



Multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects

Tengfei Shen and Wenbin Liu 

College of Sciences, China University of Mining and Technology, Xuzhou 221116, China

Received 14 July 2015, appeared 30 December 2015

Communicated by Petru Jebelean

Abstract. This paper deals with the multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects. By using critical point theory, a new result is obtained. An example is given to illustrate the main result.

Keywords: critical point theory, boundary value problems, impulsive effects, quasilinear equations.

2010 Mathematics Subject Classification: 34B15, 34B18, 34B37.

1 Introduction

Consider the following problem with impulses

$$\begin{cases} -u''(t) + a(t)u(t) - (|u(t)|^2)''u(t) = f(t, u(t)), & t \in J, \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, m, \\ u(0) = u(T) = 0, \end{cases} \quad (1.1)$$


where $t_0 = 0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $J = [0, T] \setminus \{t_1, t_2, \dots, t_m\}$, $f \in C([0, T] \times \mathbb{R}; \mathbb{R})$, $I_j \in C(\mathbb{R}; \mathbb{R})$, $a(t) \in L^\infty[0, T]$, $\Delta(u'(t_j)) = u'(t_j^+) - u'(t_j^-)$ and $u'(t_j^\pm) = \lim_{t \rightarrow t_j^\pm} u'(t)$, $j = 1, 2, \dots, m$.

This problem is derived from a class of quasilinear Schrödinger equation. When we look for the standing wave solution whose form is $\Psi(t, x) = e^{-i\omega t}u(x)$, $\omega \in \mathbb{R}$ of the following quasilinear Schrödinger equation

$$i\partial_t \Psi = -\Psi'' + W(x)\Psi - (|\Psi|^2)''\Psi - \mu|\Psi|^{q-1}\Psi, \quad x \in \mathbb{R}, \quad (1.2)$$

where $q > 1$, $\mu > 0$, we can obtain the elliptic equation of the form

$$-u'' + (W(x) - \omega)u - (|u|^2)''u = \mu|u|^{q-1}u, \quad x \in \mathbb{R}, \quad (1.3)$$

 Corresponding author. Email: cumt_equations@126.com

which was investigated by some scholars (see [2,4,9,19,22]).

It is generally known that critical point theory is a classical method to deal with the existence and multiplicity of solutions for differential equations (see [3,7,12,16,21,26,30]). Then a natural question is asked: Can we consider the multiplicity of solutions for second-order quasilinear equations with impulsive effects which are produced by the quasilinear term $(|u|^2)''u$ and u'' by using critical point theory?

Impulsive differential equations can be used to describe many evolution processes (see [5,10,11,14,17,27]). Some classical methods and theorems such as fixed point theorems, the method of lower and upper solutions and coincidence degree theory have been widely used to investigate impulsive differential equations (see [1,8,13,15,20]). Recently, critical point theory has been proved to be an effective tool to investigate boundary value problems for impulsive differential equations. Many valuable results have been obtained by some scholars (see [6,18,24,25,28,29]).

In [18], Nieto and O'Regan studied the linear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + \lambda u(t) = \sigma(t), & \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = d_j, & j = 1, 2, \dots, p, \\ u(0) = u(T) = 0, \end{cases} \quad (1.4)$$

and the nonlinear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), & \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, p, \\ u(0) = u(T) = 0. \end{cases} \quad (1.5)$$

and got some results by using critical point theory.

In [29], Zhou and Li investigated the nonlinear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + g(t)u(t) = f(t, u(t)), & \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, p, \\ u(0) = u(T) = 0. \end{cases} \quad (1.6)$$

and obtained the existence of infinitely many solutions by employing the Symmetric Mountain Pass Theorem.

However, there are few articles which considered the multiplicity of standing wave solutions for the impulsive Dirichlet boundary value problem involving the quasilinear term $(|u|^2)''u$. The impulsive effects which brought from the quasilinear term $(|u|^2)''u$ are more complicated than u'' .

Motivated by the works mentioned above, in this paper, our purpose is to investigate the multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects (1.1). Moreover, the nonlinearity f does not need to satisfy the Ambrosetti–Rabinowitz condition (see [3]). Furthermore, the impulsive terms $I_j(u)$ need to satisfy the superlinear condition rather than the sublinear condition as those in [18,23,28,29]. By making use of the variant fountain theory (see [30]), the multiplicity of solutions for the problem (1.1) are obtained.

