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Existence and uniqueness of convex monotone positive solutions for boundary value problems of an elastic beam equation with a parameter

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of convex monotone positive solutions for a boundary value problem of an elastic beam equation with a parameter. The proofs of the main results rely on a fixed point theorem and some properties of eigenvalue problems for a class of general mixed monotone operators. The results can guarantee the existence of a unique convex monotone positive solution and can be applied to construct two iterative sequences for approximating it. Moreover, we present some pleasant properties of convex monotone positive solutions for the boundary value problem dependent on the parameter. Finally, an example is given to illustrate the main results.

Keywords: existence and uniqueness, convex monotone positive solution, elastic beam equation, fixed point theorem for mixed monotone operators.

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1 Introduction

Recently, the study of fourth-order boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [1–8, 10–20, 22] and the references therein. The fourth-order problems usually characterize the deformations of an elastic beam and so they are useful for material mechanics. There are many papers discussing the existence and multiplicity of positive solutions for the elastic beam equations, by using various methods, such as the Leray–Schauder continuation method, the topological degree theory, the shooting method, fixed point theorems on cones, the critical point theory, and the lower and upper solution method; see for example the above works mentioned. For example, Webb *et al.* [17] studied the existence of positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions. By using the Krasnosel′skii fixed-point theorem of cone expansion-compression type, Yao [19] investigated the positive solutions for a fourth-order boundary value problem

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with a parameter and obtained existence and multiplicity results. Pei and Chang [15] used a monotone iterative technique and proved the existence of at least one symmetric positive solution for a fourth-order boundary value problem. Li and Zhang [11] utilized a fixed point theorem of generalized concave operators to establish the existence and uniqueness of monotone positive solutions for a fourth-order boundary value problem. In [12], Li and Zhai get the existence and uniqueness of monotone positive solutions for a fourth-order boundary value problem via two fixed point theorems of mixed monotone operators with perturbation. In [20], by using the fixed point index method, we get the existence of at least one or at least two symmetric positive solutions for a fourth-order boundary value problem. And then, by using a fixed point theorem of general α-concave operators, we also obtain the existence and uniqueness of symmetric positive solutions for the boundary value problem.

The purpose of this paper is to establish the existence and uniqueness of convex monotone positive solutions for the following nonlinear boundary value problems of an elastic beam equation with a parameter

$$\begin{aligned} u^{(4)}(t) &= \lambda f(t,u(t)),\ 0<t<1,\\ u(0)=u'(0)=u''(1)&=u^{(3)}(1)+\lambda g(u(1))=0, \end{aligned} \tag{1.1}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ and $g:[0,+\infty)\to[0,+\infty)$, $\lambda>0$ is a parameter. Here, convex monotone positive solutions mean convex increasing positive solutions. When $\lambda=1$ in boundary conditions, Wang *et al.* [16] used a fixed point theorem of cone expansion and a fixed point theorem of generalized concave operators to obtain the existence, nonexistence, and uniqueness of convex monotone positive solutions for problem (1.1). In [5], Cabada and Tersian studied the existence and multiplicity of solutions for problem (1.1) by using a three critical point theorem. Different from the above works mentioned, motivated by the work [21], we will use the main fixed point theorem and properties of eigenvalue problems for a class of general mixed monotone operators in [21] to prove the existence and uniqueness of convex monotone positive solutions for problem (1.1). Moreover, we will construct two sequences for approximating the unique solution and show that the positive solution with respect to λ has some pleasant properties.

2 Preliminaries and previous results

In this section, we present some basic concepts in ordered Banach spaces and fixed point theorems for general mixed monotone operators. For convenience of readers, we suggest that one refer to [9,21] for details.

Let $(E,\|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P\subset E$, *i.e.* $x\le y$ if and only if $y-x\in P$. If $x\le y$ and $x\ne y$, then denote $x<y$ or $y>x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P\subset E$ is a cone if it satisfies (i) $x\in P,\ r\ge 0\Rightarrow rx\in P$; (ii) $x\in P,\ -x\in P\Rightarrow x=\theta$.

