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Classification of cubic differential systems with invariant straight lines of total multiplicity eight and two distinct infinite singularities

Cristina Bujac and **Nicolae Vulpe** ✉

Institute of Mathematics and Computer Science, Academy of Science of Moldova,
5 Academiei str, Chișinău, MD-2028, Moldova

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Abstract. In this article we prove a classification theorem (Main theorem) of real planar cubic vector fields which possess two distinct infinite singularities (real or complex) and eight invariant straight lines, including the line at infinity and including their multiplicities. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of invariant polynomials. The algebraic invariants and comitants allow one to verify for any given real cubic system with two infinite distinct singularities whether or not it has invariant lines of total multiplicity eight, and to specify its configuration of lines endowed with their corresponding real singularities of this system. The calculations can be implemented on computer and the results can therefore be applied for any family of cubic systems in this class, given in any normal form.

Keywords: cubic vector fields, configuration of invariant lines, infinite and finite singularities, affine invariant polynomials.

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1 Introduction and the statement of the Main theorem

We consider here real planar differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where $P,\ Q \in \mathbb{R}[x,y]$, i.e. $P,\ Q$ are polynomials in $x,\ y$ over $\mathbb{R}$, and their associated vector fields

$$\widetilde{D} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

We say that systems (1.1) are *cubic* if $\max(\deg(P), \deg(Q)) = 3$.

✉ Corresponding author. Email: nvulpe@gmail.com

A straight line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies

$$\widetilde{D}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$$

for some polynomial $R(x,y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients u, v, w of an invariant straight line belongs to $\mathbb{C} \setminus \mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real.* Note that, since systems (1.1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

To a line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ we associate its projective completion $F(X,Y,Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbf{P}_2(\mathbb{C})$, $(x,y) \mapsto [x : y : 1]$. The line $Z = 0$ in $\mathbf{P}_2(\mathbb{C})$ is called *the line at infinity* of the affine plane $\mathbb{C}^2$. It follows from the work of Darboux (see, for instance, [11]) that each system of differential equations of the form (1.1) over $\mathbb{C}$ yields a differential equation on the complex projective plane $\mathbf{P}_2(\mathbb{C})$ which is the compactification of the differential equation $Qdx - Pdy = 0$ in $\mathbb{C}^2$. The line $Z = 0$ is an invariant manifold of this complex differential equation.

Definition 1.1. ([28]) We say that an invariant affine straight line $f(x,y) = ux + vy + w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\widetilde{D}$ has **multiplicity** m if there exists a sequence of real cubic vector fields $\widetilde{D}_k$ converging to $\widetilde{D}$, such that each $\widetilde{D}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $f_k^j = u_k^j x + v_k^j y + w_k^j = 0$, $(u_k^j, v_k^j) \neq (0,0)$, $(u_k^j, v_i^k, w_k^j) \in \mathbb{C}^3$ $(j \in \{1, \dots m\}$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).

We mention here some references on polynomial differential systems possessing invariant straight lines. For quadratic systems see [12, 25, 26, 28–31] and [32]; for cubic systems see [5–8, 16–19, 27, 35] and [36]; for quartic systems see [34] and [37]; for some more general systems see [14, 22, 23] and [24].

According to [2] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [10].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [17]. A new class of cubic systems omitted in [17] was constructed in [5].

In [7] a complete classifications of the family of cubic systems with eight invariant straight lines, including the line at infinity and including their multiplicities was done in the case of the existence of four distinct infinite singularities (real or complex). This classification was continued in [8] in the case of the existence of three distinct singularities (real and complex).

This paper is a continuation of the above mentioned two ones. More exactly, here we shall consider the family of cubic systems with invariant lines of total multiplicity eight (including the line at infinity and considering their multiplicities) in the case of the existence of two distinct singularities (real or complex).

It is well known that for a cubic system (1.1) there exist at most 4 different slopes for invariant affine straight lines, for more information about the slopes of invariant straight lines for polynomial vector fields, see [1].

Definition 1.2. ([32]). Consider a planar cubic system (1.1). We call **configuration of invariant straight lines** of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

Remark 1.3. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [24]. Thus we denote by $'(a,b)'$ the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of the multiple point.

Suppose that a cubic system (1.1) possesses 8 distinct invariant straight lines (including the line at infinity). We say that these lines form a *configuration of type* $(3,3,1)$ if there exist two triplets of parallel lines and one additional line, every set with different slopes. And we say that these lines form a *configuration of type* $(3,2,1,1)$ if there exist one triplet and one couple of parallel lines and two additional lines, every set with different slopes. Similarly are defined *configurations of types* $(3,2,2)$ and $(2,2,2,1)$ and these four types of the configurations exhaust all possible configurations formed by 8 invariant lines for a cubic system.

Note that in all configurations the invariant straight line which is omitted is the infinite one.

Suppose a cubic system (1.1) possesses 8 invariant straight lines, including the infinite one, and taking into account their multiplicities. We say that these lines form a *potential configuration of type* $(3,3,1)$ (respectively, $(3,2,2)$; $(3,2,1,1)$; $(2,2,2,1)$) if there exists a sequence of vector fields $\widetilde{D}_k$ as in Definition 1.1 having 8 distinct lines of type $(3,3,1)$ (respectively, $(3,2,2)$; $(3,2,1,1)$; $(2,2,2,1)$).

It is well known that the infinite singularities (real and/or complex) of cubic systems are determined by the linear factors of the polynomial $C_3(x,y) = yp_3(x,y) - xq_3(x,y)$ where p_3 and q_3 are the cubic homogeneities of these systems.

In this paper we consider the family of cubic systems possessing two distinct infinite singularities defined by two distinct factors of the invariant polynomial $C_3(x,y)$. Since this binary form is of degree 4 with respect to x and y we arrive at the following three possibilities concerning the factors of $C_3(x,y)$:

- two double real factors;
- two double complex factors;
- one triple and one simple factor, both real.

These three possibilities are distinguished by affine invariant criteria and in each one of the cases we indicate the corresponding canonical form of cubic homogeneities obtained via a linear transformation (see Lemma 2.9).

Our results are stated in the following theorem.

Main theorem. *Assume that a non-degenerate cubic system (i.e.* $\sum_{i=0}^{9}\mu_i^2 \neq 0$*) possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has two distinct infinite singularities, i.e. the conditions* $\mathcal{D}_1 = \mathcal{D}_3 = 0$ *and* $\mathcal{D}_2 \neq 0$ *hold. Then the following statements hold.*

(A) *This system could not have the infinite singularities defined by two double factors of the invariant polynomial* $C_3(x,y)$*.*

(B) *The system has the infinite singularities defined by one triple and one simple real factor of* $C_3(x,y)$ *(i.e.* $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$ *and* $\mathcal{D}_2 \neq 0$*) and could possess one of the 25 possible configurations Config. 8.23 – Config. 8.47 of invariant lines given in Figure 1.1.*

(C) *This system possesses the specific configuration Config. 8.j* $(j \in \{23, 24, \dots, 47\})$ *if and only if the corresponding conditions included below are fulfilled. Moreover it can be brought via an affine transformation and time rescaling to the canonical form, written below next to the configuration:*

- *Config. 8.23* $\Leftrightarrow N_2N_3 \neq 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_5 = N_6 = N_7 = 0$:
$$\begin{cases} \dot{x} = (x-1)x(1+x), \\ \dot{y} = x - y + x^2 + 3xy; \end{cases}$$
- *Config. 8.24–8.27* $\Leftrightarrow N_2 \neq 0, N_3 = 0,\ \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_4 = N_6 = N_8 = 0, N_9 \neq 0$:
$$\begin{cases} \dot{x} = x(r + 2x + x^2), \\ \dot{y} = (r+2x)y,\ r(9r-8) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.24} \Leftrightarrow N_{11} < 0\ (r<0); \\ \textit{Config. 8.25} \Leftrightarrow N_{10} > 0, N_{11} > 0\ (0<r<1); \\ \textit{Config. 8.26} \Leftrightarrow N_{10} = 0\ (r=1); \\ \textit{Config. 8.27} \Leftrightarrow N_{10} < 0\ (r>1); \end{cases}$$
- *Config. 8.28–8.30* $\Leftrightarrow N_2 \neq 0, N_3 = 0,\ \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_5 = N_8 = N_{12} = 0, N_{13} \neq 0$:
$$\begin{cases} \dot{x} = x(r - 2x + x^2), (9r-8) \neq 0 \\ \dot{y} = 2y(x-r), r(r-1) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.28} \Leftrightarrow N_{15} < 0\ (r<0); \\ \textit{Config. 8.29} \Leftrightarrow N_{14} < 0, N_{15} > 0\ (0<r<1); \\ \textit{Config. 8.30} \Leftrightarrow N_{14} > 0\ (r>1); \end{cases}$$
- *Config. 8.31, 8.32* $\Leftrightarrow N_2 = N_3 = 0,\ \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{17} = N_{18} = 0, N_{10}N_{16} \neq 0$:
$$\begin{cases} \dot{x} = x(r + x^2), \\ \dot{y} = x - 2ry,\ r \in \{-1, 1\}; \end{cases} \quad \begin{cases} \textit{Config. 8.31} \Leftrightarrow N_{10} < 0\ (r=-1); \\ \textit{Config. 8.33} \Leftrightarrow N_{10} > 0,\ (r=1); \end{cases}$$
- *Config. 8.33* $\Leftrightarrow N_2 = N_3 = 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{10} = N_{17} = N_{18} = 0, N_{16} \neq 0$: $\begin{cases} \dot{x} = x^3, \\ \dot{y} = 1 + x; \end{cases}$
- *Config. 8.34–8.38* $\Leftrightarrow N_2 = N_3 = 0, \mathcal{V}_1 = \mathcal{V}_3 = \mathcal{K}_5 = N_1 = N_{16} = N_{19} = 0, N_{18} \neq 0$:
$$\begin{cases} \dot{x} = x(r + x + x^2), \\ \dot{y} = 1 + ry, (9r-2) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.34} \Leftrightarrow N_{21} < 0\ (r<0); \\ \textit{Config. 8.35} \Leftrightarrow N_{20} > 0, N_{21} > 0\ (0<r<1/4); \\ \textit{Config. 8.36} \Leftrightarrow N_{20} = 0\ (r=1/4); \\ \textit{Config. 8.37} \Leftrightarrow N_{20} < 0\ (r>1/4); \\ \textit{Config. 8.38} \Leftrightarrow N_{21} = 0\ (r=0). \end{cases}$$
- *Config.8.39, 8.40* $\Leftrightarrow \mathcal{V}_1 = \mathcal{L}_1 = \mathcal{L}_2 = N_{22} = N_{23} = N_{24} = 0,\ \mathcal{V}_3\mathcal{K}_6 \neq 0$:
$$\begin{cases} \dot{x} = x(r + x + x^2), \\ \dot{y} = (r + 2x + 3x^2)y; \end{cases} \quad \begin{cases} \textit{Config. 8.39} \Leftrightarrow \mu_6 < 0\ (r<1/4); \\ \textit{Config. 8.40} \Leftrightarrow \mu_6 > 0\ (r>1/4). \end{cases}$$
- *Config. 8.41–8.43* $\Leftrightarrow \mathcal{V}_1 = \mathcal{L}_1 = \mathcal{L}_2 = N_{22} = N_{23} = \mathcal{K}_6 = 0,\ \mathcal{V}_3N_{24} \neq 0$:
$$\begin{cases} \dot{x} = x(r + x^2), \\ \dot{y} = 1 + ry + 3x^2y; \end{cases} \quad \begin{cases} \textit{Config. 8.41} \Leftrightarrow \mu_6 < 0\ (r<0); \\ \textit{Config. 8.42} \Leftrightarrow \mu_6 = 0\ (r=0); \\ \textit{Config. 8.43} \Leftrightarrow \mu_6 > 0\ (r>0). \end{cases}$$
- *Config. 8.44–8.47* $\Leftrightarrow \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = N_{24} = N_{25} = N_{26} = N_{27} = 0,\ \mathcal{V}_1\mathcal{V}_3 \neq 0$:
$$\begin{cases} \dot{x} = x(1+x)[r + 2 + (r+1)x], \\ \dot{y} = [r + 2 + (3+2r)x + rx^2]y; \end{cases} \quad \begin{cases} \textit{Config. 8.44} \Leftrightarrow \mu_6 < 0\ (-2<r<-1); \\ \textit{Config. 8.45} \Leftrightarrow \mu_6 > 0, N_{28} < 0\ (r<-2); \\ \textit{Config. 8.46} \Leftrightarrow \mu_6 > 0, N_{28} > 0\ (r>-1); \\ \textit{Config. 8.47} \Leftrightarrow \mu_6 = 0\ (r=-1). \end{cases}$$

Remark 1.4. If in a configuration an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant straight lines, their corresponding multiplicities.

2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv p(x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv q(x,y) \end{aligned} \tag{2.1}$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0,1,2,3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} p_0 &= a_{00}, & p_3(x,y) &= a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ p_1(x,y) &= a_{10}x + a_{01}y, & p_2(x,y) &= a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, & q_3(x,y) &= b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ q_1(x,y) &= b_{10}x + b_{01}y, & q_2(x,y) &= b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03})$ be the 20-tuple of the coefficients of systems (2.1) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03}, x, y]$.

2.1 The main invariant polynomials associated to configurations of invariant lines

It is known that on the set **CS** of all cubic differential systems (2.1) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane [28]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (2.1) with a subset of $\mathbb{R}^{20}$ via the map **CS** $\longrightarrow \mathbb{R}^{20}$ which associates to each system (2.1) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [28]. Here we shall only construct the necessary T- and CT-comitants associated to configurations of invariant lines for the family of cubic systems mentioned in the statement of Main theorem.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y], \qquad i = 0,1,2,3, \\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y], \qquad i = 1,2,3. \end{aligned}$$

which in fact are GL-comitants, see [33]. Let $f, g \in \mathbb{R}[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \, \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called *the transvectant* of index k of (f,g) (cf. [13], [20]).

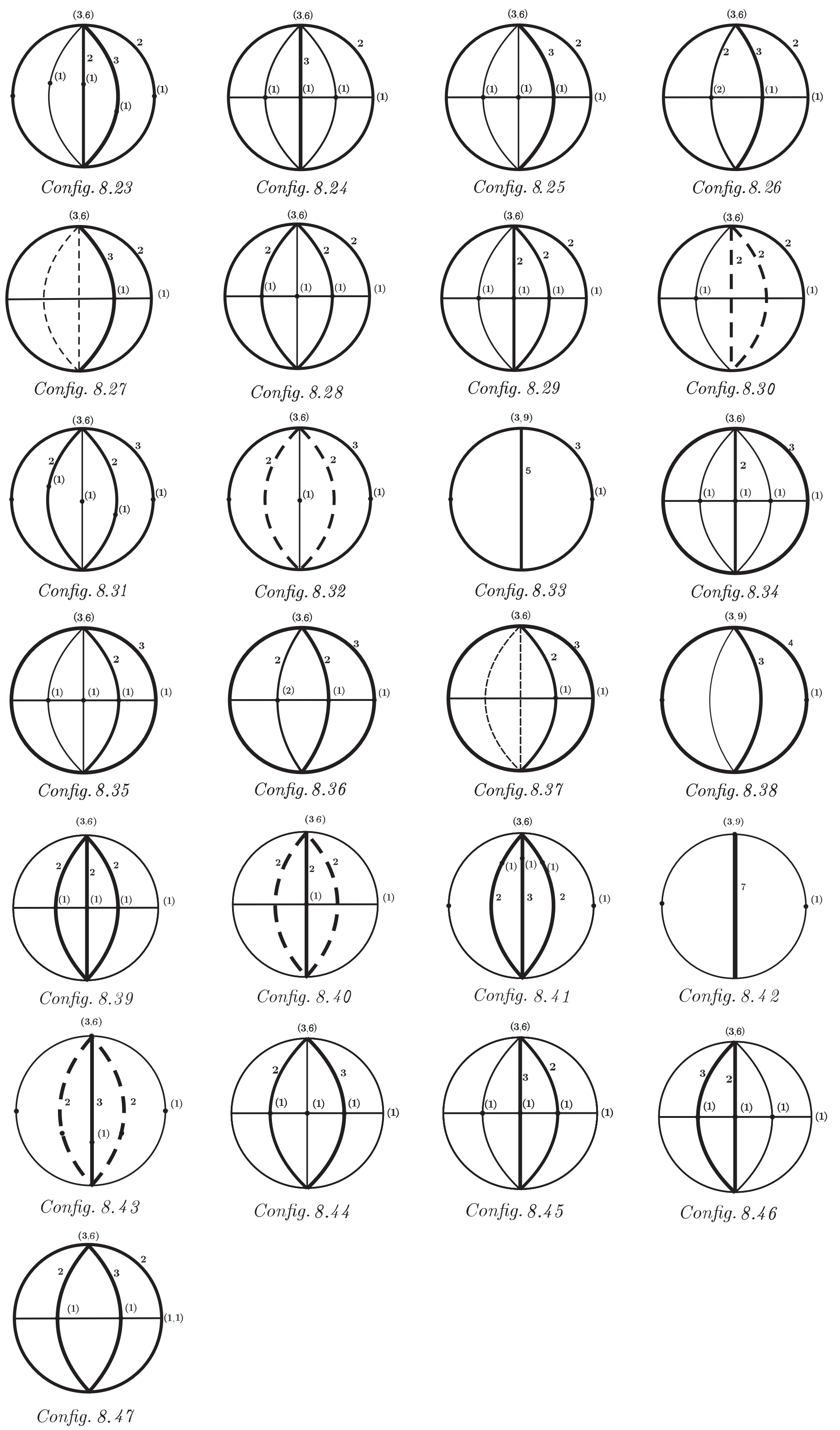


Figure 1.1: The configurations of invariant lines of total multiplicity 8 for cubic systems with 2 distinct infinite singularities

We apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(a,x,y)$ and $q(a,x,y)$ and we obtain $\tilde{p}(\tilde{a}(a,x_0,y_0),x',y') = p(a,x'+x_0,y'+y_0)$, $\tilde{q}(\tilde{a}(a,x_0,y_0),x',y') = q(a,x'+x_0,y'+y_0)$. Let us construct the following polynomials:

$$\Omega_i(a,x_0,y_0) \equiv \mathrm{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big),C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \qquad (i=1,2,3).$$

Remark 2.1. We note that the constructed polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of systems (2.1) and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{02}$ and non-homogeneous in x,y and

$$\deg_a \mathcal{G}_1 = 3, \qquad \deg_a \mathcal{G}_2 = 4, \qquad \deg_a \mathcal{G}_3 = 5,$$
$$\deg_{(x,y)} \mathcal{G}_1 = 8, \qquad \deg_{(x,y)} \mathcal{G}_2 = 10, \qquad \deg_{(x,y)} \mathcal{G}_3 = 12.$$

Notation 1. *Let $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$, i.e.*

$$\mathcal{G}_1(a,X,Y,Z) = Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),$$
$$\mathcal{G}_2(a,X,Y,Z) = Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z) = Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

and $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of the above defined affine comitants is given by the two following lemmas (see [17]).

Lemma 2.2. *The straight line $\mathcal{L}(x,y) \equiv ux+vy+w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a cubic system (2.1) if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ over $\mathbb{C}$, i.e. $\tilde{\mathcal{G}}_i(a,x,y) = (ux+vy+w)\widetilde{W}_i(x,y)$ $(i=1,2,3)$, where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.*

Lemma 2.3. *Consider a cubic system (2.1) and let $a \in \mathbb{R}^{20}$ be its 20-tuple of coefficients.*

1) If $\mathcal{L}(x,y) \equiv ux+vy+w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant straight line of multiplicity k for this system then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1,\tilde{\mathcal{G}}_2,\tilde{\mathcal{G}}_3)$ in $\mathbb{C}[x,y]$, i.e. there exist $W_i(a,x,y) \in \mathbb{C}[x,y]$ $(i=1,2,3)$ such that

$$\tilde{\mathcal{G}}_i(a,x,y) = (ux+vy+w)^k W_i(a,x,y), \qquad i=1,2,3. \tag{2.2}$$

2) If the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ then $Z^{k-1} \mid \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$, i.e. we have $Z^{k-1} \mid H(a,X,Y,Z)$.

Consider the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ constructed in [4] and acting on $\mathbb{R}[a,x,y]$, where

$$\begin{aligned}\mathbf{L}_1 = {}& 3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}}\\ &+ 3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}},\\ \mathbf{L}_2 = {}& 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}}\\ &+ 3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.\end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \text{Resultant}_x\big(p_3(a,x,y), q_3(a,x,y)\big)/y^9$ we construct the following polynomials

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \qquad i = 1,..,9,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [4]). The polynomial $\mu_i(a,x,y)$, $i \in \{0,1,\ldots,9\}$ is homogeneous of degree 6 in the coefficients of systems (2.1) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomials is revealed in the next lemma.

Lemma 2.4. ([3,4]) *Assume that a cubic system (S) with coefficients $\tilde{a}$ belongs to the family (2.1). Then the following statements hold.*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0,1,\ldots,k-1\}$ we have $\mu_i(\tilde{a},x,y) = 0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y) \neq 0$. In this case the factorization $\mu_k(\tilde{a},x,y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system (S) which "have gone" to infinity. Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have collapsed with the infinite singular point $[v_i : u_i : 0]$.

(ii) The system (S) is degenerate (i.e. $\gcd(p,q) \neq \text{const}$*) if and only if $\mu_i(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i = 0,1,\ldots,9$.*

In order to define the needed invariant polynomials we first construct the following comitants of second degree with respect to the coefficients of the initial systems:

$$\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}$$

We shall use here the following invariant polynomials constructed in [17] to characterize the family of cubic systems possessing the maximal number of invariant straight lines:

$$\mathcal{D}_1(a) = 6S_{24}^3 - \Big[(C_3, S_{23})^{(4)}\Big]^2, \qquad \mathcal{D}_2(a,x,y) = -S_{23},$$

$$\mathcal{D}_3(a,x,y) = (S_{23}, S_{23})^{(2)} - 6C_3(C_3, S_{23})^{(4)}, \qquad \mathcal{D}_4(a,x,y) = (C_3, \mathcal{D}_2)^{(4)},$$

$$\mathcal{V}_1(a,x,y) = S_{23} + 2D_3^2, \qquad \mathcal{V}_2(a,x,y) = S_{26}, \; \mathcal{V}_3(a,x,y) = 6S_{25} - 3S_{23} - 2D_3^2,$$

$$\mathcal{V}_4(a,x,y) = C_3\Big[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\Big],$$

$$\mathcal{V}_5(a,x,y) = 6T_1(9A_3 - 7A_4) + 2T_2(4T_5 - T_6) - 3T_3(3A_1 + 5A_2) + 3A_2T_4 + 36T_5^2 - 3T_{44},$$

$$\begin{aligned}
\mathcal{L}_1(a,x,y) =&\ 9C_2\,(S_{24}+24S_{27}) - 12D_3\,(S_{20}+8S_{22}) - 12\,(S_{16},D_3)^{(2)}\\
&- 3\,(S_{23},C_2)^{(2)} - 16\,(S_{19},C_3)^{(2)} + 12\,(5S_{20}+24S_{22},C_3)^{(1)}\,,\\
\mathcal{L}_2(a,x,y) =&\ 32\,(13S_{19}+33S_{21},D_2)^{(1)} + 84\,(9S_{11}-2S_{14},D_3)^{(1)}\\
&+ 8D_2\,(12S_{22}+35S_{18}-73S_{20}) - 448\,(S_{18},C_2)^{(1)}\\
&- 56\,(S_{17},C_2)^{(2)} - 63\,(S_{23},C_1)^{(2)} + 756D_3S_{13} - 1944D_1S_{26}\\
&+ 112\,(S_{17},D_2)^{(1)} - 378\,(S_{26},C_1)^{(1)} + 9C_1\,(48S_{27}-35S_{24})\,,\\
\mathcal{U}_1(a) =&\ T_{31} - 4T_{37},\ \mathcal{U}_2(a,x,y) = 6\,(T_{30}-3T_{32},T_{36})^{(1)} - 3T_{30}\,(T_{32}+8T_{37})\\
&- 24T_{36}^2 + 2C_3\,(C_3,T_{30})^{(4)} + 24D_3\,(D_3,T_{36})^{(1)} + 24D_3^2T_{37}.\\
\mathcal{K}_1(a,x,y) =&\ \left(3223T_2^2T_{140} + 2718T_4T_{140} - 829T_2^2T_{141}, T_{133}\right)^{(10)}/2,\\
\mathcal{K}_2(a,x,y) =&\ T_{74}, \qquad \mathcal{K}_4(a,x,y) = T_{13} - 2T_{11},\\
\mathcal{K}_5(a,x,y) =&\ 45T_{42} - T_2T_{14} + 2T_2T_{15} + 12T_{36} + 45T_{37} - 45T_{38} + 30T_{39},\\
\mathcal{K}_6(a,x,y) =&\ 4T_1T_8(2663T_{14} - 8161T_{15}) + 6T_8(178T_{23} + 70T_{24} + 555T_{26}) +\\
&+ 18T_9(30T_2T_8 - 488T_1T_{11} - 119T_{21}) + 5T_2(25T_{136} + 16T_{137}) -\\
&- 15T_1(25T_{140} - 11T_{141}) - 165T_{142},\\
\mathcal{K}_8(a,x,y) =&\ 10A_4T_1 - 3T_2T_{15} + 4T_{36} - 8T_{37}.
\end{aligned}$$

However, these invariant polynomials are not sufficient to characterize the cubic systems with invariant lines of the total multiplicity 8. So we construct here the following new invariant polynomials:

$$\begin{aligned}
N_1(a,x,y) =&\ S_{13}, \qquad N_2(a,x,y) = C_2D_3 + 3S_{16}, \qquad N_3(a,x,y) = T_9,\\
N_4(a,x,y) =&\ -S_{14}^2 - 2D_2^2(3S_{14} - 8S_{15}) - 12D_3(S_{14},C_1)^{(1)}\\
&+ D_2(-48D_3S_9 + 16(S_{17},C_1)^{(1)}),\\
N_5(a,x,y) =&\ 36D_2D_3(S_8 - S_9) + D_1(108D_2^2D_3 - 54D_3(S_{14} - 8S_{15}))\\
&+ 2S_{14}(S_{14} - 22S_{15}) - 8D_2^2(3S_{14} + S_{15}) - 9D_3(S_{14},C_1)^{(1)} - 16D_2^4,\\
N_6(a,x,y) =&\ 40D_3^2(15S_6 - 4S_3) - 480D_2D_3S_9 - 20D_1D_3(S_{14} - 4S_{15})\\
&+ 160D_2^2S_{15} - 35D_3(S_{14},C_1)^{(1)} + 8\left((S_{23},C_2)^{(1)},C_0\right)^{(1)},\\
N_7(a,x,y) =&\ 18C_2D_2(9D_1D_3 - S_{14}) - 2C_1D_3(8D_2^2 - 3S_{14} - 74S_{15})\\
&- 432C_0D_3S_{21}48S_7(8D_2D_3 + S_{17}) + 6S_{10}(12D_2^2 + 151S_{15}) - 51S_{10}S_{14}\\
&- 162D_1D_2S_{16} + 864D_3(S_{16},C_0)^{(1)},\\
N_8(a,x,y) =&\ -32D_3^2S_2 - 108D_1D_3S_{10} + 108C_3D_1S_{11} - 18C_1D_3S_{11} - 27S_{10}S_{11}\\
&+ 4C_0D_3(9D_2D_3 + 4S_{17}) + 108S_4S_{21},\\
N_9(a,x,y) =&\ 11S_{14}^2 - 16D_1D_3(16D_2^2 + 19S_{14} - 152S_{15}) - 8D_2^2(7S_{14} + 32S_{15})\\
&- 2592D_1^2S_{25} + 88D_2(S_{14},C_2)^{(1)},\\
N_{10}(a,x,y) =&\ -24D_1D_3 + 4D_2^2 + S_{14} - 8S_{15},\\
N_{11}(a,x,y) =&\ S_{14}^2 + D_1[16D_2^2D_3 - 8D_3(S_{14} - 8S_{15})] - 2D_2^2(5S_{14} - 8S_{15}) + 8D_2(S_{14},C_2)^{(1)},\\
N_{12}(a,x,y) =&\ -160D_2^4 - 1620D_3^2S_3 + D_1(1080D_2^2D_3 - 135D_3(S_{14} - 20S_{15}))\\
&- 5D_2^2(39S_{14} - 32S_{15}) + 85D_2(S_{14},C_2)^{(1)} + 81\left((S_{23},C_2)^{(1)},C_0\right)^{(1)} + 5S_{14}^2,
\end{aligned}$$

$$
\begin{aligned}
N_{13}(a,x,y) =\ & 2(136D_3^2S_2 - 126D_2D_3S_4 + 60D_2D_3S_7 + 63S_{10}S_{11}) \\
& - 18C_3D_1(S_{14} - 28S_{15}) - 12C_1D_3(7S_{11} - 20S_{15}) - 192C_2D_2S_{15} \\
& + 4C_0D_3(21D_2D_3 + 17S_{17}) + 3C_2(S_{14}, C_2)^{(1)}, \\
N_{14}(a,x,y) =\ & -6D_1D_3 - 15S_{12} + 2S_{14} + 4S_{15}, \\
N_{15}(a,x,y) =\ & 216D_1D_3(63S_{11} - 104D_2^2 - 136S_{15}) + 4536D_3^2S_6 + 4096D_2^4 \\
& + 120S_{14}^2 + 992D_2(S_{14}, C_2)^{(1)} - 135D_3\big[28(S_{17}, C_0)^{(1)} + 5(S_{14}, C_1)^{(1)}\big], \\
N_{16}(a,x,y) =\ & 2C_1D_3 + 3S_{10}, \qquad N_{17}(a,x,y) = 6D_1D_3 - 2D_2^2 - (C_3, C_1)^{(2)}, \\
N_{18}(a,x,y) =\ & 2D_2^3 - 6D_1D_2D_3 - 12D_3S_5 + 3D_3S_8, \\
N_{19}(a,x,y) =\ & C_1D_3(18D_1^2 - S_6) + C_0(4D_2^3 - 12D_1D_2D_3 - 18D_3S_5 + 9D_3S_8) \\
& + 6C_2D_1S_8 + 2\big(9D_2D_3S_1 - 4D_2^2S_2 + 12D_1D_3S_2 - 9C_3D_1S_6 - 9D_3(S_4, C_0)^{(1)}\big), \\
N_{20}(a,x,y) =\ & 3D_2^4 - 8D_1D_2^2D_3 - 8D_3^2S_6 - 16D_1D_3S_{11} + 16D_2D_3S_9, \\
N_{21}(a,x,y) =\ & 2D_1D_2^2D_3 - 4D_3^2S_6 + D_2D_3S_8 + D_1(S_{23}, C_1)^{(1)}, \\
N_{22}(a,x,y) =\ & T_8, \qquad N_{23}(a,x,y) = T_6, \qquad N_{24}(a,x,y) = 2T_3T_{74} - T_1T_{136}, \\
N_{25}(a,x,y) =\ & 5T_3T_6 - T_1T_{23}, \qquad N_{26} = 9T_{135} - 480T_6T_8 - 40T_2T_{74} - 15T_2T_{75}, \\
N_{27}(a,x,y) =\ & 9T_2T_9(2T_{23} - 5T_{24} - 80T_{25}) + 144T_{25}(T_{23} + 5T_{24} + 15T_{26}) \\
& - 9(T_{23}^2 - 5T_{24}^2 - 33T_9T_{76}), \\
N_{28}(a,x,y) =\ & T_3 + T_4,
\end{aligned}
$$

where

$$
A_1 = S_{24}/288, \qquad A_2 = S_{27}/72,
$$
$$
A_3 = \big(S_{23}, C_3\big)^{(4)}/2^7/3^5, \qquad A_4 = \big(S_{26}, D_3\big)^{(2)}/2^5/3^3
$$

