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Linear deterministic attributed transformations

By M. BARTHA

Introduction

This paper is based on and continues our earlier work [2] in the subject. Our point of view is close to that of the authors' of [3] inasmuch as we, too, translate an attribute grammar (or transformation) into a system of recursive definitions. Our aim was to define attributed transformations as homomorphisms between suitable algebras that can be constructed from well-known ones in a natural way. Rational algebraic theories (cf. [13]) and magmoids (cf. [1]) turned out to be the most appropriate for this purpose. Two questions may arise in connection with our new definition.

1. Why do we use these complex many sorted algebras if our aim is to map T_Σ, the free Σ-algebra, into a certain attributed structure? It would be enough to define an appropriate Σ-algebra on this structure.

Beyond the notational convenience and elegance of proofs there is one more reason. Investigating one specific attributed transformation it is generally easier to deal with Σ-algebras only. However, if we investigate e.g. the composition properties of these transformations (tree transformations here), the process of "translating" into a Σ-algebra becomes rather tedious and affected. In this case the main advantage is that we can get rid of the alphabet Σ.

2. Wouldn't it be enough to use algebraic theories only instead of magmoids?

It is true that most of the results in [4] concerning top-down tree transformations could be formulated within the framework of projective magmoids, i.e. non-degenerate algebraic theories. An attributed transformation, however, is defined by a homomorphism $h: \tilde{T}(\Sigma) \to \mathbf{D}R[k, l]$ (for the precise definitions see later), i.e. a homomorphism between (decomposable) magmoids. One might say that the homomorphism $\mathbf{T}h: T(\Sigma) \to \mathbf{TD}R[k, l]$ is already between algebraic theories. This is true, but it turns out that homomorphisms of $T(\Sigma)$ into $\mathbf{TD}R[k, l]$ generally define more complex transformations, called macro transformations (cf. [7]).

For simplicity we assume that the set of possible values is the same for all the attributes. A natural way to generalize our definition could be the introduction of "many sorted" rational theories.

1. Preliminaries

In this section we recall the basic concepts and definitions from [2] concerning attributed transformations.

A magmoid $M=(\{M(p,q)|p,q\geqq 0\}, \cdot, +, 1, 1_0)$ is a special many-sorted algebra whose sorting set consists of all pairs of nonnegative integers. $\cdot$ and $+$ denote binary operations called composition and separated sum (tensor product), respectively. Composition (rather denoted by juxtaposition) maps $M(p,q)\times M(q,r)$ into $M(p,r)$, and separated sum maps $M(p_1,q_1)\times M(p_2,q_2)$ into $M(p_1+p_2, q_1+q_2)$. $1\in M(1,1)$ and $1_0\in M(0,0)$ denote nullary operations. The following axioms must be valid in M.

(i) $(ab)c=a(bc)$ for any composable pairs $\langle a,b\rangle$ and $\langle b,c\rangle$;

(ii) $(a+b)+c=a+(b+c)$;

(iii) $(ab)+(cd)=(a+c)(b+d)$;

(iv) $a1_p=1_qa=a$ if $a\in M(p,q)$ and $1_n=\sum_{i=1}^{n} 1$ for $n\geqq 1$;

(v) $a+1_0=1_0+a=a$.

Due to (i) and (iv) M becomes a category whose objects are the nonnegative integers and the identities are the elements 1_n $(n\geqq 0)$. (For a complex categorical definition of magmoids see [11].) Therefore, $a\in M(p,q)$ is often written as $a\colon p\to q$ if M is understood.

Let $\Theta(p,q)$ denote the set of all mappings of $[p]=\{1,\ldots,p\}$ into $[q]$. Defining the composition and separated sum of mappings as it is usual, and taking the identity map of $[n]$ for 1_n we get the magmoid Θ. We denote the unique element of $\Theta(0,q)$ by 0_q $(0_0=1_0)$, and the injection $1\to p$ which picks out i from $[p]$ by π_p^i (or π_i if p is understood). For an arbitrary $\theta\in\Theta(p,q)$, $i\theta$ stands for the image of $i\in[p]$ under θ.

A magmoid M is called projective if it contains a submagmoid Θ_M isomorphic to Θ, and the following holds for every $a,b\in M(p,q)$. If $\pi_ia=\pi_ib$ for each $i\in[p]$, then $a=b$. Generally we shall assume that $\Theta_M=\Theta$, to be able to use the same notations in M as in Θ. It can be proved that for any $a_1,\ldots,a_p\colon 1\to q$ there exists a unique $a\colon p\to q$ such that $\pi_ia=a_i$ for each $i\in[p]$. This element will be denoted by $\langle\!\langle a_1,\ldots,a_p\rangle\!\rangle$. We shall use $\langle\!\langle$ and $\rangle\!\rangle$ (source-tupling) as a derived operation, extending it to the case $a_i\colon p_i\to q$ in the usual way. (In this case $\langle\!\langle a_1,\ldots,a_p\rangle\!\rangle\colon \sum_{i=1}^{p} p_i\to q$.) It was pointed out in [1] that every projective magmoid is in fact a nondegenerate algebraic theory and vice versa, depending on whether separated sum, or source tupling and the injections are considered as basic operations.

It is well-known that for every ranked alphabet $\Sigma=\bigcup_{n\geqq 0}\Sigma_n$ there exists a free projective magmoid generated by Σ, which we denote by $T(\Sigma)$. $T(\Sigma)$ has a representation by finite Σ-trees on the variables $X=\{x_1,x_2,\ldots\}$ (cf. [1]). Viewing $\sigma\in\Sigma_n$ as $\sigma(x_1,\ldots,x_n)\in T(\Sigma)(1,n)$ (which makes Σ a subsystem of $T(\Sigma)$), $T(\Sigma)$ has the property that any ranked alphabet map $h\colon\Sigma\to M$ into a projective magmoid M has a unique homomorphic extension $\bar{h}\colon T(\Sigma)\to M$. In particular, if Σ is the void alphabet, then $T(\Sigma)=\Theta$.

$T(\Sigma)$ has an important subsystem $\tilde{T}(\Sigma)$ defined as follows. $t\in\tilde{T}(\Sigma)(p,q)$ iff the frontier of t, i.e. the sequence of variables appearing at the leaves of t, is exactly

$x_1, \ldots, x_q$. $\tilde{T}(\Sigma)(1, 0) = T(\Sigma)(1, 0)$ will be denoted by T_Σ. $\tilde{T}(\Sigma)$ is a submagmoid of $T(\Sigma)$, and it is the free magmoid generated by Σ. It has the property that every $t \in T(\Sigma)$ can be uniquely written in the form $\tilde{t}\vartheta$ with $\tilde{t} \in \tilde{T}(\Sigma)$ and $\vartheta \in \Theta$.

Let $\hat{\Theta}$ denote the submagmoid of all injective mappings in Θ. $t \in T(\Sigma)$ is called linear if $\vartheta \in \hat{\Theta}$ by the decomposition $t = \tilde{t}\vartheta$ above. Clearly, the linear elements also form a submagmoid of $T(\Sigma)$, which we denote by $\hat{T}(\Sigma)$.

$\tilde{T}(\Sigma)$ is free in the important subclass of decomposable magmoids, too. A magmoid M is called decomposable if the following two conditions are satisfied:

(i) for every $a: p \to q$ $(p \geqq 2, q \geqq 0)$ and $i \in [p]$ there exists exactly one integer q_i and a_i: $1 \to q_i$ such that $a = a_1 + \ldots + a_p$;

(ii) $M(0, 0) = \{1_0\}$.

Any magmoid M can be made decomposable by the application of the functor **D**. **D** operates as follows:

(i) $\mathbf{D}M(1, q) = M(1, q)$ if $q \geqq 0$,
$1 = 1_M$,
$\mathbf{D}M(0, q) = \textit{if } q = 0 \textit{ then } \{\emptyset\} \textit{ else } \emptyset$,
if $p \geqq 2$, then $\mathbf{D}M(p, q) \subseteq (\bigcup_{r \geqq 0} M(1, r))^p$ such that $\langle a_1, \ldots, a_p \rangle \in \mathbf{D}M(p, q)$ with a_i: $1 \to q_i$ iff $\sum_{i=1}^{p} q_i = q$;

(ii) $\langle a_1, \ldots, a_{p_1} \rangle + \langle b_1, \ldots, b_{p_2} \rangle = \langle a_1, \ldots, a_{p_1}, b_1, \ldots, b_{p_2} \rangle$;

(iii) if $a = \langle a_1, \ldots, a_p \rangle$: $p \to q$ with a_i: $1 \to q_i$ and $b = \langle b_1, \ldots, b_q \rangle$: $q \to r$, then

$$a.b = \langle a_1(.)_M b^{(1)}, \ldots, a_p(.)_M b^{(p)} \rangle,$$

where $b^{(i)} = \left(\sum_{j=q^{(i)}}^{q^{(i+1)}} \right)_M b_j$ and $q^{(i)} = \sum_{j=1}^{i-1} q_j$ $(i \in [p+1])$;

(iv) if h: $M \to M'$ is a homomorphism, then

$$\mathbf{D}h(\langle a_1, \ldots, a_p \rangle) = \langle h(a_1), \ldots, h(a_p) \rangle.$$

There is a natural homomorphism ζ: $\mathbf{D}M \to M$ for which $\zeta(\langle a_1, \ldots, a_p \rangle) = a_1 + \ldots + a_p$.

Any decomposable magmoid M can be made projective by the application of the functor **T** which operates as follows:

(i) $\mathbf{T}M(p, q) = \bigcup (M(p, q') \times \Theta(q', q) \mid q' \geqq 0)$,
$1 = \langle 1_M, 1_\Theta \rangle$, $1_0 = \langle (1_0)_M, (1_0)_\Theta \rangle$;

(ii) $\langle a_1, \vartheta_1 \rangle + \langle a_2, \vartheta_2 \rangle = \langle a_1 + a_2, \vartheta_1 + \vartheta_2 \rangle$;

(iii) let a: $p \to q'$, ϑ: $q' \to q$, $b = \langle b_1, \ldots, b_q \rangle$: $q \to r$ with b_i: $1 \to r_i$ $(i \in [q])$ and φ: $r' \to r$. φ can be uniquely written in the form $\prec \varphi_1, \ldots, \varphi_q \succ$, where for each $i \in [q]$ φ_i: $r_i \to r$. Now $\langle a, \vartheta \rangle . \langle b, \varphi \rangle = \langle a(.)_M \langle b_{1\vartheta}, \ldots, b_{q'\vartheta} \rangle, \prec \varphi_{1\vartheta}, \ldots, \varphi_{q'\vartheta} \succ \rangle$;

(iv) if h: $M \to M'$ is a homomorphism between decomposable magmoids, then $\mathbf{T}h(\langle a, \vartheta \rangle) = \langle h(a), \vartheta \rangle$.

We shall also use a restriction of **T** denoted by $\hat{\mathbf{T}}$. $\langle a, \vartheta \rangle \in \hat{\mathbf{T}}M$ iff $\vartheta \in \hat{\Theta}$. It is easy to see that $\hat{\mathbf{T}}M$ is a submagmoid of $\mathbf{T}M$, so $\hat{\mathbf{T}}$ is also a functor. It is well-known that $\mathbf{T}(\tilde{T}(\Sigma)) \cong T(\Sigma)$ and $\hat{\mathbf{T}}(\tilde{T}(\Sigma)) \cong \hat{T}(\Sigma)$.

Let M be a magmoid and k an arbitrary natural number. k-dil M denotes the magmoid for which $(k\text{-dil}\, M)(p, q) = M(kp, kq)$, $1 = (1_k)_M$, $1_0 = (1_0)_M$ and the further operations are performed in it just as in M. Clearly, the operator k-dil can also be extended to a functor. Let η_k denote the inclusion function: k-dil $M \to M$. η_k is not a homomorphism, it is only a so called k-morphism. To avoid ambiguity, $M = k$-dil Θ will be the only exeption when we distinguish Θ_M from Θ, using the unique embedding $\iota_k\colon \Theta \to k$-dil Θ.

Rational algebraic theories were introduced in [13]. To remain in circles of magmoids we define this concept by means of projective magmoids, thus excluding the trivial degenerate rational theory. A rational theory R is a projective magmoid equipped with a new unary operation $^\dagger\colon R(p, p+q) \to R(p, q)$, called iteration. The carrier sets and the operations are required to satisfy the following conditions:

(i) for each $p, q \geqq 0$, $R(p, q)$ is partially ordered with minimal element $\perp_{p,q}$ ($\perp_p$ if q is understood);

(ii) separated sum and composition are monotonic, and the latter is left strict, i.e. $\perp_{p,q} a = \perp_{p,r}$ for $a\colon q \to r$;

(iii) let $a\colon p \to p+q$, and construct the sequence $(a_i\colon p \to q \mid i \geqq 0)$ as follows

$$a_0 = \perp_{p,q},\ a_{i+1} = a \langle a_i, 1_q \rangle \quad \text{for} \quad i \geqq 0.$$

Then $\bigcup\limits_{i \geqq 0} a_i$ exists and equals $a^\dagger$;

(iv) composition is both left and right continuous.

Since rational theories are ordered algebras, a homomorphism between them is required to preserve the ordering, too. It was shown in [13] that for every ranked alphabet Σ the free rational theory generated by Σ exists. This theory $R(\Sigma)$ has a representation by infinite $\Sigma_\perp$-trees on X, where $\Sigma_\perp = \Sigma \cup \{\perp\}$ and $\perp$ is a new symbol with rank 0. Reg (Σ) will denote the rational theory of all regular forests of finite Σ-trees on X.

Definition 1.1. Let R be a rational theory, $k \geqq 1$, $l \geqq 0$ integers. Define $R[k, l] = (\{R[k, l](p, q) \mid p, q \geqq 0\}, \cdot, +, 1, 1_0)$ to be the following structure. (We do not use the subscript M to indicate the magmoid in which the operations are performed if only one M is reasonable from the context.)

(i) $R[k, l](p, q) = R(kp + lq, kq + lp)$,
$1 = (1_{k+l})_R$, $1_0 = (1_0)_R$;

(ii) if $a \in R[k, l](p_1, q_1)$, $b \in R[k, l](p_2, q_2)$, then

$$a+b = \langle \mu^{kp_1}_{lq_1} + \mu^{kp_2}_{lq_2}, \nu^{kp_1}_{lq_1} + \nu^{kp_2}_{lq_2} \rangle \cdot (a+b) \cdot \langle \mu^{kq_1}_{lp_1} + \mu^{kq_2}_{lp_2}, \nu^{kq_1}_{lp_1} + \nu^{kq_2}_{lp_2} \rangle^{-1},$$

where μ^n_m (μ_n if m is understood) $= 1_n + 0_m$, ν^n_m (ν_m if n is understood) $= 0_n + 1_m$;

(iii) if $a \in R[k, l](p, q)$, $b \in R[k, l](q, r)$, then $a \cdot b = \langle \mu_{kp}, \nu_{lr} \rangle \cdot \langle a\vartheta, b\varphi \rangle^\dagger$, where $\vartheta = \nu^{kp+lq}_{kq} + \nu^{(k+l)r}_{lp}$, $\varphi = 0_{kp} + \langle \nu^{(k+l)q}_{kr}, \mu^{lq}_{kq+(k+l)r} \rangle + 0_{lp}$ (see also Fig. 1).

In [2] we proved that $R[k, l]$ is a magmoid. Let $\xi\colon R \to R'$ be a homomorphism between rational theories. Clearly, ξ defines a homomorphism $\xi[k, l]\colon R[k, l] \to R'[k, l]$, and so the operator $[k, l]$ becomes a functor.

Definition 1.2. An attributed transducer (a-transducer) is a 6-tuple $\mathbf{A} = (\Sigma, R, k, l, h, S)$, where

(i) Σ is a finite ranked alphabet, $S \notin \Sigma$;

(ii) R is a rational theory, $k \geqq 1$, $l \geqq 0$ are integers;

(iii) $h\colon \Sigma_S \to \mathbf{D}R[k, l]$ is a ranked alphabet map, where $\Sigma_S = \Sigma \cup \{S\}$ with S

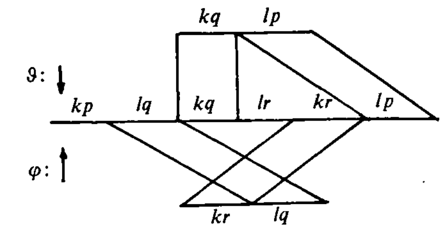


Fig. 1

having rank 1. $h(S)$ is required to be a synthesizer, i.e. $h(S)=a+0_l$ for some $a\in R(k+l, k)$.

Extend h to a homomorphism h: $\tilde{T}(\Sigma)\to\mathbf{D}R[k, l]$. $\tau_{\mathbf{A}}$: $T_\Sigma\to R(1, 0)$, the transformation defined by $\mathbf{A}$ is the following function. $\tau_{\mathbf{A}}(t)=a$, where $\pi_k^1 h(S(t))= =a+0_l$.

Let Δ be a ranked alphabet, and consider the homomorphism ε_Δ: $R(\Delta)\to\mathrm{Reg}\,(\Delta)$ for which $\varepsilon_\Delta(\delta)=\{\delta(x_1, ..., x_n)\}$ if $\delta\in\Delta_n$. Let Θ_Δ denote the congruence relation induced by ε_Δ. For simplicity we shall identify each $t\in T(\Delta)$ with its class $[t]\Theta_\Delta$.

Definition 1.3. A deterministic attributed tree transducer (a-tree transducer) from Σ into Δ is an a-transducer $\mathbf{A}=(\Sigma, R(\Delta)/\Theta_\Delta, k, l, h, S)$. In this case we consider $\tau_{\mathbf{A}}\subseteq T_\Sigma\times T_\Delta$ as a relation

$$\tau_{\mathbf{A}} = \{\langle t, u\rangle | \pi_k^1(h(S(t))) = u+0_l \quad \text{and} \quad u\in T_\Delta\}.$$

Further on a deterministic a-tree transducer from Σ into Δ will rather be denoted by the 6-tuple $(\Sigma, \Delta, k, l, h, S)$.

$\mathbf{A}=(\Sigma, \Delta, k, l, h, S)$ is called total if $h(\sigma)\in T(\Delta)$ for each $\sigma\in\Sigma_S$. Determinism, totality and linearity of tree transducers will be denoted by d, t and l, respectively. Since k-dil $T(\Delta)$ is a submagmoid of $R(\Delta)[k, 0]$, every dta-tree transducer with s-attributes only ($l=0$) is in fact a dt-top-down tree transducer and vice versa.

Let $t\in R(\Delta)[k, l](p, q)$. It is convenient to consider t as the image of a tree $u\in\tilde{T}(\Sigma)(p, q)$ under a suitable homomorphism h: $\tilde{T}(\Sigma)\to R(\Delta)[k, l]$. To underline the attributed feature of t we introduce the following notations

$$\underline{t}(r, i) = \pi_{k(r-1)+i}t \qquad \text{if} \quad r\in[p], \quad i\in[k];$$

$$\bar{t}(j, m) = \pi_{kp+l(j-1)+m}t \quad \text{if} \quad j\in[q], \quad m\in[l];$$

$$x(j, i) = x_{k(j-1)+i} \qquad \text{if} \quad j\in[q], \quad i\in[k];$$

$$y(r, m) = x_{kq+l(r-1)+m} \quad \text{if} \quad r\in[p], \quad m\in[l].$$

The intuitive meaning of these items is the following.
$\underline{t}(r, i)$: the value of the i-th synthesized attribute (s-attribute) of the r-th root;
$\bar{t}(j, m)$: the value of the m-th inherited attribute (i-attribute) of the j-th leaf;
$x(j, i)$: reference to the i-th s-attribute of the j-th leaf;
$y(r, m)$: reference to the m-th i-attribute of the r-th root.

Naturally, the roots and leaves above belong to u that we never mention explicitely. If $p=1$, then $\underline{t}(i)$ and $y(m)$ stand for $\underline{t}(1, i)$ and $y(1, m)$, respectively. We shall use these notations for $a \in R[k, l](p, q)$, too, after defining the concept of dependence on a variable in an arbitrary rational theory.

2. The composition of a dtla-tree transformation and an arbitrary a-transformation

Linear top-down tree transducers can be defined in two different ways. The original definition in [5] requires all the rules of the transducer be linear in the sense that no variable occurs more than once on the right-hand side of a rule. If we represent the transducer by a k-morphism of magmoids, say h: $T(\Sigma) \to k\text{-dil}\ T(\Delta)$ (the transducer is taken dt for simplicity), then $h(\sigma)$ $(\sigma \in \Sigma)$ resumes all the rules above in which σ appears on the left-hand side. However, the meaning of the variables in $h(\sigma)$ differs from that of the variables occuring in the rules. Therefore, if we require for all $\sigma \in \Sigma$ $h(\sigma)$ not contain two different occurences of the same variable, which is the second way to define linearity, the transducer need not be linear in the original sense, and vice versa.

Unfortunately, the original definition cannot be carried out in the case of a-tree transducers (even if the transducer is described by a set of rules as in [9]), but the second one can be adopted quite naturally.

Definition 2.1. $t \in R(\Delta)[k, l](1, q)$ is called linear if $t \in \hat{T}(\Delta_\perp)$. $t \in \mathbf{D}R(\Delta)[k, l](p, q)$ is linear if $t=1_0$, or $t=t_1+\ldots+t_p$ and each t_i: $1 \to q_i$ $(i \in [p])$ is linear. $\mathbf{A}=(\Sigma, \Delta, k, l, h, S)$ is linear if $h(\sigma)$ has a linear representant for every $\sigma \in \Sigma_S$.

Let $L_\perp(\Delta)[k, l]$ denote the system of all linear elements in $\mathbf{D}R(\Delta)[k, l]$, and $L(\Delta)[k, l]$ that of all linear and total ones.

Lemma 2.2. $L_\perp(\Delta)[k, l]$ and $L(\Delta)[k, l]$ are submagmoids of $\mathbf{D}R(\Delta)[k, l]$.

Proof. It is enough to prove the lemma for $L(\Delta)[k, l]$. Indeed, let φ: $R(\Delta_\perp) \to R(\Delta)$ be the homomorphism extending the identity map $\Delta_\perp \to \Delta \cup \{\perp\}$. If $L(\Delta_\perp)[k, l]$ is a submagmoid, then so is $L_\perp(\Delta)[k, l]$, which is the image of it under the embedding $\mathbf{D}\varphi[k, l]$.

Let $t \in R(\Delta)[k, l](1, q)$ be arbitrary, and construct the directed graph G_t as follows. The nodes of G_t are

$$\{\mathbf{rs}(i), \mathbf{ri}(m), \mathbf{ls}(j, i), \mathbf{li}(j, m) | i \in [k], m \in [l], j \in [q]\}$$

(**s**, **i**, **r** and **l** suggest synthesized, inherited, root and leaf, respectively). There is an arc from $\mathbf{rs}(i)$ to $\mathbf{ls}(j, i')(\mathbf{ri}(m))$ iff $\underline{t}(i)$ contains an occurence of $x(j, i')(y(m)$, resp.). Similarly, there is an arc from $\mathbf{li}(j, m)$ to $\mathbf{ls}(j', i)(\mathbf{ri}(m'))$ iff $\bar{t}(j, m)$ contains an occurence of $x(j', i)(y(m')$, resp.). G_t has no more arcs than those listed above. G_t is called a dependency graph. Unfortunately, the direction of the arcs is just the opposite of the direction used in most of works concerning attribute grammars (e.g. [8], [9], [12]). However, this direction is more natural from the point of view that t: $k+lq \to kq+l \in R(\Delta)$, where the arrow leads from the "components" to the "variables".

Clearly, $t \in L(\Delta)[k, l]$ iff
(i) $t \in T(\Delta)$;
(ii) there is at most one arc entering each of the nodes

$$\{\mathbf{ls}(j, i),\ \mathbf{ri}(m) | i \in [k],\ m \in [l],\ j \in [q]\}$$

in G_t (i.e. G_t is a forest).

Let $q_0 \geqq 0$ and for each $0 \leqq s \leqq q_0$, $t_s \in L(\Delta)[k, l](1, q_s)$ with $\sum_{s=1}^{q_0} q_s = q$. It suffices to prove that $t = t_0 \cdot \sum_{s=1}^{q_0} t_s \in L(\Delta)[k, l](1, q)$. Construct the graphs G_{t_s} for each $0 \leqq s \leqq q_0$, marking the nodes of G_{t_s} with a subscript s. For each $i \in [k]$, $m \in [l]$, $j \in [q_0]$ identify the node $\mathbf{ls}_0(j, i)$ with $\mathbf{rs}_j(i)$ and $\mathbf{li}_0(j, m)$ with $\mathbf{ri}_j(m)$ to get the graph G. This graph fully describes the dependence relation of the attributes while performing the composition $t_0 \cdot \sum_{s=1}^{q_0} t_s$. Therefore it is easy to see that $G_t = G^+ \setminus N_{\text{in}}$, where G^+ denotes the transitive closure of G and

$$N_{\text{in}} = \{\mathbf{ls}_0(j, i)(\equiv \mathbf{rs}_j(i)),\ \mathbf{li}_0(j, m)(\equiv \mathbf{ri}_j(m)) | i \in [k],\ m \in [l],\ j \in [q_0]\}.$$

Let us remark that, by construction, there is at most one arc entering each node of G, moreover, no arc enters the nodes

$$\{\mathbf{rs}_0(i),\ \mathbf{li}_s(j, m) | i \in [k],\ m \in [l],\ s \in [q_0],\ j \in [q_s]\}.$$

This implies that the connected subgraphs starting from these nodes are trees, so t is finite and G_t is a forest, which was to be proved.

Observe that the connected subgraphs starting from the nodes of N_{in} might be circles. This means that circularity might appear if we want to achieve the result of the composition by computing the value of all the concerning attributes, but this "inside" circularity does not affect the value of the important attributes.

Now we generalize the notion of "dependence on a variable" to projective magmoids.

Lemma 2.3. Let M be a projective magmoid with $M(1, 0) \neq \emptyset$. For any $a \in M(p, q)$ let $a = a'\vartheta$, where $a' \colon p \to q'$, $\vartheta \in \hat{\Theta}(q', q)$ and q' is minimal. The image of ϑ, $\operatorname{Im}(\vartheta)$ is then uniquely determined.

Proof. Suppose the decompositions $a = a_1'\vartheta_1 = a_2'\vartheta_2$ both satisfy the conditions of the lemma and $\operatorname{Im}(\vartheta_1) \neq \operatorname{Im}(\vartheta_2)$, e.g. $i \in \operatorname{Im}(\vartheta_1)$ but $i \notin \operatorname{Im}(\vartheta_2)$. Let $\perp \in M(1, 0)$, and consider the element $\varrho = 1_{i-1} + \perp + 0_1 + 1_{q-i} \colon q \to q$. Since $i \notin \operatorname{Im}(\vartheta_2)$, we have $\vartheta_2\varrho = \vartheta_2$, thus, $a = a_1'\vartheta_1\varrho$. Observe that $\vartheta_1\varrho = (1_{j-1} + \perp + 0_1 + 1_{q'-j})\vartheta_1$, where $i = j\vartheta_1$. On the other hand

$$1_{j-1} + \perp + 0_1 + 1_{q'-j} = (1_{j-1} + \perp + 1_{q'-j})(1_{j-1} + 0_1 + 1_{q'-j}),$$

that is

$$a_1'\vartheta_1\varrho = \left(a_1'(1_{j-1} + \perp + 1_{q'-j})\right)(1_{j-1} + 0_1 + 1_{q'-j})\vartheta_1.$$

This is a contradiction, since q' was supposed to be minimal.

We shall say that $a\colon p\to q$ depends on x_i ($i\in[q]$) if $i\in\mathrm{Im}(\vartheta)$ by the decomposition $a=a'\vartheta$ above.

Let R be a rational theory, and extend the homomorphism $\zeta\colon \mathbf{D}R[k,l]\to R[k,l]$ to a mapping $\hat{\zeta}\colon \hat{\mathbf{T}}\mathbf{D}R[k,l]\to R[k,l]$ as follows. For $\vartheta\in\hat{\Theta}(p,q)$ let

$$\hat{\zeta}(\vartheta)=\langle\!\langle \eta_k(\iota_k(\vartheta))+0_{lp},\ 0_{kq}+\varphi\rangle\!\rangle,$$

where $\varphi\colon lq\to lp$ satisfies

$$\pi_{(j-1)+m}\varphi = \textit{if}\quad j\in\mathrm{Im}(\vartheta)\quad \textit{and}\quad j=i\vartheta\quad \textit{then}\quad \pi_{l(i-1)+m}\quad \textit{else}\quad \perp_1$$

for each $j\in[q]$ and $m\in[l]$. Now, for any $a\colon n\to p$ and $\vartheta\colon p\to q$ let $\hat{\zeta}(\langle a,\vartheta\rangle)=\zeta(a).\hat{\zeta}(\vartheta)$.

Intuitively, $\hat{\zeta}(\langle a,\vartheta\rangle)$ can be obtained as follows (for simplicity let $R=R(\Delta)$ and $a\colon 1\to p$). Starting from $\zeta(a)$, any reference to an s-attribute of a leaf (say the i-th) must be pointed to the corresponding s-attribute of the $i\vartheta$-th leaf. References to i-attributes of the root remain unaltered (though the corresponding variable indices may be shifted), but the values of the i-attributes of the leaves must also be rearranged according to ϑ. The value of all the i-attributes of a "fictive" leaf is set to $\perp$.

The following example shows that, contrary to our expectations, $\hat{\zeta}$ is not a homomorphism.

Let $R=R(\Delta)$ with $\Delta=\Delta_1=\{\delta\}, k=l=1$. Consider the elements $a=\delta(y(1))\colon 1\to 0$ and $b=\langle\perp,\delta(y(1))\rangle\colon 1\to 1$ of $\mathbf{D}R(\Delta)[1,1]$. Then

$$\hat{\zeta}(\langle a,0_1\rangle.\langle b,1\rangle)=\hat{\zeta}(\langle a,0_1\rangle)=\langle\delta(y(1)),\perp\rangle,$$

but $\hat{\zeta}(\langle a,0_1\rangle).\hat{\zeta}(\langle b,1\rangle)=\langle\delta(y(1)),\delta(\perp)\rangle$.

However, it must be noticed that the only difference is between the values of the i-attrbutes of the "fictive" leaf.

Let $\hat{R}[k,l]\subseteq R[k,l]$ be the following system. $a\in\hat{R}[k,l](p,q)$ iff there exists a system $I=\{I(r)\subseteq[q]\mid r\in[p]\}$ of pairwise disjoint subsets of $[q]$ for which the following two conditions are satisfied:

(i) if $\underline{a}(r,i)$ depends on $x(j,i')$ $r\in[p], j\in[q], i,i'\in[k]$), then $j\in I(r)$, moreover, if $\underline{a}(r,i)$ depends on $y(r',m)$ ($r'\in[p], m\in[l]$), then $r=r'$;

(ii) if $\bar{a}(j,m)$ depends on $y(r,m')$, then $j\in I(r)$, moreover, if $\bar{a}(j,m)$ depends on $x(j',i)$, then for each $r\in[p]$ we have: $j\in I(r)$ iff $j'\in I(r)$.

For a fixed $a\in\hat{R}[k,l](p,q)$ there might be several systems I satisfying (i) and (ii) above. There exists, however, a minimal one I_a, in which for every $r\in[p]$, $I_a(r)$ is the least subset of $[q]$ satisfying the following two conditions:

(i) if $\underline{a}(r,i)$ depends on $x(j,i')$, then $j\in I_a(r)$ i

(ii) if $\bar{a}(j,m)$ depends on $x(j',i)$ for some; $j\in I_a(r)$, then $j'\in I_a(r)$, too.

Define the binary relation Ψ on $\hat{R}[k,l]$ as follows. For every $a,b\colon p\to q$, $a\Psi b$ iff

(i) $I_a=I_b$;

(ii) $\underline{a}(r,i)=\underline{b}(r,i)$ for each $r\in[p], i\in[k]$;

(iii) $\bar{a}(j,m)\neq\bar{b}(j,m)$ implies that $j\notin I_a(r)$ for any $r\in[p]$. We shall see that $\hat{R}[k,l]$ is a submagmoid of $R[k,l]$ and Ψ is a congruence relation. It could also be proved that $\hat{\zeta}\Psi\colon \hat{\mathbf{T}}\mathbf{D}R[k,l]\to\hat{R}[k,l]/\Psi$ is already a homomorphism.

Let us start with two easy observations.

Proposition 2.4. 1. For any appropriate $a \in \mathbf{D}R[k, l]$ and $\vartheta \in \hat{\Theta}$, $\zeta(\langle a, \vartheta \rangle) \in \hat{R}[k, l]$.

2. If $\xi: R \to R'$ is a homomorphism and $a \Psi b$ holds in $\hat{R}[k, l]$, then $\xi[k, l](a) \Psi \xi[k, l](b)$ holds in $\hat{R}'[k, l]$.

The first statement is trivial, while the second follows from the fact that the components of $\xi[k, l](a)$ and $\xi[k, l](b)$ depend on at most the same variables as the corresponding components of a and b do.

Let $k' \geq 1, l' \geq 0$, and for each $q \geq 0$ define the bijections ϱ_q and ϱ'_q as follows

$$\varrho_q = 1_{kk'} + \langle\!\langle 0_{ll'} + \langle\!\langle \sum_{j=1}^{q} \mu_{kl'}^{k'l}, \sum_{j=1}^{q} v_{kl'}^{k'l} \rangle\!\rangle, \mu_{ll'} \rangle\!\rangle,$$

$$\varrho'_q = \langle\!\langle \sum_{j=1}^{q} \mu_{ll'}^{kk'} + 1_{k'l+kl'}, \sum_{j=1}^{q} v_{ll'}^{kk'} + 0_{k'l+kl'} \rangle\!\rangle^{-1}.$$

See also Fig. 2.

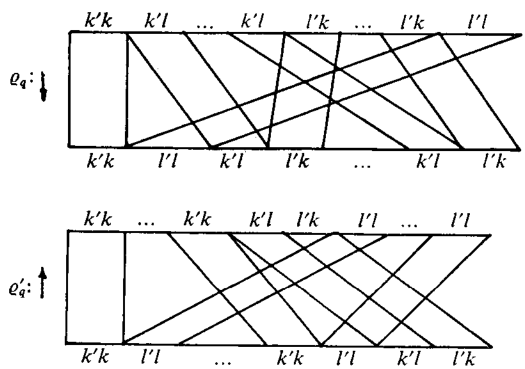


Fig. 2

Definition 2.5. $a \in R[k'k+l'l, k'l+l'k](1, q)$ is called $[k, l]$-linear if $\varrho_q a \varrho'_q \in \hat{R}[k', l'](k+lq, kq+l)$. Generally, $a \in \mathbf{D}R[k'k+l'l, k'l+l'k]$ is $[k, l]$-linear if $a = 1_0$, or $a = a_1 + \ldots + a_p$ and for each $i \in [p]$, a_i: $1 \to q_i$ is $[k, l]$-linear.

Let $a, b \in \mathbf{D}R[k'k+l'l, k'l+l'k]$ be $[k, l]$-linear elements, $a = \sum_{i=1}^{p} a_i$, $b = \sum_{i=1}^{p} b_i$ with a_i, b_i: $1 \to q_i$ $(i \in [p])$. Define $a \Phi b$ iff $\varrho_q a_i \varrho'_q \Psi \varrho_q b_i \varrho'_q$ for each $i \in [p]$.

Lemma 2.6. The $[k, l]$-linear elements form a (decomposable) submagmoid of $\mathbf{D}R[k'k+l'l, k'l+l'k]$ and Φ is a congruence relation on it.

The proof of this lemma will be given in the Appendix because of the great amount of computation it needs. The submagmoid of $[k, l]$-linear elements will be denoted by $L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]$. Taking $k=1$ and $l=0$ in the lemma we get that $\hat{R}[k', l']$ is a submagmoid of $R[k', l']$ and Ψ is a congruence relation, as we stated it before.

Let h: $\tilde{T}(\Delta)\to R[k', l']$ be a homomorphism, and define the mapping $\hat{h}[k, l]$: $L(\Delta)[k, l]\to L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]$ as follows (the notation $\hat{h}[k, l]$ is somewhat abusing here):

(i) for $t\in L(\Delta)[k, l](1, q)(=\hat{\mathbf{T}}\tilde{T}(\Delta)(k+lq, kq+l))$
$\hat{h}[k, l](t)=\varrho_q^{-1}\hat{\zeta}(\hat{\mathbf{T}}h(t))\varrho_q'^{-1}$;

(ii) for $t=t_1+\ldots+t_p\in L(\Delta)[k, l](p, q)$ $(t_i$: $1\to q_i)$
$\hat{h}[k, l](t)=\hat{h}[k, l](t_1)+\ldots+\hat{h}[k, l](t_p)$.

Lemma 2.7. $\hat{h}[k, l]\Phi$: $L(\Delta)[k, l]\to L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]/\Phi$ is a homomorphism.

This lemma, too, will be proved in the Appendix.

Now we are ready to prove the main result of this section.

Theorem 2.8. Let $\mathbf{A}_1=(\Sigma, \Delta, k, l, h_1, S_1)$ be a dtla-tree transducer, $\mathbf{A}_2=(\Delta, R, k', l', h_2, S_2)$ an arbitrary a-transducer. Then $\tau_{\mathbf{A}_1}\circ\tau_{\mathbf{A}_2}$ is also an a-transformation.

Proof. By Lemma 2.2, h_1 is in fact a homomorphism of $\tilde{T}(\Sigma_{S_1})$ into $L(\Delta)[k, l]$. Let $\Sigma_{S_1,S_2}=\Sigma\cup\{S_1, S_2\}$, where S_1 and S_2 both have rank 1, and extend h_1 to a homomorphism of $\tilde{T}(\Sigma_{S_1,S_2})$ into $L(\Delta_{S_2})[k, l]$ by $h_1(S_2)=S_2\pi_k^1+\sum_{i=1}^{k+l-1}\delta_0+0_l$. δ_0 is an arbitrary element of Δ_0. (We can suppose that δ_0 exists, because $\Delta_0=\emptyset$ would imply $\tau_{\mathbf{A}_1}=\tau_{\mathbf{A}_1}\circ\tau_{\mathbf{A}_2}=\emptyset$.) Let S be a new symbol, and define the ranked alphabet map h: $\Sigma_S\to\mathbf{D}R[k'k+l'l, k'l+l'k]$ as follows:

(i) $h(\sigma)=\hat{h}_2[k, l](h_1(\sigma))$ if $\sigma\in\Sigma_{S_1,S_2}$;

(ii) $h(S)=h(S_2)h(S_1)$.

We claim that the transducer $\mathbf{A}=(\Sigma, R, k'k+l'l, k'l+l'k, h, S)$ satisfies $\tau_{\mathbf{A}}=\tau_{\mathbf{A}_1}\circ\tau_{\mathbf{A}_2}$. Viewing h as a homomorphism of $\tilde{T}(\Sigma_{S_1,S_2})$ into $L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]$ the following diagram commutes:

$$\begin{tikzcd}
 & L(\Delta_{S_2})[k, l] \arrow[d, "\hat{h}_2{[k, l]}\Phi"] \\
\tilde{T}(\Sigma_{S_1,S_2}) \arrow[ur, "h_1"] \arrow[r, "h\Phi"'] & L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]/\Phi.
\end{tikzcd}$$

Now, for any $t\in T_\Sigma$

$$h(S(t)) = h(S_2)h(S_1)h(t)\Phi\hat{h}_2[k, l]\left(\left(S_2\pi_k^1+\sum_{i=1}^{k+l-1}\delta_0+0_l\right).h_1(S_1(t))\right) =$$

$$= \hat{h}_2[k, l]\left(S_2\pi_k^1h_1(S_1(t))+\sum_{i=1}^{k-1}\delta_0\right) = \varrho_0^{-1}\hat{\zeta}\left(\hat{\mathbf{T}}h_2\left(\langle S_2(\tau_{\mathbf{A}_1}(t)), 0_l\rangle+\sum_{i=1}^{k-1}\langle\delta_0, 0_0\rangle\right)\right)\varrho_0'^{-1} =$$

$$= \hat{\zeta}(\langle h_2(S_2(\tau_{\mathbf{A}_1}(t))), 0_l\rangle)+\sum_{i=1}^{k-1}\hat{\zeta}(\langle h_2(\delta_0), 0_0\rangle).$$

By the definition of Φ,

$$\pi_{k'k+l'l}^1 h(S(t)) = \pi_{k'k+l'l}^1(\hat{\zeta}(\langle h_2(S_2(\tau_{\mathbf{A}_1}(t))), 0_l\rangle)+$$

$$+\sum_{i=1}^{k-1}\hat{\zeta}(\langle h_2(\delta_0), 0_0\rangle)) = 0_{k'l}+(\pi_k^1.h_2(S_2(\tau_{\mathbf{A}_1}(t))))+0_{l'(k-1)} = \tau_{\mathbf{A}_2}(\tau_{\mathbf{A}_1}(t))+0_{k'l+l'k}$$

which was to be proved.

REMARK. The intuitive meaning of the above construction is the following. The attributes of the transducer **A** can be devided into four classes. These are **s**−**s**, **i**−**s**, **s**−**i** and **i**−**i** containing $k'k, k'l, l'k$ and $l'l$ attributes, respectively. To interpret the value of the four kinds of attributes let $t\in T_\Sigma$ and α a node in t. Suppose that the value of the i-th **s**-attribute (m-th **i**-attribute) of α under h_1 appears as a subtree below the node β_i (γ_m, resp.) in $S_2(\tau_{A_1}(t))$. The following table describes the value of all the attributes of α under the composite transformation.

Attribute	Index	Class	Type	Related node in $S_2(\tau_{A_1}(t))$	Value
$\langle i, i'\rangle$	$k'(i-1)+i'$	s−s	synthesized	β_i	$\mathbf{s}(\beta_i, i')$
$\langle m, i'\rangle$	$k'k+l'l+k'(m-1)+i'$	i−s	inherited	γ_m	$\mathbf{s}(\gamma_m, i')$
$\langle i, m'\rangle$	$k'k+l'l+k'l+l'(i-1)+m'$	s−i	inherited	β_i	$\mathbf{i}(\beta_i, m')$
$\langle m, m'\rangle$	$k'k+l'(m-1)+m'$	i−i	synthesized	γ_m	$\mathbf{i}(\gamma_m, m')$

In the last column, e.g. $\mathbf{s}(\beta_i, i')$ denotes the value of the i'-th **s**-attribute of β_i under h_2. If "related node in $S_2(\tau_{A_1}(t))$" does not exist, then the value of the corresponding attribute is undefined or unimportant (see the congruence Φ). It is rather surprising that the attributes of class **i**−**i** can be computed in synthesized way.

Theorem 2.9. The class of all dtla-tree transformations is closed under composition.

Proof. Let $\mathbf{A}_2$ be a dtla-tree transformation from Δ into Γ in Theorem 2.8. Then the composite transducer **A** is obviously dl, but in general not total. Let us remark, however, that for each $\sigma\in\Sigma_S$ there exists a total representant in $[h(\sigma)]\Phi$. For example, it is enough to replace the $\perp$ components of $h(\sigma)$ (which in fact correspond to the values of the **i**-attributes of the fictive leaves of $h_1(\sigma)$ under h_2) by an arbitrary $\gamma_0\in\Gamma_0$. Clearly, this modification does not change the transformation $\tau_{\mathbf{A}}$, so we are through.

Example 2.10. Let $k=l=k'=l'=1$, $\Sigma_0=\Delta_0=\{\bar a\}$, $\Sigma_1=\{a\}$, $\Delta_1=\{f, g\}$, $\Sigma=\Sigma_0\cup\Sigma_1$, $\Delta=\Delta_0\cup\Delta_1$ and $\Gamma=\Delta$. Define h_1 and h_2 as follows

$$h_1(S_1) = h_2(S_2) = \langle x(1),\ \bar a\rangle;$$

$$h_1(a) = h_2(f) = \langle f(x(1)), g(y(1))\rangle;$$

$$h_1(\bar a) = h_2(\bar a) = y(1);$$

$$h_2(g) = \langle g(x(1)), f(y(1))\rangle.$$

Clearly, $\tau_{A_1}\circ\tau_{A_2}=\{\langle a^n\,\bar a,\ f^n g^n f^n g^n \bar a\rangle \mid n\geqq 0\}$ (parenthesis are omitted for short). Following the construction of Theorem 2.8 we get the transducer $\mathbf{A}=(\Sigma, \Gamma, 2, 2, h, S)$, where

$$h(S) = [\langle x(1), \perp, x(2), \bar{a}\rangle\langle x(1), \perp, x(2), y(2)\rangle]\Theta_\Gamma =$$

$$= [\langle x(1), \perp, x(2), \bar{a}\rangle]\Theta_\Gamma \equiv \langle x(1), \bar{a}, x(2), \bar{a}\rangle(\Phi);$$

$$h(a) = \langle f(x(1)), f(x(2)), g(y(1)), g(y(2))\rangle;$$

$$h(a) = \langle y(1), y(2)\rangle.$$

It is easy to check that, indeed, $\tau_{\mathbf{A}} = \tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$.