P is called normal if there exists a constant $N>0$ such that, for all $x,y\in E$, $\theta\le x\le y$ implies $\|x\|\le N\|y\|$; in this case the infimum of such constants N is called the normality constant of P.

For all $x,y\in E$, the notation $x\sim y$ means that there exist $\mu>0$ and $\nu>0$ such that $\mu x\le y\le \nu x$. Clearly, $\sim$ is an equivalence relation. Given $h>\theta$ (*i.e.* $h\ge\theta$ and $h\ne\theta$), we denote by P_h the set $P_h=\{x\in E:\ x\sim h\}$. It is easy to see that $P_h\subset P$ is convex and $rP_h=P_h$ for all $r>0$.

Definition 2.1. An operator $A : P \times P \to P$ is said to be a mixed monotone if $A(x,y)$ is increasing in x and decreasing in y, *i.e.*, u_i, v_i $(i = 1,2) \in P$, $u_1 \le u_2, v_1 \ge v_2$ implies $A(u_1,v_1) \le A(u_2,v_2)$. An element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

Recently, in [21] Zhai and Zhang discussed the following operator equations

$$A(x,x) = x \quad \text{and} \quad A(x,x) = \lambda x,$$

where $A : P \times P \to P$ is a mixed monotone operator which satisfies the following assumptions:

(A_1) there exists $h \in P$ with $h \neq \theta$ such that $A(h,h) \in P_h$.

(A_2) for any $u,v \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1)$ such that $A(tu, t^{-1}v) \ge \varphi(t)A(u,v)$.

The authors obtained the existence and uniqueness of positive solutions for the above equations and present the following interesting results.

Lemma 2.2. *Suppose that P is a normal cone of E, and $(A_1),(A_2)$ hold. Then operator A has a unique fixed point x^* in P_h. Moreover, for any initial $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \qquad n = 1,2,\dots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Lemma 2.3. *Suppose that P is a normal cone of E, and $(A_1),(A_2)$ hold. Let x_λ $(\lambda > 0)$ denote the unique solution of the nonlinear eigenvalue equation $A(x,x) = \lambda x$ in P_h. Then we have the following conclusions:*

(B_1) *if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then x_λ is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;*

(B_2) *if there exists $\beta \in (0,1)$ such that $\varphi(t) \ge t^\beta$ for $t \in (0,1)$, then x_λ is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \to 0$;*

(B_3) *if there exists $\beta \in (0,\frac{1}{2})$ such that $\varphi(t) \ge t^\beta$ for $t \in (0,1)$, then $\lim\limits_{\lambda \to \infty} \|x_\lambda\| = 0$, $\lim\limits_{\lambda \to 0^+} \|x_\lambda\| = \infty$.*

Remark 2.4. By using Lemmas 2.2, 2.3, we can investigate the existence and uniqueness of positive solutions for many boundary value problems with a parameter, and then show some pleasant properties of positive solutions with respect to λ. For example, two-point boundary value problems, three-point boundary value problems were studied in [21], where the nonlinear terms $f(t,u,v)$ are required to be continuous. The hypothesis that f should be continuous was not stated in Theorem 3.1 of [21] but with this addition the result holds. Unfortunately, the nonlinear terms of examples 3.1–3.4 in [21] are not continuous, so those examples are not valid.

3 Existence and uniqueness of convex monotone positive solutions for problem (1.1)

In this section, we use Lemmas 2.2, 2.3 to study problem (1.1) and we present two new results on the existence and uniqueness of convex monotone positive solutions, we show that the convex monotone positive solution with respect to λ has some pleasant properties. The method is new to the literature and so is the existence and uniqueness result to fourth-order boundary value problems.

