let

$$\chi(m,t) = -e^{\frac{3m}{2}}\left(\sqrt{3}\cos\frac{\sqrt{3}m}{2} + \sin\frac{\sqrt{3}m}{2}\right) + e^{\frac{3m(t+1)}{2}}\left[\sqrt{3}\cos\frac{\sqrt{3}m(t-1)}{2} - \sin\frac{\sqrt{3}m(t-1)}{2}\right] - 2e^{\frac{3mt}{2}}\sin\frac{\sqrt{3}mt}{2}.$$

In the following steps we will verify $\chi(m,t) \geq 0$ on $(m,t) \in [-\mu,\lambda]\times[0,1]$. Since $\chi(m,0) \equiv 0$, $\chi(0,t) \equiv 0$, $\chi(m,1) = \sqrt{3}e^{\frac{3m}{2}}(e^{\frac{3m}{2}} - \cos\frac{\sqrt{3}m}{2} - \sqrt{3}\sin\frac{\sqrt{3}m}{2})$.

Set

$$w(m) \overset{\text{def}}{=} e^{\frac{3m}{2}} - \cos\frac{\sqrt{3}m}{2} - \sqrt{3}\sin\frac{\sqrt{3}m}{2}.$$

(i) In the case $(m,t) \in [-\mu,0]\times[0,1]$, the biggest negative solution of the equation $w(m) = 0$ is $m = -\mu$.

It is easy to see when $m \in [-\mu, 0]$, $w(m) \geq 0$. When $m = -\mu$, it is a unary function with respect to t. In order to get the stagnation point, let

$$\begin{cases} \dfrac{\partial \chi(m,t)}{\partial t} = 0 \\ m = -\mu. \end{cases}$$

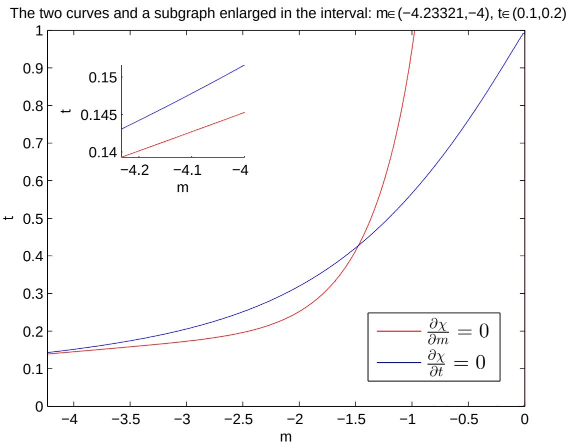

Figure 5.1

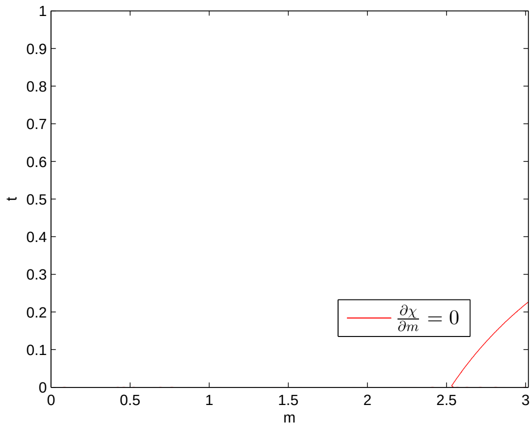

Figure 5.2

So $t_* \approx 0.1431$, and $\chi(-\mu, t_*) \approx 0.4043 > 0$. Now we will calculate the stagnation point inside the rectangle area, let

$$\begin{cases} \dfrac{\partial \chi(m,t)}{\partial m} = 0 \\ \dfrac{\partial \chi(m,t)}{\partial t} = 0, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} \frac{\partial \chi(m,t)}{\partial m} = & -2\sqrt{3}e^{\frac{3m}{2}} \cos \frac{\sqrt{3}m}{2} \\ & + e^{\frac{3m(1+t)}{2}} \left[\sqrt{3}(t+2) \cos \frac{\sqrt{3}m(t-1)}{2} - 3t \sin \frac{\sqrt{3}m(t-1)}{2} \right] \\ & - e^{\frac{3mt}{2}} t \left(\sqrt{3} \cos \frac{\sqrt{3}mt}{2} + 3 \sin \frac{\sqrt{3}mt}{2} \right), \\ \frac{\partial \chi(m,t)}{\partial t} = & -e^{\frac{3mt}{2}} m \left(\sqrt{3} \cos \frac{\sqrt{3}mt}{2} + 3 \sin \frac{\sqrt{3}mt}{2} \right) \\ & + e^{\frac{3m(1+t)}{2}} m \left[\sqrt{3} \cos \frac{\sqrt{3}m(t-1)}{2} - 3 \sin \frac{\sqrt{3}m(t-1)}{2} \right]. \end{aligned}$$