Let $v(\mathbf{A})$ denote the minimal value of the natural number K for which $\mathbf{A}$, a da-tree transducer, is K visit (for the definition of visits see e.g. [8], [12]). The complexity of $\mathbf{A}$, $c(\mathbf{A})$ is defined implicitely by the equation $v(\mathbf{A}) = \left\lceil \frac{c(\mathbf{A})}{2} \right\rceil + 1$. Now, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be dtla-tree transducers with $c(\mathbf{A}_i) = c_i$ $(i=1, 2)$, and construct the transducer $\mathbf{A}$ defining $\tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$. It can be proved that $c(\mathbf{A}) \leqq c_1 c_2 + c_1 + c_2$ and this is the best possible upper bound.

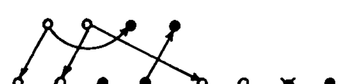
3. The composition of a dla-tree transformation and an arbitrary a-transformation

Let $\mathbf{A} = (\Sigma, \Delta, k, l, h, S)$ be a dla-tree transducer. We define a homomorphism $\mathrm{Ch}\colon \tilde{T}(\Sigma_S) \to L_\perp(\emptyset)[k+l, k+l]$ (called the trace of h) having the property that for every $t \in T_\Sigma$ and $i \in [k]$

$$\pi^i_{k+l}\mathrm{Ch}(S(t)) = \textit{if} \quad \pi^i_k h(S(t)) = \perp_{1,l} \quad \textit{then} \quad \perp_{1,k+l} \quad \textit{else} \quad \pi^i_{k+l}.$$

(Obviously, $\mathrm{Ch}(S)$ is not a synthesizer here.)

Instead of presenting a formal description we illustrate Ch via an example. Let $\sigma \in (\Sigma_S)_n$ and t the linear representant of $h(\sigma)$ having the fewest nodes. Construct the graph G_t as in Lemma 2.2. For example, let $k=l=n=2$. On Fig. 3 s- and i-attributes are represented by o-s and ·-s, respectively, in the order from left to right. The mark $\times$ indicates a $\perp$-valued attribute.

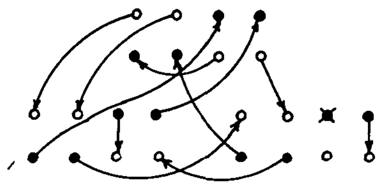


$\{\mathbf{rs}(i), \mathbf{ri}(m) | i, m \in [2]\}$

$\{\mathbf{ls}(j, i), \mathbf{li}(j, m) | j, i, m \in [2]\}$

ig. 3

Associate to each s-attribute a new i-attribute and to each i-attribute a new s-one. On Fig. 4 the nodes denoting these new attributes are placed below the corresponding old ones. The predicate, whether the value of an attribute a under h is $\perp$ or not will be expressed by: the value of a under Ch is $\perp$ or the same as the value of the associated new attribute a'. This can be achieved by checking the value of all the attributes a depends on, tracing them one after the other in an arbitrary order. In our example $\mathrm{Ch}(\sigma)$ can be represented by the graph of Fig. 4.



$\{\mathbf{rs}(i), \mathbf{ri}(m) | i, m \in [2]\}$
$\{\mathbf{ri}'(i), \mathbf{rs}'(m) | i, m \in [2]\}$
$\{\mathbf{ls}(j, i), \mathbf{li}(j, m) | j, i, m \in [2]\}$
$\{\mathbf{li}'(j, i), \mathbf{ls}'(j, m) | j, i, m \in [2]\}$

Fig. 4

To get the required result we only have to order the (old and new) attributes so that the i-th s-attribute and the i-th i-attribute ($i \in [k]$) should be the i-th old s-attribute and its associated new i-attribute, respectively.

There is a natural embedding $\otimes$: $R[k, l] \times R[k', l'] \to R[k+k', l+l']$ defined as follows. $a \otimes b = \varphi(a+b)\psi$, where $a, b\colon p \to q$ and

$$\varphi = 1_{kp} + \langle\!\langle \nu_{k'p}^{lq}, \mu_{k'p}^{lq} \rangle\!\rangle + 1_{l'q},$$
$$\psi^{-1} = 1_{kq} + \langle\!\langle \nu_{k'q}^{lp}, \mu_{k'q}^{lp} \rangle\!\rangle + 1_{l'p}.$$

It is easy to check that $\otimes$ is indeed a $1-1$ homomorphism.

Lemma 3.1. Let $\mathbf{A}_1 = (\Sigma, \Delta, k, l, h_1, S)$ be a dla-tree transducer, $\mathbf{A}_2 = (\Sigma, R, k', l', h_2, S)$ an arbitrary a-transducer. $\tau_{\mathbf{A}_2} | D\tau_{\mathbf{A}_1}$ (the restriction of $\tau_{\mathbf{A}_2}$ to the domain of $\tau_{\mathbf{A}_1}$) is an a-transformation.

Proof. Let $\mathbf{A}_S = (\Sigma_S, R, k+l+k', k+l+l', h_S, S')$ be the following a-transducer:

(i) $h_S(\sigma) = \mathrm{Ch}_1(\sigma) \otimes h_2(\sigma)$ for $\sigma \in \Sigma_S$,

(ii) $h_S(S') = \langle\!\langle \pi_1, \perp_{k+l+k'-1}, \pi_{k+l+1}, \perp_{k+l+l'-1} \rangle\!\rangle$. Observe that for every $t \in T_\Sigma$

$$\tau_{\mathbf{A}_S}(S(t)) = \textit{if } t \in D\tau_{\mathbf{A}_1} \textit{ then } \tau_{\mathbf{A}_2}(t) \textit{ else } \perp_{1,0}.$$

Now, consider the a-transducer $\mathbf{A} = (\Sigma, R, k+l+k', k+l+l', h, S)$, where $h(\sigma) = h_S(\sigma)$ for $\sigma \in \Sigma$ and $h(S) = h_S(S')h_S(S)$. Clearly, $\tau_{\mathbf{A}} = \tau_{\mathbf{A}_2} | D\tau_{\mathbf{A}_1}$.

Remark 3.2. If $\mathbf{A}_2$ is a dla-tree transducer, then so is $\mathbf{A}$.

Theorem 3.3. Let $\mathbf{A}_1 = (\Sigma, \Delta, k, l, h_1, S_1)$ be a dla-tree transducer, $\mathbf{A}_2 = (\Delta, R, k', l', h_2, S_2)$ an arbitrary a-transducer. Then $\tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$ is also an a-transformation.

Proof. Choosing any linear representant of $h_1(\sigma)$ ($\sigma \in \Sigma_{S_1}$) we get a homomorphism $h_1'\colon \tilde{T}(\Sigma_{S_1}) \to L_\perp(\Delta)[k, l]$. Since $L_\perp(\Delta)[k, l] \cong L(\Delta_\perp)[k, l]$, we can use h_1' to define the dtla-tree transducer $\mathbf{A}_1' = (\Sigma, \Delta_\perp, k, l, h_1', S_1)$. Extend h_2 to $\Delta_\perp$ by $h_2(\perp) = \perp_{k', l'}$. Theorem 2.8 implies that $\tau_{\mathbf{A}_1'} \circ \tau_{\mathbf{A}_2} = \tau_{\mathbf{A}}$ for an appropriate a-transducer $\mathbf{A}$. Clearly, $\tau_{\mathbf{A}} | D\tau_{\mathbf{A}_1} = \tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$, so the statement of the theorem follows from Lemma 3.1.

Corollary 3.4. The class of all dla-tree transformations is closed under composition.

Proof. If $\mathbf{A}_2$ is a dla-tree transducer in Theorem 3.3, then so is the composite transducer $\mathbf{A}$. Thus, the corollary follows from Remark 3.2.

4. dla-tree to string transformations

Let T be a (string) alphabet and let CF(T) denote the rational theory of all context free languages over $T\cup X$ (cf. [13]). CF(T) is in fact the "front" theory of Reg(Σ), supposing $\Sigma_0=T$. $L=\langle L_1, \dots, L_p\rangle\in$ CF$(T)(p, q)$ is called linear deterministic if each L_i $(i\in[p])$ contains at most one string and no variable occurs more than once in L.

Definition 4.1. A dla-tree to string transducer from Σ into T^* (T is finite) is an α-transducer $(\Sigma, \text{CF}(T), k, l, h, S)$, where $h(\sigma)$ is linear deterministic for each $\sigma\in\Sigma_S$.

By Lemma 2.2 the linear deterministic elements form a submagmoid of CF$(T)[k, l]$ which will be denoted by LCF$(T)[k, l]$. If **A** is a dla-tree to string transducer from Σ into T^*, we consider $\tau_{\mathbf{A}}$ as a relation, $\tau_{\mathbf{A}}\subseteq T_\Sigma\times T^*$.

Theorem 4.2. Let $\mathbf{A}=(\Sigma, \text{CF}(T), k, l, h, S)$ be a dla-tree to string transducer. Then $\tau_{\mathbf{A}}=(\varphi, K, \psi)$, where K is a regular forest, φ is a relabeling tree homomorphism (injective on K) and ψ is a dtl-top-down tree to string transformation. (Recall from [4] that the transformation defined by the bimorphism (φ, K, ψ) is $\{\langle\varphi(t), \psi(t)\rangle | t\in K\}$.)

Proof. Let ξ: CF$(T)\to$ CF$(\emptyset)$ denote the homomorphism defined by the unique homomorphism of T^* into $\emptyset^*=\{\lambda\}$. Let $A_0\subseteq$ LCF$(\emptyset)[k, l](1, 0)$ such that $a\in A_0$ iff $\pi_k^1 a=\{\lambda\}$ and $\pi_k^i a\subseteq\{\lambda\}$ for each $i\in[k]$. Since LCF$(\emptyset)[k, l]$ is a finite magmoid, the pair $(h\circ\xi[k, l], A_0)$ can be considered a deterministic finite state bottom-up tree automaton working on T_{Σ_S}, where $A=$ LCF$(\emptyset)[k, l](1, 0)$ is the set of states, $(h\circ\xi[k, l](\sigma)|\sigma\in\Sigma_S)$ describes the transitions and A_0 is the set of final states. Let **Q** denote the relabeling transducer defined by this automaton. **Q** marks each node of a tree $t\in T_{\Sigma_S}$ by a new label, which is a pair consisting of the old label $\sigma\in(\Sigma_S)_n$ and a vector of states $\langle a_0, \dots, a_n\rangle$ in which the automaton passes through the node and its sons, respectively, during the recognition (or refuse) of t. Let Σ'_S denote the ranked alphabet of these new labels.

Define $K=\tau_{\mathbf{Q}}(F_S)\subseteq T_{\Sigma'_S}$, where $F_S=\{u\in T_{\Sigma_S} | u=S(t)$ for some $t\in T_\Sigma\}$. Furthermore, let φ: $\tilde{T}(\Sigma'_S)\to T(\Sigma)$ be such that $\varphi(\langle S, \langle a_0, a\rangle\rangle)=x_1$ and $\varphi(\langle\sigma, \langle a_0, \dots, a_n\rangle\rangle)=\sigma$ if $\sigma\in\Sigma_n$. Obviously, K is regular, φ is injective on K and $\varphi(K)=D\tau_{\mathbf{A}}$.

We describe ψ as a homomorphism of $\tilde{T}(\Sigma'_S)$ into LCF$(T)[k+l, 0]$, i.e. ψ will be a $(k+l)$-state dl-top-down tree to string transformation. To avoid ambiguity we shall use the variables $Z=\{z_1, z_2, \dots\}$ $(Z\cap T=\emptyset)$ instead of X in the definition of ψ. Let \$ be a distinguished symbol in T and # a new symbol not in T. Take an arbitrary $a\in A$. If $a=\langle u_1, \dots, u_k\rangle$, then let $\bar{a}=\bar{u}_1\#\dots\#\bar{u}_k$, where $\bar{u}_i=$ *if* $u_i=\emptyset$ *then* \$ *else* u_i $(i\in[k])$. Let n denote the number of all the #-s and $y(m)$-s $(m\in[l])$ — called separating symbols — in $\bar{a}$. Clearly $n\leq k+l$. Define the mapping η_a: $((T\cup Z)^*)^{k+l}\to$ LCF$(T\cup Z)[k, l](1, 0)$ as follows. If $w=\langle w_1, \dots, w_{k+l}\rangle$, then $\eta_a(w)=\langle v_1, \dots, v_k\rangle$, where for each $i\in[k]$

(i) if $\bar{u}_i=\$$, then $v_i=\emptyset$;
(ii) if $\bar{u}_i=\lambda$, then $v_i=w_{j+1}$, where the # preceding $\bar{u}_i$ is the j-th separating symbol in $\bar{a}$.

(iii) if $\bar{u}_i = y(i_j) \dots y(i_{j+n_i})$, where $y(i_j)$ is the j-th separating symbol in a (from left to right), then $v_i = w_j y(i_j) w_{j+1} \dots y(i_{j+n_i}) w_{j+n_i+1}$.

Taking the inverse of some element under η_a we shall assume that the unnecessary components of $\eta_a^{-1}(\langle v_1, \dots, v_k \rangle) = \langle w_1, \dots, w_{k+l} \rangle$ (which do not take part in (ii) and (iii)) are set to \$.

Now, if $\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle \in (\Sigma'_S)_n$, then let

$$\psi(\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle) = \eta_{a_0}^{-1} \left(h(\sigma) \cdot \sum_{i=1}^{n} \eta_{a_i}(\langle z_{(k+l)(i-1)+1}, \dots, z_{(k+l)i} \rangle) \right).$$

ψ is obviously linear, so it is enough to prove that for every $t \in K$ with $\operatorname{root}(t) = \langle S, \langle a_0, a \rangle \rangle$ $\psi(t) = \eta_{a_0}^{-1}(h(\varphi(t)))$. This follows from the following induction.

If $\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle \in (\Sigma'_S)_n$ $(n \geqq 0)$, $t_i \in T_{\Sigma'_S}$ with $\operatorname{root}(t_i) = \langle -, \langle a_i, \dots \rangle \rangle$ and $\psi(t_i) = \eta_{a_i}^{-1}(h(\varphi(t_i)))$, then for $t = \langle \sigma, \langle a_0, \dots, a_n \rangle \rangle (t_1, \dots, t_n)$ we have $\psi(t) = \eta_{a_0}^{-1}(h(\varphi(t)))$. Really,

$$\psi(t) = \psi(\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle) \cdot \sum_{i=1}^{n} \psi(t_i) =$$

$$= \psi(\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle)[z_{(k+l)(i-1)+j} \leftarrow \pi_j \psi(t_i) | i \in [n], j \in [k+l]] =$$

$$= \eta_{a_0}^{-1} \left(h(\sigma) \cdot \sum_{i=1}^{n} \eta_{a_i}(\eta_{a_i}^{-1}(h(\varphi(t_i)))) \right) = \eta_{a_0}^{-1}(h(\varphi(t))).$$

Corollary 4.3. The surface sets of dla-tree to string transformations are the same as that of dtl-top-down tree to string transformations.

This class of languages was investigated e.g. in [10].

5. Problems

The existence of the trace homomorphism Ch described in section 3 raises the following problem. Given any regular forest $F \subseteq T_\Sigma$, is it possible to find a homomorphism $\xi: \tilde{T}(\Sigma) \to L_\perp(\emptyset)[k, l]$ such that for any $t \in T_\Sigma$, $\pi_1 \xi(t) = \textit{if } t \in F \textit{ then } \pi_1 \textit{ else } \perp_1$? The answer is positive if Σ is a unary alphabet $(\Sigma = \Sigma_0 \cup \Sigma_1)$, although a negative answer is more likely in the general case. It is also open whether it is possible to define deterministic finite state bottom-up or look-ahead tree transformations (cf. [6]) by attributed tree transducers. However, it can be shown that the classes of deterministic attributed and macro tree transformations coincide in the monadic case (i.e. if both the domain and range alphabets are unary). The proof of this result will be given in a forthcoming paper.

Appendix

To prove Lemmas 2.6 and 2.7 we need a preliminary observation.

An infinite tree $t \in R(\Delta)(p, q)$ is called local if it is determined by the sequence $\beta \in (\Delta \cup X_q)^p$ of its roots and a "successor" function χ, which for every $\delta \in \Delta_n$ $(n \geqq 0)$ specifies the sequence of labels of the sons of any node in t labeled by δ (i.e. $\chi(\delta) \in (\Delta \cup X_q)^n$). In this case we write $t = (\beta, \chi)$.

Let Ω and Λ be finite ranked alphabets, $T \in \tilde{T}(\Omega)(p, q)$ an ideal (i.e. $\pi_i T \neq x_j$ for any $i \in [p]$, $j \in [q]$) such that any two distinct nodes of T have different labels. This allows us to identify a node of T by its label. Let nds(T) denote the set of nodes (labels) of T, and for $\omega \in \mathrm{nds}(T), \omega \in \Omega_n$, let $\langle \omega(0), \ldots, \omega(n) \rangle$ denote the sequence of nodes obtained by enumerating the father of ω followed by the sons of ω. Take $\omega(0) = i$ if ω is the root of $\pi_i T$. Furthermore, let $\varkappa \colon \tilde{T}(\Omega) \to \mathbf{D}R(\Lambda)[k, l]$ be a homomorphism such that

$$\varkappa(\omega) = \nless \underline{a}(1, \omega), \ldots, \underline{a}(k, \omega), \bar{a}(1, \omega), \ldots, \bar{a}(ln, \omega) \ngtr,$$

where $n \geqq 0, \omega \in \Omega_n$ and $\{\underline{a}(i, \omega), \bar{a}(j, \omega) | i \in [k], j \in [ln]\} \subseteqq \Lambda_{kn+l}$.

It is routine to check that $\zeta(\varkappa(T)) = (\beta, \chi) \in R(\Lambda)[k, l](p, q)$ is the following local tree:

(i) $$\beta = \langle A(1), \ldots, A(kp), B(1), \ldots, B(lq) \rangle, \tag{1}$$

where $\forall r \in [p], \forall i \in [k]$

$$A\big(k(r-1)+i\big) = \underline{a}(i, \omega) \quad \text{if} \quad r = \omega(0),$$

and $\forall j \in [q], \forall m \in [l]$

$$B\big(l(j-1)+m\big) = \bar{a}\big(l(s-1)+m, \omega\big) \quad \text{if} \quad x_j = \omega(s)$$

for some $\omega \in \Omega_n, s \in [n]$;

(ii) if $\omega \in \mathrm{nds}(T), \omega \in \Omega_n$, then $\forall i \in [k], \forall j \in [ln]$

$$\chi\big(\underline{a}(i, \omega)\big) = \chi\big(\bar{a}(j, \omega)\big) = \langle A(1), \ldots, A(kn), B(1), \ldots, B(l) \rangle,$$

where $\forall s \in [n], \forall i \in [k]$

$$A\big(k(s-1)+i\big) = \mathit{Case}\ \ \omega(s) \quad \mathit{of} \quad \omega'(\in \Omega) \colon \underline{a}(i, \omega');\ x_j \colon x(j, i);$$

and $\forall m \in [l]$

$$B(m) = \mathit{Case}\ \ \omega(0) \quad \mathit{of} \quad \omega'(\in \Omega) \colon \bar{a}\big(l(s-1)+m, \omega'\big), \quad \text{where} \quad \omega'(s) = \omega;$$
$$r(\in [p]) \colon y(r, m).$$

Moreover, if $\vartheta \in \hat{\Theta}(q, q')$, then $\zeta(\hat{\mathbf{T}}\varkappa(T\vartheta)) = (\beta', \chi')$ is the following:

(i) $$\beta' \doteq \langle A(1), \ldots, A(kp), B'(1), \ldots, B'(lq') \rangle,$$

where $A(i)$ and $B(j)$ ($i \in [kp]$, $j \in [lq]$) are as in (1), and $\forall j' \in [q'], \forall m \in [l]$

$$B'\big(l(j'-1)+m\big) = \mathit{if} \quad j' \in \mathrm{Im}(\vartheta) \quad \mathit{and} \quad j' = j\vartheta \quad \mathit{then} \quad B\big(l(j-1)+m\big) \quad \mathit{else} \quad \perp;$$

(ii) for each

$$A \in \{\underline{a}(i, \omega), \bar{a}(j, \omega) | \omega \in \mathrm{nds}(T),\ \omega \in \Omega_n,\ i \in [k],\ j \in [ln]\}$$
$$\chi'(A) = \chi(A)\big[x(j, i) \leftarrow x(j', i) | i \in [k],\ j \in [lq],\ j\vartheta = j'\big].$$

Proof of Lemma 2.6. It is enough to prove the following two statements:
1. if $q_0 \geqq 0$ and for each $0 \leqq s \leqq q_0$, $w_s \in L(k, l)\mathbf{D}R[k'k+l'l, k'l+l'k](1, q_s)$ with $\sum_{s=1}^{q_0} q_s = q$, then

$$w_0 \cdot \sum_{s=1}^{q_0} w_s \in L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k](1, q);$$

2. if $w_s \Phi w_s'$ for every $0 \leqq s \leqq q_0$, then

$$w = w_0 \cdot \sum_{s=1}^{q_0} w_s \Phi w_0' \cdot \sum_{s=1}^{q_0} w_s' = w'.$$

Let $\varrho(w_s)$ stand for $\varrho_{q_s} w_s \varrho_{q_s}'$, and let

$$I_{\varrho(w_s)} = \{\underline{I}_s(1), \dots, \underline{I}_s(k), \quad \bar{I}_s(1), \dots, \bar{I}_s(lq_s)\}$$

with $\|\underline{I}_s(i)\| = \underline{n}_s(i)$, $\|\bar{I}_s(j)\| = \bar{n}_s(j)$ and $kq_s + l - \|\cup I_{\varrho(w_s)}\| = n_s$ $(i \in [k], j \in [lq_s])$. Choose injective mappings $\underline{\vartheta}_s(i)$, $\bar{\vartheta}_s(j)$ and ϑ_s, which map $[\underline{n}_s(i)]$, $[\bar{n}_s(j)]$ and $[n_s]$ into $[kq_s+l]$ such that $\operatorname{Im}(\underline{\vartheta}_s(i)) = \underline{I}_s(i)$, $\operatorname{Im}(\bar{\vartheta}_s(j)) = \bar{I}_s(j)$ and $\operatorname{Im}(\vartheta_s) = [kq_s+l] \setminus \cup I_{\varrho(w_s)}$. Let Ω consist of the symbols $\{\underline{T}_s(i), \bar{T}_s(j) | 0 \leqq s \leqq q_0, i \in [k], j \in [lq_s]\}$, where $\underline{T}_s(i) \in \Omega_{\underline{n}_s(i)}$ and $\bar{T}_s(j) \in \Omega_{\bar{n}_s(j)}$. Define Λ as the least ranked alphabet satisfying the following conditions:

(i) for every $\omega \in \Omega_n$ $(n \geqq 0)$

$$\{\underline{a}(i', \omega), \bar{a}(j', \omega) | i' \in [k'], j' \in [l'n]\} \subseteq \Lambda_{k', n+l'};$$

(ii) for each $0 \leqq s \leqq q_0$

$$\{\bar{a}(j', s), \bar{a}'(j', s) | j' \in [l'n_s]\} \subseteq \Lambda_{k'n_s}.$$

Construct local trees W_s, W_s' and $W_s^{\perp}$ $(0 \leqq s \leqq q_0)$ of $R(\Lambda)[k'k+l'l, k'l+l'k](1, q_s)$ as follows. $W_s = (\beta_s, \chi_s)$ with

(i) $\beta_s = \langle A(1), \dots, A(k'k), B(1), \dots, B(l'l), C(1, 1), \dots, C(1, k'l), D(1, 1), \dots, D(1, l'k), \dots, C(q_s, 1), \dots, C(q_s, k'l), D(q_s, 1), \dots, D(q_s, l'k)\rangle$, where $\forall i \in [k]$, $\forall i' \in [k']$

$$A(k'(i-1)+i') = \underline{a}(i', \underline{T}_s(i)),$$

$\forall m \in [l]$, $\forall m' \in [l']$

$$B(l'(m-1)+m') = \mathit{Case}\ kq_s+m \quad \mathit{of} \tag{2}$$

$$\left.\begin{array}{r} n\underline{\vartheta}_s(i): \\ n\bar{\vartheta}_s(j): \\ n\vartheta_s: \end{array}\right\rangle \bar{a}\big(l'(n-1)+m', \left\langle\begin{array}{l} \underline{T}_s(i)); \\ \bar{T}_s(j)); \\ s); \end{array}\right.$$

$\forall j \in [q_s]$, $\forall m \in [l]$, $\forall i' \in [k']$

$$C(j, k'(m-1)+i') = \underline{a}(i', \bar{T}_s(l(j-1)+m)),$$

and $\forall j' \in [q_s]$, $\forall r \in [k]$, $\forall m' \in [l']$ $D(j', l'(r-1)+m')$ is of the form (2) $[kq_s+m \leftarrow k(j'-1)+r]$, i.e., (2) with kq_s+m replaced by $k(j'-1)+r$;

(iia) $\forall i \in [k]$,

$\forall i' \in [k'], \forall j' \in [l'\underline{n}_s(i)]$

$$\chi_s(\underline{a}(i', \underline{T}_s(i))) = \chi_s(\bar{a}(j', \underline{T}_s(i))) = \langle A(1), \ldots, A(k'\underline{n}_s(i)), B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\underline{n}_s(i)]$

$$A(k'(n-1)+i') = Case\ \ n\underline{\vartheta}_s(i) \quad of \quad k(j'-1)+r(j' \in [q_s], r \in [k]):\ x(j', k(r-1)+i');$$
$$kq_s + m(m \in [l]):\ y(k'(m-1)+i'); \tag{3}$$

and $\forall m' \in [l']\ B(k'\underline{n}_s(i)+m') = y(k'l+l'(i-1)+m')$;

(iib) $\forall j \in [lq_s]$,

$\forall i' \in [k'], \forall j' \in [l'\bar{n}_s(j)]$

$$\chi_s(\underline{a}(i', \bar{T}_s(j))) = \chi_s(\bar{a}(j', \bar{T}_s(j))) = \langle A(1), \ldots, A(k'\bar{n}_s(j)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\bar{n}_s(j)]$ $A(k'(n-1)+i')$ is of the form (3) $[\underline{\vartheta}_s(i) \leftarrow \bar{\vartheta}_s(j)]$, and $\forall m' \in [l']$ $B(m') = x(j', k'k + l'(m-1)+m')$ if $j = l(j'-1)+m$ for some $j' \in [q_s]$, $m \in [l]$;

(iic) $\forall j' \in [l'n_s]$ $\chi_s(\bar{a}(j', s)) = \langle A(1), \ldots, A(k'n_s)\rangle$, where $\forall i' \in [k'], \forall n \in [n_s]$ $A(k'(n-1)+i')$ is of the form (3) $[\underline{\vartheta}_s(i) \leftarrow \vartheta_s]$.

We get W_s' and $W_s^{\perp}$ from W_s by replacing the symbols $\bar{a}(j', s)$ $(j' \in [l'n_s])$ occuring in it by $\bar{a}'(j', s)$ and $\perp$, respectively.

By construction, for any $0 \leqq s \leqq q_0$ and $i \in [k'k + l'l + (k'l + l'k)q_s]$, $\pi_i W_s$ and $\pi_i W_s'$ depend on all those variables which $\pi_i w_s$ or $\pi_i w_s'$ may depend on. Therefore, if $\pi_i W_s^{(\prime)} = \lambda\varphi$ for some $\lambda \in \Lambda$ and $\varphi \in \hat{\Theta}$, then $\pi_i w_s^{(\prime)} = a\vartheta$, where $\operatorname{Im}(\vartheta) \subseteqq \operatorname{Im}(\varphi)$. Define the ranked alphabet map $\xi\colon \Lambda \to R$ by $\xi(\lambda) = a\vartheta\varphi^{-1}$, where φ^{-1} is an arbitrary right inverse of φ. ξ is correct, since for every $\lambda \in \Lambda$ there exists exactly one s, i and φ such that $\pi_i W_s^{(\prime)} = \lambda\varphi$. Obviously $\vartheta\varphi^{-1}\varphi = \vartheta$, thus $\xi(W_s) = w_s$ and $\xi(W_s') = w_s'$ hold by the extension of ξ to a homomorphism of $R(\Lambda)$ into R.

Let $\varkappa\colon \tilde{T}(\Omega) \to \mathbf{D}R(\Lambda)[k', l']$ be the homomorphism extending the following ranked alphabet map. For every $\omega \in \Omega_n$

$$\varkappa(\omega) = \nless \underline{a}(1, \omega), \ldots, \underline{a}(k', \omega), \bar{a}(1, \omega), \ldots, \bar{a}(l'\,n, \omega) \ngtr.$$

Consider the elements

$$T_s = \nless \underline{T}_s(1)\underline{\vartheta}_s(1), \ldots, \underline{T}_s(k)\underline{\vartheta}_s(k),\ \bar{T}_s(1)\bar{\vartheta}_s(1), \ldots, \bar{T}_s(lq_s)\bar{\vartheta}_s(lq_s) \ngtr$$

of $L(\Omega)[k, l](1, q_s)$, and observe that

$$\hat{\xi}(\hat{\mathbf{T}}\varkappa(T_s)) = \varrho(W_s^{\perp})\,\Psi\varrho(W_s).$$

To complete the proof it is enough to show that

$$\varrho(W^{(\prime)}) = \varrho(W_0^{(\prime)}) \cdot \sum_{s=1}^{q_0} W_s^{(\prime)}\Psi\hat{\xi}\left[\hat{\mathbf{T}}\varkappa\left(T_0 \cdot \sum_{s=1}^{q_0} T_s\right)\right]. \tag{4}$$

Indeed, (4) shows that both W and W' are $[k, l]$-linear (see Proposition 2.4/1) and $W\Phi W'$. Thus, by Proposition 2.4/2, $w\Phi w'$.

First we compute $T=T_0\cdot\sum_{s=1}^{q_0}T_s$. Following the proof of Lemma 2.2 it is easy to see that $T\in L(\Omega)[k,l](1,q)$ is the following finite Ω-tree. With the notations of our preliminary observation

(i) $\forall i\in[k]$

$$\underline{T}_0(i)(0)=i,$$

$\forall n\in[\underline{n}_0(i)]$

$$\underline{T}_0(i)(n)=\mathit{Case}\ \ n\underline{\vartheta}_0(i)\quad \mathit{of}\quad k(s-1)+r(s\in[q_0],\,r\in[k])\colon\ \underline{T}_s(r);$$

$$kq_0+m(m\in[l])\colon\ y(m); \tag{5}$$

$\forall s\in[q_0]$, $\forall m\in[l]$

$$\overline{T}_0(l(s-1)+m)(0)=\mathit{Case}\ \ kq_s+m\quad \mathit{of}\quad n\underline{\vartheta}_s(i)\colon\ \underline{T}_s(i);\quad n\overline{\vartheta}_s(j)\colon\ \overline{T}_s(j);$$

$$n\vartheta_s\colon\ \overline{T}_0(l(s-1)+m)\notin \mathrm{nds}\,(T);$$

$\forall j\in[lq_0]$, $\forall n\in[\bar{n}_0(j)]$ $\overline{T}_0(j)(n)$ is of the form (5) $[\underline{\vartheta}_0(i)\leftarrow\overline{\vartheta}_s(j)]$;

(ii) $\forall s\in[q_0]$,
$\forall i\in[k]$

$$\underline{T}_s(i)(0)=\mathit{Case}\ \ k(s-i)+i\quad \mathit{of}\quad n\underline{\vartheta}_0(r)\colon\ \underline{T}_0(r);\quad n\overline{\vartheta}_0(j)\colon\ \overline{T}_0(j);$$

$$n\vartheta_0\colon\ \underline{T}_s(i)\notin \mathrm{nds}\,(T);$$

$\forall n\in[\underline{n}_s(i)]$

$$\underline{T}_s(i)(n)=\mathit{Case}\ \ n\underline{\vartheta}_s(i)\quad \mathit{of}$$

$$k(j'-1)+r(j'\in[q_s],\,r\in[k])\colon\ x(q^{(s)}+j',r)\left(q^{(s)}=\sum_{p=1}^{s-1}q_p\right); \tag{6}$$

$$kq_s+m(m\in[l])\colon\ \overline{T}_0(l(s-1)+m);$$

$\forall j\in[lq_s]$

$$\overline{T}_s(j)(0)=k+lq^{(s)}+j,$$

$\forall n\in[\bar{n}_s(j)]$ $\overline{T}_s(j)(n)$ is of the form (6) $[\underline{\vartheta}_s(i)\leftarrow\overline{\vartheta}_s(j)]$.

Computing $\hat{\zeta}(\hat{\mathrm{T}}\varkappa(T))$ we get a local tree (β,χ) for which

$$\text{(i)}\ \beta=\langle A(1),\dots,A(k'k),\ C(1,1),\dots,C(1,k'l),\dots,C(q,1),\dots,C(q,k'l),$$

$$D(1,1),\dots,D(1,l'k),\dots,D(q,1),\dots,D(q,l'k),\ B(1),\dots,B(l'l)\rangle, \tag{7}$$

where $\forall i\in[k]$, $\forall i'\in[k']$

$$A(k'(i-1)+i')=\underline{a}(i',\underline{T}_0(i)),$$

$\forall s\in[q_0]$,
$\forall j\in[q_s]$, $m\in[l]$, $\forall i'\in[k']$

$$C(q^{(s)}+j,\,k'(m-1)+i')=\underline{a}(i',\overline{T}_s(l(j-1)+m)),$$

$\forall i \in [k], \forall m' \in [l']$

$$D(q^{(s)}+j,\, l'(i-1)+m') = Case\ \ k(j-1)+i \quad of \tag{8}$$

$$\begin{array}{rcl} n\underline{\vartheta}_s(r): & \searrow & \underline{T}_s(r)); \\ n\bar{\vartheta}_s(j'): & \rightarrow \bar{a}(l'(n-1)+m', & \leftarrow \bar{T}_s(j')); \\ n\vartheta_s: & \perp; & \end{array}$$

$\forall m \in [l],\ \forall m' \in [l']$

$$B(l'(m-1)+m') = Case\ \ kq_0+m \quad of \tag{9}$$

$$\begin{array}{rcl} n\underline{\vartheta}_0(i): & \searrow & \underline{T}_0(i)); \\ n\bar{\vartheta}_0(j): & \rightarrow \bar{a}(l'(n-1)+m', & \leftarrow \bar{T}_0(j)); \\ n\vartheta_0: & \perp; & \end{array}$$

(iia) $\forall i \in [k]$,
$\forall i' \in [k'],\ \forall j' \in [l'\underline{n}_0(i)]$

$$\chi(\underline{a}(i', \underline{T}_0(i))) = \chi(\bar{a}(j', \underline{T}_0(i)) = \langle A(1), \ldots, A(k'\underline{n}_0(i)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'],\ \forall n \in [\underline{n}_0(i)]$

$$A(k'(n-1)+i') = Case\ \ n\underline{\vartheta}_0(i) \quad of \quad k(s-1)+r\,(s \in [q_0],\, r \in [k]):\ \underline{a}(i', \underline{T}_s(r));$$
$$kq_0+m\ \ (m \in [l]):\ x(kq+m, i'); \tag{10}$$

and $\forall m' \in [l'],\ B(m') = y(i, m')$;

(iib) $\forall j \in [lq_0]$,
$\forall i' \in [k'],\ \forall j' \in [l'\bar{n}_0(j)]$

$$\chi(\underline{a}(i', \bar{T}_0(j))) = \chi(\bar{a}(j', \bar{T}_0(j))) = \langle A(1), \ldots, A(k'\bar{n}_0(j)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'],\ \forall n \in [\bar{n}_0(j)]$ $A(k'(n-1)+i')$ is of the form (10) $[\underline{\vartheta}_0(i) \leftarrow \bar{\vartheta}_0(j)]$, and $\forall m' \in [l']$ if $j = l(s-1)+m$ for some $s \in [q_0], m \in [l]$, then

$$B(m') = Case\ \ kq_s+m \quad of$$

$$\begin{array}{rcl} n\underline{\vartheta}_s(r): & \searrow & \underline{T}_s(i)); \\ n\bar{\vartheta}_s(j'): & \rightarrow \bar{a}(l'(n-1)+m', & \leftarrow \bar{T}_s(j')); \\ n\vartheta_s: & \bar{T}_0(j) \notin \mathrm{nds}(T); & \end{array}$$

(iic) $\forall s \in [q_0],\ \forall i \in [k]$,
$\forall i' \in [k'],\ \forall j' \in [l'\underline{n}_s(i)]$

$$\chi(\underline{a}(i, \underline{T}_s(i))) = \chi(\bar{a}(j', \underline{T}_s(i))) = \langle A(1), \ldots, A(k'\underline{n}_s(i)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'],\ \forall n \in [\underline{n}_s(i)]$

$$A(k'(n-1)+i') = Case\ \ n\underline{\vartheta}_s(i) \quad of \quad k(j'-1)+r\,(j' \in [q_s],\, r \in [k]):\ x(k(q^{(s)}+j')+r, i');$$
$$kq_s+m\,(m \in [l]):\ \underline{a}(i', \bar{T}_0(l(s-1)+m)); \tag{11}$$

and $\forall m' \in [l']$

$$B(m') = Case\ k(s-1)+i\ of$$

$$\begin{array}{l} n\underline{\vartheta}_0(r): \searrow \qquad\qquad\qquad \nearrow \underline{T}_0(r)); \\ n\bar{\vartheta}_0(j): \rightarrow \bar{a}(l'(n-1)+m', \leftarrow \bar{T}_0(j)); \\ n\vartheta_0:\ \underline{T}_s(i) \notin \mathrm{nds}\,(T); \end{array}$$

(iid) $\forall s \in [q_0]$, $\forall j \in [lq_s]$,
$\forall i' \in [k']$, $\forall j' \in [l'\bar{n}_s(j)]$

$$\chi(\underline{a}(i', \bar{T}_s(j))) = \chi(\bar{a}(j', \bar{T}_s(j))) = \langle A(1), \ldots, A(k'\bar{n}_s(j)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\bar{n}_s(j)]$ $A(k'(n-1)+i')$ is of the form (11) $[\underline{\vartheta}_s(i) \leftarrow \bar{\vartheta}_s(j)]$, and $\forall m' \in [l'] B(m') = y(k+lq^{(s)}+j-1, m')$.

Now we compute $W = W_0 \cdot \sum_{s=1}^{q_0} W_s$. The result is (β', χ'), the following local tree:

(i) $\beta' = \langle A(1), \ldots, A(k'k),\ B(1), \ldots, B(l'l),\ C(1,1), \ldots, C(1, k'l),$

$D(1,1), \ldots, D(1, l'k), \ldots, C(q,1), \ldots, C(q, k'l),\ D(q,1), \ldots, D(q, l'k)\rangle,$

where all the symbols in β' are the same as the corresponding ones under (7), exept that in (8) and (9) $\perp$ must be replaced by $\bar{a}(l'(n-1)+m', s)$ and $\bar{a}(l'(n-1)+m', 0)$, respectively.

(iia) For each symbol occuring in both W and $\zeta(\hat{\mathbf{T}}\varkappa(T))$ we get $\chi'(A)$ from $\chi(A)$ by the following variable transformation:

$$\forall j \in [q], \quad \forall i \in [k], \quad \forall i' \in [k'], \quad \forall m \in [l], \quad \forall m' \in [l']$$

$$x(k(j-1)+i, i') \leftarrow x(j, k'(i-1)+i'), \quad x(kq+m, i') \leftarrow y(k'(m-1)+i'),$$

$$y(i, m') \leftarrow y(l'(i-1)+m'), \quad y(k+l(j-1)+m, m') \leftarrow x(j, k'k+l'(m-1)+m');$$

(iib) $\forall 0 \leqq s \leqq q_0$, $\forall j' \in [l'n_s]$

$$\chi'(\bar{a}(j', s)) = \langle A(1), \ldots, A(k'n_s)\rangle,$$

where $\forall i' \in [k'], \forall n \in [n_s]$ $A(k'(n-1)+i')$ is of the form (9) $[\underline{\vartheta}_0(i) \leftarrow \vartheta_0]$ or (10) $[\underline{\vartheta}_s(i) \leftarrow \vartheta_s]$ depending on $s=0$ or $s \in [q_0]$.

Finally, as it is obvious, we get W' from W by replacing $\bar{a}(j', s)$ by $\bar{a}'(j', s)$ for each $0 \leqq s \leqq q_0$, $j' \in [l'n_s]$.

It is now easy to check that (4) is true, so the lemma is proved.

Proof of Lemma 2.7. Let $q_0 \geqq 0$, and for each $0 \leqq s \leqq q_0$ let

$$t_s = \Lsh \tilde{t}_s(1)\underline{\vartheta}_s(1), \ldots, \underline{t}_s(k)\underline{\vartheta}_s(k),\ \bar{\tilde{t}}_s(1)\bar{\vartheta}_s(1), \ldots, \bar{\tilde{t}}_s(lq_s)\bar{\vartheta}_s(lq_s) \Rsh$$

be an element of $L(\Delta)[k, l](1, q_s)$ with $\underline{\tilde{t}}_s(i) \in \tilde{T}(\Delta)(1, \underline{n}_s(i)), \bar{\tilde{t}}_s(j) \in \tilde{T}(\Delta)(1, \bar{n}_s(j))$ $(i \in [k], j \in [lq_s])$. Construct the alphabets Ω and Λ, the homomorphism $\varkappa$: $\tilde{T}(\Omega) \to \mathbf{D}R(\Lambda)[k', l']$ and the trees $\{T_s | 0 \leqq s \leqq q_0\}$ as in the proof of Lemma 2.6. It is clear that any component of $\hat{h}[k, l](t_s)$ depends on at most the same variables as the corresponding component of $\hat{\mathfrak{h}}[k, l](T_s)$ does. Therefore there exists a homomorphism $\xi\colon R(\Lambda)\to R$ such that

$$\xi(\hat{\mathfrak{h}}[k, l](T_s)) = \hat{h}[k, l](t_s).$$

Let $\mu\colon \tilde{T}(\Omega)\to\tilde{T}(\Delta)$ be the homomorphism by which $\mu(\underline{T}_s(i))=\underline{\tilde{t}}_s(i)$ and $\mu(\bar{T}_s(j))=\bar{\tilde{t}}_s(j)$ for every $0\leqq s\leqq q_0, i\in[k], j\in[lq_s]$. Since $\hat{\mathbf{T}}$ is a functor and $\tilde{T}(\Omega)$ is free, the following diagram commutes:

$$\begin{tikzcd}
\hat{T}(\Omega) \arrow[r, "\hat{\mathbf{T}}\varkappa"] \arrow[d, "\hat{\mathbf{T}}\mu"] & \hat{\mathbf{T}}\mathbf{D}R(\Lambda)[k', l'] \arrow[d, "\hat{\mathbf{T}}\mathbf{D}\,\xi{[k', l']}"] \\
\hat{T}(\Delta) \arrow[r, "\hat{\mathbf{T}}h"] & \hat{\mathbf{T}}\mathbf{D}R[k', l']
\end{tikzcd}$$

This implies that for every $T\in L(\Omega)[k, l](1, q)$

$$\hat{h}[k, l](\hat{\mathbf{T}}\mu(T)) = \xi(\hat{\mathfrak{h}}[k, l](T)).$$

Thus, by (4), we get that

$$\hat{h}[k, l](t_0)\cdot\sum_{s=1}^{q_0}\hat{h}[k, l](t_s) = \hat{h}[k, l](\hat{\mathbf{T}}\mu(T_0))\cdot\sum_{s=1}^{q_0}\hat{h}[k, l](\hat{\mathbf{T}}\mu(T_s)) =$$

$$= \xi\left(\hat{\mathfrak{h}}[k, l](T_0)\cdot\sum_{s=1}^{q_0}\hat{\mathfrak{h}}[k, l](T_s)\right)\Phi\xi\left[\hat{\mathfrak{h}}[k, l]\left(T_0\cdot\sum_{s=1}^{q_0}T_s\right)\right] =$$

$$= \hat{h}[k, l]\left[\hat{\mathbf{T}}\mu\left(T_0\cdot\sum_{s=1}^{q_0}T_s\right)\right] = \hat{h}[k, l]\left(t_0\cdot\sum_{s=1}^{q_0}t_s\right),$$

what was to be proved.

