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L. KALMÁR, L. RÉDEI ET K. TANDORI

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B. SZ.-NAGY

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Elementary divisors in von Neumann rings

By ISRAEL HALPERIN in Kingston (Ontario, Canada)

1. Introduction

1. 1. Terminology. In this paper L *will always denote a complemented modular lattice* and $\mathfrak{R}$ will denote an *associative regular ring with unit element.*

We will call L an $\aleph$-*geometry* if:

(1. 1. 1) Whenever $x_\alpha \in L$ for each $\alpha \in I$ with cardinal power of $I \leqq \aleph$, the union $x = \bigcup_\alpha (x_\alpha)$ and intersection $x' = \bigcap_\alpha (x_\alpha)$ exist and for each y: if $y \cap \bigcap \left(\bigcup (x_\alpha | \alpha \in F) \right) = 0$[1]) for every *finite* subset F of I then $y \cap x = 0$; if $y \cup \left(\bigcap (x_\alpha | \alpha \in F) \right) = 1$ for every *finite* subset F of I then $y \cup x' = 1$.

If (1. 1. 1) holds for *all* $\aleph$, we will call L a *von Neumann geometry.*[2])

In every von Neumann geometry there exists a unique normalized dimension function D, vector-valued with $0 \leqq D(x) \leqq 1$ for all x in L such that $x \sim y$[3]) if and only if $D(x) = D(y)$ [9, 6]. When L is irreducible D is numerical-valued and its range of values is either $0, \frac{1}{n}, \ldots, \frac{n}{n}$ for some integer n (then L is called a *finite dimensional* or *discrete* geometry of von Neumann) or all real numbers $0 \leqq t \leqq 1$ (then L is called a *continuous*[4]) geometry of von Neumann) [9, Part I, Theorem 7. 3].

$\overline{R}_\mathfrak{R}$, $\overline{L}_\mathfrak{R}$ will denote the set of principal right (respectively, left) ideals of $\mathfrak{R}$, ordered by inclusion; $\overline{R}_\mathfrak{R}$ and $\overline{L}_\mathfrak{R}$ are complemented modular lattices [9, Part II, Theorem 2 .4]. $\mathfrak{R}$ will be called an $\aleph$-*ring* or a *von Neumann ring* if $\overline{R}_\mathfrak{R}$ (hence also $\overline{L}_\mathfrak{R}$) is an $\aleph$-geometry, respectively a von Neumann geometry.

In a von Neumann ring $\mathfrak{R}$ there exists a unique, normalized rank-function $R(a)$, vector-valued with $0 \leqq R(a) \leqq 1$ for all a in $\mathfrak{R}$, defined by: $R(a) = D((a)_r)$.[5]) If $\mathfrak{R}$ is irreducible, R is numerical-valued and $\overline{R}_\mathfrak{R}$ must be discrete or continuous; then $\mathfrak{R}$ will be called a *discrete ring*, respectively a *continuous ring* (of von Neumann).

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[1]) $\{u | \varphi(u)\}$ will denote the class of u for which $\varphi(u)$ holds.

[2]) Thus L satisfies VON NEUMANN's axioms I—V; his axiom VI (irreducibility) is not postulated [9, pages 1, 2].

[3]) In any lattice, $x \sim y$ means: x is perspective to y (that is, for some w, $x \cup w = y \cup w$ and $x \cap w = y \cap w$; $x \precsim y$ means $x \sim w$ for some $w \leqq y$.

[4]) In our terminology a continuous geometry is always *irreducible.*

[5]) $(a)_r$ and $(a)_l$ denote the principal right and the principal left ideal generated by a, respectively (since $\mathfrak{R}$ is a regular ring, $(a)_r = a\mathfrak{R}$).

A discrete ring must be of the form $\mathfrak{D}_n$[6]) with $\mathfrak{D}$ a (possibly non-commutative) division ring [9, Part II, Theorem 14.1 and page 292].

If $n>1$, $\mathfrak{R}_n$ must also be a regular ring [9, Part II, Theorem 2. 13] but if $\mathfrak{R}$ is a von Neumann ring, $\mathfrak{R}_n$ need not be a von Neumann ring (the union of a countable subset of $\bar{R}_{\mathfrak{R}_n}$ may not exist[7])); but if $\mathfrak{R}$ is an *irreducible* von Neumann ring then $\mathfrak{R}_n$ is also a von Neumann ring (see the Corollary to Lemma 3. 2 below).

The centre of $\mathfrak{R}$ will be denoted Z (if $\mathfrak{R}$ is a von Neumann ring, Z will be a commutative von Neumann ring); Z will be a division ring if and only if Z is irreducible and if and only if $\mathfrak{R}$ is irreducible.

A non-zero element x in a lattice L will be called minimal if $y_1 \leqq x$, $y_2 \leqq x$, $y_1 \sim y_2$ together imply $y_1 = y_2$.

By P we shall denote the set of all polynomials

$$p(t) = t^m + z_{m-1}t^{m-1} + \dots + z_0$$

with $m \geqq 1$ and all z_i central.[8]) p, q in P will be called *relatively prime* if $h(t)p(t) + k(t)q(t) = 1$ for some h, k of the form $t^m + z_{m-1}t^{m-1} + \dots + z_0$ with $m \geqq 0$ and all z_i central[8]). p will be called *irreducible* if p cannot be expressed as a product $p = p_1 p_2$ with p_1, p_2 in P and each of degree less than the degree of p.

If Z is not a field, Z contains a non-zero non-invertible[9]) z_0 and $p \equiv t$, $q \equiv t + z_0$ are irreducible, different but not relatively prime. This motivates the following definition.

Call p in P *pure irreducible* if for every non-zero central idempotent e, ep is irreducible in the ring $e\mathfrak{R}$. If $\mathfrak{R}$ is a von Neumann ring then for each p in P there is (obviously) a set of orthogonal non-zero central idempotents, $\{e_\lambda\}$ with $\bigcup_\lambda (e_\lambda)_r = \mathfrak{R}$ and with the property that for each λ: $e_\lambda p = e_\lambda \Pi_i q_{\lambda, i}$ with a finite set of i and with each $q_{\lambda,i}$ in P and $e_\lambda q_{\lambda,i}$ pure irreducible in $e_\lambda \mathfrak{R}$.

Let P_1 be a subset of P and let $\bar{P}_1$ consist of all $p = p_1 \dots p_m$ (all p_i in P_1). We shall call an element a *P_1-algebraic* if $p(a) = 0$ for some p in $\bar{P}_1$, *P_1-almost-algebraic* if $\bigcap((p(a))_r | p \in \bar{P}_1) = 0$. When P_1 coincides with P we omit it in this nomenclature [10, 4].

a and b are called *similar* or *conjugate* in $\mathfrak{R}$ if $b = dad^{-1}$ for some invertible d. Then for each p in P, $p(b) = dp(a)d^{-1}$, $(p(b))_r = (dp(a))_r$ and we shall show in Corollary 1 to Lemma 2. 1 below that in a von Neumann ring $(p(b))_r \sim (p(a))_r$ and hence $R(p(a) = R(p(b))$ for each p in P.

1. 2. Elementary divisors. When $\mathfrak{R} = \mathfrak{D}_n$ with $\mathfrak{D}$ a *commutative* division ring (we shall call this the *classical case*) it is known [1, page 283], [11, pages 120—124], [7, pages 92—98] that:

(1. 2. 1) *a and b are similar if and only if they have the same elementary divisors.*

---

[6]) The ring of $n \times n$ matrices with entries in $\mathfrak{S}$ will be denoted $\mathfrak{S}_n$.

[7]) This failure occurs in KAPLANSKY's example [8] where $\mathfrak{R}$ is the ring of sequences of complex numbers $a = \{a_m | m \geqq 1\}$ with all but a finite number of a_m real, with componentwise ring addition and multiplication.

[8]) An element of $\mathfrak{R}$ is called *central* if it is in the centre Z; a, b are *orthogonal* if $ab = ba = 0$.

[9]) In any ring $\mathfrak{S}$ with unit, d is called *invertible* if for some c in $\mathfrak{S}$, $dc = cd = 1$; c is called the *reciprocal* of d and denoted d^{-1}.

We give now a definition for „elementary divisors of a" in terms of the rank function, applicable in any von Neumann ring.

Note first that for b in $\mathfrak{R}$ and integer $s \geqq 0$, $(b^{s+1})_r = (b^s b)_r \leqq (b^s)_r$ and[10])

$$[(b^{s+1})_r - (b^{s+2})_r] \precsim [(b^s)_r - (b^{s+1})_r] \tag{1.2.2}$$

((1.2.2) will be proved in Lemma 2.2 below). Thus $R(b^s) \geqq R(b^{s+1})$ and $R(b^s) - R(b^{s+1}) \geqq R(b^{s+1}) - R(b^{s+2})$.

We set $R_a(p) = R(p(a))$ for each p in $eP = P(e\mathfrak{R})$ for arbitrary non-zero central idempotent e. For each integer $s \geqq 1$ we define

$$f_a(p, s) = s((R_a(p^{s-1}) - R_a(p^s)) - (R_a(p^s) - R_a(p^{s+1}))).$$

Then $f_a(p, s) \geqq 0$. The function $f_a(p, s)$ is determined by the function $R_a(p)$; the converse also holds since

$$R_a(p^{s-1}) - R_a(p^s) = \sum_{t=s}^{\infty} \frac{1}{t} f_a(p, t),$$

$$R_a(p^0) - R_a(p^s) = \sum_{i=1}^{s} \sum_{t=i}^{\infty} \frac{1}{t} f_a(p, t),$$

$$R_a(p^s) = 1 - \sum_{i=1}^{s} \sum_{t=i}^{\infty} \frac{1}{t} f_a(p, t).$$

It can be shown that if p, q are relatively prime then $1 - R_a(pq) = (1 - R_a(p)) + (1 - R_a(q))$.

Thus in any von Neumann ring the function $R_a(q)$ for all q in P is determined by the values of $f_a(ep, s)$ for all p in P with ep pure irreducible in $e\mathfrak{R}$ and e a non-zero central idempotent and all $s \geqq 1$. We shall say for each non-zero central idempotent e and p in P with ep pure irreducible in $e\mathfrak{R}$, and $f_a(ep, s) > 0$ that *$q = p^s$ is an elementary divisor of a in $e\mathfrak{R}$ occurring with normalized frequency $f_a(ep, s)$*. This definition agrees with the usual one for the classical case (there, the only possibility for e is 1) except that the normalized frequency is the usual frequency multiplied by the factor $\dfrac{s \cdot (\text{degree of } p)}{n}$. It can be shown that in every irreducible von Neumann ring

$$1 - \sum_{p,s} f_a(p, s) = R(ae_0) \geqq 0$$

where ae_0 is the transcendental part of a [4] (thus „=" holds if a is almost-algebraic, in particular for all a in the classical case).

We have noted that each p in P can be expressed „locally" as a product of pure irreducible factors. We shall call a subset P_1 of P *fully factorizable* if for each p in P_1 there are central idempotents $\{e\}$ such that $\bigcup (e)_r = \mathfrak{R}$ and such that each ep is a product $ep_1 \ldots p_m$ with each p_i in P_1 and ep_i pure irreducible in $e\mathfrak{R}$.

Clearly P itself is fully factorizable. If $\mathfrak{R}$ is irreducible then P_1 is fully factorizable if it contains all irreducible p in P.

---

[10]) If $x \geqq y$ then $[x - y]$ denotes any (fixed) w such that $y \dot{\cup} w = x$ (the dot in $\dot{\cup}$ indicates independence of the addends); such w exist in every complemented modular lattice.

1.3. Statement of main theorem. The main object of this paper is to prove the following theorem, a generalization of (1.2.1) to any von Neumann ring:

Theorem 1.1. *Let a and b be arbitrary elements of a von Neumann ring.*
(i) *For a and b to be similar it is necessary that*

(1.3.1) $R_a(q) = R_b(q)$ *for all* q *in* P.

(ii) *For a and b to be similar it is sufficient that for some fully factorizable* P_1:

(1.3.2) $R_a(p^s) = R_b(p^s)$ *for all* p *in* P_1 *and* $s \geqq 1$,

(1.3.3) *a and b are* P_1*-almost-algebraic.*[11])

(1.3.4) *Whenever* $\bar{e}$ *is central idempotent such that* $\bar{e}\mathfrak{R}$ *contains minimal elements then* $\bar{e}_1\mathfrak{R}$ *is a finite dimensional matrix ring over* $\bar{e}_1 Z$ *for some non-zero central idempotent* $\bar{e}_1$ *such that* $\bar{e}_1\bar{e} = \bar{e}_1$.

(1.3.5) $\mathfrak{R}_2$ *is a von Neumann ring.*[12])

Remark. The definition of $\mathfrak{R}_2$ is given in footnote [6]). It is shown in the Corollary to Lemma 3.2 below that $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is an irreducible von Neumann ring (equivalently, if $\overline{R}_{\mathfrak{R}}$ is a discrete or continuous geometry), more generally whenever $\mathfrak{R}$ is a direct sum of irreducible von Neumann rings.

Also, it follows from Lemma 3.1 and Lemma 3.2 below that every von Neumann ring can be expressed as a direct sum $\overline{\mathfrak{R}} \oplus \mathfrak{R}'$ in such a way that $(\mathfrak{R}')_2$ is a von Neumann ring and $\overline{\mathfrak{R}}$ is a von Neumann ring in which every idempotent is central (equivalently, $\overline{\mathfrak{R}}_{\overline{R}}$ is a Boolean algebra).

Let E be the central idempotent for which $\mathfrak{R}' = \mathfrak{R}E$; then clearly, a and b are similar in $\mathfrak{R}$ if and only if Ea, Eb are similar in $\mathfrak{R}'$ and $(1-E)a$, $(1-E)b$ are similar in $\overline{\mathfrak{R}}$. If a, b satisfy (1.3.2), (1.3.3) and (1.3.4), then at least Ea, Eb are similar in $\mathfrak{R}'$ (hence in $\mathfrak{R}$) since $\mathfrak{R}'$ satisfies (1.3.5). Thus a and b will be similar in $\mathfrak{R}$ if and only if $(1-E)a$, $(1-E)b$ are similar in the ring $\overline{\mathfrak{R}}$ ($(1-E)a$ and $(1-E)b$ satisfy (1.3.2), (1.3.3), (1.3.4) in $\overline{\mathfrak{R}}$).

In such a ring $\overline{\mathfrak{R}}$ condition (1.3.2) can be expressed in the simpler equivalent form:

(1.3.2)′ $\quad (p(a))_r = (p(b))_r$ *for all* p *in* P_1.

We shall postpone to another occasion further discussion of the case of a ring $\overline{\mathfrak{R}}$, noting here only that it is easy to see that Theorem 1.1 (ii) holds without (1.3.5), if $\overline{\mathfrak{R}}$ is the example given by Kaplansky (and described in footnote [7])).

Corollary to Theorem 1.1. *Suppose* $\mathfrak{R}$ *is a von Neumann ring which is irreducible, or more generally, is a direct sum of irreducible von Neumann rings, or more generally, has the property:* $\mathfrak{R}_2$ *is a von Neumann ring and that* (1.3.4) *holds.*

---

[11]) In the presence of (1.3.1) the condition (1.3.3) for a will imply (1.3.3) for b.

[12]) For the classical case $\mathfrak{R} = \mathfrak{D}_n$ ($\mathfrak{D}$ commutative) our proof specializes of course, to a proof of the known result (1.2.1).

If a and b in $\mathfrak{R}$ are almost algebraic then they are similar if and only if they have the same elementary divisors.

However, we shall not use rank (or dimension) functions. In (1. 3. 1) and (1. 3. 2) we shall replace equality of rank by perspectivity of corresponding principal right ideals.

1. 4. Plan of the proof of Theorem 1. 1. Corollary 1 to Lemma 2. 1 below will show that $(dp(a))_r \sim (p(a))_r$ if d is invertible. From this follows (i) of Theorem 1. 1.

To prove (ii) of Theorem 1. 1 we prove first the special case:

(1. 4. 1) *a and b are similar in an $\aleph_0$-ring $\mathfrak{R}$ if $\mathfrak{R}_2$ is an $\aleph_0$-ring, and $(a^s)_r \sim (b^s)_r$ for all $s \geqq 1$ and* $\bigcap ((a^s)_r | s \geqq 1) = 0$ (see Theorem 4. 1 below),

and then the case:

(1. 4. 2) *a and b are similar in a von Neumann ring $\mathfrak{R}$ if* (1. 3. 4), (1. 3. 5) *hold and for some pure irreducible p in P, $(p^s(a))_r \sim (p^s(b))_r$ for all $s \geqq 1$ and* $\bigcap ((p^s(a))_r | s \geqq 1) = 0$ (see Theorem 4. 2 below).

Then in the general case we show that the unit in $\mathfrak{R}$ can be decomposed into orthogonal idempotents e (not necessarily in the centre) with $\bigcup (e)_r = \mathfrak{R}$ and (using Theorem 3. 1 below) such that, for some $\bar{b} = dbd^{-1}$: for each e, $ae = ea$ and $\bar{b}e = e\bar{b}$ and ae, $\bar{b}e$ satisfy the hypotheses of (1. 4. 2) in $e\mathfrak{R}e$.

This will yield: ae and $\bar{b}e$ are similar in $e\mathfrak{R}e$. Then, using a theorem which permits „combining" such local similarities in the case that $\mathfrak{R}_2$ is a von Neumann ring (Theorem 3. 2 and its Corollary 1 below) we deduce that a and $\bar{b}$, and hence also a and b are similar.

2. Proof of (1. 2. 2) and Theorem 1. 1 (i)

If d is in $\mathfrak{R}$ we shall write d^r to denote $\{b | db = 0\}$. If $x \subset \mathfrak{R}$ we write x^r to denote $\{b | xb = 0\}$. Similarly for d^l and x^l.

Lemma 2. 1. *Suppose $d \in \mathfrak{R}$ and $x \in \bar{R}_{\mathfrak{R}}$. Let $x_0 = d^r \cap x$. If $x \cap dx = 0$ or if $\mathfrak{R}$ is an $\aleph_0$-ring, then $[x - x_0] \sim dx$.*

Corollary 1. *If also $x_0 = 0$ (in particular, if d is invertible), then $x \sim dx$ so* (i) *of Theorem* 1. 1 *holds.*

Corollary 2. *If $\mathfrak{R}$ is a von Neumann ring, then $D(x) = D(dx) + D(x_0)$.*

Proof of Lemma 2. 1. Let e, f be idempotents such that $x = (e)_r$, $d\mathfrak{R} = (f)_r$, and let $x_1 = [x - x_0]$. Then $a \in x$ implies $a = a_0 + a_1$ with $a_i \in x_i$. Thus $dx = dx_1$.

Let T denote the mapping of $0 \leqq y \leqq x_1$ onto $0 \leqq w \leqq dx_1$ defined by: $T(a)_r = (da)_r$. Then T has the properties:

(i) T is order-preserving: indeed, $(db)_r \leqq (dc)_r$ is equivalent in turn to each of: for some a in $\mathfrak{R}$, $db = dca$, $d(b - ca) = 0$, $b - ca = 0$, $(b)_r \leqq (c)_r$.

(ii) $T(a)_r \sim (a)_r$ if $T(a)_r \cap (a)_r = 0$: indeed, $(a+da)_r$ is an axis of perspectivity since

$$T(a)_r \dot{\cup} (a+da)_r = (da)_r \dot{\cup} (a+da)_r = (a)_r \dot{\cup} (a+da)_r.$$

From [4, Lemma 6. 1] it follows that $x_1 \sim dx_1 = dx$.

Proof of Corollary 2. $D(x) = D(x_0) + D(x_1)$ and $D(dx) = D(x_1)$ since $dx \sim x_1$.

Lemma 2. 2. (1. 2. 2) *holds in an* $\aleph_0$*-ring.*

Proof. Let $x^s = (b^s)_r \cap b^r$. By Lemma 2. 1, $[(b^s)_r - (b^{s+1})_r] \dot{\cup} \bar{x}^s = (b^s)_r$ for some $\bar{x}^s \sim x^s$. Since $x^{s+1} \leqq x^s$, $\bar{x}^{s+1} \precsim x^s$ (perspectivity is transitive in an $\aleph_0$-geometry [2]). Now $(b^{s+1})_r \leqq (b^s)_r$; so from [2, Lemma 6. 5] follows (1. 2. 2).

3. Lattice sums of ring elements

3. 1. Preliminary Lemmas.

Lemma 3. 1. *Suppose* $\overline{R}_{\mathfrak{R}}$ *has a basis*[13]) x_1, x_2, x_3 *with* $x_2 \sim x_1$, $x_3 \precsim x_1$. *Then if* $\mathfrak{R}$ *is an* $\aleph$*-ring (respectively von Neumann ring) so is* $\mathfrak{R}_2$.

Proof. This coincides with [5, Corollary 2 to Theorem 3. 1].

Lemma 3. 2. *Every von Neumann ring* $\mathfrak{R}$ *is a direct sum* $\overline{\mathfrak{R}} \oplus \mathfrak{R}'$ *with* $\mathfrak{R}'$ *satisfying the hypotheses of Lemma* 3. 1 *and* $\overline{\mathfrak{R}}$ *a von Neumann ring in which every idempotent is central.*[14])

Proof. If L is a von Neumann geometry then $L = \sum_{i=0}^{\infty} \oplus L_i$ where L_i has a homogeneous basis consisting of i minimal[15]) elements if $i \geqq 1$, and L_0 has the property: $0 \neq x \in L_0$ implies $0 \neq y_1 \sim y_2$ for some $y_1 \dot{\cup} y_2 \leqq x$ [9, Part III, Theorem 3.2].

There are elements $x_1^{(0)}, x_2^{(0)}$ which form a homogeneous basis for L_0: indeed take a maximal class of pairs $\{y_1^\alpha, y_2^\alpha\}$ with $\{y_1^\alpha, y_2^\alpha \mid \text{all } \alpha\} \perp$[16]) and $y_1^\alpha \sim y_2^\alpha$ for each α and set $x_1^{(0)} = \cup_\alpha y_1^\alpha$, $x_2^{(0)} = \cup_\alpha y_2^\alpha$.

For $i > 1$, L_i has a basis $x_1^{(i)}, x_2^{(i)}, x_3^{(i)}$ with $x_2^{(i)} \sim x_1^{(i)}$, $x_3^{(i)} \precsim x_1^{(i)}$: indeed, if $y_1, \ldots, y_i$ is a homogeneous basis for L_i then according as $i = 2m$ or $i = 2m+1$, take $x_1^{(i)} = y_1 \cup \ldots \cup y_m$, $x_2^{(i)} = y_{m+1} \cup \ldots \cup y_{2m}$, $x_3^{(i)} = 0$ or y_i respectively.

Let $\overline{L} = L_1$, $L' = L_0 \oplus \sum_{i=2}^{\infty} \oplus L_i$. Then $L = \overline{L} \oplus L'$ and $\overline{L}$ is a Boolean algebra whereas L' has a basis $x_j = \cup(x_j^{(i)} \mid i \neq 1)$, $j = 1, 2, 3$, with $x_2 \sim x_1$, $x_3 \precsim x_1$.

---

[13]) $x_1, \ldots x_m$ are said to be a *basis* for a lattice L if $\dot{\cup}_i x_i =$ the unit of L; the basis is called *homogeneous* if $x_i \sim x_j$ for all i, j [9].

[14]) If $\mathfrak{R}$ is a regular ring with unit, all idempotents in $\mathfrak{R}$ are central if and only if $\overline{R}_{\mathfrak{R}}$ is a Boolean algebra [9, Part II, Theorem 2.5 (Note) and Theorem 2.10].

[15]) A non-zero element x in a lattice L is called *minimal* or *locally-atomic* if $y_1 \leqq x$, $y_2 \leqq x$, $y_1 \sim y_2$ together imply $y_1 = y_2$ (for another definition, see [9, Part III, Definition 3.1]).

[16]) $\perp$ indicates „independence”.

Every direct decomposition of $L=\overline{R}_{\mathfrak{R}}^{\bullet}$ is determined by a corresponding direct decomposition of $\mathfrak{R}$ and from this follows Lemma 3. 2.

Corollary. *Suppose $\mathfrak{R}$ is a von Neumann ring. Then $\mathfrak{R}$ has the property: $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is irreducible, more generally whenever $\mathfrak{R}$ is a direct sum of irreducible von Neumann rings, more generally whenever $\mathfrak{R}$ is a direct sum of von Neumann rings $\mathfrak{R}^{\alpha}$ each of which has the property: $(\mathfrak{R}^{\alpha})_2$ is a von Neumann ring.*

Proof. Since $(\sum \oplus \mathfrak{R}^{\alpha})_2 = \sum \oplus (\mathfrak{R}^{\alpha})_2$ we need only show that $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is an irreducible von Neumann ring. But if $\mathfrak{R}$ is irreducible, then with the decomposition $\mathfrak{R} = \overline{\mathfrak{R}} \oplus \mathfrak{R}'$ of Lemma 3. 2, we must have $\mathfrak{R}=\overline{\mathfrak{R}}$ or $\mathfrak{R}=\mathfrak{R}'$. Since $(\mathfrak{R}')_2$ is a von Neumann ring (according to Lemma 3. 2), we need only prove: $\mathfrak{R}_2$ is a von Neumann ring whenever $\mathfrak{R}$ is an irreducible von Neumann ring in which every idempotent is central, equivalently, $\mathfrak{R}$ is a division ring. But in this case $\mathfrak{R}_2$ is (trivially) a discrete von Neumann ring.

Lemma 3. 3. *If $(a)_r=\bigcup_\alpha (a_\alpha)_r$ in $\overline{R}_{\mathfrak{R}}$ then $ba_\alpha=0$ for all α if and only if $ba=0$.*

Proof. [9, Part II, Corollary 2 to Lemma 2. 2.]

Lemma. 3. 4. *If $\mathfrak{R}$ is an $\aleph$-ring and $\{x_\alpha | \alpha \in I\} \perp$ in $\overline{R}_{\mathfrak{R}}$ with cardinal of $I \leqq \aleph$, there exist orthogonal idempotents $\{e_\alpha\}$ with $(e_\alpha)_r = x_\alpha$ for all α.*

Proof. Let $y=[\mathfrak{R}-\bigcup_\alpha x_\alpha]$ and choose e_α so that $(e_\alpha)_r=x_\alpha$ and $(1-e_\alpha)_r = (\bigcup_{\beta\neq\alpha} x_\beta)\cup y$.

Lemma 3. 5. *Suppose $\{e_\alpha | \alpha \in I\}$ are orthogonal idempotents with cardinal of $I \leqq \aleph$ in an $\aleph$-ring and let e be an idempotent with $(e)_r=\bigcup_\alpha (e_\alpha)_r$. Then e is the unique idempotent with also $(1-e)_r = \bigcap_\alpha (1-e_\alpha)_r$*[17]*) if and only if $(e)_l=\bigcup_\alpha (e_\alpha)_l$; then $de_\gamma=e_\gamma$, $de_\beta=0$ for $\beta\neq\gamma$ imply $de=e_\gamma$; $e_\gamma\overline{d}=e_\gamma$, $e_\beta\overline{d}=0$ for $\beta\neq\gamma$ imply $e\overline{d}=e_\gamma$.*

Proof. By [9, Part II, Lemma 2. 2, Corollary 2] $(1-e)_r = \bigcap_\alpha (1-e_\alpha)_r$ is equivalent to $(e)_l=(1-e)^l=\bigcup_\alpha (1-e_\alpha)^l=\bigcup_\alpha (e_\alpha)_l$.

Next, $(de-e_\gamma)e_\alpha=de_\alpha-e_\gamma e_\alpha=0$ for *all* α. Hence, by Lemma 3. 3, $(de-e_\gamma)e=0$, $de=e_\gamma$.

Lemma 3. 6. *Suppose $\bigcup_\alpha x_\alpha$ exists in $\overline{L}_{\mathfrak{R}}$. Then for any d in $\mathfrak{R}$, $\bigcup_\alpha (x_\alpha d)$ exists and is equal to $(\bigcup_\alpha x_\alpha)d$.*

Proof. Let $x=\bigcup_\alpha x_\alpha$. Then for each α, $xd \geqq x_\alpha d$ since $x \geqq x_\alpha$. To prove Lemma 3. 6 we need show: if $y \geqq x_\alpha d$ for all α then $y \geqq xd$.

Suppose $y \geqq x_\alpha d$ for all α. Then $y \cap (d)_l \geqq x_\alpha d$ for all α and therefore it clearly suffices to prove: $(d)_l \geqq y \geqq x_\alpha d$ for all α implies $y \geqq xd$. Now for some a in $\mathfrak{R}$, $y=(ad)_l=(a)_l d$. Let $u=(a)_l \cup d^l$. Then $y=ud$. Hence it is sufficient to prove that $u \geqq x_\alpha$ for all α; this would yield $u \geqq x$ and hence $y=ud \geqq xd$ as required.

To prove $u \geqq x_\alpha$ suppose $c \in x_\alpha$; then $cd \in ud$ since $x_\alpha d \leqq ud$, hence $cd=c_1 d$ for some $c_1 \in u$. Then $(c-c_1)d=0$, so $c-c_1 \in u$. Now u is a left ideal so $(c_1+c-c_1) \in u$, $c \in u$. Thus $x_\alpha \leqq u$, as required. This proves Lemma 3. 6.