2 Preliminaries

In this section, the following theorem will be needed in the proof of our main results. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j=k}^{\infty} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$.

Theorem 2.1 ([30, Theorem 2.2]). *The C^1 -functional $\Phi_\lambda : E \rightarrow \mathbb{R}$ defined by $\Phi_\lambda(u) = A(u) - \lambda B(u)$, $\lambda \in [1, 2]$, satisfies*

(B1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Moreover,

$$\Phi_\lambda(-u) = \Phi_\lambda(u) \text{ for all } (\lambda, u) \in [1, 2] \times E.$$

(B2) $B(u) \geq 0$; $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$ on any finite dimensional subspace of E .

(B3) There exist $\rho_k > r_k > 0$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) \text{ for all } \lambda \in [1, 2]$$

and

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \rightarrow 0 \text{ as } k \rightarrow +\infty \text{ uniformly for } \lambda \in [1, 2].$$

Then there exist $\lambda_n \rightarrow 1$, $u(\lambda_n) \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \quad \Phi_{\lambda_n}(u(\lambda_n)) \rightarrow c_k \in [d_k(2), b_k(1)] \text{ as } n \rightarrow +\infty.$$

Particularly, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k , then Φ_1 has infinitely many non-trivial critical points $\{u_k\} \in E \setminus \{0\}$ satisfying $\Phi_1(u_k) \rightarrow 0^-$ as $k \rightarrow +\infty$.

In the Sobolev space $H_0^1(0, T)$, consider the inner product

$$\langle u, v \rangle = \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u, v \in H_0^1(0, T),$$

inducing the norm

$$\|u\|_{H_0^1} = \left(\int_0^T |u(t)|^2 + |u'(t)|^2 dt \right)^{\frac{1}{2}}.$$

By Poincaré's inequality

$$\int_0^T |u(t)|^2 dt \leq \frac{1}{\sqrt{\lambda}} \int_0^T |u'(t)|^2 dt,$$

where $\lambda = \frac{\pi^2}{T^2}$ is the first eigenvalue of the problem $-u'' = \lambda u$ with Dirichlet boundary conditions, the norm $\|u\|_{H_0^1(0, T)}$ and $\|u'\|_{L^2}$ are equivalent.

But, in this paper, we define the following inner product in $H_0^1(0, T)$

$$\langle u, v \rangle_1 = \int_0^T a(t)u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u, v \in H_0^1(0, T),$$

whose norm is

$$\|u\| = \left(\int_0^T a(t)|u(t)|^2 + |u'(t)|^2 dt \right)^{\frac{1}{2}}.$$

Throughout our paper, we assume that $\text{ess inf}_{t \in [0, T]} a(t) > -\lambda$, which together with Lemma 2.1 in [29] yields that the norm $\|u\|_{H_0^1}$ and $\|u\|$ are equivalent. Thus, by the Sobolev Embedding Theorem, there exists a constant $c > 0$ such that $\|u\|_\infty := \max_{t \in [0, T]} |u(t)| \leq c\|u\|$.

For each $u \in H_0^1(0, T)$, u is absolutely continuous and $u' \in L^2(0, T)$. In this case, $\Delta u(t) = u'(t_j^+) - u'(t_j^-) = 0$ may not hold for any $t \in (0, T)$. It leads to the impulsive effects. Thus,

$$\begin{aligned} - \int_0^T (|u(t)|^2)'' u(t) v(t) dt &= - \sum_{j=0}^m \int_{t_j}^{t_{j+1}} (|u(t)|^2)'' u(t) v(t) dt \\ &= - \left(\sum_{j=0}^m 2u'(t_{j+1}^-) u^2(t_{j+1}^-) v(t_{j+1}^-) - 2u'(t_j^+) u^2(t_j^+) v(t_j^+) \right) \\ &\quad - \int_{t_j}^{t_{j+1}} 2u'^2(t) u(t) v(t) + 2u^2(t) u'(t) v'(t) dt \\ &= \sum_{j=1}^m 2\Delta u'(t_j) u^2(t_j) v(t_j) + 2u'(0) u^2(0) v(0) - 2u'(T) u^2(T) v(T) \\ &\quad + \int_0^T 2u'^2(t) u(t) v(t) + 2u^2(t) u'(t) v'(t) dt \\ &= \sum_{j=1}^m 2I_j(u(t_j)) u^2(t_j) v(t_j) + \int_0^T 2u'^2(t) u(t) v(t) + 2u^2(t) u'(t) v'(t) dt. \end{aligned}$$