are affine invariants, whereas the polynomials

$$
\begin{aligned}
& T_1 = C_3, \quad T_2 = D_3, \quad T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72, \\
& T_6 = \big[3C_1(D_3^2 - 9T_3 + 18T_4) - 2C_2(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21}) \\
& \qquad + 2C_3(2D_2^2 - S_{14} + 8S_{15})\big]/2^4/3^2, \\
& T_8 = \big[5D_2(D_3^2 + 27T_3 - 18T_4) + 20D_3S_{19} + 12\big(S_{16}, D_3\big)^{(1)} - 8D_3S_{17}\big]/5/2^5/3^3, \\
& T_9 = \big[9D_1(9T_3 - 18T_4 - D_3^2) + 2D_2(D_2D_3 - 3S_{17} - S_{19} - 9S_{21}) + 18\big(S_{15}, C_3\big)^{(1)} \\
& \qquad - 6C_2(2S_{20} - 3S_{22}) + 18C_1S_{26} + 2D_3S_{14}\big]/2^4/3^3, \\
& T_{11} = \big[\big(D_3^2 - 9T_3 + 18T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 9T_3 + 18T_4, D_2\big)^{(1)} - 12\big(S_{26}, C_2\big)^{(1)} \\
& \qquad + 12D_2S_{26} + 432(A_1 - 5A_2)C_2\big]/2^7/3^4, \\
& T_{13} = \big[27(T_3, C_2)^{(2)} - 18(T_4, C_2)^{(2)} + 48D_3S_{22} - 216(T_4, D_2)^{(1)} + 36D_2S_{26} \\
& \qquad - 1296C_2A_1 - 7344C_2A_2 + (D_3^2, C_2)^{(2)}\big]/2^7/3^4, \\
& T_{14} = \big[\big(8S_{19} + 9S_{21}, D_2\big)^{(1)} - D_2(8S_{20} + 3S_{22}) + 18D_1S_{26} + 1296C_1A_2\big]/2^4/3^3, \\
& T_{15} = 8\big(9S_{19} + 2S_{21}, D_2\big)^{(1)} + 3\big(9T_3 - 18T_4 - D_3^2, C_1\big)^{(2)} - 4\big(S_{17}, C_2\big)^{(2)} \\
& \qquad + 4\big(S_{14} - 17S_{15}, D_3\big)^{(1)} - 8\big(S_{14} + S_{15}, C_3\big)^{(2)} + 432C_1(5A_1 + 11A_2) \\
& \qquad + 36D_1S_{26} - 4D_2(S_{18} + 4S_{22})\big]/2^6/3^3, \\
& T_{21} = \big(T_8, C_3\big)^{(1)}, \qquad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \qquad T_{24} = \big(T_6, D_3\big)^{(1)}/6,
\end{aligned}
$$

$$T_{26} = \left(T_9, C_3\right)^{(1)}/4, \qquad T_{30} = \left(T_{11}, C_3\right)^{(1)}, \qquad T_{31} = \left(T_8, C_3\right)^{(2)}/24,$$
$$T_{32} = \left(T_8, D_3\right)^{(1)}/6, \qquad T_{36} = \left(T_6, D_3\right)^{(2)}/12, \qquad T_{37} = \left(T_9, C_3\right)^{(2)}/12.$$
$$T_{38} = \left(T_9, D_3\right)^{(1)}/12, \qquad T_{39} = \left(T_6, C_3\right)^{(3)}/2^4/3^2, \qquad T_{42} = \left(T_{14}, C_3\right)^{(1)}/2,$$
$$T_{44} = \left((S_{23}, C_3)^{(1)}, D_3\right)^{(2)}/5/2^6/3^3,$$
$$\begin{aligned} T_{74} = \big[&27C_0(9T_3 - 18T_4 - D_3^2)^2 + C_1\big(- 62208T_{11}C_3 - 3(9T_3 - 18T_4 - D_3^2) \\ &\times (2D_2D_3 - S_{17} + 2S_{19} - 6S_{21})\big) + 20736T_{11}C_2^2 + C_2(9T_3 - 18T_4 - D_3^2) \\ &\times (8D_2^2 + 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) - 54C_3(9T_3 - 18T_4 - D_3^2) \\ &\times (2D_1D_2 - S_8 + 2S_9) - 54D_1(9T_3 - 18T_4 - D_3^2)S_{16} \\ &- 576T_6(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21})\big]/2^8/3^4, \qquad T_{133} = \ (T_{74}, C_3)^{(1)}, \end{aligned}$$
$$T_{136} = \left(T_{74}, C_3\right)^{(2)}/24, \qquad T_{137} = \ \left(T_{74}, D_3\right)^{(1)}/6, \qquad T_{140} = \left(T_{74}, D_3\right)^{(2)}/12,$$
$$T_{141} = \left(T_{74}, C_3\right)^{(3)}/36, \qquad T_{142} = \left((T_{74}, C_3)^{(2)}, C_3\right)^{(1)}/72$$

are T-comitants of cubic systems (2.1) (see for details [28]). We note that these invariant polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [9].

2.2 Preliminary results

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i = 1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [15], [20]).

Following [17] we consider two polynomials $f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$, $g(z) = b_0z^m + b_1z^{m-1} + \cdots + b_m$, in the variable z of degree n and m, respectively. Thus the k–th *subresultant* with respect to variable z of the two polynomials $f(z)$ and $g(z)$ we shall denote by $R_z^{(k)}(f,g)$.

The geometrical meaning of the subresultant is based on the following lemma.

Lemma 2.5 ([15,20])**.** *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 2.5 the following result.

Lemma 2.6. *Two polynomials $\tilde{f}(x_1,x_2,...,x_n)$ and $\tilde{g}(x_1,x_2,...,x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f},\tilde{g}) = R_{x_j}^{(1)}(\tilde{f},\tilde{g}) = R_{x_j}^{(2)}(\tilde{f},\tilde{g}) = \cdots = R_{x_j}^{(k-1)}(\tilde{f},\tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f},\tilde{g}),$$

where $R_{x_j}^{(i)}(\tilde{f},\tilde{g}) = 0$ in $\mathbb{R}[x_1,\dots,x_{j-1},x_{j+1},\dots,x_n]$.

In paper [17] 23 configurations of invariant lines (one more configuration is constructed in [5]) are determined in the case, when the total multiplicity of these line (including the line at infinity) equals nine. For this propose in [17] there are proved some lemmas concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. In [7] these results have been completed.

Theorem 2.7 ([7]). *If a cubic system* (2.1) *possess a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*2 triplets*	$\Rightarrow$	$\mathcal{V}_1=\mathcal{V}_2=\mathcal{U}_1=0$;
(ii)	*1 triplet and 2 couples*	$\Rightarrow$	$\mathcal{V}_3=\mathcal{V}_4=\mathcal{U}_2=0$;
(iii)	*1 triplet and 1 couple*	$\Rightarrow$	$\mathcal{V}_4=\mathcal{V}_5=\mathcal{U}_2=0$;
(iv)	*one triplet*	$\Rightarrow$	$\mathcal{V}_4=\mathcal{U}_2=0$;
(v)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3=0$;
(vi)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5=0$.

In papers [7] and [8] all the possible configurations of invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity are determined for cubic systems with at least three distinct infinite singularities. In particular the next result is obtained.

Lemma 2.8 ([7]). *A cubic system with four distinct infinite singularities could not possess configuration of invariant lines of type* $(3,2,2)$. *And it possesses a configuration or potential configuration of a given type if and only if the following conditions are satisfied, respectively*

$$\begin{array}{lcl}(3,3,1) & \Leftrightarrow & \mathcal{V}_1=\mathcal{V}_2=\mathcal{L}_1=\mathcal{L}_2=\mathcal{K}_1=0,\ \mathcal{K}_2\neq 0;\\ (3,2,1,1) & \Leftrightarrow & \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=0,\ \ \mathcal{D}_4\neq 0;\\ (2,2,2,1) & \Leftrightarrow & \mathcal{V}_3=\mathcal{K}_4=\mathcal{K}_2=\mathcal{K}_8=0,\ \ \mathcal{D}_4\neq 0.\end{array}$$

Let $L(x,y)=Ux+Vy+W=0$ be an invariant straight line of the family of cubic systems (2.1). Then, we have

$$UP(x,y)+VQ(x,y)=(Ux+Vy+W)(Ax^2+2Bxy+Cy^2+Dx+Ey+F),$$

and this identity provides the following 10 relations:

$$\begin{aligned}&Eq_1=(a_{30}-A)U+b_{30}V=0, \qquad Eq_2=(3a_{21}-2B)U+(3b_{21}-A)V=0,\\ &Eq_3=(3a_{12}-C)U+(3b_{12}-2B)V=0, \qquad Eq_4=(a_{03}-C)U+b_{03}V=0,\\ &Eq_5=(a_{20}-D)U+b_{20}V-AW=0,\\ &Eq_6=(2a_{11}-E)U+(2b_{11}-D)V-2BW=0,\\ &Eq_7=a_{22}U+(b_{22}-E)V-CW=0, \qquad Eq_8=(a_{10}-F)U+b_{10}V-DW=0,\\ &Eq_9=a_{01}U+(b_{01}-F)V-EW=0, \qquad Eq_{10}=a_{00}U+b_{00}V-FW=0.\end{aligned} \tag{2.3}$$

As it was mentioned earlier, the infinite singularities (real or complex) of systems (2.1) are determined by the linear factors of the polynomial $C_3(x,y)$. So in the case of two distinct infinite singularities they are determined either by one triple and one simple real or by two double (real or complex) factors of the polynomial $C_3(x,y)$.

Lemma 2.9 ([21]). *A cubic system* (2.1) *has 2 distinct infinite singularities if and only if* $\mathcal{D}_1=\mathcal{D}_3=0$ *and* $\mathcal{D}_2\neq 0$. *The types of these singularities are determined by the following conditions:*

(i) 2 real (1 simple and 1 triple) if $\mathcal{D}_4=0$;

(ii) 2 real (2 double) if $\mathcal{D}_4\neq 0$ *and* $\mathcal{D}_2>0$;

(iii) 2 complex (2 double) if $\mathcal{D}_4\neq 0$ *and* $\mathcal{D}_2<0$.

Moreover in each one of these cases the respective homogeneous cubic parts of this system could be *brought via a linear transformation to the canonical form* (2.4)–(2.6)*, respectively:*

$$\begin{aligned} x' &= (u+1)x^3 + vx^2y + rxy^2, \\ y' &= ux^2y + vxy^2 + ry^3, \qquad C_3 = x^3y; \end{aligned} \tag{2.4}$$

$$\begin{aligned} x' &= ux^3 + qx^2y + rxy^2, \\ y' &= ux^2y + vxy^2 + ry^3, \qquad C_3 = (q-v)x^2y^2, \qquad q-v \neq 0; \end{aligned} \tag{2.5}$$

$$\begin{aligned} x' &= ux^3 + (v+1)x^2y + rxy^2 + y^3, \\ y' &= -x^3 + ux^2y + (v-1)xy^2 + ry^3, \qquad C_3 = (x^2+y^2)^2. \end{aligned} \tag{2.6}$$

3 The proof of the Main theorem

In order to prove the Main theorem we shall consider three families of systems (2.1) with the cubic homogeneities given by Lemma 2.9, i.e. homogeneities (2.4), (2.5) and (2.6).

For each family the proof of the Main theorem proceeds in 4 steps.

First we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems for which the corresponding necessary conditions provided by Theorem 2.7 in order to have the given number of triplets or/and couples of invariant parallel lines in the respective directions are satisfied.

Secondly, taking the homogeneous cubic systems $\dot{x} = \tilde{P}_3$, $\dot{y} = \tilde{Q}_3$ we add all quadratic, linear and constant terms and using the polynomial equations (2.3) as well as corresponding resultants (see Lemma 2.6) we determine these terms in order to get the needed number of invariant lines in the required configuration. Thus the second step ends with the construction of the canonical systems possessing the needed configurations.

The third step consists in the determination of the affine invariant conditions necessary and sufficient for a cubic system to belong to the family of systems (constructed at the second step) which possess the corresponding configuration of invariant lines.

And finally we construct the corresponding perturbed systems possessing 8 distinct simple invariant straight lines (including the line at infinity).

One more observation. It is clear that if for perturbed systems some condition $K(x,y) = 0$ holds, where $K(x,y)$ is an invariant polynomial, then this condition must hold also for the initial (not-perturbed) systems. So considering Lemma 2.8 we arrive at the next remark.

Remark 3.1. Assume that a cubic system with two distinct infinite singularities possesses a potential configuration of a given type. Then for this system the following conditions must be satisfied, respectively:

$$\begin{aligned} (3,3,1) &\quad\Rightarrow\quad \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0; \\ (3,2,1,1) &\quad\Rightarrow\quad \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0; \\ (2,2,2,1) &\quad\Rightarrow\quad \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0. \end{aligned}$$

3.1 Cubic systems with infinite singularities defined by two double factors of $C(x,y)$

According to Lemma 2.9 these two double factors of $C(x,y)$ are both either real or complex. We consider each of the cases separately.

3.1.1 Infinite singular points determined by two double real factors of $C(x,y)$

Taking into account Lemma 2.9 we consider the family of cubic systems

$$\begin{aligned} x' &= a + cx + dy + gx^2 + 2hxy + ky^2 + ux^3 + qx^2y + rxy^2, \\ y' &= b + ex + fy + lx^2 + 2mxy + ny^2 + ux^2y + vxy^2 + ry^3, \end{aligned} \tag{3.1}$$

for which $C_3(x,y) = (q-v)x^2y^2 \neq 0$. Hence, the infinite singular points are located at the "ends" of the straight lines $x = 0$ and $y = 0$.

Lemma 3.2. *A non-degenerate cubic system (3.1) could not have invariant straight lines of total multiplicity eight.*

Proof. We split the proof of the lemma depending on the possible types of configurations.

1) *Configuration* $(3, 3, 1)$. By Theorem 2.7 in order to have the configuration $(3, 3, 1)$ the necessary condition $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ must be fulfilled. On the other hand it was proved (see [17, Section 9.4]) that this condition cannot be satisfied, i.e. we could not have invariant straight lines of total multiplicity eight in the configuration $(3, 3, 1)$.

2) *Configuration* $(3, 2, 1, 1)$. By Theorem 2.7 in this case necessarily the condition $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ holds.

Taking into account Lemma 2.9 we consider the family of homogeneous systems

$$x' = ux^3 + qx^2y + rxy^2, \qquad y' = ux^2y + vxy^2 + ry^3 \tag{3.2}$$

with $q - v \neq 0$ and we force the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ to be satisfied. For these systems we use the following notation $\mathcal{V}_5 = \frac{32}{9}\sum_{j=0}^{4} \mathcal{V}_{5j} x^{4-j} y^j$, where

$$\begin{aligned} &\mathcal{V}_{50} = -u^2(q-v)(2q-v), \qquad \mathcal{V}_{51} = -4ru^2(q-v), \qquad \mathcal{V}_{52} = 0, \\ &\mathcal{V}_{53} = 4r^2u(q-v), \qquad\qquad\quad\; \mathcal{V}_{54} = -r^2(q-2v)(q-v). \end{aligned} \tag{3.3}$$

On the other hand for systems (3.2) we calculate $\mathcal{V}_4 = 9216\,\hat{\mathcal{V}}_4 C_3(x,y)$, where $\hat{\mathcal{V}}_4 = (q-v)(2ru+qv)$. As $C_3 = (q-v)x^2y^2 \neq 0$, we conclude that the condition $\mathcal{V}_4 = 0$ is equivalent to $2ru + qv = 0$.

We observe that the condition $\mathcal{V}_{53} = 0$ implies $ru = 0$ and without loss of generality we can consider $r = 0$ due to the changes $x \leftrightarrow y$, $r \leftrightarrow u$ and $q \leftrightarrow v$. Therefore as $q - v \neq 0$ the condition $\mathcal{V}_{50} = 0$ gives $u(2q - v) = 0$ and $\mathcal{V}_4 = 0$ implies $qv = 0$. So we get $u = 0 = r$ and due to the above changes we may assume $q = 0$ and this implies $\mathcal{V}_5 = \mathcal{V}_4 = 0$. Moreover , these conditions annulate the invariant polynomial $\mathcal{U}_2$.

Thus, considering systems (3.2) for $r = u = q = 0$ and applying a time rescaling we obtain the homogeneous system

$$\dot{x} = 0, \qquad \dot{y} = -xy^2 \tag{3.4}$$

for which $C_3 = x^2y^2$. Considering (3.4) due to a translation we may assume that for cubic systems (3.1) the condition $m = n = 0$ holds and we arrive at the family of systems

$$\dot{x} = a + cx + gx^2 + dy + 2hxy + ky^2, \qquad \dot{y} = b + ex + lx^2 + fy - xy^2. \tag{3.5}$$

Taking into account Remark 3.1 for these systems we calculate $\mathcal{K}_4 = 2x(lx^2 + hy^2)/9$ and hence the necessary condition $\mathcal{K}_4 = 0$ implies $l = h = 0$. Then we calculate

$$\mathcal{K}_5 = 20xy\big[2(4e - g^2)x^2 - 3(c - 4f)xy + 2(d + gk)y^2\big]/3$$

and the condition $\mathcal{K}_5 = 0$ is equivalent to $c = 4f$, $d = -gk$ and $e = g^2/4$. In this case we obtain

$$\mathcal{K}_6 = 2x^3y^5\big[40(79b + 712fg)x^3 - 5(268a + 1339g^2k)x^2y + 31488fkxy^2 - 7056gk^2y^3\big]/9$$

and this polynomial must vanish for any values of x and y. Obviously the condition $\mathcal{K}_6 = 0$ is equivalent to $a = -1339g^2k/268$, $b = -712fg/79$ and $fk = gk = 0$. If $k \neq 0$ then $f = g = 0$ and we get $a = b = 0$. However, this leads to degenerate systems and in what follows we assume that the condition $k = 0$ is satisfied.