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[17]) This e exists because $\bigcup_\alpha (e_\alpha)_r$ and $\bigcap_\alpha (1-e_\alpha)_r$ are complements ($\mathfrak{R}$ is an $\aleph$-ring).

3. 2. Lattice sums of ring elements. In this section, $\mathfrak{R}$ will be an $\aleph$-ring for some $\aleph$, I a set of indices α with cardinal $\leqq \aleph$.

Definition 3. 1. A set of orthogonal idempotents $\sigma = \{e_\alpha\}$ will be called a *separating system* (s. s.); then $e = e_\sigma$ will denote the unique idempotent with $(e)_r = = \bigcup_\alpha (e_\alpha)_r$, $(e)_l = \bigcup_\alpha (e_\alpha)_l$ (existing by Lemma 3. 5).

Definition 3. 2. An s. s. σ will be called a *right separating system* (r. s. s.) for $\{d_\alpha\}$ if $e_\alpha d_\alpha = d_\alpha$ for all α.

Definition 3. 3. If σ is a r. s. s. for $\{d_\alpha\}$ then $\sum_\alpha^{\sigma, r} \oplus d_\alpha$ will denote an element d such that $d \in \bigcup_\alpha (d_\alpha)_r$ and $e_\alpha d = d_\alpha$ for each α; such an element d (if existing) will be called a *σ-right lattice sum of the d_α*. Similarly for *σ-left lattice sum.*

Lemma 3. 7. *A r. s. s. σ exists for $\{d_\alpha\}$ if and only if $\{(d_\alpha)_r\} \perp$ (by Lemma 3. 4). If for some r. s. s. σ, a σ-right lattice sum of the d_α exists then its value d is unique, $d = e_\sigma d$, $(d)_l = \bigcup_\alpha (d_\alpha)_l$, and for any element b, $d_\gamma b = d$ for some γ and $d_\alpha b = 0$ for $\alpha \neq \gamma$ imply $db = d_\gamma$.*

Proof. If $e_\alpha(d - \bar{d}) = 0$ for all α, then by the right-left dual of Lemma 3. 3, $e_\sigma(d - \bar{d}) = 0$; $e_\sigma d = e_\sigma \bar{d}$. This means: the σ-right lattice sum (if existing) is unique.
Next, $(d)_l = (e_\sigma d)_l = (e_\sigma)_l d$, so by Lemma 3. 6, $(d)_l = \bigcup_\alpha (e_\alpha d)_l = \bigcup_\alpha (d_\alpha)_l$.
Finally, $e_\alpha(db - d_\gamma) = 0$ for *all* α, so by the right-left dual of Lemma 3. 3, $e_\sigma(db - d_\gamma) = 0$. Hence $db = d_\gamma$.

Definition 3. 4. If $\{(d_\alpha)_r\} \perp$, we denote by $\sum_\alpha^{r} \oplus d_\alpha$ an element d such that d is a σ-right lattice sum of the d_α for every r. s. s. σ for $\{d_\alpha\}$. This d (unique, if it exists, by Lemma 3. 7) will be called the *right lattice sum of the d_α.*

Definition 3. 5. If $\{(d_\alpha)_r\} \perp$ and $\{(d_\alpha)_l\} \perp$ we denote by $\sum_\alpha \oplus d_\alpha$ an element d such that d is a right lattice sum and a left lattice sum of the d_α. This d (unique, if it exists, by Lemma 3. 7) will be called the *lattice sum of the d_α.*

Lemma 3. 8. *If $\{(d_\alpha)_r\} \perp$ and $\{(d_\alpha)_l\} \perp$ and d is a σ-right-lattice sum of the d_α (for some r. s. s. σ) then d is a lattice sum of the d_α.*

Proof. Let $\tau = \{f_\alpha\}$ be any left separating system for $\{d_\alpha\}$ and let $f = f_\tau$. Then $e_\beta(d_\alpha - df_\alpha) = 0$ for all α, β so $e_\sigma(d_\alpha - df_\alpha) = 0$, $d_\alpha = df_\alpha$. This shows that d is a τ-left lattice sum of the d_α (by Lemma 3. 7, $(d)_l = \bigcup_\alpha (d_\alpha)_l \leqq \bigcup_\alpha (f_\alpha)_l$).
Now by right-left duality, d is a $\bar{\sigma}$-right lattice sum of the d_α for every r. s. s. $\bar{\sigma}$ for $\{d_\alpha\}$.

Lemma 3. 9. *If $\{(d_\alpha)_r | \alpha \in I\} \perp$ and I is finite then $\sum_\alpha^{r} \oplus d_\alpha$ exists and coincides with the ordinary (ring) sum $\sum_\alpha d_\alpha$.*

Proof. Obvious.

Lemma 3. 10. *If $\{e_\alpha\}$ are orthogonal idempotents then $\sum_\alpha \oplus e_\alpha$ exists and coincides with the unique idempotent e with properties: $(e)_r = \bigcup_\alpha (e_\alpha)_r$, $(e)_l = \bigcup_\alpha (e_\alpha)_l$.*

Proof. By Definition 3. 3 and Lemma 3. 5.

Corollary. *Suppose* $\{e_\alpha\}$ *are orthogonal idempotents and* $e=\sum_\alpha \oplus e_\alpha$. *If for some a in* $\mathfrak{R}$, $e_\alpha a = ae_\alpha$ *for each* α, *then* $\sum_\alpha \oplus (e_\alpha a)$ *exists and equals* $ae=ea$.

Proof. First we show $ae=ea$. We have $(e)_r = \cup_\alpha (e_\alpha)_r$ by Lemma 3. 10, and $(a-ea)e_\alpha = ae_\alpha - eae_\alpha = ae_\alpha - ee_\alpha a = ae_\alpha - e_\alpha a = 0$ for all α. By Lemma 3. 3, $(a-ea)e=0$ so $ae=eae$. By a left-right dual argument, $ea=eae$. So $ae=ea$.

Next, $\sigma=\{e_\alpha\}$ is a r. s. s. for $\{ae_\alpha\}$ and $\{(ae_\alpha)_l\}$ are independent since $(ae_\alpha)_l \leqq (e_\alpha)_l$. So by Lemma 3. 8, $ae=\sum_\alpha \oplus (ae_\alpha)$ if only ae is a σ-right lattice sum of the ae_α.

So from Definition 3. 3 we need only show (i): $e_\alpha ae = ae_\alpha$ for each α and (ii): $ae \in \cup_\alpha (ae_\alpha)_r$. But (i) holds since $e_\alpha ae = e_\alpha ea = e_\alpha a = ae_\alpha$. As for (ii), $(ae)_r \geqq (aee_\alpha)_r = (ae_\alpha)_r$ so $(ae)_r \geqq \cup_\alpha (ae_\alpha)_r$. If $(ae)_r \neq \cup_\alpha (ae_\alpha)_r$, then there exists a non-zero idempotent $g \in (ae)_r$ such that $(1-g)_r \geqq (ae_\alpha)_r$ for all α. Then $gae_\alpha = g(1-g)ae_\alpha = 0$, $(ga)e_\alpha = 0$ for all α, so by the left-right dual of Lemma 3. 3, $(ga)e=0$. But $g=aed$ for some d, so $g=gg=gaed=0$, a contradiction. Thus $(ae)_r = \cup_\alpha (ae_\alpha)_r$ so (ii) holds and the Corollary is established.

Lemma 3. 11. *It* σ *is a r. s. s. for* $\{d_\alpha\}$, *then a* σ*-right lattice sum of the* d_α *does exist if* $\mathfrak{R}_2$ *is an* $\aleph$*-ring.*[18])

Proof. Let $e=e_\sigma$ and form the matrices:

$$D_\alpha = \begin{Vmatrix} 0 & 0 \\ d_\alpha & e_\alpha \end{Vmatrix}, \quad M = \begin{Vmatrix} 0 & 0 \\ 0 & e \end{Vmatrix}.$$

$\{D_\alpha\}$ are orthogonal idempotents in $\mathfrak{R}_2$ so by Lemma 3. 5 an idempotent E in $\mathfrak{R}_2$ exists such that $(E)_r = \cup_\alpha (D_\alpha)_r$ and $(E)_l = \cup_\alpha (D_\alpha)_l$ in $\overline{R}_{\mathfrak{S}}$ with $\mathfrak{S}=\mathfrak{R}_2$. Now $MD_\alpha = D_\alpha$ for all α so $(M)_r \supset \cup_\alpha (D_\alpha)_r = (E)_r$, $ME=E$. Thus E must have the form

$$E = \begin{Vmatrix} 0 & 0 \\ d & g \end{Vmatrix}$$

with $ed=d$. Since $D_\alpha E = D_\alpha$ for all α it also follows that $e_\alpha d = d_\alpha$ for all α. Thus this element d is a σ-right lattice sum of the d_α.

Lemma 3. 12. *Suppose* $d=\sum_\alpha \oplus d_\alpha$, $c=\sum_\alpha \oplus c_\alpha$, *and some* $\sigma=\{e_\alpha\}$ *is a r. s. s. for* $\{c_\alpha\}$ *and a l. s. s. for* $\{d_\alpha\}$. *Then* $\sum_\alpha \oplus (d_\alpha c_\alpha)$ *exists and is equal to dc.*

Proof. Since $(d_\alpha c_\alpha)_r \leqq (d_\alpha)_r$, $\{(d_\alpha c_\alpha)_r\} \perp$. Similarly, $\{(d_\alpha c_\alpha)_l\} \perp$. If $\tau=\{g_\alpha\}$ is a r. s. s. for $\{d_\alpha\}$, then $g_\alpha dc = d_\alpha c = d_\alpha e_\alpha c = d_\alpha c_\alpha$.

Theorem 3. 1. *Suppose* $e=\sum_\alpha \oplus e_\alpha$ *and* $f=\sum_\alpha \oplus f_\alpha$ *for idempotents* e_α, f_α *in a von Neumann ring* $\mathfrak{R}$. *Suppose* $(e_\alpha)_r \sim (f_\alpha)_r$ *for each* α. *Then there exist* d, $\bar{d}$ *in* $\mathfrak{R}$ *such that* $d=edf$, $\bar{d}=f\bar{d}e$, $d\bar{d}=e$, $\bar{d}d=f$, $df_\alpha\bar{d}=e_\alpha$ *and* $\bar{d}e_\alpha d=f_\alpha$ *for each* α. *Moreover, if* $\cup_\alpha (e_\alpha)_r = \mathfrak{R}$ *then* $e=1=f$ *and* d *is invertible with* $\bar{d}$ *as its reciprocal.*

[18]) If $\aleph$ is an infinite cardinal, $\mathfrak{R}_2$ may fail to be an $\aleph$-ring and such d may not exist; this happens in KAPLANSKY's ring (see footnote [7])) if $e_m=(0,\dots,0,1,0,\dots)$, $d_m=(0,\dots,0,\sqrt{-1},0,\dots)$ $(m \geqq 1)$ where the non-zero components are in the m-th place.

Proof. The last statement would follow from the additivity of perspectivity in a von Neumann geometry [9, Part III], [3].

We recall VON NEUMANN's proof of Theorem 3.1 for the case I has a single index [9, Part II, Theorem 15.3 (a)]; suppose $(e)_r$ and $(f)_r$ are perspective, hence have a common complement. Then there exist idempotents e', f' such that:

$$(e)_r = (e')_r; \quad (1-e')_r = (1-f')_r; \quad (f')_r = (f)_r.$$

Define $d(e,f) = e'f$, $\bar{d}(e,f) = f'e$. Then it follows that $e' = e'f', f' = f'e', e = e'e$, $e' = ee'$, $f = f'f$, $f' = ff'$. Therefore:

$$ed(e,f)f = d(e,f);\; f\bar{d}(e,f)e = \bar{d}(e,f);$$
$$d(e,f)f\bar{d}(e,f) = d(e,f)\,\bar{d}(e,f) = e;$$
$$\bar{d}(e,f)fd(e,f) = \bar{d}(e,f)d(e,f) = f.$$

Next, if $\mathfrak{R}_2$ is also a von Neumann ring, we need only define $d = \sum_\alpha \oplus d(e_\alpha, f_\alpha)$, $\bar{d} = \sum_\alpha \oplus \bar{d}(e_\alpha, f_\alpha)$, using Lemma 3.11. and Lemma 3.12.

Finally, every von Neumann ring $\mathfrak{R}$ has a direct decomposition $\overline{\mathfrak{R}} \oplus \mathfrak{R}'$ as in Lemma 3.2 and we let E be the central idempotent for which $\mathfrak{R}' = \mathfrak{R}E$.

Then $Ea = aE$ for all a in $\mathfrak{R}$. Let a' denote aE, $\bar{a}$ denote $a(1-E)$. Then in $\overline{\mathfrak{R}}$, $(\bar{e}_\alpha)_r \sim (\bar{f}_\alpha)_r$ and since $\bar{R}_{\overline{\mathfrak{R}}}$ is a Boolean algebra, necessarily $\bar{e}_\alpha = \bar{f}_\alpha$.

In $\mathfrak{R}'$, $(e'_\alpha)_r \sim (f'_\alpha)_r$ and we can apply the argument of the preceding paragraph since $\bar{R}_{\mathfrak{S}}$ is a von Neumann ring when $\mathfrak{S} = \mathfrak{R}'_2$. Now

$$d = \left(\sum\nolimits_\alpha \oplus \bar{e}_\alpha\right) + \sum\nolimits_\alpha \oplus d(e'_\alpha, f'_\alpha),$$
$$\bar{d} = \left(\sum\nolimits_\alpha \oplus \bar{f}_\alpha\right) + \sum\nolimits_\alpha \oplus \bar{d}(e'_\alpha, f'_\alpha)$$

satisfy the requirements of Theorem 3.1.

Theorem 3.2. *Suppose $\{e_\alpha\}$ are orthogonal idempotents in a von Neumann ring $\mathfrak{R}$. Suppose for each α, $d_\alpha = e_\alpha d_\alpha = d_\alpha e_\alpha$. If $\mathfrak{R}_2$ is also a von Neumann ring, then $\sum_\alpha \oplus d_\alpha = d$ exists.*

Proof. $\sigma = \{e_\alpha\}$ is a r. s. s. for $\{d_\alpha\}$, so by Lemma 3.11, $\{d_\alpha\}$ possesses a σ-right lattice sum d. But $\{(d_\alpha)_l\} \perp$ since $(d_\alpha)_l \leqq (e_\alpha)_l$, so by Lemma 3.8, d is a lattice sum $\sum_\alpha \oplus d_\alpha$.

Corollary 1. *If in Theorem 3.2, $\{\bar{d}_\alpha\}$ satisfy $\bar{d}_\alpha = \bar{d}_\alpha e_\alpha = e_\alpha \bar{d}_\alpha$, $d_\alpha \bar{d}_\alpha = \bar{d}_\alpha d_\alpha = e_\alpha$ for each α, then $\bar{d} = \sum_\alpha \oplus \bar{d}_\alpha$ satisfies $d\bar{d} = \bar{d}d = \sum_\alpha \oplus e_\alpha$.*

Proof. Since $\sigma = \{e_\alpha\}$ is a r. s. s. for $\{\bar{d}_\alpha\}$ and l. s. s. for $\{d_\alpha\}$, it follows from Lemma 3.12 that $d\bar{d} = \sum_\alpha \oplus (d_\alpha \bar{d}_\alpha) = \sum_\alpha \oplus e_\alpha$. Similarly, $\bar{d}d = \sum_\alpha \oplus e_\alpha$.

Corollary 2. *Suppose $\{e_\alpha\}$ are orthogonal idempotents and $\{f_\alpha\}$ are orthogonal idempotents in a von Neumann ring $\mathfrak{R}$ such that $\mathfrak{R}_2$ is a von Neumann ring. Suppose for each α, $d_\alpha = e_\alpha d_\alpha = d_\alpha f_\alpha$. Then $d = \sum_\alpha \oplus d_\alpha$ exists. Moreover if $\{\bar{d}_\alpha\}$ exists such that for each α: $\bar{d}_\alpha = f_\alpha \bar{d}_\alpha = \bar{d}_\alpha e_\alpha$, $d_\alpha \bar{d}_\alpha = e_\alpha$, $\bar{d}_\alpha d_\alpha = f_\alpha$, then $\bar{d} = \sum_\alpha \oplus \bar{d}_\alpha$ exists and $d\bar{d} = \sum_\alpha \oplus e_\alpha$, $\bar{d}d = \sum_\alpha \oplus f_\alpha$.*

Proof. The argument for Theorem 3.2 and its Corollary 1 is valid in the present case.

4. Proof of the special cases (1.4.1), (1.4.2)

Lemma 4.1. *Suppose that c is in an $\aleph_0$-ring $\mathfrak{R}$ and $\cap((c^s)_r|s\geqq 1)=0$. Then $\mathfrak{R}$ can be espressed as the union of independent principal right ideals:*

(4.1.1) $$\mathfrak{R} = \dot{\cup}(x_{i,j}|1\leqq i<\infty;\ 1\leqq j\leqq i)$$

such that $cx_{i,j}=x_{i,j+1}$ and $c^r\cap x_{i,j}=0$ for $1\leqq j<i<\infty$, and $cx_{i,i}=0$ for $1\leqq i<\infty$. Then[19] *necessarily $x_{i,j}\sim x_{i,j+1}$ for $1\leqq j<i$, $c\mathfrak{R}\dot{\cup}(\cup(x_{i,1}|i\geqq 1))=\mathfrak{R}$, and for each $s\geqq 1$, $(c^s)^r\geqq x_{i,j}$ if $i-s<j\leqq i$ and $(c^s)^r\cap x_{i,j}=0$ if $j\leqq i-s$, so (by (1.1.1) and the modular law) $(c^s)^r=\cup(x_{i,j}|i\geqq 1;\ i-s<j\leqq i)$.*

Moreover any value of $[(c^i)^r-((c^i)^r\cap((c^{i-1})^r\dot{\cup} c\mathfrak{R}))]=[(c^i)^r-((c^{i-1})^r\cup((c^i)^r\cap c\mathfrak{R}))]$ may be used as $x_{i,1}$;[20] *on the other hand, any value of $[(c^r\cap c^{i-1}\mathfrak{R})-(c^r\cap c^i\mathfrak{R})]$ may be used as $x_{i,i}$.*

Proof. Suppose $x_{i,1}$ given as described and define $x_{i,j}=c^{j-1}x_{i,1}$ for $1\leqq j\leqq i$. Then for $1\leqq j<i$, $cx_{i,j}=x_{i,j+1}$. If $d\in x_{i,1}$ and $c^jd=0$ with $1\leqq j<i$ then $c^{i-1}d=0$, $d\in((c^{i-1})^r\cup((c^i)^r\cap c\mathfrak{R}))$, hence (see the definition of $x_{i,1}$) $d=0$. Thus $c^r\cap x_{i,j}=0$ for $1\leqq j<i$. Clearly $cx_{i,i}=c^ix_{i,1}=0$.

Next we show that for each $j\geqq 1$

$$\{c^j\mathfrak{R},\ x_{i,j}|i\geqq j\}\perp.$$

For suppose $c^jv=c^{j-1}v_j+\ldots+c^{j-1}v_s$ with all $v_i\in x_{i,1}$. Then we must have $c^{j-1}v_s=0$. Otherwise, $j-1<s$ and left multiplication by c^{s-j} yields: $c^sv=c^{s-1}v_s$. Then $v_s=(v_s-cv)+cv$ and $(v_s-cv)\in(c^{s-1})^r$, $cv\in(c^s)^r\cap c\mathfrak{R}$; this implies that $v_s=0$ since $v_s\in x_{s,1}$ and hence $c^{j-1}v_s=0$, after all. Repetition of this argument shows that $c^{j-1}v_i=0$ for all $i=s, s-1, \ldots, j$ and hence $c^jv=0$. This proves the assertion.

From this it follows that for each fixed $j\geqq 1$: $\{x_{i,j}|i\geqq j\}\perp$. Also $\{\dot{\cup}(x_{i,j}|i\geqq j)\,|\,j\geqq 1\}\perp$ since $\dot{\cup}(x_{i,j}|i\geqq j)\cap\dot{\cup}(x_{i,s}|i\geqq s>j)\leqq\dot{\cup}(x_{i,j}|i\geqq j)\cap c^j\mathfrak{R}=0$. This implies that $\{x_{i,j}|i\geqq 1;\ 1\leqq j\leqq i\}\perp$.

Next, by [4, Lemma 6.2], $\cap((c^j)_l|j\geqq 1)=0$ since by assumption $\cap((c^j)_r|j\geqq 1)=0$. Hence $\mathfrak{R}=(\cap((c^j)_l|j\geqq 1))^r=\cup((c^j)^r|j\geqq 1)$ (by [9, Part II, Lemma 2.2, Corollary 2]). Since $(\cup(x_{i,1}|i\geqq 1))\cup(c\mathfrak{R})\geqq(c^j)^r$ for all $j\geqq 1$ it follows that $\cup(x_{i,j}|1\leqq j\leqq i<\infty)\cup\cup c\mathfrak{R}\geqq\mathfrak{R}$. Successive left multiplication by c now gives: $\cup(x_{i,j}|1\leqq j\leqq i<\infty)\cup\cup c^m\mathfrak{R}\geqq\mathfrak{R}$ for all $m\geqq 1$, and since $\cap(c^m\mathfrak{R}|m\geqq 1)=\cap((c^m)_r|m\geqq 1)=0$, therefore $\dot{\cup}(x_{i,j}|1\leqq j\leqq i<\infty)=\mathfrak{R}$.

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[19]) $x_{i,j}\sim x_{i,j+1}$ follows from $cx_{i,j}=x_{i,j+1}$ and $c^r\cap x_{i,j}=0$ because of Corollary 1 of Lemma 2.1. Further, from $cx_{i,j}=x_{i,j+1}$ follows, because of formula (4.1.1), $c\mathfrak{R}=\dot{\cup}(x_{i,j}|1<j\leqq i)$.

[20]) When specialized to the classical case, this result yields: let T be a linear transformation of a finite dimensional vector space V into itself (V shall be finite dimensional over a division ring D but D need not be commutative) and suppose $T^p=0$ but $T^{p-1}\neq 0$ for some $p\geqq 1$. Let $N(T)=\{v|Tv=0\}$. Let $\xi_{i,1},\ldots,\xi_{i,s_i}$ be a basis for the difference-space $[N(T^i)-N(T^i)\cap(N(T^{i-1})\dot{\cup}T(V)))]$. Then $\{T^j\xi_{i,k}|i=1,\ldots,p;\ k=1,\ldots,s_i;\ j=0, 1, \ldots, i-1\}$ are a basis for V.

On the other hand, if the $x_{i,i}$ are pre-assigned as some given $[(c^r \cap c^{i-1}\mathfrak{R}) - (c^r \cap c^i\mathfrak{R})]$, set $x_{i,1} = [\{d|c^{i-1}d \in x_{i,i}\} - (c^{i-1})^r]$. We shall show that these values for $x_{i,1}$ satisfy the conditions given in the first part of Lemma 4. 1 and that $c^{i-1}x_{i,1}$ will coincide with the given $x_{i,i}$.

First, if $d \in x_{i,1}$, then $c^{i-1}d \in x_{i,i}$ and $x_{i,i} \leqq c^r$. Hence $c^i d = 0$. This proves: $x_{i,1} \leqq (c^i)^r$.

Next, $x_{i,1}$ is a relative complement of $(c^i)^r \cap ((c^{i-1})^r \cup (c\mathfrak{R}))$ with respect to $(c^i)^r$; to show this we must prove: (i) $x_{i,1} \cap ((c^{i-1})^r \cup c\mathfrak{R}) = 0$, (ii) $x_{i,1} \cup ((c^{i-1})^r \cup c\mathfrak{R}) \geqq (c^i)^r$.

To prove (i), suppose $d \in x_{i,1}$ and $d = u + cv$ with $c^{i-1}u = 0$. Then $c^{i-1}d = c^i v \in x_{i,i}$ and $c(c^i v) = 0$. Hence $c^i v \in (c^r \cap c^i\mathfrak{R})$ so, from the definition of $x_{i,i}$ it follows that $c^i v = 0$. Thus $c^{i-1}d = 0$. Now we have $d \in (c^{i-1})^r$, so from the definition of $x_{i,1}$ it follows that $d = 0$. This proves (i).

To prove (ii), we remark that from the definition of $x_{i,1}: x_{i,1} \cup (c^{i-1})^r = \{d|c^{i-1}d \in x_{i,i}\}$. Hence

$$x_{i,1} \cup ((c^{i-1})^r \cup c\mathfrak{R}) = \{d\,|\,c^{i-1}d \in x_{i,i}\} \cup c\mathfrak{R}.$$

Now suppose $u \in (c^i)^r$. Then $c^{i-1}u \in c^r$ so $c^{i-1}u \in (c^r \cap c^{i-1}\mathfrak{R})$. Then from the definition of $x_{i,i}: c^{i-1}u = v + w$ for some $v \in x_{i,i}$ and some $w \in (c^r \cap c^i\mathfrak{R})$. Now $w = c^i q$ for some q. Therefore $u = d + cq$ where $d = u - cq$ has the property: $c^{i-1}d = c^{i-1}u - c^i q = v \in x_{i,i}$. Hence $u \in (x_{i,1} \cup ((c^{i-1})^r \cup c\mathfrak{R}))$, which implies (ii).

Finally, if $d \in x_{i,1}$, then $c^{i-1}d \in x_{i,i}$, so $c^{i-1}x_{i,1} \leqq x_{i,i}$; on the other hand, if $u \in x_{i,i}$, then $u = c^{i-1}w$ for some w, so $w = d + v$ for some $d \in x_{i,1}$ and some $v \in (c^{i-1})^r$. So $u = c^{i-1}d \in c^{i-1}x_{i,1}$. Thus $x_{i,i} \leqq c^{i-1}x_{i,1}$. Hence $x_{i,i} = c^{i-1}x_{i,1}$ as stated.

Now all parts of Lemma 4. 1 are established.

Remark. If c is an element in an arbitrary regular ring $\mathfrak{R}$ with unit and $c^h = 0$ for some integer h, then the proof of Lemma 4. 1 is valid; moreover the range of i may be restricted to $1 \leqq i \leqq h$ (the appeal to [4, Lemma 6. 2] and [9, Part II, Lemma 2. 2, Corollary 2]) is unnecessary here since $\mathfrak{R} = \cup((c^j)^r | j \geqq 1)$ is an immediate consequence of $c^h = 0$, $(c^h)^r = \mathfrak{R}$.

Lemma 4. 2. *Suppose the hypotheses of Lemma 4. 1 hold and that* $c = p(a)$ *for some element* a *and some pure irreducible* p *in* P, $p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$.

If z_0 *is invertible, in particular if* $m > 1$, *then the element* a *is invertible. In every case if* $\mathfrak{R}$ *is a von Neumann ring*[21]) *and* (1. 3. 4) *holds* $\mathfrak{R}$ *has a decomposition as described in Lemma 4. 1 with the additional properties: For each* $i \geqq 1$,

(i) $x_{i,i} = \dot{\cup}(a^j x_i | 0 \leqq j < m)$ *for some* x_i;

(ii) $x_{i,1} = \cup(a^j y_i | 0 \leqq j < m)$ *for some* y_i *with* $c^{i-1}y_i = x_i$;

(iii) $(a^j)^r \cap y_i = 0$ *so* $a^j y_i \sim y_i$ *for* $0 \leqq j < mi$.

---

[21]) Lemma 4.1 (and Lemma 4.2 for the case $m = 1$) hold if $\mathfrak{R}$ is any $\aleph_0$-ring. But if $m > 1$ our proof of Lemma 4.2 uses transfinite induction (or Zorn's Lemma) and requires $\mathfrak{R}$ to be a von Neumann ring in which (1. 3. 4) holds.

Then (necessarily implied by (i), (ii), (iii) *in any* $\aleph_0$*-ring)*

(iv) $\bigcup(a^jy_i\,|\,0\leqq j<ms) = \dot{\bigcup}(x_{i,j}\,|\,1\leqq j\leqq s)$ *for* $1\leqq s\leqq i$ *and* $\{a^jy_i\,|\,0\leqq j<mi\}\perp$;

(v) $\dot{\bigcup}(a^jy_i\,|\,i\geqq 1, 0\leqq j<mi)=\mathfrak{R}$.

Proof. Suppose that $d\in a^r$. Then

$$0 = \bigcap((p^s(a))_r|s\geqq 1)\geqq\bigcap((p^s(a)d)_r|s\geqq 1) = \bigcap((z_0^sd)_r|s\geqq 1) = (d)_r$$

if z_0 is invertible. Then $a^r=0$, $(a)_l=\mathfrak{R}$. Thus if z_0 is invertible then (in any $\aleph_0$-ring by [4, Lemma 6.2]) a is invertible.

Suppose $m>1$. Suppose e is a non-zero central idempotent. If $ez_0=0$, then $ep(t)=et(t^{m-1}+\ldots+z_1)$ which is impossible since p is pure irreducible. Hence $ez_0\neq 0$ for every non-zero central idempotent e. But $z_0\mathfrak{R}=e_0\mathfrak{R}$ for some central idempotent e_0 [9, Part II, Theorem 2.5], and $(1-e_0)z_0\in(1-e_0)e_0\mathfrak{R}=0$. This forces $1-e_0$ to be 0, so $e_0=1$, $z_0\mathfrak{R}=\mathfrak{R}$. This shows that z_0 is invertible, and therefore, by the preceding paragraph, a also is invertible.