Similarly, we have

$$- \int_0^T u''(t) v(t) dt = \sum_{j=1}^m I_j(u(t_j)) v(t_j) + \int_0^T u'(t) v'(t) dt.$$

Define the functional $\Phi : H_0^1(0, T) \rightarrow \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2} \|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} (2t^2 + 1) I_j(t) dt + \int_0^T u'^2(t) u^2(t) dt - \int_0^T F(t, u(t)) dt,$$

where $F(t, u) = \int_0^u f(t, s) ds$. Clearly, $\Phi \in C^1(H_0^1(0, T), \mathbb{R})$,

$$\begin{aligned} \langle \Phi'(u), v \rangle &= \int_0^T u'(t) v'(t) + a(t) u(t) v(t) dt + \int_0^T 2u'^2(t) u(t) v(t) + 2u^2(t) u'(t) v'(t) dt \\ &\quad + \sum_{j=1}^m (2u^2(t_j) + 1) I_j(u(t_j)) v(t_j) - \int_0^T f(t, u(t)) v(t) dt. \end{aligned}$$

Definition 2.2. A function $u \in H_0^1(0, T)$ is a weak solution of the problem (1.1), if it is a critical point of Φ .

Next, let

$$\Phi_\lambda(u) := A(u) - \lambda B(u),$$

where

$$\begin{aligned} A(u) &:= \frac{1}{2} \|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} (2t^2 + 1) I_j(t) dt + \int_0^T u'^2(t) u^2(t) dt, \\ B(u) &:= \int_0^T F(t, u(t)) dt, \end{aligned}$$

$\lambda \in [1, 2]$. Clearly, the critical points of $\Phi_1(u) = \Phi(u)$ correspond to the weak solutions of the problem (1.1). In $H_0^1(0, T)$, we can choose a completely orthonormal basis e_j and set $X_j = \mathbb{R}e_j$. Thus, Z_k and Y_k can be defined.

3 Main result

Theorem 3.1. *Assume that $F(t, u)$ is even about u and the following conditions are satisfied.*

(H1) $I_j(u)$ are odd about u and $I_j(u)u \geq 0$, ($j = 1, 2, \dots, m$).

(H2) There exist constants $b_j > 0$ and $\gamma_j \in [1, \infty)$ such that $|I_j(u)| \leq b_j |u|^{\gamma_j}$.

(H3) $F(t, u) = o(|u|^\nu)$ as $|u| \rightarrow 0$ uniformly on $[0, T]$.

(H4) There exist constants $l_1, L > 0$ such that

$$|f(t, u)| \leq l_1 |u|^p, \quad |u| \geq L, \quad p \in [0, 1), \quad t \in [0, T].$$

(H5) There exist constants $l_2, l_3 > 0$ such that

$$F(t, u) \geq l_2 |u|^\theta + l_3 |u|^\nu, \quad \theta, \nu \in [1, 2), \quad t \in [0, T].$$

Then the problem (1.1) has infinitely many solutions.

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Under the assumptions of Theorem 3.1, there exists a ρ_k small enough such that $a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0$ and $d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \rightarrow 0$ as $k \rightarrow +\infty$ uniformly for any $\lambda \in [1, 2]$.*

Proof. Let $\Gamma_k := \sup_{u \in Z_k, \|u\| = 1} \|u\|_\infty$. Then $\Gamma_k \rightarrow 0$ as $k \rightarrow +\infty$. By (H3), for given $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$F(t, u) \leq \epsilon_1 |u|^\nu, \quad |u| \leq \delta_1, \quad t \in [0, T].$$