As it was mentioned earlier the infinite singularities of systems (3.5) are located at the "end" of the lines $x = 0$ and $y = 0$. We shall examine these directions using the equations (2.3) and applying the next remark.

Remark 3.3. Assume that the infinite singularities of a cubic system are located on the "ends" of the axis $x = 0$ and $y = 0$. Then the invariant affine lines must be either of the form $Ux + W = 0$ or $Vy + W = 0$. Therefore we can assume $U = 1$ and $V = 0$ (for the direction $x = 0$) and $U = 0$ and $V = 1$ (for the direction $y = 0$). In this case, considering W as a parameter, six equations among (2.3) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non zero determinant) and we can determine their values, which annulate some of the equations (2.3). So in what follows we will examine only the non-zero equations containing the last parameter W.

Considering the relations $k = a = 0$ and $b = -712fg/79$ we obtain the family of systems

$$\dot{x} = x(4f + gx), \qquad \dot{y} = -712fg/79 + g^2x/4 + fy - xy^2 \tag{3.6}$$

for which we calculate $H(X, Y, Z) = 2^{-4}79^{-2}XZ(gX + 4fZ)$ (see Notation 1).

Considering Lemmas 2.2 and 2.3 we conclude that systems (3.6) possess invariant straight lines of total multiplicity 4: two affine line ($x = 0$, $gx + 4f = 0$) and the infinite line $Z = 0$, which is double. However for having invariant lines of total multiplicity 8 the polynomial $H(X, Y, Z)$ must have degree 7 in X, Y and Z. So we have to find out the conditions in order to increase its degree.

We begin with the examination of the direction $x = 0$ ($U = 1, V = 0$). So, considering (2.3) and Remark 3.3 for systems (3.6) we have:

$$Eq_i = 0, \quad i = 1, \dots, 9, \qquad Eq_{10} = -4fW + gW^2$$

As regard the direction $y = 0$ (i.e. $U = 0, V = 1$) we have

$$Eq_8 = (g - 2W)(g + 2W)/4, \qquad Eq_{10} = -f(712g + 79W)/79,$$
$$R_W^{(0)}(Eq_8, Eq_{10}) = -2021535f^2g^2/24964.$$

According to Lemma 2.6 the above equations have a common solution if and only if $fg = 0$. However for $f = 0$ we get degenerate systems and hence $f \neq 0$ and $g = 0$. In this case we get the systems

$$\dot{x} = 4fx, \qquad \dot{y} = fy - xy^2,$$

for which $H(X, Y, Z) = 4fXYZ^2$ and $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = -1584f^6Y^6Z^{12} \neq 0$ as $f \neq 0$. So we could not increase the degree of $H(X, Y, Z)$, i.e. the above systems could only possess invariant lines of total multiplicity five. □

3) *Configuration* (2, 2, 2, 1). According to Theorem 2.7, if a cubic system possesses 7 invariant straight lines in the configuration (2, 2, 2, 1), then necessarily the condition $\mathcal{V}_3 = 0$ holds. A straightforward computation of the value of $\mathcal{V}_3$ for systems (2.5) yields: $\mathcal{V}_3 = -\frac{1}{32}\sum_{j=0}^{4}\mathcal{V}_{3j}x^{4-j}y^j$, where

$$\begin{gathered}\mathcal{V}_{30} = u^2, \qquad \mathcal{V}_{31} = 2u(2q-v), \qquad \mathcal{V}_{33} = -2r(q-2v),\\ \mathcal{V}_{32} = -2q^2 + 2ru + 5qv - 2v^2, \qquad \mathcal{V}_{34} = r^2.\end{gathered} \tag{3.7}$$

The condition $\mathcal{V}_{30} = \mathcal{V}_{34} = 0$ yields $r = u = 0$ and we obtain $\mathcal{V}_{32} = (2v-q)(2q-v) = 0$. Without loss of generality we can assume $q = 2v$ due to the change $x \leftrightarrow y$ and $q \leftrightarrow v$. Therefore after a suitable time rescaling we obtain the system

$$\dot{x} = 2x^2y, \qquad \dot{y} = xy^2. \tag{3.8}$$

Considering this system due to a translation we may assume that for cubic systems (3.1) the condition $n = g = 0$ holds and we arrive at the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + 2x^2y, \qquad \dot{y} = b + ex + fy + 2mxy + lx^2 + xy^2. \tag{3.9}$$

In what follows by Lemma 2.8 we force the conditions $\mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0$ to be satisfied. So we calculate

$$\mathcal{K}_4 = -2(2lx^3 + 4mx^2y - hxy^2 + ky^3)/9 = 0$$

and this implies $l = m = h = k = 0$. In this case we calculate $\mathcal{K}_2 = -3x^4y^4(bx - ay)$ which vanishes if and only if $a = b = 0$. Then $\mathcal{K}_8 = -8(2c - 5f)x^2y^2 = 0$ and this gives the condition $c = 5f/2$. Thus considering the above conditions we arrive at the family of systems

$$\dot{x} = (5fx + 2dy + 4x^2y)/2, \qquad \dot{y} = ex + fy + xy^2, \tag{3.10}$$

for which we examine the directions $x = 0$ and $y = 0$.

Considering (2.3), (3.10) and Remark 3.3 in the case of the direction $x = 0$ we have

$$Eq_9 = d + 2W^2, \qquad Eq_{10} = -\frac{5}{2}fW, \qquad R_W^{(0)}(Eq_9, Eq_{10}) = 25df^2/4 = 0,$$

whereas in the case of the direction $y = 0$ we get

$$Eq_8 = e + W^2, \qquad Eq_{10} = -fW, \qquad R_W^{(0)}(Eq_8, Eq_{10}) = ef^2 = 0.$$

So in order to have invariant lines in the direction $x = 0$ (respectively $y = 0$) the condition $df = 0$ (respectively $ef = 0$) must be satisfied. We consider two cases: $f = 0$ and $f \neq 0$.

If $f = 0$ then we obtain systems

$$\dot{x} = (d + 2x^2)y, \qquad \dot{y} = x(e + y^2). \tag{3.11}$$

for which we calculate $H(X,Y,Z) = (2X^2 + dZ^2)(Y^2 + eZ^2)$ and

$$\begin{aligned}R_X^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) &= -9d^3Y^8Z^6(Y^2 + eZ^2)^2 = 0,\\ R_Y^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) &= 9d^2e^3X^8Z^{10} = 0,\\ R_Z^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) &= 9d^2X^6Y^{10} = 0.\end{aligned}$$

Therefore the condition $d = 0$ is necessary in order to have an additional factor (in X, Y, Z) of the polynomials $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. However this condition leads to degenerate systems.

Assume now $f \neq 0$. Then we must have $d = 0$ and $d^2 + e^2 \neq 0$, otherwise systems (3.10) become degenerate.

If $d = 0$ and $e \neq 0$ for systems (3.10) we calculate $H(X, Y, Z) = X/4$ and

$$R_X^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) = -129600f^{12}Y^3Z^{24}(Y^2 + eZ^2)^2(Y^2 + 5eZ^2)^6 \neq 0,$$
$$R_Y^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) = 10368e^2f^4X^{15}Z^{24}(16eX^2 + 5f^2Z^2)^6 \neq 0,$$
$$R_Z^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) = 74649600f^6X^{24}Y^{24}(4eX - fY)^{12} \neq 0,$$

whereas for $e = 0$ and $d \neq 0$ obtain $H(X, Y, Z) = Y/4$ and

$$R_X^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) = -2073600d^2f^2Y^{15}Z^{22}(2dY^2 - 5f^2Z^2)(2dY^2 - f^2Z^2)^6 \neq 0,$$
$$R_Y^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) = -64800f^{12}X^5Z^{24}(X^2 - 2dZ^2)^6(2X^2 + dZ^2) \neq 0,$$
$$R_Z^{(0)}(\mathcal{G}_1/H, \mathcal{G}_2/H) = 74649600f^6X^{24}Y^{24}(fX + 2dY)^12 \neq 0.$$

So in both cases we could not have more than 2 invariant lines, including the infinite one.

3.1.2 Infinite singular points determined by two double complex factors of $C(x, y)$

In this case according to Lemma 2.9 we consider the family of systems

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + ux^3 + (v + 1)x^2y + rxy^2 + y^3, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) - x^3 + ux^2y + (v - 1)xy^2 + ry^3, \end{aligned} \tag{3.12}$$

for which $C_3(x, y) = (x^2 + y^2)^2$. By Theorem 2.7 the condition $\mathcal{V}_1 = 0$ is necessary to have two triplets of parallel invariant lines. However calculating this T-comitant $\mathcal{V}_1 = 16\sum_{j=0}^{4} \mathcal{V}_{1j}x^{4-j}y^j$ we get the contradiction, since $\mathcal{V}_{10} = u^2 + 3 \neq 0$.

Assume now that we have three couples of parallel invariant lines, i.e. the condition $\mathcal{V}_3 = 0$ holds. For systems (3.12), denoting $\mathcal{V}_3 = 32\sum_{j=0}^{4} \mathcal{V}_{3j}x^{4-j}y^j$, we calculate $\mathcal{V}_{30} + \mathcal{V}_{34} = -(u^2 + r^2 + 18) \neq 0$ and again obtain a contradiction with Theorem 2.7.

In order to have two couples of invariant straight lines $\big($to get the configuration $(3, 2, 1, 1)\big)$ we examine the condition $\mathcal{V}_5 = 0$. Denoting $\mathcal{V}_5 = \frac{9}{32}\sum_{j=0}^{4} \mathcal{V}_{5j}x^{4-j}y^j$ we calculate $\mathcal{V}_{54} - \mathcal{V}_{50} = -2v\big[(r - u)^2 + v^2\big] = 0$, which is equivalent to $v = 0$. In this case, $\mathcal{V}_{51} + \mathcal{V}_{53} = -4(r - u)^3 = 0$ and for $r = u$ and $v = 0$ we obtain $\mathcal{V}_5 = 0$ and $\mathcal{V}_4 = 16(1 + u^2) \neq 0$. This contradicts Theorem 2.7.

So we proved the next result.

Lemma 3.4. *A cubic system (3.12) could not have invariant straight lines of total multiplicity eight.*

3.2 Cubic systems with infinite singularities defined by one triple and one simple real factors of $C(x, y)$

According to Lemma 2.9 in this case we consider the following family of cubic systems:

$$\begin{aligned} x' &= p_0 + p_1(x, y) + p_2(x, y) + (u + 1)x^3 + vx^2y + rxy^2, \\ y' &= q_0 + q_1(x, y) + q_2(x, y) + ux^2y + vxy^2 + ry^3 \end{aligned} \tag{3.13}$$

with $C_3 = x^3y$. Hence, the infinite singular points are situated at the "ends" of the straight lines $x = 0$ and $y = 0$.

3.2.1 Construction of the corresponding cubic homogeneities

The cubic homogeneous systems in the case under consideration were constructed in [6, Proposition 9]. More precisely considering the proof of Proposition 9 [6] we have the next result.

Lemma 3.5 ([6]). *Assume that for a cubic homogeneous system the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$ and $\mathcal{D}_2 \neq 0$ hold. Then via a linear transformation and time rescaling this system can be brought to one of the canonical form $(\tilde{P}_3, \tilde{Q}_3)$ indicated bellow if and only if the corresponding conditions are satisfied. Moreover the cubic systems with these homogeneities could have only the configurations of invariant lines of the type given next to each homogeneity:*

- $\mathcal{V}_1 = \mathcal{V}_3 = 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = (x^3, 0) \;\Rightarrow\; (3,3,1), (3,2,1,1), (2,2,2,1)$;
- $\mathcal{V}_1 = 0,\ \mathcal{V}_3 \neq 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = (x^3, 3x^2y) \;\Rightarrow\; (3,3,1)$;
- $\mathcal{V}_1 \neq 0,\ \mathcal{V}_3 = 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = (2x^3, 3x^2y) \;\Rightarrow\; (2,2,2,1)$;
- $\mathcal{V}_1\mathcal{V}_3 \neq 0,\ \mathcal{V}_5 = \mathcal{U}_2 = 0 \;\Rightarrow\; (\tilde{P}_3, \tilde{Q}_3) = \big((u+1)x^3, ux^2y\big),\ u(u+3)(2u+3) \neq 0 \Rightarrow (3,2,1,1)$.

In [6] the family of cubic systems with homogeneities $(x^3, 0)$ was completely investigated. More exactly it was proved that a system belonging to this family could have only one of the configurations of invariant lines *Config. 8.23* – *Config. 8.38* given in Figure 1.1. Moreover the necessary and sufficient affine invariant conditions for the realization of each one of these configurations are constructed as it is stated in the Main theorem.

In what follows we consider the remaining three families of systems, the cubic homogeneities of which are given by the above lemma. Since each one of these cubic homogeneities leads to the given type of configurations (see Lemma 3.5), in order to construct the corresponding canonical form we could apply first the respective necessary conditions given by Lemma 2.8.

Remark 3.6. Any invariant line of the form $x + \alpha = 0$ (i.e. in the direction $x = 0$) of cubic systems (2.1) must be a factor of the polynomials $P(x, y)$, i.e. $(x + \alpha) \mid P(x, y)$.

Indeed, according to the definition, for an invariant line $ux + vy + w = 0$ we have $uP + vQ = (ux + vy + w)R(x, y)$, where the cofactor $R(x, y)$ generically is a polynomial of degree two. In our particular case (i.e. $u = 1, v = 0, w = \alpha$) we obtain $P(x) = (x + \alpha)R(x)$, which means that $(x + \alpha)$ divides $P(x)$.

This remark could be applied for any cubic system when we examine the direction $x = 0$. Similarly, for an invariant line $y + \beta = 0$ in the direction $y = 0$ it is necessary $(y + \beta) \mid Q(x, y)$.

3.2.2 Construction of the normal form with cubic homogeneities $(x^3, 3x^2y)$

Considering the cubic homogeneities $(x^3, 3x^2y)$ due to a translation we may assume that for cubic systems (3.1) the condition $g = l = 0$ holds and we arrive at the family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \qquad \dot{y} = b + ex + fy + 2mxy + ny^2 + 3x^2y \tag{3.14}$$

with $C_3 = -2x^3y$.

Taking into account Remark 3.1 for these systems we calculate

$$\mathcal{L}_1 = 20736x^2\left[(2h + n)x + ky\right]$$

and thus, the condition $\mathcal{L}_1 = 0$ is equivalent to $n = -2h$ and $k = 0$. Then we have $\mathcal{K}_1 = 0$ and

$$\mathcal{L}_2 = 20736x[(21d - 8hm) + 48h^2y] = 0$$

which implies the conditions $h = d = 0$. So for $n = k = h = d = 0$ we get the following family of systems:

$$\dot{x} = a + cx + x^3 \equiv P(x), \qquad \dot{y} = b + ex + fy + 2mxy + 3x^2y \equiv Q(x,y) \tag{3.15}$$

for which we calculate

$$\begin{gathered} H(X,Y,Z) = X^3 + cXZ^2 + aZ^3, \qquad \mathcal{G}_1/H = F_1(X,Y,Z), \\ \mathcal{G}_2/H = F_2(X,Y,Z)(X^3 + cXZ^2 + aZ^3) = F_2(X,Y,Z)P^*(X,Z), \\ \mathcal{G}_3/H = -48Q^*(X,Y,Z)[P^*(X,Z)]^2. \end{gathered} \tag{3.16}$$

where $P^*(X,Z)$ (respectively $Q^*(X,Y,Z)$) is the homogenization of the polynomial $P(x)$ (respectively $Q(x,y)$) of systems (3.15) and $F_1(X,Y,Z)$, $F_2(X,Y,Z)$ are homogeneous polynomials in X, Y and Z of the degree five and four, respectively. It is clear that these systems are degenerate if and only if the polynomials $P(x)$ and $Q(x,y)$ have a non-constant common factor (depending on x), i.e. the following condition must hold:

$$\Phi(y) \equiv R_x^{(0)}(P(x), Q(x,y)) \neq 0. \tag{3.17}$$

Systems (3.15) possess invariant lines of total multiplicity 4, including the infinite one , but we need 8 invariant lines (considered with their multiplicities), i.e. additionally we have to obtain a common factor of fourth degree of the polynomials $\mathcal{G}_i/H$, $i = 1,2,3$. In order to reach this situation we examine the directions $x = 0$ and $y = 0$.