Next, suppose (i), (ii) and (iii) hold. Then, by (ii), $c^{i-1}(a^jy_i)=a^jc^{i-1}y_i=a^jx_i$ so $a^jy_i\sim a^jx_i$ if $0\leqq j<m$ by Corollary 1 to Lemma 2.1 (since $(c^{i-1})^r\cap x_{i,1}=0$). But $x_{i,1}\sim x_{i,i}$ so, by (i), $\bigcup(a^jy_i|0\leqq j<m)\sim\dot{\bigcup}(a^jx_i|0\leqq j<m)$. This forces: $\{a^jy_i|0\leqq j<m\}\perp$ by [2, Lemmas 6.15, 4.4], in any $\aleph_0$-ring. The same argument applies to inclusion relation

$$\bigcup(a^jy_i\,|\,0\leqq j<ms)\geqq\dot{\bigcup}(c^ka^jy_i\,|\,0\leqq j<m; 0\leqq k\leqq s-1) = \dot{\bigcup}(x_{i,j}\,|\,1\leqq j\leqq s)$$

and forces the addends on the left to be independent and the inclusion to be equality. Thus (i), (ii), (iii) imply (iv) and hence (v).

We need now only show that (i), (ii) and (iii) can be satisfied.

If $m=1$, choose $x_{i,i}$ and $x_{i,1}$ as in Lemma 4.1. Let $x_i=x_{i,i}$, $y_i=x_{i,1}$. Then (i) and (ii) hold obviously ($m=1$). Suppose for some j with $0\leqq j<i$ and some $d\in y_i$ that $a^jd=0$; then $(c-z_0)^jd=0$ so $c^jd\in(x_{i,1}\cup\ldots\cup x_{i,j})\cap x_{i,j+1}=0$, hence $d=0$. Thus (iii) holds by Corollary 1 to Lemma 2.1.

We may therefore suppose $m>1$. Let $A_i=c^r\cap c^{i-1}\mathfrak{R}$. Then $aA_i\leqq A_i$ so $aA_i=A_i$ (since a is invertible).

Now since p is pure irreducible and (1.3.4) holds, an argument of von Neumann [4, Lemma 5.1] applies here[21]) and shows, by transfinite induction that for some x_i: $A_{i+1}\dot{\cup}(\dot{\bigcup}(a^jx_i|0\leqq j<m))=A_i$. Hence we may use $\bigcup(a^jx_i|0\leqq j<m)$ as the pre-assigned $x_{i,i}$ in Lemma 4.1 and (i) will hold.

Let $B_i=\{d\,|\,c^{i-1}d\in x_i\}\leqq(c^i)^r$ and define $y_i=[B^i-(c^{i-1})^r]$. Then $c^{i-1}y_i=x_i$. Also

$$\left(\dot{\bigcup}(a^jy_i\,|\,0\leqq j<m)\right)\dot{\cup}(c^{i-1})^r=\{d\,|\,c^{i-1}d\in x_{i;i}\}$$

so we may also (in the proof of Lemma 4.1) choose $\bigcup(a^jy_i|0\leqq j<m)$ as $x_{i,1}$. Then (ii) holds.

As for (iii), since $m>1$, a is invertible and $(a^j)^r=0$; thus (iii) does hold.

This completes the proof of Lemma 4.2.

Lemma 4.3. *Suppose a and b are elements in a regular ring $\mathfrak{R}$ with unit and suppose m is an integer $\geqq 1$. Suppose $x_1, \ldots, x_m$ is a basis for $\bar{R}_{\mathfrak{R}}$ such that $ax_i=x_{i+1}=bx_i$ for $1\leqq i<m$ and $a^r=b^r=x_m$. Then a and b are similar.*

Proof. We may suppose $m \geqq 2$ (if $m=1$ then $a=b=0$ and so $b=dad^{-1}$ with $d=1$).

Since $x_1, \ldots, x_m$ is a basis for $\bar{R}_{\mathfrak{R}}: \dot{\bigcup}(x_i | 1 \leqq i \leqq m) = \mathfrak{R}$, in particular $x_i \cap x_j = 0$ if $i \neq j$. But if $1 \leqq i < m$, then $ax_i = x_{i+1}$ and $a^r \cap x_i = x_m \cap x_i = 0$, so by Corollary 1 to Lemma 2. 1, $x_i \sim x_{i+1}$. Hence $x_1, \ldots, x_m$ is a homogeneous basis for $\bar{R}_{\mathfrak{R}}$. Then by [9, Part II, Lemma 3. 6] there exist matrix units s_{ij} $(i, j = 1, \ldots, m)$ with $(s_{ii})_r = x_i$ for all i. Finally, the proof of [9, Part II, Theorem 3. 3] (note especially [9, page 99, lines 13, 14]) shows that $\mathfrak{R} = \mathfrak{S}_m$ with $\mathfrak{S}_1 = s_{11}\mathfrak{R}s_{11}$.

We shall call $c = (c_{ij})$ *off-diagonal* if (i) $c_{ij} = 0$ except when $i = j+1$ and (ii) $c_{j+1,j}$ is invertible (in $s_{11}\mathfrak{R}s_{11}$) for $1 \leqq j < m$. Let c_0 be the off-diagonal element with non-zero entries all $1 (= s_{11})$.

Now the hypotheses of Lemma 4. 3 force a and b to be off-diagonal; so it is sufficient to prove a and c_0 are similar. Thus we need only find an invertible $d = (d_{ij})$ such that $ad = dc_0$. For this purpose choose $d_{ij} = 0$ for $i \neq j$, and $d_{11} = 1$, $d_{ii} = a_{i,i-1}a_{i-1,i-2} \ldots a_{21}$ for $i > 1$; then $ad = dc_0$.

This completes the proof of Lemma 4. 3.

Theorem 4. 1. *Suppose that a, b are in an $\aleph_0$-ring, $\mathfrak{R}$ such that $\mathfrak{R}_2$ is an $\aleph_0$-ring, and $(a^s)_r \sim (b^s)_r$ for $s \geqq 1$ and $\cap((a^s)_r | s \geqq 1) = 0$. Then a and b are similar.*

Proof. Since $\cap((b^s)_r | s \geqq 1) \precsim (a^m)_r$ for all $m \geqq 1$ it follows by [2, Lemma 6. 11] that $\cap((b^s)_r | s \geqq 1) = 0$.

Let $x^a_{i,j}$ and $x^b_{i,j}$ be determined for a, b respectively as in Lemma 4. 1. First we shall show that $x^a_{i,j} \sim x^b_{i,j}$. We have: $(a^s)_r \sim (b^s)_r$ for $s \geqq 1$, hence $[(a^{s-1})_r - (a^s)_r] \sim [(b^{s-1})_r - (b^s)_r]$ for $s \geqq 1$. Then by Lemma 4. 1, $\dot{\bigcup}(x^a_{i,s} | i \geqq s) \sim \dot{\bigcup}(x^b_{i,s} | i \geqq s)$.

Since $x^a_{i,s} \sim x^a_{i,1}$ for each $i \geqq s$, $\dot{\bigcup}(x^a_{i,s} | i \geqq s) \sim \dot{\bigcup}(x^a_{i,1} | i \geqq s)$. Hence $\dot{\bigcup}(x^a_{i,1} | i \geqq s) \sim \dot{\bigcup}(x^b_{i,1} | i \geqq s)$, and so by subtraction, $x^a_{i,1} \sim x^b_{i,1}$ for all $i \geqq 1$. Then $x^a_{i,j} \sim x^a_{i,1} \sim x^b_{i,1} \sim x^b_{i,j}$ so $x^a_{i,j} \sim x^b_{i,j}$ for all $1 \leqq j \leqq i < \infty$, as stated.

Now let $\{e_{i,j}\}$, $\{f_{i,j}\}$ be families of orthogonal idempotents such that $(e_{i,j})_r = x^a_{i,j}$ and $(f_{i,j})_r = x^b_{i,j}$. Then by Theorem 3. 1, $df_{i,j}d^{-1} = e_{i,j}$ for some invertible d.

The element $c = dbd^{-1}$ has the property: $(c^s)_r = (db^s)_r \sim (b^s)_r$ for $s \geqq 1$ (use Corollary 1 to Lemma 2. 1), so $(c^s)_r \sim (a^s)_r$. Hence $\cap((c^s)_r | s \geqq 1) = 0$ (the argument used above for b applies to c also). Finally $(df_{i,j}d^{-1})_r$ may be used as $x^c_{i,j}$ since the mapping: $(u)_r \to (dud^{-1})_r = (du)_r$ is a lattice automorphism of $\bar{R}_{\mathfrak{R}}$.

So we may suppose $x^c_{i,j} = (e_{i,j})_r = x^a_{i,j}$ and clearly, we need only prove a and c are similar. In other words, we may assume $x^a_{i,j} = x^b_{i,j} = x_{i,j}$ (say).

Let $\{e_i\}$ be orthogonal idempotents with $(e_i)_r = \dot{\bigcup}(x_{i,j} | 1 \leqq j \leqq i)$, $\sum_i \oplus e_i = 1$. Then $ae_i = e_ia$, $be_i = e_ib$ for all i and the hypotheses of Lemma 4. 3 are satisfied in the ring $e_i\mathfrak{R}e_i$ by ae_i, be_i and $x_{i,j}e_i$ $(1 \leqq j \leqq i)$.

Thus for some d_i, $\bar{d}_i$ in $e_i\mathfrak{R}e_i$, $d_i\bar{d}_i = e_i = \bar{d}_id_i$ and $e_ib = d_iae_i\bar{d}_i$.

Now $d = \sum_i \oplus d_i$, $\bar{d} = \sum_i \oplus \bar{d}_i$ exist by Lemma 3. 11; and by Lemma 3. 12, $d\bar{d} = \bar{d}d = 1$, $b = da\bar{d}$. Thus Theorem 4. 1 is established.[22])

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[22]) Theorem 4.1 together with Lemma 4.3, yields a „canonical" representation for any a in $\mathfrak{R}$ for which $\cap((a^s)_r | s \geqq 1) = 0$.

Lemma 4.4. *Suppose a and b are invertible elements in a regular ring $\mathfrak{R}$ with unit. Suppose $m \geqq 1$ and $p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$ is in P and $p(a) = p(b) = 0$. Suppose $x, ax, \ldots, a^{m-1}x$ is a basis for $\overline{R}_{\mathfrak{R}}$ and $a^i x = b^i x$ for $1 \leqq i < m$. Then a and b are similar.*

Proof. We may suppose $m \geqq 2$ (if $m = 1$ then $a = -z_0 = b$ and $b = dad^{-1}$ with $d = 1$).

Then a is invertible and Corollary 1 to Lemma 2.1 shows that $x, ax, \ldots, a^{m-1}x$ is a homogeneous basis for $\overline{R}_{\mathfrak{R}}$. Hence $\mathfrak{R}$ possesses matrix units s_{ij} $(i, j = 1, \ldots, m)$ with $(s_{ii})_r = a^{i-1}x$ for $1 \leqq i \leqq m$.

Call $c = (c_{ij})$ *p-off-diagonal* if:

(i) $c_{i+1,i}$ is invertible (in $s_{11}\mathfrak{R}s_{11}$) for $1 \leqq i < m$,
(ii) $c_{i,m}c_{m,m-1}c_{m-1,m-2}\ldots c_{i+1,i} = -z_{i-1}$ for $1 \leqq i \leqq m$, and
(iii) $c_{ij} = 0$ for all other i, j.

Let c_0 be the p-off-diagonal element with $c_{i+1,i} = 1$ for $1 \leqq i < m$.

The hypotheses of Lemma 4.4 force a and b to be p-off-diagonal. Hence, we need only show that $ad = dc_0$ for some invertible d. For this purpose take $d_{11} = 1$, $d_{ii} = a_{i,i-1}\ldots a_{21}$ for $1 < i \leqq m$ and $d_{ij} = 0$ for $i \neq j$. This completes the proof of Lemma 4.4.

Theorem 4.2. *Suppose that a and b are elements in a von Neumann ring $\mathfrak{R}$ and that (1.3.4), (1.3.5) hold, and $m \geqq 1$ and $p(t) = t^m + z_{m-1}t^{m-1} + \ldots + z_0$ is in P and pure irreducible. Suppose $(p^s(a))_r \sim (p^s)b))_r$ for all $s \geqq 1$ and $\bigcap((p^s(a))_r | s \geqq 1) = 0$. Then a and b are similar.*

Proof. Theorem 4.1 applies to $p(a)$ and $p(b)$ and shows that $p(a) = dp(b)d^{-1} = p(dbd^{-1})$ for some invertible d. If $m = 1$, then $b + z_0 = d(a + z_0)d^{-1}$ so $b = dad^{-1}$ for some invertible d, as required. Thus we may assume $m \geqq 2$. Since we need only show that a and dbd^{-1} are similar, we may now assume that $p(b) = p(a)$.

Now Lemma 4.2 can be applied to yield elements y_i^a, y_i^b for a, b respectively, as described in Lemma 4.2. The corresponding values of $x_{i,1}$ (as described in Lemma 4.1) $x_{i,1}^a, x_{i,1}^b$ may not be the same but they are of the form $[x - \bar{x}]$ for the same $x, \bar{x}$; hence they are perspective. So, by (ii) of Lemma 4.2: $\dot{\bigcup}(a^j y_i^a | 0 \leqq j < m) \sim \dot{\bigcup}(b^j y_i^b | 0 \leqq j < m)$. Moreover, by (v) of Lemma 4.2, the elements in each of these unions form an independent family, and by (iii) of Lemma 4.2, the elements in the same family are mutually perspective.

Now $y_i^a \sim y_i^b$ follows from the theorem that $u_1 \sim v_1$ in a von Neumann geometry whenever $\dot{\bigcup}(u_i | 1 \leqq i \leqq m) \sim \dot{\bigcup}(v_i | 1 \leqq i \leqq m)$ with $\{u_i\}$ mutually perspective and $\{v_i\}$ mutually perspective (in the terminology of [9, Part III, page 272]: if $mA = mB$ with $m \geqq 1$ then $A = B$). To prove this theorem assume if possible that $u_1 \sim v_1$ is false. Then for some w in the centre of the geometry:

$w \cap u_1 \sim v_1^0$ where $v_1^0 \leqq v_1$ but $v_1^0 \neq v_1$ (here we use [9, Part III, Theorem 2.7], and interchange u_1, v_1 if necessary). Then there exist elements v_i^0 such that

$$w \cap \dot{\bigcup_{i=1}^{m}} u_i = \dot{\bigcup_{i=1}^{m}} (w \cap u_i) \sim \dot{\bigcup_{i=1}^{m}} (w \cap v_i^0) = w \cap \dot{\bigcup_{i=1}^{m}} v_i^0 \leqq$$

$$\leqq w \cap \dot{\bigcup_{i=1}^{m}} v_i \quad \text{but with} \quad w \cap \dot{\bigcup_{i=1}^{m}} v_i^0 \neq w \cap \dot{\bigcup_{i=1}^{m}} v_i^0.$$

On the other hand, $\left(w \cap \bigcup_{i=1}^{m} u_i\right) \sim \left(w \cap \bigcup_{i=1}^{m} v_i\right)$ by [9, Part III, Theorem 1. 4, (*a*), with $w = \bar{a}$, $\bigcup_{i=1}^{m} u_i = a$, $\bigcup_{i=1}^{m} v_i = b$]. So by the transitivity of perspectivity the lattice element $c = w \cap \bigcup_{i=1}^{m} v_i$ satisfies: $c \sim c_1$ with $c_1 \leqq c$, $c_1 \neq c$. But this is impossible. Hence $u_1 \sim v_1$ must hold, and so $y_i^a \sim y_i^b$.

Then $a^j y_i^a \sim b^j y_i^b$ for all j. Now by Theorem 3. 1 there exists a similarity mapping which maps $b^j y_i^b$ onto $a^j y_i^a$ for all $0 \leqq j < mi$. Hence, in proving Theorem 4.2, we may suppose $b^j y_i^b = a^j y_i^a$ for all $0 \leqq j < mi$.

Now set $Y_i = \bigcup (a^j y_i | 0 \leqq j < mi)$. Then $\dot{\bigcup}_i Y_i = \mathfrak{R}$ and $aY_i = bY_i$ for all $i \geqq 1$. By Lemma 3. 4 there exist orthogonal idempotents F_i with $(F_i)_r = Y_i$, $aF_i = F_i a$, $bF_i = F_i b$.

The hypotheses of Lemma 4. 4 are satisfied in the ring $F_i \mathfrak{R} F_i$ by aF_i and bF_i and $\{a^j y_i F_i | 0 \leqq j < mi\}$. Hence aF_i and bF_i are similar in the ring $F_i \mathfrak{R} F_i$ and, as in the proof of Theorem 4. 1, Lemmas 3. 11 and 3. 12 can be used to derive: a and b are similar in $\mathfrak{R}$.[23])

5. Proof of the Main Theorem

We suppose $\mathfrak{R}$ is a von Neumann ring satisfying (1. 3. 4), (1. 3. 5) and need only prove Theorem 1. 1 (ii). It will be sufficient to prove the following „augmentation" lemma.

Lemma 5. 1. *Suppose* P_1, a, b *satisfy the hypotheses* (1. 3. 2) *and* (1. 3. 3). *Suppose* $S^a = \{e_\alpha^a, p_\alpha | \alpha \in I\}$ *and* $S^b = \{e_\alpha^b, p_\alpha | \alpha \in I\}$ *have the properties:*[24])

(5. 1. 1) e_α^a, e_α^b *are non-zero idempotents with* $\bar{e}_\alpha^a = \bar{e}_\alpha^b = \bar{e}_\alpha$ (say) *for each* $\alpha \in I$, $p_\alpha \in P_1$ *and* $\bar{e}_\alpha p_\alpha$ *is pure irreducible in* $\bar{e}_\alpha \mathfrak{R}$*;*

(5. 1. 2) $\bar{e}_\alpha \left(\bigcap ((p_\alpha^s(a))_r | s \geqq 1)\right) = (\bar{e}_\alpha - e_\alpha^a)_r$,
$\bar{e}_\alpha \left(\bigcap ((p_\alpha^s(a))_l | s \geqq 1)\right) = (\bar{e}_\alpha - e_\alpha^a)_l$*; similarly for* b *in place of* a.

(5. 1. 3) $\bar{e}_\alpha \bar{e}_\beta p_\alpha \neq \bar{e}_\alpha \bar{e}_\beta p_\beta$ if $\bar{e}_\alpha \bar{e}_\beta \neq 0$.

Then:

(5. 1. 4) $(e_\alpha^a)_r \sim (e_\alpha^b)_r$ *for each* $\alpha \in I$, $(\Sigma_\alpha \oplus e_\alpha^a)_r \sim (\Sigma_\alpha \oplus e_\alpha^b)_r$*;*

(5. 1. 5) $\{e_\alpha^a | \alpha \in I\}$, $\{e_\alpha^b | \alpha \in I\}$ *are sets of orthogonal idempotents,* $ae_\alpha = e_\alpha a$, $e_\alpha b = be_\alpha$*;*

(5. 1. 6) *If* $\sum_\alpha \oplus e_\alpha \neq 1$ *it is possible to augment* S^a, S^b *by pairs* (e^a, p), (e^b, p) *preserving* (5. 1. 1), (5. 1. 2) *and* (5. 1. 3).

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[23]) Theorem 4.2 together with Lemma 4.4 yields a „canonical" representation for any a in $\mathfrak{R}$ for which $\bigcap ((p^s(a))_r | s \geqq 1) = 0$ for some *pure irreducible* p in P.

[24]) For any idempotent e in a von Neumann ring $\mathfrak{R}$, we write $\bar{e}$ to denote the central cover of e, that is, the central idempotent $\bar{e}$ with the properties: $\bar{e}e = e$ and for any central idempotent f, $fe = e$ implies $f\bar{e} = \bar{e}$.

Remark. Theorem 1. 1 (ii) can be deduced from Lemma 5. 1. To see this, note that by transfinite induction (or ZORN's Lemma) it is possible to choose S^a, S^b to be maximal with the properties (5. 1. 1), (5. 1. 2), (5. 1. 3). Then from (5. 1. 6) it will follow that $\sum_\alpha \oplus e_\alpha^a = 1$ and hence $\sum_\alpha \oplus e_\alpha^b = 1$.

Then by Theorem 3. 2 there will exist d, d' in $\mathfrak{R}$ such that $dd' = d'd = 1$ (so $d' = d^{-1}$) and $de_\alpha^b d' = e_\alpha^a$, $d' e_\alpha^a d = e_\alpha^b$ for each α. The mapping $u \to dud^{-1}$ is a ring isomorphism of $e_\alpha^b \mathfrak{R} e_\alpha^b$ onto $e_\alpha^a \mathfrak{R} e_\alpha^a$.

Let $c = dbd^{-1}$. Then $de_\alpha^b bd^{-1} = dbe_\alpha^b d^{-1} = e_\alpha^a c = ce_\alpha^a$ and (5. 1. 1), (5. 1. 2), (5. 1. 3) hold if b is replaced by c and each e_α^b is replaced by $e_\alpha^a = e_\alpha$ (say).

In each ring $e_\alpha \mathfrak{R} e_\alpha$, the elements $e_\alpha c$, $e_\alpha a$ satisfy the hypothesis of Theorem 4. 2. Hence g_α, g'_α exist in $e_\alpha \mathfrak{R} e_\alpha$ such that $g_\alpha g'_\alpha = g'_\alpha g_\alpha = e_\alpha$ and $g_\alpha e_\alpha c g'_\alpha = e_\alpha a$.

If now $\mathfrak{R}_2$ is also a von Neumann ring then, by Corollary 1 to Theorem 3. 2, the elements $g = \sum \oplus g_\alpha$, $g' = \sum_\alpha \oplus g'_\alpha$ exist and satisfy: $gg' = g'g = 1$ (so $g' = g^{-1}$). Then by the Corollary to Lemma 3. 10, $c = \sum_\alpha \oplus e_\alpha c$ and by Lemma 3. 12, $gcg^{-1} = \sum_\alpha \oplus (g_\alpha (e_\alpha c) g_\alpha^{-1}) = \sum_\alpha \oplus (e_\alpha a) = a$.

Thus *if* $\mathfrak{R}_2$ *is a von Neumann ring*, c and a are similar, hence b and a are similar, which establishes Theorem 1. 1 (ii).

Thus we need only prove Lemma 5. 1 to complete the proof of Theorem 1. 1.

Proof of Lemma 5. 1. The hypotheses of Theorem 1. 1 (ii) imply that (recall the definition of $\bar{e}_\alpha$ given in footnote [24]):

$$\bar{e}_\alpha\big(\cap (p_\alpha^s(a))_r \,|\, s \geqq 1\big) \sim \bar{e}_\alpha\big(\cap (p_\alpha^s(b))_r \,|\, s \geqq 1\big).$$

Hence (5. 1. 2) implies $(e_\alpha^a)_r \sim (e_\alpha^b)_r$. Now (5. 1. 5) follows from [4, § 7. 1]. Then $\dot{\bigcup} (e_\alpha^a)_r \sim \dot{\bigcup} (e_\alpha^b)_r$, by the additivity of perspectivity in von Neumann geometries [3]. So (5. 1. 5) and (5. 1. 4) both hold.

Finally, we establish (5. 1. 6). Suppose $E = 1 - \sum_\alpha \oplus e_\alpha^a \neq 0$. Then $Ee_\alpha^a = e_\alpha^a E = 0$ and the Corollary to Lemma 3. 10 shows that $aE = Ea$ so $p(a)E = Ep(a)$ for all $p \in P$.

Now a is assumed to be P_1-almost algebraic, so $\cap_p (p(a))_r = 0$ when p varies over all products of factors from P_1. Hence $\cap_p (Ep(a))_r = \cap_p (p(a)E)_r = 0$. Thus for some such p, $(Ep(a))_r \neq (E)_r$.

Since P_1 is fully factorizable there is a set of orthogonal non-zero central idempotents $\{\bar{e}\}$ such that $\bigcup (\bar{e})_r = \mathfrak{R}$ and each $\bar{e}p$ is a product $\bar{e}p_1 \ldots p_m$ with all p_i in P_1 and $\bar{e}p_i$ pure irreducible in $\bar{e}\mathfrak{R}$.

Now for at least one of these $\bar{e}$ we have $(\bar{e}E)_r \neq (\bar{e}Ep(a))_r$ since for every c in $\mathfrak{R}$: $(c)_r = \bigcup (\bar{e}c)_r$ (use the Corollary to Lemma 3. 10). Hence with this $\bar{e}$: $(\bar{e}Ep(a))_r = (\bar{e}Ep_1(a) \ldots p_m(a))_r \neq (\bar{e}E)_r$ where the p_i are all in P_1 and each $\bar{e}p_i$ is pure irreducible in $\bar{e}\mathfrak{R}$. If $\bar{e}Ep_i(a)b_i = \bar{e}E$ were to hold for some b_i for $i = 1, \ldots, m$ we would have

$$\bar{e}Ep_1(a) \ldots p_m(a) b_m \ldots b_1 = \bar{e}Ep_1(a) \ldots p_{m-1}(a) \bar{e}Eb_{m-1} \ldots b_1 =$$
$$= \bar{e}E\bar{e}Ep_1(a) \ldots p_{m-1}(a) b_{m-1} \ldots b_1 = \ldots = \bar{e}E\bar{e}E \ldots \bar{e}E = \bar{e}E,$$

a contradiction. Thus, if p is replaced by a suitable p_i, we can assert: p is in P_1, $\bar{e}p$ is pure irreducible in $\bar{e}\mathfrak{R}$ and $(\bar{e}Ep(a))_r \neq (\bar{e}E)_r$. For the rest of this proof we keep p fixed with this value.

Now we apply the well known method of „exhaustion”. Let $\{\bar{f}\}$ be a set of orthogonal non-zero central idempotents maximal with the property: $\bar{f}\bar{e} = \bar{f}$ and

$(\bar{f}Ep(a))_r = (\bar{f}E)_r$. Let $\bar{f}_0 = \sum \oplus (\bar{f})$. Then $\bar{f}_0\bar{e} = \bar{f}_0$ and, using the Corollary to Lemma 3. 10, we deduce $(\bar{f}_0Ep(a))_r = (\bar{f}_0E)_r$.

Thus, if $\bar{e}$ is replaced by $\bar{e} - \bar{f}_0$ we can assert: $(\bar{g}Ep(a))_r \neq (\bar{g}E)_r$ whenever $\bar{g}$ is a non-zero central idempotent with $\bar{g} = \bar{e}\bar{g}$. For the rest of this proof we keep $\bar{e}$ fixed with this value (clearly, $\bar{e} \neq 0$).

Applying [4, § 7. 1] to the ring $\bar{e}\mathfrak{R}$, we choose e^a to be the unique idempotent with $e^a = \bar{e}e^a$ and

$$(\bar{e} - e^a)_r = \bigcap ((\bar{e}p^s(a))_r \mid s \geqq 1),$$

and

$$(\bar{e} - e^a)_l = \bigcap ((\bar{e}p^s(a))_l \mid s \geqq 1);$$

similarly, with $p^s(b)$ in place of $p^s(a)$, we choose e^b.

Since we assume (1. 3. 2) and (1. 3. 3) it follows that for each $s \geqq 1$, $(\bar{e}p^s(a))_r \sim \sim (\bar{e}p^s(b))_r$, hence

$$\bigcap ((\bar{e}p^s(a))_r | s \geqq 1) \sim \bigcap ((\bar{e}p^s(b))_r | s \geqq 1)$$

(use [2] or [3]). Hence, by subtraction: $(e^a)_r \sim (e^b)_r$, and so $\overline{e^a} = \overline{e^b}$ (use [9, Part III, Theorem 1. 4 (*d*)]).

We now prove that if S^a, S^b are augmented by the pairs (e^a, p), (e^b, p) then (5. 1. 1), (5. 1. 2), (5. 1. 3) are preserved.

First, we shall show that $\overline{e^a} = \bar{e}$. If this were false then, since $e^a\bar{e} = e^a$ it follows that $\overline{\bar{g}e^a} = 0$ (and hence $\bar{g}e^a = 0$) for some $\bar{g} = \bar{g}\bar{e} \neq 0$. But our choice of e^a implies, by [4, § 7. 1], that $((\bar{e} - e^a)p(a))_r = (\bar{e} - e^a)_r$ so $(\bar{g})_r = (\bar{g}(\bar{e} - e^a))_r = (\bar{g}(\bar{e} - e^a)p(a))_r = = (\bar{g}p(a))_r$; $(\bar{g})_r = (\bar{g}p(a))_r$. But also, by our choice of $\bar{e}$: $(\bar{g}E)_r \neq (\bar{g}Ep(a))_r$. This is a contradiction, for if $\bar{g}p(a)c = \bar{g}$, then $\bar{g}Ep(a)c = E\bar{g}p(a)c = E\bar{g} = \bar{g}E$. This contradiction shows that $\bar{e}^a = \bar{e}$. Since $\bar{e} \neq 0$, it follows that $e^a \neq 0$ and so (5. 1. 1) and (5. 1. 2) continue to hold.