Based on (H4), we have

$$F(t, u) \leq l_1 |u|^{p+1}, \quad |u| \geq L, \quad t \in [0, T].$$

From the continuity of $F(t, u)$, for $(t, |u|) \in [0, T] \times [\delta_1, L]$, there exists $M > 0$ such that

$$F(t, u) \leq \epsilon_1 |u|^\nu + l_1 |u|^{p+1} + M.$$

So, we have

$$F(t, u) \leq \epsilon_1 |u|^\nu + (M\delta_1^{-1-p} + l_1) |u|^{p+1}, \quad u \in \mathbb{R}, \quad t \in [0, T]. \quad (3.1)$$

Based on (H1), for any $u \in Z_k$ and $\|u\|$ small enough, we have

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2} \|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} (2t^2 + 1) I_j(t) dt + \int_0^T u'^2(t) u^2(t) dt - \lambda \int_0^T F(t, u(t)) dt \\ &\geq \frac{1}{2} \|u\|^2 - \lambda \epsilon_1 \int_0^T |u|^\nu dt - \lambda (M\delta_1^{-1-p} + l_1) \int_0^T |u|^{p+1} dt \\ &\geq \frac{1}{2} \|u\|^2 - 2\epsilon_1 T \Gamma_k^\nu \|u\|^\nu - 2T \Gamma_k^{p+1} (M\delta_1^{-1-p} + l_1) \|u\|^{p+1} \\ &\geq \frac{1}{8} \rho_k^2 \geq 0, \end{aligned}$$

where $\|u\| = \rho_k := (16T \Gamma_k^{p+1} (M\delta_1^{-1-p} + l_1) + 16\epsilon_1 T \Gamma_k^\nu)^{\frac{1}{1-p}}$ (without loss of generality, assume that $\nu \geq p + 1$). It is easy to find that $\rho_k \rightarrow 0$ as $k \rightarrow +\infty$. Thus, we can obtain that $a_k(\lambda) \geq 0$ and $d_k(\lambda) \rightarrow 0$ as $n \rightarrow +\infty$ uniformly for $\lambda \in [1, 2]$. \square

Lemma 3.3. *Under the assumptions of Theorem 3.1, there exists a r_k small enough such that $b_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0$ for $\lambda \in [1, 2]$.*

Proof. Let $M_1 = \max\{b_1, b_2, b_3, \dots\}$. For any $u \in Y_k$, by the equivalence of the norms on the finite-dimensional space Y_k and (H5), we have

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2} \|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} (2t^2 + 1) I_j(t) dt + \int_0^T u'^2(t) u^2(t) dt - \lambda \int_0^T F(t, u(t)) dt \\ &\leq \frac{1}{2} \|u\|^2 + 2M_1 \sum_{j=1}^m c^{3+\gamma_j} \|u\|^{3+\gamma_j} + M_1 \sum_{j=1}^m c^{1+\gamma_j} \|u\|^{1+\gamma_j} + c^2 c_1^2 \|u\|^4 \\ &\quad - \lambda l_2 \int_0^T |u|^\theta dt - \lambda l_3 \int_0^T |u|^\nu dt \\ &\leq \frac{1}{2} \|u\|^2 + 2M_1 \sum_{j=1}^m c^{3+\gamma_j} \|u\|^{3+\gamma_j} + M_1 \sum_{j=1}^m c^{1+\gamma_j} \|u\|^{1+\gamma_j} + c^2 c_1^2 \|u\|^4 \\ &\quad - \lambda l_2 c_2 \|u\|^\theta - \lambda l_3 c_3 \|u\|^\nu, \end{aligned}$$

which together with $\theta, \nu \in [1, 2)$ yields that $\Phi_\lambda(u) < 0$ for $\|u\| := r_k < \rho_k$ small enough and $\lambda \in [1, 2]$. \square

Lemma 3.4. *Under the assumptions of Theorem 3.1, there exist $\lambda_n \rightarrow 1$, $u(\lambda_n) \in Y_n$ such that*

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \quad \Phi_{\lambda_n}(u(\lambda_n)) \rightarrow c_k \in [d_k(2), b_k(1)] \quad \text{as } n \rightarrow +\infty.$$