In our considerations we will work in the Banach space $C[0,1]$, the space of all continuous functions on $[0,1]$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Notice that this space can be equipped with a partial order given by

$$x,\ y \in C[0,1],\ x \le y \Leftrightarrow x(t) \le y(t) \text{ for } t \in [0,1].$$

Set $P = \{x \in C[0,1] :\ x(t) \ge 0,\ t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in $C[0,1]$ and the normality constant is 1.

From [2], if f, g are continuous, then problem (1.1) is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s))ds + \lambda g(u(1))\phi(t),\ t \in [0,1],$$

where

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), & 0 \le s \le t \le 1, \\ t^2(3s-t), & 0 \le t \le s \le 1, \end{cases}$$

and $\phi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3$.
From [11], we give the following properties of the Green's function $G(t,s)$ and $\phi(t)$.

Lemma 3.1. *For any $t, s \in [0,1]$, we have*

$$\frac{1}{3}s^2t^2 \le G(t,s) \le \frac{1}{2}st^2, \qquad \frac{1}{3}t^2 \le \phi(t) \le \frac{1}{2}t^2.$$

The following conclusion is simple, so we omit its proof.

Lemma 3.2. *If $u \in C^4[0,1]$ satisfies*

$$\begin{cases} u^{(4)}(t) \ge 0,\ t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0,\ u^{(3)}(1) \le 0, \end{cases}$$

then: (i) $u(t)$ is monotone increasing on $[0,1]$; (ii) $u''(t) \ge 0,\ t \in [0,1]$, that is, $u(t)$ is a convex function on $[0,1]$.

Theorem 3.3. *Assume that*

(H_1) *$f : [0,1] \times [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to [0,+\infty)$ are continuous;*

(H_2) *$f(t,x)$ is increasing in $x \in [0,+\infty)$ for each $t \in [0,1]$ with $f(t,0) \not\equiv 0$, and $g(x)$ is decreasing in $x \in [0,+\infty)$;*

(H_3) *for $\eta \in (0,1)$, there exist $\varphi_i(\eta) \in (\eta,1)$ $(i = 1,2)$ such that*

$$f(t,\eta x) \ge \varphi_1(\eta) f(t,x), \quad g(\eta x) \le \frac{1}{\varphi_2(\eta)} g(x), \quad \forall\, t \in [0,1],\ x \in [0,+\infty).$$

Then, for any given $\lambda > 0$, problem (1.1) has a unique convex monotone positive solution u_λ^* in P_h, where $h(t) = t^2,\ t \in [0,1]$. Moreover, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences:

$$x_n = \lambda \int_0^1 G(t,s) f(s, x_{n-1}(s)) ds + \lambda g(y_{n-1}(1)) \phi(t),$$
$$y_n = \lambda \int_0^1 G(t,s) f(s, y_{n-1}(s)) ds + \lambda g(x_{n-1}(1)) \phi(t), \qquad n = 1, 2, \ldots,$$

we have $x_n \to u_\lambda^*, y_n \to u_\lambda^*$ as $n \to +\infty$. Further, (i) if $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ for $t \in (0,1)$, then u_λ^* is strictly increasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* < u_{\lambda_2}^*$; (ii) if there exists $\beta \in (0,1)$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$; (iii) if there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\lim_{\lambda \to 0^+} \|u_\lambda^*\| = 0, \lim_{\lambda \to +\infty} \|u_\lambda^*\| = +\infty$.

Proof. For any $u, v \in P$, we define

$$A(u,v)(t) = \int_0^1 G(t,s) f(s, u(s)) ds + g(v(1)) \phi(t), \qquad t \in [0,1].$$

Evidently, u is the solution of problem (1.1) if and only if $u = \lambda A(u,u)$. Noting that $f(t,x),\ g(x) \geq 0$ and $G(t,s) \geq 0$, it is easy to check that $A : P \times P \to P$. In the sequel we check that A satisfies all assumptions of Lemma 2.2.