Next we show that (5. 1. 3) also continues to hold. We suppose for some α that $\bar{g} = \bar{e}_\alpha\bar{e} \neq 0$ and we need only show that $\bar{g}p_\alpha \neq \bar{g}p$. It is sufficient to show that $\bar{g}p_\alpha(a) \neq \bar{g}p(a)$.

Since $e^a_\alpha E = 0$ it follows from (5. 1. 2) that

$$(\bar{e}_\alpha E)_r = ((\bar{e}_\alpha - e^a_\alpha)E)_r \leqq \bigcap ((p^s_a(a))_r \mid s \geqq 1) \geqq (p_\alpha(a))_r,$$

so $E\bar{e}_\alpha = p_\alpha(a)c$ for some c in $\mathfrak{R}$. Then $\bar{g}Ep_\alpha(a)c = \bar{g}E\bar{e}_\alpha = \bar{g}E$ so $(\bar{g}Ep_\alpha(a))_r = (\bar{g}E)_r$. But by our choice of $\bar{e}$, since $\bar{g} \neq 0$ and $\bar{g} = \bar{e}\bar{g}$: $(\bar{g}Ep(a))_r \neq (\bar{g}E)_r$. Hence $\bar{g}p_\alpha(a) \neq \neq \bar{g}p(a)$, as required to show that (5. 1. 3) continues to hold.

This completes the proof of Lemma 5. 1 and so Theorem 1. 1 is established.

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Die Jordan—Dedekindsche Bedingung im direkten Produkt von geordneten Mengen

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Es sei S eine geordnete (=teilweise geordnete) Menge, $a, b \in S$, $a \leqq b$, $[a, b] = \{x | x \in S, a \leqq x \leqq b\}$. Bezeichnen wir mit $\mathfrak{R}$ (a, b) das System aller maximalen Ketten von $[a, b]$. In [1] (vgl. auch [2], S. 11) wurde die folgende Bedingung untersucht (Jordan—Dedekindsche Bedingung):

(JD_1) Ist $u, v \in S$, $u \leqq v$, $R_1, R_2 \in \mathfrak{R}(u, v)$, so sind die Ketten R_1, R_2 isomorph.

Eine andere Bedingung über die Ketten in S wurde von G. SZÁSZ [3] eingeführt:

(JD_2) Ist $u, v \in S, u \leqq v, R_1, R_2 \in \mathfrak{R}(u, v)$, so gilt kard $R_1 =$ kard R_2.[1])

Für geordnete Mengen von endlicher Länge sind die Bedingungen (JD_1), (JD_2) äquivalent. Die Bedingung (JD_2) wurde auch in [4]—[7] behandelt.

In dieser Arbeit gehen wir aus einer von G. BIRKHOFF gestellten Frage [2, S. 11, Ex. 6] aus: „Prove (or disprove) that the cardinal product of any two partly ordered sets of finite length which satisfy the Jordan—Dedekind chain condition also satisfies it." Ein kurzer Beweis hierfür steht in 2. Ferner untersuchen wir die Bedingung (JD_2) für das direkte Produkt von geordneten Mengen A, B, wobei die Längen von A, B nicht endlich zu sein brauchen.

Es seien A, B nichtleere geordnete Mengen, $S = AB$,

$$s_i = (a_i, b_i) \quad (i = 1, 2), \quad a_i \in A, \quad b_i \in B, \quad s_1 < s_2.$$

1. Es ist klar, daß das Element s_1 genau dann ein unterer Nachbar[2]) von s_2 ist, wenn eine der folgenden Bedingungen erfüllt ist:

a) a_1 ist ein unterer Nachbar von a_2 und $b_1 = b_2$,

b) $a_1 = a_2$ und b_1 ist ein unterer Nachbar von b_2.

2. *Wenn für jedes* $R_1 \in \mathfrak{R}(a_1, a_2)$ *und jedes* $R_2 \in \mathfrak{R}(b_1, b_2)$ kard $R_1 = n_1$ *bzw.* kard $R_2 = n_2$ *ist (wobei* n_1, n_2 *natürliche Zahlen bedeuten), so gilt* kard $R = n_1 + n_2 - 1$ *für jedes* $R \in \mathfrak{R}(s_1, s_2)$.

Beweis: durch Induktion in Bezug auf die natürliche Zahl $n_1 + n_2 \geqq 2$. Der Fall $n_1 + n_2 = 2$ (d. h. $s_1 = s_2$) ist trivial. Es sei $n_1 + n_2 > 2$, d. h. $s_1 < s_2$. In R gibt es einen unteren Nachbar s von s_2. Nach 1 und nach der Induktionsvoraussetzung gilt kard $R' = n_1 + n_2 - 2$ für jedes $R' \in \mathfrak{R}(s_1, s)$. Also ist kard $R =$ kard $R' + 1 = n_1 + n_2 - 1$.

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[1]) Für eine beliebige Menge M bezeichnen wir mit kard M die Mächtigkeit von M.

[2]) Vgl. [8], S. 6.

3. Wir nennen eine geordnete Menge S *diskret*, wenn jede beschränkte Kette in S endlich ist. Aus 2 folgt:

Satz 1. *Wenn die diskreten geordneten Mengen A, B die Bedingung* (JD_2) *erfüllen, dann erfüllt auch $S=AB$ diese Bedingung.*

Bemerkung. Durch Induktion kann Satz 1 für ein direktes Produkt $S=A_1 A_2 \ldots A_n$ von n Faktoren verallgemeinert werden. Für das vollständige direkte Produkt ΠA_i (wobei die Anzahl der direkten Faktoren A_i unendlich ist), gilt ein analoger Satz nicht. (Vgl. [6].)

4. Wir werden S *k-vollständig* nennen, wenn aus $u, v \in S$, $u<v$, $R \in \mathfrak{R}(u, v)$ folgt, daß die Kette R ein vollständiger Verband ist. (Es ist leicht zu zeigen, daß eine k-vollständige gerichtete Menge kein Verband zu sein braucht.)

Satz 2. *Es seien A, B k-vollständige geordnete Mengen, welche die Bedingung* (JD_2) *erfüllen. Dann erfüllt auch $S=AB$ die Bedingung* (JD_2).

Beweis. Wir setzen voraus, daß für jede Kette $R_1 \in \mathfrak{R}(a_1, a_2)$ und $R_2 \in \mathfrak{R}(b_1, b_2)$, $\operatorname{kard} R_1 = n_1$ bzw. $\operatorname{kard} R_2 = n_2$ ist. Wenn n_1 und n_2 endlich sind, so ist nach 2 $\operatorname{kard} R = n_1 + n_2 - 1$ für jede Kette $R \in \mathfrak{R}(s_1, s_2)$.

Wenn wenigstens eine der Mächtigkeiten n_1, n_2 unendlich ist, bezeichnen wir $n = \max(n_1, n_2)$. Es sei $R \in \mathfrak{R}(s_1, s_2)$; R_A sei die Menge aller $a \in A$, für die es ein $b' \in B$ gibt, so daß $(a, b') \in R$; die Bedeutung von R_B ist analog. Offensichtlich sind R_A und R_B Ketten, $R_A \subset [a_1, a_2]$, $R_B \subset [b_1, b_2]$, also ist $\operatorname{kard} R_A \leqq n_1$, $\operatorname{kard} R_B \leqq n_2$. Wir setzen $C = \{(a, b) | a \in R_A, b \in R_B\}$. Es ist $\operatorname{kard} C = \operatorname{kard} R_A$. $\operatorname{kard} R_B \leqq n_1 n_2 = n$; aus $R \subset C$ folgt dann $\operatorname{kard} R \leqq n$.

Es sei z. B. $n = n_1$. Aus dem Auswahlsaxiom folgt, daß es Ketten $R_1 \in \mathfrak{R}(a_1, a_2)$, $R_2 \in \mathfrak{R}(b_1, b_2)$ gibt, so daß $R_A \subset R_1$, $R_B \subset R_2$. Anderseits wählen wir ein Element $a_0 \in R_1$, $a_1 < a_0 < a_2$ aus. Es sei B_1 die Menge aller $b \in R_2$, für die es ein $a \in A$, $a \leqq a_0$ gibt, so daß $(a, b) \in R$. Da B k-vollständig ist, existiert in R_2 das Element $b_0 = \sup B_1$. Bezeichnen wir $s_0 = (a_0, b_0)$ und sei $s = (a, b)$ ein beliebiges Element von R. Für jedes $a \in R_1$ sind die Elemente a, a_0 vergleichbar. Wenn $a \leqq a_0$ gilt, so ist (nach der Definition von b_0) $b \leqq b_0$ und folglich $s \leqq s_0$. Es sei $a > a_0$. Wir wollen zeigen, daß dann $b \geqq b_0$ ist. Wäre nämlich $b < b_0$, so gäbe es ein Element $b' \in B_1$ mit $b < b' \leqq b_0$, und zu diesem b' könnte man ein Element $a' \in A$ finden, so daß $s' = (a', b') \in R$, $a' \leqq a_0$. Die Elemente s, s' wären dann aber unvergleichbar, was unmöglich ist. Es gilt also $b \geqq b_0$. Das Element s_0 ist mit allen $s' \in R$ vergleichbar, also ist $s_0 \in R$. Daraus folgt $R_1 \subset R_A$, so daß $R_1 = R_A$, $\operatorname{kard} R \geqq \operatorname{kard} R_1 = n$ ist; nach der oben gewonnenen Ungleichung gilt also $\operatorname{kard} R = n$.

Bemerkung. Die Voraussetzung über die k-Vollständigkeit kann in Satz 2 nicht weggelassen werden. (Vgl. Satz 2 und 3.)

5. Wenn A oder B die Bedingung (JD_2) nicht erfüllt, so erfüllt auch ihr direktes Produkt $S=AB$ diese Bedingung nicht.

Die „einfachsten" geordneten Mengen, welche die Bedingung (JD_2) (trivialerweise) erfüllen, sind die wohlgeordneten Mengen. Nehmen wir an, daß A die Bedingung (JD_2) erfüllt; es stellt sich die Frage, unter welchen Umständen diese Eigenschaft von A auch für jedes direkte Produkt AB erhalten bleibt, wobei B eine beliebige wohlgeordnete Menge ist.

Es sei $\mathfrak{m}$ irgendeine Mächtigkeit. Ferner sei B eine wohlgeordnete Menge mit kleinstem Element b_1 und mit größtem Element b_2, kard $B=\mathfrak{m}$. Unter diesen Voraussetzungen gilt der folgende

Satz 3. *Wenn* $[a_1, a_2]\subset A$, $R\in\mathfrak{R}(a_1, a_2)$, kard $R=\mathfrak{n}\leqq\mathfrak{m}$ *und R kein vollständiger Verband ist, so gibt es zu jeder Mächtigkeit* $\mathfrak{n}'$ $(\mathfrak{n}\leqq\mathfrak{n}'\leqq\mathfrak{m})$ *eine Kette* $R'\in\mathfrak{R}(s_1, s_2)$ *mit* kard $R'=\mathfrak{n}'$.

Beweis. Nach der Voraussetzung gibt es in R ein Ideal $A_1\neq\emptyset$ ohne größtes Element und ein duales Ideal $A_2\neq\emptyset$ ohne kleinstes Element, so daß $A_1\cap A_2=\emptyset$, $A_1\cup A_2=R$. Es sei $\mathfrak{n}'$ eine Mächtigkeit mit $\mathfrak{n}\leqq\mathfrak{n}'\leqq\mathfrak{m}$. Es gibt ein Element $b_0\in B$, so daß kard $B_0=\mathfrak{n}'$, wobei $B_0=[b_1, b_0]$. Wir bezeichnen

$$R_1=\{(a_1, b)|b\in B_0\},\quad R_2=\{(a, b)|a\in A_1\},$$

$$R_3=\{(a, b_2)|a\in A_2\},\quad R'=R_1\cup R_2\cup R_3.$$

Offenbar ist R' eine Kette und kard $R_1=\mathfrak{n}'$, kard $R_2\leqq\mathfrak{n}$, kard $R_3\leqq\mathfrak{n}$, also ist kard $R'=\mathfrak{n}'$. Es sei $s=(a, b)\in[s_1, s_2]$; nehmen wir an, daß s mit allen Elementen der Kette R' vergleichbar ist. Wir wollen zeigen, daß $R'\in\mathfrak{R}(s_1, d_2)$ gilt, d. h. daß s zu R' gehört. Offensichtlich ist $a\in R$.

a) Es sei $a=a_1$. Nach der Voraussetzung gibt es ein $a'\in A_1$ mit $a'>a_1$. Wir bezeichnen $s'=(a', b_0)$. Da $s'\in R_2$ ist, sind die Elemente s, s' vergleichbar; aus der Beziehung $a'>a_1$ folgt daher $b_0\geqq b$, also ist $s\in R_1$.

b) Es sei $a>a_1$, $a\in A_1$. Da s und (a_1, b_0) vergleichbar sind, gilt $b\geqq b_0$. Ferner gibt es ein $a'\in A_1$, $a<a'$. Aus der Vergleichbarkeit von s und (a', b_0) bekommen wir dann $b\leqq b_0$. Also ist $b=b_0$, $s\in R_2$.

c) Es sei $a\in A_2$. Es gibt ein $a'\in A_2$, $a'<a$. Da s und (a', b_2) vergleichbar sind, gilt $b=b_2$, also ist $s\in R_3$.

Aus den Sätzen 2 und 3 folgt unmittelbar der

Satz 4. *Es sei A eine geordnete Menge, welche die Bedingung* (JD_2) *erfüllt. Dann sind die folgenden Bedingungen äquivalent:*

a) *es gibt eine wohlgeordnete Menge B derart, daß $S=AB$ die Bedingung* (JD_2) *nicht erfüllt,*

b) *A ist nicht k-vollständig.*

Satz 5. *Es sei*

$$R_1\in\mathfrak{R}(a_1, a_2),\quad R_2\in\mathfrak{R}(b_1, b_2),$$

$$c_1, c_2\in R_2,\quad R_3=[b_1, c_1]\cap R_2,\quad R_4=[b_1, c_2]\cap R_2,$$

wobei R_1 kein vollständiger Verband und kard $R_1=\mathfrak{n}$, $\mathfrak{n}\leqq$ kard $R_3<$ kard R_4 *ist. Dann ist im direkten Produkt $S=AB$ die Bedingung* (JD_2) *nicht erfüllt.*

Beweis. In analoger Weise wie im Beweis des Satzes 3 können maximale Ketten R', R'' konstruiert werden, derart, daß kard $R'=$ kard R_3, kard $R''=$ kard R_4 ist.

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Ein hinreichendes und notwendiges Kriterium für die ZPE-Eigenschaft in kommutativen, regulären Halbgruppen

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In Verallgemeinerung des ZERMELOschen Beweises für den Fundamentalsatz der elementaren Zahlentheorie hat HASSE in [1] ein hinreichendes Kriterium dafür angegeben, daß in einem Integritätsbereich $\mathfrak{R}$ jede Nichteinheit ($\neq o$) als Produkt von Primelementen[1]) geschrieben werden kann, welches von KRULL in [2] zu einem hinreichenden und notwendigen Kriterium verschärft wird. Beide Kriterien (vgl. auch KRULL [3], § 5) arbeiten mit einer Art reellwertigen Bewertung und beruhen auf der Existenz gewisser Linearkombinationen, womit sie von der Ringeigenschaft von $\mathfrak{R}$ wesentlichen Gebrauch machen. In der vorliegenden Note wird gezeigt, daß sich der ZERMELOsche Beweisgedanke sogar zu einem allgemeinen Kriterium für eine Zerlegung in Primelemente (ZPE) in Halbgruppen ausbauen läßt, welches die oben genannten Kriterien als Spezialfälle enthält. Die allgemeinere Formulierung des Kriteriums macht dabei seinen Beweis einfacher und beeinträchtigt auch nicht die Handlichkeit seiner Anwendung, wofür wir abschließend einige Beispiele geben.

Für eine (kommutative und reguläre) Halbgruppe $\mathfrak{H}$ mit Einselement e [2]) lassen sich die Begriffe Teiler, Einheit, assoziierte Elemente usw. wie üblich erklären. Insbesondere verstehen wir unter einer echten Zerlegung

$$a = a_1 a_2 \ldots a_n$$

von $a \in \mathfrak{H}$ eine solche, für die $n \geqq 2$ gilt und kein $a_i \in \mathfrak{H}$ eine Einheit ist; eine Nichteinheit $r \in \mathfrak{H}$ ohne echte Zerlegungen heißt irreduzibel und sogar prim, wenn aus $r|ab$ stets $r|a$ oder $r|b$ folgt. Die Eindeutigkeit jeder Zerlegung in irreduzible Elemente (bis auf assoziierte Faktoren) ist gleichwertig damit, daß auch umgekehrt jedes irreduzible Element von $\mathfrak{H}$ prim ist. Damit läuft die Möglichkeit, jede Nichteinheit von $\mathfrak{H}$ in ein Produkt von Primelementen zu zerlegen, auf das gleiche hinaus wie die Existenz und Eindeutigkeit der Zerlegung aller Nichteinheiten von $\mathfrak{H}$ in irreduzible Elemente. Wir sagen dann, daß $\mathfrak{H}$ eine ZPE-Halbgruppe ist und zeigen, daß für diese Eigenschaft von $\mathfrak{H}$ das folgende Kriterium hinreichend und notwendig ist:

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[1]) Entsprechend der nachstehenden Festlegung der Begriffe „irreduzibel" und „prim" (im Einklang mit KRULL [2], [3] bzw. RÉDEI [4], § 79) sind die Faktoren einer solchen Zerlegung bis auf assoziierte Elemente eindeutig bestimmt.

[2]) Übrigens gelten alle Überlegungen auch für Halbgruppen ohne Einselement, wenn man einfach alle sich auf Einheiten bzw. Assoziiertheit beziehenden Formulierungen wegfallen läßt.

Für die Elemente von $\mathfrak{F}$ *läßt sich eine irreflexive, asymmetrische und transitive Relation* $a \prec b$ *erklären, die den nachstehenden Bedingungen genügt:*

(1) *In jeder Teilmenge von* $\mathfrak{F}$ *existiert ein in bezug auf diese Relation minimales Element.*
(2) *Ist* b *keine Einheit, so gilt* $a \prec ab$ *für alle* $a \in \mathfrak{F}$.
(3) *Aus* $a \prec b$ *folgt* $ac \prec bc$ *für alle* $c \in \mathfrak{F}$.
(4) *Zu je zwei nicht assoziierten irreduziblen Elementen* r *und* s *von* $\mathfrak{F}$ *gibt es ein Element* $z \in \mathfrak{F}$, *welches*

$$\alpha)\ z \prec r, \quad \beta)\ s \nmid z, \quad \gamma)\ \text{aus } s \mid ra \text{ folgt } s \mid za,$$

oder die gleichen Bedingungen mit Vertauschung von r *und* s *erfüllt.*

In der Tat ist dieses Kriterium hinreichend für die ZPE-Eigenschaft von $\mathfrak{F}$. Wir gehen indirekt vor und betrachten zunächst die Menge aller Nichteinheiten von $\mathfrak{F}$, für die keine Zerlegung in irreduzible Elemente existieren sollte. In dieser Menge gibt es nach (1) ein minimales Element c, welches also nicht irreduzibel ist. Aus $c = ab$ folgt aber gemäß (2) $a \prec c$ und $b \prec c$, so daß es nach Wahl von c für a und b und damit für c Zerlegungen in irreduzible Elemente gibt. Entsprechend sei nun c ein minimales Element mit zwei wesentlich verschiedenen Zerlegungen

$$c = r_1 r_2 \ldots r_n = s_1 s_2 \ldots s_m.$$

Dabei kann kein r_i zu einem der s_j assoziiert sein, da sonst nach (2) und Wahl von c beide Zerlegungen bis auf assoziierte Elemente übereinstimmten. Wir können also etwa auf r_1 und s_1 (4) anwenden. Mit $z \prec r_1$ gilt nach (3)

$$za = zr_2 \ldots r_n \prec r_1 r_2 \ldots r_n = r_1 a = c$$

und wir erhalten

$$s_1 \mid zr_2 \ldots r_n, \quad s_1 \nmid z, \quad s_1 \nmid r_2, \ldots, s_1 \nmid r_n$$

im Widerspruch dazu, daß die Zerlegung von $zr_2 \ldots r_n \prec c$ in irreduzible Elemente nach der Wahl von c eindeutig ist.

Ist umgekehrt $\mathfrak{F}$ eine ZPE-Halbgruppe, so ist durch die Anzahl der in den Zerlegungen auftretenden Primelemente (worunter für Einheiten die Zahl 0 zu verstehen ist) eine Halbordnungsrelation gegeben, die ersichtlich die Bedingungen (1), (2) und (3) unseres Kriteriums erfüllt. Für (4) stellen wir sogleich allgemeiner fest:

Ist für eine ZPE-Halbgruppe $\mathfrak{F}$ *irgendeine Halbordnungsrelation erklärt, die den Bedingungen* (1), (2) *und* (3) *genügt, so ist auch* (4) *erfüllt, und zwar sogar für beliebige Elemente* r *und* s *aus* $\mathfrak{F}$ *mit* $r \nmid s$ *und* $s \nmid r$.

Wir brauchen nämlich für z nur den größten gemeinsamen Teiler von r und s zu nehmen, für den man leicht die Aussagen $\alpha)$, $\beta)$ und $\gamma)$ nachprüft.

Abschließend wenden wir unser Kriterium zum Beweis einiger wichtigen, bekannten ZPE-Aussagen an. Für den Halbring N der natürlichen Zahlen ist es ersichtlich mit der üblichen Ordnungsrelation und $z = r - s$ für $r > s$ erfüllt. Ist $\mathfrak{R}$ ein euklidischer Ring mit der Zuordnung $a \to g(a)$, so bedeute $a \prec b$ einfach $g(a) < g(b)$, und man wählt $z = r - sq$ mit $g(z) < g(s) \leqq g(r)$; zum Nachweis von

(3) benötigt man allerdings, daß $g(ac) < g(bc)$ aus $g(a) < g(b)$ folgt.[3]) Für einen Hauptidealring $\mathfrak{R}$ setzt man $a \prec b$ genau dann, wenn für die zugehörigen Ideale $(a) \supset (b)$ gilt; mit z aus $(r, s) = (z)$ ergeben sich sofort die Bedingungen unseres Kriteriums.

Weiterhin sei $\mathfrak{R}$ der Halbring[4]) aller Ideale eines Ringes $\mathfrak{R}$, dessen Ideale dem Oberkettensatz und dem Faktorsatz genügen. Letzterer besagt also, daß jedes Oberideal $\mathfrak{a}$ eines Ideals $\mathfrak{b}$ auch Faktor von $\mathfrak{b}$, und damit in unserem Sinne Teiler von $\mathfrak{b}$ ist[5]). Mit $\mathfrak{a} \prec \mathfrak{b}$ genau dann, wenn $\mathfrak{a} \supset \mathfrak{b}$ gilt, erhält man eine Halbordnungsrelation, die auch den Forderungen (1), (2) und (3) unseres Kriteriums genügt; (1) ist nämlich die dem Oberkettensatz entsprechende Maximalbedingung, (3) ist gleichwertig mit der Regularität der Idealmultiplikation, die sich aus dem Faktorsatz ergibt, und (2) folgt aus (3) und $\mathfrak{R} \prec \mathfrak{b}$ für jedes $\mathfrak{b} \neq \mathfrak{R}$. Zum Beweis von (4) setzt man $\mathfrak{z} = \mathfrak{r} + \mathfrak{s}$: Wegen des Faktorsatzes gilt mit $\mathfrak{r} \nmid \mathfrak{s}$ und $\mathfrak{s} \nmid \mathfrak{r}$ auch $\mathfrak{r} \not\supseteq \mathfrak{s}$ und $\mathfrak{s} \not\supseteq \mathfrak{r}$, also $\mathfrak{z} \supset \mathfrak{r}$ und $\mathfrak{z} \supset \mathfrak{s}$ und damit bereits $\alpha)$ und $\beta)$, während sich $\gamma)$ sofort aus $\mathfrak{z}\mathfrak{a} = \mathfrak{r}\mathfrak{a} + \mathfrak{s}\mathfrak{a}$ ergibt.

Schließlich kann auch die Übertragung der ZPE-Eigenschaft von einem Ring $\mathfrak{R}$ auf den Polynomring $\mathfrak{R}[x]$ mit unserem Kriterium gezeigt werden, wobei nur der sog. Hilfssatz von Gauss über das Produkt primitiver Polynome als Hilfsmittel verwendet wird. Dazu erweitern wir eine unserem Kriterium genügende Relation für die Elemente des ZPE-Ringes $\mathfrak{R}$ gemäß

$$\sum_{\nu=0}^{n} a_\nu x^\nu \prec \sum_{\mu=0}^{m} b_\mu x^\mu \quad (a_n \neq o,\ b_m \neq o)$$

$$\text{genau dann, wenn entweder } n < m$$

$$\text{oder } n = m \text{ und } a_n \prec b_m,$$

zu einer Relation für die Elemente von $\mathfrak{R}[x]$, die ersichtlich irreflexiv, asymmetrisch und transitiv ist und auch wieder den Bedingungen (1), (2) und (3) genügt. Den Nachweis von (4) führen wir für irreduzible Elemente und unterscheiden die Fälle:

$r \in \mathfrak{R}$, $s \in \mathfrak{R}$: Hier leistet das schon in $\mathfrak{R}$ existierende Element z auch für $\mathfrak{R}[x]$ das Verlangte.

$r(x) \in \mathfrak{R}[x]$, $s \in \mathfrak{R}$: Für das Einselement $e = z$ ist $e \prec r(x)$, $s \nmid e$ und aus $s | r(x) a(x)$ folgt $s | a(x)$, da $r(x)$ als irreduzibles Polynom den Inhalt e hat.

$r(x) \in \mathfrak{R}[x]$, $s(x) \in \mathfrak{R}[x]$: Die Division mit Rest liefert in der Form

$$c r(x) = s(x) q(x) + z(x), \quad c \in \mathfrak{R},$$

ein Element $z(x) \in \mathfrak{R}[x]$ mit einem kleineren Grad als dem von $s(x)$ und dem von $r(x)$. Dieses Element $z(x)$ ist ungleich o (sonst wäre c der Inhalt von $q(x)$ und damit $s(x)$ Teiler von $r(x)$) und erfüllt $\alpha)$, $\beta)$ und $\gamma)$.

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[3]) Für die üblicherweise betrachteten Beispiele euklidischer Ringe ist diese Bedingung ebenso wie die meist nur geforderte Aussage $g(a) \leqq g(ab)$ erfüllt; zum Nachweis der Hauptidealringeigenschaft sind ohnehin beide entbehrlich.

[4]) Für die Begriffsbildung des Halbringes und weitere Literatur vgl. Weinert [5].

[5]) Man beachte, daß der für beliebige Halbgruppen erklärte Begriff des Teilers sich definitionsgemäß zunächst mit dem idealtheoretischen Begriff des Faktors und nicht mit dem des Oberideals deckt. Auch ist ein irreduzibles Element $\mathfrak{r}$ von $\mathfrak{R}$ hier als multiplikativ unzerlegbares Ideal erklärt, welches aber auf Grund des Faktorsatzes maximal und damit auch Primideal von $\mathfrak{R}$ ist.

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Zur Holomorphentheorie der Ringe

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§ 1

In seiner Arbeit [2] hat L. RÉDEI den zum Begriff des Holomorphs einer Gruppe analogen Begriff der Holomorphe eines Ringes geschaffen, der es in der Tat ermöglicht, zu einer Reihe von gruppentheoretischen Sätzen entsprechende Sätze für Ringe aufzustellen. Ein auffälliger Unterschied zur Gruppentheorie ist dabei, daß es Ringe gibt, die mehr als ein Holomorph haben[1]). So wird bereits in [2] gezeigt, daß ein Zeroring R dann und nur dann genau ein Holomorph besitzt, wenn der Endomorphismenring $\mathscr{E}(R^+)$ von R^+ kommutativ ist[2]). Da gemäß SZELE—SZENDREI [4] verhältnismäßig wenig Moduln kommutative Endomorphismenringe besitzen, könnte man vermuten, daß auch allgemein die Ringe mit mehreren Holomorphen vorherrschen. Die nachfolgenden Ergebnisse berechtigen aber wohl zu der Feststellung, daß die weitaus wichtigsten Klassen von Ringen nur ein Holomorph besitzen.

Zunächst sind nach unserer Kenntnis die folgenden drei hinreichenden Bedingungen für die Einzigkeit des Holomorphs eines Ringes R bekannt:

I) R hat ein Einselement (RÉDEI [2]).

II) $R = R^2$, wobei mit R^2 das von allen Produkten von Elementen aus R erzeugte Ideal bezeichnet wird (LEEUWEN [1]).

III) R ist nullteilerfrei (LEEUWEN [1], im kommutativen Fall bereits SZENDREI [5]).

Für die Durchführung des Beweises in [1] reicht es aber schon aus, daß R überhaupt ein links- bzw. rechtsreguläres Element enthält. In weiterer Verschärfung werden wir sogar *die Einzigkeit des Holomorphs von R aus folgender Bedingung ableiten:*

IV. *Für den Annullator* $\mathfrak{n}$ *von* R *gilt* $\mathfrak{n} = (0)$.

Damit genügt es, die alles übrige umfassenden Bedingungen II und IV zu beachten, zu denen noch die ersichtlich hinreichende Bedingung V hinzukommt.