Abstract

We define an interesting subclass of deterministic attributed tree transducers. The importance of this subclass lies in its nice closure properties with respect to composition. It is proved that a deterministic and linear attributed tree transformation can be composed by any attributed transformation without leaving the class of attributed transformations. Moreover, the class of linear deterministic attributed tree transformations is closed under composition. Finally we show that the surface sets of linear deterministic attributed tree to string transformations are the same as the surface sets of linear deterministic top-down ones.

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On ν_i-products of automata

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In this paper we introduce a family of compositions and investigate it from the point of view of isomorphic completeness. Using results concerning well-known types of compositions, we give necessary and sufficient conditions for a system of automata to be isomorphically complete with respect to these products.

By an *automaton* we mean a finite automaton without output. For any nonvoid set X let us denote by X^* the free monoid generated by X. Furthermore, denote by X^+ the free semigroup generated by X. Considering an automaton $\mathbf{A}=(X, A, \delta)$, the transition function δ can be extended to $A \times X^* \to A$ in the following way: $\delta(a, \lambda)=a$ and $\delta(a, p)=\delta(\delta(a, p'), x)$ for any $a \in A, p=p'x \in X^*$, where λ denotes the empty word of X^*. Further on we shall use the notation $ap_{\mathbf{A}}$ for $\delta(a, p)$. If there is no danger of confusion then we omit the index $\mathbf{A}$ in $ap_{\mathbf{A}}$. Let M be an arbitrary nonvoid set. Denote by $P(M)$ the set of all subsets of M.

Let $\mathbf{A}_t=(X_t, A_t, \delta_t)$ $(t=0, \ldots, n-1)$ be a system of automata. Moreover let X be a finite nonvoid set, φ a mapping of $A_0 \times \ldots \times A_{n-1} \times X$ into $X_0 \times \ldots \times X_{n-1}$ and γ a mapping of $\{0, \ldots, n-1\}$ into $P(\{0, \ldots, n-1\})$ such that φ can be given in the form

$$\varphi(a_0, \ldots, a_{n-1}, x) = (\varphi_0(a_0, \ldots, a_{n-1}, x), \ldots, \varphi_{n-1}(a_0, \ldots, a_{n-1}, x))$$

where each φ_t $(0 \leqq t \leqq n-1)$ is independent of states, which have indices not contained in the set $\gamma(t)$. We say that $\mathbf{A}=\left(X, \prod_{t=0}^{n-1} A_t, \delta\right)$ is a ν_i-*product* of $\mathbf{A}_t$ $(t=0, \ldots, n-1)$ with respect to X, φ and γ if $|\gamma(t)| \leqq i$ $(t=0, \ldots, n-1)$ and for any $(a_0, \ldots, a_{n-1}) \in \prod_{t=0}^{n-1} A_t$ and $x \in X$

$$\delta((a_0, \ldots, a_{n-1}), x) =$$

$$= (\delta_0(a_0, \varphi_0(a_0, \ldots, a_{n-1}, x)), \ldots, \delta_{n-1}(a_{n-1}, \varphi_{n-1}(a_0, \ldots, a_{n-1}, x))).$$

For this product we use the notation $\prod_{t=0}^{n-1} \mathbf{A}_t(X, \varphi, \gamma)$.

It is clear that the ν_0-product is the same as the quasi-direct product. Therefore, we consider the case $i \geqq 1$ only. Furthermore, it is interesting to note that if $n=2, i=1, \gamma(0)=\{1\}, \gamma(1)=\{0\}$ then we obtain the cross product (see [2]) as a special case of the ν_1-product. Finally, observe that the ν_i-product is rearrangable, i.e. changing the order of components of a ν_i-product $\prod_{t=0}^{n-1} \mathbf{A}_t(X, \varphi, \gamma)$ and choosing suitable mappings φ', γ' we get such a ν_i-product which is isomorphic to the original one.

Let Σ be a system of automata. Σ is called *isomorphically complete* with respect to the ν_i-product if any automaton can be embedded isomorphically into a ν_i-product of automata from Σ. Furthermore, Σ is called a *minimal* isomorphically complete system if Σ is isomorphically complete and for arbitrary $\mathbf{A} \in \Sigma$ the system $\Sigma \setminus \{\mathbf{A}\}$ is not isomorphically complete.

For any natural number $n \geqq 1$ denote by $\mathbf{D}_n = (X_n, \{1, \ldots, n\}, \delta_n)$ the automaton for which $X_n = \{x_{rs}\colon 1 \leqq r, s \leqq n\}$ and

$$\delta_n(t, x_{rs}) = \begin{cases} s & \text{if } t = r, \\ t & \text{otherwise} \end{cases}$$

for any $t \in \{1, \ldots, n\}$ and $x_{rs} \in X_n$.

The following theorem holds for the ν_i-products if $i \geqq 1$.

Theorem 1. A system Σ of automata is isomorphically complete with respect to the ν_i-product $(i \geqq 1)$ if and only if for any natural number $n \geqq 1$, there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product of $\mathbf{A}$ with a single factor.

Proof. Theorem 1 can be proved in a similar way as the corresponding statement for the α_i-products in [4]. The sufficiency follows from Theorem 2 in [4], but it is not difficult to see directly. In order to prove the necessity we show that for any $n \geqq 1$ if $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product of automata from Σ then there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{D}_{[\sqrt[i+1]{n}]}$ can be embedded isomorphically into a ν_i-product of $\mathbf{A}$ with a single factor, where $[\sqrt[i+1]{n}]$ denotes the largest integer less than or equal to $\sqrt[i+1]{n}$.

If $n=1$ then the statement is obvious. Now let $n>1$ and assume that $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product $\mathbf{B} = \prod_{t=0}^{k} \mathbf{A}_t(X_n, \varphi, \gamma)$ of automata $\mathbf{A}_t = (X_t', A_t, \delta_t) \in \Sigma$ $(t=0, \ldots, k)$. Let us denote by μ such an isomorphism and for any $t \in \{1, \ldots, n\}$ denote by $(a_{t0}, \ldots, a_{tk})$ the image of t under μ. We distinguish two cases depending on the sets $\gamma(t)$ $(t=0, \ldots, k)$. If $\gamma(t) = \emptyset$ for all $t \in \{0, \ldots, k\}$ then $\mathbf{B}$ is a quasi-direct product. Since the quasi-direct product can be considered as a special α_{i+1}-product we have that $\mathbf{D}_n$ can be embedded isomorphically into an α_{i+1}-product $\prod_{t=1}^{k} \mathbf{A}_t(X_n, \varphi)$ of automata from Σ. From this, by the proof of Theorem 2 in [4], it follows that there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{D}_{[\sqrt[i+1]{n}]}$ can be embedded isomorphically into an α_{i+1}-product of $\mathbf{A}$ with a single factor. Since an α_{i+1}-product with a single factor is a ν_i-product with a single factor we have proved the statement for this case.

Now assume that $\gamma(t)\neq\emptyset$ for some $t\in\{0, \ldots, k\}$. By the rearrangability of ν_i-products, without loss of generality we may suppose that $\gamma(0)\neq\emptyset$. We show that $\mathbf{D}_n$ can be embedded isomorphically into a ν_{i+1}-product of automata from $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ with at most $i+1$ factors. If $k\leqq i$ then we are ready. Assume that $k>i$. We may suppose that there exist natural numbers $r\neq s$ $(1\leqq r, s\leqq n)$ such that $a_{r0}\neq a_{s0}$ since otherwise $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product of automata from $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ with k factors. Let $\gamma(0)=\{j_1, \ldots, j_w\}$. By the definition of the ν_i-product, we have that $w\leqq i$ and

$$\varphi_0(a_0, \ldots, a_k, x)=\varphi_0(a_{j_1}, \ldots, a_{j_w}, x) \quad \text{for any} \quad (a_0, \ldots, a_k)\in \prod_{t=0}^{k} A_t \quad \text{and} \quad x\in X_n.$$

We prove that the elements $(a_{t0}, a_{tj_1}, \ldots, a_{tj_w})$ $(t=1, \ldots, n)$ are pairwise different. Indeed, assume that $a_{u0}=a_{v0}$ and $a_{ut}=a_{vt}$ $(t=j_1, \ldots, j_w)$ for some $u\neq v$ $(1\leqq u, v\leqq n)$. Then $\varphi_0(a_{uj_1}, \ldots, a_{uj_w}, x)=\varphi_0(a_{vj_1}, \ldots, a_{vj_w}, x)$ for any $x\in X_n$. Therefore, in the ν_i-product $\mathbf{B}$ the automaton $\mathbf{A}_0$ obtains the same input signal in the states a_{u0} and a_{v0} for any $x\in X_n$. Since μ is isomorphism, $u\neq v$ and $a_{u0}=a_{v0}$, thus the automaton $\mathbf{A}_0$ goes from the state a_{u0} into the state a_{t0} and from the state a_{v0} it goes into the state a_{v0} for any x_{ut} $(t=1, \ldots, n)$. This implies $a_{v0}=a_{t0}$ $(t=1, \ldots, n)$ which contradicts our assumption $a_{r0}\neq a_{s0}$. Therefore, we have that the elements $(a_{t0}, a_{tj_1}, \ldots, a_{tj_w})$ $(t=1, \ldots, n)$ are pairwise different. Now take the following ν_{i+1}-product $\mathbf{C}=\mathbf{A}_0\times\mathbf{A}_{j_1}\times\ldots\times\mathbf{A}_{j_w}(X_n, \psi, \bar{\gamma})$ where for any $t\in\{0, \ldots, w\}$ $\bar{\gamma}(t)=\{0, 1, \ldots, w\}$ and

$$\psi_t(b_0, \ldots, b_w, x)=\begin{cases}\varphi_0(a_{r0}, \ldots, a_{rk}, x) & \text{if} \quad t=0 \quad \text{and} \quad \text{there exists} \quad 1\leqq r\leqq n \\ \quad \text{such that} \quad b_0=a_{r0}, \quad b_s=a_{rj_s} \quad (s=1, \ldots, w), & \\ \varphi_{j_t}(a_{r0}, \ldots, a_{rk}, x) & \text{if} \quad t\neq 0 \quad \text{and} \quad \text{there exists} \quad 1\leqq r\leqq n \\ \quad \text{such that} \quad b_0=a_{r0}, \quad b_s=a_{rj_s} \quad (s=1, \ldots, w), & \\ \text{otherwise arbitrary input signal from } X_0' \quad \text{if} & \\ \quad t=0 \quad \text{and} \quad \text{from} \quad X_{j_t}' \quad \text{if} \quad t\neq 0, & \end{cases}$$

for all $(b_0, \ldots, b_w)\in A_0\times A_{j_1}\times\ldots\times A_{j_w}$ and $x\in X_n$. It is not difficult to see that the correspondence $\mu'\colon t\to(a_{t0}, a_{tj_1}, \ldots, a_{tj_w})$ $(t=1, \ldots, n)$ is an isomorphism of $\mathbf{D}_n$ into $\mathbf{C}$. Therefore, we have that $\mathbf{D}_n$ can be embedded isomorphically into a ν_{i+1}-product of automata from $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ with at most $i+1$ factors. But a ν_{i+1}-product with at most $i+1$ factors is an α_{i+1}-product and thus, in a similar way as in the first case, we obtain that $\mathbf{D}_{[\sqrt[i+1]{n}]}$ can be embedded isomorphically into a ν_i-product of $\mathbf{A}_t$ with a single factor for some $0\leqq t\leqq k$. This ends the proof of Theorem 1.

Observe that $\mathbf{D}_m$ can be embedded isomorphically into a ν_0-product of $\mathbf{D}_n$ with a single factor for any $m\leqq n$. Using this fact, by Theorem 1, we get the following

COROLLARY. There exists no system of automata which is isomorphically complete with respect to the ν_i-product $(i\geqq 1)$ and minimal.

In [1] F. Gécseg has introduced the concepts of the generalized α_i-product and the simulation and characterized the isomorphically and homomorphically complete systems with respect to them. Further on we shall introduce the concept of the generalized ν_i-product and investigate the isomorphically complete systems with respect to this product and the simulation.

We say that an automaton $\mathbf{A}=(X, A, \delta)$ *isomorphically simulates* $\mathbf{B}=(Y, B, \delta')$ if there exist one-to-one mappings $\mu: B\to A$ and $\tau: Y\to X^+$ such that $\mu(\delta'(b, y))= =\delta(\mu(b), \tau(y))$ for any $b\in B$ and $y\in Y$. The following obvious observation holds for the isomorphic simulation.

Lemma 1. If $\mathbf{A}$ can be simulated isomorphically by $\mathbf{B}$ and $\mathbf{B}$ can be simulated isomorphically by $\mathbf{C}$ then $\mathbf{C}$ isomorphically simulates $\mathbf{A}$.

Let $\mathbf{A}_t=(X_t, A_t, \delta_t)$ $(t=0, ..., n-1)$ be a system of automata. Moreover let X be a finite nonvoid set, φ a mapping of $A_0\times...\times A_{n-1}\times X$ into $X_0^+\times...\times X_{n-1}^+$ and γ a mapping of $\{0, ..., n-1\}$ into $P(\{0, ..., n-1\})$ such that φ can be given in the form

$$\varphi(a_0, ..., a_{n-1}, x)=(\varphi_0(a_0, ..., a_{n-1}, x), ..., \varphi_{n-1}(a_0, ..., a_{n-1}, x))$$

where each $\varphi_t(0\leqq t\leqq n-1)$ is independent of states, which have indices not contained in the set $\gamma(t)$. We say that $\mathbf{A}=\left(X, \prod_{t=0}^{n-1} A_t, \delta\right)$ is a *generalized ν_i-product* of $\mathbf{A}_t$ $(t=0,..., n-1)$ with respect to X, φ and γ if $|\gamma(t)|\leqq i$ $(t=0, ..., n-1)$ and for any $(a_0, ..., a_{n-1})\in \prod_{t=0}^{n-1} A_t$ and $x\in X$ $\delta((a_0, ..., a_{n-1}), x)=(\delta_0(a_0, \varphi_0(a_0, ..., a_{n-1}, x)), ...$ $..., \delta_{n-1}(a_{n-1}, \varphi_{n-1}(a_0, ..., a_{n-1}, x)))$.

A system Σ of automata is called *isomorphically S-complete* with respect to the generalized ν_i-product if any automaton can be simulated isomorphically by a generalized ν_i-product of automata from Σ.

Observe that in the definitions of the simulation and the generalized ν_i-product all input words are different from the empty word. Therefore, further on, by an input word we mean a nonempty word. Also the following notation will be used. If k, s are integers and t is a natural number then $k+s \pmod t$ denotes the least nonnegative residue of $k+s$ modulo t. Furthermore, for any $n\geqq 1$ denote by $\mathbf{T}_n=(T_n, \{0, ..., n-1\}, \delta_n)$ the automaton for which T_n is the set of all transformations of $\{0, ..., n-1\}$ and $\delta_n(k, t)=t(k)$ for any $k\in\{0, ..., n-1\}$ and $t\in T_n$.

Lemma 2. If $\mathbf{T}_n$ can be simulated isomorphically by a generalized α_0-product $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi)$ then $\mathbf{T}_n$ can be simulated isomorphically by $\mathbf{A}_j$ for some $j\in\{0, ..., k\}$.

Proof. Lemma 2 follows from the proof of Theorem 1 in [1]. Now we give another proof. Obviously it is enough to prove the statement for the generalized α_0-product of two factors. Indeed, assume that $\mathbf{T}_n$ can be simulated isomorphically by the generalized α_0-product $\mathbf{A}\times\mathbf{B}(X, \varphi)$ under μ and τ. Let us denote by (a_t, b_t) the image of t under μ $(t=0, ..., n-1)$. If $a_0=a_t$ for all $t\in\{1, ..., n-1\}$ then the elements b_t $(t=0, ..., n-1)$ are pairwise different. Now define the mapping τ' in the following way: for any $t_u\in T_n$ $\tau'(t_u)=\varphi_1(a_0, y_1)...\varphi_1(a_0, y_s)$ if $\tau(t_u)=y_1...y_s$. Let us denote by μ' the mapping determined by $\mu'(t)=b_t$ $(t=0, ..., n-1)$. It is not difficult to see that $\mathbf{B}$ isomorphically simulates $\mathbf{T}_n$ under μ' and τ'. Now assume that there exist natural numbers $r\neq s$ $(0\leqq r, s\leqq n-1)$ such that $a_r\neq a_s$. In this case we show that the states a_t $(t=0, ..., n-1)$ are pairwise different. Suppose that $a_u=a_v$ for some $u\neq v$ $(0\leqq u, v\leqq n-1)$. Let us denote by t_{ij} the element of T_n for which $t_{ij}(i)=j$ and $t_{ij}(k)=k$ if $k\neq i$ $(k=0, 1, ..., n-1)$ for all

i, j $(0 \leqq i, j \leqq n-1)$. Now let $w \in \{0, \ldots, n-1\}$ be arbitrary. Then $t_{uw}(u) = w$ and $t_{uw}(v) = v$. By isomorphic simulation, $(a_u, b_u)\tau(t_{uw}) = (a_w, b_w)$ and $(a_v, b_v)\tau(t_{uw}) = (a_v, b_v)$. Let $\tau(t_{uw}) = y_1 \ldots y_m$. Then $a_u \varphi_0(y_1) \ldots \varphi_0(y_m) = a_w$ and $a_v \varphi_0(y_1) \ldots \varphi_0(y_m) = a_v$. Therefore, by $a_u = a_v$, we obtain $a_w = a_v$. Since w was arbitrary we got that $a_t = a_v$ for all $t \in \{0, \ldots, n-1\}$ which contradicts our assumption $a_r \neq a_s$. Now we have that the states a_t $(t = 0, \ldots, n-1)$ are pairwise different. In this case it is not difficult to see that $\mathbf{A}$ isomorphically simulates $\mathbf{T}_n$ under μ' and τ' where $\mu'(t) = a_t$ $(t = 0, \ldots, n-1)$ and for any $t_u \in T_n$ $\tau'(t_u) = \varphi_0(y_1) \ldots \varphi_0(y_s)$ if $\tau(t_u) = y_1 \ldots y_s$.

Lemma 3. If $\mathbf{T}_n$ can be simulated isomorphically by a generalized ν_1-product $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ then $\mathbf{T}_n$ can be simulated by a generalized ν_1-product $\prod_{t=0}^{r} \mathbf{B}_t(X, \varphi', \gamma')$ where $r \leqq k$, $\mathbf{B}_t \in \{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ and $\gamma'(t) = \{t-1 \pmod{(r+1)}\}$ for any $t \in \{0, \ldots, r\}$.

Proof. We proceed by induction on the number of components of the generalized ν_1-product. If $k=0$ then the statement is obvious. Now let $k>0$ and assume that the statement is valid for any l less than k. Moreover, suppose that $\mathbf{T}_n$ can be simulated isomorphically by a generalized ν_1-product $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$. Define the binary relation ϱ on the set $\{0, \ldots, k\}$ as follows: $i \varrho j$ if and only if $i = j$ or $\gamma(i) = \{j\}$ or $\gamma(j) = \{i\}$ for any $i, j \in \{0, \ldots, k\}$. Denote by $\hat{\varrho}$ the transitive closure of ϱ. Then $\hat{\varrho}$ is an equivalence relation on $\{0, \ldots, k\}$. Depending on $\hat{\varrho}$, we shall distinguish three cases.

First assume that the partition induced by $\hat{\varrho}$ has at least two blocks. Let us denote by $\hat{\varrho}(j)$ the block containing j. By the rearrangability of the ν_i-product, we may assume that $\hat{\varrho}(0) = \{0, \ldots, m-1\}$. From this, using the fact that $\bigcup_{s \in \hat{\varrho}(t)} \gamma(s) \subseteq \hat{\varrho}(t)$ holds for any $t \in \{0, \ldots, k-1\}$, we obtain that $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ is isomorphic to a quasi-direct product of two automata $\mathbf{C}_1$ and $\mathbf{C}_2$ where $\mathbf{C}_1$ is a generalized ν_1-product of $\mathbf{A}_0, \ldots, \mathbf{A}_{m-1}$ and $\mathbf{C}_2$ is a generalized ν_1-product of $\mathbf{A}_m, \ldots, \mathbf{A}_k$. Therefore, by Lemma 1, Lemma 2 and our induction hypothesis, we get that the statement is valid.

Now let us suppose that the partition induced by $\hat{\varrho}$ has one block only and there exists an $u \in \{0, \ldots, k\}$ with $u \notin \bigcup_{t=0}^{k} \gamma(t)$. By the rearrangability of ν_i-product, we may suppose that $u = k$. Then observe that $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ is isomorphic to a generalized α_0-product of two automata $\mathbf{C}_1$ and $\mathbf{A}_k$ where $\mathbf{C}_1$ is a generalized ν_1-product of $\mathbf{A}_0, \ldots, \mathbf{A}_{k-1}$. From this, by Lemma 1, Lemma 2 and induction hypothesis, the statement follows.

Finally, assume that the partition induced by $\hat{\varrho}$ has one block only and $\bigcup_{t=0}^{k} \gamma(t) = \{0, \ldots, k\}$. Consider the mapping f determined as follows: for any $t \in \{0, \ldots, k\}$ $f(t) = j$ if and only if $j \in \gamma(t)$. By the definition of $\hat{\varrho}$ and our assumption on $\hat{\varrho}$, it can be seen that f is a cyclic permutation of the set $\{0, \ldots, k\}$. Now rearrange

$\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ in the form $\prod_{t=0}^{k} \mathbf{A}_{f_{(0)}^{k-t-1}}(X, \varphi', \gamma')$. Then, by the rearrangability of v_i-product and Lemma 1, we obtain that $\mathbf{T}_n$ can be simulated isomorphically by $\prod_{t=0}^{k} \mathbf{A}_{f_{(0)}^{k-t-1}}(X, \varphi', \gamma')$. On the other hand, it is not difficult to see that $\prod_{t=0}^{k} \mathbf{A}_{f_{(0)}^{k-t-1}}(X, \varphi', \gamma')$ satisfies the condition of our statement. This ends the proof of Lemma 3.

Now we are ready to study the generalized v_1-product. We have

Theorem 2. A system Σ of automata is isomorphically S-complete with respect to the generalized v_1-product if and only if one of the following three conditions is satisfied by Σ:

(1) for any natural number $n>1$ there exists an automaton in Σ having n different states a_t $(t=0, \dots, n-1)$ and input words q_t $(t=0, \dots, n-1)$ such that $a_t q_t = a_{t+1 (\mathrm{mod}\, n)}$ $(t=0, \dots, n-1)$,

(2) Σ contains an automaton which has two different states a, b and input words p, q, r such that $ap=br=a$ and $aq=bp=b$,

(3) there exists an automaton in Σ which has two different states a, b and input words p, q, r such that $ap \neq bp$, $apq=bpq=a$ and $ar=b$.

Proof. In order to prove the sufficiency of conditions (1)—(3) we use the following observation.

For any automaton $\mathbf{A}=(X, A, \delta)$, $\mathbf{A}$ can be simulated isomorphically by $\mathbf{T}_n$ with $n \geqq \max(2, |A|)$. Therefore, by Lemma 1, if for any $n \geqq 2$ the automaton $\mathbf{T}_n$ can be simulated isomorphically by a generalized v_1-product of automata from Σ then Σ is isomorphically S-complete with respect to the generalized v_1-product. On the other hand, take the following elements t_1, t_2 and t_3 of T_n

$$t_1(k) = k+1 \ (\mathrm{mod}\, n) \quad (k=0, \dots, n-1),$$

$$t_2(0) = 1,\ t_2(1) = 0,\ t_2(k) = k \quad (k=2, \dots, n-1),$$

$$t_3(0) = t_3(1) = 0 \quad \text{and} \quad t_3(k) = k \quad (k=2, \dots, n-1).$$

It can be proved (see [3]) that the mappings t_1, t_2, t_3 generate the complete transformation semigroup over the set $\{0, \dots, n-1\}$. Therefore, the automaton $\mathbf{T}_n$ can be simulated isomorphically by the automaton $\mathbf{T}'_n=(\{t_1, t_2, t_3\}, \{0, \dots, n-1\}, \delta'_n)$ where $\delta'_n=\delta_n|_{\{0, \dots, n-1\} \times \{t_1, t_2, t_3\}}$. From this we obtain that if for any $n \geqq 2$ the automaton $\mathbf{T}'_n$ can be simulated isomorphically by a generalized v_1-product of automata from Σ then Σ is isomorphically S-complete with respect to the generalized v_1-product.

First suppose that Σ satisfies (1). Then it is not difficult to see that for any automaton $\mathbf{A}$ there exists an automaton $\mathbf{B} \in \Sigma$ such that $\mathbf{A}$ can be simulated isomorphically by a generalized v_1-product of $\mathbf{B}$ with a single factor.

Now assume that Σ satisfies (2) by $\mathbf{A} \in \Sigma$. Let $n \geqq 5$ be arbitrary and take the generalized v_1-product $\mathbf{A}^n(X, \varphi, \gamma)$ where

$$X = \{u_i\colon 1 \leqq i < n\} \cup$$
$$\cup \{v_i\colon 0 \leqq i < n\} \cup \{x_i\colon 1 < i < n\} \cup \{y_i\colon 1 \leqq i < n-1\} \cup \{v, x, y, z, w\}$$

and the mappings γ and φ are defined in the following way: for any $t\in\{0, \dots, n-1\}$

$$\gamma(t) = t-1 (\text{mod } n),$$

$$\varphi_t(a, u_i) = p, \quad \varphi_t(b, u_i) = \begin{cases} q & \text{if} \quad t = i, \\ p & \text{otherwise} \quad (i = 1, \dots, n-1), \end{cases}$$

$$\varphi_t(a, v_i) = \begin{cases} r & \text{if} \quad t = i, \\ p & \text{otherwise}, \end{cases} \quad \varphi_t(b, v_i) = \begin{cases} r & \text{if} \quad 0 < t < i, \\ p & \text{otherwise} \quad (i = 0, \dots, n-1), \end{cases}$$

$$\varphi_t(a, x_i) = p, \quad \varphi_t(b, x_i) = \begin{cases} r & \text{if} \quad i \leqq t \leqq n-1, \\ p & \text{otherwise} \quad (i = 2, \dots, n-1), \end{cases}$$

$$\varphi_0(a, y_i) = p, \qquad \varphi_0(b, y_i) = q,$$

$$\varphi_t(a, y_i) = p, \quad \varphi_t(b, y_i) = \begin{cases} r & \text{if} \quad 1 \leqq t < i, \quad i \neq 2, \\ p & \text{otherwise} \quad (i = 1, \dots, n-2 \quad \text{and} \quad t \geqq 1) \end{cases}$$

$$\varphi_t(a, v) = p, \quad \varphi_t(b, v) = \begin{cases} r & \text{if} \quad 1 \leqq t \leqq n-2, \\ p & \text{otherwise}, \end{cases}$$

$$\varphi_0(a, x) = p, \quad \varphi_0(b, x) = r, \quad \varphi_t(a, x) = \varphi_t(b, x) = p \quad (t \geqq 1),$$

$$\varphi_0(a, z) = p, \quad \varphi_0(b, z) = r, \quad \varphi_1(a, z) = r, \quad \varphi_1(b, z) = p,$$

$$\varphi_2(a, z) = \varphi_2(b, z) = p, \quad \varphi_t(a, z) = p, \quad \varphi_t(b, z) = r \quad (t > 2),$$

$$\varphi_0(a, w) = q, \quad \varphi_0(b, w) = p, \quad \varphi_t(a, w) = p, \quad \varphi_t(b, w) = r \quad (t \geqq 1),$$

$$\varphi_0(a, y) = q, \quad \varphi_0(b, y) = \varphi_t(a, y) = \varphi_t(b, y) = p \quad (t \geqq 1).$$

Take the mappings

$$\mu: \begin{array}{ll} 0 & \to (b, a, \dots, a), \\ \vdots & \\ n-1 & \to (a, a, \dots, b), \end{array}$$

$$\tau: \begin{array}{l} t_1 \to q_1 \dots q_{n-1}, \\ t_2 \to u_3 \dots u_{n-1} y_1 z u_1 \dots u_{n-1} y x_2 u_3 \dots u_{n-1} v_0 x_3 u_2 \dots u_{n-1} y x_2, \\ t_3 \to u_3 \dots u_{n-1} y_1 z u_1 \dots u_{n-1} w, \end{array}$$

where

$$q_1 = u_1 \dots u_{n-2} v_{n-1} u_1 \dots u_{n-1} v y,$$

$$q_2 = u_1 \dots u_{n-3} v_{n-2} v_0 u_1 \dots u_{n-2} x_{n-1} y_{n-2} u_{n-1} y,$$

$$q_3 = u_1 \dots u_{n-4} v_{n-3} v_0 x_{n-1} x u_{n-1} u_1 \dots u_{n-3} x_{n-2} u_{n-1} y_{n-3} x_{n-1} u_{n-2} u_{n-1} y,$$

$$q_i = u_1 \dots u_{n-i-1} v_{n-i} v_0 x_{n-i+2} u_{n-i+3} \dots u_{n-1} x x_{n-i+3} u_{n-i+2} \dots u_{n-1}$$
$$u_1 \dots u_{n-i} x_{n-i+1} u_{n-i+2} \dots u_{n-1} y_{n-i} x_{n-i+2} u_{n-i+1} \dots u_{n-1} y$$

if $4 \leqq i < n-1$ and

$$q_{n-1} = v_1 x_2 u_4 \dots u_{n-1} x x_4 u_3 \dots u_{n-1} v_0 x_3 u_2 \dots u_{n-1} y x_2.$$

Now we show that $\mathbf{T}'_n$ can be simulated isomorphically by $\mathbf{A}^n(X, \varphi, \gamma)$ under μ and τ. The validity of the equations $\mu(\delta'_n(j, t_l)) = \delta_{\mathbf{A}^n}(\mu(j), \tau(t_l))$ $(l=2, 3)$ $(j=0, \ldots, n-1)$ can be checked by a simple computation.

Introduce the following notation

$$u_{jt}^{(i)} = \begin{cases} b & \text{if } \; j = t, \;\; j \leqq n-i-1 \;\; \text{or} \;\; t = 1, \;\; j > n-i-1 \\ & \qquad \text{or} \;\; t > n-i-1, \;\; t > j, \\ a & \text{otherwise}, \end{cases}$$

$1 \leqq i < n-2$, $0 \leqq t \leqq n-1$ and $0 \leqq j \leqq n-1$. It can be proved by induction on i that $\mu(j) q_1 \ldots q_i = (u_{j0}^{(i)}, \ldots, u_{jn-1}^{(i)})$ for any $j \in \{0, \ldots, n-1\}$ and $1 \leqq i < n-2$. On the other hand $(u_{j0}^{(n-3)}, \ldots, u_{jn-1}^{(n-3)}) q_{n-2} q_{n-1} = \mu(j+1 \pmod n)$ for any $j \in \{0, \ldots, n-1\}$. Therefore, $\mu(\delta'_n(j, t_1)) = \mu(j+1 \pmod n) = (u_{j0}^{(n-3)}, \ldots, u_{jn-1}^{(n-3)}) q_{n-2} q_{n-1} = \mu(j) q_1 \ldots q_{n-1} = \delta_{\mathbf{A}^n}(\mu(j), \tau(t_1))$ for any $j \in \{0, \ldots, n-1\}$. This ends the proof of the sufficiency of condition (2).

Now suppose that Σ satisfies (3) by $\mathbf{A} \in \Sigma$. Then there exist states $a \neq b$ of $\mathbf{A}$ and input words p, q, r such that $ap \neq bp$, $apq = bpq = a$ and $ar = b$. Observe that it is enough to prove the sufficiency of (3) for the case $a \notin \{ap, bp\}$. Indeed, assume that $a \in \{ap, bp\}$. We distinguish two cases. If $b \in \{ap, bp\}$ then p is a permutation of the set $\{a, b\}$ and thus the automaton $\mathbf{A}$ has the property required in (2). If $b \notin \{ap, bp\}$ then introducing the notations $a' = b$, $b' = a$, $p' = p$, $q' = qr$, $r' = pq$ we obtain that $a' \neq b'$, $a'p' \neq b'p'$, $a'p'q' = b'p'q' = a'$, $a'r' = b'$ and $a' \notin \{a'p', b'p'\}$. Therefore, without loss of generality we may assume that $a \notin \{ap, bp\}$. Now let $n \geqq 6$ be arbitrary and take the generalized ν_1-product $\mathbf{A}^n(X, \varphi, \gamma)$ where $X = \{x_1, \ldots, x_8\}$ and the mappings γ, φ are defined in the following way: for any $t \in \{0, \ldots, n-1\}$

$$\gamma(t) = \{t-1 \pmod n\}$$

$$\varphi_t(a, x_1) = pq, \quad \varphi_t(b, x_1) = r,$$

$$\varphi_t(a, x_2) = \begin{cases} p & \text{if } \; t = 1, \\ pqp & \text{otherwise}, \end{cases} \quad \varphi_t(b, x_2) = \begin{cases} p & \text{if } \; t = 2, \\ rp & \text{otherwise}, \end{cases}$$

$$\varphi_t(ap, x_3) = q, \quad \varphi_t(bp, x_3) = qr,$$

$$\varphi_t(a, x_4) = p, \quad \varphi_t(b, x_4) = \begin{cases} pq & \text{if } \; t = 1, \\ p & \text{otherwise}, \end{cases}$$

$$\varphi_t(a, x_5) = \begin{cases} qp & \text{if } \; b \neq ap, \\ p & \text{if } \; b = ap, \end{cases} \quad \varphi_t(ap, x_5) = q, \quad \varphi_t(bp, x_5) = \begin{cases} r & \text{if } \; t = 1, \\ qr & \text{if } \; t \neq 1, \end{cases}$$

$$\varphi_t(a, x_6) = p, \quad \varphi_t(b, x_6) = \begin{cases} q & \text{if } \; t = 2, \\ p & \text{otherwise}, \end{cases}$$

$$\varphi_t(ap, x_6) = \begin{cases} pq & \text{if } \; b \neq ap, \\ \varphi_t(b, x_6) & \text{otherwise}, \end{cases} \quad \varphi_t(bp, x_6) = \begin{cases} pq & \text{if } \; b = ap, \\ \varphi_t(b, x_6) & \text{otherwise}, \end{cases}$$

$$\varphi_t(a, x_7) = \begin{cases} p & \text{if} \quad b \neq ap, \quad t = 3, \\ qp & \text{if} \quad b \neq ap, \quad t \neq 3, \\ rp & \text{if} \quad b = ap, \quad t = 3, \\ qrp & \text{if} \quad b = ap, \quad t \neq 3, \end{cases}$$

$$\varphi_t(ap, x_7) = q, \quad \varphi_t(bp, x_7) = \begin{cases} r & \text{if} \quad t = 2, \\ qr & \text{otherwise}, \end{cases}$$

$$\varphi_t(a, x_8) = \begin{cases} p & \text{if} \quad t = 3, \\ pqp & \text{otherwise}, \end{cases} \quad \varphi_t(b, x_8) = \begin{cases} qp & \text{if} \quad t = 3, \\ p & \text{if} \quad t = 4, \\ rp & \text{otherwise}, \end{cases}$$

$$\varphi_t(ap, x_8) = \begin{cases} qrp & \text{if} \quad b \neq ap, \quad t = 4, \\ p & \text{if} \quad b \neq ap, \quad t = 5, \\ \varphi_t(b, x_8) & \text{if} \quad b = ap, \\ \text{an arbitrary input word otherwise}, \end{cases}$$

$$\varphi_t(bp, x_8) = \begin{cases} qrp & \text{if} \quad b = ap, \quad t = 4, \\ p & \text{if} \quad b = ap, \quad t = 5, \\ \varphi_t(b, x_8) & \text{if} \quad b \neq ap, \\ \text{an arbitrary input word otherwise}, \end{cases}$$

and in all other cases φ_t is defined arbitrarily. Take the following mappings

$$\mu: \begin{array}{l} 0 \;\;\;\;\; \to (b, a, \dots, a) \\ \vdots \\ n-1 \to (a, \dots, a, b) \end{array} \qquad \tau: \begin{array}{l} t_1 \to x_1, \\ t_2 \to x_4 x_5 x_6 x_7 x_8 x_3 x_1^{n-4}, \\ t_3 \to x_2 x_3 x_1^{n-2}. \end{array}$$

Distinguishing the cases $b=ap$ and $b\neq ap$ it can be seen easily that $\mu(\delta_n'(j), t_l)=\delta_{A^n}(\mu(j), \tau(t_l))$ for any $j\in\{0, \dots, n-1\}$ and $l\in\{1, 2, 3\}$ which yields the sufficiency of (3).

In order to prove the necessity assume that none of conditions (1)—(3) is satisfied by Σ and Σ is isomorphically S-complete with respect to the generalized ν_1-product. Since Σ does not satisfy (1) there exists a natural number $m>2$ such that Σ does not contain an automaton having the property required in (1) for any $n\geqq m$. Let $n>m^{\binom{m}{2}}$ be an arbitrary fixed natural number. By the assumption on the isomorphic S-completeness of Σ, there exists a generalized ν_1-product $\mathbf{B}=\prod_{t=0}^{k-1} \mathbf{A}_t(X, \varphi, \gamma)$ of automata from Σ such that $\mathbf{T}_n$ can be simulated isomorphically by $\mathbf{B}$ under suitable μ and τ. By Lemma 3, we may suppose that $\gamma(t)=\{t-1 \pmod k\}$ $(t=0, \dots, k-1)$. Let us denote by $(a_{l0}, \dots, a_{lk-1})$ the image of l under μ for any $l\in\{0, \dots, n-1\}$. Consider an arbitrary nonvoid subset $\Gamma=\{j_1, \dots, j_r\}$ of the set $\{0, \dots, k-1\}$. Define a relation π_Γ on $\prod_{t=0}^{k-1} A_t$ in the following way: $(a_0, \dots, a_{k-1})\pi_\Gamma(b_0, \dots, b_{k-1})$ if and only if $a_{j_s-\binom{m}{2}+u \pmod k}=$

$=b_{j_s-\binom{m}{2}+u(\mathrm{mod}\, k)}$ $(u=1, \ldots, \binom{m}{2})$, $(s=1, \ldots, r)$ for any $(a_0, \ldots, a_{k-1}), (b_0, \ldots, b_{k-1}) \in \in \prod_{t=0}^{k-1} A_t$. It is clear that π_Γ is an equivalence relation on $\prod_{t=0}^{k-1} A_t$. Now let us denote by $\bar{B}$ the set $\{(a_{l0}, \ldots, a_{lk-1}) \colon 0 \leqq l \leqq n-1\}$ and let $\bar{\pi}_\Gamma = \pi_\Gamma \cap (\bar{B} \times \bar{B})$.

We shall show that $(a_0, \ldots, a_{k-1}) \bar{\pi}_\Gamma (b_0, \ldots, b_{k-1})$ implies $(a_0, \ldots, a_{k-1}) \tau(t) \bar{\pi}_{\Gamma'}$ $(b_0, \ldots, b_{k-1}) \tau(t)$ for any $t \in T_n$ and $(a_0, \ldots, a_{k-1}), (b_0, \ldots, b_{k-1}) \in \prod_{t=0}^{k-1} A_t$, where $\Gamma' = \{j_s + |\tau(t)| (\mathrm{mod}\ k) \colon 1 \leqq s \leqq r\}$. Indeed, assume that $(a_0, \ldots, a_{k-1})\ \bar{\pi}_\Gamma (b_0, \ldots, b_{k-1})$ and let $t \in T_n$ be arbitrary. Since $\mathbf{T}_n$ can be simulated isomorphically by $\mathbf{B}$ there exist $t_1, t_2, t_3 \in T_n$ such that

$$(a_0, \ldots, a_{k-1}) \tau(t) \tau(t_1) = (b_0, \ldots, b_{k-1}) \tau(t) \tau(t_1),$$

$$(a_0, \ldots, a_{k-1}) \tau(t) \tau(t_1) \tau(t_2) = (b_0, \ldots, b_{k-1}),$$

$$(b_0, \ldots, b_{k-1}) \tau(t) \tau(t_1) \tau(t_3) = (a_0, \ldots, a_{k-1}).$$

Let $\tau(t) = x_1 \ldots x_j$, $\tau(t_1) = x_{j+1} \ldots x_{j+u}$, $\tau(t_2) = y_1 \ldots y_v$ and $\tau(t_3) = z_1 \ldots z_w$. Introduce the following notations

$$q_{1t}^{(1)} = \varphi_t(a_{t-1(\mathrm{mod}\, k)}, x_1) \quad (t = 0, \ldots, k-1),$$

$$q_{lt}^{(1)} = \varphi_t(a_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(1)} \ldots q_{l-1t-1(\mathrm{mod}\, k)}^{(1)}, x_l) \quad (t = 0, \ldots, k-1), \quad (2 \leqq l \leqq j+u),$$

$$q_{1t}^{(2)} = \varphi_t(b_{t-1(\mathrm{mod}\, k)}, x_1) \quad (t = 0, \ldots, k-1),$$

$$q_{lt}^{(2)} = \varphi_t(b_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(2)} \ldots q_{l-1t-1(\mathrm{mod}\, k)}^{(2)}, x_l) \quad (t = 0, \ldots, k-1), \quad (2 \leqq l \leqq j+u),$$

$$p_{1t} = \varphi_t(a_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(1)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(1)}, y_1) \quad (t = 0, \ldots, k-1),$$

$$p_{lt} = \varphi_t(a_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(1)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(1)} p_{1t-1(\mathrm{mod}\, k)} \ldots p_{l-1t-1(\mathrm{mod}\, k)}, y_l)$$
$$(t = 0, \ldots, k-1), \quad (2 \leqq l \leqq v),$$

$$r_{1t} = \varphi_t(b_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(2)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(2)}, z_1) \quad (t = 0, \ldots, k-1),$$

$$r_{lt} = \varphi_t(b_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(2)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(2)} r_{1t-1(\mathrm{mod}\, k)} \ldots r_{l-1t-1(\mathrm{mod}\, k)}, z_l)$$
$$(t = 0, \ldots, k-1), \quad (2 \leqq l \leqq w).$$

Then, by the above equations, we have that for any $t \in \{0, \ldots, k-1\}$

(i) $$a_t q_{1t}^{(1)} \ldots q_{j+ut}^{(1)} = b_t q_{1t}^{(2)} \ldots q_{j+ut}^{(2)},$$

(ii) $$a_t q_{1t}^{(1)} \ldots q_{j+ut}^{(1)} p_{1t} \ldots p_{vt} = b_t,$$

(iii) $$b_t q_{1t}^{(2)} \ldots q_{j+ut}^{(2)} r_{1t} \ldots r_{wt} = a_t.$$

Now let us denote by $(a_0^{(0)}, \ldots, a_{k-1}^{(0)})$, $(b_0^{(0)}, \ldots, b_{k-1}^{(0)})$ the states $(a_0, \ldots, a_{k-1})$, $(b_0, \ldots, b_{k-1})$ and $(a_0^{(i)}, \ldots, a_{k-1}^{(i)})$, $(b_0^{(i)}, \ldots, b_{k-1}^{(i)})$ the states $(a_0, \ldots, a_{k-1}) x_1 \ldots x_i$, $(b_0, \ldots, b_{k-1}) x_1 \ldots x_i$ $(i = 1, \ldots, j)$, respectively. To prove our statement we show that $(a_0, \ldots, a_{k-1})\ \bar{\pi}_\Gamma (b_0, \ldots, b_{k-1})$ implies $(a_0^{(i)}, \ldots, a_{k-1}^{(i)}) \pi_{\Gamma_i} (b_0^{(i)}, \ldots, b_{k-1}^{(i)})$ for any $0 \leqq i \leqq j$, where $\Gamma_i = \{j_s + i\, (\mathrm{mod}\ k) \colon 1 \leqq s \leqq r\}$. We proceed by induction on i. $(a_0^{(0)}, \ldots, a_{k-1}^{(0)}) \pi_{\Gamma_0} (b_0^{(0)}, \ldots, b_{k-1}^{(0)})$ obviously holds. Now assume that our statement has been proved for $i-1$ $(1\leqq i\leqq j)$. Then from $(a_0^{(i-1)}, \ldots, a_{k-1}^{(i-1)})\pi_{\Gamma_{i-1}}(b_0^{(i-1)}, \ldots$ $\ldots, b_{k-1}^{(i-1)})$ it follows that

$$a^{(i-1)}_{j_s-\binom{m}{2}+l+i-1(\mathrm{mod}\,k)} = b^{(i-1)}_{j_s-\binom{m}{2}+l+i-1(\mathrm{mod}\,k)} \quad \left(l = 1, \ldots, \binom{m}{2}\right), \quad (s = 1, \ldots, r).$$

Therefore, by the definition of $q_{il}^{(1)}, q_{il}^{(2)}$ we have that

$$q^{(1)}_{ij_s-\binom{m}{2}+l+i-1(\mathrm{mod}\,k)} = q^{(2)}_{ij_s-\binom{m}{2}+l+i-1(\mathrm{mod}\,k)} \quad \left(l = 2, \ldots, \binom{m}{2}+1\right), \quad (s=1, \ldots, r)$$

and thus $a^{(i)}_{j_s-\binom{m}{2}+l+i(\mathrm{mod}\,k)}=b^{(i)}_{j_s-\binom{m}{2}+l+i(\mathrm{mod}\,k)}\left(l=1, \ldots, \binom{m}{2}-1\right), (s=1, \ldots, r)$.