Firstly, we prove that A is a mixed monotone operator. In fact, for $u_i, v_i \in P,\ i = 1,2$ with $u_1 \geq u_2,\ v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t),\ v_1(t) \leq v_2(t),\ t \in [0,1]$ and by (H_2),

$$\begin{aligned} A(u_1, v_1)(t) &= \int_0^1 G(t,s) f(s, u_1(s)) ds + g(v_1(1)) \phi(t) \\ &\geq \int_0^1 G(t,s) f(s, u_2(s)) ds + g(v_2(1)) \phi(t) = A(u_2, v_2)(t). \end{aligned}$$

That is, $A(u_1, v_1) \geq A(u_2, v_2)$.

Next we show that A satisfies the condition (A_2). From (H_3), for $\eta \in (0,1)$, we have $g(\eta^{-1} x) \geq \varphi_2(\eta) g(x),\ \forall\ x \in [0, +\infty)$. Let $\varphi(t) = \min\{\varphi_1(t),\ \varphi_2(t)\},\ t \in (0,1)$. Then $\varphi(t) \in (t, 1)$. From (H_3), for any $\eta \in (0,1)$ and $u, v \in P$, we obtain

$$\begin{aligned} A(\eta u, \eta^{-1} v)(t) &= \int_0^1 G(t,s) f(s, \eta u(s)) ds + g(\eta^{-1} v(1)) \phi(t) \\ &\geq \varphi_1(\eta) \int_0^1 G(t,s) f(s, u(s)) ds + \varphi_2(\eta) g(v(1)) \phi(t) \\ &\geq \varphi(\eta) \left[\int_0^1 G(t,s) f(s, u(s)) ds + g(v(1)) \phi(t) \right] \\ &= \varphi(\eta) A(u,v)(t), \qquad t \in [0,1]. \end{aligned}$$

Hence, $A(\eta u, \eta^{-1} v) \geq \varphi(\eta) A(u,v),\ \forall\ u, v \in P,\ \eta \in (0,1)$. So the condition (A_2) in Lemma 2.2 is satisfied. Now we show that $A(h,h) \in P_h$. On one hand, it follows from (H_2) and Lemma 3.1 that

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s) f(s, h(s)) ds + g(h(1)) \phi(t) \\ &= \int_0^1 G(t,s) f(s, s^2) ds + g(1) \phi(t) \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 \frac{1}{3}t^2 s^2 f(s,0)ds + g(1)\frac{1}{3}t^2 \\
&= \frac{1}{3}\left[\int_0^1 s^2 f(s,0)ds + g(1)\right]t^2, \qquad t \in [0,1].
\end{aligned}$$

On the other hand, also from (H_2) and Lemma 3.1, we obtain

$$\begin{aligned}
A(h,h)(t) &\leq \int_0^1 \frac{1}{2}t^2 s f(s,1)ds + g(1)\frac{1}{2}t^2 \\
&= \frac{1}{2}\left[\int_0^1 s f(s,1)ds + g(1)\right]t^2, \qquad t \in [0,1].
\end{aligned}$$

Let

$$r_1 = \frac{1}{3}\left[\int_0^1 s^2 f(s,0)ds + g(1)\right], \qquad r_2 = \frac{1}{2}\left[\int_0^1 s f(s,1)ds + g(1)\right].$$

Since $f(t,x)$ is continuous and increasing in x with $f(t,0) \not\equiv 0$, $g(1) \geq 0$, we can get

$$0 < r_1 = \frac{1}{3}\left[\int_0^1 s^2 f(s,0)ds + g(1)\right] \leq \frac{1}{2}\left[\int_0^1 s f(s,1)ds + g(1)\right] = r_2.$$