V. Der Endomorphismenring $\mathscr{E}(R^+)$ von R^+ ist kommutativ.

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[1]) In diesem Falle ist es auch nicht möglich, die Eindeutigkeit des Holomorphs eines solchen Ringes R etwa durch eine geeignete Auswahl oder mit Hilfe des Durchschnitts der maximalen Ringe befreundeter Doppelhomothetismen von R erreichen zu wollen, da nichtbefreundete Doppelhomothetismen von R nur durch verschiedene Everettsche Erweiterungen induziert werden können. Auch überlegt man sich leicht, daß verschiedene Holomorphe von R keine äquivalenten Erweiterungen von R sein können.

[2]) In der Bezeichnung richten wir uns weitgehend nach RÉDEI [3]. Insbesondere war das dort auf Seite 204 formulierte Problem Anlaß zu der vorliegenden Note. Die wichtigsten Begriffsbildungen fassen wir übrigens am Anfang von § 2 auch noch einmal zusammen.

Wir werden jedoch zeigen, daß es auch noch weitere Ringe mit nur einem Holomorph gibt, die also keiner der bisher genannten hinreichenden Bedingungen genügen. Dabei verwenden wir Untersuchungen über diejenigen Endomorphismen, die als Komponenten der Doppelhomothetismen auftreten können. In diesem Zusammenhang ergeben sich weitere Kriterien für die Einzigkeit des Holomorphs eines Ringes R, auf die sich alle anderen Aussagen zurückführen lassen[3]). Insbesondere verweisen wir hier auf die in § 2 formulierten Sätze 3 und 4.

Abschließend wenden wir uns noch der Frage nach der Kommutativität der Holomorphe zu, wo wir bisher bekannte Ergebnisse entsprechend ergänzen. Auch hier ist ja die Tatsache, daß es zahlreiche Ringe mit kommutativen Holomorphen gibt, ohne Analogon zur Gruppentheorie, da das Holomorph einer Gruppe G (vom trivialen Fall der Gruppen 1. und 2. Ordnung abgesehen) nichtkommutativ ist.

§ 2

Es sei R ein Ring mit den Elementen $\alpha, \beta, \ldots$, und $\mathscr{E}(R^+)$ der Endomorphismenring des als Modul R^+ aufgefaßten Ringes R. Die direkte Summe

$$\mathscr{E}_2(R^+) = \mathscr{E}(R^+) \oplus \mathscr{E}^\circ(R^+)$$

von $\mathscr{E}(R^+)$ und dem zu ihm entgegengesetzten Ring $\mathscr{E}^\circ(R^+)$ besteht dann aus allen Doppelendomorphismen $a=(a_1, a_2)$ von R^+, d. h. den Doppelabbildungen

$$\alpha \to a\alpha = a_1\alpha, \qquad \alpha \to \alpha a = a_2\alpha,$$

für deren Nacheinanderanwendung also gilt:

$$\text{(1)} \qquad \begin{aligned} \alpha &\to (ab)\alpha = (a_1 b_1)\alpha = a_1(b_1\alpha), \\ \alpha &\to \alpha(ab) = (a_2 \circ b_2)\alpha = (b_2 a_2)\alpha = b_2(a_2\alpha). \end{aligned}$$

Insbesondere heißt ein solcher Doppelendomorphismus $a=(a_1, a_2)$ ein Doppelhomothetismus von R, wenn er erfüllt:

$$\text{(2)} \qquad \begin{aligned} a(\alpha\beta) &= (a\alpha)\beta, \quad \text{d. h.} \quad a_1(\alpha\beta) = (a_1\alpha)\beta, \\ (\alpha\beta)a &= \alpha(\beta a), \quad \text{d. h.} \quad a_2(\alpha\beta) = \alpha(a_2\beta), \end{aligned}$$

$$\text{(3)} \qquad (\alpha a)\beta = \alpha(a\beta), \quad \text{d. h.} \quad (a_2\alpha)\beta = \alpha(a_1\beta),$$

$$\text{(4)} \qquad (a\alpha)a = a(\alpha a), \quad \text{d. h.} \quad a_2 a_1\alpha = a_1 a_2\alpha.$$

Schließlich nennt man Doppelhomothetismen $a=(a_1, a_2)$ und $b=(b_1, b_2)$ befreundet, wenn gilt:

$$\text{(5)} \qquad \begin{aligned} (a\alpha)b &= a(\alpha b), \quad \text{d. h.} \quad b_2 a_1\alpha = a_1 b_2\alpha, \\ (b\alpha)a &= b(\alpha a), \quad \text{d. h.} \quad a_2 b_1\alpha = b_1 a_2\alpha. \end{aligned}$$

Da Differenz und Produkt befreundeter Doppelhomothetismen a und b von R wieder Doppelhomothetismen von R sind, welche darüber hinaus mit jedem zu a

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[3]) Der Beweis von IV kann allerdings ebensogut auch unabhängig davon geführt werden, vgl. Fußnote 4.

und b befreundeten Doppelhomothetismus c befreundet sind, kann man von den in $\mathscr{E}_2(R^+)$ enthaltenen Ringen befreundeter Doppelhomothetismen sprechen. Nach Rédei [3] liegt jede Menge befreundeter Doppelhomothetismen (insbesondere also jeder Doppelhomothetismus) von R in einem maximalen Ring $\mathfrak{D}$ dieser Art. Die zugehörigen faktorenfreien Everettschen Erweiterungen $\mathfrak{D} \dotplus R$ sind die Holomorphe von R, und die Einzigkeit des Holomorphs von R läuft auf die Existenz nur eines maximalen Ringes befreundeter Doppelhomothetismen hinaus.

Wir stellen nun fest, daß die Bedingung (2_1) unter allen Endomorphismen von R^+ diejenigen auswählt, die überhaupt als erste Komponente eines Doppelhomothetismus von R in Frage kommen. Wie man leicht nachrechnet, bilden diese Endomorphismen $a_1, b_1, \ldots$ einen Unterring $\mathscr{H}_1$ von $\mathscr{E}(R^+)$. Entsprechend bilden diejenigen Endomorphismen $a_2, b_2, \ldots$, die (2_2) erfüllen, einen solchen Unterring $\mathscr{H}_2$.

Der Durchschnitt $\mathscr{H}^* = \mathscr{H}_1 \cap \mathscr{H}_2$ enthält jedenfalls den Nullendomorphismus $o = o^*$, den identischen Automorphismus $e = e^*$ sowie alle die Endomorphismen, die von einem Zentrumselement ϱ von R gemäß

$$\alpha \to \varrho\alpha = \alpha\varrho$$

induziert werden. Ist allgemein a^* ein Element von $\mathscr{H}^*$, so ist $a = (a^*, a^*)$ wegen

$$(a^*\alpha)\beta = a^*(\alpha\beta) = \alpha(a^*\beta)$$

und

$$a^*a^*\alpha = a^*a^*\alpha$$

ein Doppelhomothetismus von R. Da zwei solche Doppelhomothetismen $a = (a^*, a^*)$, $b = (b^*, b^*)$ genau dann befreundet sind, wenn

$$a^*b^*\alpha = b^*a^*\alpha$$

gilt, erhalten wir

Satz 1. *Für die Einzigkeit des Holomorphs von R ist notwendig, daß der Ring* $\mathscr{H}^* \subseteq \mathscr{E}(R^+)$ *kommutativ ist.*

Weiterhin betrachten wir die Menge $\mathscr{H}_1'$ derjenigen Endomorphismen $a_1 \in \mathscr{H}_1$, die tatsächlich als erste Komponente in wenigstens einem Doppelhomothetismus von R auftreten, zu denen es also ein $a_2 \in \mathscr{H}_2$ gibt, so daß für a_1 und a_2 die Bedingungen (3) und (4) erfüllt sind. Entsprechend definieren wir die Menge $\mathscr{H}_2'$. Aus obigem geht hervor, daß dann sogar

$$\mathscr{H}^* = \mathscr{H}_1' \cap \mathscr{H}_2'$$

gilt, jedoch sind $\mathscr{H}_1'$ und $\mathscr{H}_2'$ im allgemeinen keine Unterringe von $\mathscr{E}(R^+)$. Da zwei Doppelhomothetismen $a = (a_1, a_2)$ und $b = (b_1, b_2)$ genau dann befreundet sind, wenn gemäß (5) stets

$$b_2a_1\alpha = a_1b_2\alpha \quad \text{und} \quad a_2b_1\alpha = b_1a_2\alpha$$

gilt, kommen wir zu folgendem Kriterium:

Satz 2. *Für die Einzigkeit des Holomorphs von R ist notwendig und hinreichend, daß jeder Endomorphismus aus* $\mathscr{H}_1'$ *mit jedem Endomorphismus aus* $\mathscr{H}_2'$ *vertauschbar ist, d. h., daß alle Kommutatoren*

$$[a_1, b_2] = a_1b_2 - b_2a_1 \quad \textit{mit} \quad a_1 \in \mathscr{H}_1', \; b_2 \in \mathscr{H}_2'$$

gleich dem Nullendomorphismus sind.

Insbesondere sind dann auch $\mathscr{H}_1'$ *und* $\mathscr{H}_2'$ *Unterringe von* $\mathscr{E}(R^+)$.

Zusatz. *Allgemein sind diese Kommutatoren* $[a_1, b_2]$ *Elemente von* $\mathfrak{H}^*$, *die* R *in den Annullator* $\mathfrak{n}$ *von* R *und* R^2 *auf das Nullelement von* R *abbilden.*

Es bleibt die Ringeigenschaft von $\mathfrak{H}_1'$ bzw. $\mathfrak{H}_2'$ und der Zusatz zu beweisen; wir begnügen uns mit letzterem. Zunächst gilt $([a_1, b_2]\alpha)\beta = 0$ wegen

$$((a_1 b_2)\alpha)\beta = (a_1(b_2\alpha))\beta = a_1((b_2\alpha)\beta) = a_1(\alpha(b_1\beta)) = (a_1\alpha)(b_1\beta)$$

und

$$((b_2 a_1)\alpha)\beta = (b_2(a_1\alpha))\beta = (a_1\alpha)(b_1\beta)$$

(wobei wir von (2_1) und (3) mit einem zu b_2 korrespondierenden b_1 Gebrauch gemacht haben) und entsprechend $\beta([a_1, b_2]\alpha) = 0$ für alle β aus R, so daß stets $[a_1, b_2]\alpha \in \mathfrak{n}$ erfüllt ist. Analog folgt

$$[a_1, b_2](\alpha\beta) = (a_1 b_2)(\alpha\beta) - (b_2 a_1)(\alpha\beta) = (a_1\alpha)(b_2\beta) - (a_1\alpha)(b_2\beta) = 0,$$

woraus man nun auch die übrigen Behauptungen des Zusatzes erhält.

Wie man sieht, ergibt sich aus Satz 2 und dem Zusatz sofort, daß die Bedingungen II, IV und V für die Einzigkeit des Holomorphs von R hinreichend sind, wobei die II und IV betreffenden Aussagen im wesentlichen auch bereits den Zusatz ergeben.[4]) Weiterhin gilt bei kommutativer Multiplikation von R bereits $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}^*$, und wir erhalten in Ergänzung zu unseren Sätzen das

Korollar. *Für die Einzigkeit des Holomorphs eines kommutativen Ringes* R *ist die Kommutativität des Ringes* $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}^* \subseteqq \mathscr{E}(R^+)$ *notwendig und hinreichend.*

Wir kommen nun zu dem angekündigten Beispiel eines Ringes R mit nur einem Holomorph, der jedoch alle in § 1 angegebenen hinreichenden Bedingungen verletzt. Es sei $R = R_1 \oplus R_2$ die direkte Summe eines Körpers $R_1 = \langle 0, \varepsilon\rangle$ der Ordnung 2 und eines Zeroringes $R_2 = \langle 0, \nu\rangle$ der Ordnung 2. Dann hat R diesen Zeroring als Annullator, es gilt $R^2 = R_1 \subset R$ und der Endomorphismenring $\mathscr{E}(R^+)$ enthält die zur vollen Permutationsgruppe von drei Elementen isomorphe Automorphismengruppe von R^+, ist also nichtkommutativ. Trotzdem hat R auf Grund des Korollars nur ein Holomorph, denn $\mathfrak{H}_1$ besteht aus folgenden Endomorphismen, deren Kommutativität man leicht nachrechnet:

Bild von	bei s_0	bei s_1	bei s_2	bei s_3
0	0	0	0	0
ε	0	ε	ε	0
ν	0	ν	0	ν
$\varepsilon + \nu = a$	0	a	ε	ν

Wie wir nur bemerken wollen, besteht der maximale Ring $\mathfrak{D}$ der Doppelhomothe-

[4]) Selbstverständlich lassen sich diese Aussagen auch ohne explizite Verwendung der Komponenten der Doppelhomothetismen ableiten, so etwa IV gemäß:

$$(a(ab))\beta = a((ab)\beta) = a(a(b\beta)) = (aa)(b\beta) = ((aa)b)\beta,$$

also $(a(ab) - (aa)b)\beta = 0$ und entsprechend $\beta(a(ab) - (aa)b) = 0$, so daß aus $\mathfrak{n} = (0)$ die Einzigkeit des Holomorphs von R folgt.

tismen aus folgenden Elementen:

$$(s_0, s_0)\ (s_1, s_1)\ (s_2, s_2)\ (s_3, s_3)$$
$$(s_0, s_3)\ (s_3, s_0)\ (s_1, s_2)\ (s_2, s_1).$$

Mit $\mathfrak{H}^*$ ist natürlich auch $\mathfrak{D}$, nicht aber das Holomorph $\mathfrak{D} + R$ kommutativ, wie aus dem weiter unten zitierten Kriterium von LEEUWEN hervorgeht (vgl. § 3).

Anderseits gibt es aber tatsächlich auch Ringe, die keine Zeroringe sind und trotzdem mehrere Holomorphe besitzen. Ist etwa $R = R_1 \oplus R_2 \oplus R_3$ die direkte Summe des Ringes $R_1 = \Gamma$ der ganzen Zahlen und zweier Zeroringe R_2 und R_3 mit Γ^+ als Modul, so hat R als Endomorphismenring den vollen Matrizenring $\mathfrak{M}_3(\Gamma)$. Wie man leicht nachrechnet, besteht $\mathfrak{H}_1$ gerade aus allen Matrizen der Form

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix},$$

so daß R nach dem Korollar (oder auch schon nach Satz 1) mehrere Holomorphe besitzt.

Schließlich können wir diese und ähnliche Beispiele folgenden allgemeinen Aussagen unterordnen:

Satz 3. *Es sei $R = R_1 \oplus R_2$ ein Ring, in dem ein Zeroring R_2 (der damit Unterring des Annullators $\mathfrak{n}$ von R ist) als direkter Summand auftritt. Dann hat R mehrere Holomorphe, wenn dies für R_2 zutrifft, wenn also der Endomorphismenring $\mathscr{E}(R_2^+)$ von R_2^+ nichtkommutativ ist.*

Beweis. Nach Voraussetzung gibt es Endomorphismen s_1 und s_2 von R_2^+ und ein Element $\varrho_2 \in R_2^+$ mit

$$(s_1 s_2 - s_2 s_1)\,\varrho_2 \neq 0.$$

Gemäß $s_i(\alpha_1 + \alpha_2) = s_i\alpha_2$ mit $\alpha_1 \in R_1$, $\alpha_2 \in R_2$ läßt sich jeder dieser Endomorphismen von R_2^+ zu einem Endomorphismus von R^+ fortsetzen. Wegen

$$s_i((\alpha_1 + \alpha_2)(\beta_1 + \beta_2)) = s_i(\alpha_1\beta_1 + 0) = 0$$
$$(s_i(\alpha_1 + \alpha_2))(\beta_1 + \beta_2) = (s_i\alpha_2)(\beta_1 + \beta_2) = 0$$
$$(\alpha_1 + \alpha_2)(s_i(\beta_1 + \beta_2)) = (\alpha_1 + \alpha_2)(s_i\beta_2) = 0$$

liegen diese fortgesetzten Endomorphismen s_1 und s_2 in $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \mathfrak{H}^* \subseteqq \mathscr{E}(R^+)$, so daß $\mathfrak{H}^*$ nichtkommutativ ist und damit R nach Satz 1 mehrere Holomorphe besitzt.

Leider können wir dieses Kriterium nur mit einer Verschärfung der Voraussetzung über R auch als notwendig nachweisen. Immerhin enthält der nachfolgende Satz rein formal alle in § 1 angegebenen hinreichenden Kriterien.

Satz 4. *Es sei $R = R^2 \oplus \mathfrak{n}$ die direkte Summe des Ideals R^2 und seines Annullators $\mathfrak{n}$. Dann ist für die Einzigkeit des Holomorphs von R notwendig und hinreichend, daß $\mathfrak{n}$ nur ein Holomorph besitzt, also der Endomorphismenring $\mathscr{E}(\mathfrak{n}^+)$ kommutativ ist.*

Beweis. Nach dem voranstehenden Satz ist nur zu zeigen, daß R unter den angegebenen Bedingungen nur ein Holomorph besitzt. Anderenfalls gäbe es Endomorphismen $a_1 \in \mathfrak{H}_1'$ und $b_2 \in \mathfrak{H}_2'$ und ein Element $\alpha_1 + \alpha_2 \in R$ mit

$$(a_1 b_2 - b_2 a_1)(\alpha_1 + \alpha_2) \neq 0.$$

Wegen $\alpha_1 \in R^2$, $\alpha_2 \in \mathfrak{n}$ folgt aus dem Zusatz zu Satz 2, daß dabei

$$(a_1 b_2 - b_2 a_1)(\alpha_1 + \alpha_2) = (a_1 b_2 - b_2 a_1)\alpha_2 = \nu$$

mit $\nu \neq 0$ aus $\mathfrak{n}$ gilt. Andererseits bildet ganz allgemein ein Endomorphismus $a_1 \in \mathfrak{H}_1'$ jedes Element $\alpha_2 \in \mathfrak{n}$ wieder auf ein Element aus $\mathfrak{n}$ ab, wie sich aus (2_1) bzw. (3) gemäß

$$a_1(\alpha_2 \beta) = a_1(0) = 0 = (a_1 \alpha_2)\beta$$
$$(a_2 \beta)\alpha_2 = 0 = \beta(a_1 \alpha_2)$$

mit beliebigen $\beta \in R$ ergibt; das gleiche gilt für jeden Endomorphismus b_2 aus $\mathfrak{H}_2'$. Damit induzieren die Endomorphismen von R^+ aus $\mathfrak{H}_1' \cup \mathfrak{H}_2'$ Endomorphismen von $\mathfrak{n}^+$, wobei also insbesondere die oben angegebenen Endomorphismen a_1 und b_2 nichtkommutative Endomorphismen von $\mathfrak{n}^+$ liefern.

§ 3

Leeuwen hat in [1] gezeigt, daß alle Holomorphe eines Ringes R genau dann kommutativ sind, wenn für jeden Doppelhomothetismus a stets $a\alpha = \alpha a$ gilt, also jeder Doppelhomothetismus die Form $a = (a^*, a^*)$ mit $a^* \in \mathfrak{H}^*$ hat. Insbesondere ist dann der Ring R selbst kommutativ. Daraus folgt (vgl. [1]), daß das eindeutig bestimmte Holomorph eines nullteilerfreien Ringes R bzw. eines Ringes R mit $R^2 = R$ genau dann kommutativ ist, wenn dies für R zutrifft. Die erste dieser Aussagen läßt sich verallgemeinern:

Satz 5. *Ist R ein Ring mit dem Annullator $\mathfrak{n} = (0)$, so ist das eindeutig bestimmte Holomorph von R dann und nur dann kommutativ, wenn R kommutativ ist.*

Es gilt dann nämlich wegen

$$(\alpha a)\beta = \alpha(a\beta) = (a\beta)\alpha = a(\beta\alpha) = a(\alpha\beta) = (a\alpha)\beta$$

auch schon $(\alpha a - a\alpha)\beta = 0$ für alle β, also stets $\alpha a = a\alpha$.

Jedoch ist es nicht allgemein richtig, daß sich für Ringe mit eindeutig bestimmtem Holomorph die Kommutativität von R auf das Holomorph überträgt. Ein entsprechendes Gegenbeispiel war bereits in § 2 im Anschluß an das Korollar aufgetreten.

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Characterizations of congruence lattices of abstract algebras

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INTRODUCTION

In this paper we deal with the characterization problem of the lattice $\Theta(A)$ of *all* congruence relations of an abstract algebra A (briefly, congruence lattice). In § 1 of the Introduction we summarize our results concerning the general characterization problem, the solution of which answers Problem 50 of G. BIRKHOFF [1], originally proposed by BIRKHOFF and FRINK [2]. In § 2 we show that the representation theorems of WHITMAN and JONSSON are easy consequences of our results; we also solve the problem of complete representation. Concerning congruence lattices of type 2 and 3 we are able to prove more than the results stated in § 1. These results are summarized in § 3 in the form of embedding theorems for abstract algebras. In the next section we outline the method of the paper based on the systematic study of partial abstract algebras. The contents of the paper are sketched in the same section.

§ 1. Congruence lattices

An element x of the complete lattice L is called *compact* if $x \leqq \bigvee (x_\lambda; \lambda \in \Lambda)$ implies $x \leqq \bigvee (x_\lambda; \lambda \in \Lambda')$ for some finite $\Lambda' \subseteqq \Lambda$. A lattice L is *compactly generated* if it is complete and every element of L is the complete join of compact elements.

If A is an abstract algebra, $a, b \in A$, then there is a least $\Theta \in \Theta(A)$ such that $a \equiv b(\Theta)$; this is denoted by Θ_{ab}. Every Θ_{ab} as an element of $\Theta(A)$ is easily shown to be compact and thus *every congruence lattice is compactly generated.*[1])

The question whether or not every compactly generated lattice is isomorphic to a congruence lattice was proposed by BIRKHOFF and FRINK [2], again in BIRKHOFF [1] as Problem 50. One of our principal results is to answer this problem affirmatively.

Theorem I. *To any compactly generated lattice L there corresponds an abstract algebra A for which $\Theta(A)$, the lattice of all congruence relations of A, is isomorphic to L.*

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[1]) This assertion was first observed by BIRKHOFF and FRINK [2]; the conditions they have used are equivalent to, yet different from, those used above. The notion of compact element goes back to BÜCHI [3] and NACHBIN [10]. In [7], HASHIMOTO proves that every congruence lattice is isomorphic to the lattice of all ideals of a semilattice, a statement again equivalent to the above one.

One may hope to get a stronger form of Theorem I, so as to impose further conditions on A. In order to do this, consider $\Theta, \Phi \in \Theta(A)$ and $x, y \in A$. It is known that $x \equiv y(\Theta \cup \Phi)$ if and only if there exists a sequence $x = z_0, z_1, \ldots, z_m, z_{m+1} = y$ of elements of A such that $z_i \equiv z_{i-1}(\Theta)$ or $z_i \equiv z_{i-1}(\Phi)$ $(i = 1, 2, \ldots, m+1)$. We say A is *of type n* if, for every x, y, Θ, Φ $(x \equiv y(\Theta \cup \Phi))$, the sequence $\{z_i\}$ may be chosen so that $m = n$. This means, that while in an arbitrary abstract algebra, corresponding to a fixed quadruple x, y, Θ, Φ, the least m may be arbitrarily large, in algebras of type n, m may not exceed n; e. g. a ring or a group is always of type 1.

It is easy to prove that if A is of type 1 or 2 then $\Theta(A)$ is modular. Hence, from this point of view we get the best possible result if we can replace A of Theorem I by one of type 3. This is done in

Theorem II. *Let L be a compactly generated lattice. Then there exists an abstract algebra A of type* 3 *such that L and* $\Theta(A)$ *are isomorphic.*

As we said above, if A is of type 2 then $\Theta(A)$ is modular. This raises the question: which lattices are isomorphic to such a $\Theta(A)$? This is answered by

Theorem III. *Every compactly generated modular lattice is isomorphic to the congruence lattice of a suitable abstract algebra of type* 2.

§ 2. Representations

If H is a set then the set $\mathscr{E}(H)$ of all equivalence relations of H is a complete lattice and $\mathscr{E}(H) \cong \Theta(H)$ if H is considered as an abstract algebra without operations.

By a *representation* of the lattice L we mean an ordered pair $\langle F, H \rangle$, where H is a set and

$$x \to F(x)$$

is an isomorphism of L into $\mathscr{E}(H)$. If this isomorphism preserves complete join and meet, then the representation is called *complete*.

It is well known that $\langle F, A \rangle$,

$$F(\Theta) = \Theta,$$

is a complete representation of $\Theta(A)$; this will be called the *natural representation* of $\Theta(A)$. Further, it is easily shown that a lattice having a complete representation is compactly generated. Hence Theorem I implies at once

Corollary I. 1. *A complete lattice L has a complete representation if and only if L is compactly generated.*

This is the analogue of Whitman's fundamental theorem [11], asserting that every lattice has a representation. In fact, Whitman's theorem is a trivial consequence of Corollary I. 1. Indeed, if L_1 is a lattice then we extend it to L_2 by adding a zero element. Then we define L as the lattice of all ideals of L_2. Obviously, L is compactly generated, hence by Corollary I. 1 it has a representation $\langle F, H \rangle$ which is at the same time a representation of L_1. Thus

Corollary I. 2. (Whitman [11].) *Every lattice has a representation.*

Jonsson [8] defined the concept of representation of type n. If $x, y \in L$ and if $\langle F, H \rangle$ is a representation of L, then define $F(x); F(y)$ as the relation theoretic product of $F(x)$ and $F(y)$, i. e. $u \equiv v(F(x); F(y))$ $(u, v \in H)$ if and only if there is a $w \in H$ such that $u \equiv w(F(x))$ and $w \equiv v(F(y))$. Then $F(x) \cup F(y)$ is the join of the ascending series

$$F(x); F(y), \quad F(x); F(y); F(x), \quad F(x); F(y); F(x); F(y), \ldots .$$

If this series terminates at its n-th member for all $x, y \in L$ then the representation $\langle F, H \rangle$ of L is said to be of type n.

It is obvious that *an abstract algebra A is of type n if and only if the natural representation of $\Theta(A)$ is of type n.* Thus we get

Corollary II. 1. *A complete lattice L has a complete representation of type* 3 *if and only if it is compactly generated.*

Corollary II. 2. (JONSSON [8].) *Every lattice has a representation of type* 3.

And, similarly, the consequences of Theorem III are:

Corollary III. 1. *A complete lattice L has a complete representation of type* 2, *if and only if L is modular and compactly generated.*

Corollary III. 2. (JONSSON [8].) *Every modular lattice has a representation of type* 2, *and conversely.*

*

Another type of representation is obtained by means of subgroups of a group. A *subgroup representation* $\langle F, G \rangle$ of a lattice L consists of a group G and an isomorphism F of L into $L(G)$, the lattice of all subgroups of G. The subgroup representation is *complete,* if the isomorphism preserves complete joins and meets.

From Theorem I we conclude easily

Corollary I. 3. *A complete lattice L has a complete subgroup representation if and only if L is compactly generated.*

Corollary I. 4. (WHITMAN [11].) *Every lattice has a subgroup representation.*

§ 3. Embedding of abstract algebras

To prove Theorem II and III it is enough to construct only one abstract algebra A satisfying the hypotheses. In fact, we can prove much more. Given an arbitrary abstract algebra A we embed it in an abstract algebra B, such that $\Theta(A) \cong \Theta(B)$ and B is of type 3, or of type 2 if $\Theta(A)$ is modular. These — together with Theorem I — are much more than Theorems II and III. For a precise formulation of these new theorems we need a definition of embedding, because in these constructions A is not a subalgebra of B.

We say that the algebra B is an *extension* of the algebra A if [2])

1. $A \subseteqq B$;

2. to every operation f of A there corresponds an operation $\bar{f}$ of B (the extension of f), such that $f(a_1, a_2, \ldots, a_n) = \bar{f}(a_1, a_2, \ldots, a_n)$ if $a_1, a_2, \ldots, a_n \in A$.

If B is an extension of A and $\overline{\Theta}$ is a congruence relation of B then it includes a congruence relation Θ on A: let $a \equiv b(\Theta)$, $a, b \in A$ if and only if $a \equiv b(\overline{\Theta})$. If $\overline{\Theta} \to \Theta$

[2]) $\subseteqq$ is the set theoretical inclusion.

is an isomorphism between $\Theta(B)$ and $\Theta(A)$ then we say that $\Theta(B)$ and $\Theta(A)$ are isomorphic *in the natural way.*

Theorem II′. *Every abstract algebra A may be extended to an abstract algebra B of type* 3, *such that $\Theta(A)$ is isomorphic to $\Theta(B)$ in the natural way.*

Theorem III′. *Let A be an abstract algebra such that $\Theta(A)$ is modular. Then A has an extension B of type* 2, *such that $\Theta(A)$ is isomorphic to $\Theta(B)$ in the natural way.*

§ 4. The method and lay-out of the paper

To prove the theorems listed above we have to construct abstract algebras; to carry out these constructions seems to be rather difficult. But if we dispense with the assumption that an operation ,of an abstract algebra must be defined for every n-tuple (n depending on the operation), thus getting the definition of partial abstract algebra, then the task is fairly easy. The difficulty lies in the next step: we want to extend the partial abstract algebra to an abstract algebra so that the „good" properties should not be altered. E. g. such a property is that $\Theta(A)$ be isomorphic to L, where L is fixed.