Proof. Clearly, Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Since $F(t, u)$ is even about u and $I_j(u)$ are odd about u , we have $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times H_0^1(0, T)$. Furthermore, by (H5) and the equivalence of the norms on the finite-dimensional space on $H_0^1(0, T)$, there exist two positive constants c_4, c_5 such that $B(u) \geq l_2 c_4 \|u\|^\theta + l_3 c_5 \|u\|^\nu$. So, $B(u) \geq 0$, $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$. Thus, (B1) and (B2) are satisfied. By Lemma 3.2 and 3.3, (B3) holds. In view of Theorem 2.1, we can obtain Lemma 3.4. \square

Next, we show the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $u(\lambda_n) := u_n \in Y_n$. First, we will prove that $\{u_n\}$ is bounded on $H_0^1(0, T)$. Based on Lemma 3.4, there exist $\lambda_n \rightarrow 1$, $u_n \in Y_n$ such that $\Phi'_{\lambda_n}|_{Y_n}(u_n) = 0$, $\Phi_{\lambda_n}(u_n) \rightarrow c_k \in [d_k(2), b_k(1)]$ as $n \rightarrow +\infty$. Thus, we have

$$\begin{aligned} \Phi_{\lambda_n}(u_n) &= \frac{1}{2} \|u_n\|^2 + \sum_{j=1}^m \int_0^{u_n(t_j)} (2t^2 + 1) I_j(t) dt + \int_0^T u_n'^2 u_n^2 dt - \lambda_n \int_0^T F(t, u_n) dt \\ &\geq \frac{1}{2} \|u_n\|^2 - 2 \int_0^T F(t, u_n) dt. \end{aligned}$$

By the same way as Lemma 3.2, we have

$$\Phi_{\lambda_n}(u_n) \geq \frac{1}{2} \|u_n\|^2 - 2Tc^{p+1} (M\delta_1^{-1-p} + l_1) \|u_n\|^{p+1} - 2\epsilon_1 Tc^\nu \|u_n\|^\nu,$$

which implies that $\{u_n\}$ is bounded on $H_0^1(0, T)$. Then there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that $u_n \rightharpoonup u$ in $H_0^1(0, T)$ and $u_n \rightarrow u$ uniformly in

$C[0, T]$. Thus,

$$\begin{aligned}
 & \langle \Phi'_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u), u_n - u \rangle \rightarrow 0, \\
 & \int_0^T (u_n'^2(t)u_n(t) - u'^2(t)u(t))(u_n(t) - u(t))dt \rightarrow 0, \\
 & \sum_{j=1}^m (I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \rightarrow 0, \\
 & \sum_{j=1}^m (I_j(u_n(t_j))u_n^2(t_j) - I_j(u(t_j))u^2(t_j))(u_n(t_j) - u(t_j)) \rightarrow 0, \\
 & \int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t))dt \rightarrow 0,
 \end{aligned}$$

as $n \rightarrow +\infty$. Moreover,

$$\begin{aligned}
 & \int_0^T (u_n'^2(t)u_n'(t) - u'^2(t)u'(t))(u_n'(t) - u'(t))dt \\
 &= \int_0^T [(u_n'^2(t) - u'^2(t))u_n'(t) + u'^2(t)(u_n'(t) - u'(t))](u_n'(t) - u'(t))dt \\
 &= \int_0^T u_n'(t)(u_n'(t) - u'(t))(u_n'^2(t) - u'^2(t))dt + \int_0^T u'^2(t)|u_n'(t) - u'(t)|^2dt.
 \end{aligned}$$

Since

$$\int_0^T u_n'(t)(u_n'(t) - u'(t))(u_n'^2(t) - u'^2(t))dt \rightarrow 0$$

as $n \rightarrow +\infty$, we have

$$\begin{aligned}
 & \langle \Phi'_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u), u_n - u \rangle \\
 &= \|u_n - u\|^2 + 2 \int_0^T (u_n'^2(t)u_n(t) - u'^2(t)u(t))(u_n(t) - u(t))dt \\
 &\quad + 2 \int_0^T (u_n'^2(t)u_n'(t) - u'^2(t)u'(t))(u_n'(t) - u'(t))dt \\
 &\quad + \sum_{j=1}^m (I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \\
 &\quad + 2 \sum_{j=1}^m (I_j(u_n(t_j))u_n^2(t_j) - I_j(u(t_j))u^2(t_j))(u_n(t_j) - u(t_j)) \\
 &\quad - \lambda_n \int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t))dt \\
 &= \|u_n - u\|^2 + 2 \int_0^T u'^2(t)|u_n'(t) - u'(t)|^2dt + o(1),
 \end{aligned}$$