Consequently,

$$A(h,h)(t) \geq r_1 h(t), \qquad A(h,h)(t) \leq r_2 h(t), \qquad t \in [0,1].$$

So we have

$$r_1 h \leq A(h,h) \leq r_2 h.$$

Hence $A(h,h) \in P_h$, the condition (A_1) in Lemma 2.2 is satisfied. Therefore, by Lemma 2.3, there exists a unique $u_\lambda^* \in P_h$ such that $A(u_\lambda^*, u_\lambda^*) = \frac{1}{\lambda}u_\lambda^*$. That is, $u_\lambda^* = \lambda A(u_\lambda^*, u_\lambda^*)$, and then

$$u_\lambda^*(t) = \lambda \int_0^1 G(t,s) f(s, u_\lambda^*(s))ds + \lambda g(u_\lambda^*(1))\phi(t), \qquad t \in [0,1].$$

It is easy to check that u_λ^* is a unique positive solution of the problem (1.1) for given $\lambda > 0$. In view of $u_\lambda^{*(4)}(t) = \lambda f(t, u_\lambda^*(t))$, $0 < t < 1$ and $u_\lambda^*(0) = u_\lambda^{*\prime}(0) = u_\lambda^{*\prime\prime}(1) = u_\lambda^{*(3)}(1) + \lambda g(u_\lambda^*(1)) = 0$, then from Lemma 3.2, $u_\lambda^*(t)$ is increasing and convex on $[0,1]$. Further, if $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$. Lemma 2.3 (B_1) means that u_λ^* is strictly decreasing in $\frac{1}{\lambda}$. So u_λ^* is strictly increasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^\beta$ for $t \in (0,1)$. Lemma 2.3 (B_2) means that u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^\beta$ for $t \in (0,1)$. Lemma 2.3 (B_3) means $\lim_{\lambda \to 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda \to \infty} \|u_\lambda^*\| = \infty$.

Let $A_\lambda = \lambda A$, then A_λ also satisfies all the conditions of Lemma 2.2. By Lemma 2.2, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences $x_{n+1} = A_\lambda(x_n, y_n)$, $y_{n+1} = A_\lambda(y_n, x_n)$, $n = 0,1,2,\ldots$, we have $x_n \to u_\lambda^*$, $y_n \to u_\lambda^*$ as $n \to \infty$. That is,

$$\begin{aligned}
x_{n+1} &= \lambda \int_0^1 G(t,s) f(s, x_n(s))ds + \lambda g(y_n(1))\phi(t) \to u_\lambda^*(t), \\
y_{n+1} &= \lambda \int_0^1 G(t,s) f(s, y_n(s))ds + \lambda g(x_n(1))\phi(t) \to u_\lambda^*(t),
\end{aligned}$$

as $n \to \infty$. □

Theorem 3.4. *Assume* (H_1) *and*

(H_4) $f(t,x)$ *is decreasing in* $x \in [0,+\infty)$ *for each* $t \in [0,1]$ *with* $f(t,1) \not\equiv 0$, *and* $g(x)$ *is increasing in* $x \in [0,+\infty)$;

(H_5) *for* $\eta \in (0,1)$, *there exist* $\varphi_i(\eta) \in (\eta,1)$ $(i=1,2)$ *such that*

$$f(t,\eta x) \leq \frac{1}{\varphi_1(\eta)} f(t,x), \;\; g(\eta x) \geq \varphi_2(\eta) g(x), \qquad \forall\, t \in [0,1],\; x \in [0,+\infty).$$

Then, for any given $\lambda > 0$, *problem* (1.1) *has a unique convex monotone positive solution* u_λ^* *in* P_h, *where* $h(t) = t^2$, $t \in [0,1]$. *Moreover, for any initial values* $x_0,\ y_0 \in P_h$, *constructing successively the sequences:*

$$x_n = \lambda \int_0^1 G(t,s) f(s, y_{n-1}(s)) ds + \lambda g(x_{n-1}(1))\phi(t),$$
$$y_n = \lambda \int_0^1 G(t,s) f(s, x_{n-1}(s)) ds + \lambda g(y_{n-1}(1))\phi(t), \qquad n = 1,2,\ldots,$$