Now, if $a^{(i)}_{j_s+i(\mathrm{mod}\,k)}=b^{(i)}_{j_s+i(\mathrm{mod}\,k)}$ for all $1\leqq s\leqq r$ then we get that $(a_0^{(i)}, \ldots, a_{k-1}^{(i)})\pi_{\Gamma_i}(b_0^{(i)}, \ldots, b_{k-1}^{(i)})$ and so we are ready. In the opposite case there exists an index $s\in\{1, \ldots, r\}$ such that $a^{(i)}_{j_s+i(\mathrm{mod}\,k)}\neq b^{(i)}_{j_s+i(\mathrm{mod}\,k)}$. Let us denote by f the index $j_s+i\,(\mathrm{mod}\,k)$. Then $a_f^{(i)}\neq b_f^{(i)}$. From this, by $q_{if}^{(1)}=q_{if}^{(2)}$, it follows that $a_f^{(i-1)}\neq b_f^{(i-1)}$ and $a_f^{(i-1)}q_{if}^{(1)}\neq b_f^{(i-1)}q_{if}^{(1)}$. Now let $h=\min\left(j+u-i, \binom{m}{2}-1\right)$. Then, by $a^{(i)}_{f-\binom{m}{2}+l(\mathrm{mod}\,k)}= b^{(i)}_{f-\binom{m}{2}+l(\mathrm{mod}\,k)}$ $\left(l=1, \ldots, \binom{m}{2}-1\right)$, we have that $q^{(1)}_{i+lf}=$ $=q^{(2)}_{i+lf}$ $\left(l=1, \ldots, \binom{m}{2}-1\right)$. Therefore, $q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}=q^{(2)}_{i+1f}\ldots q^{(2)}_{i+hf}$. Now we show that $a_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}=b_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}$. Indeed, if $h=i+u-i$ then we get the required equality from (i). If $h=\binom{m}{2}-1$ then let us consider the sets M_l $(l=0, \ldots, h)$ defined by $M_0=\{a_f^{(i)}, b_f^{(i)}\}$ and $M_l=M_{l-1}q^{(1)}_{i+lf}$ $(l=1, \ldots, h)$. If $|M_l|=1$ for some $l\in\{1, \ldots, h\}$ then $a_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+lf}=b_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+lf}$ and thus $a_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}=$ $=b_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}$. Therefore, it is enough to consider the case for which $|M_l|=2$ for all $l\in\{0, \ldots, h\}$. If $M_g=M_l$ for some $0\leqq g<l\leqq h$ then $M_g p=M_l$ where $p=q^{(1)}_{i+g+1f}\ldots q^{(1)}_{i+lf}$. But in this case it can be seen easily that the automaton $\mathbf{A}_f$ has the property required in (2) which is a contradiction. Now consider the case for which $|M_l|=2$ for all $l\in\{0, \ldots, h\}$ and the sets M_l $(l=0, \ldots, h)$ are pairwise different. It is not difficult to see that from (ii) and (iii) it follows that for any $a, b\in \bigcup_{l=0}^{h} M_l$ there exists an input word p of $\mathbf{A}_f$ with $ap=b$. From this, by the definition m, we obtain that $\left|\bigcup_{l=0}^{h} M_l\right| =m'<m$. Thus we got that a set with cardinality $m'(<m)$ has $\binom{m}{2}$ pairwise different subsets of two elements which is a contradiction. Therefore, we have proved that $a_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}=b_f^{(i)}q^{(1)}_{i+1f}\ldots q^{(1)}_{i+hf}$. In this case, by (i), (ii), (iii), it can be seen easily that the automaton $\mathbf{A}_f$ with the states $a_f^{(i-1)}, b_f^{(i-1)}$ has the property required in (3) which is a contradiction. So we get a contradiction from the assumption $a^{(i)}_{j_s+i(\mathrm{mod}\,k)}\neq b^{(i)}_{j_s+i(\mathrm{mod}\,k)}$ for some $s\in\{1, \ldots, r\}$. Therefore, $a^{(i)}_{j_s+i(\mathrm{mod}\,k)}=b^{(i)}_{j_s+i(\mathrm{mod}\,k)}$ for all $s\in\{1, \ldots, r\}$ and thus $(a_0^{(i)}, \ldots, a_{k-1}^{(i)})\pi_{\Gamma_i}(b_0^{(i)}, \ldots, b_{k-1}^{(i)})$. From this, by $i=j$ we obtain that $(a_0, \ldots, a_{k-1})x_1\ldots x_j\pi_{\Gamma_j}(b_0, \ldots, b_{k-1})x_1\ldots x_j$ i.e. $(a_0, \ldots, a_{k-1})\tau(t)\pi_{\Gamma'}(b_0, \ldots, b_{k-1})\tau(t)$. On the other hand $(a_0, \ldots, a_{k-1})\tau(t)$, $(b_0, \ldots, b_{k-1})\tau(t)\in\bar{B}$ and thus $(a_0, \ldots, a_{k-1})\tau(t)\bar{\pi}_{\Gamma'}(b_0, \ldots, b_{k-1})\tau(t)$ which ends the proof of the statement.

Since $n>m^{\binom{m}{2}}$ there exists a subset $\Gamma\subsetneqq\{0, \ldots, k-1\}$ such that $\bar{\pi}_\Gamma\neq\Delta_B$, where Δ_B denotes the identity relation on $\bar{B}$. Therefore, the set $C=\{\Gamma\colon \Gamma\subsetneqq$ $\subsetneqq\{0, \ldots, k-1\}, \Gamma\neq\emptyset, \bar{\pi}_\Gamma\neq\Delta_B\}$ is nonempty. Then let us denote by $\Gamma=\{j_1, \ldots, j_r\}$ such an element of C for which $|\Gamma|$ is maximal. Since $\bar{\pi}_\Gamma\neq\Delta_B$ there exist $u\neq$ $\neq v\in\{0, \ldots, n-1\}$ with $\mu(u)\bar{\pi}_\Gamma\mu(v)$. Consider the element $t_1\in T_n$ defined by $t_1(u)=v$, $t_1(v)=u$ and $t_1(l)=l$ if $l\in\{0, \ldots, n-1\}\setminus\{u, v\}$. By the isomorphic simulation, we have that $\mu(u)\tau(t_1)=\mu(v)$, $\mu(v)\tau(t_1)=\mu(u)$ and $\mu(l)\tau(t_1)=\mu(l)$ if $l\in\{0,\dots,n-1\}\setminus\setminus\{u,v\}$. On the other hand $\mu(u)\bar{\pi}_\Gamma\mu(v)$ and thus $\mu(u)\tau(t_1)\bar{\pi}_{\Gamma'}\mu(v)\tau(t_1)$, where $\Gamma'=\{j_s+|\tau(t_1)|(\mathrm{mod}\ k): 1\leqq s\leqq r\}$. Therefore, $\mu(u)\bar{\pi}_{\Gamma'}\mu(v)$. It is clear that the mapping $\beta_1: t\to t+|\tau(t_1)|(\mathrm{mod}\ k)$ $(t=0,\dots,k-1)$ is a permutation of the set $\{0,\dots,k-1\}$ and thus $|\Gamma|=|\Gamma'|$. By the maximality of $|\Gamma|$ we have that $\Gamma'\subseteq\Gamma$ and thus $\Gamma=\Gamma'$. This means that the mapping β_1 fixes the set Γ, i.e. $\beta_1(\Gamma)=\Gamma$, where $\beta_1(\Gamma)$ denotes the set $\{\beta_1(t): t\in\Gamma\}$. On the other hand it is not difficult to see that β_1 fixes a subset M of the set $\{0,\dots,k-1\}$ if and only if

$$M=\{i,\ i+|\tau(t_1)|(\mathrm{mod}\ k),\ \dots,\ i+(f-1)|\tau(t_1)|(\mathrm{mod}\ k)\}$$

for some $i\in\{0,1,\dots,\text{g.c.d.}\,(k,|\tau(t_1)|)-1\}$ or M is equal to an union of such sets, where g.c.d. $(k,|\tau(t_1)|)$ denotes the greatest common divisor of the numbers $k,|\tau(t_1)|$ and $f=k/\text{g.c.d}\,(k,|\tau(t_1)|)$. Furthermore, it is clear that the considered sets $m_i=\{i,\ i+|\tau(t_1)|(\mathrm{mod}\ k),\dots,i+(f-1)|\tau(t_1)|(\mathrm{mod}\ k)\}$ form a partition of $\{0,\dots,k-1\}$. Thus assume that $\Gamma=\bigcup_{t=1}^{g} m_{i_t}$. Now consider the set $\bar{B}\setminus\{\mu(u),\mu(v)\}$. Since $n\geqq 3$ there exists an element $w\in\{0,\dots,n-1\}$ such that $\mu(w)\in\bar{B}\setminus\{\mu(u),\mu(v)\}$. Let us denote by t_2 a cyclic permutation from T_n with $t_2(u)=v$ and $t_2(v)=w$. By the isomorphic simulation we have that $\mu(u)\tau(t_2)=\mu(v)$ and $\mu(v)\tau(t_2)=\mu(w)$. On the other hand $\mu(u)\bar{\pi}_\Gamma\mu(v)$. Therefore, $\mu(u)\tau(t_2)\bar{\pi}_{\Gamma'}\mu(v)\tau(t_2)$ where $\Gamma'=\{j_s+|\tau(t_2)|(\mathrm{mod}\ k): 1\leqq s\leqq r\}$. Since the mapping $\beta_2: t\to t+|\tau(t_2)|(\mathrm{mod}\ k)$ $(t=0,\dots,k-1)$ is a permutation of $\{0,\dots,k-1\}$ we obtain that $|\Gamma|=|\Gamma'|$. Now we distinguish two cases.

First assume that $\Gamma=\Gamma'$. Then it is not difficult to see that $\mu(u)\bar{\pi}_\Gamma\mu(l)$ holds for any $l\in\{0,\dots,n-1\}$ which contradicts the maximality of $|\Gamma|$.

Now assume that $\Gamma\neq\Gamma'$. Observe that $\Gamma'=\bigcup_{t=1}^{g}\beta_2(m_{i_t})$ and $\beta_2(m_{i_t})=m_{i_t+|\tau(t_2)|(\mathrm{mod\ g.c.d.}\,(k,|\tau(t_1)|))}$. Therefore, from $|\Gamma|=|\Gamma'|$ and $\Gamma\neq\Gamma'$ it follows that there exists an index $j\in\{0,\dots,\text{g.c.d.}\,(k,|\tau(t_1)|)-1\}$ with $m_j\cap\Gamma=\emptyset$ and $m_j\subseteq\Gamma'$. On the other hand $\mu(v)\bar{\pi}_{\Gamma'}\mu(w)$ and thus $\mu(v)\tau(t_1)\bar{\pi}_{\Gamma''}\mu(w)\tau(t_1)$ where $\Gamma''=\beta_1(\Gamma')$. By $\mu(v)\tau(t_1)=\mu(u)$ and $\mu(w)\tau(t_1)=\mu(w)$ we obtain that $\mu(u)\bar{\pi}_{\Gamma''}\mu(w)$. Since β_1 fixes the sets m_i $(i=0,\dots,\text{g.c.d.}\,(k,|\tau(t_1)|)-1)$ we have that $m_j\subseteq\Gamma''$. Then $j\in\Gamma'$ and $j\in\Gamma''$ and thus

$$a_{vj-\binom{m}{2}+l(\mathrm{mod}\,k)}=a_{wj-\binom{m}{2}+l(\mathrm{mod}\,k)}\quad \left(l=1,\dots,\tbinom{m}{2}\right),$$

$$a_{wj-\binom{m}{2}+l(\mathrm{mod}\,k)}=a_{uj-\binom{m}{2}+l(\mathrm{mod}\,k)}\quad \left(l=1,\dots,\tbinom{m}{2}\right).$$

From this it follows that $j\in\Gamma$ which is a contradiction. This ends the proof of the necessity.

The next theorem holds for the generalized ν_i-product if $i>1$.

Theorem 3. A system Σ of automata is isomorphically S-complete with respect to the generalized ν_i-product $(i>1)$ *if and only if* Σ *contains an automaton which has two different states* a,b *and input words* p,q *such that* $ap=b$ *and* $bq=a$.

Proof. The necessity is obvious. Conversely, assume that Σ satisfies the condition of Theorem 3 by $\mathbf{A}$. Let $n\geqq 3$ be arbitrary and take the generalized ν_2-product

$\mathbf{A}^n(X, \varphi, \gamma)$ where $X = \{x_1, \ldots, x_6\}$ and the mappings γ, φ are defined in the following way: for any $t \in \{0, \ldots, n-1\}$

$$\gamma(t) = \{t, t-1 \,(\mathrm{mod}\, n)\},$$

$$\varphi_t(a, a, x_1) = pq, \ \varphi_t(a, b, x_1) = q, \varphi_t(b, a, x_1) = p,$$

$$\varphi_0(a, a, x_2) = \varphi_0(b, a, x_2) = p, \varphi_0(a, b, x_2) = q, \ \varphi_1(a, a, x_2) = pq, \ \varphi_1(a, b, x_2) = q,$$

$$\varphi_1(b, a, x_2) = p, \quad \varphi_t(u, v, x_2) = \begin{cases} pq & \text{if} \quad v = a, \\ qp & \text{if} \quad v = b, \quad (t = 2, \ldots, n-1), \end{cases}$$

$$\varphi_t(u, v, x_3) = \begin{cases} pq & \text{if} \quad v = a, \\ qp & \text{if} \quad v = b, \quad (t = 0, 1), \end{cases}$$

$$\varphi_t(u, v, x_3) = \begin{cases} p & \text{if} \quad v = a, \quad u = b, \\ pq & \text{if} \quad v = a, \quad u = a, \\ qp & \text{if} \quad v \neq a \quad (t = 2, \ldots, n-1), \end{cases}$$

$$\varphi_0(a, a, x_4) = \varphi_0(b, a, x_4) = pq, \quad \varphi_0(a, b, x_4) = qp, \quad \varphi_0(b, b, x_4) = q,$$

$$\varphi_t(u, v, x_4) = \begin{cases} pq & \text{if} \quad v = a, \\ qp & \text{if} \quad v = b, \quad (t = 1, \ldots, n-1), \end{cases}$$

$$\varphi_t(u, v, x_5) = \begin{cases} pq & \text{if} \quad v = a, \\ qp & \text{if} \quad v = b, \quad (t = 0, 1), \end{cases}$$

$$\varphi_t(u, v, x_5) = \begin{cases} q & \text{if} \quad u = v = b, \\ qp & \text{if} \quad u = a, \quad v = b, \\ pq & \text{if} \quad v = a, \quad (t = 2, \ldots, n-1), \end{cases}$$

$$\varphi_0(a, a, x_6) = \varphi_0(b, a, x_6) = p, \ \varphi_0(a, b, x_6) = qp,$$

$$\varphi_1(a, a, x_6) = \varphi_1(b, a, x_6) = pq, \quad \varphi_1(a, b, x_6) = q,$$

$$\varphi_t(u, v, x_6) = \begin{cases} pq & \text{if} \quad v = a, \\ qp & \text{if} \quad v = b, \quad (t = 2, \ldots, n-1). \end{cases}$$

In the remaining cases $\varphi_t(u, v, x_j)$ is an arbitrary input word from $\{p, q\}$. Now consider the mappings:

$$\mu\colon \begin{array}{rl} 0 & \to (b, a, \ldots, a), \\ 1 & \to (a, b, \ldots, a), \\ \vdots & \\ n-1 & \to (a, a, \ldots, b), \end{array} \qquad \tau\colon \begin{array}{l} t_1 \to x_1, \\ t_2 \to x_2 x_3^{n-3} x_4 x_5, \\ t_3 \to x_6 x_3^{n-3} x_4 x_5. \end{array}$$

It is not difficult to see that the automaton $\mathbf{T}'_n$ can be simulated isomorphically by $\mathbf{A}^n(X, \varphi, \gamma)$ under μ and τ.

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Decomposition results concerning K-visit attributed tree transducers

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The concept of attributed tree transducer was introduced in [1], [4] and [6]. On the other hand, the 1-visit, pure K-visit and simple K-visit classes of attributed grammars were defined in [3] and [5]. In this paper, we formulate these properties for deterministic attributed tree transducers defined in [6] and prove some decomposition results. Namely, we show that each tree transformation induced by a pure K-visit attributed tree transducer can be induced by a bottom-up tree transducer followed by an 1-visit attributed tree transducer. Here, the bottom-up tree transducer can be substituted by a top-down one. Moreover, each tree transformation induced by a simple K-visit attributed tree transducer can be induced by a deterministic bottom-up tree transducer followed by an 1-visit attributed tree transducer.

1. Notions and notations

By a type we mean a finite set F of the form $F=\bigcup_{n<\omega} F_n$ where the sets F_n are pairwise disjoint and $F_0 \neq \emptyset$.

For an arbitrary type F and set S the set of trees over S of type F is the smallest set $T_F(S)$ satisfying:

(i) $F_0 \cup S \subseteq T_F(S)$,

(ii) $f(p_1, \ldots, p_n) \in T_F(S)$ whenever $f \in F_n, p_1, \ldots, p_n \in T_F(S)$ $(n>0)$. If $S=\emptyset$ then $T_F(S)$ is written T_F.

The set of all positive integers is denoted by N. Let N^* denote the free monoid generated by N, with identity λ.

For a tree $p(\in T_F(S))$ the depth $(\mathrm{dp}\,(p))$, root $(\mathrm{root}\,(p))$, the set of subtrees $(\mathrm{sub}\,(p))$ of p and paths $(\mathrm{path}\,(p))$ of p as a subset of N^* are defined as follows:

(i) $\mathrm{dp}\,(p)=0$, $\mathrm{sub}\,(p)=\{p\}$, $\mathrm{root}\,(p)=p$, $\mathrm{path}\,(p)=\{\lambda\}$ if $p \in F_0 \cup S$,

(ii) $\mathrm{dp}\,(p)=1+\max\,\{\mathrm{dp}\,(p_i) \mid 1 \leqq i \leqq n\}$, $\mathrm{root}\,(p)=f$, $\mathrm{sub}\,(p)=\{p\} \cup \left(\bigcup (\mathrm{sub}\,(p_i) \mid 1 \leqq i \leqq n)\right)$, $\mathrm{path}\,(p)=\{\lambda\} \cup \{iv \mid 1 \leqq i \leqq n,\ v \in \mathrm{path}\,(p_i)\}$ if $p=f(p_1, \ldots, p_n)$ $(n>0, f \in F_n)$.
Subtrees of height 0 of a tree $p(\in T_F(S))$ are called leaves of p.

For each $p(\in T_F(S))$, $w(\in \text{path}(p))$ there is a corresponding label $\text{lb}_p(w)$ $(\in F \cup S)$ and a subtree $\text{str}_p(w)$ $(\in \text{sub}(p))$ in p which are defined by induction on the length of w:

(i) $\text{lb}_p(w) = \text{root}(p)$, $\text{str}_p(w) = p$ if $w = \lambda$,

(ii) $\text{lb}_p(w) = \text{lb}_{p_i}(v)$, $\text{str}_p(w) = \text{str}_{p_i}(v)$ if $w = iv$, $p = f(p_1, \dots, p_n)$, $1 \leqq i \leqq n$.

In the rest of this paper, F, G and H always mean types, moreover, the set of auxiliary variables $Z = \{z_0, z_1, \dots\}$ and its subsets $Z_n = \{z_1, \dots, z_n\}$ $(n = 0, 1, \dots)$ are kept fixed. Observe that $Z_0 = \emptyset$. Let $n \geqq 0$ and $p \in T_F(Z_n)$. Substituting the elements $s_1, \dots, s_n$ of a set S for $z_1, \dots, z_n$ in p, respectively, we have another tree, which is in $T_F(S)$ and denoted by $p(s_1, \dots, s_n)$. There is a distinguished subset $\hat{T}_F(Z_n)$ of $T_F(Z_n)$ defined as follows: $p \in \hat{T}_F(Z_n)$ if and only if each z_i $(1 \leqq i \leqq n)$ appears in p exactly once.

We now turn to the definition of tree transducers. The terminology used here follows [2].

Subsets of $T_F \times T_G$ are called tree transformations. The domain of a tree transformation $\tau(\subseteqq T_F \times T_G)$ is denoted by $\text{dom}\,\tau$ and defined by $\text{dom}\,\tau = \{p \in T_F | (p, q) \in \tau$ for some $q \in T_G\}$. The composition $\tau_1 \circ \tau_2$ of the tree transformations $\tau_1(\subseteqq T_F \times T_G)$ and $\tau_2(\subseteqq T_G \times T_H)$ is defined by $\tau_1 \circ \tau_2 = \{(p, q) | (p, r) \in \tau_1,\ (r, q) \in \tau_2$ for some $r\}$. If $\mathscr{C}_1$ and $\mathscr{C}_2$ are classes of tree transformations then their composition $\mathscr{C}_1 \circ \mathscr{C}_2$ is the class $\mathscr{C}_1 \circ \mathscr{C}_2 = \{\tau_1 \circ \tau_2 | \tau_1 \in \mathscr{C}_1, \tau_2 \in \mathscr{C}_2\}$.

By a bottom-up tree transducer we mean a system $\mathbf{A} = (F, A, G, A', P)$ where A is a nonempty finite set, the set of states, $A'(\subseteqq A)$ is the set of final states, moreover, P is a finite set of rewriting rules of the form $f(a_1 z_1, \dots, a_k z_k) \to aq$ where $k \geqq 0$, $f \in F_k$, $a, a_1, \dots, a_k \in A$, $q \in T_G(Z_k)$. $\mathbf{A}$ is said to be deterministic if different rules in P have different left sides. P can be used to define a binary relation $\underset{\mathbf{A}}{\Rightarrow}$ on the set $T_F(A \times T_G)$. The reflexive, transitive closure of $\underset{\mathbf{A}}{\Rightarrow}$ is denoted by $\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}$ and called derivation. The exact definition can be found in [2]. The tree transformation induced by $\mathbf{A}$ is a relation $\tau_{\mathbf{A}}(\subseteqq T_F \times T_G)$ defined by

$$\tau_{\mathbf{A}} = \{(p, q) | p \overset{*}{\underset{\mathbf{A}}{\Rightarrow}} aq \quad \text{for some} \quad a(\in A')\}.$$

A top-down tree transducer is again a system $\mathbf{A} = (F, A, G, A', P)$ which differs from the bottom-up one only in the form of the rewriting rules. Here, P is a finite set of rules of the form $af(z_1, \dots, z_k) \to q(a_1 z_{i_1}, \dots, a_l z_{i_l})$ where $k, l \geqq 0$, $f \in F_k$, $a, a_1, \dots, a_l \in A$, $1 \leqq i_1, \dots, i_l \leqq k$, $q \in \hat{T}_G(Z_l)$. Moreover, A' is called the set of initial states. The relation $\underset{\mathbf{A}}{\Rightarrow}$ can now be defined on the set $T_G(A \times T_F)$ and its reflexive, transitive closure is again denoted by $\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}$ (c.f. [2]). The tree transformation induced by $\mathbf{A}$ is a relation $\tau_{\mathbf{A}}(\subseteqq T_F \times T_G)$ defined by

$$\tau_{\mathbf{A}} = \{(p, q) | ap \overset{*}{\underset{\mathbf{A}}{\Rightarrow}} q \quad \text{for some} \quad a(\in A')\}.$$

The following concept of attributed tree transducer was defined in [6]. We repeat this definition, with a slightly different formalism, because this new one seems to be simpler. Moreover, we allow not only the completely defined but the partially defined case as well.

By a deterministic attributed tree transducer, or shortly DATT, we mean a system $\mathbf{A}=(F, A, G, a_0, P, \mathrm{rt})$ defined as follows:

(a) A is a finite set, the set of attributes, which is the union of the disjoint sets A_s and A_i where A_s is called the set of synthesized attributes, A_i is called the set of inherited attributes;

(b) $a_0 \in A_s$;

(c) rt is a partial mapping from A_i to T_G;

(d) P is a finite set of rewriting rules of the form

$$af(z_1, \ldots, z_k) \leftarrow \bar{q}(a_1 z_{j_1}, \ldots, a_l z_{j_l}) \tag{1}$$

where $k, l \geqq 0$, $f \in F_k$, $\bar{q} \in \hat{T}_G(Z_l)$, $a \in A_s$, $0 \leqq j_1, \ldots, j_l \leqq k$, $a_r \in A_i$ if $j_r = 0$ and $a_r \in A_s$ if $1 \leqq j_r \leqq k$ $(r=1, \ldots, l)$ as well as rules of the form

$$a(z_j, f) \leftarrow \bar{q}(a_1 z_{j_1}, \ldots, a_l z_{j_l}) \tag{2}$$

where $f \in F_k$ for some $k(\geqq 1)$, $l \geqq 0$, $a \in A_i$, $1 \leqq j \leqq k$, $\bar{q} \in \hat{T}_G(Z_l)$, $0 \leqq j_1, \ldots, j_l \leqq k$ and a_r is the same as above $(r=1, \ldots, l)$. Any two different rules of P are required to have different left sides.

From now on, for the sake of convenience we shall use the following notation for each element x of the set $N \cup \{0\}$

$$\bar{x} = \begin{cases} x & \text{if } x \in N \\ \lambda & \text{if } x = 0. \end{cases}$$

Let $p \in T_F$. We can define the relation $\underset{p,\mathbf{A}}{\Longleftarrow}$ on the set $T_G(A \times \mathrm{path}(p))$ in the following way. For $q, r (\in T_G(A \times \mathrm{path}(p)))$ $q \underset{p,\mathbf{A}}{\Longleftarrow} r$ if r is obtained from q by substituting the tree $\bar{q}((a_1, v_1), \ldots, (a_l, v_l))$ for some leaf $(a, w) (\in A \times \mathrm{path}(p))$ of q if either the condition (a) or (b) holds:

(a) (i) $a \in A_s$,
(ii) $\mathrm{lb}_p(w) = f (\in F_k$ for some $k \geqq 0)$,
(iii) the rule (1) is in P,
(iv) $v_r = w\bar{j}_r$ $(r=1, \ldots, l)$;

(b) (i) $a \in A_i$,
(ii) $w = vj$ for some $j (\in N)$,
(iii) $\mathrm{lb}_p(v) = f (\in F_k$ for some $k \geqq 1)$,
(iv) the rule (2) is in P,
(v) $v_r = v\bar{j}_r$ $(r=1, \ldots, l)$.

Observe that a leaf of q which is in $A_i \times \{\lambda\}$ can never be substituted because, for such a leaf, neither (a) nor (b) can hold. Therefore we define the relation "$\underset{p,\mathbf{A}}{\Longleftarrow}$ concerning rt" which contains $\underset{p,\mathbf{A}}{\Longleftarrow}$ in the following manner: $q \underset{p,\mathbf{A}}{\Longleftarrow} r$ concerning rt if either $q \underset{p,\mathbf{A}}{\Longleftarrow} r$ or r is obtained from q by substituting $\mathrm{rt}(a)$ (if it exists) for a leaf $(a, \lambda) (\in A_i \times \{\lambda\})$ of q. Let the n-th power, transitive closure, reflexive, transitive closure of $\underset{p,\mathbf{A}}{\Longleftarrow}$ be denoted by $\overset{n}{\underset{p,\mathbf{A}}{\Longleftarrow}}$, $\overset{+}{\underset{p,\mathbf{A}}{\Longleftarrow}}$, $\overset{*}{\underset{p,\mathbf{A}}{\Longleftarrow}}$, respectively, and similarly for the relation $\underset{p,\mathbf{A}}{\Longleftarrow}$ concerning rt. We can now define the tree transformation $\tau_{\mathbf{A}}(\subseteq T_F \times T_G)$ induced by $\mathbf{A}$ in the following way

$$\tau_{\mathbf{A}} = \{(p, q) | (a_0, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q \text{ concerning rt}\}.$$

An example for a DATT can be found in [6]. The relation $\overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}}$ is called derivation. The length $\operatorname{lt}(\alpha)$ of a derivation $\alpha = q \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} r$ is defined as the integer n for which $q \overset{n}{\underset{p, \mathbf{A}}{\Leftarrow}} r$.

In the rest of this paper, by a DATT we always mean a noncircular DATT (see [6]).

Before going on, we make an observation which will often be used without reference. Let $p \in T_F$, $w \in \operatorname{path}(p)$, $l \geq 0$, $q \in \hat{T}_G(Z_l)$, $a \in A_s$, $a_1, \dots, a_l \in A_i$ and let $\operatorname{str}_p(w)$ be denoted by p_w.

Suppose that

$$(a, w) \overset{n}{\underset{p, \mathbf{A}}{\Leftarrow}} q((a_1, w), \dots, (a_l, w)) \tag{3}$$

and there is no step in (3), in which, a leaf in $A_i \times \{w\}$ is substituted. Then

$$(a, \lambda) \overset{n}{\underset{p_w, \mathbf{A}}{\Leftarrow}} q((a_1, \lambda), \dots, (a_l, \lambda))$$

and the converse also holds.

The classes of all tree transformations induced by top-down tree transducers, (deterministic) bottom-up tree transducers, deterministic attributed tree transducers are denoted by $\mathcal{T}$, $(\mathcal{D})\mathcal{B}$, $\mathcal{DA}$, respectively.

2. K-visit attributed tree transducers

Let $\mathbf{A}(=(F, A, G, a_0, P, \text{rt}))$ be a DATT and let $K(\geq 1)$ be an integer.

By a partition of A we mean a sequence $((I_1, S_1), \dots, (I_l, S_l))$ where I_j (S_j) are pairwise disjoint subsets of A_i (A_s) whose union is A_i (A_s). Let $\Phi_K(A)$ denote the set of all partitions of A with $l \leq K$.

Now let $f \in F_k$ $(k \geq 0)$, $\mathbf{e}^i \in \Phi_K(A)$ with $\mathbf{e}^i = ((I_1^i, S_1^i), \dots, (I_{l_i}^i, S_{l_i}^i))$ $(i = 0, 1, \dots, k)$. The oriented graph $D_f(\mathbf{e}^0, \mathbf{e}^1, \dots, \mathbf{e}^k)$ is defined as follows. Its nodes are the symbols I_j^λ, S_j^λ $(j = 1, \dots, l_0)$ and the symbols I_j^i, S_j^i $(i = 1, \dots, k, j = 1, \dots, l_i)$. Edges are oriented for each

(i) $j(=1, \dots, l_0)$ from I_j^λ to S_j^λ;

(ii) $j(=1, \dots, l_0 - 1)$ from S_j^λ to I_{j+1}^λ;

(iii) $i(=1, \dots, k)$, $j(=1, \dots, l_i)$ from I_j^i to S_j^i;

(iv) $i(=1, \dots, k)$, $j(=1, \dots, l_i - 1)$ from S_j^i to I_{j+1}^i;

(v) $j(=1, \dots, l_0)$, $a(\in S_j^0)$ from $X_r^{i_s}$ to S_j^λ if there is a rule $af(z_1, \dots, z_k) \leftarrow q(a_1 z_{i_1}, \dots, a_l z_{i_l})$ in P for which $a_s \in X_r^{i_s}$ under some $s(=1, \dots, l)$, $r(=1, \dots, l_{i_s})$, $X \in \{I, S\}$;

(vi) $i(=1, \dots, k)$, $j(=1, \dots, l_i)$, $a(\in I_j^i)$ from $X_r^{i_s}$ to I_j^i if there is a rule $a(z_i, f) \leftarrow q(a_1 z_{i_1}, \dots, a_l z_{i_l})$ in P with $a_s \in X_r^{i_s}$ under some s, r, X defined as in (v).

The graph $D_f(\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^k)$ corresponds to the concept of partition graph for a production of an attribute grammar, which concept was introduced in [5].

Let $p(=f(p_1, \ldots, p_k)) \in T_F$ $(k>0, f \in F_k)$ and consider a mapping π: path $(p) \to \Phi_K(A)$. The mappings π^i: path $(p_i) \to \Phi_K(A)$ are defined by $\pi^i(w)=\pi(iw)$ $(i=1, \ldots, k,\ w \in \text{path}(p_i))$.

Now, let again $p \in T_F$ and π: path $(p) \to \Phi_K(A)$. The oriented graph $D_p(\pi)$ is defined by induction on $\text{dp}(p)$:

(i) if $p=f(\in F_0)$ with $\pi(\lambda)=\mathbf{e}$ then $D_p(\pi)=D_f(\mathbf{e})$;

(ii) if $p=f(p_1, \ldots, p_k)$ $(k>0,\ f \in F_k)$ with $\pi(\lambda)=\mathbf{e}$, $\pi(i)=\mathbf{e}^i$ $(i=1, \ldots, k)$ then $D_p(\pi)=D_f(\mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k) \cup (\cup (D'_{p_i}(\pi^i) \mid 1 \leqq i \leqq k))$ where $D'_{p_i}(\pi^i)$ is obtained from $D_{p_i}(\pi^i)$ by "multiplying its nodes by i", that is, the nodes of $D'_{p_i}(\pi^i)$, are the symbols X_r^{iw} where X_r^w are nodes of $D_{p_i}(\pi^i)$, moreover, there is an edge from X_r^{iw} to Y_s^{iv} in $D'_{pi}(\pi^i)$ iff there is an edge from X_r^w to Y_s^v in $D_{p_i}(\pi^i)$. Nodes and edges of graphs are combined as sets.

Definition 1. We say that $\mathbf{A}$ is pure K-visit, if for each $p(\in \text{dom}\, \tau_{\mathbf{A}})$ there exists a π: path $(p) \to \Phi_K(A)$ with acyclic $D_p(\pi)$.

To support this definition, the following observation can be made. If $D_p(\pi)$ is acyclic then a computation sequence (see in [5] for attribute grammars) can be constructed, which induces a K-visit tree-walking attribute evaluation strategy on p.

Definition 2. Suppose that to each $f(\in F)$ there corresponds an element $\mathbf{e}^f$ of $\Phi_K(A)$ and let $\Pi_K=\{\mathbf{e}^f \mid f \in F\}$. $\mathbf{A}$ is said to be simple K-visit concerning Π_K if for each $p(\in \text{dom}\, \tau_{\mathbf{A}})$ there exists a π: path $(p) \to \Pi_K$ for which the following two conditions hold:

(i) if $\text{lb}_p(w)=f$ then $\pi(w)=\mathbf{e}^f$ $(w \in \text{path}(p))$,

(ii) $D_p(\pi)$ is acyclic.

$\mathbf{A}$ is simple K-visit, if it is simple K-visit concerning some Π_K.

The classes of all tree transformations induced by pure, simple K visit DATTs are denoted by $\mathscr{DA}_{PK}$, $\mathscr{DA}_{SK}$, respectively. Observe, that $\Phi_1(A)=\{(A_i, A_s)\}$ so, in the particular case $K=1$, the two properties defined above are identical. Therefore $\mathscr{DA}_{P1}=\mathscr{DA}_{S1}$ and they can be denoted by $\mathscr{DA}_1$.

Theorem 3. For each $K(\geqq 1)$, $\mathscr{DA}_{PK} \subset \mathscr{B} \circ \mathscr{DA}_1$.

Proof. Let $\mathbf{A}(=(F, A, G, a_0, P, \text{rt}))$ be a pure K-visit DATT. Consider the bottom-up tree transducer $\mathbf{B}(=(F, B, \bar{F}, B', P'))$ where

(a) $B=B'=\Phi_K(A)$;

(b) for each $m(\geqq 0)$, $\bar{F}_m$ is defined as follows $\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle \in \bar{F}_m$ if and only if

(i) $f \in F_k$ for some $k(\geqq 0)$,

(ii) $\mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k \in \Phi_K(A)$,

(iii) $m=l_1+\ldots+l_k$ where l_i is the number of components of $\mathbf{e}^i$ $(i=1, \ldots, k)$,

(iv) $D_f(\mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k)$ is acyclic;

(c) for each $m(\geqq 0)$, $\langle f;\ \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle$ $(\in \bar{F}_m)$ the rule

$$f(\mathbf{e}^1 z_1, \ldots, \mathbf{e}^k z_k) \to \mathbf{e}\langle f;\ \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle(\overbrace{z_1, \ldots, z_1}^{l_1 \text{ times}}, \ldots, \overbrace{z_k, \ldots, z_k}^{l_k \text{ times}})$$

is in P'.

Moreover, let the DATT $\mathbf{C}=(\bar{F}, C, G, c_0, P'', \mathrm{rt}'')$ be defined as follows

(a) $C_s=A_s$, $C_i=A_i$, $c_0=a_0$, $\mathrm{rt}''=\mathrm{rt}$;

(b) P'' is constructed in the following way. Let $m\geqq 0$, $\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle \in \bar{F}_m$ with $\mathbf{e}=((I_1, S_1), \ldots, (I_l, S_l))$ and $\mathbf{e}^j=((I_1^j, S_1^j), \ldots, (I_{l_j}^j, S_{l_j}^j))$ $(1\leqq j\leqq k)$. For each $a(\in C_s)$ let the rule $a\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle\,(z_1, \ldots, z_m)\leftarrow q(a_1 z_{i_1}, \ldots, a_s z_{i_s})$ be in P'' if the following conditions hold:

(i) $$af(z_1, \ldots, z_k) \leftarrow q(a_1 z_{j_1}, \ldots, a_s z_{j_s})\in P,$$

(ii) $$i_r = \begin{cases} j_r(=0) & \text{if } \ a_r\in A_i \qquad\qquad\qquad (r=1, \ldots, s) \\ l_1+\ldots+l_{j_r-1}+n & \text{if } \ a_r\in S_n^{j_r} \ \text{ for some } \ n(=1, \ldots, l_{j_r}). \end{cases}$$

Moreover, for each $j(=1, \ldots, k)$, $n(=1, \ldots, l_j)$, $a(\in I_1^j\cup\ldots\cup I_n^j)$ let the rule $a(z_i, \langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle)\leftarrow q(a_1 z_{i_1}, \ldots, a_s z_{i_s})$ be in P'' if

(i) $$a(z_j, f) \leftarrow q(a_1 z_{j_1}, \ldots, a_s z_{j_s})\in P,$$

(ii) $$i = l_1+\ldots+l_{j-1}+n,$$

$$i_r = \begin{cases} j_r(=0) & \text{if } \ a_r\in A_i \qquad\qquad\qquad (r=1, \ldots, s) \\ l_1+\ldots+l_{j_r-1}+u & \text{if } \ a_r\in S_u^{j_r} \ \text{ for some } \ u=(1, \ldots, l_{j_r}). \end{cases}$$

The 1-visit property of $\mathbf{C}$ can be shown in the following manner. In [3], it was proved that an attributed grammar is 1-visit iff each of its brother graphs is acyclic. We can formulate the concept of the brother graph for DATTs and can easily show that each brother graph of $\mathbf{C}$ is acyclic.

The proof of the next lemma can be performed by a simple induction on $\mathrm{dp}\,(p)$.

Lemma 4. Let $p\in T_F$, $\mathbf{e}\in B$. Then $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ for some $\bar{q}(\in T_{\bar{F}})$ if and only if there exists a $\pi\colon \mathrm{path}\,(p)\to\Phi_K(A)$ with $\pi(\lambda)=\mathbf{e}$ and acyclic $D_p(\pi)$.

Lemma 5. Let $p\in T_F, \bar{q}\in T_{\bar{F}}, q\in \hat{T}_G(Z_s), a_1, \ldots, a_s\in A_i, \mathbf{e}\in B$ with $\mathbf{e}=((I_1, S_1), \ldots$ $\ldots, (I_l, S_l))$ and let $a\in S_j$ for some $j(=1, \ldots, l)$. Suppose that $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ and $(a, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q((a_1, \lambda), \ldots, (a_s, \lambda))$. Then $a_1, \ldots, a_s\in I_1\cup\ldots\cup I_j$.

Proof. It follows from the previous lemma that there exists a $\pi\colon \mathrm{path}\,(p)\to$ $\to\Phi_K(A)$ with $\pi(\lambda)=\mathbf{e}$ and acyclic $D_p(\pi)$. Suppose that, say, $a_1\in I_k$ where $k>j$. Then, by the definition of $D_p(\pi)$, there is a path from I_k^λ to S_j^λ in $D_p(\pi)$ due to the dependency edges of $D_p(\pi)$. On the other hand, there is a path from S_j^λ to I_k^λ in $D_p(\pi)$ because $k>j$, which contradicts the fact that $D_p(\pi)$ is acyclic.

Lemma 6. Let $a\in A_s$, $p\in T_F$, $\bar{q}\in T_{\bar{F}}$, $q\in T_G(A_i\times\{\lambda\})$, $\mathbf{e}\in B$. Suppose that $(a, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q$ and $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$. Then $(a, \lambda) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q$.

Proof. The proof can be performed by induction on $\mathrm{dp}\,(p)$.

(a) Let $\mathrm{dp}\,(p)=0$ i.e. $p=f(\in F_0)$. Then by supposition, $af\leftarrow q'(a_1 z_0, \ldots, a_s z_0)\in P$ $(s\geqq 0, q'\in\hat{T}_G(Z_s), a_1, \ldots, a_s\in A_i)$, $q=q'((a_1, \lambda), \ldots, (a_s, \lambda))$, moreover, $f\to\mathbf{e}\langle f; \mathbf{e}\rangle\in P'$ and $\bar{q}=\langle f; \mathbf{e}\rangle$. Therefore, by the definition of $\mathbf{C}$, $a\langle f; \mathbf{e}\rangle\leftarrow q'(a_1 z_0, \ldots, a_s z_0)\in P''$.

(b) Now let $\mathrm{dp}(p)>0$ that is $p=f(p_1, \dots, p_k)$ $(k>0, f\in F_k)$. Here, $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ can be written in the form

$$p = f(p_1, \dots, p_k) \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} f(\mathbf{e}^1\bar{q}_1, \dots, \mathbf{e}^k\bar{q}_k) \underset{\mathbf{B}}{\Rightarrow}$$

$$\mathbf{e}\langle f;\ \mathbf{e}, \mathbf{e}^1, \dots, \mathbf{e}^k\rangle (\overbrace{\bar{q}_1, \dots, \bar{q}_1}^{l_1 \text{ times}}, \dots, \overbrace{\bar{q}_k, \dots, \bar{q}_k}^{l_k \text{ times}}) = \mathbf{e}\bar{q}$$

with

$$\mathbf{e}^j = ((I_1^j, S_1^j), \dots, (I_{l_j}^j, S_{l_j}^j)) \quad (j = 1, \dots, k).$$

First we can prove the following

STATEMENT. Let $1\leqq j\leqq k$, $1\leqq n\leqq l_j$, $b\in I_1^j\cup\dots\cup I_n^j$, $t\in T_G(A_i\times\{\lambda\})$ and suppose that the relation $\beta=(b, j) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t$ holds. Then $(b, i) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} t$ where $i=l_1+\dots+l_{j-1}+n$.