We use two methods to bypass these difficulties: the first is the extension of a partial algebra to a free algebra; and the second is a procedure which identifies the „new" congruence relations of the free algebra with the congruence relations of the partial algebra.

It is not surprising that on proving theorems for abstract algebras the key role is played by partial abstract algebras, for partial algebras are nothing but generating systems considered *in abstracto*. This was kept in mind when the analogues of the notions of abstract algebras were defined for partial abstract algebras.

In the Introduction only the most important results are listed. All the theorems of the paper are numbered by arabic numerals; these are related to the results mentioned in the Introduction as follows: Theorem I is essentially Theorem 10; Theorem II is part of the Corollary to Theorem 14; Theorem II′ is part of Theorem 14; Theorem III is contained in the Corollary to Theorem 15; Theorem III′ is contained in Theorem 15.

The contents of the paper are the following: In Chapter I the notion of partial abstract algebra and the free algebra generated by a partial algebra are introduced and some of their properties are examined. The most important result of this part is Theorem5 which states that every congruence relation of a partial algebra may be extended to the free algebra generated by the partial algebra. In Chapter II contructions are developed in order to prove Theorem 10 (Theorem I). In the last section several applications of Theorem 10 are proved. In Chapter III our first task is to modify the construction in Chapter II in order to prove Theorem 14 (Theorem II). Finally, an analysis of the proof of Theorem 14 shows how to make further modifications which lead us to Theorem 15 (Theorem III).

Some open questions are mentioned in the last section of Chapter III.

CHAPTER I

PARTIAL ABSTRACT ALGEBRAS

§ 1. Some notions and notations

Set theoretical join and meet of the sets A, B will be designated by $A \vee B$, $A \wedge B$ and by $\bigvee A_\alpha$, $\bigwedge A_\alpha$, if α runs over an index set. $A \setminus B$ stands for the set theoretical difference if $A \supseteqq B$, i. e. $B \vee (A \setminus B) = A$, $B \wedge (A \setminus B) = \emptyset$ (the void set).

Let a set A be given. A *partial operation* f on A is a function which maps a part of $A \times A \times \ldots \times A$ (n times) into A. The domain of f will be denoted by $D(f, A)$ $(\subseteqq A \times A \times \ldots \times A)$. If $D(f, A) = A \times A \times \ldots \times A$, then f is an *operation*. If $D(f, A) = \emptyset$ then f is called *trivial*.

A *partial abstract algebra* (briefly: *partial algebra*) is a set A and a set $P(A)$ of partial operations defined on A. Let $P^*(A)$ denote the set of all non trivial operations of A. We say that the partial algebra B is the *homomorphic image* of the partial algebra A, if there is a many-one mapping η of A onto B and a one-to-one correspondence $f \to g$ between $P^*(A)$ and $P^*(B)$ such that the usual property

$$\eta f(a_1, a_2, \ldots, a_n) = g(\eta a_1, \eta a_2, \ldots, \eta a_n) \quad (a_1, a_2, \ldots, a_n) \in D(f, A)$$

holds true. It is an *isomorphism* if η is one-to-one. We should like to point out that in the definition of homomorphism and isomorphism the trivial operations are dispensed with. Endomorphisms and automorphisms are defined as usual.

According to the definition of homomorphism, an equivalence relation Θ of A is called a congruence relation if $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in D(f, A)$, $a_i \equiv b_i(\Theta)$ $(i = 1, 2, \ldots, n)$, $f \in P(A)$ imply $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)$ (Θ). Under the usual partial ordering the congruence relations of A form a complete lattice $\Theta(A)$ called the *congruence lattice* of A.

Theorem 1. *If A is a partial algebra, then $\Theta(A)$ is a compactly generated lattice*[3]).

Proof. The proof of the similar assertion for algebras uses the well known description of the complete join in $\Theta(A)$. Although this fails to be true in case of partial algebras, the following weaker analogue is true: if $x \equiv y (\bigvee \Theta_\lambda)$ $(x, y \in A)$ then there exists a finite subset $\{\Theta_i\}$ of the $\{\Theta_\lambda\}$ such that $x \equiv y \left(\bigvee_{i=1}^{n} \Theta_i \right)$. Using this weaker assertion one can prove that the congruence relation Θ is compact if and only if it is of the form $\bigvee_{i=1}^{n} \Theta_{a_i b_i}$, where Θ_{ab} $(a, b \in A)$ denotes the least congruence relation under wich $a \equiv b$. From this the assertion of the theorem follows as usual.

Let A be a partial algebra and H a subset of A and P a subset of $P(A)$. If f is a partial operation of A beloging to P then it may be also considered as a partial operation [4]) of H: $(h_1, \ldots, h_n)$ $(h_i \in H)$ is in the domain of f if $(h_1, \ldots, h_n) \in D(f, A)$

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[3]) The notion of compactly generated lattice is defined in § 1 of this Introduction.

[4]) There is no danger of confusion, therefore we do not introduce notation for the restricted operation..

and $f(h_1, \dots, h_n) \in H$. With this definition H is a partial algebra and $P = P(H) \subseteq \subseteq P(A)$. In this case A will be called an *extension* of H. (Or we may say that H is a *restriction* of A.) Using this construction of partial algebras one says that a generating systems of an algebra may always be considered as a partial algebra. The converse of this statement is the

Theorem 2. *Every partial algebra may be extended to an algebra.*

Proof. The assertion is trivial: if A is a partial algebra then let $B = \{A, p\}$ where p is a new element and if $f \in P(A)$ and $(u_1, \dots, u_n) \notin D(f, A)$ $(u_1, \dots, u_n \in B)$ then define $f(u_1, \dots, u_n) = p$. Obviously, B is an algebra and it is an extension of A.

§ 2. Free algebras

In the proof of Theorem 2 the least extension of a partial algebra to an algebra has been constructed. Nevertheless, this construction fails to have the property that every congruence relation of the partial algebra may be extended to the algebra, which is a very important property in this paper. Therefore we confine now our attention to the construction of an extension having this additional property.

It is much simpler to perform this construction if on the partial algebra only partial operations of one variable are defined. Since in this and in the next chapter only such partial algebras are dealt with we suppose that this is the case.

Let S be a partial algebra such that $P(S)$ consists of partial operations of one variable. In this case if $\varphi \in P(S)$ then $D(\varphi, S) \subseteq S$. Further, let $\varphi(H)$, $H \subseteq D(\varphi, S)$ denote the set of all $\varphi(x)$, $x \in H$. If $\varphi, \psi \in P(S)$ we put $\varphi\psi(x) = \varphi(\psi(x))$. Similarly, we use the notation $\varphi_1 \dots \varphi_n(x)$ $(\varphi_1, \dots, \varphi_n \in P(S), x \in S)$.

We fix a $\varphi \in P(S)$ and to every $x \in S \setminus D(\varphi, S)$ we define a new element $\bar{x}$, such that $\bar{x} \notin S$ and $x \neq y$, $x, y \in S \setminus D(\varphi, S)$ imply $\bar{x} \neq \bar{y}$. The set formed by S and all the $\bar{x}$ is denoted by $S[\varphi]$. We define partial operations on $S[\varphi]$:

1. Let every partial operation ψ of S different from φ be a partial operation of $S[\varphi]$ with an unchanged domain: $D(\psi, S) = D(\psi, S[\varphi])$;

2. φ is a partial operation of $S[\varphi]$; on $D(\varphi, S)$ it is defined as it was; if $x \in S \setminus D(\varphi, S)$ then $\varphi(x) = \bar{x}$; $\varphi(x)$ is defined for no $x \in S[\varphi] \setminus S$.

$S[\varphi]$ with the partial operations defined under 1 and 2 is a partial algebra; it is an extension of S. The element $\bar{x}$ $(x \in S \setminus D(\varphi, S))$ will be denoted by $\varphi(x)$.

To every $\varphi \in P(S)$ we construct $S[\varphi]$ such that if $\varphi \neq \psi$ then $S[\varphi] \wedge S[\psi] = S$. We define S_1 as the join of the $S[\varphi]$:

$$S_1 = \bigvee (S[\varphi];\ \varphi \in P(S)).$$

S_1 as the set theoretical join of partial algebras is itself a partial algebra. We may write also $P(S) = P(S_1)$, for every partial operation of S_1 is the extension of a partial operation of S. Thus S_1 is an extension of S. In a similar way we define

$$S_2 = \bigvee (S_1[\varphi];\ \varphi \in P(S)), \dots, S_n = \bigvee (S_{n-1}[\varphi];\ \varphi \in P(S)).$$

The partial algebras $S_1, S_2, \dots$ form an ascending chain, all of them are extensions of S, indeed, S_n is an extension of S_{n-1}; thus their join $\bar{S}$ is also a partial algebra and it is also an extension of S, and $P(\bar{S}) = P(S)$.

Theorem 3. *$\bar{S}$ as constructed above is an algebra, and $\bar{S}$ is generated by S. The algebra $\bar{S}$ is free in the following sense: if the algebra S^* is generated by the partial algebra S', $P(S')=P(S^*)$ and $x \rightarrow x'$ is an isomorphism between S and S' then $x \rightarrow x'$ may be extended to a homomorphism of $\bar{S}$ onto S^*.*

Proof: trivial.

§ 3. Extension of congruence relations

Let the partial algebra B be an extension of the partial algebra A. We say that the congruence relation Φ of B is the *extension* of the congruence relation Θ of A if $x \equiv y(\Theta)$ and $x \equiv y(\Phi)$ are equivalent whenever $x, y \in A$. If Θ has an extension, then it has, obviously, a least extension, which will be denoted by $\overline{\Theta}$.

Theorem 4. *Every congruence relation of S may be extended to $S[\varphi]$.*

Supplement. *If $\Theta \in \Theta(S)$ and $\overline{\Theta}$ is the least extension of Θ to $S[\varphi]$ then $\overline{\Theta}$ may be described as follows: $u \equiv v(\overline{\Theta})$ $(u, v \in S[\varphi])$ if and only if one of the following conditions hold:*

I. *$u, v \in S$ and $u \equiv v(\Theta)$;*

II. *$u, v \in S[\varphi] \setminus S$, i. e. $u=\varphi(x)$, $v=\varphi(y)$, where $x, y \in S \setminus D(\varphi, S)$ and either 1. $x \equiv y(\Theta)$ or 2. there exist $a=\varphi(x_0)$, $b=\varphi(y_0) \in S$ such that $x \equiv x_0(\Theta)$, $y \equiv y_0(\Theta)$, $a \equiv b(\Theta)$;*

III. *$u \in S$, $v \in S[\varphi] \setminus S$ (or symmetrically, interchanging u and v), i. e. $v=\varphi(y)$, $y \in S \setminus D(\varphi, S)$ and there exists an $a=\varphi(y_0) \in S$, for which $u \equiv a(\Theta)$ and $y \equiv y_0(\Theta)$.*

Fig. 1 helps to visualize case I III.

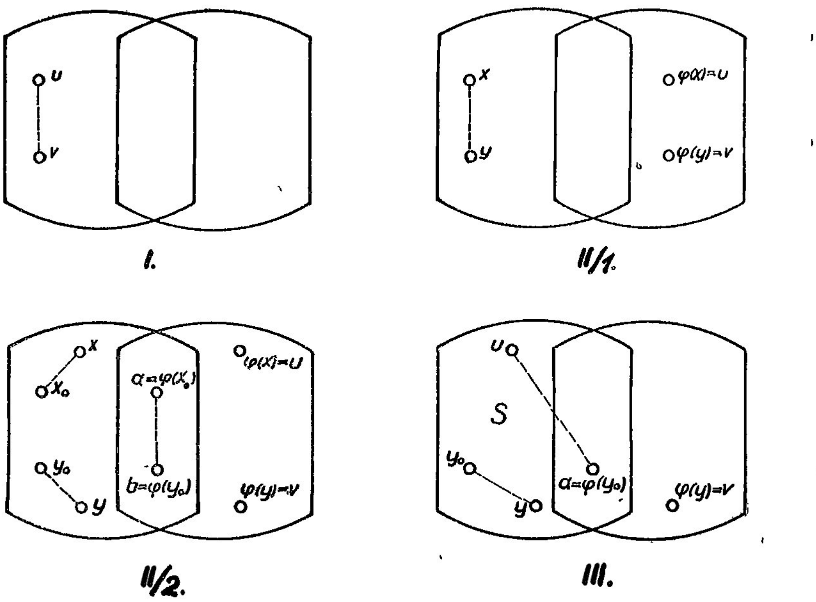

Fig. 1

In Fig. 1 a broken line connecting two elements means that the two elements are congruent modulo Θ.

Proof. Let $\Theta[\varphi]$ be the relation defined by I—III of Supplement to Theorem 4. It is enough to prove that it is a congruence relation, for the relation $\Theta[\varphi]=\overline{\Theta}$ is then obvious.

Owing to I and II/1 we get that $\Theta[\varphi]$ is reflexive and, by the symetry of I—III in u and v, it is also symmetric. The substitution property may be proved as follows: Let $\psi\in P(S[\varphi]) = P(S)$, $x\equiv y(\Theta[\varphi])$ and $x, y\in D(\psi, S[\varphi])$. We distinguish two cases:

(1) $\varphi\neq\psi$. Then $x, y\in D(\psi, S[\varphi]) = D(\psi, S)$ and by I we get $x\equiv y(\Theta)$ and so $\psi(x)\equiv\psi(y)(\Theta)$, and again by I $\psi(x)\equiv\psi(y)(\Theta[\varphi])$.

(2) $\varphi=\psi$. Then necessarily $x, y\in S$. We want to prove $\varphi(x)\equiv\varphi(y)(\Theta[\varphi])$; this follows from I if $x, y\in D(\varphi, S)$, from III with $a=\varphi(x)$ if $x\in D(\varphi, S)$, $y\notin D(\varphi, S)$ (and in the symmetrical case), from II/1 if $x, y\notin D(\varphi, S)$.

It remains to prove that $\Theta[\varphi]$ is transitive.

Let $u\equiv v(\Theta[\varphi])$, $v\equiv w(\Theta[\varphi])$; we have to prove $u\equiv w(\Theta[\varphi])$. We will distinguish 8 cases.

(α) $u, v, w\in S$. In this case $u\equiv w(\Theta[\varphi])$ is clear owing to I and the transitivity of Θ.

(β) $u, v\in S$; $w\in S[\varphi]\setminus S$; i. e. $w=\varphi(x)$, $x\in S$. By I $u\equiv v(\Theta)$; from III we conclude the existence of an $a=\varphi(x_0)\in S$ satisfying $v\equiv a(\Theta)$, $x_0\equiv x(\Theta)$. Thus $u\equiv a(\Theta)$ and $x_0\equiv x(\Theta)$, $a=\varphi(x_0)\in S$, i. e. by III we get $u\equiv w(\Theta[\varphi])$.

(β') $v, w\in S$; $u\in S[\varphi]\setminus S$. The proof is the same as under (β).

(γ) $u, w\in S$; $v\in S[\varphi]\setminus S$; i. e. $v=\varphi(x)$, $x\in S$. By III $u\equiv v(\Theta[\varphi])$ means the existence of an $a=\varphi(x_0)\in S$ such that $u\equiv a(\Theta)$, $x_0\equiv x(\Theta)$. Similarly, there exists a $b=\varphi(y_0)\in S$ with $w\equiv b(\Theta)$, $y_0\equiv x(\Theta)$. Thus $x_0\equiv y_0(\Theta)$, i. e. $a=\varphi(x_0)\equiv\varphi(y_0)= =b(\Theta)$; consequently, $u\equiv a(\Theta)$, $a\equiv b(\Theta)$, $b\equiv w(\Theta)$, so $u\equiv w(\Theta)$, and by I we get $u\equiv w(\Theta[\varphi])$.

(δ) $u\in S$; $v, w\in S[\varphi]\setminus S$; i. e. $v=\varphi(x)$, $w=\varphi(y)$. Owing to III we get that with suitable $a=\varphi(x_0)\in S$ the congruences $u\equiv a(\Theta)$, $x_0\equiv x(\Theta)$ hold. The congruence $v\equiv w(\Theta[\varphi])$ means that either

1. $x\equiv y(\Theta)$, or that
2. there exist $a'=\varphi(x_0')$ and $b=\varphi(y_0)$ such that $x_0'\equiv x(\Theta)$, $y_0\equiv y(\Theta)$, and $a'\equiv b(\Theta)$.

In the first case $x_0\equiv y(\Theta)$ and $a=\varphi(x_0)\equiv\varphi(y)=w(\Theta[\varphi])$. But $u\equiv a(\Theta)$. Thus owing to III we get $u\equiv w(\Theta[\varphi])$.

In the second case $x_0\equiv x_0'(\Theta)$, thus $a=\varphi(x_0)\equiv\varphi(x_0')=a'(\Theta)$ implying $a\equiv b(\Theta)$ and so $u\equiv b(\Theta)$. But $y_0\equiv y(\Theta)$, resulting — by III — $u\equiv w(\Theta[\varphi])$.

(δ') $w\in S$; $u, v\in S[\varphi]\setminus S$. The proof is the same as under (δ).

(ε) $v\in S$; $u, w\in S[\varphi]\setminus S$; thus $u=\varphi(x)$, $w=\varphi(y)$.

Owing to III we get the existence of $a=\varphi(x_0)$, $b=\varphi(y_0)\in S$ such that $v\equiv a(\Theta)$, $x_0\equiv x(\Theta)$, $v\equiv b(\Theta)$ and $y_0\equiv y(\Theta)$. We get from these $a\equiv b(\Theta)$, and thus owing to II/2 we get $u\equiv w(\Theta[\varphi])$.

(φ) $u, v, w\in S[\varphi]\setminus S$, thus $u=\varphi(x)$, $v=\varphi(y)$, $w=\varphi(z)$. Let $a=\varphi(x_0)$, $b=\varphi(y_0)$, $c=\varphi(z_0)$, $d=\varphi(v_0)$ be suitable elements of S. $u\equiv v(\Theta[\varphi])$ means either

a/1 $x\equiv y(\Theta)$,

or a/2 $x\equiv x_0(\Theta)$, $a\equiv b(\Theta)$, $y_0\equiv y(\Theta)$.

$v \equiv w(\Theta[\varphi])$ is equivalent to either
b/1 $y \equiv z(\Theta)$
or b/2 $y \equiv z_0(\Theta)$, $c \equiv d(\Theta)$, $v_0 \equiv z(\Theta)$.

If a/1 and b/1 hold then $x \equiv z(\Theta)$, thus — by II/1 — $u \equiv w(\Theta[\varphi])$ holds.

If a/1 and b/2 hold then $x \equiv z_0(\Theta)$, thus II implies $u \equiv w(\Theta[\varphi])$. The case when a/2 and b/1 hold is similar.

If a/2 and b/2 hold then $a \equiv b(\Theta)$, $y_0 \equiv y(\Theta)$, $y \equiv z_0(\Theta)$, $c \equiv d(\Theta)$, i. e. $a \equiv d(\Theta)$, thus $u \equiv w(\Theta[\varphi])$. The proof of Theorem 4 is finished.

Based on Theorem 4 we prove

Theorem 5. *Let S be a partial algebra and $\bar{S}$ be the free algebra generated by S* (as defined in § 2). *Every congruence relation of S may be extended to $\bar{S}$.*

Before proving this theorem, we need

Lemma 1. *Let be given a partial algebra S and a set of partial algebras $\{S_\alpha\}$, for which*

1. *S_α is an extension of S, $(P(S)=P(S_\alpha))$;*
2. *$S_\alpha \wedge S_\beta = S$ if $\alpha \neq \beta$;*
3. *$x \in S_\alpha$, $\varphi \in P(S)$, $\varphi(x) \in S_\beta$ and $\alpha \neq \beta$ imply $\varphi(x) \in S$;*
4. *every congruence relation of Θ may be extended to every S_α.*

Then $S^* = \bigvee S_\alpha$ is a partial algebra containing S, S^* is an extension of S, and every congruence relation of S may be extended to S^*.

Proof. Only the last assertion calls for proof. Let Θ_α be the extension of Θ to S_α. We define the relation Φ:

I. $x \equiv y(\Phi)$, $x, y \in S_\alpha$ is equivalent to $x \equiv y(\Theta_\alpha)$;

II. $x \equiv y(\Phi)$, $x \in S_\alpha$, $y \in S_\beta$, $\alpha \neq \beta$ if and only if with a suitable $a \in S$ we have $x \equiv a(\Theta_\alpha)$, $a \equiv y(\Theta_\beta)$.

It is routine to check that Φ is a congruence relation and, obviously, it is an extension of Θ to S^*.

Now we prove Theorem 5. Let $\Theta \in \Theta(S)$. Theorem 4 guarantees the extendability of Θ to the $S[\varphi_\alpha]$, $\varphi_\alpha \in P(S)$. The set of the $S[\varphi_\alpha]$ satisfies the hypotheses of Lemma 1, thus Θ may be extended to S_1 (which is the S^* of Lemma 1). In a similar way we get that Θ may be extended to $S_2, S_3, \ldots$ and hence to $\bar{S}$, finishing the proof of Theorem 5.

CHAPTER II

COMPACTLY GENERATED LATTICES AS CONGRUENCE LATTICES

§ 1. Preliminary constructions

Our principal aim in this chapter is to prove Theorem I (Theorem 10). This will be done in § 3 while in §§ 1 and 2 some preparations are made.

Let S be a partial algebra, $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x) \in P(S)$, $D(\varphi_1, S) = \{a\}$, $D(\varphi_2, S) = \varnothing$, $D(\varphi_3, S) = \{b\}$, $a, b \in S$ and $\varphi_1(a) = c$, $\varphi_3(b) = d$, $c, d \in S$. In the partial algebra

$\bigvee_{i=1}^{3} S[\varphi_i]$ we identify $\varphi_1(b)$ with $\varphi_2(b)$ and $\varphi_2(a)$ with $\varphi_3(a)$ getting the partial algebra T' (see Fig. 2).

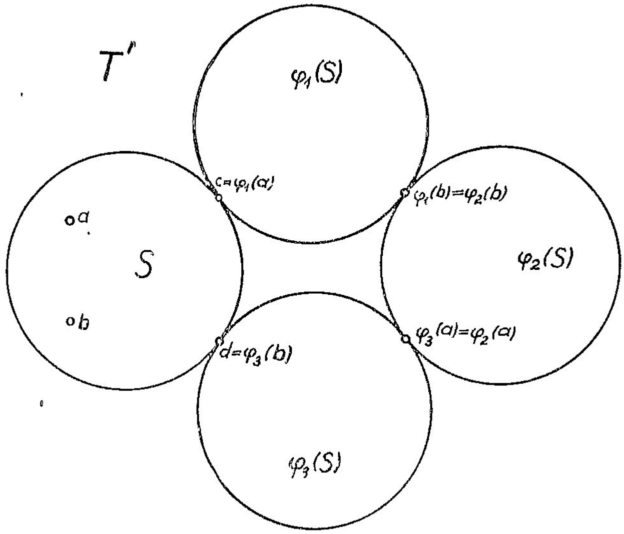

Fig. 2

T' is an extension of S but it is not necessarily true that every congruence relation of S may be extended to T'. Call a congruence relation Θ of S *admissible* if it satisfies one of the following conditions:

$$A_1: a \not\equiv b(\Theta);$$

$$A_2: a \equiv b(\Theta) \text{ and } c \equiv d(\Theta).$$

Roughly speaking, Θ is admissible if $a \equiv b(\Theta)$ implies $c \equiv d(\Theta)$.

Now suppose that Θ may be extended to T' and let $\overline{\Theta}$ be an extensions of Θ. If $a \equiv b(\Theta)$ then $a \equiv b(\overline{\Theta})$ and $c = \varphi_1(a) \equiv \varphi_1(b)(\overline{\Theta})$, $\varphi_2(a) \equiv \varphi_2(b)(\overline{\Theta})$, $\varphi_3(a) \equiv \varphi_3(b) = d(\overline{\Theta})$, thus the assumptions $\varphi_1(b) = \varphi_2(b)$ and $\varphi_2(a) = \varphi_3(a)$ imply that $c \equiv d(\overline{\Theta})$, consequently $c \equiv d(\Theta)$.

This proves that if a congruence relation is extensible then it is admissible. This and the converse of this statement is contained in

Theorem 6. *The congruence relation Θ of S is admissible if and only if it may be extended to T'.*

Proof. We have to prove the „only if” part of the theorem. Suppose that Θ is admissible and define a relation Θ^* of T' as follows: let $u \equiv v(\Theta^*)$ mean for $u, v \in S$ that $u \equiv v(\Theta)$ and for $u, v \in \varphi_i(S)$ ($\varphi_i(S)$ denotes the set of all $\varphi_i(x)$, $x \in S$) that $u = \varphi_i(x), v = \varphi_i(y), x, y \in S$ and $x \equiv y(\Theta)$, otherwise let $u \not\equiv v(\Theta^*)$. Then Θ^* is a symmetric and reflexive relation having the substitution property. Let $\overline{\Theta}$ denote the transitive extension of Θ^*. The relation $\bar\Theta$ is trivially a congruence relation of T'. We prove that on S and on the $\varphi_i(S)$ the relations $\bar\Theta$ and Θ^* coincide. It is enough to prove this for S, a similar reasoning applies then to $\varphi_i(S)$. *Per definitionem* $u\equiv v(\bar\Theta)$, $u, v\in S$ if and only if there exists a sequence $u=x_0, x_1, \dots, x_n=v$ of elements of T' such that $x_{i-1}\equiv x_i(\Theta^*)$ $(i=1, 2, \dots, n)$. If all the $x_i\in S$ then $u\equiv v(\Theta)$, thus $u\equiv v(\Theta^*)$ is obvious. S and a $\varphi_i(S)$ ($\varphi_i(S)$ and a $\varphi_j(S)$, $i\neq j$) have at most *one* element in common. This if we impose the natural condition on the sequence $x_0, \dots, x_n$ that no element may occur more than once, then we see that the sequence must contain elements from all the $\varphi_i(S)$. It is easy to see that such a sequence may be substituted by the following simpler one: $u=x_0, x_1=c, x_2=\varphi_1(b), x_3=\varphi_2(a)$, $x_4=d, x_5=v$ (or interchanging u with v). $x_1\equiv x_2(\Theta^*)$ implies $a\equiv b(\Theta)$, and by A_2 we get $c\equiv d(\Theta)$; thus $u\equiv v(\Theta)$ and $u\equiv v(\Theta^*)$, proving that $\bar\Theta$ and Θ^* are equivalent on S, finishing the proof of this theorem. We proved a little more than required; we have exhibited at the same time a well-described extension of an admissible congruence relation.

* * *

Now let S be a partial algebra; the operations of S will be denoted by $\omega^\nu(x)$ $(\nu\in\Omega_1)$ and the partial operations by $\varphi_i^\mu(x)$ $(\mu\in\Omega_2, i=1, 2, 3)$; we suppose that $D(\varphi_1^\mu, S)=\{a^\mu\}$, $D(\varphi_2^\mu, S)=\emptyset$, $D(\varphi_3^\mu, S)=\{b^\mu\}$ and $\varphi_1^\mu(a^\mu)=c^\mu$, $\varphi_3^\mu(b^\mu)=d^\mu$ $(a^\mu, b^\mu, c^\mu, d^\mu\in S)$. To each μ the φ_i^μ are of the type described at the beginning of the section, thus the corresponding T' — which now will be denoted by T_μ — may be constructed. We also suppose that $\mu\neq\mu'$ implies $T_\mu\cap T_{\mu'}=S$. Further, let $T=\bigvee_{\mu} T_\mu$ and $\bar T$ the free algebra generated by T.

The congruence relation Θ of S is called *admissible* if it is admissible for any fixed $\mu\in\Omega_2$ (i. e. if for $\mu\in\Omega_2$ the congruence $a^\mu\equiv b^\mu(\Theta)$ holds, then $c^\mu\equiv d^\mu(\Theta)$).

Let $\Theta\in\Theta(S)$; then there exists a unique admissible congruence relation Θ' which is minimal with respect to $\Theta'\geqq\Theta$. Indeed, let Ω_2^1 denote the set of those $\mu\in\Omega_2$ for which $a^\mu\equiv b^\mu(\Theta)$, and define $\Theta_1=\Theta\cup\bigvee(\Theta_{c^\mu d^\mu};\ \mu\in\Omega_2^1)$, if Θ_{n-1} is defined, set $\Theta_n=(\Theta_{n-1})_1$ and $\Theta'=\bigvee\limits_{n=1}^{\infty}\Theta_n$. Obviously, Θ' is admissible and the least admissible congruence relation $\geqq\Theta$.

A central result of this paper is

Theorem 7. *The congruence relation Θ of S may be extended to $\bar T$ if and only if it is admissible. To every pair u, v of elements of $\bar T$, there exists a uniquely determined least admissible congruence relation Θ such that under $\bar\Theta$ (the minimal extension of Θ to $\bar T$) u and v are congruent.*

The first assertion of the theorem is obvious from Theorems 5 and 6 and Lemma 1. The second assertion is rather involved; as a preparation we will prove Lemmas 2 and 3.