we have $x_n \to u_\lambda^*$, $y_n \to u_\lambda^*$ *as* $n \to +\infty$. *Further, (i) if* $\varphi_i(t) > t^{\frac{1}{2}}$ $(i=1,2)$ *for* $t \in (0,1)$, *then* u_λ^* *is strictly increasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^* < u_{\lambda_2}^*$; *(ii) if there exists* $\beta \in (0,1)$ *such that* $\varphi_i(t) \geq t^\beta$ $(i=1,2)$ *for* $t \in (0,1)$, *then* u_λ^* *is continuous in* λ, *that is,* $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ *implies* $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$; *(iii) if there exists* $\beta \in (0,\frac{1}{2})$ *such that* $\varphi_i(t) \geq t^\beta$ $(i=1,2)$ *for* $t \in (0,1)$, *then* $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = 0, \lim_{\lambda\to+\infty} \|u_\lambda^*\| = +\infty$.

Proof. For any $u, v \in P$, we define

$$A(u,v)(t) = \int_0^1 G(t,s) f(s,v(s)) ds + g(u(1))\phi(t), \qquad t \in [0,1].$$

Evidently, u is the solution of problem (1.1) if and only if $u = \lambda A(u,u)$. Similar to the proof of Theorem 3.3, from (H_4), we obtain that $A : P \times P \to P$ is a mixed monotone operator.

Next we show that A satisfies the condition (A_2). From (H_5), for $\eta \in (0,1)$, we have $f(t,\eta^{-1}x) \geq \varphi_1(\eta) f(t,x)$, $\forall\ x \in [0,+\infty)$. Let $\varphi(t) = \min\{\varphi_1(t), \varphi_2(t)\}$, $t \in (0,1)$. Then $\varphi(t) \in (t,1)$. From (H_5), for any $\eta \in (0,1)$ and $u, v \in P$, we obtain

$$\begin{aligned} A(\eta u, \eta^{-1} v)(t) &= \int_0^1 G(t,s) f(s, \eta^{-1} v(s)) ds + g(\eta u(1))\phi(t) \\ &\geq \varphi_1(\eta) \int_0^1 G(t,s) f(s,v(s)) ds + \varphi_2(\eta) g(u(1))\phi(t) \\ &\geq \varphi(\eta) \left[\int_0^1 G(t,s) f(s,v(s)) ds + g(u(1))\phi(t) \right] \\ &= \varphi(\eta) A(u,v)(t), \qquad t \in [0,1]. \end{aligned}$$

Hence, $A(\eta u, \eta^{-1} v) \geq \varphi(\eta) A(u,v)$, $\forall\, u,v \in P,\ \eta \in (0,1)$. So the condition (A_2) in Lemma 2.2 is satisfied. Now we show that $A(h,h) \in P_h$. On one hand, it follows from (H_4) and Lemma 3.1 that

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s) f(s,h(s)) ds + g(h(1))\phi(t) \\ &= \int_0^1 G(t,s) f(s,s^2) ds + g(1)\phi(t) \end{aligned}$$

$$\geq \int_0^1 \frac{1}{3} t^2 s^2 f(s,1) ds + g(1) \frac{1}{3} t^2$$
$$= \frac{1}{3} \left[\int_0^1 s^2 f(s,1) ds + g(1) \right] t^2, \qquad t \in [0,1].$$

On the other hand, also from (H_3) and Lemma 3.1, we obtain

$$A(h,h)(t) \leq \int_0^1 \frac{1}{2} t^2 s f(s,0) ds + g(1) \frac{1}{2} t^2$$
$$= \frac{1}{2} \left[\int_0^1 s f(s,0) ds + g(1) \right] t^2, \qquad t \in [0,1].$$

Let

$$r_3 = \frac{1}{3} \left[\int_0^1 s^2 f(s,1) ds + g(1) \right], \qquad r_4 = \frac{1}{2} \left[\int_0^1 s f(s,0) ds + g(1) \right].$$