The proof of this statement can be done by an induction on $\mathrm{lt}(\beta)$. When $\mathrm{lt}(\beta)=1$ then $b(z_j, f)\leftarrow t'(b_1z_0, \dots, b_sz_0)\in P$ $(s\geqq 0, t'\in T_G(Z_s), b_1, \dots, b_s\in A_i)$ and $t=t'((b_1, \lambda), \dots, (b_s, \lambda))$ so, by the definition of $\mathbf{C}$, $b(z_i, f)\leftarrow t'(b_1z_0, \dots, b_sz_0)\in P''$.
When $\mathrm{lt}(\beta)>1$ then β can be written in the following form

$$(b, j) \underset{p,\mathbf{A}}{\Leftarrow} t'((b_1, j_1), \dots, (b_s, j_s)) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t'(t_1, \dots, t_s)=t$$

where

$s\geqq 0$, $t'\in \hat{T}_G(Z_s)$, $b_1, \dots, b_s\in A$, $t_1, \dots, t_s\in T_G(A_i\times\{\lambda\})$, $b(z_j, f)\leftarrow t'(b_1z_{j_1}, \dots, b_sz_{j_s})\in P$.
Then, by the definition of $\mathbf{C}$, $b(z_i, \langle f; \mathbf{e}, \mathbf{e}^1, \dots, \mathbf{e}^k\rangle)\leftarrow t'(b_1z_{i_1}, \dots, b_sz_{i_s})\in P''$ where

$$i_r = \begin{cases} j_r(=0) & \text{if } \ b_r\in A_i \qquad\qquad (r = 1, \dots, s)\\ l_1+\dots+l_{j_r-1}+v & \text{if } \ b_r\in S_v^{j_r} \ \text{ for some } \ v(=1, \dots, l_{j_r}).\end{cases}$$

Now let $r(=1, \dots, s)$ be such an index for which $b_r\in S_v^{j_r}$ and so $1\leqq j_r\leqq k$. Then the relation $(b_r, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r$ can be written in the form $(b_r, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r'((c_1, j_r), \dots$ $\dots, (c_u, j_r)) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r'(\bar{t}_1, \dots, \bar{t}_u)=t_r$ for some $u(\geqq 0)$, $t_r'(\in\hat{T}_G(Z_u))$, $c_1, \dots, c_u(\in A_i)$, $\bar{t}_1, \dots$ $\dots, \bar{t}_u(\in T_G(A_i\times\{\lambda\})$ and we can suppose that the derivation $(b_r, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r'((c_1, j_r), \dots$ $\dots, (c_u, j_r))$ has no such a step, in which, a leaf in $A_i\times\{j_r\}$ substituted. Then $(b_r, \lambda) \overset{*}{\underset{p_{j_r},\mathbf{A}}{\Leftarrow}} t_r'((c_1, \lambda), \dots, (c_u, \lambda))$ so, by the induction hypothesis concerning $\mathrm{dp}(p)$, we have $(b_r, \lambda) \overset{*}{\underset{\bar{q}_{j_r},\mathbf{C}}{\Leftarrow}} t_r'((c_1, \lambda), \dots, (c_u, \lambda))$ which means that $(b_r, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} t_r'((c_1, i_r), \dots$ $\dots, (c_u, i_r))$ because $\mathrm{lb}_{\bar{q}}(i_r)=\bar{q}_{j_r}$. On the other hand, by Lemma 5, $c_1, \dots, c_u\in I_1^{j_r}\cup\dots$ $\dots\cup I_v^{j_r}$, moreover, the length of each of the derivations $(c_1, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} \bar{t}_1, \dots$ $\dots, (c_u, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} \bar{t}_u$ is less than $\mathrm{lt}(\beta)$ so we have $(c_1, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} \bar{t}_1, \dots, (c_u, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} \bar{t}_u$, that is $(b_r, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} t_r'(\bar{t}_1, \dots, \bar{t}_u)=t_r$.

If r is such an index for which $b_r\in A_i$ and so $j_r=0$ then $t_r=(b_r, \lambda)$, therefore

$(b_r, \bar{l}_r) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} t_r$ again. All that means that

$$(b, i) \underset{\bar{q}, \mathbf{C}}{\Leftarrow} t'((b_1, \bar{l}_1), \ldots, (b_s, \bar{l}_s)) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} t'(t_1, \ldots, t_s) = t$$

proving our statement.

Now we return to the induction step of the lemma. The relation $(a, \lambda) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q$ can be written in the form

$$(a, \lambda) \underset{p, \mathbf{A}}{\Leftarrow} q'((a_1, j_1), \ldots, (a_s, j_s)) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q'(q_1, \ldots, q_s) = q$$

where $s \geqq 0$, $q' \in \hat{T}_G(Z_s)$, $a_1, \ldots, a_s \in A$, $q_1, \ldots, q_s \in T_G(A_i \times \{\lambda\})$ and $af(z_1, \ldots, z_k) \leftarrow q'(a_1 z_{j_1}, \ldots, a_s z_{j_s})$ is in P. Then, by the definition of $\mathbf{C}$, the rule $a\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle (z_1, \ldots, z_m) \leftarrow q(a_1 z_{i_1}, \ldots, a_s z_{i_s})$ is in P'' where $m = l_1 + \ldots + l_k$ and

$$i_r = \begin{cases} j_r (= 0) & \text{if} \quad a_r \in A_i \qquad\qquad (r = 1, \ldots, s) \\ l_1 + \ldots + l_{j_r - 1} + n & \text{if} \quad a_r \in S_n^{j_r} \quad \text{for some} \quad n (= 1, \ldots, l_{j_r}). \end{cases}$$

Let $r (= 1, \ldots, s)$ be an index for which $a_r \in S_n^{j_r}$ for some $n (= 1, \ldots, l_{j_r})$ and so $1 \leqq j_r \leqq k$. Then the relation $(a_r, \bar{j}_r) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q_r$ can be written in the form $(a_r, j_r) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q_r'((b_1, j_r), \ldots, (b_u, j_r)) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q_r'(\bar{q}_1, \ldots, \bar{q}_u) = q_r$ for some $u \geqq 0$, $q_r' \in \hat{T}_G(Z_u)$, $b_1, \ldots, b_u \in A_i$, $\bar{q}_1, \ldots, \bar{q}_u \in T_G(A_i \times \{\lambda\})$. We can again suppose, that there is no step in the derivation $(a_r, j_r) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q_r'((b_1, j_r), \ldots, (b_u, j_r))$, in which, a leaf in $A_i \times \{j_r\}$ is substituted. Therefore $(a_r, \lambda) \underset{p_{j_r}, \mathbf{A}}{\overset{*}{\Leftarrow}} q_r'((b_1, \lambda), \ldots, (b_u, \lambda))$ from which, by Lemma 5, $b_1, \ldots, b_u \in I_1^{j_r} \cup \ldots \cup I_n^{j_r}$ and, by the induction hypothesis on $\mathrm{dp}(p)$, we get $(a_r, \lambda) \underset{\bar{q}_{j_r}, \mathbf{C}}{\overset{*}{\Leftarrow}} q_r'((b_1, \lambda), \ldots, (b_u, \lambda))$ that is $(a_r, i_r) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} q_r'((b_1, i_r), \ldots, (b_u, i_r))$. On the other hand, by the statement, we have $(b_1, i_r) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} \bar{q}_1, \ldots, (b_u, i_r) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} \bar{q}_u$ which means that $(a_r, i_r) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} q_r'(\bar{q}_1, \ldots, \bar{q}_u) = q_r$.

If $r (= 1, \ldots, s)$ is such an index for which $a_r \in A_i$ and so $j_r = 0$ then it is clear that $q_r = (a_r, \lambda)$, therefore $(a_r, \bar{l}_r) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} q_r$ again. The two cases of r and $a\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle (z_1, \ldots, z_m) \leftarrow q'(a_1 z_{i_1}, \ldots, a_s z_{i_s}) \in P''$ together prove that

$$(a, \lambda) \underset{\bar{q}, \mathbf{C}}{\Leftarrow} q'((a_1, \bar{l}_1), \ldots, (a_s, \bar{l}_s)) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} q'(q_1, \ldots, q_s) = q.$$

This ends the proof of Lemma 6.

The proof of the next lemma is essentially the converse of the previous one.

Lemma 7. Let $a \in A_s$, $p \in T_F$, $\bar{q} \in T_F$, $q \in T_G(A_i \times \{\lambda\})$, $\mathbf{e} \in B$. Suppose that $p \underset{\mathbf{B}}{\overset{*}{\Rightarrow}} \mathbf{e}\bar{q}$ and $(a, \lambda) \underset{\bar{q}, \mathbf{C}}{\overset{*}{\Leftarrow}} q$. Then $(a, \lambda) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q$.

Now we are ready to prove our theorem. Suppose that $(p, q) \in \tau_{\mathbf{A}}$ that is $(a_0, \lambda) \underset{p, \mathbf{A}}{\overset{*}{\Leftarrow}} q$ concerning rt. Because $\mathbf{A}$ is K-visit, by Lemma 4, there exist $\bar{q} \in T_F$

and $\mathbf{e} \in B$ with $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$, therefore, by Lemma 6, $(a_0, \lambda) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q$ concerning rt″, hence $(p, q) \in \tau_{\mathbf{B}} \circ \tau_{\mathbf{C}}$. Conversely, by $(p, q) \in \tau_{\mathbf{B}} \circ \tau_{\mathbf{C}}$ we have a $\bar{q} \in T_F$ for which $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ under some $\mathbf{e}(\in B)$ and $(a_0, \lambda) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q$ concerning rt″. Then, by Lemma 7, we have $(a_0, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q$ concerning rt. The fact, that the inclusion is strict follows from the proof of Theorem 4.1 of [6]. This ends the proof of Theorem 3.

After studying the proof of the previous theorem two observation can be made. On the one hand, instead of the bottom-up tree transducer **B** we can have a top-down one which can be constructed by reversing the rewriting rules of **B**. Although this top-down one does not induce the same tree transformation as **B**, the following will be valid.

Corollary 8. $\mathscr{D}\mathscr{A}_{PK} \subset \mathscr{T} \circ \mathscr{D}\mathscr{A}_1.$

On the other hand it also seems that if **A** is simple K-visit then a deterministic bottom-up tree transducer can be constructed, so we have

Corollary 9. $\mathscr{D}\mathscr{A}_{SK} \subset \mathscr{D}\mathscr{B} \circ \mathscr{D}\mathscr{A}_1.$

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On a representation of deterministic uniform root-to-frontier tree transformations

By F. GÉCSEG

The concepts of products and complete systems of finite automata can be generalized for ascending algebras in a natural way (see [4]). Results in finite automata theory imply that for most types of products there are no finite complete systems of ascending algebras. Therefore, it is reasonable to investigate a weaker form of completeness to be called m-completeness when tree transformations are represented up to a finite but not bounded height. In this paper we give necessary and sufficient conditions under which a system of ascending algebras is m-complete for the class of all deterministic uniform root-to-frontier tree transformations with respect to different kinds of products. Moreover, we show the existence of such finite m-complete systems.

1. Notions and notations

The terms "node of a tree" and "subtree at a given node of a tree" will be used in an informal and obvious way.

The symbol R will stand for a nonvoid finite rank type with $0 \notin R$.

By a *path* of rank type R we mean a word over $U(R) = \bigcup(\{(m, 1), \dots, (m, m)\} | m \in R)$. The set of all paths with rank type R will be denoted by $\mathrm{pt}(R)$.

Take a ranked alphabet Σ of rank type R, a tree $p \in F_\Sigma(X_n)$ and a path $u \in \mathrm{pt}(R)$. Then the *realization* $u(p)$ of u in p (if it exists) is defined in the following way:

1. if $u=e$ then $u(p)=e$ and u *ends* in p at the root of p,
2. if $u=u_1(m, i)$, $u_1(p)$ exists, u_1 ends in p at the node d of p labelled by σ and $\sigma \in \Sigma_m$ then $u(p)=u_1(p)(\sigma, i)$ and u ends in p at the i^{th} descendent of d.

For $U \subseteq \mathrm{pt}(R)$ and $T \subseteq F_\Sigma(X_n)$ $(n \geqq 1)$ let $U(T) = \{u(p) | u \in U, p \in T\}$. One can easily see, that for arbitrary $n \geqq 1$, $\mathrm{pt}(R)(F_\Sigma(X_n)) = U(\Sigma)^*$, where $U(\Sigma) = \bigcup(\{(\sigma, 1), \dots, (\sigma, m)\} | \sigma \in \Sigma_m, m > 0)$.

Let Σ be an operator domain with $\Sigma_0 = \emptyset$. A (deterministic) *ascending Σ-algebra* $\mathcal{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an m-ary ascending operation $\sigma^{\mathcal{A}}: A \to A^m$, where m is the arity of σ. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitely, but we write $\mathcal{A}=(A, \Sigma)$. If Σ is not specified then we speak about an ascending algebra. The ascending Σ-algebra $\mathcal{A}$ is finite if both A and Σ are finite. Moreover, $\mathcal{A}$ has *rank type* R if Σ is of rank type R. The class of all finite ascending Σ-algebras of rank type R will be denoted by $K(R)$. If there is no danger of confusion then we omit $\mathcal{A}$ in $\sigma^{\mathcal{A}}$.

In this paper by an algebra we mean a finite deterministic ascending algebra.

A *(deterministic) root-to-frontier* ΣX_n*-recognizer* or a $(D)R\Sigma X_n$-recognizer, for short, is a system $\mathbf{A}=(\mathcal{A}, a_0, X_n, \mathbf{a})$, where

(1) $\mathcal{A}=(A, \Sigma)$ is a finite Σ-algebra,
(2) $a_0 \in A$ is the *initial state,*
(3) $\mathbf{a}=(A^{(1)}, \ldots, A^{(n)}) \in P(A)^n$ is the *final-state vector.*

Next we recall the concept of a tree transducer.

A *root-to-frontier tree transducer (R-transducer)* is a system $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, A', P)$, where

(1) Σ and Ω are ranked alphabets,
(2) X_n and Y_m are the *frontier alphabets,*
(3) A is a ranked alphabet consisting of unary operators, the *state set* of $\mathfrak{A}$. (It is assumed that A is disjoint with all other sets in the definition of $\mathfrak{A}$, except A'.)
(4) $A' \subseteq A$ is the set of *initial states,*
(5) P is a finite set of *productions* of the following two types:
(i) $ax_i \to q$ $(a \in A,\ x_i \in X_n,\ q \in F_\Omega(Y_m))$,
(ii) $a\sigma \to q$ $(a \in A,\ \sigma \in \Sigma_l,\ l \geqq 0,\ q \in F_\Omega(Y_m \cup A\Xi_l))$. $(\Xi = \{\xi_1, \xi_2, \ldots\}$ is the set of auxiliary variables.)

The *transformation* induced by $\mathfrak{A}$ will be denoted by $\tau_{\mathfrak{A}}$.

The R-transducer $\mathfrak{A}$ is *deterministic* if $A'=\{a_0\}$ is a singleton and there are no distinct productions in P with the same left side. Moreover, the R-transducer $\mathfrak{A}$ is *uniform* if each production $a\sigma \to q$ $(a \in A,\ \sigma \in \Sigma_l,\ l \geqq 0,\ q \in F_\Omega(Y_m \cup A\Xi_l))$ can be written in the form $a\sigma \to \bar{q}(a_1\xi_1, \ldots, a_l\xi_l)$ for some $\bar{q} \in F_\Omega(Y_m \cup \Xi_l)$. In this paper by a transducer we shall mean a deterministic uniform R-transducer. One can easily see that for every transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, a_0, P)$ there exists a transducer $\mathfrak{B}=(\Sigma, X_n, B, \Omega', Y_m, b_0, P')$ such that (i) for arbitrary $b \in B$ and $\sigma \in \Sigma_m$ with $m>0$ there is exactly one production in P' with left side $b\sigma$, and (ii) $\tau_{\mathfrak{B}}=\tau_{\mathfrak{A}}$. In the sequel we shall confine ourselves to transducers having property (i) and $\Sigma_0=\emptyset$.

To a transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, a_0, P)$ we can correspond an $R\Sigma X_n$-recognizer $\mathbf{A}=(\mathcal{A}, a_0, X_n, \mathbf{a})$ with $\mathcal{A}=(A, \Sigma)$ and $\mathbf{a}=(A^{(1)}, \ldots, A^{(n)})$, where

(1) for arbitrary $l>0$, $\sigma \in \Sigma_l$, $a \in A$ and $(a_1, \ldots, a_l) \in A^l$ if $(a_1, \ldots, a_l)=\sigma^{\mathcal{A}}(a)$ then $a\sigma \to q(a_1\xi_1, \ldots, a_l\xi_l) \in P$ for some $q \in F_\Omega(Y_m \cup \Xi_l)$,

(2) $a \in A^{(i)}$ $(1 \leqq i \leqq n)$ if and only if $ax_i \to q \in P$ for some $q \in F_\Omega(Y_m)$.

The class of all recognizers obtained from $\mathfrak{A}$ in the above way will be denoted by $\operatorname{rec}(\mathfrak{A})$.

Now take an $R\Sigma X_n$-recognizer $\mathbf{A}=(\mathscr{A}, a_0, X_n, \mathbf{a})$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(A^{(1)}, \ldots, A^{(n)})$. Define a transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, a_0, P)$ by

$$P = \{ax_i \to q^{(a,i)} \mid a\in A^{(i)},\ q^{(a,i)}\in F_\Omega(Y_m),\ i = 1, \ldots, n\} \cup$$
$$\cup \{a\sigma \to q^{(a,\sigma)}(a_1\xi_1, \ldots, a_l\xi_l) \mid a\in A,\ \sigma\in\Sigma_l,\ l > 0,$$
$$(a_1, \ldots, a_l) = \sigma^{\mathscr{A}}(a),\ q^{(a,\sigma)}\in F_\Omega(Y_m\cup \Xi_l)\},$$

where the ranked alphabet Ω, the integer m and the trees on the right sides of the productions in P are fixed arbitrarily. Denote by $\mathrm{tr}(\mathbf{A})$ the class of all transducers obtained from $\mathbf{A}$ in the above way. Obviously, for arbitrary transducer $\mathfrak{A}$ and $\mathbf{A}\in \mathrm{rec}(\mathfrak{A})$ the inclusion $\mathfrak{A}\in \mathrm{tr}(\mathbf{A})$ holds. Therefore, we have

Statement 1. For every transducer $\mathfrak{A}$ there exists a recognizer $\mathbf{A}$ such that $\mathfrak{A}\in \mathrm{tr}(\mathbf{A})$.

Next we recall the concept of a product of ascending algebras (see [4]).

Let $\Sigma, \Sigma^1, \ldots, \Sigma^k$ be ranked alphabets of rank type R, and consider the Σ^i-algebras $\mathscr{A}_i=(A_i, \Sigma^i)$ $(i=1, \ldots, k)$. Furthermore, let

$$\psi = \{\psi_m\colon A_1\times\ldots\times A_k\times\Sigma_m \to \Sigma^1_m\times\ldots\times\Sigma^k_m \mid m\in R\}$$

be a family of mappings. Then by the *product* of $\mathscr{A}_1, \ldots, \mathscr{A}_k$ with respect to ψ we mean the Σ-algebra $\psi(\mathscr{A}_1, \ldots, \mathscr{A}_k, \Sigma)=\mathscr{A}=(A, \Sigma)$ with $A=A_1\times\ldots\times A_k$ and for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $\mathbf{a}\in A$

$$\sigma^{\mathscr{A}}(\mathbf{a}) = \big((\mathrm{pr}_1(\sigma_1^{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{a}))), \ldots, \mathrm{pr}_1(\sigma_k^{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{a})))), \ldots$$
$$\ldots, (\mathrm{pr}_m(\sigma_1^{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{a}))), \ldots, \mathrm{pr}_m(\sigma_k^{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{a}))))\big),$$

where $(\sigma_1, \ldots, \sigma_k)=\psi_m(\mathbf{a}, \sigma)$ and $\mathrm{pr}_i(\mathbf{a})$ $(1\leqq i\leqq k)$ denotes the i^{th} component of $\mathbf{a}$.

To define special types of products let us write ψ_m in the form $\psi_m=(\psi_m^{(1)}, \ldots, \psi_m^{(k)})$ where for arbitrary $\mathbf{a}\in A$ and $\sigma\in\Sigma_m$, $\psi_m(\mathbf{a}, \sigma)=(\psi_m^{(1)}(\mathbf{a}, \sigma), \ldots, \psi_m^{(k)}(\mathbf{a}, \sigma))$. We say that $\mathscr{A}$ is an α_i*-product* $(i=0, 1, \ldots)$ if for arbitrary j $(1\leqq j\leqq k)$ and $m\in R$, $\psi_m^{(j)}$ is independent of its u^{th} component if $i+j\leqq u\leqq k$. If $\Sigma^1=\ldots=\Sigma^k=\Sigma$ and $\psi_m(\mathbf{a}, \sigma)=(\sigma, \ldots, \sigma)$ for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $\mathbf{a}\in A$ then $\mathscr{A}$ is the *direct product* of $\mathscr{A}_1, \ldots, \mathscr{A}_k$. In the case of an α_i-product in $\psi_m^{(j)}$ we shall indicate only those variables on which $\psi_m^{(j)}$ may depend.

One can see easily that the formation of the product, α_0-product and direct product is associative. (This is not true for the α_i-product with $i>0$.)

Let $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a_0, P)$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, b_0, P')$ be two transducers and $m\geqq 0$ an integer. We write $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ if $\tau_{\mathfrak{A}}(p)=\tau_{\mathfrak{B}}(p)$ for every $p\in F_\Sigma^m(X_u)$, where $F_\Sigma^m(X_u)$ denotes the set of all trees from $F_\Sigma(X_u)$ with height less than or equal to m.

Take a class K of algebras of rank type R. We say that K is *metrically complete* (*m-complete*, for short) with respect to the product (α_i-product) if for arbitrary transducer $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a_0, P)$ and integer $m\geqq 0$ there exist a product (α_i-product) $\mathscr{B}=(B, \Sigma)$ of algebras from K, an element $b_0\in B$ and a vector $\mathbf{b}\in P(B)^u$ such that $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ for some $\mathfrak{B}\in \mathrm{tr}(\mathbf{B})$, where $\mathbf{B}=(\mathscr{B}, b_0, X_u, \mathbf{b})$.

Let $\mathscr{A}=(A, \Sigma)$ be an arbitrary algebra from $K(R)$. We correspond to $\mathscr{A}$ a semiautomaton $s(\mathscr{A})=(I_{\mathscr{A}}, A, \delta_{\mathscr{A}})$, where $I_{\mathscr{A}}=U(\Sigma)$ and for arbitrary $a\in A$ and $(\sigma, i)\in I_{\mathscr{A}}$, $\delta_{\mathscr{A}}(a, (\sigma, i))=\mathrm{pr}_i(\sigma^{\mathscr{A}}(a))$.

Take a Σ-algebra $\mathscr{A}=(A, \Sigma)\in K(R)$, an element $a\in A$ and an integer $m\geqq 0$. We say that the system $(\mathscr{A}, a)$ is *m-free* if the initial semiautomaton $s(\mathscr{A}, a)=(I_{\mathscr{A}}, A, a, \delta_{\mathscr{A}})$ is m-free. (For the definition of m-free semiautomata, see [1]. In [1] initial semiautomata are called initial automata. Moreover, here it is not supposed that $s(\mathscr{A}, a)$ is connected.)

For the system $(\mathscr{A}, a)$ and integer $m\geqq 0$ set $A_a^{(m)}=\{\delta_{\mathscr{A}}(a, p)|p\in I_{\mathscr{A}}^*, |p|\leqq m\}$, where $|p|$ denotes the length of p. Moreover, $\delta_{\mathscr{A}}(a, e)=a$ and $\delta_{\mathscr{A}}(a, p(\sigma, i))=\delta_{\mathscr{A}}(\delta_{\mathscr{A}}(a, p), (\sigma, i))$ $(p\in I_{\mathscr{A}}^*, (\sigma, i)\in I_{\mathscr{A}})$.

Let $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ be two systems with $\mathscr{A}=(A, \Sigma)$, $\mathscr{B}=(B, \Sigma)\in K(R)$. A mapping φ of $A_a^{(m)}$ onto $B_b^{(m)}$ is an *m-homomorphism* of $(\mathscr{A}, a)$ onto $(\mathscr{B}, b)$ if it satisfies the following conditions:

(1) $\varphi(a)=b$,
(2) $\varphi(\sigma^{\mathscr{A}}(a'))=\sigma^{\mathscr{B}}(\varphi(a'))$ $(a'\in A_a^{(m-1)}, \sigma\in\Sigma_l, l>0)$.

If the above φ is also one-to-one then we speak about an *m-isomorphism* and say that $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ are *m-isomorphic*. In notation, $(\mathscr{A}, a)\overset{m}{\simeq}(\mathscr{B}, b)$.

One can easily prove the following statements.

Statement 2. Let $\mathscr{A}=(A, \Sigma)$, $\mathscr{B}=(B, \Sigma)\in K(R)$ and $a\in A$, $b\in B$ be arbitrary. For an integer $m\geqq 0$, $(\mathscr{B}, b)$ is an m-homomorphic image of $(\mathscr{A}, a)$ if and only if $s(\mathscr{B}, b)$ is an m-homomorphic image of $s(\mathscr{A}, a)$.

Statement 3. Let $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ be the systems of Statement 2. For arbitrary $m\geqq 0$,

(1) if $(\mathscr{A}, a)$ is m-free then $(\mathscr{B}, b)$ is an m-homomorphic image of $(\mathscr{A}, a)$,
(2) if $(\mathscr{A}, a)$ is m-free and m-isomorphic to $(\mathscr{B}, b)$ then $(\mathscr{B}, b)$ is also m-free, and
(3) if both $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ are m-free then they are m-isomorphic.

The next statement is also obvious.

Statement 4. Take two systems $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ $(\mathscr{A}=(A, \Sigma), \mathscr{B}=(B, \Sigma)\in K(R), a\in A, b\in B)$. Moreover, let $m\geqq 0$ be an integer. If $(\mathscr{B}, b)$ is an m-homomorphic image of $(\mathscr{A}, a)$ then for arbitrary $u\geqq 0$, $\mathbf{b}\in P(B)^u$, $\mathbf{B}=(\mathscr{B}, b, X_u, \mathbf{b})$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, b, P')\in\mathrm{tr}\,(\mathbf{B})$ there exist an $\mathbf{a}\in P(A)^u$, an $\mathbf{A}=(\mathscr{A}, a, X_u, \mathbf{a})$ and an $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a, P)\in\mathrm{tr}\,(\mathbf{A})$ such that $\tau_{\mathfrak{B}}\overset{m}{=}\tau_{\mathfrak{A}}$.

Let $(\mathscr{A}, a)$ be a system with $\mathscr{A}=(A, \Sigma)\in K(R)$ and $a\in A$ an element. We say that for an integer $m\geqq 0$ the algebra $\mathscr{B}=(B, \Sigma)$ *m-isomorphically represents* $(\mathscr{A}, a)$ if there exists a $b\in B$ such that $(\mathscr{A}, a)\overset{m}{\simeq}(\mathscr{B}, b)$.

The α_i-product and the α_j-product $(i, j\geqq 0)$ will be called *metrically equivalent (m-equivalent)* provided that a system of algebras is m-complete with respect to the α_i-product if and only if it is m-complete with respect to the α_j-product. The m-equivalence between an α_i-product and the product is defined similarly.

Finally, we shall suppose that every finite index set $I=\{i_1, \ldots, i_k\}$ is given together with a (fixed) ordering of its elements. Furthermore, for arbitrary system $\{a_{i_j}|i_j\in I\}$, $(a_{i_j}|i_j\in I)$ is the vector $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ if $i_1<i_2<\ldots<i_k$ is the ordering of I.

For terminology not defined here, see [2] and [3].

2. Metrically complete systems

In this section we give necessary and sufficient conditions for a system of ascending algebras to be m-complete with respect to the α_i-products $(i=0, 1, \ldots)$ and the product. We shall see that the α_i-products are m-equivalent to each other and they are m-equivalent to the product.

We start with

Theorem 1. A system $K \subseteq K(R)$ is m-complete with respect to the product (α_i-product) if and only if for every $m \geqq 0$ each m-free system $(\mathcal{A}, a)$ with $\mathcal{A} \in K(R)$ can be represented m-isomorphically by a product (α_i-product) of algebras from K.

Proof. The sufficiency is obvious by Statements 3 and 4.

To prove the necessity take an arbitrary m-free system $(\mathcal{A}, a_0)$ with $\mathcal{A} = (A, \Sigma) \in K(R)$. Consider the transducer $\mathfrak{A} = (\Sigma, X_n, A, \Omega, A \times X_n, a_0, P)$, where $n > 1$ is an arbitrary natural number, $\Omega_l = A \times \Sigma_l$ $(l > 0)$ and P consists of the following productions:

(1) $ax_i \to (a, x_i)\ (a \in A,\ x_i \in X_n)$,

(2) $a\sigma \to (a, \sigma)(a_1\xi_1, \ldots, a_l\xi_l)\ (a \in A,\ \sigma \in \Sigma,\ l > 0,\ \sigma^{\mathcal{A}}(a) = (a_1, \ldots, a_l))$.

Let $\mathcal{B} = (B, \Sigma)$ be a product (α_i-product) of algebras from K such that for a $\mathfrak{B} = (\Sigma, X_n, B, \Omega, A \times X_n, b_0, P') \in \mathrm{tr}(\mathbf{B})$ we have $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$, where $\mathbf{B} = (\mathcal{B}, b_0, X_n, \mathbf{b})$ $(b_0 \in B,\ \mathbf{b} \in P(B)^n)$. We show that $(\mathcal{B}, b_0)$ is m-free. This, by Statement 3, will imply that $(\mathcal{A}, a_0) \overset{m}{\cong} (\mathcal{B}, b_0)$.

First of all observe that $\mathfrak{A}$ is a totally defined, linear, nondeleting transducer inducing a one-to-one transformation. Moreover, in a tree $\tau_{\mathfrak{A}}(p)$ with $h(p) \leqq m$ no subtree occurs more than once. Therefore, by $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$, all productions occurring in a derivation $b_0 p \Rightarrow^* q$ $(p \in F_\Sigma(X_n),\ q \in F_\Omega(X_n \times A))$ with $h(p) \leqq m$ are linear and nondeleting. Thus, we have the following relation between derivations in $\mathfrak{A}$ and $\mathfrak{B}$. Let $u \in \mathrm{pt}(R)$ be a path with $|u| \leqq m$. Take a tree $p \in F_\Sigma(X_n)$ with $h(p) \leqq m$, and assume that $u(p)$ is defined, it ends in p at the node d, p' is the subtree of p at d, $\bar{p}(\xi_1)$ is obtained from p by replacing the occurrence of p' at d by ξ_1, $\delta_{\mathcal{A}}(a_0, u(p)) = a$ and $\delta_{\mathcal{B}}(b_0, u(p)) = b$. Then the following derivations are valid:

$$a_0 p = a_0 \bar{p}(p') \Rightarrow^*_{\mathfrak{A}} q_1(ap') \Rightarrow^*_{\mathfrak{A}} q_1(q') = q$$

and

$$b_0 p = b_0 \bar{p}(p') \Rightarrow^*_{\mathfrak{B}} q_2(bp') \Rightarrow^*_{\mathfrak{B}} q_2(q'') = q,$$

where $a_0\bar{p}(\xi_1) \Rightarrow^*_{\mathfrak{A}} q_1(a\xi_1)$, $b_0\bar{p}(\xi_1) \Rightarrow^*_{\mathfrak{B}} q_2(b\xi_1)$ $(q_1, q_2 \in F_\Omega(A \times X_n \cup \xi_1))$ and $ap' \Rightarrow^*_{\mathfrak{A}} q'$, $bp' \Rightarrow^*_{\mathfrak{B}} q''$ $(q', q'' \in F_\Omega(A \times X_n))$. (Observe that ξ_1 occurs exactly once in q_1 and q_2.) Furthermore, if $v_1 \in \mathrm{pt}(R)$ is the path such that $v_1(q_1)$ ends in q_1 at the node labelled by ξ_1 and $v_2 \in \mathrm{pt}(R)$ is the path for which $v_2(q_2)$ ends in q_2 at the node labelled by ξ_1 then $v_2(q_2)$ is a subword of $v_1(q_1)$.

Now assume that $(\mathcal{B}, b_0)$ is not m-free, that is there are two distinct words $u, v \in I^*_{\mathcal{B}}$ $(=I^*_{\mathcal{A}})$ such that $|u|, |v| \leqq m$ and $\delta_{\mathcal{B}}(b_0, u) = \delta_{\mathcal{B}}(b_0, v) = b$. Let $\bar{u}, \bar{v} \in \mathrm{pt}(R)$ be paths and $p_1, p_2 \in F_\Sigma(X_n)$ trees such that $\bar{u}(p_1) = u$, $\bar{v}(p_2) = v$, $h(p_1)$, $h(p_2) \leqq m$, u ends in p_1 at the node d_1 and v ends in p_2 at the node d_2. Replace in p_1 and p_2 the subtrees at d_1 resp. d_2 by x_1, and denote by $\bar{p}_1$ resp. $\bar{p}_2$ the resulting trees. Moreover, let $\delta_{\mathscr{A}}(a_0, u)=a_1$ and $\delta_{\mathscr{A}}(a_0, v)=a_2$. (Note that $a_1 \neq a_2$ since $u \neq v$ and $(\mathscr{A}, a_0)$ is m-free.) Then, by the choice of $\mathfrak{A}$, if $q_1, q_2 \in F_\Omega(A \times X_n)$ are obtained by the derivations $a_0\bar{p}_1 \Rightarrow_{\mathfrak{A}}^* q_1$ and $a_0\bar{p}_2 \Rightarrow_{\mathfrak{A}}^* q_2$ then $\bar{u}(q_1)$ ends in q_1 at a node labelled by (a_1, x_1) and $\bar{v}(q_2)$ ends in q_2 at a node labelled by (a_2, x_1). Moreover, by $\tau_{\mathfrak{B}} \overset{m}{=} \tau_{\mathfrak{A}}, b_0\bar{p}_1 \Rightarrow_{\mathfrak{B}}^* q_1$ and $b_0\bar{p}_2 \Rightarrow_{\mathfrak{B}}^* q_2$ hold also. From this, taking into consideration our observation concerning the relation between derivations in $\mathfrak{A}$ and $\mathfrak{B}$, we get that at the ends of $\bar{u}(q_1)$ and $\bar{v}(q_2)$ the same label should occur which is a contradiction.

The next theorem gives necessary conditions for a system of ascending algebras to be m-complete with respect to the product.

Theorem 2. Let $K \subseteq K(R)$ be a system which is m-complete with respect to the product. Then the following conditions are satisfied:

(i) for arbitrary integer $m \geqq 0$, path $\bar{u} \in \mathrm{pt}(R)$ with $|\bar{u}|=m$, rank $l \in R$ and natural number $1 \leqq i \leqq l$ there exist an $\mathscr{A}=(A, \Sigma') \in K$, an $a_0 \in A$, $\sigma_1, \sigma_2 \in \Sigma'_l$ and a $u \in \bar{u}(F_{\Sigma'}(X_1))$ such that $\delta_{\mathscr{A}}(a_0, u(\sigma_1, i)) \neq \delta_{\mathscr{A}}(a_0, u(\sigma_2, i))$,

(ii) for arbitrary integer $m \geqq 0$, path $\bar{u} \in \mathrm{pt}(R)$ with $|\bar{u}|=m$, rank $l \in R$ $(l>1)$ and integers $1 \leqq i<j \leqq l$ there exist an $\mathscr{A}=(A, \Sigma) \in K$, an $a_0 \in A$, a $\sigma \in \Sigma_l$ and a $u \in \bar{u}(F_\Sigma(X_1))$ such that $\delta_{\mathscr{A}}(a_0, u(\sigma, i)) \neq \delta_{\mathscr{A}}(a_0, u(\sigma, j))$.

Proof. We start with the necessity of (i). Assume that there are $m \geqq 0, \bar{u} \in \mathrm{pt}(R)$ with $|\bar{u}|=m$, $l \in R$ and $1 \leqq i \leqq l$ such that for arbitrary $\mathscr{A}=(A, \Sigma') \in K$, $a_0 \in A$, $\sigma_1, \sigma_2 \in \Sigma'_l$ and $u \in \bar{u}(F_{\Sigma'}(X_1))$ the equation $\delta_{\mathscr{A}}(a_0, u(\sigma_1, i))=\delta_{\mathscr{A}}(a_0, u(\sigma_2, i))$ holds. Take a ranked alphabet Σ of rank type R such that Σ_l contains two distinct elements σ and σ'. Moreover, consider a product $\mathscr{B}=(B, \Sigma)=\psi(\mathscr{A}_1, \ldots, \mathscr{A}_k, \Sigma)$ $(\mathscr{A}_i= =(A_i, \Sigma^i) \in K, i=1, \ldots, k)$ and an element $\mathbf{b}_0 \in B$. We show that the system $(\mathscr{B}, \mathbf{b}_0)$ is not $(m+1)$-free.

First of all let us introduce a notation. Consider the above product $\mathscr{B}$ and define the mappings $\psi^i\colon B \times F_\Sigma(X_n) \to F_{\Sigma^i}(X_n)$ $(i=1, \ldots, k;\ n \geqq 0)$ in the following way: for arbitrary $\mathbf{b} \in B$ and $p \in F_\Sigma(X_n)$

(1) if $p=x_j$ $(1 \leqq j \leqq n)$ then $\psi^i(\mathbf{b}, p)=x_j$,

(2) if $p=\sigma(p_1, \ldots, p_l)$ then $\psi^i(\mathbf{b}, p)=\sigma_i(\psi^i(\mathbf{b}_1, p_1), \ldots, \psi^i(\mathbf{b}_l, p_l))$, where $(\sigma_1, \ldots, \sigma_k)=\psi_l(\mathbf{b}, \sigma)$ and $(\mathbf{b}_1, \ldots, \mathbf{b}_l)=\sigma^{\mathscr{B}}(\mathbf{b})$.

One can see easily that for arbitrary $\mathbf{b} \in B$, $p \in F_\Sigma(X_n)$ and $\bar{u} \in \mathrm{pt}(R)$ the equation $\delta_{\mathscr{B}}(\mathbf{b}, \bar{u}(p))=(\delta_{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{b}), \bar{u}(\psi^1(\mathbf{b}, p))), \ldots, \delta_{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{b}), \bar{u}(\psi^k(\mathbf{b}, p))))$ holds.

Now take two trees $p, q \in F_\Sigma(X_1)$ such that $(\bar{u}(l, i))(p)=u(\sigma, i)$ and $(\bar{u}(l, i))(q)= =u(\sigma', i)$. For every $j(=1, \ldots, k)$ let $(\bar{u}(l, i))(\psi^j(\mathbf{b}_0, p))=u_j(\sigma^{(j)}, i)$ and $(\bar{u}(l, i))(\psi^j(\mathbf{b}_0, q))=v_j(\bar{\sigma}^{(j)}, i)$. By the definition of the product, the equations $u_j=v_j$ $(j=1, \ldots, k)$ obviously hold. Moreover,

$$\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma, i)) = (\delta_{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{b}_0),\ u_1(\sigma^{(1)}, i)), \ldots, \delta_{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{b}_0),\ u_k(\sigma^{(k)}, i)))$$

and

$$\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma', i)) = (\delta_{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{b}_0),\ u_1(\bar{\sigma}^{(1)}, i)), \ldots, \delta_{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{b}_0),\ u_k(\bar{\sigma}^{(k)}, i))).$$

But, by our assumptions, $\delta_{\mathscr{A}_j}(\mathrm{pr}_j(\mathbf{b}_0), u_j(\sigma^{(j)}, i))=\delta_{\mathscr{A}_j}(\mathrm{pr}_j(\mathbf{b}_0), u_j(\bar{\sigma}^{(j)}, i))$ for every j $(1 \leqq j \leqq k)$, i.e., $\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma, i))=\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma', i))$. Therefore, $(\mathscr{B}, \mathbf{b}_0)$ is not $(m+1)$-free which, by Theorem 1, implies that K is not m-complete with respect to the product.

The necessity of (ii) can be shown in a similar way.

Theorem 3. If a system $K \subseteq K(R)$ satisfies the conclusions of Theorem 2 then K is m-complete with respect to the α_0-product.

Proof. Let Σ be a fixed ranked alphabet of rank type R. We shall show by induction on m that for every integer $m \geqq 0$ there are an α_0-product $\mathscr{B}=(B, \Sigma)$ of algebras from K and an element $\mathbf{b} \in B$ such that $(\mathscr{B}, \mathbf{b})$ is m-free. This, by Theorem 1, will end the proof of Theorem 3.

If $m=0$ then our claim is obviously valid. Let us suppose that our statement has been proved for an $m \geqq 0$, and take a product $\mathscr{A}=(A, \Sigma)$ of algebras from K and an element $a \in A$ such that $(\mathscr{A}, a)$ is m-free. By our assumption, for every $\bar{u}=\bar{u}_1(l, i)$ $(\bar{u}_1 \in \mathrm{pt}(R), l \in R, 1 \leqq i \leqq l)$ there are an $\mathscr{A}^{(\bar{u})}=(A^{(\bar{u})}, \Sigma^{(\bar{u})}) \in K$, an $a^{(\bar{u})} \in A^{(\bar{u})}$, two operators $\sigma_1, \sigma_2 \in \Sigma_l^{(\bar{u})}$ and a $u_1 \in \bar{u}_1(F_\Sigma(X_1))$ such that $\delta_{\mathscr{A}^{(\bar{u})}}(a^{(\bar{u})}, u_1(\sigma_1, i)) \neq \delta_{\mathscr{A}^{(\bar{u})}}(a^{(\bar{u})}, u_1(\sigma_2, i))$. Moreover, for arbitrary $\bar{u}=\bar{u}_1(l, i)$, $\bar{v}=\bar{u}_1(l, j)$ $(\bar{u}_1 \in \mathrm{pt}(R)$, $l \in R$, $l>1$, $1 \leqq i<j \leqq l)$ there are an $\mathscr{A}^{(\bar{u}, \bar{v})}=(A^{(\bar{u}, \bar{v})}, \Sigma^{(\bar{u}, \bar{v})})$, an $a^{(\bar{u}, \bar{v})} \in A^{(\bar{u}, \bar{v})}$, a $u_1 \in \bar{u}_1(F_\Sigma(X_1))$ and a $\bar{\sigma} \in \Sigma_l^{(\bar{u}, \bar{v})}$ such that $\delta_{\mathscr{A}^{(\bar{u}, \bar{v})}}(a^{(\bar{u}, \bar{v})}, u_1(\bar{\sigma}, i)) \neq \delta_{\mathscr{A}^{(\bar{u}, \bar{v})}}(a^{(\bar{u}, \bar{v})}, u_1(\bar{\sigma}, j))$. Consider an index set I consisting of all pairs (u, v) where $u, v \in U(\Sigma)^*$, $u \neq v$, $|u|=m+1$ and $|v| \leqq m+1$. For the pair (u, v) with $u=u'(\sigma, i) \in \bar{u}(F_\Sigma(X_1))$ and $v=v'(\sigma^*, j)$ if $u' \neq v'$ or $\sigma \neq \sigma^*$ take the α_0-product $\mathscr{A}^{(u, v)}=\psi^{(u, v)}(\mathscr{A}, \mathscr{A}^{(\bar{u})}, \Sigma)=(A^{(u, v)}, \Sigma)$, where $\psi^{(u, v)}$ is defined in the following way. For every $s \in R$, $\psi_s^{(u, v)(1)}$ is the identity mapping on Σ_s. If $w=w_1(\sigma', j)$ $(\sigma' \in \Sigma_k)$ is a proper subword of u' and $w'=w_1'(\sigma'', j)$ is the subword of u_1 with $|w'|=|w|$ then let

$$\psi_k^{(u, v)(2)}(\delta_{\mathscr{A}}(a, w_1), \sigma')=\sigma''.$$

In all other cases, except $\psi_l^{(u, v)(2)}(\delta_{\mathscr{A}}(a, u'), \sigma)$, $\psi_s^{(u, v)(2)}$ $(s \in R)$ is given arbitrarily in accordance with the definition of the α_0-product. Since $u' \neq v'$ or $\sigma \neq \sigma^*$ and $(\mathscr{A}, a)$ is m-free $\delta_{\mathscr{A}^{(u, v)}}((a, a^{(\bar{u})}), v)$ is defined. Now let

$$\psi_l^{(u, v)(2)}(\delta_{\mathscr{A}}(a, u'), \sigma)=\begin{cases}\sigma_1 & \text{if } \delta_{\mathscr{A}^{(u, v)}}((a, a^{(\bar{u})}), v)=(a_1, a_2) \\ & \text{and } \delta_{\mathscr{A}^{(\bar{u})}}(a^{(\bar{u})}, u_1(\sigma_1, i)) \neq a_2 \\ \sigma_2 & \text{otherwise.}\end{cases}$$

Obviously, $(\mathscr{A}^{(u, v)}, a^{(u, v)})$ with $a^{(u, v)}=(a, a^{(\bar{u})})$ is m-free and $\delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, u) \neq \delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, v)$.