Lemma 2. *Let S and T' be as in Theorem 6. Then to every $u, v\in T'$ there exists a least admissible $\Theta\in\Theta(S)$ such that $u\equiv v(\bar\Theta)$.*

Proof. If $u, v\in S$ and $a\not\equiv b(\Theta_{uv})$ (resp. $a\equiv b(\Theta_{uv})$), then Θ_{uv} (resp. $\Theta_{uv}\cup\Theta_{cd}$) is the least admissible congruence relation. Θ may be found similarly if $u, v\in\varphi_i(S)$.

If u and v are not both in S or in $\varphi_i(S)$ then it is not simple to find Θ. We will show how to construct Θ in a typical case, the complete discussion will be left to the reader.

Let $u \in S$, $v \in \varphi_2(S)$; i. e. $v = \varphi_2(x)$, $x \in S$. We state $\Theta = \Theta_{ax} \cup \Theta_{bx} \cup \Theta_{uc} \cup \Theta_{cd}$. This Θ is admissible for $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$. Further, $u \equiv d(\Theta)$, $b \equiv a(\Theta)$, $a \equiv x(\Theta)$, so $u \equiv d(\overline{\Theta})$, $\varphi_3(b) \equiv \varphi_3(a) = \varphi_2(a)\ (\overline{\Theta})$, $\varphi_2(a) \equiv \varphi_2(x)\ (\overline{\Theta})$; consequently $u \equiv v = \varphi_2(x)(\overline{\Theta})$. Finally, we have to prove that if $\Phi \in \Theta(S)$, Φ is admissible, and $u \equiv v(\overline{\Phi})$, then $\Phi \geqq \Theta$. Indeed, $u \equiv v(\overline{\Phi})$ (by the proof of Theorem 6) implies that either

1. $u \equiv d(\Phi^*)$, $d = \varphi_3(b) \equiv \varphi_3(a) = \varphi_2(a)(\Phi^*)$, $\varphi_2(a) \equiv \varphi_2(x)(\Phi^*)$

or

2. $u \equiv c(\Phi^*)$, $c = \varphi_1(a) \equiv \varphi_1(b) = \varphi_2(b)(\Phi^*)$, $\varphi_2(b) \equiv \varphi_2(x)(\Phi^*)$,

where Φ^* is the relation defined in the proof of Theorem 6.

Let us consider the first possibility. By the definition of Φ^* we get from the relations of 1 the congruences $u \equiv d(\Phi)$, $b \equiv a(\Phi)$, $a \equiv x(\Phi)$. Consequently, $\Theta_{ud} \cup \Theta_{ba} \cup \Theta_{ax} \leqq \Phi$. Thus $a \equiv b(\Phi)$; hence by A_2 we get $c \equiv d(\Phi)$, i. e. $\Theta_{cd} \leqq \Phi$. So $\Theta_{ud} \cup \Theta_{ab} \cup \Theta_{ax} \cup \Theta_{cd} \leqq \Phi$. But $\Theta = \Theta_{ud} \cup \Theta_{ab} \cup \Theta_{ax} \cup \Theta_{cd}$ is obvious, thus in the first case $\Theta \leqq \Phi$ is proved. The second case may be proved in the same way, thus the proof is finished.

Lemma 3. *Let S and T be as in Theorem 7. Then to every $u, v \in T$ there exists a least admissible congruence relation $\Theta \in \Theta(S)$ such that $u \equiv v(\overline{\Theta})$.*

Proof. Let $u, v \in T = \bigvee T_\mu$, it is enough to consider the case $u \in T_\mu \setminus S$, $v \in T_\nu \setminus S$, $\mu \neq \nu$, for the other cases were treated in Lemma 2.

There are nine cases to be distinguished; from these we pick out a typical one, the others may be treated similarly.

Let $u \in \varphi_3^\mu(S) \setminus S$ and $v \in \varphi_2^\nu(S)$, i. e. $u = \varphi_3^\mu(x)$, $v = \varphi_2^\nu(y)$, $x, y \in S$. Let Φ be admissible such that $u \equiv v(\overline{\Phi})$. Then one of the following conditions 1—4 holds;

1. $$u = \varphi_3^\mu(x) \equiv \varphi_3^\mu(b^\mu) = d^\mu(\Phi^*),\ d^\mu \equiv c^\nu = \varphi_1^\nu(a^\nu)(\Phi^*),$$
$$\varphi_1^\nu(a^\nu) \equiv \varphi_1^\nu(b^\nu)(\Phi^*),\ \varphi_1^\nu(b^\nu) = \varphi_2^\nu(b^\nu) \equiv \varphi_2^\nu(y) = v(\Phi^*)$$

from which we get

$$\Theta_1 = \Theta_{xb^\mu} \cup \Theta_{d^\mu c^\nu} \cup \Theta_{a^\nu b^\nu} \cup \Theta_{b^\nu y} \leqq \Phi.$$

2. $$u = \varphi_3^\mu(x) \equiv \varphi_3^\mu(b^\mu) = d^\mu(\Phi^*),\ d^\mu \equiv d^\nu = \varphi_3^\nu(b^\nu)(\Phi^*),$$
$$\varphi_3^\nu(b^\nu) \equiv \varphi_3^\nu(a^\nu)(\Phi^*),\ \varphi_3^\nu(a^\nu) = \varphi_2^\nu(a^\nu) \equiv \varphi_2^\nu(y) = v(\Phi^*)$$

from which we get

$$\Theta_2 = \Theta_{xb^\mu} \cup \Theta_{d^\mu d^\nu} \cup \Theta_{b^\nu a^\nu} \cup \Theta_{a^\nu y} \leqq \Phi.$$

3.-4. $$u = \varphi_3^\mu(x) \equiv \varphi_3^\mu(a^\mu) = \varphi_2^\mu(a^\mu)(\Phi^*),\ \varphi_2^\mu(a^\mu) \equiv$$
$$\equiv \varphi_2^\mu(b^\mu)(\Phi^*);\ \varphi_2^\mu(b_\mu) = \varphi_1^\mu(b^\mu) \equiv \varphi_1^\mu(a^\mu) = c^\mu\ (\Phi^*),$$

further in case 3 $c^\mu \equiv c^\nu(\Phi^*)$, $c^\nu = \varphi^\nu(a^\nu) \equiv \varphi^\nu(b^\nu) = \varphi_2^\nu(b^\nu)(\Phi^*)$, $\varphi_2^\nu(b^\nu) \equiv \varphi_2^\nu(y) = v(\Phi^*)$ and in case 4 $c^\mu \equiv d^\nu(\Phi^*)$, $d^\nu = \varphi_3^\nu(b^\nu) \equiv \varphi_3^\nu(a^\nu) = \varphi_2^\nu(a^\nu)(\Phi^*)$, $\varphi_2^\nu(a^\nu) \equiv$

$\equiv \varphi_2^\nu(y) = v(\Phi^*)$, and so we get respectively

$$\Theta_3 = \Theta_{xa^\mu} \cup \Theta_{b^\mu a^\mu} \cup \Theta_{c^\mu c^\nu} \cup \Theta_{a^\nu b^\nu} \cup \Theta_{b^\nu y} \leqq \Phi,$$

$$\Theta_4 = \Theta_{xa^\mu} \cup \Theta_{b^\mu a^\mu} \cup \Theta_{c^\mu d^\nu} \cup \Theta_{b^\nu a^\nu} \cup \Theta_{a^\nu y} \leqq \Phi.$$

We prove that $\Theta = \Theta_1'$ (the notation was introduced before Theorem 7). It is enough to show that $\Theta_1' \leqq \Theta_i'$ for $i = 2, 3, 4$. But $(\Phi')' = \Phi'$ holds for every $\Phi \in \Theta(A)$, thus it is enough to prove $\Theta_1 \leqq \Theta_i'$ $(i = 2, 3, 4)$.

The case $i = 2$ is trivial because of $\Theta_1' = \Theta_2'$. (This follows from the special choice of u and v.) Now we prove $\Theta_1 \leqq \Theta_3'$ as follows: obviously

$$\Theta_{xb^\mu} \leqq \Theta_{xa^\mu} \cup \Theta_{b^\mu a^\mu},$$

further

$$\Theta_{d^\mu c^\nu} \leqq (\Theta_{b^\mu a^\mu} \cup \Theta_{c^\mu d^\nu} \cup \Theta_{a^\nu b^\nu})';$$

thus the relation

$$\Theta_1 \leqq \Theta_3'$$

is obvious. The last relation $\Theta_1 \leqq \Theta_4'$ may be proved similarly finishing the proof of Lemma 3.

Now we are going to prove Theorem 7. Let $u, v \in \overline{T}$, $u = \gamma_1 \dots \gamma_n(x)$, $v = \delta_1 \dots \delta_m(y)$, $\gamma_i, \delta_i \in P(S)$, $x \notin D(\gamma_n, S)$, $y \notin D(\delta_m, S)$. Now we use the assumption that all the partial operations of S are either operations (the $\omega^\nu(x)$, $\nu \in \Omega_1$) or of the special type φ_i^μ. It follows that γ_n and δ_m are of type φ_i^μ.

Let T^p denote the set of all elements of $\overline{T}$ which may be represented in the form

$$\gamma_1 \dots \gamma_n(x), \quad n \leqq p, \quad x \in S, \quad x \notin D(\gamma_n, S), \quad \gamma_1, \dots, \gamma_n \in P(S).$$

Then

$$S = T^0 \subseteqq T^1 = T \subseteqq T^2 \dots$$

and

$$\overline{T} = \bigcup T^i.$$

We suppose $u, v \in T^p$ and prove our assertions by induction on p.

The case $p = 1$ was settled in Lemmas 2 and 3. Let us suppose that we have proved the assertion for all $k < p$. The set $T^p \setminus T^{p-1}$ is the join of sets of the form

$$H_\alpha = \bigcup_{i=1}^{3} \lambda_1 \dots \lambda_{p-1} \varphi_i^\mu(S)$$

(α depending on $\lambda_1, \dots \lambda_{p-1}$, μ and i). If both u and v are in T^{p-1} then the assertion follows from the induction hypothesis. So we may suppose that $u \notin T^{p-1}$, thus $u \in H_\alpha$ for some α.

Now we may repeat the chain of thoughts of Lemmas 2 and 3; the role of S is taken by T^{p-1}, that of T_ν by H_α. The only difference is that for S the assertion was trivial; now, for T^{p-1} it is the induction hypothesis. This is essential when we are looking for the least admissible congruence relation, under whose extension e. g. c^μ and d^ν are congruent.

§ 2. Compactly generated lattices

Before proving Theorem I we need two easy theorems on compactly generated lattices the first of which is probably well-known while the second is due to NACHBIN.

Theorem 8. *Let L be a compactly generated lattice and H a complete sublattice of L. Then H is also compactly generated.*

Proof. A principal ideal of a compactly generated lattice is obviously compactly generated. Thus we may suppose that the unit element of H is the unit element of L. Now let u be an arbitrary element of L, and define $a(u)$ as the meet of all $h \in H$ with $h \geqq u$,

$$a(u) = \bigwedge (h; h \in H, h \geqq u).$$

H is a complete sublattice, thus $a(u) \in H$; in fact $a(u)$ is the least element of H which is $\geqq u$. It is routine to check that if u is compact in L then $a(u)$ is compact in H. From this the assertion follows easily.

Let F be a semilattice with O, i. e. let be defined on F a binary operation $\cup$, which is idempotent, commutative and associative, further, $x \cup O = x$ for all $x \in F$. A subset I of F is called an ideal, if it is non-void and $x \cup y \in I (x, y \in F)$ if and only if x and $y \in F$. A natural partial ordering of F is: $x \leqq y$ if and only if $x \cup y = y$; then $x \cup y$ is the least upper bound of x and y. Now, I is an ideal if and only if 1. $x, y \in I$ imply $x \cup y \in I$; 2. $x \in I$, $y \in F$, $y \leqq x$ imply $y \in I$. The set $I(F)$ of all ideals of F form a complete lattice if the partial ordering is the set-inclusion.

Theorem 9. (NACHBIN [10].) *A lattice L is compactly generated if and only if L is isomorphic to the lattice of all ideals of a semilattice F with O. In fact, if L is compactly generated then F is isomorphic to the semilattice of all compact elements of L. Further, the compact elements of $I(F)$ are the principal ideals.*

A sketch of the proof. Let L be the compactly generated lattice and F the semilattice with zero of the compact elements of L. First, one has to prove that F is really a semilattice, i. e. the join of two compact elements is again compact. Then take an $a \in L$ and define I_a as the set of all $x \in F$ with $x \leqq a$. The correspondence $a \to I_a$ is an isomorphism between L and $I(F)$. The only non-trivial step is to prove that if I is an ideal of F and $a = \bigvee (x; x \in I)$, where the complete join is in L, then $I_a = I$. Indeed, if $y \in I_a$, then $y \leqq \bigvee (x; x \in I)$. Thus by the compactness of y we get the existence of a finite subset I' of I such that $y \leqq \bigvee (x; x \in I')$, i. e. $y \in I$. We proved $I_a \subseteq I$ while $I \subseteq I_a$ is trivial, thus $I = I_a$ as required.

§ 3. A characterization theorem

Now we are ready to prove Theorem I.

Theorem 10. *A lattice L is compactly generated if and only if there exists an abstract algebra A such that L is isomorphic to $\Theta(A)$.*

Proof. It is known that $\Theta(A)$ is compactly generated (e. g. it follows easily from Theorem 8).

Now suppose that L is a compactly generated lattice with more than 2 elements. Then there exists a semilattice F such that L is isomorphic to $I(F)$. By using this fact, we construct first a partial algebra B with $\Theta(B) \cong L$.

The elements of B are the finite subsets of $F \setminus \{O\}$. The void set is also an element of B if we identify it with the element O of F. Therefore, it will be denoted by O. We define operations and partial operations on B ($\vee$ and $\wedge$ denote the set theoretical union and intersection, i. e. the operations of B; $\cup$ denotes the only operation of F):

1. to every $u \in B$ let be assigned two operations

$$\varphi_u(x) = u \vee x \quad \text{and} \quad \psi_u(x) = u \wedge x,$$

2. to any $a, b, c \in B$ with $c \leqq a \cup b$ let a partial operation $\alpha_{abc}(x)$ be defined, whose domain is O and $\{a, b\}$: let $\alpha_{abc}(0) = 0$, $\alpha_{abc}(\{a, b\}) = \{c\}$.

We assert that $\Theta(B) \cong I(F)$. First observe that B is a generalized Boolean algebra endowed with the partial operations $\alpha_{abc}(x)$; in fact, the join and meet operation of B was given in such a way that one variable was fixed. Thus every congruence relation Θ is completely determined by $I(\Theta) = \{x; x \equiv 0(\Theta)\}$. Every element of B is a finite join of atoms, thus $I(\Theta)$ is completely determined by $I\{\Theta\}$, the set of atoms contained in $I(\Theta)$. The elements of $I\{\Theta\}$ are of the form $\{a\}$, where $a \in F$. Let $\bar{I}\{\Theta\}$ denote a subset of F consisting of 0 and of all a for which $\{a\} \in \in I\{\Theta\}$.

We prove that $\Theta \to \bar{I}\{\Theta\}$ is an isomorphism between $\Theta(B)$ and I(F).

First we prove that $\bar{I}\{\Theta\}$ is an ideal of F. If $a, b \in \bar{I}\{\Theta\}$ then $\{a\}$ and $\{b\} \in I\{\Theta\}$, thus $\{a, b\} \in I(\Theta)$. But applying $\alpha_{a,b\,a\cup b}$ we get $\alpha_{a,b,\,a\cup b}(\{a, b\}) \equiv \alpha_{a,b,\,a\cup b}(0)(\Theta)$, i. e. $\{a \cup b\} \in I(\Theta)$ and so $a \cup b \in \bar{I}\{\Theta\}$. On the other hand, if $c \leqq a \in \bar{I}\{\Theta\}$, then $\{a\} \in \in I\{\Theta\}$; thus $\{a\} \equiv 0$ (Θ) and then $\alpha_{aac}(\{a\}) \equiv \alpha_{aac}(0)$ (Θ) i. e. $\{c\} \equiv 0(\Theta)$ and we reached $c \in \bar{I}\{\Theta\}$, as required.

Now let $I \in I(F)$, we prove that there exists a $\Theta \in \Theta(B)$ such that $I = \bar{I}\{\Theta\}$. On defining Θ it is enough to give a criteria for an element x of B to be congruent to 0. This is the following: let $x \equiv 0$ (Θ) if and only if $x = 0$ or x is the join of atoms $\{a\}$ such that $a \in I$. It is routine to check that Θ is a congruence relation and $\bar{I}\{\Theta\} = I$.

Thus $\Theta \to \bar{I}\{\Theta\}$ is a one-to-one order preserving correspondence between $\Theta(B)$ and $I(F)$, so this is an isomorphism.

To make possible the application of the results developed so far we change B to B'. This new partial algebra B' is essentially the same as B only every operation $\alpha_{abc}(x)$ is replaced by three operations: $\overset{i}{\alpha}_{abc}(x)$ $(i = 1, 2, 3)$. Let

$$D(\overset{1}{\alpha}_{abc}, B') = \{\{a, b\}\}, \quad D(\overset{2}{\alpha}_{abc}, B') = \emptyset, \quad D(\overset{3}{\alpha}_{abc}, B') = \{0\},$$

and

$$\overset{1}{\alpha}_{abc}(\{a, b\}) = \{c\}, \quad \overset{3}{\alpha}_{abc}(0) = 0.$$

Obviously, B' has more congruence relations than B had, but using the notion of admissible congruence relations, as defined before Theorem 7, we see that a congruence relation Θ of B' is a congruence relation of B if and only if it is admissible.

Now we apply the construction of Theorem 7 (we may do so, for every partial operation of B' is either an operation, or one of the type φ_i^μ, $i = 1, 2, 3$, $\mu \in \Omega_2$;

here Ω_2 is the set of all triples a, b, c of F, for which $c \leqq a \cup b$), leading to an algebra B_1 (which was $\overline{T}$ in Theorem 7). Now, according to Theorem 7, every admissible congruence relation Θ of B' may be extended to a congruence relation $\overline{\Theta}$ of B_1; further, to every pair u, v of elements of B_1, there exists a smallest admissible congruence relation Θ, such that $u \equiv v(\overline{\Theta})$. Denoting by Φ' the smallest admissible congruence relation $\geqq \Phi$, it is obvious that $\Theta = \Theta'_{a(u,v)\,0}$ with a suitable $a(u, v) \in B'$. But $\Theta'_{a(u,v)\,0} = \Theta_{a(u,v)\,0}$ (this is perhaps the most important property of B'!) thus we can associate with Θ an element $a(u, v)$ of B'. If we require that $a(u, v)$ be an atom, then it is uniquely determined.

Now we define for every $u, v \in B$, three partial operations $\alpha^i_{uv}(x)$, such that

$$D(\overset{1}{\alpha}_{uv}, B_1) = \{u\}, \quad D(\overset{2}{\alpha}_{uv}, B_1) = \emptyset, \quad D(\overset{3}{\alpha}_{uv}, B_1) = \{v\},$$

and

$$\overset{1}{\alpha}_{uv}(u) = a(u, v), \quad \overset{3}{\alpha}_{uv}(v) = 0.$$

If we consider B_1 together with these new partial operations, we get B_1'.

We assert that a congruence relation Θ of B_1 is admissible if and only if it is the extension of an admissible congruence relation of B'.

First, let Φ be an admissible congruence relation of B_1, and let Θ denote the congruence relation of B' which is induced by Φ (i. e. $x \equiv y(\Theta)$, $x, y \in B'$ if and only if $x \equiv y(\Phi)$). Let $u \equiv v(\Phi)$, $u, v \in B_1'$. Φ is admissible, so $a(u, v) \equiv 0(\Phi)$; thus $a(u, v) \equiv 0(\Theta)$. We get that in B' the relation $\Theta_{a(u,v)\,0} \leqq \Theta$ holds true. By definition

$$u \equiv v(\overline{\Theta}_{a(u,v)0}),$$

thus

$$u \equiv v(\overline{\Theta}),$$

and we see that $\overline{\Theta} = \Phi$. On the other hand, if $\Phi = \overline{\Theta}$ with a suitable $\Theta \in \Theta(B')$, and $u \equiv v(\Phi)$, then $\Theta_{a(u,v)0} \leqq \Theta$ by the definition of $a(u, v)$, and so $a(u, v) \equiv 0(\Phi)$; i. e., Φ is admissible.

Now, we construct from B_1' an algebra B_2 by the method of Theorem 7, and proceeding so we get B_2', $B_3, \ldots$ and so on.

We have constructed an ascending sequence (of type ω) of algebras

$$B' \subset B_1 \subset B_2 \subset \ldots$$

Let A be the union of these:

$$A = \bigcup_{i=1}^{\infty} B_i.$$

A is obviously an algebra. Every admissible congruence relation of B' may be extended to B_1, from B_1 to B_2 and so forth to A. We assert that A has no other congruence relation. Of course, a congruence relation Φ of A induces a congruence relation Φ_n of B_n ($n = 1, 2, \ldots$). But Φ_n may be extended to B_{n+1} (in fact, Φ_{n+1} is such an extension), thus — as we have proved above — Φ is an extension of an admissible congruence relation of B'. Thus $\Theta(A)$ is isomorphic to the lattice of all admissible congruence relations of B', which is isomorphic to L, completing the proof of Theorem 10.

§ 4. Applications

In this section we will draw some conclusions from Theorem 10.

Corollary 1. *To every finite lattice L, there corresponds an abstract algebra A such that $L \cong \Theta(A)$.*

More generally:

Corollary 2. *Let L be a lattice with zero element and satisfying the ascending chain condition.**) *Then there exists an abstract algebra A with $L \cong \Theta(A)$.*

The assertion of Corollary 2 is obvious from Theorem 10, for if L satisfies the hypotheses of Corollary 2, then every ideal of L is a principal one, thus $L \cong I(L)$; Theorem 10 gives an algebra A with $\Theta(A) \cong I(L)$; hence we get $L \cong \Theta(A)$, as asserted.

Corollary 3. *A lattice L has a complete representation if and only if L is compactly generated.*

This is now obvious, for $\mathcal{E}(H)$ (see the notation in § 2 of the Introduction) is compactly generated and by Theorem 8 every complete sublattice of a compactly generated lattice is itself compactly generated. Thus if L has a complete representation $\langle F, H \rangle$ then the sublattice of $\mathcal{E}(H)$ formed by the $F(x)$, $x \in L$ is compactly generated and so is L. Conversely, if L is compactly generated, then by Theorem 10 there exists an algebra A with $L \cong \Theta(A)$; let $\varphi: x \to x\varphi \in \Theta(A)$ be this isomorphism. If $\langle F, A \rangle$ is the natural (complete) representation of $\Theta(A)$ (see § 2 of Introduction) then $\langle F\varphi, A \rangle$ is a complete representation of L, where $F\varphi$ denotes the product of the mappings F and φ.

Corollary 4. (Whitman [11].) *Every lattice has a representation.*

* * *

We get an other type of application if we consider the special properties of the algebra A, constructed in the proof of Theorem 10.

In our paper [6] we have proved the following theorem:

To every abstract algebra C there exists an abstract algebra D such that $\Theta(C) \cong \Theta(D)$ and every compact congruence relation of D is of the form Θ_{ab}.

The question arises whether or not it is possible to choose such a D where the element a may be fixed. An answer is given in

Corollary 5. *To every abstract algebra C there exists an abstract algebra D and a fixed element o of D such that $\Theta(C) \cong \Theta(D)$, and every compact congruence relation of D is of the form $\Theta_{oa} (a \in D)$.*

Let $L = \Theta(C)$ and $D = A$, where A, is the algebra constructed in Theorem 10 if we start with L. Then $A = D$ has the property stated with $o = 0$. The easy proof is left to the reader.

Let $G(A)$ denote the automorphism group of A. The question arises what relation has the structure of $G(A)$ to $\Theta(A)$. We will prove that already the simplest $G(A)$ allows $\Theta(A)$ to be arbitrary.

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*) This means that if $x_1, x_2, \ldots$ are elements of L such that $x_1 \leq x_2 \leq \ldots$, then there exists an integer n such that $x_n = x_{n+1} = \ldots$.

Corollary 6. *The algebra A constructed in* § 3 *has a trivial automorphism group, i. e.* $G(A)\cong 1$.

Proof. The reader should remember that there is a subset B' of A such that B' generates A; there is an operation φ_0 which is the identity operation on B', i. e. $\varphi_0(x)=x$ for all $x\in B'$. But if $x\notin B'$, then by free generation $\varphi_0(x)\neq x$; thus

(i) $x\in B'$ if and only if $\varphi_0(x)=x$, where φ_0 is a fixed operation of A and B' is a generating system of A.

Suppose $\alpha\in G(A)$ and $x\in B'$ then $\varphi_0(\alpha x)=\alpha\varphi_0(x)=\alpha x$ and thus by (i) we get $\alpha x\in B'$. On the other hand if $x\in A$ and $\alpha x\in B'$ then $x=\alpha^{-1}(\alpha x)\in B'$. We get the following result:

(ii) α (restricted to B') is an automorphism of B'.

By free generation this implies

(iii) the automorphism groups of B' and A are isomorphic.

B' is a generating system of the whole A; it follows that if $\alpha\neq\beta$ are automorphisms of A then their restrictions to B' are different automorphisms of B'; we conclude:

(iv) if $G(B')=1$ then $G(A)=1$.

Thus Corollary 6 is proved if $G(B')=1$.

Now suppose $G(B')\neq 1$, i. e. $\alpha\in G(B')$, $x\in B'$ and $\alpha x\neq x$. It is no restriction to suppose x is an atom. Obviously, there exist in B_1 elements u, v such that $a(u, v)=x$, i. e. there is a partial operation β of B_1' which is defined only at u and $\beta(u)=x$. This implies $\alpha(\beta(u))\neq\beta(u)$, i. e. $\beta(\alpha u)\neq\beta(u)$, thus $\alpha u\neq u$ and $\beta(\alpha u)=$ $=\alpha x\in B'$. But $\beta(a)$ is in B' if and only if $a=u$ or $a=v$ thus $\alpha u=v$, and we reach $\alpha x=0$, a contradiction.

* * *

Finally we mention

Corollary 7. *A complete lattice L has a complete subgroup representation if and only if L is compactly generated.*

An application of Corollary 3 shows that it is enough to prove that $\mathscr{E}(H)$, the lattice of all equivalence relations of A, has a complete subgroup representation. It is a result of G. Birkhoff that $\mathscr{E}(H)$ has a subgroup representation (see [11], where the proof is reproduced). But his proof gives, in fact, a complete subgroup representation of $\mathscr{E}(H)$, as may be easily checked. Thus Corollary 7 is proved.

CHAPTER III

ABSTRACT ALGEBRAS OF TYPE 2 AND 3

§ 1. Preliminary results

If we want to prove Theorems II′ and III′ then it is not enough to have the theory of free algebras developed only for algebras with unitary operations. Therefore we now formulate these results for arbitrary algebras.

Let S be a partial algebra and $\varphi\in P(S)$. $D(\varphi, S)$ denotes the n-tuples $(a_1,\dots,a_n)$ for which φ is defined. We assigne to every n-tuple $(u_1,\dots,u_n)\notin D(\varphi, S)$ a new element $X_{u_1,\dots,u_n}$ such that if $(u_1,\dots,u_n)\neq(v_1,\dots,v_n)$ then $X_{u_1,\dots,u_n}\neq X_{v_1,\dots,v_n}$. $S[\varphi]$ denotes the set S together with the new elements. We define operations on $S[\varphi]$:

1. $\psi(a_1,\dots,a_m)$ is defined for a $\psi\neq\varphi$ if and only if $(a_1,\dots,a_m)\in D(\psi,S)$.

2. $\varphi(a_1,\dots,a_n)$ is unchanged if $(a_1,\dots,a_n)\in D(\varphi,S)$; if all the $a_i\in S$ but $(a_1,\dots,a_n)\notin D(\varphi,S)$ then $\varphi(a_1,\dots,a_n)=X_{a_1,\dots,a_n}$; for other n-tuples φ is not defined.

Now construct $S[\varphi]$ for all $\varphi\in P(S)$ such that if $\varphi\neq\psi$ then $S[\varphi]\wedge S[\psi]=S$; define $S_1=\bigvee(S[\varphi];\ \varphi\in P(S))$, $S_2=\bigvee(S_1[\varphi];\ \varphi\in P(S))$ and so on and $\bar S=\bigvee_{i=1}^{\infty}S_i$.

The same proof as those of Theorem 3, 4, 5 applies to get the following result.

Theorem 11. *$\bar S$ is the free algebra generated by S. Every congruence relation of S may be extended to $\bar S$.*

* * *

Let S be a partial algebra, whose partial operations are either operations $\omega^\nu(x_1,\dots,x_n)$ $(\nu\in\Omega_1)$ or of the type $\varphi_i^\mu(x)$: $i=1,2,3$, $\mu\in\Omega_2$ and $D(\varphi_1^\mu,S)=\{a^\mu\}$, $D(\varphi_2^\mu,S)=\emptyset$, $D(\varphi_3^\mu,S)=\{b^\mu\}$. The congruence relation Θ is called admissible if for every $\mu\in\Omega_2$, $a^\mu\equiv b^\mu(\Theta)$ implies $\varphi_1^\mu(a^\mu)\equiv\varphi_3^\mu(b^\mu)\,(\Theta)$.