Since in the direction $x = 0$ we already have 3 invariant lines $x^3 + cx + a = 0$ (which could coincide), we consider the equations (2.3) only for the direction $y = 0$. Considering systems (3.15) and Remark 3.3 we have

$$Eq_5 = -3W, \qquad Eq_8 = e - 2mW, \qquad Eq_{10} = b - fW.$$

Evidently that the above equations could have only one common solution ($W_0 = 0$) and for this it is necessary and sufficient $e = b = 0$. So in what follows we examine two cases: $e^2 + b^2 = 0$ (i.e. the existence of a line in the direction $y = 0$) and $e^2 + b^2 \neq 0$ (i.e. the non-existence of a line in the direction $y = 0$).

1) *The case* $e^2 + b^2 = 0$. Then we get the family of systems

$$\dot{x} = a + cx + x^3, \qquad \dot{y} = (f + 2mx + 3x^2)y \equiv \tilde{Q}(x)y \tag{3.18}$$

for which we calculate

$$\begin{gathered} H(X,Y,Z) = Y(X^3 + cXZ^2 + aZ^3), \qquad \mathcal{G}_1/H = \tilde{F}_1(X,Z), \\ \mathcal{G}_2/H = \tilde{F}_2(X,Z)P^*(X,Z), \qquad \mathcal{G}_3/H = -48\tilde{Q}^*(X,Z)[P^*(X,Z)]^2. \end{gathered}$$

The polynomial H is of degree 4 and thus we need to increase its degree up to 7. In order to reach this situation we have to obtain a common factor of degree 3 of the above polynomials. We observe that the polynomials $\mathcal{G}_i/H$, $i = 1,2,3$ do not depend on Y, i.e. they could not have common factors in Y. This means that systems (3.18) have only one invariant line in the direction $y = 0$. Moreover since $\mathcal{G}_1/H \mid_{(Z=0)} = -6X^4$ we also could not have Z as a common factor. Therefore all three polynomials could have only factors of the form $X+\alpha$, which must be factors of the polynomial $P^*(X,Z)$ (see Remark 3.6).

Thus, in order to get a common factor of the third degree of the mentioned polynomials, the following condition must hold:

$$R_X^{(0)}([P^*]^2,\tilde{F}_1)=R_X^{(1)}([P^*]^2,\tilde{F}_1)=R_X^{(2)}([P^*]^2,\tilde{F}_1)=0.$$

Considering systems (3.18) we calculate

$$R_X^{(2)}([P^*]^2,\tilde{F}_1)=[\Psi(a,c,f,m)]^2Z^8=0$$

where $\Psi(a,c,f,m)=3(c-f)(3c-f)-36am+8(5c-3f)m^2+16m^4$.

***a*)** Assume first $m=0$. Then the last condition is equivalent to $(3c-f)(c-f)=0$, i.e. we need to examine two cases: $f=3c$ and $f=c$.

Assuming $f=3c$ we calculate $R_X^{(0)}([P^*]^2,\tilde{F}_1)=46656a^4c^6Y^6Z^{24}=0$ and considering (3.17) we have $\Phi=R_x^{(0)}(P(x),\tilde{Q}(x))=27a^2\neq 0$. Thus we arrive at the condition $c=0$ which implies $f=0$ and so we obtain systems

$$\dot{x}=a+x^3,\qquad \dot{y}=3x^2y. \tag{3.19}$$

We remark that this family of systems is a subfamily of (3.20) and we will examine it together with the family of systems (3.20).

Now we consider $f=c\neq 0$ which implies $\Phi=(27a^2+4c^3)\neq 0$. In this case we obtain $R_X^{(0)}([P^*]^2,\tilde{F}_1)=R_X^{(1)}([P^*]^2,\tilde{F}_1)=R_X^{(2)}([P^*]^2,\tilde{F}_1)=0$ and $R_X^{(3)}([P^*]^2,\tilde{F}_1)=-216aZ^3\neq 0$ (i.e. the condition $a\neq 0$ is necessary, otherwise we get invariant lines of the total multiplicity 9). As a result we arrive at the family of systems

$$\dot{x}=a+cx+x^3,\qquad \dot{y}=(c+3x^2)y \tag{3.20}$$

for which $H(X,Y,Z)=3Y(X^3+cXZ^2+aZ^3)^2$, i.e. these systems possess invariant line of total multiplicity 8.

We observe that systems (3.19) belong to this family for $c=0$. So we allow the parameter c to be zero in order to include (3.19) in (3.20).

It is clear that the polynomial $a+cx+x^3$ has at least one real solution, say x_0. Therefore due to the translation of the origin of coordinates to the singular point $(x_0,0)$ systems (3.20) become of the form

$$\dot{x}=x(e+gx+x^2),\qquad \dot{y}=(e+2gx+3x^2)y \tag{3.21}$$

where $e=c+3x_0^2$ and $g=3x_0$ and we calculate $H(X,Y,Z)=X^2Y(X^2+gXZ+eZ^2)^2$. On the other hand considering systems (3.21) we calculate $R_X^{(0)}(\mathcal{G}_2/H,\mathcal{G}_1/H)=24g(9e-2g^2)Z^3$ and therefore the condition $g(9e-2g^2)\neq 0$ guaranties us to have no 9 invariant lines considered with their multiplicities. Taking into consideration that $g\neq 0$ due to the rescaling $(x,y,t)\mapsto(gx,y,t/g^2)$ we can set $g=1$ and we obtain

$$\dot{x}=x(r+x+x^2),\qquad \dot{y}=(r+2x+3x^2)y. \tag{3.22}$$

We also observe that systems (3.22) possess 3 finite singularities: $(0,0)$ and $\left(\frac{-1\pm\sqrt{1-4r}}{2},0\right)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 2.4 for these systems we calculate:

$$\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0,\qquad \mu_6=r^2(4r-1)x^6.$$

If $r(4r-1) \neq 0$ by Lemma 2.4 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Moreover by the same lemma systems (3.22) become degenerate only if either $r = 0$ or $r = 1/4$ and in both cases we indeed get degenerate systems.

Thus, systems (3.21) possess 7 affine invariant lines and the type of some of these lines depends on the polynomial $1-4r = \text{Discriminant}[x^2+x+r, x]$. Therefore we have the following two possibilities:

- *The possibility* $1-4r > 0$. Then we denote $1-4r = u^2 \neq 0$ $\big($i.e. $r = (1-u^2)/4 \neq 0\big)$ and considering (3.21) we get the systems

$$\dot{x} = x(1+2x-u)(1+2x+u)/4, \qquad \dot{y} = (1-u^2+8x+12x^2)y/4 \tag{3.23}$$

  with $H(X,Y,Z) = 2^{-6}X^2Y(2X+Z-uZ)^2(2X+Z+uZ)^2$. So in this case we obtain 1 simple and 3 double invariant straight lines, all real and distinct. Evidently that the condition $r(4r-1) \neq 0$ is equivalent to $u(1-u^2) \neq 0$.

- *The possibility* $1-4r < 0$. Then denoting $1-4r = -u^2 \neq 0$ (i.e. $r = (1+u^2)/4 \neq 0$) we arrive at the systems

$$\dot{x} = x[(2x+1)^2+u^2]/4, \qquad \dot{y} = (1+u^2+8x+12x^2)y/4 \tag{3.24}$$

  for which we have $H(X,Y,Z) = 2^{-6}X^2Y(4X^2+4XZ+Z^2+u^2Z^2)^2$. Clearly, in this case we obtain the following types of invariant straight lines: one double real, two double complex and one simple real, all distinct.

More exactly, systems (3.22) possess the configuration *Config. 8.39* if $\Delta > 0$ and *Config. 8.40* in the case of $\Delta < 0$.

b) *The subcase* $m \neq 0$. We may set $m = 1$ $\big($due to the rescaling $(x,y,t) \mapsto (mx, y, t/m^2)\big)$ and considering systems (3.18) we calculate

$$R_X^{(0)}([P^*]^2, \tilde{F}_1) = [\Psi_1(a,c,f)]^2[\Psi_2(a,c,f)]^2Z^{24} = 0, \qquad \Phi(a,c,f) = \Psi_2(a,c,f) \neq 0,$$
$$\Psi_1(a,c,f) = 8a + (c-f)[c(4+c) - 2cf + f^2].$$

Clearly the condition $\Psi_1(a,c,f) = 0$ is necessary and sufficient to have a common factor of $[P^*]^2$ and $\tilde{F}_1$ for non-degenerate systems (3.18). Then $a = (f-c)(4c+c^2-2cf+f^2)/8$ and we have

$$R_X^{(1)}([P^*]^2, \tilde{F}_1) = (4+3c-3f)(4c+3c^2-6cf+3f^2)^3(16+16c+3c^2-6cf+3f^2)^2Z^{15}/64,$$
$$\Phi = (4c+3c^2-6cf+3f^2)^2(16+16c+3c^2-6cf+3f^2)/64 \neq 0.$$

So the equality $R_X^{(1)}([P^*]^2, \tilde{F}_1) = 0$ implies $f = (4+3c)/3$ and then $R_X^{(2)}([P^*]^2, \tilde{F}_1) = 256(4+3c)^2Z^8/9 = 0$ which contradicts $\Phi = 4(4+3c)^3/27 \neq 0$. So in the case $m \neq 0$ systems (3.18) could not have invariant lines of multiplicity 8.

2) *The case* $e^2+b^2 \neq 0$. We again consider systems (3.15) which already possess 3 lines in the direction $x = 0$. Taking into consideration that we are in the case of non-existence of an invariant line in the direction $y = 0$, in order to increase the degree of the polynomial H we need a common factor of the degree 4 of the polynomials $\mathcal{G}_i/H, i = 1,2,3$. By Lemma 2.6 this happens if and only if $R_X^{(0)}\left([P^*]^2, F_1\right) = R_X^{(1)}\left([P^*]^2, F_1\right) = R_X^{(2)}\left([P^*]^2, F_1\right) = R_X^{(3)}\left([P^*]^2, F_1\right) = 0$. We calculate

$$R_X^{(3)}\left([P^*]^2, F_1\right) = -8[27a - 2m(18c - 9f + 8m^2)]Y^3Z^3 + 12(3ce + 6bm - 4em^2)Y^2Z^4 + e^3Z^6 = 0$$

and so the above condition is equivalent to $e = bm = 27a - 2m(18c - 9f + 8m^2) = 0$. Since $e^2 + b^2 \neq 0$ we obtain $e = m = a = 0$ and $b \neq 0$. In this case we calculate

$$R_X^{(0)}\left([P^*]^2, F_1\right) = (c - f)^6 Z^{24}(3cY - fY - bZ)^4(fY + bZ)^2 = 0$$

and since $b \neq 0$ it results $f = c$. Consequently we obtain the family of systems

$$\dot{x} = x(c + x^2), \qquad \dot{y} = b + cy + 3x^2y \tag{3.25}$$

with $H(X, Y, Z) = 3X^3(X^2 + cZ^2)^2$ and after the rescaling $(x,\ y) \to (x,\ by)$ we arrive at the one-parameter family of systems

$$\dot{x} = x(c + x^2), \qquad \dot{y} = 1 + cy + 3x^2y. \tag{3.26}$$

Here we may assume $c = \{-1, 0, 1\}$ due to the rescaling $(x, y, t) \mapsto (|c|^{1/2}x, |c|^{-1}y, |c|^{-1}t)$.

Considering Lemma 2.4 for these systems we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = 4c^3x^6.$$

If $c \neq 0$ systems (3.26) possess 3 finite singularities: $(0, -1/c)$ and $\left(\pm\sqrt{-c}, 1/(2c)\right)$ and by Lemma 2.4 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0, 1, 0]$ located on the "end" of the invariant line $x = 0$.

If $c = 0$ then $\mu_6 = \mu_7 = \mu_8 = 0$ and $\mu_9 = 9x^9$. So the system is non-degenerate and all 9 finite singularities have gone to infinity and collapsed with the same singular point.

Thus in the case $c \neq 0$ a system (3.26) possesses three distinct invariant affine lines (one triple and two doubles), and namely:

$$L_{1,2,3} = x, \qquad L_{4,5} = x - \sqrt{-c}, \qquad L_{6,7} = x + \sqrt{-c}.$$

Moreover, for $c < 0$ we have real invariant straight lines whereas for $c > 0$ we get two complex invariant lines. As a result we obtain the configuration *Config. 8.41* in the case $c = -1$ and *Config. 8.43* in the case $c = 1$.

In the case $c = 0$ the invariant affine line $x = 0$ becomes of multiplicity 7 and we arrive at the configuration *Config. 8.42*.

The above results lead as to the following proposition.

Proposition 3.7. *Systems (3.15) possess invariant lines of total multiplicity eight if and only if the following set of conditions holds:*

$$e = m = f - c = 0 \quad \text{and either:} \quad b = 0 \text{ and } a(27a^2 + 4c^3) \neq 0 \quad \text{or} \quad b \neq 0 \text{ and } a = 0. \tag{3.27}$$

3.2.3 Construction of the normal form with cubic homogeneities $(2x^3, 3x^2y)$

Here we consider systems (3.13) with the homogeneous cubic part $(2x^3, 3x^2y)$ for which we may assume $g = l = 0$ due to a translation.. Therefore we arrive at the next family of systems

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + 2x^3, \qquad \dot{y} = b + ex + fy + 2mxy + ny^2 + 3x^2y. \tag{3.28}$$

First of all for these systems we force the condition $\mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0$ to be satisfied, which by Remark 3.1 corresponds to the type of configuration $(2,2,2,1)$. We calculate $\mathcal{K}_4 = -2x^2(hx + ky)$ and therefore $\mathcal{K}_4 = 0$ implies $k = h = 0$. Then we have

$$\text{Coefficient}[\mathcal{K}_2, x^9] = 9(em - 3b) = 0 \quad \text{and} \quad \text{Coefficient}[\mathcal{K}_2, x^5y^4] = -3n^3 = 0.$$

Clearly, the above conditions are equivalent to $b = em/3$ and $n = 0$ and in this case we get

$$\mathcal{K}_2 = (27a - 9cm + 18fm - 8m^3)x^8y - 18dmx^7y^2, \qquad \mathcal{K}_8 = 198dx^4.$$

Therefore, the condition $\mathcal{K}_2 = \mathcal{K}_8 = 0$ implies $d = 0$ and $a = m(9c - 18f + 8m^2)/27$. So considering the conditions

$$k = h = n = d = 0,\ b = em/3, \qquad a = m(9c - 18f + 8m^2)/27$$

we arrive at the family of systems

$$\dot{x} = m(9c - 18f + 8m^2)/27 + cx + 2x^3, \qquad \dot{y} = em/3 + fy + 3x^2y + x(e + 2my) \tag{3.29}$$

for which we have $H(X,Y,Z) = 3^{-9}[54X^3 + 27cXZ^2 + (9cm - 18fm + 8m^3)Z^3]$.

Considering Remark 3.3 we examine the direction $y = 0$:

$$Eq_5 = -3W, \qquad Eq_8 = e - 2mW, \qquad Eq_{10} = \frac{1}{3}(em - 3fW)$$

and we observe that the equations $Eq_5 = Eq_8 = Eq_{10} = 0$ could have a unique solution $(W_0 = 0)$ if and only if $e = 0$. Therefore, we arrive at the next conclusion: if $e = 0$ then we have a line in the direction $y = 0$, whereas if $e \neq 0$ then we could not have a line in this direction. So, in what follows, we examine these two possibilities.