Now assume that $u'=v'$ and $\sigma=\sigma^*$; that is $u=u'(\sigma, i) \in \bar{u}(F_\Sigma(X_1))$ and $v=u'(\sigma, j) \in \bar{v}(F_\Sigma(X_1))$. Take the α_0-product $\mathscr{A}^{(u, v)}=\psi^{(u, v)}(\mathscr{A}, \mathscr{A}^{(\bar{u}, \bar{v})}, \Sigma)=(A^{(u, v)}, \Sigma)$, where $\psi^{(u, v)}$ is given as follows. Again for every $s \in R$, $\psi_s^{(u, v)(1)}$ is the identity mapping on Σ_s. If $w=w_1(\sigma', t)$ $(\sigma' \in \Sigma_k)$ is a proper subword of u' and $w'=w_1'(\sigma'', t)$ is the subword of u_1 with $|w'|=|w|$ then let $\psi_k^{(u, v)(2)}(\delta_{\mathscr{A}}(a, w_1), \sigma')=\sigma''$. Moreover, $\psi_l^{(u, v)(2)}(\delta_{\mathscr{A}}(a, u'), \sigma)=\bar{\sigma}$. In any other cases $\psi_s^{(u, v)(2)}$ $(s \in R)$ is given arbitrarily in accordance with the definition of the α_0-product. Since $(\mathscr{A}, a)$ is m-free $\mathscr{A}^{(u, v)}$ is well defined. Again, $(\mathscr{A}^{(u, v)}, a^{(u, v)})$ with $a^{(u, v)}=(a, a^{(\bar{u}, \bar{v})})$ is m-free and $\delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, u) \neq \delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, v)$.

Finally, take the direct product $\mathscr{B}=(B, \Sigma)=\Pi(\mathscr{A}^{(u, v)} \mid (u, v) \in I)$ and the vector $\mathbf{b}=(a^{(u, v)} \mid (u, v) \in I)$. Then $(\mathscr{B}, \mathbf{b})$ is $(m+1)$-free. Indeed, for two different words $u, v \in U(\Sigma)^*$ if $|u|, |v|<m+1$ then $\delta_{\mathscr{B}}(\mathbf{b}, u) \neq \delta_{\mathscr{B}}(\mathbf{b}, v)$ since they differ in all of their components, and if $|u|=m+1$ and $|v| \leqq m+1$ then $\delta_{\mathscr{B}}(\mathbf{b}, u)$ and $\delta_{\mathscr{B}}(\mathbf{b}, v)$

are different at least in their $(u, v)^{\text{th}}$ components. Since the direct product is a special α_0-product and the formation of the α_0-product is associative $\mathcal{B}$ is an α_0-product of algebras from K.

From Theorems 2 and 3 we get

Corollary 4. For arbitrary $i, j \geqq 0$ the α_i-product and the α_j-product are m-equivalent to each other and they are m-equivalent to the product.

We now give an algorithm to decide for a finite $K \subseteqq K(R)$ whether K is m-complete with respect to the product.

Take an algebra $\mathcal{A}=(A, \Sigma) \in K$. For arbitrary $l \in R$ and $1 \leqq i \leqq l$ set $A^{(l,i)}=\{a \in A \mid \mathrm{pr}_i(\sigma_1^{\mathcal{A}}(a)) \neq \mathrm{pr}_i(\sigma_2^{\mathcal{A}}(a))$ for some $\sigma_1, \sigma_2 \in \Sigma_l\}$. Moreover, for every $a \in A$ let $L_a^{(l,i)}$ be the language recognized by the automaton $\mathcal{A}_a^{(l,i)}=(I_{\mathcal{A}}, A, a, \delta_{\mathcal{A}}, A^{(l,i)})$. Furthermore, let $L_{\mathcal{A}}^{(l,i)}=\bigcup(L_a^{(l,i)} \mid a \in A)$ and $L^{(l,i)}=\bigcup(L_{\mathcal{A}}^{(l,i)} \mid \mathcal{A} \in K)$. For arbitrary $l \in R$ $(l>1)$ and $1 \leqq i<j \leqq l$ define $L^{(l,i,j)}$ in a similar way with $A^{(l,i,j)}=\{a \in A \mid \mathrm{pr}_i(\sigma^{\mathcal{A}}(a)) \neq \mathrm{pr}_j(\sigma^{\mathcal{A}}(a))$ for some $\sigma \in \Sigma_l\}$ instead of $A^{(l,i)}$. Finally, denote by $\bar{\Sigma}$ the union of all ranked alphabets belonging to algebras from K, and take the language homomorphism $\varphi\colon U(\bar{\Sigma})^* \to U(R)^*$ given by $\varphi(\sigma, i)=(k, i)$ $(\sigma \in \bar{\Sigma}, r(\sigma)=k)$, where $r(\sigma)$ denotes the rank of σ. Then, by Theorems 2 and 3, K is m-complete with respect to the product if and only if

(1) for arbitrary $l \in R$ and $1 \leqq i \leqq l$, $\varphi(L^{(l,i)})=U(R)^*$,
(2) for arbitrary $l \in R$ $(l>1)$ and $1 \leqq i<j \leqq l$, $\varphi(L^{(l,i,j)})=U(R)^*$.

The validity of these equations is decidable effectively.

Finally, for a given rank type R we give a one-element system which is m-complete with respect to the product. Let Σ be a ranked alphabet of rank type R such that for every $l \in R$, $\Sigma_l=\{\sigma_1^{(l)}, \sigma_2^{(l)}\}$. Assume that the greatest natural number in R is n. Take the Σ-algebra $\mathcal{A}=(A, \Sigma)$, where $A=\{a_0, \ldots, a_n\}$, $\sigma_1^{(l)}(a_i)=(a_{i+1(\mathrm{mod}\, n+1)}, \ldots, a_{i+1(\mathrm{mod}\, n+1)})$ $(l \in R, i=0, 1, \ldots, n)$, $\sigma_2^{(l)}(a_n)=(a_n, a_{n-1}, \ldots, a_{n-l+1})$ $(l \in R)$ and for arbitrary $l \in R$ and a_i with $i \neq n$, $\sigma_2^{(l)}(a_i)$ is defined arbitrarily. $(i+1 \,(\mathrm{mod}\, n+1)$ denotes the least residue of $i+1$ modulo $n+1.)$ One can see easily that the system $K=\{\mathcal{A}\}$ satisfies the conclusions of Theorem 2.

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On cofinal and definite automata

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1. Introduction

Cofinal or directable automata were introduced in [1] and further investigated in [2, 7, 8, 9]. Cofinal automata are automata whose states can be directed to a single state by a suitable input word. We will call a cofinal automaton definite if there is an integer n such that all input words of length greater than or equal n direct the state set to a single state. Perles et al. [10] investigated definite events and definite automata. In particular they used shift registers, a special type of definite automata, in their discussion of the synthesis problem. Moreover, Stoklosa [12, 13] investigated these automata from an algebraic point of view. In section 2 of this paper we will prove a graph theoretic property of shift registers, namely that the transition diagram of a shift register contains a hamiltonian circle. In section 3 we apply this result in order to investigate the determination whether an arbitrary automaton is cofinal or not. In section 4 we determine the structure of all strongly definite automata with the aid of shift registers. Finally, in section 5, we characterize the general structure of definite automata. Let us give precise definitions first.

Definition 1.1. An *automaton* (more exactly, an *X-automaton*) A, denoted by $A=(S, X)$, consists of the following data: (i) S is a nonempty finite *set of states*. (ii) X is a nonempty finite *set of inputs*. (iii) There exists a function M_A of $S\times X^*$ into S, called a *state transition function,* such that $M_A(s, pq)=M_A(M_A(s, p), q)$ and $M_A(s, e)=s$ for all $s\in S$ and all $p, q\in X^*$, where X^* is the free monoid over X and e is its identity.

Note that in the following sp^A will often be used to denote $M_A(s, p)$.

Definition 1.2. An automaton $A=(S, X)$ is said to be *cofinal* (or *directable* in [1, 2]) if there exists $p\in X^*$ such that $Sp^A=\{sp^A|s\in S\}$ is a singleton.

Definition 1.3. An automaton $A=(S, X)$ is called a *definite* automaton if there exists an integer $n\geqq 0$ such that $|Sp^A|=1$ holds for all $p\in X^*$ with $|p|\geqq n$. If A is a definite automaton, then the least integer n such that the above condition holds is called the *degree of A* and denoted by $d(A)$.

A definite automaton is cofinal. The class of definite automata A with $d(A)=0$ is exactly the class of all one-state automata. Furthermore, if $d(A)=n\geqq 1$ for a definite automaton A, then there exists a $q \in X^*$ with $|q| = n-1$ and $|Sq^A| > 1$. A definite automaton $A = (S, X)$ is called a *strongly definite* automaton if it is *strongly connected*, i.e., if for all $s, s' \in S$ there exists $p \in X^*$ such that $sp^A = s'$ holds. If $|X| = 1$ for a strongly definite automaton, then $|S| = 1$ holds too.

Definition 1.4. Let n be a nonnegative integer, X a finite set and X^n the set of all words over X of length n. Then the automaton $A(n) = (X^n, X)$ whose state transition function is defined by $(yp)x^{A(n)} = px$ for all $(y, p, x) \in X \times X^{n-1} \times X$ if $n \geqq 1$ and $ex^{A(n)} = e$ for all $x \in X$ if $n = 0$ is called an *n-stage shift register without feedback* (or briefly an nSR).

Obviously, n-stage shift registers are strongly definite automata.

2. A graph theoretic property of nSR's

The purpose of this section is to prove the following theorem. (For the notion of a hamiltonian circle in a directed graph see [5].)

Theorem 2.1. There exists a hamiltonian circle in the state transition diagram of an nSR.

Note that the state transition diagram of an nSR is the directed graph whose vertices are states and where there is a directed edge from p to q, labelled by x, iff $px^{A(n)} = q$ for $(p, x, q) \in X^n \times X \times X^n$. If $n = 0$, the theorem holds trivially. Therefore we assume $n \geqq 1$ for the rest of this section. Before proving the theorem, we need the following definition.

Definition 2.1. Let $r \geqq 1$. A sequence $p_1, p_2, \ldots, p_r$ of distinct elements of X^n with $p_i x_i^{A(n)} = p_{i+1}$ with $x_i \in X$ for $1 \leqq i \leqq r-1$ is called a *chain of length* $r-1$ and denoted by $p_1 \xrightarrow[x_1]{} p_2 \xrightarrow[x_2]{} p_3 \xrightarrow[x_3]{} \cdots \xrightarrow[x_{r-2}]{} p_{r-1} \xrightarrow[x_{r-1}]{} p_r$ (or briefly $p_1 \to p_2 \to p_3 \to \ldots \to p_{r-1} \to p_r$).

Now we first provide some lemmata.

Lemma 2.1. Let $p_1 \to p_2 \to \ldots \to p_r$ with $p_i = y_i q_i$, $(y_i, q_i) \in X \times X^{n-1}$, for $1 \leqq i \leqq r$, be a chain of length $r-1$. Then there exists a $p \in X^n$ such that $p_1 \to p_2 \to \ldots \to p_{r-1} \to p_r \to p$ iff there exists an $x \in X$ such that $q_r x \notin \{p_1, p_2, \ldots, p_{r-1}, p_r\}$.

The proof is easy and thus omitted.

Lemma 2.2. Let $p_1 \to p_2 \to \ldots \to p_r$. If there is no $p \in X^n$ such that $p_1 \to p_2 \to \ldots \to p_{r-1} \to p_r \to p$ holds, then there exists some $x \in X$ such that $p_r \xrightarrow[x]{} p_1$, i.e., we have a circle $\langle p_1, p_2, \ldots, p_{r-1}, p_r \rangle$ in the state transition diagram of $A(n)$.

Proof. Let $p_i = y_i q_i$ with $(y_i, q_i) \in X \times X^{n-1}$ for $1 \leqq i \leqq r$ and $p_i \xrightarrow[x_i]{} p_{i+1}$ for $1 \leqq i \leqq r-1$. By Lemma 2.1, we have $q_r x \in \{p_1, p_2, \ldots, p_r\}$ for all $x \in X$. This means that $q_r X = \{q_r x | x \in X\} \subseteq \{p_1, p_2, \ldots, p_r\}$. Let $Xq_r = \{xq_r | x \in X\}$. It is obvious that $|q_r X| = |Xq_r| = |X|$ holds. Now assume $p_1 = y_1 q_1 \notin q_r X$. This implies $q_r X \subseteq \{p_2, \ldots, p_r\}$. Furthermore we have $p_i = y_i q_i \in q_r X$ iff $p_{i-1} = y_{i-1} q_{i-1} \in Xq_r$ for all i with $2 \leqq i \leqq r$. Therefore the set $\{p_1, p_2, \ldots, p_{r-1}\}$ contains $|q_r X|$ elements of Xq_r. Together with $p_r \in Xq_r$ we obtain $|Xq_r| \geqq |q_r X| + 1$ in contradiction to the fact $|Xq_r|=|q_r X|$. Hence $p_1 \in q_r X$. Since $p_r=y_r q_r$, there exists some $x \in X$ such that $p_r \xrightarrow[x]{} p_1$.

Lemma 2.3. Let $p_1 \to p_2 \to \ldots \to p_r$ and $\{p_1, p_2, \ldots, p_r\} \neq X^n$. Then there exists a $p'_1 \to p'_2 \to \ldots \to p'_r \to p'_{r+1}$ such that $\{p_1, p_2, \ldots, p_r\} \subseteqq \{p'_1, p'_2, \ldots, p'_r, p'_{r+1}\}$.

Proof. If we have $p_1 \to p_2 \to \ldots \to p_{r-1} \to p_r \to p$ for some $p \in X^n$, there is nothing to do. Now, assume that there does not exist such $p \in X^n$. By Lemma 2.2, we have a circle $\langle p_1, p_2, \ldots, p_r \rangle$ in the state transition diagram of $A(n)$. Let $p \in X^n - \{p_1, p_2, \ldots, p_r\}$. Then it is easy to see that $pp_1^{A(n)}=p_1$. From this it can easily be shown that there exist some $p', p'' \in X^*, x \in X$ and i with $1 \leqq i \leqq r$ such that $p_1=p'xp''$, $pp'^{A(n)} \in X^n - \{p_1, p_2, \ldots, p_r\}$ and $(pp'^{A(n)})x^{A(n)}=p_i$. It is obvious that in this case we have $pp'^{A(n)} \to p_i \to p_{i+1} \to \ldots \to p_1 \to p_2 \to \ldots \to p_{i-2} \to p_{i-1}$ and $\{p_1, p_2, \ldots, p_r\} \subseteqq \{pp'^{A(n)}, p_i, p_{i+1}, \ldots, p_1, p_2, \ldots, p_{i-1}\}$. This completes the proof of the lemma.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $p_1 \to p_2 \to \ldots \to p_r$ be one of the longest chains in the transition diagram of $A(n)$. Then, by Lemma 2.3, we have $X^n=\{p_1, p_2, \ldots, p_r\}$. Moreover, by Lemma 2.2, $\{p_1, p_2, \ldots, p_r\}$ forms a circle $\langle p_1, p_2, \ldots, p_{r-1}, p_r \rangle$, i.e., the state transition diagram of $A(n)$ has a hamiltonian circle.

Remark 2.1. Note that the previous results provide an algorithm for obtaining a hamiltonian circle in the state transition diagram of an nSR.

3. Cofinal automata and cofinal congruences

We will now apply the foregoing theorem to investigate the determination whether an arbitrary automaton is cofinal or not and to give a characterization of the minimal cofinal congruence of an arbitrary automaton. In this section all automata are assumed to be automata over a fixed alphabet X. Let us first give

Definition 3.1. Let n be a positive integer and $\mathscr{A}(n)=\{A=(S, X) \mid |S|=n$ and A is cofinal$\}$. Then by $\delta(n)$ we denote the value $\max\limits_{A \in \mathscr{A}(n)} \min\{|p| \mid p \in X^*$ and $|Sp^A|=1\}$.

In [1, 11], $\delta(n)$ is investigated. Černy et al. [1] conjectured that $\delta(n)=(n-1)^2$. However at present only $(n-1)^2 \leqq \delta(n) \leqq 0(n^3)$ is known. The following result is obvious.

Proposition 3.1. Let $A=(S, X)$ be an automaton such that $|S|=n$. Then A is cofinal iff there exists a $p \in X^{\delta(n)}$ such that $|Sp^A|=1$. ($X^{\delta(n)}$ is the set of all words over X with length $\delta(n)$.)

To test whether or not an automaton $A=(S, X)$ with n states is cofinal, we have to check whether or not Sp^A is a singleton for each $p \in X^{\delta(n)}$. Another more economical method would be to merge all $p \in X^{\delta(n)}$ in a single word w and to check the property "cofinal" with this word w. We first introduce some notions. Let $u, w \in X^*$. u is called a *subword* of w iff $w=u'uu''$ for some $u', u'' \in X^*$. Now let $w \in X^*$ such that every $u \in X^{\delta(n)}$ is a subword of w. Then w is called a *merged word of* $X^{\delta(n)}$ [3]. Obviously we have:

Proposition 3.2. Let $A=(S, X)$ be an automaton with $|S|=n$. Then A is cofinal iff $|Sw^A|=1$, where w is a merged word of $X^{\delta(n)}$.

It can easily be seen that the length of a merged word of $X^{\delta(n)}$ is greater than or equal to $|X|^{\delta(n)}+\delta(n)-1$. Moreover, with the aid of Theorem 2.1, we can show:

Lemma 3.1. There exists a merged word w of $X^{\delta(n)}$ such that $|w|=|X|^{\delta(n)}+$ $+\delta(n)-1$.

Proof. By Theorem 2.1, the state transition diagram of $A(\delta(n))=(X^{\delta(n)}, X)$ has a hamiltonian circle $\langle p_1, p_2, \ldots, p_r\rangle$ with $r=|X|^{\delta(n)}$. Let $p_1 \xrightarrow[x_1]{} p_2 \xrightarrow[x_2]{} p_3 \xrightarrow[x_3]{} \ldots$ $\ldots \xrightarrow[x_{r-2}]{} p_{r-1} \xrightarrow[x_{r-1}]{} p_r$ and put $w=p_1x_1x_2\ldots x_{r-2}x_{r-1}$. This proves the lemma.

Now we can state the following:

Theorem 3.1. There exists a $w\in X^*$ satisfying the following conditions:

(i) $|w|=|X|^{\delta(n)}+\delta(n)-1$.
(ii) For each automaton $A=(S, X)$ with $|S|=n$, A is cofinal iff $|Sw^A|=1$.

Remark 3.1. In [3], Dömösi discussed a general method to obtain the shortest merged word w of L, where L is a finite subset of X^*.

We will now use Lemma 3.1 to characterize the minimal cofinal congruence of an arbitrary automaton. To this end, we first recall the following notions. Let $A=(S, X)$ be an automaton. An equivalence relation ϱ on S is called *congruence on A* if $(s, s')\in\varrho$ implies $(sx^A, s'x^A)\in\varrho$ for all $s, s'\in S$ and $x\in X$. Let ϱ, ϱ' be congruences on A. Then $\varrho\wedge\varrho'$ and $\varrho\vee\varrho'$, the *product* and *sum* of ϱ and ϱ', are defined as usual (see e.g. [6]). $R(A)$, the set of all congruences on A, forms a lattice w.r.t. $\wedge$ and $\vee$. We now define:

Definition 3.2. Let $A=(S, X)$ be an automaton. A congruence ϱ on A is said to be *cofinal* if for all $s, s'\in S$ there exists a $p\in X^*$ such that $(sp^A, s'p^A)\in\varrho$ holds.

Let π_ϱ denote the partition of S induced by ϱ and $\pi_\varrho(s)$ the block of π_ϱ containing $s\in S$. We have:

Lemma 3.2. Let $A=(S, X)$ be an automaton and ϱ a congruence on A. Then ϱ is cofinal iff there exist a $p\in X^*$ and an $s_0\in S$ with $Sp^A\subseteq\pi_\varrho(s_0)$.

Proof. The "if part" is obvious. Conversely, let ϱ be cofinal and T a maximal subset of S such that there exist a $p\in X^*$ and an $s_0\in S$ with $Tp^A\subseteq\pi_\varrho(s_0)$. Assume $T\neq S$ and let $s\in S-T$. Then we have $(sp^A, s_0)\notin\varrho$. Since ϱ is cofinal, there exists a $p'\in X^*$ such that $(sp^Ap'^A, s_0p'^A)\in\varrho$. Since ϱ is a congruence, we have $(T\cup\{s\})(pp')^A\subseteq\pi_\varrho(s_0p'^A)$. This contradicts the minimality of T, hence $S=T$.

By $R_{cf}(A)$ we denote the set of all cofinal congruences on A. Let $\varrho, \varrho'\in R_{cf}(A)$. By Lemma 3.2, there exist $p, p'\in X^*$ such that $(sp^A, s'p^A)\in\varrho$ and $(sp'^A, s'p'^A)\in\varrho'$ for all $s, s'\in S$. This implies $(s(pp')^A, s'(pp')^A)\in\varrho\wedge\varrho'$ for all $s, s'\in S$. Therefore, $\varrho\wedge\varrho'\in R_{cf}(A)$. $\varrho\vee\varrho'\in R_{cf}(A)$ can be shown in a similar way. Thus $R_{cf}(A)$ forms a sublattice of $R(A)$. We now give

Definition 3.3. Let $A=(S, X)$ be an automaton. The minimal element of $R_{cf}(A)$, denoted by ϱ_{cf}, is called the *minimal cofinal congruence on A*.

Now we will characterize ϱ_{cf}.

Theorem 3.2. Let $A=(S, X)$ be an automaton with $|S|=n$ and ϱ a congruence on A. Let w be a merged word of $X^{\delta(n)}$. Then $\varrho=\varrho_{cf}$ iff ϱ is the minimal congruence on A such that $Sw^A \subseteqq \pi_\varrho(s_0)$ for some $s_0 \in S$.

Proof. The assertion follows from Proposition 3.2 and the fact that ϱ is cofinal iff the quotient automaton A/ϱ is cofinal.

Remark 3.2. We can develop further properties of cofinal congruences and their quotient automata along the line of [4], where similar notions for commutative congruences were introduced.

4. The structure of strongly definite automata

In this section we consider homomorphic images of nSR's in order to characterize strongly definite automata. We have:

Theorem 4.1. Let $A=(S, X)$ be an automaton and let n be a positive integer. Then A is a strongly definite automaton with $d(A)\leqq n$ iff A is a homomorphic image of $A(m)=(X^m, X)$ for all integers m with $m\geqq n$.

Proof. It is easy to see that $A(m)$ is a strongly definite automaton of degree m. Let A be a homomorphic image of $A(m)$. Then A is a strongly definite automaton with $d(A)\leqq d(A(m))=m$. This completes the proof of the "if" part. Now let A be a strongly definite automaton with degree $d(A)\leqq n$ and $m\geqq n$. Let h be the following mapping of X^m into $S: h(p)=Sp^A$ for all $p\in X^m$. Since $d(A)\leqq m$, this mapping is well defined. Note that a singleton Sp^A is considered as an element of S. We prove that h is surjective. Let $s\in S$ and $p'\in X^m$. Since A is strongly connected, there exists a $q\in X^*$ such that $(Sp'^A)q^A=s$. Let $p'q=p''p$ with $p\in X^m$. Then we have $s=S(p'q)^A=S(p''p)^A=(Sp''^A)p^A=Sp^A$. Finally, we prove that h is a homomorphism of $A(m)$ onto A. Let $p=x'p'$ with $x'\in X, p'\in X^{m-1}$ and $x\in X$. Then we have $h(px^{A(m)})=h((x'p')x^{A(m)})=h(p'x)=S(p'x)^A=Sx'^A(p'x)^A= =S(x'p')^Ax^A=(Sp^A)x^A=h(p)x^A$. This completes the proof of the "only if" part.

Remark 4.1. We can prove that the homomorphism h in the above proof is the unique homomorphism of $A(m)$ onto A. In general, if there exists a homomorphism of a strongly cofinal automaton onto another automaton, it is uniquely determined. For this, see [8].

The following corollary is obvious. Note that the inequality $|S|\geqq d(A)+1$ follows directly from Theorem 7 of [10].

Corollary 4.1. Let A be a strongly definite automaton. Then we have $|X|^{d(A)}\geqq|S|\geqq d(A)+1$. Moreover, $|X|^{d(A)}=|S|$ iff A is isomorphic to $A(d(A))$.

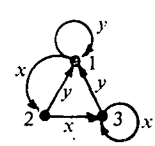

Fig. 1

Example 4.1. Let A be given by the diagram of Fig. 1. If A is a strongly definite automaton, then $2^{d(A)}\geqq 3\geqq d(A)+1$, hence $d(A)=2$. On the other hand, we have $\{1, 2, 3\}(xx)^A=3, \{1, 2, 3\}(xy)^A= =1, \{1, 2, 3\}(yx)^A=2$ and $\{1, 2, 3\}(yy)^A=1$. This shows that A is really a strongly definite automaton with degree 2. Furthermore, A is not isomorphic to $A(2)$. Finally, the homomorphism h of $A(2)$ onto A is given as follows: $h(xx)=3$, $h(xy)=1$, $h(yx)=2$ and $h(yy)=1$.

Remark 4.2. In Theorem 2.1 we proved that the state transition diagram of a shift register has a hamiltonian circle. Moreover, in Theorem 4.1 we proved that the set of all homomorphic images of shift registers coincide with the set of all strongly definite automata. It seems to be interesting to consider the following problem: Under what conditions may the state diagram of a strongly definite automaton have a hamiltonian circle?

5. The structure of definite automata

In [10], Perles et al. discussed the synthesis problem of definite automata. In this section we will also deal with this problem. Strongly definite automata are given as homomorphic images of shift registers, and a method to obtain all homomorphic images of a given automaton is well known [6]. Therefore it remains to determine the structure of definite automata which are not necessarily strongly connected. Let us first give

Definition 5.1. Let $A=(S, X)$ be a definite automaton. Then the subset $U=\{Sp^A | p\in X^{d(A)}\}$ of S is called the *core of* S.

Lemma 5.1. For all $x\in X$ we have $Ux^A\subseteq U$.

Proof. The lemma obviously holds for $d(A)=0$. Assume $d(A)\geqq 1$ and let $s\in U$. Then there exists a $p\in X^{d(A)}$ such that $s=Sp^A$. Let $p=x'p'$ with $x'\in X$ and $p'\in X^{d(A)-1}$. Then, for all $x\in X$, we have $sx^A=(Sp^A)x^A=(Sx'^Ap'^A)x^A=$ $=(Sx'^A)(p'x)^A=S(p'x)^A$, where $p'x\in X^{d(A)}$. Consequently, we have $sx^A\in U$.

Lemma 5.2. Let $C=(U, X)$, where $sx^C=sx^A$ for all $(s, x)\in U\times X$. Then C is a strongly definite automaton and $d(C)\leqq d(A)$.

Proof. Let $s\in U$. There exists a $p\in X^{d(A)}$ such that $s=Sp^A$. Therefore $s=$ $=Sp^A=Up^A=Up^C$. This shows that C is a strongly connected automaton. Obviously, C is definite with $d(C)\leqq d(A)$.

Definition 5.2. $C=(U, X)$ is called the *core of* A. Moreover, $d(C)$ is the *radius of the core* and denoted by $r_c(A)$.

Definition 5.3. Let $A=(S, X)$ be a definite automaton and $C=(U, X)$ its core. Then $S-U$ is called the *shell of* S. Moreover, $\max\{|px|\,|s\in S-U, p\in X^*, x\in X, sp^A\in S-U$ and $s(px)^A\in U\}$ is called the *thickness of the shell* and denoted by $t_s(A)$.

The following result is obvious.

Proposition 5.1. $t_s(A)\leqq d(A)\leqq t_s(A)+r_c(A)$ and $r_c(A)\leqq d(A)$.

We characterize definite automata by means of $r_c(A)$ and $t_s(A)$.

Let $A=(S, X)$ be a definite automaton and $C=(U, X)$ its core. Let $T_0=U$ and $T_1=\{s\in S | sx^A\in T_0$ for all $x\in X\}$. We have:

Lemma 5.3. $T_0\subseteq T_1$ and if $S-U\neq\emptyset$ then $T_1-T_0\neq\emptyset$.

Proof. $T_0\subseteq T_1$ is obvious. Suppose that for all $s\in S-U$ we have $s\notin T_1$. Then for all $s\in S-U$ there exists some $x_1\in X$ such that $sx_1^A\notin T_0=U$. Since

$sx_1^A \in S-U$, by the same reason as above, there exists some $x_2 \in X$ such that $(sx_1^A)x_2^A \in T_0 = U$. By continuing this procedure, we have an infinite sequence $x_1, x_2, \ldots$ $\ldots, x_k, \ldots$ of elements of X such that $s(x_1 x_2 \ldots x_k)^A \notin U$ for any positive integer k. This contradicts the definiteness of A. Hence $T_1 - T_0 \neq \emptyset$.

Now suppose that T_i is defined and $T_{i-1} \subseteq T_i$. Set $T_{i+1} = \{s \in S \mid sx^A \in T_i$ for all $x \in X\}$. Then, by the same way as in the proof of the above lemma, we obtain:

Lemma 5.4. $T_i \subseteq T_{i+1}$ and if $S - T_i \neq \emptyset$ then $T_{i+1} - T_i \neq \emptyset$.

It is obvious that there exists some positive integer i such that $T_i = T_{i+1}$ and $T_k = T_i$ for all $k \geqq i$. This means that in the case $S - U \neq \emptyset$ there exists a minimal positive integer n such that $T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_{n-1} \subset T_n = T_{n+1} = \ldots$ and $S = T_n$.

Definition 5.4. Let $A = (S, X)$ be a definite automaton and $\{T_i | 0 \leqq i \leqq n\}$ the set defined as above. Let $L_i = T_i - T_{i-1}$ for all i with $1 \leqq i \leqq n$. Then $\{L_i | 1 \leqq i \leqq n\}$ is called the *set of layers of the shell.*

Lemma 5.5. The number of layers coincides with $t_s(A)$.

Proof. Let $s \in S - U$. Then there exists some i with $1 \leqq i \leqq n$ such that $s \in L_i$. It is obvious that $sp^A \in U$ holds for all $p \in X^i$. This means that $t_s(A) \leqq n$. Now let $s \in L_n$. Then, by the definition of L_n, there exists some $x_n \in X$ such that $sx_n^A \in L_{n-1}$. By the same way as above, there exists some $x_{n-1} \in X$ such that $(sx_n^A)x_{n-1}^A \in L_{n-2}$. By the same procedure, we have a sequence $x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1$ of elements in X such that $s(x_n x_{n-1} x_{n-2} \ldots x_{k+1} x_k)^A \notin U$ for $2 \leqq k \leqq n$ and $s(x_n \ldots x_2 x_1)^A \in U$. Consequently we have $t_s(A) \leqq n$. Thus $t_s(A) = n$.

Now we are ready to prove the following theorem.

Theorem 5.1. Let $A = (S, X)$ be a definite automaton with $(r_c(A), t_s(A)) = (r, t)$. Then S can be partitioned in $\{U(=L_0), L_i | 1 \leqq i \leqq t\}$ such that:

(i) $C = (U, X)$ is a strongly definite automaton with degree r, where $sx^C = sx^A$ for all $(s, x) \in U \times X$.
(ii) $sx^A \in U \cup L_1 \cup L_2 \cup \ldots \cup L_{i-1}$ for all $(s, x) \in L_i \times X$ with $1 \leqq i \leqq t$.
(iii) For all $s \in L_i$ with $1 \leqq i \leqq t$ there exists an $x_i \in X$ such that $sx_i^A \in L_{i-1}$.

Conversely, let $C = (U, X)$ be a strongly definite automaton with degree r and let $\{U(=L_0), L_i | 1 \leqq i \leqq t\}$ be a partition of a finite set S. Then each automaton $A = (S, X)$ whose state transition function satisfies the above conditions (i)—(iii) is a definite automaton with $(r_c(A), t_s(A)) = (r, t)$.

Proof. Let $C = (U, X)$ be the core of A and $\{L_i | 1 \leqq i \leqq t\}$ the set of layers fo the shell. The first part of the theorem is now obvious. The second part is obvious too.

In Proposition 5.1 inequalities were given. We show that there is no relationship among $d(A)$, $r_c(A)$ and $t_s(A)$ beside these inequalities.

Proposition 5.2. Let d, r and t be nonnegative integers such that $t \leqq d \leqq t + r$ and $r \leqq d$. Then for all alphabets X with $|X| \geqq 2$ there exists a definite automaton $A = (S, X)$ such that $d = d(A)$, $r = r_c(A)$ and $t = t_s(A)$.

Proof. Let S be the disjoint union $S=V\cup V^{(1)}\cup\ldots\cup V^{(t)}\cup\{t_1, t_2, \ldots, t_n\}$. Here $V=X^r$, $V^{(i)}=\{v^{(i)}|\, v\in V\}$ are copies of V for $1\leqq i\leqq t$ and $t_1, \ldots, t_n$ are $n=d-r$ additional states. Choose $x_0\in X$ and define the state transition function of A as follows:

(i) For all $(v, x)\in V\times X=X^r\times X$ set $vx^A=vx^{A(r)}$.
(ii) For all $x\in X$ and $2\leqq i\leqq n$ set $t_i x^A=t_{i-1}$ and furthermore set $t_1 x^A= =x_0^r=x_0 \ldots x_0\in X^r$.
(iii) For all $(v, x)\in V\times X$ and $2\leqq i\leqq t$ set $v^{(i)}x^A=(vx^{A(r)})^{(i-1)}$ and furthermore set $v^{(1)}x^A=vx^{A(r)}$.

This situation is depicted in Fig. 2. Obviously, A is a definite automaton. Let us first show $d=d(A)$. The case $d=0$ is trivial. Let $d\geqq 1$. If now $r=0$, then $V=\{e\}$, $n=t=d$ and $d=d(A)$. If now $t=0$, then $r=d$, $n=0$ and $d=d(A)$. Hence we can assume $t, d, r\geqq 1$. If now $n=0$, then $d=r$. Since $|X|\geqq 2$, we have $d=d(A)$. Let now $n\geqq 1$ and $p=p'x\in X^n$ with $p'\in X^{n-1}$, $x\in X$ and $x=x_0$. Then there exists a $p''\in X^{r-1}$ such that $(\{t_n\}\cup X^r)p^A\supseteq\{x_0^r\}\cup\{p''x\}$. It is easy to see that $|(\{x_0^r\}\cup\{p''x\})q^A|\neq 1$ for all $q\in X^{r-1}$. Consequently, $|S(pq)^A|>1$. This means that $d(A)>|pq|=n+r-1= =d-1$. Now let $p\in X^d$. Then $p=p'p''$ with $p'\in X^n$ and $p''\in X^r$. From this $(\{t_1, t_2, \ldots, t_n\}\cup V)p^A\subseteq Vp''^A=p''$ follows. On the other hand, since $d\geqq t$, we have $V^{(i)}p^A=Vp^A=p''$ for all i with $1\leqq i\leqq t$. Therefore $Sp^A=p''$. This means that $d(A)\leqq d$. Hence $d(A)=d$. The core of A coincides with $A(r)$, hence $r_c(A)=r$, and since $n=d-r\leqq t$, we have $t_s(A)=t$.

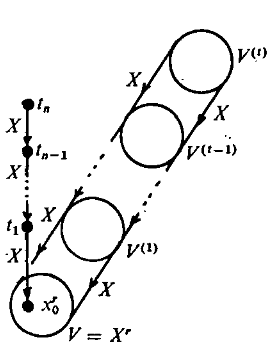

Fig. 2

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Basic theoretical treatment of fuzzy connectives

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Introduction

One of the most interesting problems in the theory of fuzzy sets is the choice of the fuzzy connective operations, i.e. the union and the intersection.

Definition 1. The fuzzy set μ is an arbitrary function

$$\mu\colon X \to [0, 1] \tag{1}$$

interpreted on the non-empty universal discourse X.

In such a sense, the characteristic function of the common sets is a special fuzzy set.

Zadeh (1965) [24] extended the intersection and union of the subsets of the common sets in the following way

$$\mu_{A\cup B}(x) = \max(\mu_A(x),\ \mu_B(x)) \quad \text{for all} \quad x\in X \quad \text{and}$$
$$\mu_{A\cap B}(x) = \min(\mu_A(x),\ \mu_B(x)) \quad \text{for all} \quad x\in X, \tag{2}$$

where $\mu_{A\cup B}$ and $\mu_{A\cap B}$ are the fuzzy sets corresponding to $A\cup B$ and $A\cap B$, respectively.

Below we shall survey in broad outlines the development of the views relating to fuzzy operations. Historical survey of fuzzy operations:

Besides operations (1), others also have been proposed for the generalization of the operations in set theory [24], [17]. Some examples are

$$\mu_{A\cap B}(x) = \mu_A(x)\cdot\mu_B(x) \quad \text{and}$$
$$\mu_{A\cup B}(x) = \mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x) \tag{3}$$

or

$$\mu_{A\cap B}(x) = \max(\mu_A(x)+\mu_B(x)-1,\ 0) \quad \text{and}$$
$$\mu_{A\cup B}(x) = \min(\mu_A(x)+\mu_B(x),\ 1). \tag{4}$$

All this reveals the arbitrary nature of the definitions. This arbitrariness can be resolved with a basis on the axiom system general in mathematics. Strivings in this direction were first made in defence of the min and max operations [3], [12], [9].

In effect, this merely involved the characterization of operations (1) with other properties.

Subsequently, other axiom systems were created [11], [12], [14], which were not represented by operations (1); there were publications in which algebraic structures were investigated without representation [2], [13], [15]. Here the emphasis was on the rational establishment of the axioms.

A whole series of axiomatic examinations arose for the most varied operations; however, these were unable to unify the views relating to the operations, but rather made the problem more ramified. Study of the mutual interrelations between the axiom systems might have led to a solution, but very great difficulty was caused by the fact that it was impossible to compare the axioms. Only one such study has been made [10].

One possibility was to return to the bases, i.e. to base the rational nature of the axioms not on opinions, but on empirical examinations. The first such examination did not relate exactly to this, but to the question of whether the created operations correspond to practice [21]. The result was that they do not.

Further, it is not advisable to make a mathematical theory dependent on narrow empirical examinations; rather, operation classes must be produced from which the appropriate operation can be selected in a manner adequate to the practical requirements.

The operations should if possible be made flexible. Parameter-dependent operation series were produced by Yager [23] and by Hamacher [11], but these were as individual as the earlier operations. Although operation classes were defined, a practical interpretation of the parameters did not materialize.

The next period was characterized by the appearance of monographs on operations and axiom systems [6], [22].

These works ensured a possibility for the discovery of the common properties of operations and axiom systems and for the selection of a minimal axiom system [4]. However, only a narrow range of the examined operations could be characterized with these axiom systems.

The axioms of this minimal system are the strict monotonity of the operations, the holding of the correspondence principle, associativity and continuity. The adoption of these axioms can be based rationally in the following way:

The correspondence principle is satisfied by all fuzzy operations, i.e. their restriction to the characteristic function is a classical set-theory operation. The associativity holds for every operation examined so far, and in addition a possibility is created for the extension of two-variant operations to multi-variant ones. The lack of continuity terminates the homogeneous effect of the operation.

Strict monotonity is not satisfied by every operation; its condition rather served the realization of the representation. However, the condition of monotonity exists for all operations.

Thus, it is advisable to carry out an examination of not strictly monotonous operations. Hence, we must obtain, for example, (4) and (1).

The main result in the paper is the giving of representations of all operations of such type, as functions of various conditions.

The study relies on the theory of ordered semigroups [8], [20] and the associative function equations [1].

1. Fuzzy algebra

Let I be the closed interval [0, 1] of the real numbers. This notation partly serves to simplify the description, and partly refers to the generalizability of the theorems and definitions.

The set of all the fuzzy sets (1) is

$$F(X) = \{\mu | \mu\colon X \to I\} \tag{5}$$

(we shall denote $F(X)$ briefly by F). Let Ch be a set of common characteristic functions.

Definition 2. The fuzzy sets μ and ν are said to be equal if

$$\mu(x) = \nu(x) \quad \text{for all} \quad x \in X. \tag{6}$$

The fuzzy set μ precedes the fuzzy set ν if

$$\mu(x) \leqq \nu(x) \quad \text{for all} \quad x \in X. \tag{7}$$

Theorem 1. The relation $\leqq$ is a partial ordering on F. Let us consider an n-ary algebraic operation

$$*\colon F^n \to F \quad (n = 1, 2, \ldots) \tag{8}$$

on the set F of fuzzy sets.

Definition 3. The operation $*$ is isotonic (antitonic) if it follows from the inequalities

$$\mu_i \leqq \nu_i \quad (i = 1, 2, \ldots, n)$$

that

$$\mu_1 * \ldots * \mu_n \leqq \nu_1 * \ldots * \nu_n \;\; (\mu_1 * \ldots * \mu_n \geqq \nu_1 * \ldots * \nu_n) \tag{9}$$

for all $(\mu_1, \ldots, \mu_n), (\nu_1, \ldots, \nu_n) \in F^n$. The isotonic and antitonic operations together are said to be monotonic.

The ordering relation $\leqq$ interpreted on the fuzzy sets is a generalization of the partial ordering defined by the entailment interpreted on the common sets.

Definition 4. By fuzzy algebra [5] (the algebra of fuzzy sets) we understand all those algebraic structures interpreted on F for which it holds that

(A1) all of its operations are monotonic.

Fuzzy algebra is said to be "ordinary" if the following condition also holds:

(A2) the restriction of all of its algebraic operations to Ch agrees with some set-theory operation with the same number of variables.

In our work we shall examine those ordinary fuzzy algebras $\langle F, * \rangle$ (in the following simply fuzzy algebra) which satisfy the following conditions:

(F1) $*$ is a binary connective operation, i.e. its restriction to Ch is either intersection interpreted on the normal sets, or union.

Let us consider those fuzzy algebraic operations for which there is a function $f\colon I \times I \to I$ such that

$$(\mu * \nu)(x) = f(\mu(x), \nu(x)) \quad \text{for all} \quad x \in X. \tag{10}$$

The attribution $* \to f$ is mutually unambiguous. Let us denote the set of fuzzy algebraic operations with this property by Z.

(F2) Let $*$ be an operation belonging to Z.

Theorem 2. Let $*: F\times F\to F$ be an operation in Z. The algebraic structure $\langle F, *\rangle$ satisfying condition F1 is fuzzy algebra if, and only if, it holds for the function f ascribed to $*$ that

(i) f is monotonic in the sense agreeing with $*$;

(ii) $f(0, 0)=0, f(1, 1)=1$, and further, if the restriction of the operation $*$ to Ch is intersection (union); then $f(0, 1)=f(1, 0)=0$ $(f(0, 1)=f(1, 0)=1)$.

Proof. Let $\langle F, *\rangle$ be the fuzzy algebra satisfying condition F1.