Apparently, $\{(m,t) \mid m = 0, t \in [0,1]\}$ solve the system (5.2), while there may exist other solutions. In Figure 5.1, by computing we obtain $t_* \approx 0.4274818$, $m_* \approx -1.47531$, and $\chi(m_*, t_*) \approx 0.4797 > 0$. Consequently, $\chi(m,t) \geq 0$ on $(m,t) \in [-\mu, 0] \times [0,1]$.

(ii) When $(m,t) \in [0,\lambda] \times [0,1]$ it is easy to prove that if $m > 0$, then $w(m) > 0$. When $m = \lambda$, we get a unary function with respect to t. So, let

$$\begin{cases} \dfrac{\partial \chi(m,t)}{\partial t} = 0 \\ m = \lambda. \end{cases}$$

By calculating, we gain $\frac{\partial \chi(m,t)}{\partial t}\big|_{t=0} = \sqrt{3} m e^{\frac{3m}{2}} g(m)$, and the stagnation point $t_* = 0$, hence $\chi(m,0) \equiv 0$. It only remains to verify the stagnation point inside the rectangle region. Therefore, we also use the system (5.2).

Obviously, $\{(m,t) \mid m = 0, t \in [0,1]\}$ solve the system (5.2) while there may be other solutions. From the Figure 5.2 we know that there exists no other solution. Thus, we can get $\chi(m,t) \geq 0$ on $(m,t) \in [0,\lambda] \times [0,1]$.

In summary, when $(m,t) \in [-\mu,\lambda] \times [0,1]$, we get $\chi(m,t) \geq 0$. Therefore, $B \geq 0$. But when $m = \lambda$, the denominator $R(t)$ vanishes. In the case $m = \lambda$, $B \geq 0$ ($B = 0$ if and only if $t = 0$), $C \geq 0$ ($C = 0$ if and only if $t = 0$), hence $B + C \geq 0$. So $m \neq \lambda$. When $m \in [-\mu,\lambda)$, from the property of the function $g(m)$, we know that the denominator $\sqrt{3} m e^{3m/2} g(m) \geq 0$, (the equality holds if and only if $m = 0$) then $\frac{B}{\sqrt{3} m e^{3m/2} g(m)} \geq 0$. By the property of the function $h(m)$, we can see that if $t \in [0,1], m \in [-\mu,\lambda)$, then $C \geq 0$, therefore $\frac{C}{\sqrt{3} m e^{3m/2} g(m)} \geq 0$, $R(t) \geq 0$. At last, we can get $u(t) \geq 0$. □

Remark 5.2. A principle of comparison has been proved in the reference [11], it says the following.

Assume $0 < N \leq 2$, if $q(t) \in C^2[0,1]$ satisfies

$$q''(t) \geq N \int_0^t q(s)\,\mathrm{d}s, \qquad t \in [0,1],\ q(0) \leq 0,\ q(1) \leq 0,$$

then $q(t) \leq 0,\ t \in [0,1]$.

That is to say, when $-2 \leq M < 0$, if $u(t) \in C^3[0,1]$ satisfies

$$\begin{cases} u'''(t) + Mu(t) \leq 0, \quad t \in [0,1] \\ u(0) = 0,\ u'(0) \geq 0,\ u'(1) \geq 0, \end{cases}$$

then $u(t) \geq 0$ for arbitrary $t \in [0,1]$.

By the above Lemma, we generalize the result from $-2 \leq M < 0$ to $-\mu^3 \leq M < \lambda^3$.

6 Application

In this section we will utilize Theorem 5.1 to study the solvability of the boundary problem (6.1):

$$\begin{cases} u'''(t) + Mu(t) = f(t,u(t)) \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{6.1}$$

Definition 6.1. If there exist $\alpha, \beta \in C^3[0,1]$ satisfying

$$\begin{cases} \alpha'''(t) + M\alpha(t) \geq f(t,\alpha(t)), \quad t \in [0,1] \\ \alpha(0) \leq 0,\ \alpha'(0) \leq 0,\ \alpha'(1) \leq 0, \end{cases} \tag{6.2}$$

$$\begin{cases} \beta'''(t) + M\beta(t) \le f(t, \beta(t)), & t \in [0,1] \\ \beta(0) \ge 0, \ \beta'(0) \ge 0, \ \beta'(1) \ge 0, \end{cases} \tag{6.3}$$

where $m \in [-\mu, \lambda)$, $(M = m^3)$. Then, α, β are respectively called the lower and upper solutions of the boundary value problem (6.1).