1) *The case* $e = 0$. Then we obtain the systems

$$\dot{x} = m(9c - 18f + 8m^2)/27 + cx + 2x^3 \equiv P(x), \qquad \dot{y} = (f + 2mx + 3x^2)y \equiv \tilde{Q}(x)y \tag{3.30}$$

for which we calculate

$$H(X,Y,Z) = 3^{-8}Y[54X^3 + 27cXZ^2 + (9cm - 18fm + 8m^3)Z^3],$$
$$\mathcal{G}_1/H = F_1(X,Z), \quad \mathcal{G}_2/H = F_2(X,Z)P^*(X,Z), \quad \mathcal{G}_3/H = -2^3 3^6 \tilde{Q}^*(X,Z)[P^*(X,Z)]^2.$$

The polynomial H is of degree four, but must be of degree 7. In order to reach this situation we have to determine a common factor of the third degree of $\mathcal{G}_i/H, i = 1,2,3$. So the following condition must hold: $R_X^{(0)}([P^*]^2, F_1) = R_X^{(1)}([P^*]^2, F_1) = R_X^{(2)}([P^*]^2, F_1) = 0$. We calculate

$$R_X^{(0)}([P^*]^2, F_1) = 3^{20}(3f - m^2)^2(9c - 6f + 8m^2)^4[\Upsilon_1(c,f,m)]^2 Z^{24} = 0,$$
$$R_X^{(1)}([P^*]^2, F_1) = 3^{20}2^2 m(3f - m^2)(9c - 6f + 8m^2)^3 \Upsilon_2(c,f,m) Z^{15} = 0,$$
$$\Phi(y) = R_x^{(0)}(P(x), Q(x,y)) = (3f - m^2)(9c - 6f + 8m^2)^2 y^3/27 \neq 0,$$

where

$$\Upsilon_1 = 9(c-f)(c+2f)^2 + 9(5c-6f)(c+2f)m^2 + 48(c-2f)m^4 + 64m^6,$$
$$\Upsilon_2 = 9(2c+f)(c+2f)^2 + 9(c+2f)(17c+18f)m^2 + 48(7c+10f)m^4 + 64m^6$$

and clearly due to $\Phi(y) \neq 0$ the conditions either $m = \Upsilon_1 = 0$ or $m \neq 0$ and $\Upsilon_1 = \Upsilon_2 = 0$ are necessary.

We detect that in the case $m = 0$ we have $\Upsilon_1 = 9(c-f)(c+2f)^2 = 0$ and this implies either the existence of 9 invariant lines (if $c+2f = 0$) or degenerate systems (if $c+f = 0$ and $R_X^{(2)}([P^*]^2, F_1) = 0$).

Assume now $m \neq 0$. We observe that the solution of the equation $\Upsilon_1 = 0$, as well as the one of the equation $\Upsilon_2 = 0$ must depend on the parameter m. Hence for non-degenerate systems (3.30) the polynomials Υ_1 and Υ_2 vanish if and only if they have a common factor depending on the parameter m. So by Lemma 2.6 the condition

$$R_m^{(0)}(\Upsilon_1, \Upsilon_2) = 2^{42}3^{12}(c-f)^2(c+2f)^{14}(c+14f)^2 = 0$$

must hold and considering $c = -2f$ or $c = -14f$ we get that in both cases the polynomials Υ_1 and Υ_2 possess the same common factor, and namely $3f - m^2$. In the case $c = f$ we get $3f + 8m^2$ as a common factor of these polynomials. So the condition $(3f - m^2)(3f + 8m^2) = 0$ must hold in order to have a common solution of the equations $\Upsilon_1 = 0$ and $\Upsilon_2 = 0$. However we arrive at the contradiction because $\Phi(y) = (3f - m^2)(3f + 8m^2)^2 y^3/27 \neq 0$.

2) *The case $e \neq 0$.* We assume $e = 1$ due to the rescaling $(x, y, t) \mapsto (ex, y, t/e^2)$ and therefore we obtain systems for which we have the polynomial $H(X,Z) = 3^{-9}[54X^3 + 27cXZ^2 + (9cm - 18fm + 8m^3)Z^3]$ of the degree 3. So we have to increase this degree up to 7, i.e. we need a common factor of the forth degree of the polynomials $\mathcal{G}_i/H, i = 1,2,3$. More precisely, for non-degenerate systems we need to find out a common factor of the forth degree of the polynomials F_1 and P^*. By Lemma 2.6 the condition $R_X^{(0)}([P^*]^2, F_1) = R_X^{(1)}([P^*]^2, F_1) = R_X^{(2)}([P^*]^2, F_1) = R_X^{(3)}([P^*]^2, F_1) = 0$ must be satisfied. For the systems (3.29) with $e = 1$ we calculate Coefficient$[R_X^{(3)}([P^*]^2, F_1), Z^6] = 2^2 3^{18} \neq 0$ which leads us to the conclusion that in this case we could not obtain systems with 8 invariant straight lines, considering the infinite one. So we proved the next result.

Proposition 3.8. *A cubic system (3.28) cannot have invariant straight lines of total multiplicity eight.*

3.2.4 Construction of the normal form with cubic homogeneities $((u+1)x^3, ux^2y)$

In this case, considering (3.13) and the cubic homogeneity $\big((u+1)x^3, ux^2y\big)$ via a translation of the origin of coordinates we can consider $l = m = 0$ (since $u \neq 0$) and therefore we get the cubic systems

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + (1+u)x^3, \\ \dot{y} &= b + ex + fy + ny^2 + ux^2y. \end{aligned} \tag{3.31}$$

Now we force the necessary conditions $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$ (see Remark 3.1) which correspond to the type of configuration (3, 2, 1, 1) to be satisfied. We calculate

$$\text{Coefficient}[\mathcal{K}_4, x^2y] = ku(3-u)/9 = 0, \quad \text{Coefficient}[\mathcal{K}_5, xy^3] = -40k^2(6+u)(3+4u)/3 = 0$$

which leads to $k = 0$ and then we obtain

$$\mathcal{K}_4 = -u(3n + 2hu + nu)x^3/9 = 0.$$

Since $u \neq 0$ the last condition is equivalent to $h = -n(3+u)/(2u)$ which implies

$$\begin{aligned}\text{Coefficient}[\mathcal{K}_5, x^3y] &= 10n^2(-54 - 90u - 33u^2 + u^3)/u^2,\\ \text{Coefficient}[\mathcal{K}_6, x^8y^3] &= 2n^3\big(13150u^6 - 125874 - 492669u - 774792u^2\\ &\quad - 638868u^3 - 268688u^4 - 23699u^5\big)/(9u^2).\end{aligned}$$

It is easy to check that Coefficient$[\mathcal{K}_5, x^3y]$ = Coefficient$[\mathcal{K}_6, x^8y^3] = 0$ if and only if $n = 0$ (which imply $h = 0$) and in this case we get $\mathcal{K}_5 = 10du^2 = 0$ which gives $d = 0$. In this case it remains to examine only the condition $\mathcal{K}_6 = 0$.

Thus taking into consideration the conditions $k = h = n = d = 0$ systems (3.31) become

$$\dot{x} = a + cx + gx^2 + (u+1)x^3, \qquad \dot{y} = b + ex + fy + ux^2y \tag{3.32}$$

for which we calculate

$$\mathcal{K}_6 = 40u^4(81fg - 369au + 68cgu + 136fgu - 96au^2)x^{11}/27 = 0$$

and $\mathcal{V}_1\mathcal{V}_3 = -512u^2(3+u)(3+2u)x^8 \neq 0$ (see Lemma 3.5), i.e. for these systems the following conditions hold:

$$\psi \equiv 81fg - 369au + 68cgu + 136fgu - 96au^2 = 0 = \mathcal{K}_6, \quad u(u+3)(2u+3) \neq 0. \tag{3.33}$$

In addition, considering (2.3) and Remark 3.3 for systems (3.31) we examine the direction $y = 0$:

$$Eq_5 = -uW, \qquad Eq_8 = e, \qquad Eq_{10} = b - fW.$$

We see that the equations Eq_8 and Eq_{10} could have only one common solution ($W_0 = 0$) and for this it is necessary and sufficient $b = e = 0$. So in what follows we examine two cases: $e^2 + b^2 = 0$ and $e^2 + b^2 \neq 0$.

The case of the existence of an invariant line in the direction $y = 0$. Then $b = e = 0$ and we get the systems

$$\dot{x} = a + cx + gx^2 + (u+1)x^3 \equiv P(x), \quad \dot{y} = y(f + ux^2) \equiv y\tilde{Q}(x) \tag{3.34}$$

for which the conditions (3.33) hold. We consider two possibilities: $u + 1 \neq 0$ and $u + 1 = 0$.

The subcase $u + 1 \neq 0$. Then for the above systems we calculate

$$\begin{gathered}H(X,Y,Z) = Y[X^3(1+u) + gX^2Z + cXZ^2 + aZ^3] \equiv XP^*(X,Z),\\ \mathcal{G}_1/H = F_1(X,Z), \qquad \mathcal{G}_2/H = P^*(X,Z)F_2(X,Z), \qquad \mathcal{G}_3/H = 24\tilde{Q}^*(X,Z)\big[P^*(X,Z)\big]^2,\end{gathered}$$

where $P^*(X,Z)$ (respectively $Q^*(X,Z)$) is the homogenization of the polynomial $P(x)$ (respectively $Q(x)$). It is clear that these systems are non-degenerate if and only if

$$\varphi(a,c,f,g,u) \equiv R_X^{(0)}\big(P(x), \tilde{Q}(x)\big) \neq 0.$$

Since the polynomials $\mathcal{G}_i/H$, $i = 1,2,3$ do not depend on Y we conclude, that we could not have invariant lines in the direction $y = 0$ except the existent invariant line $y = 0$. Moreover due to $\mathcal{G}_1/H\mid_{(Z=0)} = uX^4 \neq 0$ (as $u \neq 0$) we obtain that Z could not be a a common factor of these polynomials. Therefore the degree of the polynomial $H(x,y)$ could be increased up to seven only with the factors of the form $X+\alpha Z$. Considering Remark 3.6 we deduce that these factors must be factors of the polynomial $P^*(X,Z)$. So $F_1(X,Z)$ must have a common factor of degree 3 with $\left[P^*(X,Z)\right]^2$. Therefore by Lemma 2.6 the conditions $R_X^{(0)}(F_1,[P^*(X,Z)]^2) = R_X^{(1)}(F_1,[P^*(X,Z)]^2) = R_X^{(2)}(F_1,[P^*(X,Z)]^2) = 0$ and $R_X^{(3)}(F_1,[P^*(X,Z)]^2) \neq 0$ must hold. We calculate

$$R_X^{(0)}(F_1,[P^*]^2) = [\varphi]^2\Gamma_1^2 Z^{24}, \qquad R_X^{(1)}(F_1,[P^*]^2) = -2\varphi\Gamma_2 Z^{15}, \qquad R_X^{(2)}(F_1,[P^*]^2) = \Gamma_3 Z^8,$$

where $\Gamma_1(a,c,f,g,u)$, $\Gamma_2(a,c,f,g,u)$ and $\Gamma_3(a,c,f,g,u)$ are some polynomials of total degree 5, 11 and 12, respectively. Evidently since $\varphi \neq 0$ from the above conditions it results $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$.

We claim that for non-degenerate systems (3.34) the polynomials Γ_1 and Γ_2 vanish if and only if they have a common factor depending on the parameter a. Indeed, we observe that Γ_1 is of degree two with respect to a and moreover Coefficient$[\Gamma_1,a^2] = (3+2u)^3 \neq 0$. This means that the solution of the equation $\Gamma_1 = 0$ must depend on a.

On the other hand the degree of Γ_2 with respect to a equals 3 and Coefficient$[\Gamma_2,a^3] = u^2(u+1)(3+2u)(u+2) = 0$ if and only if $u = -2$. In this case we obtain

$$\begin{aligned}\Gamma_2 = (f-2g^2)[&8a^2g + a(3f^2 - 2cf - 12cg^2 + 14fg^2 - 8g^4) + g(2c^2f - 7cf^2\\ &+ 4f^3 + 4c^2g^2 - 2cfg^2 + 2f^2g^2 - 4fg^4)]\end{aligned}$$

and hence for $u = -2$ and $f = 2g^2$ we have $\Gamma_2 = 0$ and this solution does not depend of a. However in this case the condition $\Gamma_1 = -(a - cg + 2g^3)^2 = 0$ gives $a = g(c - 2g^2)$ and this leads to degenerate systems.

If $(f - 2g^2) \neq 0$ then in order to impose the polynomial Γ_2 to vanish for any value of the parameter a it is necessary $g = 0$ and then we calculate

$$\Gamma_1 = cf^2 - f^3 - a^2, \qquad \Gamma_2 = -a(2c-3f)f^2, \qquad \varphi = 8(cf^2 - f^3 - a^2) + (2c-3f)^2 f.$$

Clearly the conditions $(2c - 3f)f = 0$ and $\Gamma_1 = 0$ implies $\varphi = 0$, i.e. systems become degenerate and this completes the proof of our claim.

Thus the polynomials Γ_1 and Γ_2 must have a common factor depending on a and by Lemma 2.5 the condition $R_a^{(0)}(\Gamma_1,\Gamma_2) = 0$ must hold. We calculate

$$R_a^{(0)}(\Gamma_1,\Gamma_2) = (3+2u)^4 \Upsilon_1(c,f,g,u)[\Upsilon_2(c,f,g,u)]^4 \Upsilon_3(c,f,g,u)\Upsilon_4(c,f,g,u) = 0$$

where

$$\begin{aligned}
\Upsilon_1(c,f,g,u) &= f(3+u)^2 - 9c + 3g^2 - 6cu - cu^2,\\
\Upsilon_2(c,f,g,u) &= f(1+u)(3+2u) + 6c - 2g^2 + 7cu + 2cu^2,\\
\Upsilon_3(c,f,g,u) &= 4fg^2u + (3f - cu + 3fu)^2,\\
\Upsilon_4(c,f,g,u) &= u^2[g^2 - c(2+u)^2]^2 + f^2(1+u)^2[3+u(3+u)]^2\\
&\quad - 2fu(1+u)[c(2+u)^2(3+u(3+u)) - g^2(5+u(5+u))].
\end{aligned}$$

1) *The case* $\Upsilon_1(c,f,g,u)=0$. Then $f=(9c-3g^2+6cu+cu^2)/(3+u)^2$ which implies the existence of the common factor $\bar{\psi}(a,c,g,u)=a(3+u)^3-g(9c-2g^2+6cu+cu^2)$ of the polynomials $\Gamma_i,\ i=1,2$. From $\bar{\psi}=0$ it results $a=g(9c-2g^2+6cu+cu^2)/(3+u)^3$ and this gives $\Gamma_1=\Gamma_2=0$ and

$$\Gamma_3(c,g,u)=(9c-3g^2+6cu+g^2u+cu^2)^2[c(3+u)^2(3+2u)^2-g^2(27+27u+8u^2)]/(3+u)^8.$$

Since $\varphi(c,g,u)=(9c-3g^2+6cu+g^2u+cu^2)^3/(3+u)^6\neq 0$ and in addition $(3+u)(3+2u)\neq 0$, the condition $\Gamma_3=0$ gives $c=g^2(27+27u+8u^2)/[(3+u)^2(3+2u)^2]$. Therefore considering the relations

$$f=-\frac{g^2u(9+4u)}{(3+u)^2(3+2u)^2},\qquad a=\frac{3g^3}{(3+u)^2(3+2u)^2},\qquad c=\frac{g^2(27+27u+8u^2)}{(3+u)^2(3+2u)^2}\tag{3.35}$$

with $g(2+u)\neq 0$ (otherwise we get degenerate systems) due to the transformation $(x,y,t)\mapsto\big(-g(3+2u+2ux)/[(3+u)(3+2u)],y,t(3+u)^2(3+2u)^2/[4g^2u^2]\big)$ the last systems could be brought to the 1-parameter family of systems

$$\dot{x}=x(1+x)\big[u+2+(u+1)x\big],\qquad \dot{y}=y\big[u+2+(3+2u)x+ux^2\big]\tag{3.36}$$

for which $H=X^3Y(X+Z)^2\big[(u+1)X+(u+2)Z\big]$.

Thus these systems possess 3 finite singularities: $(-1,0)$, $(0,0)$ and $(-(2+u)/(1+u),0)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 2.4 for these systems we calculate:

$$\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=0,\qquad \mu_6=(u+1)(2+u)^3x^6.$$

Since $(u+1)(2+u)\neq 0$, by Lemma 2.4 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$.

Thus a system (3.36) possesses four distinct invariant affine lines: three in the direction $x=0$ (one triple, one doubles and one simple) and one line in the direction $y=0$, and namely:

$$L_{1,2,3}=x,\qquad L_{4,5}=x+1,\qquad L_6=(u+1)x+(u+2),\qquad L_7=y.$$

Comparing the lines $x=-1$ and $x=-(u+2)/(u+1)$ with $x=0$ we detect the following possibilities which depend on the value of the parameter u:

- *The possibility* $u<-2$. Then the simple invariant line is located on the domain between the triple and the double ones and we obtain the configuration *Config. 8.44*;
- *The possibility* $-2<u<-1$. Then the triple invariant line is located on the domain between the simple and the double ones and we have *Config. 8.45*;
- *The possibility* $u>-1$. The double invariant line is located on the domain between the simple and the triple ones and we get the configuration *Config. 8.46*.