(i) Let us assume that $*$ is isotonic. Let $x_1, x_2, y_1, y_2\in I$, so that $x_1\leqq x_2$ and $y_1\leqq y_2$. Let us consider the fuzzy sets

$$\mu_1(x) = x_1,\ \mu_2(x) = x_2,\ \nu_1(x) = y_1,\ \nu_2(x) = y_2 \quad \text{for every} \quad x\in X.$$

For these it holds that

$$\mu_1 \leqq \mu_2 \quad \text{and} \quad \nu_1 \leqq \nu_2.$$

It follows from the isotonity of operation $*$ that

$$\mu_1 * \nu_1 \leqq \mu_2 * \nu_2.$$

Taking F2 into consideration:

$$f(\mu_1(x),\ \nu_1(x)) \leqq f(\mu_2(x),\ \nu_2(x)) \quad \text{for all} \quad x\in X.$$

It therefore follows from $x_1\leqq x_2$ and $y_1\leqq y_2$ that

$$f(x_1, y_1) \leqq f(x_2, y_2),$$

i.e. f is isotonic. The postulate can be demonstrated similarly for the antitonic case.

(ii) The postulate arises simply from consideration of A2 or F1 and F2.

Proof of the inverse of the postulate is likewise simple.

Consequence: with the operation f ascribed to $*$ I is an ordered algebraic structure.

Theorem 2 ensures that study of the representations of the algebraic structure determined by the operation f ascribed to the operation $*$ is sufficient for examination of the representations of the fuzzy algebras $\langle F, *\rangle$ satisfying conditions F1 and F2.

As concerns f, let us assume that

(F3) f is associative;

(F4) f is continuous on $I\times I$.

It can readily be seen that the operation $*$ determined by such f is associative and continuous from point to point, i.e. if the series of fuzzy sets $\{\mu_n\}$ and $\{\nu_n\}$ converge from point to point to the fuzzy sets μ and ν, then the series of fuzzy sets $\{\mu_n * \nu_n\}$ converges from point to point to the fuzzy set $\mu * \nu$.

In the following section postulates will be given for the case when the restriction to Ch of the operation $*$ determined by f is the normal set-theory intersection. In this case we denote the determining function by c. The function corresponding to the union is denoted by d. The postulates for c and their proofs can be applied appropriately to d.

2. Representation theorem

Let us first summarize the properties having by the function $c: I \times I \to I$ defined in section 2.

(T1) c is monotonous;
(T2) $c(0, 0)=0$, $c(1, 1)=1$, $c(0, 1)=c(1, 0)=0$;
(T3) c is associative;
(T4) c is continuous.

Theorem 3. If T1 and T3 hold for c, then
(T1′) c is isotonic [8].
Thus, the set I forms a semigroup completely ordered with operation c.

Definition 5. The function h is said to be Archimedean in the interval $[a, b]$ if

$$h(x, x) < x \quad \text{for all} \quad x \in (a, b). \tag{11}$$

The representation theorem relating to the Archimedean case was proved by Ling [16] by means of elementary analysis. The theorem can be derived from the earlier result of Mostert and Shields [18].

We shall make use of this theorem in the following.

Theorem 4. Let J be a closed interval $[a, b]$ of real numbers, and h the function $h: J \times J \to J$. h has the properties that
(i) h is monotonous;
(ii) h is associative;
(iii) h is continuous;
(iv) $h(a, a)=a$, $h(b, b)=b$, $h(b, x)=h(x, b)=x$ $(x \in X)$; (12)
(v) h is Archimedean
if and only if there exists a continuous, strictly monotonously decreasing function g, mapping the interval $[a, b]$ into the interval $[0, \infty]$ for which $g(b)=0$ such that h may be represented in the form

$$h(x, y) = g^{(-1)}(g(x)+g(y)) \tag{13}$$

where $g^{(-1)}$ is the pseudo-inverse of g

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if} \quad g(b) \leqq x \leqq g(a), \\ a & \text{if} \quad g(a) \leqq x, \end{cases} \tag{14}$$

where g^{-1} is the normal inverse of function g in $[g(b), g(a)]$.

Function g is termed the additive generator of the Archimedean operation h, and g is unambiguously determined apart from a positive constant, i.e. $\alpha \cdot g$ $(\alpha > 0)$ likewise generates h.

It should be noted that the theorem can also be stated in such a way that the generator function g' maps the interval $[a, b]$ into $[-\infty, 0]$, it increases strictly monotonously, and $g(b)=0$. In this case the definition of the pseudo-inverse is modified appropriately.

Function c with properties T1—T4 satisfies conditions (i)—(iv) of Theorem 4. In the following we shall not restrict our considerations to the Archimedean case. Mostert and Shields have carried out similar examinations relating to semigroups [18].

Let us consider the set of the idempotent points of the interval I

$$N = \{x \mid x \in I, \quad c(x, x) = x\}. \tag{15}$$

Theorem 5. N is a closed set.

Proof. We see that N contains every accumulation point. Let x_0 be an optional accumulation point of N. A point series x_n may then be selected from N such that

$$\lim_{n \to \infty} x_n = x_0.$$

Since $x_n \in N$ for every n, we have $c(x_n, x_n) = x_0$, and c is continuous (T4), so that

$$c(x_0, x_0) = \lim_{x_n \to x_0} c(x_n, x_n) = \lim_{x_n \to x_0} x_n = x_0,$$

and thus $x_0 \in N$.

Let $M = I \setminus N$. M is a restricted and open point set. Let us assume that M is not empty.

Theorem 6. M can be constructed as the combination of a finite or infinitely large number of open intervals, not projecting into one another in pairs, the end-points of which do not belong to M [21].

Therefore M has the form

$$M = \bigcup_{i \in P} M_i \tag{16}$$

where P is a finite or an infinite index set and $M_i = (a_i, b_i)$, for which, if $x \in (a_i, b_i)$,

$$c(x, x) \neq x, \tag{17}$$

while $c(a_i, a_i) = a_i$ and $c(b_i, b_i) = b_i$.

Let us select an optional region $[a_i, b_i] \times [a_i, b_i]$. In this region it holds too that c is isotonic (T1′), associative (T3) and continuous (T4). For determination of the properties corresponding to T2, let us consider the following theorems:

Theorem 7. For every $x \in [a_i, b_i]$:

(i)
$$c(a_i, x) = c(x, a_i) = a_i, \tag{18}$$

(ii)
$$c(b_i, x) = c(x, b_i) = x. \tag{19}$$

Proof. First, we see that

$$c(1, x) = c(x, 1) = x \quad \text{for all} \quad x \in I. \tag{20}$$

On the basis of (T2), $c(0, 1) = 0$ and $c(1, 1) = 1$, and with consideration of the continuity (T4) the function $c(x, 1)$ therefore maps I on I. Then, for any $y \in I$ there exists an $x \in I$ such that $c(x, 1) = y$. Utilizing this fact and the associativity (T3).

$$c(y, 1) = c(c(x, 1), 1) = c(x, c(1, 1)) = c(x, 1) = y \quad \text{for all} \quad y \in I.$$

Part (i) of the theorem is a simple consequence of the isotonity (T1′) and (20)

$$a_i = c(a_i, a_i) \leqq c(x, a_i) \leqq c(1, a_i) = a_i.$$

The proof of part (ii) is the application of that of (20) to $[a_i, b_i]$.

Theorem 8. For every $x \in (a_i, b_i)$, $c(x, x) < x$.

Proof. As a consequence of the isotonity (T1′) and (19)

$$c(x, x) \leqq c(b_i, x) = x \quad \text{for all} \quad x \in [a_i, b_i].$$

If $x \in (a_i, b_i)$, then $c(x, x) \neq x$, so that

$$c(x, x) < x \quad \text{for all} \quad x \in (a_i, b_i).$$

Theorem 9. For every $(x, y) \in [a_i, b_i] \times [a_i, b_i]$

$$c(x, y) \leqq \min(x, y). \tag{21}$$

Proof. As a consequence of (19)

$$c(x, y) \leqq c(x, b_i) = x \quad \text{and} \quad c(x, y) \leqq c(b_i, y) = y,$$

therefore,

$$c(x, y) \leqq \min(x, y).$$

Theorem 10. Let $H = I^2 \setminus \bigcup_{i \in P} M_i^2$. Then

$$c(x, y) = \min(x, y) \quad \text{for all} \quad (x, y) \in H. \tag{22}$$

Proof. Let us assume that $x \leqq y$. Let $(x, y) \in H$.
(i) If $x \in N$ and $y \in I$, then

$$x = c(x, x) \leqq c(x, y) \leqq c(x, 1) = x.$$

(ii) If $x \in (a_i, b_i) \subseteqq M$ and $y \notin (a_i, b_i)$, then

$$x = c(x, b_i) \leqq c(x, y) \leqq c(x, 1) = x.$$

In both cases $c(x, y) = x = \min(x, y)$.

Let c_i be the restriction of the function c to the region $[a_i, b_i] \times [a_i, b_i]$. As a consequence of the equalities $c(a_i, a_i) = a_i$ and $c(b_i, b_i) = b_i$ as well as the isotonity (T1′) and continuity (T4) of c, c_i maps the region $[a_i, b_i] \times [a_i, b_i]$ on $[a_i, b_i]$.

To summarize, c_i satisfies conditions T1′, T3, T4 and T2′

(T2′) $$c_i(a_i, a_i) = a_i, \; c_i(b_i, b_i) = b_i,$$
$$c_i(a_i, b_i) = c_i(b_i, a_i) = a_i$$

and by Theorem 8 it is Archimedean. From the Ling theorem, therefore, for every $i \in P$ there exists a generator function g_i additive in $[a_i, b_i]$ to c_i.

Thus, the following theorem holds for c:

Theorem 11. Let c be the function $c: I \times I \to I$. c satisfies conditions T1—T4 if and only if c has the form

$$c(x, y) = \begin{cases} g_i^{(-1)}(g_i(x) + g_i(y)), & \text{if } (x, y) \in M_i^2 = (a_i, b_i)^2 \;\; i \in P \\ \min(x, y), & \text{if } (x, y) \in I^2 \setminus \bigcup_{i \in P} M_i^2 \end{cases} \tag{23}$$

where $\{M_i\}_{i\in P}$ is the sum of a finite or infinitely large number of open intervals belonging to I, not projecting into one another in pairs. g_i is a function mapping the closed interval $[a_i, b_i]$ into the interval $[0, \infty]$, which is a continuous, strictly monotonously decreasing function, and $g_i(b_i)=0$. $g^{(-1)}$ is the pseudo-inverse of g_i. (It should be noted that P may be empty.)

Proof. (i) Let us assume that T1—T4 hold for the function $c\colon I\times I\to I$. If every point of I is idempotent, i.e. $I=N$, then on the basis of Theorem 10, $c(x, y)=\min(x, y)$ for all $(x, y)\in I^2$. If $N\subset I$, then as a consequence of Theorems 5 and 6 in I^2 there are regions $[a_i, b_i]\times[a_i, b_i]$ not projecting into one another in pairs, in which the functions c_i satisfy the conditions of Theorem 4 (Ling) by Theorems 7—10. In the given region, therefore, there exist generator functions g_i additive for c_i-s. Outside such regions, from Theorem 10: $c(x, y)=\min(x, y)$.

(ii) Let us assume that the function $c\colon I\times I\to I$ exists in the form (23). If P is empty, then $c(x, y)=\min(x, y)$ for all $(x, y)\in I^2$. Therefore T1—T4 hold.

If P is not empty, then by Theorem 4 (Ling) the function c is isotonic, associative and continuous separately both in the regions $\{M_i^2\}$ $(i\in P)$ and outside these regions.

Because of (12), at the limit of the regions M_i^2, $c(x, y)=\min(x, y)$, and c therefore has no breakpoint. Thus, c is continuous (T4) in I^2. T2 similarly follows from these arguments.

The proof of the isotonity (T1′) and the associativity (T3) is lengthy, and accordingly we do not present it here.

Without proof, we list some of the consequences of Theorem 11.

Theorem 12. Every function $c\colon I\times I\to I$ satisfying conditions T1—T4 is commutative.

Definition 6. The function $t\colon I\times I\to I$ is said to be a t norm [19] if

(i) $t(0, 0)=0$, $t(x, 1)=t(1, x)=x$ for all $x\in I$,
(ii) t is isotonic,
(iii) t is commutative, and
(iv) t is associative.

Definition 7. The function $t\colon I\times I\to I$ is said to be a strict t norm if (i) and (iv) hold, and

(v) t is continuous, and
(vi) t is strictly isotonic, i.e.

$$t(x_1, y) < t(x_2, y) \quad \text{if} \quad 0 < x_1 < x_2 \leqq 1,$$

$$t(x, y_1) < t(x, y_2) \quad \text{if} \quad 0 < y_1 < y_2 \leqq 1.$$

Theorem 13. Every function $c\colon I\times I\to I$ satisfying conditions T1—T4 is a continuous t norm.

If we assume strict monotonity instead of T1 for function c, then it is a strict t norm and Archimedean in I.

Studies relating to continuous t norms have been performed by Schweizer and Sklar [19], [20].

Finally, let us examine the possibility of constructing the min (x, y) function by means of a generator function. By Theorem 4 (Ling) there is no additive generator of form (13), as it is not Archimedean. Ling studied this problem in some detail [16].

Theorem 14. Let J be the closed interval $[a, b]$ of the real number straight line. If $c(x, y)=\min(x, y)$ for all $(x, y)\in[a, b]\times[a, b]$, then there does not exist a continuous function $g: [a, b]\to[0, \infty]$ such that c can be represented in the form

$$\min(x, y) = g^*(g(x)+g(y)) \tag{24}$$

where it holds for (the not unconditionally continuous) g^* that $g^*(g(x))=x$ for all $x\in[a, b]$.

Theorem 15. Assume that J and c satisfy the conditions of Theorem 14. Then, there does not exist a strictly monotonously decreasing function $g: [a, b]\to\to[0, \infty]$ such that c can be represented in the form

$$\min(x, y) = g^*(g(x)+g(y))$$

where g^* is the function defined in Theorem 14.

A connection may be created between the generator functions and $\min(x, y)$ from another aspect. Let $g(x)$ be the additive generator function of $c(x, y)$.

Theorem 16. $g^\lambda(x)$ $(\lambda>0)$ also has the properties of the generator functions.

Theorem 17. If $c_\lambda(x, y)$ is an operation determined by the generator function $g^\lambda(x)$, then

$$\lim_{\lambda\to\infty} c_\lambda(x, y) = \min(x, y).$$

Theorems 16 and 17 for strictly monotonous functions $c(x, y)$ have been proved by Dombi [4].

3. Examples

(i) Zadeh [24]

$$c(x, y) = \min(x, y),$$
$$(c(x, x) = x,\ x\in I).$$

(ii) Lukasievicz [17]

$$c(x, y) = \max(x+y-1, 0)$$
$$g(x) = \begin{cases} 1-x, & \text{if } x \leqq 1,\\ 0, & \text{if } x > 1\end{cases}$$

(not strictly monotonous, Archimedean).

(iii) [24]

$$c(x, y) = x\cdot y,$$
$$g(x) = -\log x$$

(strictly monotonous).

(iv) Dubois [7]

$$c(x, y) = \frac{x\cdot y}{\max(x, y, \lambda)} = \begin{cases}\dfrac{x\cdot y}{\lambda} & \text{if } \lambda > x, y,\\ \min(x, y), & \text{otherwise,}\end{cases}$$

$$g(x) = -\log\frac{x}{\lambda}, \quad \text{if } x > 0,$$

(v) Hamacher [12]

$$c(x, y) = \frac{\lambda \cdot x \cdot y}{1-(1-\lambda)\cdot(x+y-x\cdot y)},$$

$$g(x) = -\log \frac{\lambda \cdot x}{1+(\lambda-1)\cdot x}.$$

(vi) Yager [23]

$$c(x, y) = \begin{cases} 1-((1-x)^\lambda+(1-y)^\lambda)^{1/\lambda}, & \text{if} \quad (1-x)^\lambda+(1-y)^\lambda < 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} (1-x)^\lambda, & \text{if} \quad x < 1, \\ 0, & \text{if} \quad x \geqq 1. \end{cases}$$

(vii) Dombi [4]

$$c(x, y) = \frac{1}{1+\left(\left(\frac{1}{x}-1\right)^\lambda + \left(\frac{1}{y}-1\right)^\lambda\right)^{1/\lambda}},$$

$$g(x) = \left(\frac{1}{x}-1\right)^\lambda$$

4. Conclusion

The objective outlined in the Introduction has been attained. The square resolution existing in the general case is based on the non-Archimedean nature. If we do not desire such a resolution, then the operations must be restricted to the Archimedean case.

Modification of other conditions means the possibility of a further step in the investigations. An example is the study of the non-continuous case, e.g.

$$t(x, y) = \begin{cases} x, & \text{if} \quad y = 1, \\ y, & \text{if} \quad x = 1, \\ 0, & \text{if} \quad x \neq 1 \quad \text{and} \quad y \neq 1, \end{cases}$$

which otherwise satisfies T1—T3.

Setting out from the generator functions, another research area is the characterization of the possible operation classes, or the study of the connection between various operations, e.g. generalization of the DeMorgan laws.

The question still remains of what connection exists between the empirical examinations and the fuzzy algebraic operations. The research up to date has not provided a satisfactory answer to this.

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Run length control in simulations and performance evaluation and elementary Gaussian processes

By M. ARATÓ[1]

1. Introduction

This paper discusses some statistical problems which arise in analyzing the results of experiments involving the measurement evaluation and comparison of the performance of computing systems, and simulation of such processes, as well. These sequences are generally correlated and in most cases contain a portion which is nonstationary. It is widely accepted that a computer system is operating under a stochastic load and generates stochastic response sequences which are assumed stationary. Such sequences include system response times, utilizations, throughputs (measured e.g. in transactions/sec.), device waiting times, etc. The properties of these output sequences are unknown and the system is being measured in order to estimate characteristics of the specific sequences. As an example the experimenter might be interested in the mean, covariance function of the response times (or even in the response time distribution) and in the utilizations of the major system components (CPU, memory, disks, etc.). Furthermore, the experimenter is often interested in estimating the above quantities as a function of some input parameter such as the number of terminals or transaction rate and in comparing these estimated functions for alternative system configurations. The output sequences are correlated (often strongly) and hence the usual statistical procedures which assume independent observations do not apply.

Let us consider a database system (see e.g. [8], [9]), where transaction response time and transaction rate are particularly important. These have been chosen as the major criteria for evaluating an alternative system. There were made modifications to the operating system so that certain supervisory functions which account for a substantial amount of processor utilization are executed on a separate processor.

A typical time series of transaction response times and its sample correlation function is given in Figure 1.

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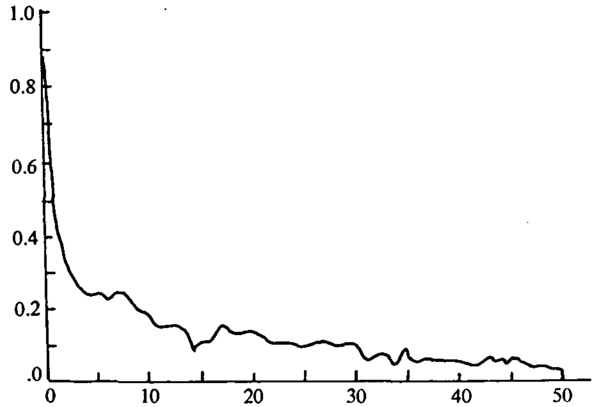

Fig. 1
Sample covariance function of transaction response time

The problem of getting confidence intervals for the mean of a stationary output sequence from a discrete event simulation has an upgrowing literature and program packages (see e.g. [9], [10], [12] and [14]). This problem is connected with a run length control procedure which is designed to terminate the simulation when a confidence interval of a prespecified relative width has been generated or to continue the run to a maximum length.

This paper is concerned with the above mentioned problems for the following practical point of view. Instead of using the spectral analysis techniques, which assume indirectly the asymptotic normality, we are using the stochastic difference and differential equation method, which enables us to calculate the confidence limits in advance, to get exact results in the Gaussian case and, at the same time, good approximations for non-Gaussian sequences.

The results are in good agreement with those of the simulation (see [9], [10]), though the calculations can be carried out on a small calculator, using the tables of the known exact distribution of the maximum likelihood estimator of the damping parameter of an autoregressive (AR) process.

There exist many approaches to the problem of generating confidence intervals for the mean of dependent sequences of random variables and for determining the length of a steady-state simulation. In our method we get the same results by simple calculations based on the concept of sufficient statistics and on the approximation of discrete time process by continuous time process. It is remarkable that explicit results can be gotten and carried out only in the continuous time case.

The main novelty in our method is not only its simplicity, but in the direct estimation of the correlation and giving sufficient statistics. Indeed, instead of the tedious calculations of spectral densities we are using only the first covariances and the boundary random variables which keep the storage requirements of the method extremely low.

Using two estimates

$$\overline{X}_N = \frac{1}{N}\sum_{1}^{N} X_i, \quad \overline{X}_0 = \frac{X_1+X_N}{2} \tag{1}$$

for the unknown mean $\mu=EX_i$ in the correlated case it is not known which of them is better. Let, for simplicity, X_i be the following time series $X_i=y_i+\mu$, where

$$y_i = \varrho y_{i-1}+\varepsilon_i, \quad \left(E\varepsilon_i = 0,\, \sigma_\varepsilon^2 = (1-\varrho^2)\sigma_y^2\right). \tag{2}$$

Then $\bar{X}_N$ is not uniformly, in $0<\varrho<1$, a better estimate than $\bar{X}_0$, in the sense that $\operatorname{Var}\bar{X}_N \geqq \operatorname{Var}\bar{X}_0$, if $\varrho \sim 1$ (see [3], [4]) and compare with (10) below.

Finally, let us point out that in constructing confidence bounds by the spectral method and by the normal approximation, one can find a gap in the earlier proofs, because the authors do not care about the question of uniform (in $0<f_x(0)<\infty$, where $f_x(\lambda)$ is the spectral density of process x) normal approximation when the number of observations $N\to\infty$. Nevertheless, it can easily be seen that uniform approximation does not hold even in the above mentioned special case (2), if $0\leqq\varrho<1$ (see [2], [4]).

2. Preliminary results

The sample covariance functions of waiting time and response time experiments show an exponentially decaying and never an oscillating character, which allows us not to be interested in checking hidden periodicities. In this case, all the roots of the characteristic equation of a higher order AR process are real and negative (in the continuous time case), or less than, in moduli, 1 (in the discrete time case).

This makes possible to assume that the process or one of his derivatives has a simple structure. Our method can be used for higher order autoregressive schemes too, after simple transformations and assuming that the roots of the characteristic polynomial are real.

On the basis of the sample covariance function we may assume that the sequence of observations $X(1), X(2), \ldots$ forms a realization of a one dimensional stationary, Markovian and Gaussian process $\xi(n)$ (called elementary Gaussian), with unknown parameters $\mu=E\xi(n)$, $\sigma_\xi^2=D^2\xi(n)=\operatorname{Var}\xi(n)$ and

$$\operatorname{corr}\left(\xi(n), \xi(n-1)\right) = \varrho, \quad \text{i.e.,}$$

$$\left(\xi(n)-\mu\right) = \varrho\left(\xi(n-1)-\mu\right)+\varepsilon(n), \tag{3}$$

where $\varepsilon(n)$ is a Gaussian white noise with $E\varepsilon(n)=0$, $\sigma_\varepsilon^2=(1-\varrho^2)\sigma_\xi^2$.

We are interested for instance in the construction of confidence limits for the parameter μ, or if we denote the process of the base system by $\xi_1(n)$ and the alternative system, after certain functional redistribution by $\xi_2(n)$ then the main question is that whether the difference of sample means

$$\bar{X}_{N,1}-\bar{X}_{N,2}$$

differs significantly from 0 or not. N is the sample size and

$$\bar{X}_{N,i} = \frac{1}{N}\sum_{n=1}^{N} X_i(n), \quad i = 1, 2. \tag{4}$$

Let us recall the following results (see e.g. [4] or [13]). The spectral density function, $f_\xi(\lambda)$, of the process $\xi(n)$ has the form

$$f_\xi(\lambda)=\frac{1}{2\pi}\frac{\sigma_\varepsilon^2}{|1-\varrho e^{-i\lambda}|^2}=\frac{(1-\varrho^2)\sigma_\xi^2}{2\pi}\frac{1}{(1-\varrho\cos\lambda)^2+\varrho^2\sin^2\lambda}, \tag{5}$$

$$f_\xi(0)=\frac{\sigma_\xi^2}{2\pi}\cdot\frac{1+\varrho}{1-\varrho},\quad 0\leqq\varrho<1.$$

If ϱ and σ_ξ^2 are known the maximum likelihood estimator of μ is the following $\left(\text{where } \frac{X_1+X_N}{2}, \sum_1^N X_i \text{ form a system of sufficient statistics}\right)$,

$$\hat{\mu}=\frac{x_1+x_N+(1-\varrho)\sum_2^{N-1}x_i}{2+(1-\varrho)(N-2)}, \tag{6}$$

which is normally distributed with parameters

$$\left(\mu,\ \sigma_\xi^2\frac{1+\varrho}{2+(1-\varrho)(N-2)}\right). \tag{7}$$

Assuming that $\xi(n)$ is the discrete variant of the continuous process $\xi(t)$ with the differential

$$d\xi(t)=-\lambda\xi(t)\,dt+\sigma_w\cdot dw(t),\quad \varrho=e^{-\lambda\Delta t}, \tag{8}$$

where $w(t)$ is the standard Wiener process, then it is known that σ_w can be estimated exactly and $2\lambda\sigma_\xi^2=\sigma_w^2$. The damping (or decaying) parameter λ (and so ϱ, too) can be estimated poorly and this is the reason why μ has fairly wide confidence intervals. The maximum likelihood estimator of λ is approximately normally distributed if $\lambda T\cong 1000$. Tables of the distribution of the maximum likelihood estimator of the parameter λ can be found in [4], or [5], [6]. In the continuous time case the sufficient statistics of the unknown parameter μ are $\xi(0)+\xi(T)$, $\int_0^T\xi(t)\,dt$ and the maximum likelihood estimator has the form

$$\hat{\mu}=\frac{\xi(0)+\xi(T)+\lambda\int_0^T\xi(t)\,dt}{2+\lambda T}, \tag{9}$$

with variance $2\sigma_\xi^2/(2+\lambda T)$. Note that for $T=1$, $\sigma_w^2=1$ we have

$$D^2\left(\frac{\xi(0)+\xi(1)}{2}\right)=\frac{1+e^{-\lambda}}{4\lambda}<D^2\left(\int_0^1\xi(t)\,dt\right)=\frac{\lambda+e^{-\lambda}-1}{\lambda^3},\quad\text{if}\quad\lambda<2 \tag{10}$$

i.e., depending on λT the mean of two observations can be a better estimate for μ than $\frac{1}{T}\int_0^T\xi(t)\,dt$, and of course better than $\frac{1}{N+1}\sum_{i=0}^N\xi\left(\frac{Ti}{N}\right)$.

The sufficient statistics for λ are

$$s_1^2 = \frac{1}{2}[\xi^2(0)+\xi^2(T)], \quad s_2^2 = \frac{1}{T}\int_0^T \xi^2(t)\,dt, \tag{11}$$

and the maximum likelihood estimator has the form $(\sigma_w^2=1)$

$$\hat{\lambda} = \frac{-[s_1^2-T/2]+\sqrt{[s_1^2-T/2]^2+Ts_2^2}}{2Ts_2^2}. \tag{12}$$

3. Confidence interval construction

Using advantage of the table given in [5] (or [6]) and the approximate variance of $\hat{\mu}$ getting from (7)

$$\frac{\sigma_\xi^2}{N}\frac{1+\varrho}{1-\varrho}, \tag{13}$$

the following approximate confidence intervals can be used for μ/σ_ξ, having the upper $\hat{\varrho}_{\cdot 95}$ and lower $\hat{\varrho}_{\cdot 05}$ confidence bounds for ϱ at the levels 0.95 and 0.05

$$-1.645\sqrt{\frac{1+\hat{\varrho}}{N(1-\hat{\varrho})}} < \frac{\mu}{\sigma_\xi}-\frac{\hat{\mu}}{\sigma_\xi} < 1.645\sqrt{\frac{1+\hat{\varrho}}{N(1-\hat{\varrho})}} = \tilde{\sigma}_{\hat{\varrho}}(\cdot 9), \tag{14}$$

and $\tilde{\sigma}_{\hat{\varrho}}(0.9)$ we call the half confidence interval width at the level $p=0.9$.

Table 1 contains the lower and upper estimates of ϱ for different sample size and the half confidence interval width at level $p=0.9$ and for all the values $\varrho, \hat{\varrho}_{\cdot 95}, \hat{\varrho}_{\cdot 05}$.

From Table 1, one can get estimation for the run length control too, in the sense that the required half-width is attained or not. At given ϱ and ε (half-width) with $\varrho_{\cdot 05}$ one can get the maximum value $\underline{N}(\varrho)$ for which

$$1.645\sqrt{\frac{1+\hat{\varrho}_{\cdot 05}}{N(1-\hat{\varrho}_{\cdot 05})}} < \varepsilon, \tag{15}$$

and the minimal value $\overline{N}(\varrho)$

$$1.645\sqrt{\frac{1+\hat{\varrho}_{\cdot 95}}{N(1-\hat{\varrho}_{\cdot 95})}} < \varepsilon, \tag{16}$$

e.g. for $\varrho=.99=1-\frac{1}{100}$ and $\varepsilon=0.33$ (when $N=5000$) one can get

$$\underline{N}\left(1-\frac{1}{100}\right) = 4320, \quad \overline{N}\left(1-\frac{1}{100}\right) = 7680.$$

Note that in the case when ϱ, σ, m are all unknown, it does not exist such a statistic with known distribution as Student's t in the independent observation case. With this respect we recall the following results (see [2], [3], [4]).

Let us assume for simlicity that $T=1$ and $\sigma_w=1$. Let us take a positive functional $\varkappa(\xi)$ for the lower confidence limit of λ, and $\bar{\mu}(\xi)$, $\underline{\mu}(\xi)$ real-valued functionals as upper and lower confidence limits for μ. We assume that all these func-

Table 1

N	100	500	1000	5000	10000	50000
		$\varrho=0.98$				
λ	2.020	10.101	20.203	101.014	202.03	1010.14
$\hat{\varrho}_{.95}$	0.9996	0.995	0.991	0.985	0.984	0.981
$\hat{\varrho}_{.05}$	0.956	0.969	0.971	0.976	0.977	0.979
$\hat{\sigma}_{.9}(\varrho)$	1.673	0.732	0.518	0.231	0.164	0.979
$\hat{\sigma}_{.9}(\hat{\varrho}_{.95})$	11.630	1.469	0.774	0.268	0.183	0.075
$\hat{\sigma}_{.9}(\hat{\varrho}_{.05})$	1.097	0.586	0.429	0.211	0.153	0.071
		$\varrho=0.99$				
λ	1.010	5.025	10.050	50.252	100.50	502.52
$\hat{\varrho}_{.95}$	0.9999*	0.9993	0.9976	9.9934	0.9924	0.9911
$\hat{\varrho}_{.05}$	0.9750	0.9816	0.9841	0.9869	0.9879	0.9891
$\hat{\sigma}_{.9}(\varrho)$	2.321	1.038	0.734	0.328	0.232	0.104
$\hat{\sigma}_{.9}(\hat{\varrho}_{.95})$	23.263*	3.032	1.501	0.404	0.266	0.110
$\hat{\sigma}_{.9}(\hat{\varrho}_{.05})$	1.426	0.763	0.581	0.287	0.211	0.099
		$\varrho=0.995$				
λ	0.5012	2.506	5.013	25.063	50.125	250.63
$\hat{\varrho}_{.95}$	0.9999*	0.9999	0.9996	0.9973	0.9967	0.9959
$\hat{\varrho}_{.05}$	0.9852	0.9893	0.9908	0.9928	0.9934	0.9942
$\hat{\sigma}_{.9}(\varrho)$	3.286	1.469	1.039	0.465	0.329	0.147
$\hat{\sigma}_{.9}(\hat{\varrho}_{.95})$	23.263*	23.263	3.678	0.633	0.405	0.162
$\hat{\sigma}_{.9}(\hat{\varrho}_{.05})$	1.905	1.003	0.765	0.387	0.286	0.136
		$\varrho=0.998$				
λ	0.202	1.001	2.002	10.010	20.020	100.10
$\hat{\varrho}_{.95}$	0.9999*	0.9999*	0.9999	0.9995	0.9991	0.9985
$\hat{\varrho}_{.05}$	0.9925	0.9950	0.9955	0.9968	0.9971	0.9975
$\hat{\sigma}_{.9}(\varrho)$	5.199	2.325	1.644	0.735	0.520	0.233
$\hat{\sigma}_{.9}(\hat{\varrho}_{.95})$	23.263*	23.263*	23.263	1.471	0.775	0.269
$\hat{\sigma}_{.9}(\hat{\varrho}_{.05})$	2.681	1.469	1.095	0.581	0.432	0.208
		$\varrho=0.999$				
λ	0.100	0.500	1.001	5.003	10.005	50.03
$\hat{\varrho}_{.95}$	0.99999*	0.99999*	0.99999*	0.99993	0.99976	0.99933
$\hat{\varrho}_{.05}$	0.99700	0.99710	0.99748	0.99815	0.99840	0.99869
$\hat{\sigma}_{.9}(\varrho)$	7.35	3.289	2.326	1.040	0.735	0.329
$\hat{\sigma}_{.9}(\hat{\varrho}_{.95})$	73.566*	73.566*	73.566*	3.932	1.502	0.402
$\hat{\sigma}_{.9}(\hat{\varrho}_{.05})$	4.244	1.931	1.410	0.765	0.581	0.287

The half confidence interval width $\hat{\sigma}_p(\varrho)=1.645\sqrt{(1+\varrho)/N(1-\varrho)}$ at level p for $\mu/\sigma_\xi\cdot$ $\hat{\varrho}_\beta$ means the β level confidence bound of ϱ, $\varrho=e^{-\lambda/N}$, $\lambda=-N\log\varrho$, N is the sample size.

* In the cases marked by * the upper confidence bound for ϱ is equal to 1 and the confidence interval width is ∞ (see section 4).

tionals are continuous on R_ξ in the $C[0, 1]$ metric, but $\bar{\mu}$ and $\underline{\mu}$ may assume values $+\infty$ and $-\infty$. The continuity of functionals assuming infinite values is to be understood as continuity induced by the topology of the real line, closed by points $-\infty$ and ∞. First we have the following assertion, which says that no nonzero lower limit can be constructed for the parameter λ with any degree of confidence.

Theorem 1. *Let $\beta>0$, and let $\varkappa(\xi)$ be a positive functional defined in the space R_ξ and continuous in the $C[0, 1]$ metric, with the property that $\varkappa(\xi)\to\infty$ if $\sup|\xi(t)|\to\infty$. Let it satisfy for any μ and λ the condition $P\{\lambda>\varkappa(\xi)\}>\beta$. Then*

$$P\{\varkappa(\xi)=0\} \geqq g(\lambda, \beta) \tag{17}$$

where the positive function $g(\cdot)$ does not depend on the choice of functional and $g(\lambda, \beta)\to 1$ as $\lambda\to 0$.

For parameter μ the following statement says that if μ, λ are unknown it is impossible to construct finite confidence intervals using continuous functions. We assume that $\bar{\mu}$ and $\underline{\mu}$ has the property that for a real value c

$$\bar{\mu}(\xi+c) = \bar{\mu}(\xi)+c, \quad \underline{\mu}(\xi+c) = \underline{\mu}(\xi)+c. \tag{18}$$

Theorem 2. *Let $\beta>1/2$, and let $\underline{\mu}(\xi), \bar{\mu}(\xi)$ be real valued functionals (which may assume values $-\infty$ or $+\infty$) on the space R_ξ, which are continuous in the $C[0, 1]$ metric and which satisfy the conditions*

$$\begin{aligned} P\{\mu \geqq \underline{\mu}(\xi)\} &\geqq \beta, \\ P\{\mu < \bar{\mu}(\xi)\} &\geqq \beta, \end{aligned} \tag{19}$$

for any μ and $\lambda(-\infty<\mu<\infty, \lambda>0)$. Then

$$\begin{aligned} P\{\bar{\mu}(\xi) = \infty\} &\geqq f(\lambda, \beta), \\ P\{\underline{\mu}(\xi) = -\infty\} &\geqq f(\lambda, \beta), \end{aligned} \tag{20}$$

where $f(\lambda, \beta)$ does not depend on the choice of these functionals, and $f(\lambda, \beta)\to 1/2$ as $\lambda\to 0$.

Simulation results were given in [6] to illustrate the situation and to have a picture on the function $g(\lambda, \beta)$, where the following estimators $(T=1, \sigma_w^2=1)$ were taken:

$$\tilde{m}_1 = \frac{1}{N}\sum_1^N \xi_i, \quad \tilde{\lambda}_1 = \frac{1}{\frac{2}{N}\sum_1^N (\xi_i-\tilde{m}_1)^2},$$

$\tilde{m}_2, \tilde{\lambda}_2$ the maximum likelihood estimators,

$$\tilde{m}_3 = \frac{\xi(0)+\xi(1)}{2}, \quad \tilde{\lambda}_3 = \frac{2}{(\xi(1)-\xi(0))^2},$$

where $\xi_1=\xi(i/N)$, $(i=1, 2, \ldots, N)$, $\xi_0=\xi(0)$. N was taken between 60 and 100 and n (the number of samples) was 1000. We have the following approximations:

$$\begin{aligned}
&g(\lambda, 0.05) \approx 1 \quad \text{if} \quad \lambda<0.5, \text{ (i.e., } P(\varkappa(\xi)=0) = 1 \text{ if } \hat{\lambda}\leqq 0.5 \text{ on level } \beta=0.05),\\
&g(\lambda, 0.05) \approx 1 \quad \text{if} \quad \lambda<4,\\
&g(\lambda, 0.9) \approx 1 \quad \text{if} \quad \lambda<9,\\
&g(\lambda, 0.95) \approx 1 \quad \text{if} \quad \lambda<12.
\end{aligned}$$

It seems that

$$g(\lambda, \beta) \approx e^{-c_\beta \lambda}, \quad \text{when} \quad \lambda \to 0,$$

but this statement is not proved.

Theorem 2 can be reworded as follows: *When the parameters μ and λ of a stationary Gaussian Markov process are unknown, it is impossible to construct finite confidence intervals for μ using continuous functionals.*

From the proof provided in [2] it can be seen that for any $\varepsilon>0$ there exists a $\Lambda(\varepsilon)$ such that for small values of λ

$$\sup_{\lambda<\lambda_0, \mu} P_{\lambda,\mu}\{\bar{\mu}(\xi) > \mu\} \leqq 1/2+\Lambda \cdot \lambda_0^{(1/2)-\varepsilon}.$$

4. Run length control and sequential estimation

Running a simulation less than its length would not provide the information needed, while running it longer would be a waste of time, so it has great practical meaning for the experimenter to have some preliminary estimation about the accuracy requirements. We shall assume further, that this accuracy requirement is specified by the half-width of the confidence interval of the mean value, μ, devided by the standard deviation, σ_ξ, of the process $\xi(t)$. In this section we will describe the incorporation of the method of sections 2 and 3 into a sequential estimation procedure. We shall show that one possible approach is that, when using the approximation with continuous time we estimate the decay parameter λ (and so ϱ) by given accuracy. This procedure uses the same amount of storage required earlier but uses some new random time moments (the Markov moments) and requires only a small amount of computing per output element.

Let us denote by ε the required relative half-width of the ratio μ/σ_ξ, and by $p=1-\beta$ the given confidence level, and $x_{1-(\beta/2)}$ the $1-(\beta/2)$-quantile of the Gaussian distribution.

For given α, where $1-\alpha$ means the confidence level for ϱ, to make small the difference

$$\frac{x_{1-(\beta/2)}}{\sqrt{N}}\left[\sqrt{\frac{1+\hat{\varrho}_{1-(\alpha/2)}}{1-\hat{\varrho}_{1-(\alpha/2)}}}-\sqrt{\frac{1+\hat{\varrho}_{\alpha/2}}{1-\hat{\varrho}_{\alpha/2}}}\right] < \varepsilon_1, \tag{21}$$

we shall take advantage of sequential estimation of ϱ. For given α and c let us take H in such a way that (x_α denotes the α quantile of normal distribution).

$$H < \frac{c^2}{(x_{1-(\alpha/2)}-x_{\alpha/2})^2}. \tag{22}$$

Further, let us denote by

$$\tau(H)=\inf\left\{t\colon \int_0^t \xi^2(s)\,ds \geqq H(\varepsilon,\alpha)\right\}, \tag{23}$$

the Markov moment and take

$$\lambda(H)=-\frac{1}{H}\int_0^{\tau(H)} \xi(t)\,d\xi(t)=-\frac{\xi^2(\tau(H))-\xi^2(0)-\tau(H)}{2\int_0^{\tau_H}\xi^2(t)\,dt}. \tag{24}$$

Then the following statement is true (see Liptser, Shiryaev [13], Arató [4]).

Theorem 3. *The sequential estimator $\lambda(H)$ is normally distributed with parameters*

$$E_\lambda\lambda(H)=\lambda,\quad D^2(\lambda(H))=\frac{1}{H}, \tag{25}$$

and it is efficient, i.e., it has minimal variance.

The calculated H depends on C, α and from the realization getting $\tau(H)$ for given ε_1, β it is possible to check (compare with (21)).

$$\frac{x_\beta}{\sqrt{\tau(H)}}\left[\sqrt{\frac{1+\hat{\varrho}_{1-(\alpha/2)}}{1-\hat{\varrho}_{1-(\alpha/2)}}}-\sqrt{\frac{1+\hat{\varrho}_{\alpha/2}}{1-\hat{\varrho}_{\alpha/2}}}\right]<\varepsilon_1, \tag{26}$$

where $\hat{\varrho}=e^{-\hat{\lambda}\cdot\Delta/N}$, $\hat{\lambda}_{1-(\alpha/2)}=\lambda(H)+x_{1-(\alpha/2)}/\sqrt{H}$. After the fulfilment of (26) one can construct confidence limits for the unknown mean μ.

To get some approximations for $\tau(H)$ one has to turn to the papers of Novikov [15]—[17] (see also Liptser—Shiryaev [13]).

Theorem 3 remains valid (under some natural conditions on $a(t,\xi)$) if we regard the process

$$d\xi(t)=\lambda a(t,\xi(t))\,dt+dw(t),$$

(see Liptser—Shiryaev [13] § 17.5).

A natural question arises whether the advantages of sequential estimators are consequences of a rather long mean observation time $E_\lambda(\tau(H))$. For general $a(t,\xi)$ this question is unsolved. The following statement is true (see Novikov [17]).

Theorem 4. *For $\lambda\geqq 0$ as $T\to\infty$,*

$$P_\lambda(\tau(H)\geqq T)=4\left(\frac{H}{\pi T^2}\right)^{1/2}\exp\left\{-\frac{\lambda^2 H}{2}-\frac{T^2}{8H}+\frac{\lambda T}{2}\right\}(1+o(1)), \tag{27}$$

$$E_\lambda\tau(H)\leqq 2\left[\lambda H+2\sqrt{H}\right]+\sqrt{8(\lambda^2H^2+4\lambda H)+2H}. \tag{28}$$

Further, if $\lambda^2 H\to\infty$, then

$$E_\lambda\tau(H)=2\lambda H\left(1+\frac{3}{4\lambda^2 H}+o\left(\frac{1}{\lambda^2 H}\right)^2\right), \tag{29}$$

and if $\lambda^2 H\to 0$, then

$$E_\lambda\tau(H)=H^{1/2}[2.09+0.856\lambda H^{1/2}+o(\lambda^2 H)]. \tag{30}$$

Note that remarkable fact that these results are in good agreement of those simulation results which are published in HEIDELBERGER, WELCH [9], [10] or HEIDELBERGER [8].

Tables based on Theorems 3 and 4, one can construct easily.

Abstract

This paper intends to show that the method proposed by Kolmogorov in constructing confidence limits for diffusion type processes gives a more simple and straightforward tool in run length control of output sequences of stationary series than the spectral method. There exists an upgrowing literature of the spectral method for construction confidence limits (see e. g. the survey paper HEIDELBERGER, WELCH [9]), and even software program packages were constructed on this basis. We show that the Gaussian processes, when the computational requirements and storage remain low, can be used as good approximations with the advantage that instead of simulation one can get exact formulas. The connection between run length control and sequential estimation methods are found and some results of Novikov can be used.