Note that the equation (6.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) \, ds.$$

So we define an operator $T\colon C[0,1] \to C[0,1]$ as

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s)) \, ds.$$

$u(t)$ is a solution of (6.1) if and only if $u(t)$ is a fixed point of T.

Lemma 6.2. *If α, β are respectively the lower and upper solutions of the boundary value problem (6.1), then $\alpha \le T\alpha$, $T\beta \le \beta$.*

Proof. Let $\alpha(t)$ be a lower solution of (6.1), $u(t)$ be a solution of the following problem

$$\begin{cases} u'''(t) + Mu(t) = f(t, \alpha(t)) \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{6.4}$$

then

$$u(t) = \int_0^1 G(t,s) f(s, \alpha(s)) \, ds = T\alpha(t).$$

Since

$$\alpha'''(t) + M\alpha(t) \ge f(t, \alpha(t)) = u'''(t) + Mu(t),$$

we obtain

$$\begin{cases} (u-\alpha)'''(t) + M(u-\alpha)(t) \le 0 \\ (u-\alpha)(0) = 0, \ (u-\alpha)'(0) \ge 0, \ (u-\alpha)'(1) \ge 0. \end{cases}$$

By Theorem 5.1, we know $(u-\alpha)(t) \ge 0$, thus $T\alpha(t) = u(t) \ge \alpha(t)$. Following the same lines we can get $T\beta \le \beta$. □

Theorem 6.3. *Assume that $f \in C([0,1] \times R, R)$ is continuous and is increasing with respect to the second variable. If problem (6.1) has an upper solution $\beta(t) \in C^3[0,1]$ and a lower solution $\alpha(t) \in C^3[0,1]$ such that $\alpha(t) \le \beta(t)$, $(t \in [0,1])$, then it has at least one solution in D, where*

$$D = \{x \in C[0,1] \mid \alpha(t) \le x(t) \le \beta(t), \quad t \in [0,1]\},$$

provided $m \in [-\mu, \lambda)$ $(M = m^3)$.

Proof. If $\alpha(t)$, $\beta(t)$ are the lower and upper solutions of problem (6.1), then by Lemma 6.2, we have $\alpha \le T\alpha$, $T\beta \le \beta$. It is easy to prove that

(1) T is an increasing operator and maps D in D;

(2) T is compact.

(6.1) has a solution in D. □

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References

[1] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin-New York, 1971. MR0460785

[2] A. Cabada, The method of lower and upper solutions for second, third, fourth and higher order boundary value problems, *J. Math. Anal. Appl.* **185**(1994), 302–320. MR1283059; url

[3] Y. Feng, S. Liu, Multiplicity of solutions for a nonlinear third-order boundary value problem (in Chinese), *Chinese J. Engrg. Math.* **20**(2003), No. 4, 109–112. MR2035434

[4] M. Greguš, *Third order linear differential equations*, Mathematics and its Applications (East European Series), Vol. 22, D. Reidel Publishing Co., Dordrecht, 1987. MR882545

[5] L. K. Jackson, Existence and uniqueness of solutions of boundary value problems for third order differential equations *J. Differential Equations* **13**(1973), 432–437. MR0335925

[6] G. Klaasen, Differential inequalities and existence theorems for second and third order boundary value problems, *J. Differential Equations* **10**(1971), 529–537. MR0288397

[7] J. Mawhin, Topological degree and boundary value problems for nonlinear differential equations, in: M. Furi, P. Zecca (Eds.), *Topological methods for ordinary differential equations*, Lecture Notes in Mathematics, Vol. 1537, Springer, New York/Berlin, 1993, pp. 74–142. MR1226930

[8] R. Ma, Disconjugacy and extremal solutions of nonlinear third-order equations, *Commun. Pure Appl. Anal.* **13**(2014), No. 3, 1223–1236. MR3177697

[9] P. J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Differential Equations* **190**(2003), 643–662. MR1970045; url

[10] J. R. Ward, Jr., Periodic solutions of ordinary differential equations with bounded nonlinearities, *Topol. Method Nonl. Anal.* **19**(2002), No. 2, 275–282. MR1921049

[11] Q. Yao, Y. Feng, The existence of solutions for a third order two-point boundary value problem, *Appl. Math. Lett.* **15**(2002), 227–232. MR1880762; url