2) *The case* $\Upsilon_2(c,f,g,u)=0$. Then $f=-(6c-2g^2+7cu+2cu^2)/[(1+u)(3+2u)]$ and therefore we get

$$\Gamma_1(a,c,g,u)=[a(1+u)(3+2u)^3-g(9c-2g^2+12cu+4cu^2)]^2/[(1+u)^2(3+2u)^3]=0.$$

Since $(1+u)(3+2u)\neq 0$ the last condition gives

$$a=[g(9c-2g^2+12cu+4cu^2)]/[(1+u)(3+2u)^3]$$

which implies $\Gamma_1 = \Gamma_2 = 0$ and

$$\Gamma_3(c,g,u) = 4(2+u)^2[\phi_1(c,g,u)]^3 \times [c(3+u)^2(3+2u)^2 - g^2(27+27u+8u^2)]/[(1+u)(3+2u)^6],$$
$$\varphi(c,g,u) = [\phi_1(c,g,u)]^2\phi_2(c,g,u)/[(1+u)(3+2u)^6] \neq 0$$

where ϕ_1 and ϕ_2 are polynomials of degree 3 and 4, respectively. Therefore the condition $\Gamma_3 = 0$ is equivalent to

$$c = g^2(27+27u+8u^2)/[(3+u)^2(3+2u)^2]$$

and considering the above conditions we have $f = -\frac{g^2u(9+4u)}{(3+u)^2(3+2u)^2}$, $a = \frac{3g^3}{(3+u)^2(3+2u)^2}$ which all together are equivalent to the conditions (3.35).

3) *The case* $\Upsilon_3(c,f,g,u) = 0$. Suppose first $g \neq 0$. Then denoting $3f - cu + 3fu = c_1$ (i.e. $c = (3f + 3fu - c_1)/u$) the condition $\Upsilon_3 = 0$ gives $f = -c_1^2/(4g^2u)$. In this case the polynomial $\tilde{\psi} = c_1^3(1+u) + g^2u(c_1^2 - 4agu^2)$ is a common factor of Γ_1 and Γ_2 and it must be zero. However calculations yield: $\varphi = \tilde{\psi}\tilde{\psi}_1/(g^6u^3) \neq 0$ (where $\tilde{\psi}_1$ is a polynomial) and we get a contradiction.

Assume now $g = 0$. In this case we get $\Upsilon_3 = (3f - cu + 3fu)^2 = 0$ which implies $c = 3f(1+u)/u$. Then the common factor of the polynomials Γ_1 and Γ_2 is $a^2u^3 + 4f^3(1+u)^2 = \varphi \neq 0$, i.e. we again arrive at a contradiction.

4) *The case* $\Upsilon_4(c,f,g,u) = 0$. We calculate

$$\text{Discriminant}[\Upsilon_4, c] = -16fg^2u^3(1+u)^2(2+u)^4 \geq 0.$$

It was proved earlier (see page 26) that in the case $u = -2$ the condition $\Gamma_1 = \Gamma_2 = 0$ leads to degenerate systems. So we assume $u + 2 \neq 0$ and we examine two subcases: $g \neq 0$ and $g = 0$.

Assume first $g \neq 0$. Then $fu \leq 0$ and we set a new parameter: $fu^3 = -v^2$ (i.e. $f = -v^2/f^3$) and we calculate $\Upsilon_4 = \Phi^{\pm} = 0$, where

$$\Phi^{\pm} = (1+u)(3+3u+u^2)v^2 \pm 2gu^2(1+u)v + u^4[c(2+u)^2 - g^2].$$

It is clear that we could consider only the case $\Phi^- = 0$ (due to the change $v \to -v$) and this condition gives us

$$c = [g^2u^4 + 2gu^2(1+u)v - (1+u)(3+3u+u^2)v^2]/[u^4(2+u)^2].$$

Substituting the expressions for the parameters f and c in the polynomials Γ_1 and Γ_2 we detect that the common factor of these polynomials is again $\varphi \neq 0$.

Suppose now $g = 0$. Then we have $\Upsilon_4 = f(1+u)(3+3u+u^2) - cu(2+u)^2$ and as $(1+u)(3+3u+u^2) \neq 0$ the condition $\Upsilon_4 = 0$ gives $f = cu(2+u)^2/[(1+u)(3+3u+u^2)]$ Herein we calculate $\Gamma_1 = (3+2u)^3\varphi/u^3$ and due to $\varphi \neq 0$ we obtain $\Gamma_1 \neq 0$.

Thus we proved that in the case either $\Upsilon_3 = 0$ or $\Upsilon_4 = 0$ systems (3.34) could not have invariant lines of total multiplicity eight.

The subcase $u + 1 = 0$. Therefore we have $u = -1$ and systems (3.34) become

$$\dot{x} = a + cx + gx^2 \equiv \tilde{P}(x), \qquad \dot{y} = y(f - x^2) \equiv y\tilde{Q}(x), \tag{3.37}$$

for which we calculate

$$\begin{aligned}
H(X,Y,Z) &= YZ(gX^2+cXZ+aZ^2) \equiv Y\tilde{P}^*(X,Z),\\
\mathcal{G}_1/H &= -X^4+(c+2f)X^2Z^2+2(a+fg)XZ^3+(c-f)fZ^4 \equiv F_1(X,Z),\\
\mathcal{G}_2/H &= -\big[2X^3-(c+2f)XZ^2-(a+fg)Z^3\big]\tilde{P}^*(X,Z) \equiv F_2(X,Z)\tilde{P}^*(X,Z),\\
\mathcal{G}_3/H &= 24Z^2[\tilde{Q}^*(X,Z)]\big[\tilde{P}^*(X,Z))\big]^2.
\end{aligned}$$

So we need to determine a common factor of degree 3 of the polynomials $\mathcal{G}_i/H$, $i=1,2,3$, which in fact must contains only the factors of the polynomial $\tilde{P}^*(X,Z)$ (see Remark 3.6).

On the other hand we observe that for non-degenerate systems (3.37) the polynomials $f-x^2$ and $\tilde{P}(x)$ have no common factors, i.e. the following condition must holds:

$$\varphi(a,c,f,g) \equiv R_x^{(0)}(\tilde{P}(x),\tilde{Q}(x)) = (a+fg)^2-c^2f \neq 0.$$

Thus considering the structure of the polynomial $\mathcal{G}_3/H$ we deduce that the polynomial $\tilde{P}^*(X,Z)$ must be a factor of the polynomial F_1. So the following conditions are necessary: $R_X^{(0)}\big(F_1,\tilde{P}^*(X,Z)\big) = R_X^{(1)}\big(F_1,\tilde{P}^*(X,Z)\big) = 0$. We calculate

$$R_X^{(1)}\big(F_1,\tilde{P}^*(X,Z)\big) = (c-g^2)(2ag-c^2+2fg^2)Z^3 = 0. \tag{3.38}$$

We observe that $c-g^2\neq 0$, otherwise supposing $c=g^2$ we obtain $R_X^{(0)}\big(\mathcal{G}_1/H,\tilde{P}^*(X,Z)\big) = \varphi^2Z^8 \neq 0$. Thus for non-degenerate systems the condition (3.38) gives $2ag-c^2+2fg^2=0$, where $g\neq 0$, otherwise we get $g=c=0$ which contradicts $c-g^2\neq 0$. So we obtain $a=\frac{c^2-2fg^2}{2g}$ and calculations yield

$$\begin{aligned}
R_X^{(0)}\big(F_1,\tilde{P}^*(X,Z)\big) &= \frac{1}{16g^4}c^2(c-2g^2)^2(c^2-4fg^2)^2Z^8 = 0,\\
\varphi(c,f,g) &= c^2(c^2-4fg^2)/(4g^2) \neq 0.
\end{aligned}$$

Therefore $c=2g^2$ and considering (3.33) and the relations $u=-1$ and $a=\frac{c^2-2fg^2}{2g}$ we calculate

$$R_X^{(0)}\big(F_1(X,Z)/\tilde{P}^*,\tilde{P}^*(X,Z)\big) = 4g^2(4f-5g^2)Z^4,\ \psi = -82g(4f-5g^2) = 0.$$

Hence due to $g\neq 0$ we get the unique condition $f=5g^2/4$ and this leads to the family of systems

$$\dot{x} = g(g+2x)(3g+2x)/4, \qquad \dot{y} = y(5g^2-4x^2)/4,$$

which via the changing $(x,y,t)\mapsto(g(x-1/2),y,t/g^2)$ could be brought to the system

$$\dot{x} = x(1+x), \qquad \dot{y} = y(1+x-x^2). \tag{3.39}$$

For this system we have $H(X,Y,Z) = X^3YZ(X+Z)^2$, i.e. it possesses invariant lines of total multiplicity eight. We observe that system (3.39) has 2 finite singularities: $(-1,0)$, $(0,0)$ which are located on the invariant line $y=0$. On the other hand considering Lemma 2.4 for these systems we calculate:

$$\mu_0=\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=\mu_6=0, \qquad \mu_7=-x^6y.$$

By Lemma 2.4 all other 7 finite singular points have gone to infinity. Moreover, according to this lemma, six of them collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$ and the last one collapses with the singular point $[1,0,0]$ located on the "end" of the invariant line $y = 0$.

Thus besides the double infinite line a system (3.39) possesses three distinct invariant affine lines: two in the direction $x = 0$ (one triple and one double) and a line in the direction $y = 0$, and namely:

$$L_{1,2,3} = x, \qquad L_{4,5} = x+1, \qquad L_6 = y.$$

Therefore we get the configuration *Config. 8.47*.

The case of the non-existence of an invariant line in the direction $y = 0$. In this case we consider systems (3.31), i.e. the systems

$$\begin{aligned} \dot{x} &= a + cx + gx^2 + (1+u)x^3 \equiv P(x),\\ \dot{y} &= b + ex + fy + ux^2y \equiv Q(x,y), \end{aligned} \tag{3.40}$$

for which the conditions $b^2 + e^2 \neq 0$ and (3.33) hold. For these systems we calculate

$$\begin{aligned} H(X,Y,Z) \equiv P^*(X,Z), \qquad \mathcal{G}_1/H \equiv F_1(X,Y,Z),\\ \mathcal{G}_2/H \equiv F_2(X,Y,Z)P^*(X,Z), \qquad \mathcal{G}_3/H = 24[Q^*(X,Y,Z)]\left[P^*(X,Z)\right]^2, \end{aligned} \tag{3.41}$$

where $P^*(X,Z)$ (respectively $Q^*(X,Y,Z)$) is the homogenization of the polynomial $P(x)$ $\big($respectively $Q(x,y)\big)$. Since $u \neq 0$ and $F_1|_{Z=0} = uX^4Y$ we deduce that Z could not be a a common factor of these polynomials. Therefore the degree of the polynomial $H(X,Y,Z)$ could be increased up to seven only with the factors of the form $X + \alpha Z$.

On the other hand for non-degenerate systems (3.40) the polynomials $Q(x,y)$ and $P(x)$ have no common factors, i.e. the following condition must holds:

$$\varphi(y) \equiv R_x^{(0)}\big(P(x), Q(x,y)\big) \neq 0.$$

We claim that systems (3.40) could not possess invariant lines of total multiplicity eight. In order to prove this we split our examination in two subcases: $u+1 \neq 0$ and $u+1 = 0$.

The subcase $u+1 \neq 0$. So in order to increase the degree of H we have to find out the conditions which imply the existence of a common factor of the forth degree of $\mathcal{G}_i/H$, $i = 1,2,3$. Therefore by Lemma 2.6 the following condition must hold:

$$R_X^{(0)}\left(F_1, [P^*]^2\right) = R_X^{(1)}\left(F_1, [P^*]^2\right) = R_X^{(2)}\left(F_1, [P^*]^2\right) = R_X^{(3)}\left(F_1, [P^*]^2\right) = 0.$$

We calculate Coefficient$[R_X^{(3)}\left(F_1, [P^*]^2\right), Z^6] = -e^3(1+u)^2(2+u)^3 = 0$ and since $u+1 \neq 0$ we examine either the condition $e = 0$ (then $b \neq 0$) or $u = -2$.

1) *The possibility* $e = 0$. Then we have

$$\begin{aligned} \text{Coefficient}[R_X^{(3)}\left(F_1, [P^*]^2\right), Y^2Z^4] &= -2bgu^2(1+u)(4+3u) = 0,\\ \psi(f,g,u) = \ & g(81f + 68cu + 136fu) - 3au(123+32u) = 0 \end{aligned}$$

and since $ub(u+1) \neq 0$ we get the condition $g(4+3u) = 0$.

Assuming $g = 0$ we obtain $R_X^{(3)}\left(F_1, [P^*]^2\right) = 2au^3(1+u)(2+u)Y^3Z^3 = 0$ and taking into consideration the above expression of $\psi(f,g,u)$ in this case, we conclude that $a = 0$. Then we have Coefficient$[R_X^{(2)}\left(\mathcal{G}_1/H, [P^*]^2\right), Z^{12}] = b^4(1+u)^4(3+2u)^4 \neq 0$.

Now we suppose $u = -4/3$ and $g \neq 0$ and we get $R_X^{(3)}\left(F_1, [P^*]^2\right) = 128(2a - 6cg + 3fg)Y^3Z^3/243 = 0$, i.e. $a = 3(2c - f)g/2$. However the last condition implies

$$\text{Coefficient}[R_X^{(2)}\left(F_1, [P^*]^2\right), Z^{12}] = b^4/6561 \neq 0$$

(since $b \neq 0$). Thus in the case $e = 0$ we could not have systems with invariant lines of total multiplicity eight.

2) *The possibility* $u = -2$, $e \neq 0$. Then $R_X^{(3)}\left(F_1, [P^*]^2\right) = 4(ef - 4bg + 2eg^2)Y^2Z^4 = 0$ and this gives $f = 2g(2b - eg)/e$. In this case we obtain $\text{Coefficient}[R_X^{(2)}\left(\mathcal{G}_1/H, [P^*]^2\right), Z^{12}] = (eg - b)^4 = 0$, i.e. $g = b/e$ which implies

$$\text{Coefficient}[R_X^{(2)}\left(F_1, [P^*]^2\right), Y^2Z^{10}] = 4(2b^3 - bce^2 + ae^3)^2/e^4 = 0,$$
$$\varphi = (2b^3 - bce^2 + ae^3)\tilde{\varphi}(y) \neq 0$$

where $\tilde{\varphi}(y)$ is a polynomial of degree 3 and therefore we arrive at a contradiction.

The subcase $u + 1 = 0$. Then we get systems

$$\dot{x} = a + cx + gx^2 \equiv P(x), \quad \dot{y} = b + ex + fy - x^2y \equiv Q(x, y) \tag{3.42}$$

for which we calculate $H(X, Y, Z) \equiv ZP^*(X, Z)$ and $\mathcal{G}_i/H$, $i = 1, 2, 3$ have the same factors as in (3.41). So we need to determine a common factor of degree 4 of the polynomials $\mathcal{G}_i/H$, $i = 1, 2, 3$, which in fact must contains only the factors of the polynomial $P^*(X, Z)$ (see Remark 3.6). Therefore the following conditions must hold:

$$R_X^{(0)}\left(\mathcal{G}_1/H, [P^*]^2\right) = R_X^{(1)}\left(\mathcal{G}_1/H, [P^*]^2\right) = R_X^{(2)}\left(\mathcal{G}_1/H, [P^*]^2\right) = R_X^{(3)}\left(\mathcal{G}_1/H, [P^*]^2\right) = 0.$$

For systems (3.42) we calculate $R_X^{(3)}\left(\mathcal{G}_1/H, [P^*]^2\right) = -gZ(2cY + egZ) = 0$. If $g = 0$ then we obtain $R_X^{(2)}\left(\mathcal{G}_1/H, [P^*]^2\right) = c^4Y^2Z^4 = 0$ which yields $c = 0$. However in this case we get $R_X^{(0)}\left(\mathcal{G}_1/H, [P^*]^2\right) = a^8Y^4Z^{16} \neq 0$ due to $\varphi = a^2y^2 \neq 0$.

If $g \neq 0$ then $c = e = 0$ and then $b \neq 0$ (as $b^2 + e^2 \neq 0$). These conditions imply $R_X^{(2)}\left(\mathcal{G}_1/H, [P^*]^2\right) = g^2Z^4(2aY + 2fgY + bgZ)^2 \neq 0$ (since $bg \neq 0$). This completes the proof of our claim, i.e. systems (3.40) could not possess invariant lines of total multiplicity eight.