which implies that $u_n \rightarrow u$ in $H_0^1(0, T)$. Then Φ_1 has infinitely many nontrivial critical points $\{u^k\} \in H_0^1(0, T) \setminus \{0\}$ satisfying $\Phi_1(u^k) \rightarrow 0^-$ as $k \rightarrow +\infty$. Thus, the problem (1.1) has infinitely many solutions. \square

Example 3.5.

$$\begin{cases} -u''(t) + a(t)u(t) - (|u(t)|^2)''u(t) = f(t, u(t)), & t \in J, \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $a(t) = 1$, $t_1 = \frac{1}{2}$, $F(t, u) = |u|\ln(1 + |u|^{\frac{1}{2}}) + |u|^{\frac{5}{4}}(\sin |u|^{\frac{1}{2}} + 3)$, $I_j(u) = u^3$, $v = 1$. Clearly, the conditions of (H1), (H2), (H3) and (H5) are satisfied. Moreover,

$$|f(t, u)| \leq \ln(1 + |u|^{\frac{1}{2}}) + \frac{|u|^{\frac{1}{2}}}{2(1 + |u|^{\frac{1}{2}})} + 5|u|^{\frac{1}{4}} + \frac{1}{2}|u|^{\frac{3}{4}} \leq 2|u|^{\frac{4}{5}}, \quad |u| \geq L,$$

where L should be large enough. Thus, (H4) holds. Then Example 3.5 has infinitely many solutions.

Acknowledgements

The authors really appreciate the referee's valuable suggestions and comments, which improved the former version of this paper. This research is supported by the National Natural Science Foundation of China (No. 11271364).

References

- [1] R. P. AGARWAL, D. O'REGAN, Multiple nonnegative solutions for second order impulsive differential equations, *Appl. Math. Comput.* **114**(2000), 51–59. [MR1775121](#); [url](#)
- [2] C. O. ALVES, O. H. MIYAGAKI, S. H. M. SOARES, On the existence and concentration of positive solutions to a class of quasilinear elliptic problems on \mathbb{R} , *Math. Nachr.* **284**(2011), 1784–1795. [MR2838282](#); [url](#)
- [3] A. AMBROSETTI, P. H. RABINOWITZ, Dual variational methods in critical point theory and applications, *J. Func. Anal.* **14**(1973), 349–381. [MR0370183](#); [url](#)
- [4] A. AMBROSETTI, Z. Q. WANG, Positive solutions to a class of quasilinear elliptic equation on \mathbb{R} , *Discrete Contin. Dyn. Syst.* **9**(2003), 55–68. [MR1951313](#); [url](#)
- [5] H. BAEK, Extinction and permanence of a three-species Lotka–Volterra system with impulsive control strategies, *Discrete Dyn. Nat. Soc.* **2008**, Art. ID 752403, 18 pp. [MR2452458](#); [url](#)
- [6] L. BAI, B. X. DAI, Three solutions for a p -Laplacian boundary value problem with impulsive effects, *Appl. Math. comput.* **217**(2011), 9895–9904. [MR2806376](#); [url](#)
- [7] T. BARTSCH, Z. Q. WANG, Existence and multiplicity results for some superlinear elliptic problems on \mathbb{R}^N , *Comm. Partial Differential Equations* **20**(1995), 1725–1741. [MR1349229](#); [url](#)
- [8] M. BENCHOHRA, J. HENDERSON, S. NTOUYAS, *Impulsive differential equations and inclusions*, Contemporary Mathematics and Its Applications, Vol. 2, Hindawi Publishing Corporation, New York, 2006. [MR2322133](#)
- [9] H. CHEN, S. LIU, Standing waves with large frequency for 4-superlinear Schrödinger–Poisson systems, *Ann. Mat. Pura. Appl.* **194**(2015), 43–53. [MR3303004](#); [url](#)
- [10] M. CHOISY, J. F. GUÉGAN, P. ROHANI, Dynamics of infectious diseases and pulse vaccination: Teasing apart the embedded resonance effects, *Phys. D* **223**(2006), 26–35. [MR2304823](#); [url](#)