Since f is continuous and $f(t,1) \not\equiv 0,\ g(1) \geq 0$, we can get

$$0 < r_3 = \frac{1}{3} \left[\int_0^1 s^2 f(s,1) ds + g(1) \right] \leq \frac{1}{2} \left[\int_0^1 s f(s,0) ds + g(1) \right] = r_4.$$

Consequently,

$$A(h,h)(t) \geq r_3 h(t), \qquad A(h,h)(t) \leq r_4 h(t), \qquad t \in [0,1].$$

So we have

$$r_3 h \leq A(h,h) \leq r_4 h.$$

Hence $A(h,h) \in P_h$, the condition (A_1) in Lemma 2.2 is satisfied. Therefore, by Lemma 2.3, there exists a unique $u_\lambda^* \in P_h$ such that $A(u_\lambda^*, u_\lambda^*) = \frac{1}{\lambda} u_\lambda^*$. That is, $u_\lambda^* = \lambda A(u_\lambda^*, u_\lambda^*)$, and then

$$u_\lambda^*(t) = \lambda \int_0^1 G(t,s) f(s, u_\lambda^*(s)) ds + \lambda g(u_\lambda^*(1)) \phi(t),\ t \in [0,1].$$

It is easy to check that u_λ^* is a unique positive solution of the problem (1.1) for given $\lambda > 0$. In view of $u_\lambda^{*(4)}(t) = \lambda f(t, u_\lambda^*(t)),\ 0 < t < 1$ and $u_\lambda^*(0) = u_\lambda^{*\prime}(0) = u_\lambda^{*\prime\prime}(1) = u_\lambda^{*(3)}(1) + \lambda g(u_\lambda^*(1)) = 0$, then from Lemma 3.2, $u_\lambda^*(t)$ is increasing and convex on $[0,1]$. Further, if $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$. Lemma 2.3 (B_1) means that u_λ^* is strictly decreasing in $\frac{1}{\lambda}$. So u_λ^* is strictly increasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*,\ u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^\beta$ for $t \in (0,1)$. Lemma 2.3 (B_2) means that u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\varphi(t) \geq t^\beta$ for $t \in (0,1)$. Lemma 2.3 (B_3) means $\lim_{\lambda \to 0^+} \|u_\lambda^*\| = 0,\ \lim_{\lambda \to \infty} \|u_\lambda^*\| = \infty$.

Let $A_\lambda = \lambda A$, then A_λ also satisfies all the conditions of Lemma 2.2. By Lemma 2.2, for any initial values $x_0,\ y_0 \in P_h$, constructing successively the sequences $x_{n+1} = A_\lambda(x_n, y_n),\ y_{n+1} = A_\lambda(y_n, x_n),\ n = 0,1,2,\ldots,$ we have $x_n \to u_\lambda^*, y_n \to u_\lambda^*$ as $n \to \infty$. That is,

$$x_{n+1} = \lambda \int_0^1 G(t,s) f(s, y_n(s)) ds + \lambda g(x_n(1)) \phi(t) \to u_\lambda^*(t),$$
$$y_{n+1} = \lambda \int_0^1 G(t,s) f(s, x_n(s)) ds + \lambda g(y_n(1)) \phi(t) \to u_\lambda^*(t),$$

as $n \to \infty$. □

Remark 3.5. Comparing Theorems 3.3–3.4 with the main results in [11,12], we provide some alternative approaches to study the similar type of problems under different conditions. Our analysis relies on some results of operator equation $A(x,x) = \lambda x$, where A is general mixed monotone. So the functions f, g in the problem (1.1) can have two different monotonicity. In [11], the method used there is a theorem of operator equation $Ax = x$, where A is increasing. So in [11] the functions f, g of fourth-order boundary value problems only have stationary monotonicity. In [12], the authors used some results of operator equation $A(x,x) + Bx = x$ to study fourth-order boundary value problems. From [21], we know that $A(x,x) + Bx$ was proved to be general mixed monotone. Therefore, the main results in [12] are special cases of Theorems 3.3–3.4. In addition, our results can guarantee the existence of a unique convex monotone positive solution and can be applied to construct two iterative sequences for approximating it. We present some pleasant properties of convex monotone positive solutions for the boundary value problem dependent on the parameter. Because the operator equation in Lemma 2.3 is only concerned with one parameter, we only study the problem (1.1) which the parameter in the equation is the same as in the boundary condition, and thus our method can not be applied to some boundary value problems with several different parameters.