In such a way in Subsection 3.2.4 we proved the next result.

Proposition 3.9. *Systems* (3.32) *possess invariant lines of total multiplicity eight if and only if the following set of conditions holds:*

$$\begin{aligned} &e = b = 0, \qquad a = \frac{3g^3}{(3+u)^2(3+2u)^2}, \qquad c = \frac{g^2(27 + 27u + 8u^2)}{(3+u)^2(3+2u)^2}, \\ &f = \frac{-g^2u(9+4u)}{(3+u)^2(3+2u)^2}, \qquad g(u+2) \neq 0 \end{aligned} \tag{3.43}$$

4 Invariant conditions for the configurations *Config. 8.39–8.47*

By Lemma 2.9 the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$, $\mathcal{D}_2 \neq 0$ are necessary and sufficient for a cubic system to have two real distinct infinite singularities which are determined by one triple and one simple factors of $C_3(x, y)$. After a linear transformation such a cubic system could be brought to the form (3.13). According to Lemma 3.5 the above mentioned cubic systems could have one of the four cubic homogeneities given by this lemma. Moreover, the invariant polynomials which distinguish different configurations of invariant lines in the case of cubic systems with homogeneity $(x^3, 0)$ have been constructed in [6]. Since for the homogeneity $(2x^3, 3x^2y)$ we get no configurations (see Subsection 3.2.3) we restrict our attention to the remaining two cubic homogeneities: $(x^3, 3x^2y)$ and $\big((u+1)x^3, ux^2y\big)$. Thus considering Lemma 3.5 the conditions $\mathcal{V}_1 = 0$, $\mathcal{V}_3 \neq 0$ and respectively $\mathcal{V}_5 = \mathcal{U}_2 = 0$, $\mathcal{V}_1\mathcal{V}_3 \neq 0$ lead us to systems (3.14) and respectively (3.31) (via a linear transformation and time rescaling). Additionally for a system (3.14) (respectively (3.31)) we applied Remark 3.1 and we proved that the condition $\mathcal{L}_1 = \mathcal{L}_2 = 0$ (respectively $\mathcal{K}_4 = \mathcal{K}_5 = 0$) is equivalent to $n = k = h = d = 0$ which leads to systems (3.15) (respectively (3.32)).

In what follows considering systems (3.15) and (3.32) we find out the invariant conditions which are equivalent to the conditions mentioned in Propositions 3.7 and 3.9.

1) Systems (3.15) For these systems we calculate $N_{22} = mx^5$ and it is evident that the condition $N_{22} = 0$ is equivalent to $m = 0$ and in this case we calculate

$$N_{23} = -3ex^6 + 3(c-f)x^5y, \qquad N_{24} = 216bx^{13}, \qquad \mathcal{K}_6 = 162000ax^{11}.$$

Thus $N_{23} = 0$ implies $e = c - f = 0$ and therefore, according to Proposition 3.7, next we split our examination in two cases: $b = 0$, $a \neq 0$, i.e. $N_{24} = 0$, $\mathcal{K}_6 \neq 0$ and $b \neq 0$, $a = 0$, i.e. $N_{24} \neq 0$, $\mathcal{K}_6 = 0$.

a) If $N_{24} = 0$ and $\mathcal{K}_6 \neq 0$ then we arrive at systems (3.20) for which $\mu_6 = (27a^2 + 4c^3)x^6$ should be non-zero in order to have non-degenerate systems. Moreover due to a transformation the last systems became of the form (3.22) with $\mu_6 = r^2(4r-1)x^6 \neq 0$.

So if for systems (3.15) the conditions $N_{22} = N_{23} = N_{24} = 0$, $\mathcal{K}_6 \neq 0$ hold then we get either *Config. 8.39* if $\mu_6 < 0$ or *Config. 8.40* if $\mu_6 > 0$.

b) Assume now $N_{24} \neq 0$ and $\mathcal{K}_6 = 0$. Then applying a rescaling we arrive at systems (3.26) for which $\mu_6 = 4c^3x^6 \neq 0, c = \{-1, 0, 1\}$. Therefore if for a system (3.32) the conditions $N_{22} = N_{23} = \mathcal{K}_6 = 0$, $N_{24} \neq 0$ are satisfied then we have *Config. 8.41* if $\mu_6 < 0$; *Config. 8.42* if $\mu_6 = 0$ and *Config. 8.43* if $\mu_6 > 0$.

2) Systems (3.32) For these systems we calculate $N_{24} = 2bu^4x^{13}/3$, $N_{25} = 5eu^2x^{10}/3$. It is evident that the condition $N_{24} = N_{25} = 0$ implies $b = e = 0$ (since $u \neq 0$, see Lemma 3.5). In this case we have $N_{26} = 20u^4(9a - cg - 2fg + 3au)x^{10}y/9$ and

$$N_{27} = 40u^4[3(c-f)(12c+15f-4g^2) - 3u(c-f)^2 - 16g^2u(c+2f) + 4(c-f)(c+2f)u^2]/9.$$

Taking into consideration the conditions (3.33), it is easy to verify that $N_{26} = N_{27} = \mathcal{K}_6 = 0$ lead us to the conditions (3.43). Since $u(3+u)(3+2u) \neq 0$ (due to $\mathcal{V}_1\mathcal{V}_3 \neq 0$) and $g \neq 0$ (as systems are non-degenerate) applying the corresponding transformation to systems (3.32) with the conditions (3.43) (mentioned on page 27) we arrive at systems (3.36) for which we have $u + 2 \neq 0$ (otherwise these systems become degenerate). For systems (3.36) we calculate

$$\mu_i = 0, \quad i = 0, 1, \ldots, 5, \qquad \mu_6 = (u+1)(u+2)^3x^6, \qquad N_{28} = -2(3+2u)x^4.$$

If $\mu_6 \neq 0$ we obtain $\operatorname{sign}(\mu_6) = \operatorname{sign}\big((u+1)(u+2)\big)$. Therefore if $\mu_6 < 0$ then $-2 < u < -1$ and we get *Config. 8.44*, whereas in the case $\mu_6 > 0$ we have either *Config. 8.45* for $N_{28} < 0$ or *Config. 8.46* for $N_{28} > 0$. In the case $\mu_6 = 0$, i.e. $u = -1$ we arrive at systems (3.39) and we get *Config. 8.47*.

5 Perturbations of normal forms

To finish the proof of the Main theorem it remains to construct for the normal forms, given in the statement of this theorem, the corresponding perturbations, which prove that the respective invariant straight lines have the indicated multiplicities. In this section we construct such perturbations and for each configuration *Configs. 8.j*, $j = 39, 40, \ldots, 47$ we give:

- the corresponding normal form and its invariant straight lines;
- the respective perturbed normal form and its invariant straight lines;
- the configuration *Config.* $8.j_\varepsilon$, $j = 39, 40, \ldots, 47$ which corresponds to the perturbed systems.

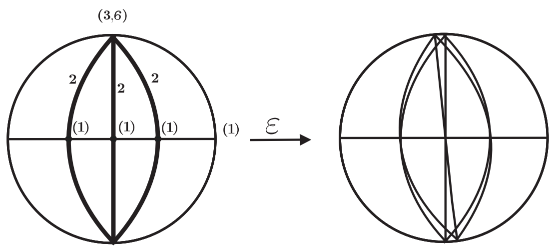
Config. 8.39: $\begin{cases} \dot{x} = x(1+x)(v+x), \\ \dot{y} = (v+2x+2vx+3x^2)y; \end{cases}$

Invariant lines: $L_{1,2} = x,\ L_{3,4} = x+1,\ L_{5,6} = x+v,\ L_7 = y;$

We remark that these systems are obtained from (3.23) due to the transformation $(x,y,t) \mapsto \big((1-u)x/2, y, 4t/(u-1)\big)$ and notation $v = (1+u)/(1-u)$, where $v \neq \{0,1\}$ (since $u(1-u^2) \neq 0$).

Config. 8.39_ε: $\begin{cases} \dot{x} = x(1+x)(v+x), \\ \dot{y} = y[v+2x+2vx+3x^2+\varepsilon(y+vy+3xy+y^2\varepsilon)]; \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = x+y\varepsilon,\ L_3 = x+1,\ L_4 = x+1+y\varepsilon,\ L_5 = x+v, \\ L_6 = x+v+y\varepsilon,\ L_7 = y. \end{cases}$



Config. 8.39 *Config.* 8.39_ε

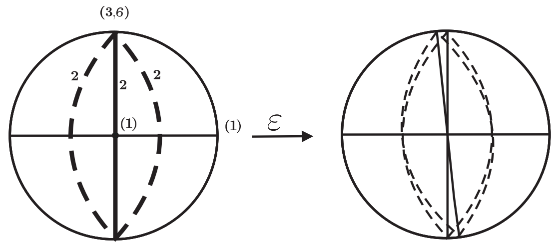
Config. 8.40: $\begin{cases} \dot{x} = (x-1)(v^2+x^2), \\ \dot{y} = (v^2-2x+3x^2)y; \end{cases}$

Invariant lines : $L_{1,2} = x-1,\ L_{3,4} = x-vi,\ L_{5,6} = x+vi,\ L_7 = y;$

We remark that these systems are obtained from (3.24) due to the transformation $(x,y,t) \mapsto \big((x-1)/2, y, 4t\big)$ and changing u by v.

Config. 8.40_ε: $\begin{cases} \dot{x} = (x-1)(v^2+x^2), \\ \dot{y} = y[v^2-2x+3x^2-2\varepsilon(y-3xy-2y^2\varepsilon)]; \end{cases}$

Invariant lines: $\begin{cases} L_1 = x-1,\ L_2 = x-1+2y\varepsilon,\ L_3 = x-vi,\ L_4 = x-vi+2y\varepsilon, \\ L_5 = x+vi,\ L_6 = x+vi+2y\varepsilon,\ L_7 = y. \end{cases}$



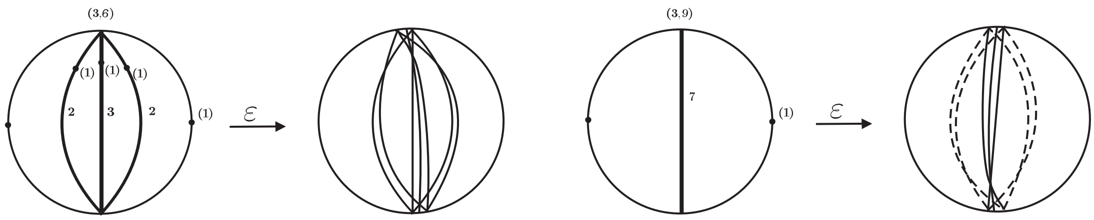
Config. 8.40 *Config. 8.40*$_\varepsilon$

Config. 8.41–8.43 : $\begin{cases} \dot{x} = x(r + x^2), \\ \dot{y} = 1 + ry + 3x^2y; \end{cases}$

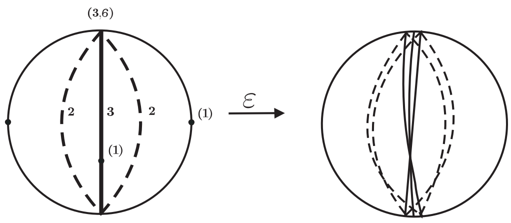
Invariant lines: $L_{1,2,3} = x,\ L_{4,5} = x + \sqrt{-r},\ L_{6,7} = x - \sqrt{-r};$

Config. 8.41$_\varepsilon$*–8.43*$_\varepsilon$: $\begin{cases} \dot{x} = rx + x^3 - \varepsilon + x\varepsilon, \\ \dot{y} = 1 + ry + 3x^2y + \varepsilon(y + 6xy^2 + 4y^3\varepsilon); \end{cases}$

Invariant lines:
$$L_1 = x + y\varepsilon,\ L_{2,4,6} = rx + x^3 - \varepsilon + x\varepsilon,$$
$$L_{3,5,7} = rx + x^3 + \varepsilon(1 + x + 2ry + 6x^2y + 2y\varepsilon + 12xy^2\varepsilon + 8y^3\varepsilon^2).$$



Config. 8.41 *Config. 8.41*$_\varepsilon$ *Config. 8.42* *Config. 8.42*$_\varepsilon$



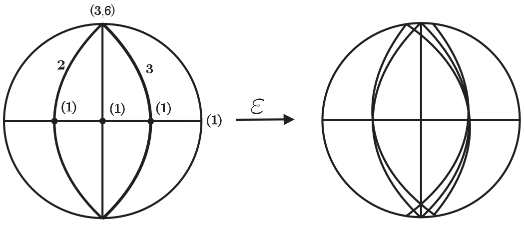
Config. 8.43 *Config. 8.43*$_\varepsilon$

Config. 8.44–8.46 : $\begin{cases} \dot{x} = x(1 + x)[u + 2 + (u + 1)x], \\ \dot{y} = y[u + 2 + (3 + 2u)x + ux^2]; \end{cases}$

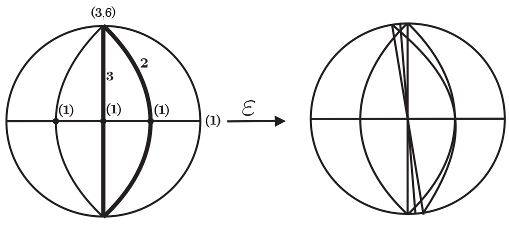
Invariant lines: $L_{1,2,3} = x,\ L_{4,5} = x + 1,\ L_6 = (u + 1)x + (u + 2),\ L_7 = y;$

Config. 8.44$_\varepsilon$*-8.46*$_\varepsilon$: $\begin{cases} \dot{x} = x(1 + x)[u + 2 + (u + 1)x], \\ \dot{y} = y[(2 + u) + (3 + 2u)x + ux^2 - (3 + u)xy\varepsilon - (2 + u)y^2\varepsilon^2]; \end{cases}$

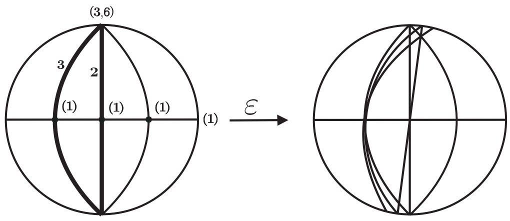
Invariant lines:
$$L_1 = x,\ L_2 = x + y\varepsilon,\ L_3 = x + 2y\varepsilon + uy\varepsilon,\ L_4 = x + 1,$$
$$L_5 = 1 + x + y\varepsilon,\ L_6 = (u + 1)x + (u + 2),\ L_7 = y.$$



Config. 8.44 *Config. 8.44*$_\varepsilon$



Config. 8.45 *Config. 8.45*$_\varepsilon$



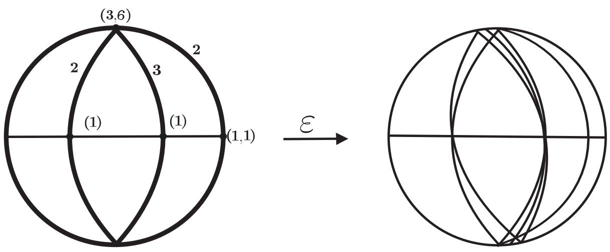
Config. 8.46 *Config. 8.46*$_\varepsilon$

Config. 8.47: $\begin{cases} \dot{x} = x(x+1), \\ \dot{y} = y(1+x-x^2); \end{cases}$

Invariant lines: $L_{1,2,3} = x,\ L_{4,5} = x+1,\ L_6 = y,\ L_7 : Z = 0;$

Config. 8.47$_\varepsilon$: $\begin{cases} \dot{x} = x(1+x-2x\varepsilon)(1-\varepsilon-x\varepsilon+2x\varepsilon^2)/(2\varepsilon-1)^2, \\ \dot{y} = xy + xy^2(\varepsilon-2)\varepsilon - x^2y(1+\varepsilon) - y(\varepsilon-1)/(2\varepsilon-1)^2 + y^3(\varepsilon-1)\varepsilon^2; \end{cases}$

Invariant lines: $L_1 = x,\ L_2 = x+y\varepsilon,\ L_3 = x+y\varepsilon-y\varepsilon^2,\ L_4 = 2x\varepsilon - y\varepsilon + 2y\varepsilon^2 - 1 - x,$
$L_5 = x+1,\ L_6 = y,\ L_7 = 1-\varepsilon-x\varepsilon+2x\varepsilon^2.$



Config. 8.47 *Config. 8.47*$_\varepsilon$

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