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A Theory of Finite Functions, Part I.
On finite trees associated to certain finite functions

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1. Introduction

1.1. Let A be a set of cardinality l, $l\in\omega$, $l\geqq 2$. For $n, m\in\omega$ we set $O_A^{(n,m)}=\{f|f\colon A^n\to A^m\}$ and $O_A^{(m)}=\bigcup_{n\in\omega} O_A^{(n,m)}$. Certain subsets of $O_A^{(m)}$, in particular, of $O_A^{(1)}$, are interesting for the very different mathematical theories of algebra, logic and computer science. For example, the celebrated result of I. Rosenberg picks up some subsets when enumerating maximal closed classes of $O_A^{(1)}$ [9]. Several special types of functions such as monotone (unate) and symmetric ones play a role in the theory of logic design [10], and in other applications of finite functions (cf. e.g. Dedekind's problem on freely generated lattices generated by finitely many generators). In the common part of logic and computer science, e.g. in the theory of theorem-proving and of semantics for programming languages, certain restrictions to logical formulae with prescribed forms seem to help in increasing efficiency [8].

1.2. One possible method for investigation the properties of these subsets is to associate special finite algebras (or more precisely finite graphs and trees) to the elements of $O_A^{(m)}$. There is a very common way of doing this: let the "parse tree" be associated to each function. By this correspondence several remarkable results have been established. The parse tree, however, mirrors mostly the syntactical features of the function at hand and very little can be learnt about the "semantics" of the mapping by the parse tree only. Here we suggest another tree-representation of finite functions — the valuation tree — and show the use by examples. Valuation trees are compressed forms of valuation tables (generalized truth tables) of functions (for $l=2$, see [5]). It should be mentioned that a more compact representation in graph forms can also be given, cf. [1] for $l=2$. Trees, however, seem to be more tractable in spite of or thanks to their redundancies. Clearly, valuation trees are completely semantically oriented and designed to contain all information about the action of a function.

1.3. The natural question arises what kinds of trees are associated to certain interesting subsets of $O_A^{(m)}$. Our main contribution in this first part of a series of papers is to present a uniform graphical property, the level-homogenity, to answer this question. As an illustration we apply the method for three well-known pre-primal subsets of $O_A^{(1)}$. In this paper we do not assume any algebraic structure on A except the ordering relation. From the second part, however, we shall endow some more operations to A, in fact, we suppose that A is a Post-algebra of order l and apply the results obtained in Part I to this case. Actually, we shall develop some optimization techniques for synthesizing Post formulae. Later parts are devoted partly to complexity questions where several estimates are established concerning the methods of Part I and II, and partly to different problems concerning finite functions.

1.4. The organization of this paper is as follows. In Section 2, we overview the notations used in this series of papers. In Section 3 we deal with trees and introduce several notions and notations concerning them. Some notions of this section will be used only in later parts, but is presented here for the sake of uniformity. Key notion of these considerations, the level-homogeneous tree, will be introduced in Section 4. This section deals with some auxiliary concepts, too. Finite functions enter in Section 5 where, after a general representation theorem, we investigate degenerate, order-preserving, value-preserving and permutation-preserving functions in terms of trees.

We note that this paper is selfcontained, i.e. no preliminary knowledge is assumed.

2. Preliminaires

2.1. Let ω be the set of finite ordinals, $\emptyset$ is the empty set. If $m\in\omega$, then we make use of the following notations: $\{m\}=\{0, 1, ..., m-1\}$, $[m]=\{1, 2, ..., m\}$, $[0]=\emptyset$, $[\omega]=\{1, 2, ...\}$. We shall fix $2\leq l<\omega$; $n, m\in\omega$ and the set A of cardinality l. Since A is finite, it can be identified with $\{l\}$. We shall usually use this identification. From now on, the letters l, m, n, A will always refer to these fixed sets. Let $<$ be the well-known total ordering on $\{l\}$ (and thus on A). We extend $<$ to the elements of $\{l\}^n$ (hence to A^n) componentwise. The elements of the set $O_A^{(n,m)}=\{f|f\colon A^n\to A^m\}$ will be called n-ary A-functions with m output. We make this concept independent of arity by setting $O_A^{(m)}=\bigcup_{n\in\omega} O_A^{(n,m)}$. If $f\in O_A^{(m)}$, then

$$f=(f_1, ..., f_m) \quad \text{where} \quad f_i\in O_A^{(1)} \quad \text{for all} \quad i\in[m]; \quad \text{i.e.} \quad O_A^{(m)}=(O_A^{(1)})^m. \tag{1}$$

If $g\notin O_A^{(m)}$ and g is a function (a meta-function) of n arguments, then $e_1e_2 ... e_ng$ will denote the application of g to the arguments $e_1, e_2, ..., e_n$. This is to be distinguished from any application of a function $f\in O_A^{(m)}$ which will be displayed as $fe_1 ... e_n$.

Let $f\in O_A^{(n,1)}$. By $f(x_i/\alpha)$ we mean a function in $O_A^{(n-1,1)}$ which is obtained from f by substituting α for each occurrence of x_i provided x_i occurs in f, otherwise let $f(x_i/\alpha)=f$, (and hence in $O_A^{(n,1)}$). $f^*(x_1/\alpha_1, ..., x_n/\alpha_n)$ denotes the value of f under substituting its variables $x_1, ..., x_n$ by $\alpha_1, ..., \alpha_n$ in due course. In Part II we shall give a more detailed method for computing this value (by assuming that A is a Post algebra).

If $f \in O_A^{(n,m)}$, then we always assume that an ordering of variables occurring in f, say $x_1, x_2, \ldots, x_n$, is fixed. This convention will be essential from Section 5.

Let $f \in O_A^{(n,m)}$. Then, for every $n_1 > n$, f can be considered as a function of n_1 variables, i.e. $f \in O_A^{(n_1,m)}$ (cf. subsection 5.2).

The cardinality of a set H is denoted by card H. $\mathscr{P}H$ is the powerset of H. If f is a function defined on H and $H' \subset H$, then $f \restriction H'$ is the restriction of f onto H'. Range f and Dom f denote the range and domain of the function f, respectively.

We shall omit all indices without any remark unless confusion can occur. In this paper $O_A^{(n,m)}$ will be denoted by $O_l^{(n,m)}$ to emphasize that no algebraic operations are present on A. All considerations apply for arbitrary $m > 0$, however, for the sake of simplicity, we often give definitions and assertions in the case $m = 1$, only. If generalization for larger m is not straightforward then we shall explicitly discuss it.

3. Trees

3.1. Let V be an arbitrary set and $\varrho: V \to \{l+1\}$. The pair (V, ϱ) is called an l-ary *pretree* (ranked set). We set $E_{V,\varrho} = \{(v, i) \mid v \in V \wedge i \in [v\varrho]\}$. The function ϱ is the *rank function* of the pregraph; $v\varrho$ is the *rank* of v in (V, ϱ) provided $v \in V$. $E_{V,\varrho}$ is the set of *edges*.

The triplet $T = ((V, \varrho), \sigma, (\varepsilon_1, \ldots, \varepsilon_m))$ is an *m-rooted l-ary tree* if and only if (V, ϱ) is an l-ary pretree; $\sigma: E \to V$; $\varepsilon_1, \ldots, \varepsilon_m \in V$ and the following (Peano-like) conditions are satisfied:

(i) σ is a bijection.

(ii) Range $\sigma \cap \{\varepsilon_i \mid i \in [m]\} = \emptyset$.

(iii) If $V' \subset V$ is such that $\{\varepsilon_i \mid i \in [m]\} \subset V'$ and $[V']_\sigma \subset V$, where $[V']_\sigma$ denotes the closure of V' under σ, then $V' = V$.

The elements of V are called *points* of T; the point ε_i $(i \in [m])$ is the i-th *root* and σ is the *successor function* of T.

Note, that $m = 0$ implies $V = \emptyset$. We shall use the name *leaf* for an element of $0\varrho^{-1}$ (of a given tree), where $a\varrho^{-1}$ denotes the total inverse of ϱ on a. Clearly, card $V \in [\omega]$ entails $0\varrho^{-1} \neq \emptyset$. From now on, we always assume that card $V \in [\omega]$ and $m \neq 0$.

We remark, that m-rooted trees are usually defined in a different way (cf. [2]). The definition presented here is originated from C. C. Elgot et al. and is proved equivalent to the more common one used in the literature in [6].

3.2. We define the *immediate successors* vD_T^1 and the successors vD_T of v in T as follows:

$$vD_T^1 = \{v' \mid v' \in V \wedge \exists i (i \in [v\varrho] \wedge v' = (v, i)\sigma)\} \tag{2}$$

and

$$vD_T = \{v' \mid v' \in V \wedge (\exists n \in \omega, \exists f\colon [n+1] \to V)(1f = v \wedge (n+1)f = v' \wedge \wedge (\forall j \in [n])((j+1)f \in (jf)D_T^1))\}. \tag{3}$$

In particular, $v \in vD_T$, i.e. $\varepsilon_i D_T \neq \emptyset$ provided $i \in [m]$, and for all $v \in V$, there exists a unique $i \in [m]$ such that $v \in \varepsilon_i D_T$. The following assertion is immediate by definitions.

Lemma 1. *Let T be a tree and $\varepsilon_i \in V$, $i \in [m]$, furthermore assume that $v \in \hat{\varepsilon}_i D_T$. Then, there exist exactly one n ($n \in \omega$) and exactly one f such that*

$$1f = \varepsilon_i \wedge (n+1)f = v \wedge (\forall j \in [n])\big((j+1)f \in (jf) D_T^1\big) \tag{4}$$

holds.

If the conditions of Lemma 1 are fulfilled for v, then n and f, determined uniquely by (4) and the remark preceeding the assertion, are called the *level of* v and the *derivation function of* v, respectively. We shall use the notations, $v\lambda_T$ for the level of v and vd_T for the sequence $(1f, 2f, \dots, (n+1)f)$, the *derivation* of v in T. By a *path* we mean a derivation of a leaf $v \in 0\varrho^{-1}$. We shall denote the set of all paths of T by P_T. Clearly, card P_T=card $0\varrho^{-1}$. For each $p=(1f, 2f, \dots$ $\dots, (n+1)f) \in P_T$, there exist a unique i and a sequence $(k_1, \dots, k_n)$ such that $1f=\varepsilon_i$ and for all $j \in [n]$, $(jf, k_j)\sigma = (j+1)f$, hence we can use the pair $(i, (k_1, \dots, k_n))$ to identify paths. Note, that the set of paths in T completely determines T, thus P_T and T can be identified and is actually done at several points of this paper.

3.3. We define $T\lambda$, the *level of* T, as follows:

$$T\lambda = n \Leftrightarrow (\forall v \in 0\varrho^{-1})\big((v\lambda_T \leqq n) \wedge (\exists v \in 0\varrho^{-1})(v\lambda_T = n)\big)$$

i.e., $T\lambda$ is the least element of ω such that every leaf of T has level less than or equal to n. The tree T is exactly of level n if and only if

$$(\forall v \in 0\varrho^{-1})(v\lambda_T = n).$$

3.4. The m-rooted l-ary tree T exactly of level n is *complete* if and only if $(\forall v \in V)(v\varrho = l)$. It follows that in an m-rooted l-ary complete tree T,

$$\text{card}\, P_T = m \cdot l^n.$$

The following observation is trivial but very useful.

Lemma 2. *Let T_1 and T_2 be arbitrary m-rooted l-ary complete trees of level n. Then T_1 and T_2 are isomorphic.*

Let T be an m-rooted l-ary tree of level n and let $h \in [n]$. We say that T is *complete on level h* if and only if $(\forall v \in V_T)(v\lambda = h \Rightarrow v\varrho = l)$.

3.5. Let T_1 and T_2 be two m-rooted l-ary trees exactly of level n. We say T_1 is a *subtree* of T_2 if and only if $P_{T_1} \subset P_{T_2}$ and for all $p=(1f, 2f, \dots, (n+1)f) \in P_{T_1}$, if for some $i \in [m]$, $(n+1)f \in \varepsilon_i D_{T_2}$, then $(n+1)f \in \varepsilon_i D_{T_1}$. Note, that if T_1 is a subtree of T_2, then it may well happen that T_1 is not a subalgebra of T_2, and vica versa. If there is a subtree T' in T_2 such that T' is isomorphic to T_1, then we say T_1 is *embeddable* in T_2. Obviously, every m-rooted l-ary tree exactly of level n is embeddable in an (m-rooted l-ary) complete tree. The embedding is, up to isomorphism, unique by definition and Lemma 2.

3.6. Let T be an m-rooted l-ary tree exactly of level n and let $P \subset P_T$. P defines, in the natural way, an m-rooted l-ary tree exactly of level n which is a subtree of T, *the subtree of T determined by P*. This subtree is unique and we denote it by T_P.

Let T be an m-rooted l-ary tree exactly of level n. Let $p=(i,(k_1,\ldots,k_n))\in P_T$, $q=(j,(h_1,\ldots,h_n))\in P_T$. We let $p\sim q\Leftrightarrow(\forall s\in[n])(k_s=h_s)$. Clearly, $\sim$ is an equivalence relation. Set $\tilde{p}=\{q\,|\,q\in P_T\wedge p\sim q\}$ and $\tilde{P}_T=\{\tilde{p}\,|\,p\in P_T\}$. The 1-rooted l-ary tree exactly of level n determined by $\tilde{P}_T$ is named the *compressed form of* T and is denoted by T^c. Let $P\subset P_T$, then the subtree $T^c_{\tilde{P}}$ of T^c determined by P is called the *compressed-subtree of T determined by* P. Note, however, that this name is a somewhat misleading: $T^c_{\tilde{P}}$ is not a subtree of T in the very sense of 3.5.

3.7. Let $T=((V,\varrho),\sigma,(\varepsilon_1,\ldots,\varepsilon_m))$ be an m-rooted l-ary tree of level n, $v\in 0\varrho^{-1}$. Let us suppose that $v\lambda_T=h$, $h<n$. Let V_1 be a set of new points with cardinality $\sum_{i\in[n-h]} l^i$. The tree $T^E_v=((V\cup V_1,\varrho'),\sigma',(\varepsilon_1,\ldots,\varepsilon_m))$ is defined as follows: $\varrho'\upharpoonright V=\varrho$, and for all $w\in V_1$, $w\varrho'=l$; $\sigma'\upharpoonright V=\sigma$ and σ' is extended to V_1 in such a way that T^E_v is a tree ($f\upharpoonright H$ denotes the restriction of the function of f to the set H). Roughly speaking, the tree T^E_v is obtained from T by identifying the root of a 1-rooted l-ary complete tree of level $n-h$ to v. Let $\{v_1,\ldots,v_s\}\subset 0\varrho^{-1}$ be that set of leaves, the level of which is strictly less than n. Let $T_0=T$ and for every $r\in[s]$, $T_r=(T_{r-1})^E_{v_r}$. Then, T_s is unique up to isomorphism and is called *the extended form of* T, in notation T^E. Clearly, T^E is an m-rooted l-ary tree exactly of level n.

3.8. Let T be a complete m-rooted l-ary tree of level n and define the *index function* $\delta\colon P_T\to\{ml^n\}$ by the formula

$$p\delta=(i-1)l^n+\sum_{j\in[n]}k_j\cdot l^{n-j} \tag{5}$$

where p is determined by the pair $(i,(k_1,\ldots,k_n))$. Clearly, δ is a bijection, hence for each $k\in\{m\cdot l^n\}$, there exists $p\in P_T$ such that $p\delta=k$. If p is determined by the pair $(i,(k_1,\ldots,k_n))$ then we shall make use of the following notations $p=k\delta^{-1}$, $(i,(k_1,\ldots,k_n))\Delta=k$, $k\Delta^{-1}=(i,(k_1,k_2,\ldots,k_n))$. We use also the *compressed index function* $\delta^c\colon P_T\to\{l^n\}$ defined by

$$p\delta^c=\sum_{j\in[n]}k_jl^{n-j}. \tag{6}$$

If T is not complete but is exactly of level n, then $\delta=\delta'\upharpoonright P_T$, where δ' is the index function defined on the complete tree in which T is embeddable. It is obvious, that P_T determines a unique subset of $\{m\cdot l^n\}$; the notations introduced above apply in the natural way. If T is of level n but is not exactly of level n, then we extend δ as follows: $\delta^E\colon P_T\to\mathcal{P}\{ml^n\}$; for $p=(1f,2f,\ldots,(h+1)f)\in P_T$, let $p\delta^E=\{p'\delta\,|\,p'\in P_{T^E}\wedge p'=(s_1,s_2,\ldots,s_{n+1})$ such that for all $j\in[h+1],s_j=jf\}$. It follows, that if $h=n$, then $p\delta^E=p\delta$.

δ^E, the *extended index function,* is well defined since δ is a bijection. It follows that δ^E is injective as well and thus we can employ the natural generalizations of $(\delta^E)^{-1}$, Δ^{-1}, Δ to those k which are in the range of δ^E.

3.9. Let T be an m-rooted l-ary tree. The pair (T,τ) is called a *terminated* (m-rooted, l-ary) tree if and only if $\tau\colon P_T\to\{l\}$.

3.10. Let us define the following function $\xi^{(l)}\colon\omega\to\omega^{l-1}$; for $k\in\omega$ let $k\xi^{(l)}=(\xi^{(l)}_1,\xi^{(l)}_2,\ldots,\xi^{(l)}_{l-1})$ where $\xi^{(l)}_i$, $i\in[l-1]$ is the number of occurrences of i in the l-ary expansion of k. Let T be an m-rooted l-ary tree exactly of level n. The pair (T, ξ) is called a *ξ-augmented* (m-rooted, l-ary) *tree* if and only if $\xi\colon P_T \to \omega^{l-1}$ is defined by $p\xi = (p\delta^c)\xi^{(l)}$. The following assertion can be proved by an easy induction.

Lemma 3. *Let (T, ξ) be a ξ-augmented m-rooted l-ary tree, $p = (1f, \dots, (n+1)f) \in P_T$ and $p\xi = (\xi_1^{(l)}, \dots, \xi_{l-1}^{(l)})$. Then, for all $s \in [l-1]$,* card $\{(jf, s) | j \in [n] \wedge (jf, s)\sigma = (j+1)f\} = \xi_s^{(l)}$. In other words, if $p = (i, (k_1, \dots, k_n))$, then $\xi_s^{(l)}$ gives the number of k_j such that $k_j = s$.

4. Homogeneous trees

4.1. Let T be a 1-rooted l-ary tree exactly of level n; $j \in [n]$. T is called *λ-homogeneous* (to shorten the term level-homogeneous) *on level j* if and only if

$$(\forall v_1, v_2 \in V,\ \forall h, k \in [l])\big((v_1\lambda = v_2\lambda = j \wedge v_1 \neq v_2 \wedge (v_1, h) \in E) \Rightarrow \Rightarrow ((v_2, k) \in E \Leftrightarrow k = h)\big). \tag{7}$$

An equivalent formalization of (7) is the following

$$(\forall v_1, v_2 \in V,\ \forall k \in [l])\big((v_1\lambda = v_2\lambda = j \wedge v_1 \neq v_2) \Rightarrow \Rightarrow ((v_1, k) \in E \Leftrightarrow (v_2, k) \in E)\big). \tag{8}$$

T is *λ-homogeneous* if and only if, for all $j \in [n]$, T is λ-homogeneous on level j. Clearly, any path $p \in P_T$ considered as a tree, any complete tree and any tree exactly of level 1 is λ-homogeneous.

Let r be a binary relation on $\{l\}$ and T a 1-rooted l-ary tree exactly of level n. We can extend r to paths of T by defining $(p, q) \in \bar{r} \Leftrightarrow$ for all $j \in [n]$, $(p_j, q_j) \in r$, where $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n) \in P_T$.

The following assertion, although it is trivial, gives some insight into the very nature of λ-homogeneous trees.

Lemma 4. *Let r be an arbitrary binary relation on $\{l\}$, let T be a 1-rooted l-ary tree exactly of level n, and let $\bar{r}$ denote the extension of r to P_T defined as above. Then for every $p \in P_T$, the set $\{p' | p' \in P_T \wedge (p, p') \in \bar{r}\}$ uniquely determines a λ-homogeneous subtree of T.*

Proof. It follows that $\{p' | p' \in P_T \wedge (p, p') \in \bar{r}\}$ defines a unique subtree of T; let $T_{r;p}$ denote this subtree, and let $p = (k_1, \dots, k_n)$. Let us suppose, that $v_1, v_2 \in V_{T_{r,p}}$ such that $v_1 \neq v_2$ and $v_1\lambda = v_2\lambda = h$ for some $h \in [n]$. Then, $v_1\varrho = v_2\varrho$ and for all $j \in [v_1\varrho]$, $(v_1, j) \in E \Leftrightarrow (j, k_h) \in r \Leftrightarrow (v_2, j) \in E$. But then $((v_1, j) \in E \Rightarrow (v_2, j) \in E) \Leftrightarrow (j, k_h) \in r$, hence $T_{r,p}$ is λ-homogeneous on level h. Being h arbitrary, we have that $T_{r,p}$ is λ-homogeneous.

If r is nonempty and total (i.e. $\forall x \exists y((x, y) \in r)$), then $T_{r,p}$ is not empty. We also note, that the converse of the lemma is not true; more precisely, if T_1 is a λ-homogeneous subtree of T, then it may well happen that there is no binary relation r on $\{l\}$ such that $T_1 = T_{r,p}$ for an appropriate $p \in P_T$.

In particular, if r is a partial ordering or is a non-trivial equivalence or $r = \{(x, \pi x) | x \in \{l\}\}$ where π is a permutation of $\{l\}$ with l/q cycles of the same prime length q, then $T_{r,p}$ is λ-homogeneous by Lemma 4. All of these relations are total so if $r \neq \emptyset$, then $T_{r,p} \neq \emptyset$ for any $p \in P_T$. This observation establishes some links between λ-homogeneous trees and (three) types of maximal closed classes exhibited by Rosenberg's completeness theorem. The main interest of this paper is, however, to use λ-homogeneous subtrees of a tree to portrait some elementary properties of the function to which the tree at hand is associated by Theorem 13 below, hence we do not provide similar results for the other (three) types of maximal closed classes. Instead, we study further λ-homogeneous trees. The following lemmata are immediate.

Lemma 5. *Let T be a 1-rooted l-ary tree exactly of level n. If T_1 is a λ-homogeneous subtree of T, then there exists a maximal λ-homogeneous subtree T_2 of T containing T_1; i.e. $P_{T_1} \subset P_{T_2} \subset P_T$ and T_2 is not a subtree of any λ-homogeneous subtree of T containing T_1 other than T_2.*

Note, that T_2 is not unique in general.

Lemma 6. *Let T be a 1-rooted l-ary tree exactly of level n, let r be a nonempty reflexive binary relation on $\{l\}$. Then, for every $p \in P_T$, the tree $T_{r,p}$ is the unique maximal λ-homogeneous subtree of T which contains p.*

Proof. Since r is reflexive, $p \in P_{T_{r,p}}$. λ-homogenity and uniqueness follow from Lemma 4. It remains to prove that $T_{r,p}$ is maximal. It is, however, trivial by definition since if for some $p' \in P_T$, $(p, p') \in \bar{r}$ then $p' \in P_{T_{r,p}}$, hence no λ-homogeneous subtree of T exists which contains p and $T_{r,p}$ properly.

4.2. Let T be a terminated m-rooted l-ary tree and let $t \subset \{l\}$. T is said to be *τ-homogeneous with respect to* (in short w.r.t.) t if and only if $(\forall p \in P_T)(p\tau \in t)$. T is called *quasi τ-homogeneous w.r.t.* t if and only if

$$(\exists p \in P_T)\big(p\tau \notin t \wedge (\forall p' \in P_T)(p'\tau \notin t \Rightarrow p = p')\big).$$

In particular, if $t \in \{l\}$, then T is τ-homogeneous w.r.t. t if and only if $(\forall p \in P_T)$ $(p\tau = t)$ and T is quasi τ-homogeneous w.r.t. t if and only if for all but one p in P_T, $p\tau = t$.

Let T be a terminated m-rooted l-ary tree and let r be a partial ordering on $\{l\}$; $\bar{r}$ is the expansion of r to P_T. T is *τ-increasing w.r.t.* r if and only if

$$(\forall p, p')\big((p, p') \in \bar{r} \Rightarrow (p\tau, p'\tau) \in r\big).$$

Lemma 7. *Let T be a 1-rooted l-ary terminated tree exactly of level n. Let T_1 be a λ-homogeneous subtree of T which is τ-homogeneous w.r.t. some $t \subset \{l\}$. Then there exists a maximal λ-homogeneous subtree of T which contains T_1 and is τ-homogeneous w.r.t. t.*

Lemma 8. *Let T be a 1-rooted l-ary terminated tree exactly of level n; let r be a partial ordering on $\{l\}$. Then, for every $p \in P_T$, there exists a maximal λ-homogeneous subtree T_1 of T such that*

(i) $p \in P_{T_1} \subset P_{T_{r,p}} \subset P_T$,

(ii) *T_1 is τ-increasing w.r.t. r.*

Lemma 9. *Let T be a 1-rooted terminated l-ary tree exactly of level n; let r be a nontrivial equivalence relation on $\{l\}$. Then, for every $p \in P_T$, there exists a maximal λ-homogeneous subtree T_1 of T such that*

(i) $p\in P_{T_1}\subset P_{T_{r,p}}\subset P_T$,
(ii) T_1 *is τ-homogeneous w.r.t. r;* i.e. $(\forall p, p'\in P_{T_1})((p, p')\in\bar{r}\Rightarrow(p\tau, p'\tau)\in r)$.

Note, that T_1 is not unique in general in either of the above three lemmata. Proofs are immediate by finiteness of trees.

Lemma 10. *Let T be a 1-rooted l-ary terminated tree exactly of level n. Let T_1 be a λ-homogeneous subtree of T which is τ-homogeneous w.r.t. some $t\in\{l\}$, and assume that for some $v\in V_{T_1}$, $v\lambda=j$ and for $k_1, k_2\in\{l\}$, $k_1\neq k_2$, we have both $(v, k_1)\in E_{T_1}$ and $(v, k_2)\in E_{T_1}$. Let $p=(b_1, \ldots, b_{j-1}, k_1, b_{j+1}, \ldots, b_n)$ and $q=(b_1, \ldots, b_{j-1}, k_2, b_{j+1}, \ldots, b_n)$. Then, $p\in P_{T_1}\Leftrightarrow q\in P_{T_1}$.*

Proof. Let $v_1, v_2\in V_{T_1}, v_1\lambda=v_2\lambda=h$. Let $p=(1f, 2f, \ldots, (n+1)f)$, $q=(1g, 2g, \ldots, (n+1)g)$. Let us suppose that, $hf=v_1, hg=v_2$. Let the root of the tree be ε. Then $1f=\varepsilon=1g$, moreover for $h<j$, $hf=hg$ by simple induction. For $h=j$, we have $v_1=v_2$ and $(v_1, k_1)\in E_{T_1}$, $(v_2, k_2)\in E_{T_1}$ by assumption. For $h>j$, we have $(v_1, b_h)\in E_{T_1}\Leftrightarrow(v_2, b_h)\in E_{T_1}$ by λ-homogenity.

Lemma 11. *Let T be a 1-rooted l-ary terminated tree exactly of level n. Let T_1 be a λ-homogeneous subtree of T which is τ-homogeneous w.r.t. some $t\in\{l\}$ and is complete on level j for some $j\in[n]$. Then every path of the form $(b_1, \ldots, b_{j-1}, k, b_{j+1}, \ldots, b_n)$ with fixed $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n\in\{l\}$ and arbitrary $k\in\{l\}$ is in P_{T_1}.*

Proof. It follows from Lemma 10 by an easy induction.

Let $t\subset\{l\}$ and define r_t by

$$(\forall l_1, l_2\in\{l\})\left((l_1, l_2)\in r_t \Leftrightarrow l_1\in t\wedge l_2\in t\right).$$

Clearly, r_t is an equivalence relation. The following assertion is immediate.

Lemma 12. *Let T be a 1-rooted l-ary tree exactly of level n. Let us fix $t\subset\{l\}$ and let $p\in P_T, q\in P_T$. Then $T_{r_t,p}=T_{r_t,q}\Leftrightarrow(p, q)\in\bar{r}_t$.*

It follows from Lemmata 4, 5, 12, that r_t determines a unique maximal λ-homogeneous subtree of T. We shall denote it by T_{r_t}.

4.3. Let T be a terminated ξ-augmented m-rooted l-ary tree. T is *ξ-homogeneous* if and only if

$$(\forall a\in\omega^{l-1})(\exists t\in\{l\})(a\xi^{-1}\subset t\tau^{-1}).$$

4.4. Some further considerations concerning different types of homogenity will appear in later parts. In particular, the notions of anti-λ-homogeneous trees and of combs will be introduced and investigated.

5. Representation of finite functions by terminated trees

5.1. Let $f\in O_l^{(n,m)}$ and let T be an m-rooted l-ary complete tree of level n. We define a terminated tree for f, $T_f=(T, \tau)$, as follows. Let $k\in\{ml^n\}$ be arbitrary and $k\Delta^{-1}=(i, (k_1, \ldots, k_n))$. Then, let

$$(k\delta^{-1})\tau = f^*(x_1/k_1, \ldots, x_n/k_n). \tag{9}$$

By Lemmata 1, 2, the definition (9) is correct.

Theorem 13. *Let $f \in O_l^{(n,m)}$. Then every m-rooted l-ary complete terminated tree $T_f = (T, \tau)_f$ for f is isomorphic to a terminated tree $T'_f = (T', \tau')_f$ with $V \subset O_f^{(n,1)}$.*

Proof. It follows from Lemma 2 that any two m-rooted l-ary complete terminated trees of level n for f are isomorphic. It is sufficient therefore to prove that there exists a terminated tree $(T', \tau')_f$ for f with $V \subset O_l^{(n,1)}$, which is m-rooted, l-ary, complete and of level n. We define (T', τ') by recurrence. Let $i \in [m]$ and $\varepsilon_i = f_i(x_1, \ldots, x_n)$ where f_i is the i-th component of f. If $g(x_1/k_1, \ldots, x_{h-1}/k_{h-1})x_h x_{h+1} \ldots x_n$ is defined as a point of $V \cap \varepsilon_i D_T$ on level h, then let

$$g(x_1/k_1, \ldots, x_{h-1}/k_{h-1})x_h x_{h+1} \ldots x_n \varrho = l \quad \text{and} \quad g(x_1/k_1, \ldots, x_{h-1}/k_{h-1})x_h x_{h+1} \ldots x_n D^1_{T'} =$$
$$= \{g(x_1/k_1, \ldots, x_{h-1}/k_{h-1}, x_h/k)x_{h+1} \ldots x_n \,|\, k \in \{l\}\}$$

and for all $k \in \{l\}$,

$$\big(g(x_1/k_1, \ldots, x_{h-1}/k_{h-1})x_h x_{h+1} \ldots x_n, k\big)\sigma = g(x_1/k_1, \ldots, x_h/k)x_{h+1} \ldots x_n.$$

We stop this recursion on level n, where no point depends on any variables; i.e. eveiy points on level n is of the form $g(x_1/k_1, \ldots, x_n/k_n)$. The leaves of the tree obtained are the points on level n. If p is a path in this tree, then $p\tau'$ is defined by (9). It is not hard to see that V hence $T' = ((V, \varrho), \sigma, (\varepsilon_1, \ldots, \varepsilon_m))$ are well defined. Clearly, T' is m-rooted, l-ary complete tree of level n, and (T', τ') is for f.

The terminated tree T'_f, defined uniquely up to isomorphism by Theorem 13 is called the *tree associated to* f (recall that T'_f is defined after fixing an ordering of the variables of f; it is clear that T'_f depends heavily on this ordering). In the sequel we simply write T_f to denote the tree associated to f.

From now on in this section we shall assume that $m=1$. The general case can be treated in a similar way at the expense of some complication of technical details.

5.2. Let $f \in O_l^{(n,1)}$. $fx_1 \ldots x_j \ldots x_n$ is *partially degenerate in* x_j if and only if for arbitrary $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n \in \{l\}$, there exist $k_1, k_2 \in \{l\}$, $k_1 \neq k_2$ such that $f^* b_1 \ldots b_{j-1} k_1 b_{j+1} \ldots b_n = f^* b_1 \ldots b_{j-1} k_2 b_{j+1} \ldots b_n$. If for all $k_1, k_2 \in \{l\}$ this equation holds, then f is called *degenerate in* x_j. Let $PD_j^{(n,1)}$ and $D_j^{(n,1)}$ denote the sets of functions (in $O_l^{(n,1)}$) partially degenerate and degenerate in x_j, respectively. The set of nondegenerate functions is defined by $ND^{(n,1)} = O_l^{(n,1)} - \bigcup_{j \in [n]} D_j^{(n,1)}$.

Theorem 14. *Let $f \in O_l^{(n,1)}$ and let $(T, \tau) = T_f$. Then, the following two assertions are equivalent. For $j \in [n]$,*

(i) *$f \in PD_j^{(n,1)}$.*

(ii) *For every maximal λ-homogeneous subtree T_1 of T_f, which is τ-homogeneous w.r.t. some $t \in \{l\}$, there exist $k_1, k_2 \in \{l\}$, $k_1 \neq k_2$ such that T_1 contains the edges (v, k_1) and (v, k_2) for all $v \in V_{T_1}$, $v\lambda = j$.*

Proof. Let $f \in PD_j^{(n,1)}$ and assume that T_1 is a maximal λ-homogeneous subtree of T_f which is τ-homogeneous w.r.t. some $t \in \{l\}$. By definition, for all

$p, p' \in P_{T_1}$, we have $f^*(p) = f^*(p')$. Let $p = (b_1, ..., b_j, ..., b_n) \in P_{T_1}$. Since f is partially degenerate in x_j and T_1 is maximal, there exists an $a \in \{l\}$ such that $a = b_j$ and $p' = (b_1, ..., a, ..., b_n) \in P_{T_1}$. Let $k_1 = b_j, k_2 = a$. Then we obtain, that for some v on level j, (v, k_1) and (v, k_2) are in E_{T_1}. T_1 is λ-homogeneous, hence for all $v' \in V_{T_1}, v'\lambda = j$ we have $(v', k_1) \in E_{T_1}$ and $(v', k_2) \in E_{T_2}$.

Conversely, assume that for every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$, there exist $k_1, k_2 \in \{l\}$ such that $k_1 \neq k_2$ and E_{T_1} contains (v, k_1) and (v, k_2) for all $v \in V_{T_1}$ on level j. Let $p = (b_1, ..., b_j, ..., b_n) \in P_{T_f}$ be arbitrary. By Lemma 7, there exists a maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. $p\tau$. By assumption, there exist $k_1, k_2 \in \{l\}$ such that $k_1 \neq k_2$ and $(v, k_1) \in E_{T_1}$, $(v, k_2) \in E_{T_2}$ for all $v \in V_{T_1}$ on level j. By Lemma 10, $p' = (b_1, ..., k_1, ..., b_n)$ and $p'' = (b_1, ..., k_2, ..., b_n)$ are in P_{T_1}. Then, by τ-homogenity, $f^* b_1 ... k_1 ... b_n = f^* b_1 ... k_2 ... b_n$, hence $f \in PD_j^{(n,1)}$.

Theorem 15. *Let $f \in O_l^{(n,1)}$ and let $(T, \tau) = T_f$. Then, the following two assertions are equivalent. For $j \in [n]$,*

(i) $f \in D_j^{(n,1)}$.

(ii) *Every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$ is complete on level j.*

Proof. Let $f \in D_j^{(n,1)}$. Then, by definition, for arbitrary fixed $b_1, ..., b_{j-1}$, $b_{j+1}, ..., b_n \in \{l\}$, and for all $k_1, k_2 \in \{l\}, k_1 \neq k_2$ we have $f^* b_1 ... k_1 ... b_n = f^* b_1 ... k_2 ... b_n$. Consider all paths of the form $(b_1, ..., k, ..., b_n)$ where k varies over $\{l\}$. It is easily seen, that these paths gives rise to a λ-homogeneous subtree T_1 of T which is complete on level j. Clearly, any maximal λ-homogeneous subtree T_2 of T containing T_1 is again complete on level j, hence those maximal λ-homogeneous subtrees of T which are τ-homogeneous w.r.t. some $t \in \{l\}$, namely w.r.t. $f^* b_1 ... k_1 ... b_n$ and contain T_1 are complete on level j. Since $b_1, b_2, ..., b_{j-1}$, $b_{j+1}, ..., b_n$ are chosen arbitrarily, it follows that every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$ is complete on level j.

Conversely, assume that every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$ is complete on level j. Consider all paths of the form $(b_1, ..., k, ..., b_n)$ for fixed $b_1, ..., b_{j-1}, b_{j+1}, ..., b_n \in \{l\}$ and for all $k \in \{l\}$. These paths form a λ-homogeneous subtree T_1 of T which is complete on level j. But T_1 is contained in a maximal λ-homogeneous subtree of T which is τ-homogeneous w.r.t. some t and complete on level j by Lemma 11. It follows, that $f^* b_1 ... b_{j-1} k b_{j+1} ... b_n = t$ for all $k \in \{l\}$.

Corollary 16. *Let $f \in O_l^{(n,1)}$ and let $(T, \tau) = T_f$. Then, the following two assertions are equivalent:*

(i) $f \in ND^{(n,1)}$.

(ii) *No naximal λ-homogeneous subtree T_1 of T exists such that T_1 is τ-homogeneous w.r.t. some $t \in \{l\}$ and complete on some level $j, j \in [n]$.*

Degenerate and partially degenerate functions will be investigated further in the next part [3].

5.3. Let $f \in O_l^{(n,1)}$. Let r be a partial ordering on $\{l\}$. $fx_1 ... x_j ... x_n$ is *r-preserving in* x_j if and only if for arbitrary $b_1, ..., b_{j-1}, b_{j+1}, ..., b_n \in \{l\}$ and for all $k_1, k_2 \in \{l\}$, $(k_1, k_2) \in r$ entails

$$(f^* b_1 \dots b_{j-1} k_1 b_{j+1} \dots b_n, f^* b_1 \dots b_{j-1} k_2 b_{j+1} \dots b_n) \in r.$$

If $X \subset \{x_1, \dots, x_n\}$, then we say that f is *r-preserving in* X if and only if f is r-preserving in x_j for all $x_j \in X$. f is called *r-preserving* if and only if f is r-preserving in $\{x_1, \dots, x_n\}$. We shall denote by $M_{j,r}^{(n,1)}$, $M_{X,r}^{(n,1)}$, $M_r^{(n,1)}$, the sets of functions which are r-preserving in x_j, in X and in $x_1, \dots, x_n$, respectively.

Theorem 17. *Let $f \in O_l^{(n,1)}$ and let r be a partial ordering on $\{l\}$. Then, the following two assertions are equivalent. For $j \in [n]$,*

(i) $f \in M_{j,r}^{(n,1)}$.

(ii) *For every $p = (p_1, \dots, p_j, \dots, p_n) \in P_{T_f}$, the subtree generated by $\{q | q = (p_1, \dots, q_j, \dots, p_n) \wedge (p_j, q_j) \in r \wedge q_j \in \{l\}\}$ is τ-increasing w.r.t. r.*

Proof. Trivial.

Theorem 18. *Let $f \in O_l^{(n,1)}$, $X \subset \{x_1, \dots, x_n\}$, $X = \{x_{j_1}, \dots, x_{j_k}\}$ and let r be a partial ordering on $\{l\}$. The following two assertions are equivalent:*

(i) $f \in M_{X,r}^{(n,1)}$.

(ii) *For every $p = (p_1, \dots, p_{j_1-1}, p_{j_1}, p_{j_1+1}, \dots, p_{j_k-1}, p_{j_k}, p_{j_k+1}, \dots, p_n) \in P_{T_f}$, the subtree generated by $\{q | q = (p_1, \dots, p_{j_1-1}, q_{j_1}, p_{j_1+1}, \dots, p_{j_k-1}, q_{j_k}, q_{j_k+1}, \dots, p_n) \wedge q_{j_1}, \dots, q_{j_k} \in \{l\} \wedge (\forall s \in [k]) \ ((p_{j_s}, q_{j_s}) \in r)\}$ is τ-increasing w.r.t. r.*

Proof. It follows from Theorem 17, by easy induction.

Theorem 19. *Let $f \in O_l^{(n,1)}$ and let r be a partial ordering on $\{l\}$. Then, the following two assertions are equivalent:*

(i) $f \in M_r^{(n,1)}$.

(ii) *For every $p \in P_{T_f}$, the (unique) maximal λ-homogeneous subtree of T_f generated by p and r is τ-increasing w.r.t. r.*

Proof. By Lemma 4, $T_{r,p}$ is λ-homogeneous and is obviously maximal. Taking $X = \{x_1, \dots, x_n\}$, Theorem 19 follows from Theorem 18 since if $p = (p_1, \dots, p_n)$, then $T_{r,p} = \{q | q = (q_1, \dots, q_n) \wedge (\forall i \in [n]) \ (q_i \in \{l\} \wedge (p_i, q_i) \in r)$.

This characterization of r-preserving functions will be used later to estimate the cardinality of $M_r^{(n,1)}$ [4].

5.4. Let $t \subset \{l\}$ and define

$$T_t^{(n,1)} = \{f | f \in O_l^{(n,1)} \wedge (\forall a)(f^* a \in t) \tag{10}$$

the set of *t-valued* functions. We have immediately:

Theorem 20. *Let $f \in O_l^{(n,1)}$, $t \subset \{l\}$. Then, the following two assertions are equivalent:*

(i) $f \in T_t^{(n,1)}$.

(ii) *T_f is τ-homogeneous w.r.t. t.*

Theorem 20 will be used in later parts to establish strong decidability of some finite-valued sentential calculi in which the elements of t are designated and, using some additional arguments, to prove the strong completeness of some finite-valued predicate logics.

Let $t\subset\{l\}$, and define

$$Q_t^{(n,1)}=\{f|f\in O_l^{(n,1)}\wedge(\exists a\in\{l\}^n)(f^*a\notin t)\wedge(\forall b\in\{l\}^n)(f^*b\notin t\Leftrightarrow b=a)\}$$

the set of *quasi t-valued* functions. Elements of $Q_t^{(n,1)}$ are natural generalizations of functions associated to conditional sentences (Horn sentences, or quasi-equations) of the two-valued propositional logic. They have almost all of the nice properties of the two-valued functions associated to Horn sentences and hence it is of some interest to characterize them by trees. We have immediately

Theorem 21. *Let $f\in O_l^{(n,1)}$ and $t\subset\{l\}$. Then, the following two assertions are equivalent:*

(i) $f\in Q_t^{(n,1)}$.

(ii) T_f *is quasi τ-homogeneous w.r.t. r.*

Let $t\subset\{l\}$ and r_t be the equivalence relation generated by t. We set

$$P_t^{(n,1)}=\{f|f\in O_l^{(n,1)}\wedge(\forall a,b\in\{l\}^n)((a,b)\in\bar r_t\Rightarrow(f^*a,f^*b)\in r_t)\},$$

the set of *t-preserving* functions. The following claim is trivial.

Theorem 22. *Let $f\in O_l^{(n,1)}$ and $t\subset\{l\}$. Then, the following two assertions are equivalent:*

(i) $f\in P_t^{(n,1)}$.

(ii) T_{r_t}, *the subtree of T_f determined by r_t is τ-homogeneous w.r.t. r_t.*

5.5. Let $f\in O_l^{(n,1)}$ and π be a permutation of the set $[n]$. f *preserves* π if and only if for all $a_1,\ldots,a_n\in\{l\}$, we have $f^*a_1\ldots a_n=f^*a_{\pi(1)}\ldots a_{\pi(n)}$. Let $S^{(n,1)}=\{f|f\in O_l^{(n,1)}$ and f preserves all permutations π of the set $[n]\}$.

Theorem 23. *Let $f\in O_l^{(n,1)}$ and let (T,τ) be a terminated ξ-augmented* 1*-rooted l-ary tree associated to f. Then the following two assertions are equivalent:*

(i) $f\in S^{(n,1)}$.

(ii) (T,τ) *is ξ-homogeneous.*

Proof. The theorem follows immediately from the well-known fact [7], that

$$f\in S^{(n,1)}\Leftrightarrow(\forall a\in\omega^{l-1},\forall p_1,p_2\in a\xi^{-1})(p_1\tau=p_2\tau).$$

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