To illustrate how our main results can be used in practice we present a simple example.

Example 3.6. Consider the following fourth-order boundary value problem:

$$\begin{aligned} &u^{(4)}(t) = \lambda\{[u(t)]^{\frac{1}{4}} + a(t)\},\ 0 < t < 1, \\ &u(0) = u'(0) = u''(1) = u^{(3)}(1) + \lambda[u(1) + b]^{-\frac{1}{6}} = 0, \end{aligned} \tag{3.1}$$

where $b > 0$, $a : [0,1] \to [0,+\infty)$ is continuous with $a \not\equiv 0$. Evidently, problem (3.1) fits the framework of problem (1.1). In this example, let

$$f(t,x) = x^{\frac{1}{4}} + a(t),\ g(x) = [x+b]^{-\frac{1}{6}}.$$

Obviously, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $g : [0,+\infty) \to [0,+\infty)$ is continuous. And it easy to see that $f(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, and $g(x)$ is decreasing in $x \in [0,+\infty)$. Moreover, from the given condition, we have $f(t,0) = a(t) \not\equiv 0$. Set $\varphi_1(\eta) = \eta^{\frac{1}{4}}$, $\varphi_2(\eta) = \eta^{\frac{1}{6}}$, $\eta \in (0,1)$. Then $\varphi_1(\eta)$, $\varphi_2(\eta) \in (\eta, 1)$ and

$$f(t,\eta x) = \eta^{\frac{1}{4}} x^{\frac{1}{4}} + a(t) \geq \varphi_1(\eta) f(t,x),\ g(\eta x) = [\eta x + b]^{-\frac{1}{6}} \leq \frac{1}{\varphi_2(\eta)} g(x),$$

for $t \in [0,1]$, $x \geq 0$. Hence, all the conditions of Theorem 3.3 are satisfied. An application of Theorem 3.3 implies that problem (3.1) has a unique convex monotone positive solution u^*_λ in $P_h = P_{t^2}$, and for any initial values x_0, $y_0 \in P_{t^2}$, constructing successively the sequences

$$\begin{aligned} x_n &= \lambda \int_0^1 G(t,s)[x^{\frac{1}{4}}_{n-1}(s) + a(s)]ds + \lambda[y_{n-1}(1) + b]^{-\frac{1}{6}} \phi(t), \\ y_n &= \lambda \int_0^1 G(t,s)[y^{\frac{1}{4}}_{n-1}(s) + a(s)]ds + \lambda[x_{n-1}(1) + b]^{-\frac{1}{6}} \phi(t), \qquad n = 1,2,\ldots, \end{aligned}$$

we have $x_n(t) \to u^*_\lambda(t)$, $y_n(t) \to u^*_\lambda(t)$ as $n \to \infty$, where $G(t,s)$ is given as in Lemma 3.1. Moreover, note that $\varphi_1(t)$, $\varphi_2(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then from Theorem 3.3, u^*_λ is strictly increasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u^*_{\lambda_1} \leq u^*_{\lambda_2}$, $u^*_{\lambda_1} \neq u^*_{\lambda_2}$. Take $\beta = \frac{1}{4}$ and applying Theorem 3.3, we know that u^*_λ is continuous in λ and $\lim_{\lambda\to 0^+} \|u^*_\lambda\| = 0$, $\lim_{\lambda\to\infty} \|u^*_\lambda\| = \infty$.

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