- [11] A. D'ONOFRIO, On pulse vaccination strategy in the SIR epidemic model with vertical transmission, *Appl. Math. Lett.* **18**(2005), 729–732. [MR2144719](#); [url](#)
- [12] L. C. EVANS, *Partial differential equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 1998. [MR1625845](#)
- [13] D. FRANCO, J. J. NIETO, Maximum principle for periodic impulsive first order problems, *J. Comput. Appl. Math.* **88**(1998), 149–159. [MR1609074](#) ; [url](#)
- [14] V. LAKSHMIKANTHAM, D. D. BAINOV, P. S. SIMEONOV, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, Vol. 6, World Scientific, Teaneck, NJ, 1989. [MR1082551](#)
- [15] X. N. LIN, D. Q. JIANG, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, *J. Math. Anal. Appl.* **321**(2006), 501–514. [MR2241134](#); [url](#)
- [16] J. MAWHIN, M. WILLEM, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Vol. 74, Springer-Verlag, New York, 1989. [MR0982267](#)
- [17] S. I. NENOV, Impulsive controllability and optimization problems in population dynamics, *Nonlinear Anal.* **36**(1999), 881–890. [MR1682836](#); [url](#)
- [18] J. J. NIETO, D. O'REGAN, Variational approach to impulsive differential equations, *Nonlinear Anal.* **10**(2009) 680–690. [MR2474254](#); [url](#)
- [19] M. POPPENBERG, K. SCHMITT, Z. Q. WANG, On the existence of soliton solutions to a quasilinear Shrodinger equations, *Calc. Var. Partial Differ. Equ.* **14**(2002), 329–344. [MR1899450](#); [url](#)
- [20] D. QIAN, X. LI, Periodic solutions for ordinary differential equations with sublinear impulsive effects, *J. Math. Anal. Appl.* **303**(2005), 288–303. [MR2113882](#); [url](#)
- [21] P. H. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, Washington DC, American Mathematical Society, 1986. [MR0845785](#)
- [22] Z. P. SHEN, Z. Q. HAN, Existence of solutions to quasilinear Schrödinger equations with indefinite potential, *Electron. J. Differential Equations* **2015**, No. 91, 1–9. [MR3337868](#)
- [23] J. T. SUN, H. B. CHEN, Multiplicity of solutions for a class of impulsive differential equations with Dirichlet boundary conditions via variant fountain theorems, *Nonlinear Anal.* **11**(2010) 4062–4071. [MR2683856](#); [url](#)
- [24] Y. TIAN, J. HENDERSON, Three anti-periodic solutions for second-order impulsive differential inclusions via nonsmooth critical point theory, *Nonlinear Anal.* **72**(2012), 6496–6505. [MR2965234](#); [url](#)
- [25] J. XIAO, J. J. NIETO, Z. G. LUO, Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods, *Commun. Nonlinear Sci. Numer. Simulat.* **17**(2012), 426–432. [MR2826020](#); [url](#)

- [26] E. ZEIDLER, *Nonlinear functional analysis and its applications*, Vol. III, Variational methods and optimization, Translated from the German by Leo F. Boron, Springer-Verlag, New York, 1985. [MR0768749](#)
- [27] H. ZHANG, L. CHEN, J. J. NIETO, A delayed epidemic model with stage-structure and pulses for pest management strategy, *Nonlinear Anal.* **9**(2008), 1714–1726. [MR2422575](#); [url](#)
- [28] Z. H. ZHANG, R. YUAN, An application of variational methods to Dirichlet boundary value problem with impulses, *Nonlinear Anal.* **11**(2010), 155–162. [MR2570535](#); [url](#)
- [29] J. W. ZHOU, Y. K. LI, Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, *Nonlinear Anal.* **71**(2009), 2856–2865. [MR2532812](#); [url](#)
- [30] W. ZOU, Variant fountain theorems and their applications, *Manuscripta Math.* **104**(2001), 343–358. [MR1828880](#); [url](#)