Theorem 12. *S may be extended to an algebra S^1 such that a congruence relation Θ of S may be extended to a congruence relation $\bar\Theta$ of S^1 if and only if Θ is admissible. Further, if Φ is a congruence relation of S^1 then there exists an admissible congruence relation Θ of S such that $\Phi=\bar\Theta$. Finally, the relations $\varphi_1^\mu(b^\mu)=\varphi_2^\mu(b^\mu)$, $\varphi_2^\mu(a^\mu)=\varphi_3^\mu(a^\mu)$, $\mu\in\Omega_2$ hold true in S^1.*

* * *

We need also a new form of the result of our paper [6].

Theorem 13. *Every abstract algebra A may be extended to an abstract algebra A_1 such that*

1. *every congruence relation Θ of A may be extended to a congruence relation $\bar\Theta$ of A_1;*

2. *$\Theta\to\bar\Theta$ is an isomorphism between $\Theta(A)$ and $\Theta(A_1)$ i. e. to every $\Phi\in\Theta(A_1)$ there exists a $\Theta\in\Theta(A)$ such that $\Phi=\bar\Theta$;*

3. *every compact congruence relation of A_1 is minimal;*

4. *if $a,b,c,d\in A$ then there exists $e,f,g\in A_1$ such that $\Theta_{ab}=\Theta_{ef}$, $\Theta_{cd}=\Theta_{fg}$, $\Theta_{ab}\cup\Theta_{cd}=\Theta_{eg}$.*

Remark. Conditions 1 and 2 mean that $\Theta(A)$ and $\Theta(A_1)$ are isomorphic in the natural way.

The theorem stated in [6] is weaker than our Theorem 13, but we actually proved Theorem 13 for algebras with unitary operations; a slight modification of the construction of [6] gives the result of Theorem 13.[5])

[5]) In [6] we used the fact that the algebra has only unitary operations only at the step, when we constructed A_1 from A, in § 3. If A has operations f of more than one variable, then we define its extension on A_1 as follows: $f(a_1,\dots,a_n)=f(b_1,b_2,\dots,b_n)$ where $a_i=b_i$, if $a_i\in A$, $b_i=a$ otherwise. One can easily that with this definition one can carry out the proof of the theorem.

§ 2. Abstract algebras of type 3

We will prove the following theorem:

Theorem 14. *To every abstract algebra A there corresponds an abstract algebra B such that the following conditions are satisfied:*

1. *B is an extension of A;*
2. *every congruence relation Θ of A may be extended to a congruence relation $\bar{\Theta}$ of B;*
3. *$\Theta \to \bar{\Theta}$ sets up an isomorphism between $\Theta(A)$ and $\Theta(B)$;*
4. *B is of type 3;*
5. *every compact congruence relation of B is minimal.*

Remark. Conditions 2 and 3 mean that $\Theta(A)$ and $\Theta(B)$ are isomorphic in the natural way.

One can see that Theorem 14 contains Theorem II′ of § 3 of the Introduction. Further, according to Theorem 10, for every compactly generated lattice L there exists an algebra A with $L \cong \Theta(A)$. Now if we construct the algebra B of Theorem 14 corresponding to this algebra A, then we get that there exists an algebra B with $L \cong \Theta(B)$ and B is of type 3. Summing up we get the following.

Corollary. *The following conditions on a lattice L are equivalent:*

1. *L is compactly generated;*
2. *L has a complete representation;*
3. *L has a complete representation of type 3;*
4. *there exists an abstract algebra A with $L \cong \Theta(A)$;*
5. *there exists an abstract algebra A of type 3 with $L \cong \Theta(A)$.*

Now we are going to prove Theorem 14. We start with the algebra $A_0 = A$ and we extend A_0 to A_0^1 according to Theorem 13. Let $x, y, u, v \in A_0^1$ such that $x \equiv y(\Theta_{uv})$; then we define three partial operations $\varphi_1, \varphi_2, \varphi_3$ on A_0^1:

$$D(\varphi_1, A_0^1) = \{u\}, \quad D(\varphi_2, A_0^1) = \emptyset, \quad D(\varphi_3, A_0^1) = \{v\}$$

and $\varphi_1(u) = x$, $\varphi_3(v) = y$. Let A_0^2 be defined as the partial algebra which we get if the φ_i are defined on A_0^1 for every quadruple $x, y, u, v (x \equiv y(\Theta_{uv}))$.

Every congruence relation of A_0^2 is admissible; it further satisfies all the assumptions we have made in Theorem 12, therefore we can extend A_0^2 to an algebra A_1, such that A_1 already satisfies conditions 1, 2, 3 of Theorem 14. Now we construct A_2 from A_1, A_3 from A_2, and so on, in the same way as A_1 has been constructed from A_0, The algebras $A_0, A_1, \ldots$ form an ascending chain, therefore $B = \bigvee_{i=1}^{\infty} A_1$ is an algebra. Since all the A_i satisfy 1, 2, 3, and 5 of Theorem 14, therefore so does B. It remains only to verify condition 4. Let $x \equiv y(\Theta \cup \Phi)$, then there exist compact congruence relations $\Theta_1 \leqq \Theta$ and $\Phi_1 \leqq \Phi$ such that $x \equiv y(\Theta_1 \cup \Phi_1)$. By condition 5 $\Theta_1 = \Theta_{ab}$ and $\Phi_1 = \Theta_{cd}$ with suitable elements a, b, c, d of B. There exists an integer n with $x, y, a, b, c, d \in A_n$. By condition 4 of Theorem 13, there exist elements e, f, g of A_n^1 such that $\Theta_{ab} = \Theta_{ef}$, $\Theta_{cd} = \Theta_{fg}$ and $\Theta_{ab} \cup \Theta_{cd} = \Theta_{eg}$. Thus $x \equiv y(\Theta_{eg})$. Therefore A_n^2 has operations $\varphi_1, \varphi_2, \varphi_3$ such that $\varphi_1(e) = x$, $\varphi_1(g) = \varphi_2(g)$, $\varphi_2(e) = \varphi_3(e)$, $\varphi_3(g) = y$.[6] Then $z_0 = x$, $z_1 = \varphi_1(f)$, $z_2 = \varphi_2(f)$,

[6] See the construction in § 1 of Ch. II and Theorem 12.

$z_3=\varphi_3(f)$, $z_4=y$ is a sequence of elements such that $z_0\equiv z_1(\Theta_{ab})$, $z_1\equiv z_2(\Theta_{cd})$, $z_2\equiv z_3(\Theta_{ab})$, $z_3\equiv z_4(\Theta_{cd})$. Indeed, $e\equiv f(\Theta_{ab})$ (for $\Theta_{ab}=\Theta_{ef}$), thus $z_0=\varphi_1(e)\equiv \equiv\varphi_1(f)=z_1(\Theta_{ab})$. Similarly, $z_1=\varphi_1(f)\equiv\varphi_1(g)(\Theta_{cd})$ (for $\Theta_{fg}=\Theta_{cd}$) and $\varphi_1(g)= =\varphi_2(g)\equiv\varphi_2(f)=z_2(\Theta_{cd})$ thus $z_1\equiv z_2(\Theta_{cd})$, and so on.

To sum up, whenever $x\equiv y(\Theta\cup\Phi)$ $(x, y\in A,\ \Theta, \Phi\in\Theta(A))$ we can find elements $x=z_0, z_1, z_2, z_3, z_4=y$ such that $z_0\equiv z_1(\Theta)$, $z_1\equiv z_2(\Phi)$, $z_2\equiv z_3(\Theta)$, $z_3\equiv z_4(\Phi)$ (we take into consideration that $\Theta_{ab}\leqq\Theta_1\leqq\Theta$, $\Theta_{cd}\leqq\Phi_1\leqq\Phi$), wihch is the definition of algebra of type 3, Thus condition 4 of Theorem 14 is also verified.

§ 3. Abstract algebras of type 2

The analogue of Theorem 14 for modular lattices is the following:

Theorem 15. *Let A be an abstract algebra such that $\Theta(A)$ is modular. Then there exists an abstract algebra B such that the following conditions are satisfied:*

1. *B is an extension of A;*
2. *every congruence relation Θ of A may be extended to a congruence relation $\overline{\Theta}$ of B;*
3. *$\Theta\to\overline{\Theta}$ sets up an isomorphism between $\Theta(A)$ and $\Theta(B)$;*
4. *B is of type 2;*
5. *every compact congruence relation of B is minimal.*

Remark. Conditions 2 and 3 mean that $\Theta(A)$ and $\Theta(B)$ are isomorphic in the natural way.

Of course in Theorem 15 the essential conditions are that $\Theta(A)\cong\Theta(B)$ and that B is of type 2.

Again, combining Theorem 15 with Theorem 10 we get the

Corollary. *The following conditions on a lattice L are equivalent:*

1. *L is compactly generated and modular;*
2. *L has a complete representation of type 2;*
3. *there exists an abstract algebra A of type 2 such that $L\cong\Theta(A)$.*

For the Corollary the only thing we must verify is that condition 2 implies condition 1; it is enough to prove that if L has a representation of type 2 then L is modular; this is a theorem of [8][7].

For the proof of Theorem 15 we need some preliminary results. The proof of Theorem 15 will be given after Theorem 18.

The crucial point of the proof of Theorem 14 was the following: we can prove that B is of type 3 because the construction given at the beginning of § 1 of Chapter II and which is perfomed in the construction of B several times gives rise to a sequence of elements which guarantee that B is of type 3. In the construction in ques-

[7]) For completeness' sake we prove this. Let L have a representation $\langle F, A\rangle$ of type 2, $a, b, c\in L$, $a\geqq c$. Then $a\cap(b\cup c)\geqq(a\cap b)\cup c$ holds always, hence it is enough to prove that $p, q\in A$. $p\equiv q(F(a\cap(b\cup c)))$ imply $p\equiv q(F((a\cap b)\cup c))$. Indeed, if $p\equiv q(F(a\cap(b\cup c)))$ then $p\equiv q(F(a)\cap F(b\cup c))$, thus $p\equiv q(F(b\cup c))$ and $p\equiv q(F(a))$. We have a representation of type 2, thus $p\equiv q(F(b\cup c))$ implies the existence of r and s such that $p\equiv r(F(c))$, $r\equiv s(F(b))$, $s\equiv q(F(c))$. Then $c\leqq a$ implies that $r\equiv p\equiv q\equiv s$ $(F(a))$, thus $r\equiv s(F(a\cap b))$. We get $p\equiv q(F(a\cap b)\cup F(c))$ that is $p\equiv q(F((a\cap b)\cup c))$, which was to be proved.

tion we start from a partial algebra S and we take three further copies of S, and we identify some elements. One can easily seen that if we want to get an algebra of type 2 then we must reduce the number of new copies of S to 2. This is the main difficulty. Of course, the analogue of Theorem 6 for this modified construction may be proved easily, but Theorem 7 is already not true. We have to introduce some new operations — using the modularity of $\Theta(A)$ — to enforce the existence of the least admissible congruence relation, the existence of which is the main statement of Theorem 7.

So first we modify the construction of § 1 of Chapter II. Let S be a partial algebra, $\varphi_1(x), \varphi_2(x) \in P(S), D(\varphi_1, S) = \{a\}$, $D(\varphi_2, S) = \{b\}$, $\varphi_1(a) = c$, $\varphi_2(a) = d$. We identify in $S[\varphi_1] \cup S[\varphi_2]$ the elements $\varphi_1(b)$ and $\varphi_2(b)$ (see Fig. 3), getting the partial algebra T'. The congruence relation Θ of S is called *admissible* again if either $a \not\equiv b(\Theta)$ or if $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$ (i. e. if $a \equiv b(\Theta)$ „implies” $c \equiv d(\Theta)$). Then

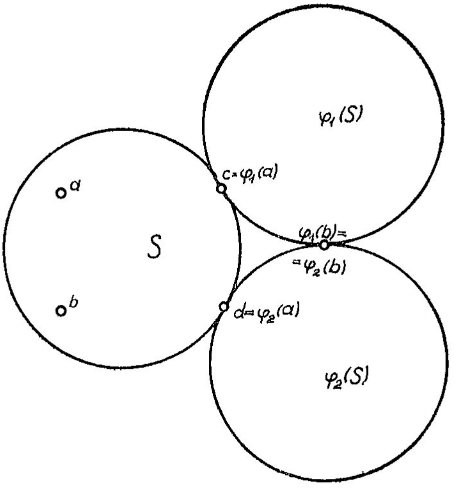

Fig. 3

Theorem 16. *The congruence relation Θ of S is admissible if and only if it may be extended to T'. The minimal extension $\overline{\Theta}$ of Θ is the transitive extension of Θ^*, where Θ^* is identical with Θ on S, and $\varphi_i(x) \equiv \varphi_i(y)$ (Θ^*), if and only if $x \equiv y(\Theta)$ $(x, y \in S)$. The relations Θ^* and $\overline{\Theta}$ are identical on S, on $\varphi_1(S)$ and on $\varphi_2(S)$.*

Proof. Copy the proof of Theorem 6.

Now we want to see what can be said about the congruence relation Θ of S for which $u \equiv v(\overline{\Theta})$, with $u, v \in T'$ fixed. To do this we make three assumptions on S: 1. $\Theta(S)$ is modular, 2. the compact congruence relations of S are minimal; 3. every congruence relation of S is admissible. We distinguish several cases.

A. $u, v \in S$. Obviously[8]), $\Theta = \Theta'_{uv}$ is the smallest admissible congruence relation for which $u \equiv v(\overline{\Theta})$.

B. $u \in S$, $v \in \varphi_1(S)$, i. e. $v = \varphi_1(x)$, $x \in S$. Let Θ be admissible, $u \equiv v(\overline{\Theta})$. Then either

$$(a) \quad u \equiv c(\Theta), \quad a \equiv x(\Theta),$$

or

$$(b) \quad u \equiv d(\Theta), \quad a \equiv b(\Theta), \quad b \equiv x(\Theta).$$

Thus the two congruences

$$\Theta_1 = \Theta_{uc} \cup \Theta_{ax}, \quad \Theta_2 = \Theta_{ud} \cup \Theta_{ab} \cup \Theta_{bx}$$

[8]) The reader should remember that if Θ is a congruence relation of S then Θ' denotes the least admissible congruence relation $\geqq \Theta$ (see the text of § 1 of Chapter II, before Theorem 7).

have the property that either $\Theta_1' \leqq \Theta$ or $\Theta_2' \leqq \Theta$. If we prove $\Theta_1' \leqq \Theta_2'$, then we are through. Indeed, $a \equiv x(\Theta_{ab} \cup \Theta_{bx})$, thus $\Theta_{ax} \leqq \Theta_2'$; further $a \equiv b(\Theta_2)$, thus $c \equiv d(\Theta_2')$; we get $\Theta_{uc} \leqq \Theta_{ud} \cup \Theta_{dc} \leqq \Theta_2'$. Hence $\Theta_1 \leqq \Theta_2'$, so $\Theta_1' \leqq (\Theta_2')' = \Theta_2'$, q. e. d.

C. $u \in S$, $v \in \varphi_2(S)$. Proof as in case B.

D. $u, v \in \varphi_1(S)$, i. e. $u = \varphi_1(x)$, $v = \varphi_1(y)$, $x, y \in S$. Then $u \equiv v(\overline{\Theta})$ implies either

$$(a) \quad x \equiv y(\Theta),$$

or

$$(b) \quad x \equiv a(\Theta), \quad c \equiv d(\Theta), \quad a \equiv b(\Theta), \quad b \equiv y(\Theta),$$

or

$$(c) \quad y \equiv a(\Theta), \quad c \equiv d(\Theta), \quad a \equiv b(\Theta), \quad b \equiv x(\Theta).$$

This shows that, obviously, $\Theta = \Theta_{xy}'$ is the smallest admissible congruence relation under which $u \equiv v(\overline{\Theta})$.

E. $u, v \in \varphi_2(S)$. Proof as in case D.

We see that the three conditions imposed on S have not yet been used.

F. $u \in \varphi_1(S)$, $v \in \varphi_2(S)$ i. e. $u = \varphi_1(x)$, $v = \varphi_2(y)$, $x, y \in S$ (or symmetrically, interchanging u and v). If $u \equiv v(\overline{\Theta})$, then either

$$(a) \quad x \equiv b(\Theta), \quad y \equiv b(\Theta),$$

or

$$(b) \quad x \equiv a(\Theta), \quad c \equiv d(\Theta), \quad a \equiv y(\Theta).$$

Let $\Theta_1 = \Theta_{xb} \cup \Theta_{yb}$, $\Theta_2 = \Theta_{xa} \cup \Theta_{cd} \cup \Theta_{ya}$. Then, if any, Θ_1' or Θ_2' should be the smallest admissible congruence relation Θ such that $u \equiv v(\overline{\Theta})$. But it turns out that neither $\Theta_1' \leqq \Theta_2'$ nor $\Theta_2' \leqq \Theta_1'$ hold in general. Now we use conditions 1—3.

Let $\Theta_3 = \Theta_{xa} \cup \Theta_{ya}$. Then $\Theta_1 \cup \Theta_2 = \Theta_1 \cup \Theta_3$ and $\Theta_3 \leqq \Theta_2$. Thus by the modularity of $\Theta(S)$ we get

$$\Theta_2 = \Theta_2 \cap (\Theta_1 \cup \Theta_3) = (\Theta_2 \cap \Theta_1) \cup \Theta_3.$$

Θ_2 and Θ_3 are compact congruence relations, therefore we can find a $\Theta_4 \leqq \Theta_2 \cap \Theta_3$ such that Θ_4 is compact and $\Theta_4 \cup \Theta_3 = \Theta_2$. Because of $\Theta_{xy} \leqq \Theta_1 \cap \Theta_2$ we may choose Θ_4 such that $\Theta_{xy} \leqq \Theta_4$ is true.

Every compact congruence relation is minimal, therefore $\Theta_4 = \Theta_{ef}$ $(c, f \in S)$. Of course, e and f are not uniquely determined by u and v; already Θ_4 is not unique, but if it were, we could, in general, choose several e and f. But let us fix a pair e, f; we may write $e = e(u, v)$, $f = f(u, v)$.

Suppose that to every $u \in \varphi_1(S)$, $v \in \varphi_2(S)$ we have found e and f. Then we assign to every u, v a new pair of partial operations $\alpha_1(x)$ and $\alpha_2(x)$ such that

$$D(\alpha_1, T') = \{e\}, \quad D(\alpha_2, T') = \{f\}, \quad \alpha_1(e) = u, \quad \alpha_2(f) = v.$$

Let T'' denote the partial algebra T' endowed with these new operations.

Theorem 17. *T'' is an extension of S. A congruence relation of S may be extended to T'' if and only if it is admissible. To every $u, v \in T''$ there exists a least admissible congruence relation Θ of S such that $u \equiv v(\overline{\Theta})$.*

Proof. Let Ψ be an admissible congruence relation of S. It is in general not true that $\overline{\Psi}$ (the extension of Ψ to T') is a congruence relation of T''. The extendability of Ψ to T'' means that extending Ψ to T'' we do not get new congruence relations in S. The extension Ψ_1 of Ψ to T'' may be defined as the transitive extension of Ψ^*, where Ψ^* is a relation equivalent to Ψ on S, $\varphi_i(x) \equiv \varphi_i(y)$ (Ψ^*) $(x, y \in S)$ if and only if $x \equiv y(\Psi)$, and $u = \varphi_1(x) \equiv \varphi_2(y) = v(\Psi^*)$ if and only if $\Theta_{ef} \leqq \Psi$ $(e = e(u, v), f = f(u, v))$.

We have the following remark: let $u = \varphi_i(x)$ $(i = 1$ or $2)$ $v = \varphi_j(y)$ $(j = 1$ or $2)$ and $u \equiv v(\Psi^*)$. Then $x \equiv y(\Psi)$. Indeed, if $i = j$, then this is true by definition. If $i \neq j$ then $\Theta_{ef} \leqq \Psi$. But e and f were chosen so that $\Theta_{xy} \leqq \Theta_{ef}$. Thus $\Theta_{xy} \leqq \Psi$ is obvious. The transitive extension Ψ_1 of Ψ^* gives rise to new congruences in S if and only if $c \equiv d(\Psi_1)$ while $c \not\equiv d(\Psi)$. We prove that this is impossible. Indeed $c \equiv d(\Psi_1)$ means the existence of a sequence $c = z_0, z_1, \ldots, z_n = d$, all the z_i being in $\varphi_1(S) \vee \vee \varphi_2(S)$, such that $z_{i-1} \equiv z_i(\Psi^*)$, $i = 1, 2, \ldots, n$. Let $z_i = \varphi_j(u_i)$ where j is either 1 or 2. Then by the remark of the last but one paragraph we have $u_0 \equiv u_1(\Psi)$, $u_1 \equiv u_2(\Psi), \ldots, u_{n-1} \equiv u_n(\Psi)$ i. e. $u_0 \equiv u_n$ (Ψ). But $\varphi_1(u_0) = c$, $\varphi_2(u_n) = d$; thus $u_0 = a$, $u_n = b$ and we have $a \equiv b(\Psi)$. Now we use that Ψ is admissible, therefore $c \equiv d(\Psi)$, contrary to the hypothesis. Q. e. d.

Now we generalize Theorem 17.

Theorem 18. *Let S be a partial abstract algebra with the following properties: the partial operations of S are $\varphi_i^\mu(x)$, $i = 1, 2$ $\mu \in \Omega$, where $D(\varphi_1^\mu, S) = \{a^\mu\}$, $D(\varphi_2^\mu, S) = \{b^\mu\}$, $\varphi_1^\mu(a^\mu) = c^\mu$, $\varphi_2^\mu(a^\mu) = d^\mu$; all other partial operations of S are operations; if Θ is a compact congruence relation then so is*[9] *Θ'; every compact congruence relation of S is minimal; the admissible congruence relations of S form a modular lattice*[10]*.*

Then there exists an abstract algebra S^* such that

I. *S^* is an extension of S;*

II. *every admissible congruence relation Θ of S may be extended to a congruence relation $\overline{\Theta}$ of S^*;*

III. *$\Theta \to \overline{\Theta}$ is an isomorphism between the lattice of admissible congruence relations of S and $\Theta(S^*)$.*

Proof. Copy the proof of Theorem 7 and use the construction of Theorem 18 rather than that of Theorem 6.

Now we are ready to prove Theorem 15. We apply the same procedure as in the proof of Theorem 14, the only difference is that we use Theorem 18 rather than Theorem 16. The algebra B will be of type 2 because the construction given before Theorem 16 uses only two new copies of S, therefore whenever $x \equiv y$ $(\Theta \cup \Phi)$ we can find a sequence $x = z_0, z_1, z_2, z_3 = y$ such that $z_0 \equiv z_1(\Theta)$, $z_1 \equiv z_2(\Phi)$, $z_2 \equiv z_3(\Theta)$. The construction of the z_i is also the same as in the proof of Theorem 14.

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[9]) Θ' denotes the least admissible congruence relation $\geqq \Theta$. Now a congruence relation Φ is admissible if for every $\mu \in \Omega$ the relations $a^\mu \equiv b^\mu(\Phi)$, $c^\mu \equiv d^\mu(\Phi)$ are equivalent.

[10]) The admissible congruence relations of S always form a complete lattice, which is in general not a sublattice of $\Theta(S)$.

§ 4. Problems

The first main result of this paper is that to every compactly generated lattice L there exists an abstract algebra A such that $L \cong \Theta(A)$. But the algebra A which is constructed in the proof is pathological. Therefore the problem arises as to whether or not it is possible to construct an A which belongs to certain known classes.

Problem 1. *Is it true that to every compactly generated lattice there conesponds an abstract algebra A such that $L \cong \Theta(A)$ and every operation of A is binary and assoticative (A is a superposition of semigroups)?* Or the same problem, requiring A to be a semi-group.

In other words, characterize the congruence lattices of semigroups.

* * *

If L is finite the construction used gives rise to a countable A.

Problem 2. *Is it possible to represent every finite lattice in the form $\Theta(A)$, where A is a finite abstract algebra?*

This problem seems to be an extremely difficult one. Its solution should imply an answer in affirmative to Problem 48 of [1] asking whether or not every finite lattice is embeddable in a finite partition lattice. A variant of our Problem 2, the solution of which does not imply the solution of BIRKHOFF's problem, is the following.

Problem 2'. *Let $\mathfrak{A}_1$ be the class of all (finite) lattices which may be represented as $\Theta(A)$, where A is a finite abstract algebra; let $\mathfrak{A}_2$ be the class of all (finite) lattices which may be represented as sublattices of finite partition lattices. Is $\mathfrak{A}_1 = \mathfrak{A}_2$ true?*

* * *

Let $\mathfrak{A}_L$ be the class of all compactly generated lattices, $\mathfrak{A}_G$ the class of all lattices which are isomorphic to the lattice of all subgroups of a group, $\mathfrak{A}^G$ the class of lattices which are isomorphic to a complete sublattice of a lattice from $\mathfrak{A}_G$; similarly let $\mathfrak{A}_R$ be the class of lattices which are isomorphic to the lattice of all subrings of a ring and $\mathfrak{A}^R$ the class of lattices which are complete sublattices of a lattice from $\mathfrak{A}_R$. The relations $\mathfrak{A}_G \supseteqq \mathfrak{A}^G$and $\mathfrak{A}_R \supseteqq \mathfrak{A}^R$ are trivial. We have proved $\mathfrak{A}_L = \mathfrak{A}^G$.

Problem 3. *Find the proper relations between $\mathfrak{A}_L (= \mathfrak{A}^G)$, $\mathfrak{A}_G$, $\mathfrak{A}_R$ and $\mathfrak{A}^R$. Are all identical?*

* * *

In this paper we have completed the argument of [6] to show that every abstract algebra A may be extended to an abstract algebra B such that $\Theta(A) \cong \Theta(B)$ and every compact congruence relation of B is of the form Θ_{ab}. And we proved that for every abstract algebra A there exists an abstract algebra B such that $\Theta(A) \cong \Theta(B)$, and every compact congruence relation of B is of the form Θ_{oa}, where o is a *fixed element* of B. Can these two results be combined?

Problem 4.[11]) *Prove that every abstract algebra can be extended to an abstract*

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[11]) *Added in proof* (May 9, 1963): We have proved the following result.

Theorem. *Every algebra A can be extented to an algebra B such that $\Theta(A)$ and $\Theta(B)$ are isomorphic in the natural way, further, any compact congruence relation Θ is of the form Θ_{oa} where o is an arbitrary element of B (a depending on Θ and o).*

algebra B such that every compact congruence relation of B is of the form Θ_{oa}, where o is a fixed element of B.

* * *

The two main results of Chapter III may be formulated as follows: If L is compactly generated and L has a representation of type $i (i=2, 3)$ then $L \cong \Theta(A)$ where A is of type i. We could not prove (or disprove) the similar result for $i=1$. It is the following:

Problem 5. *Prove that to every compactly generated lattice L which has a representation of type 1, there exists an abstract algebra A such that $L \cong \Theta(A)$ and any two congruence relations of A are permutable (i. e. if $x \equiv y(\Theta)$, $y \equiv z(\Phi)$ then there exists a w such that $x \equiv w(\Phi)$, $w \equiv z(\Theta)$).*

* * *

G. BIRKHOFF has proved that to every group G there corresponds an abstract algebra A such that G is isomorphic to the group of all automorphisms of A. Let A be an abstract algebra; we assigne to A a couple $(G^{(A)}, L^{(A)})$, where $G^{(A)}$ is the automorphism group of A and $L^{(A)}$ the congruence lattice of A. BIRKHOFF's result states that every G occurs in the first place in a couple (G, L). We have proved that a lattice L occurs in the second place if and only if it is compactly generated. And what is more, we showed that if this is the case, then L already occurs in a couple $(1, L)$ where 1 denotes the group of one element. These results suggest that the first and second components of a couple are independent. More precisely:

Problem 6. *Let G be an arbitrary group and L a compactly generated lattice. Prove that there exists an abstract algebra such that $(G^{(A)}, L^{(A)})$ is identical with (G, L).*

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The generation of affine hulls[1])

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0. Introduction. Throughout this paper, E denotes a vector space over a field Φ of characteristic zero, the case of special interest being that in which Φ is the real number field R. A subset A of E is called *affine* iff $\alpha u+\beta v \in A$ whenever $u, v \in A$, $\alpha, \beta \in \Phi$, and $\alpha+\beta=1$. (When $\Phi=R$, this requires that A contain each line determined by any two of its points.) Each intersection of affine sets is affine, and the *affine hull* ($\operatorname{aff} X$) of a set X is defined as the intersection of all affine sets containing X. Equivalently, $\operatorname{aff} X$ is the set of all *affine combinations* of X, these being points of the form $\sum_1^n \alpha_i x_i$ with $n \in N$ (natural numbers), $x_i \in X$, $\alpha_i \in \Phi$, and $\sum_1^n \alpha_i=1$. This relationship between *blank hulls* and *blank combinations* is valid not only when *blank* means *affine*, but also when it means *linear*, *positive*, or *convex* (where, for the last two, Φ should be an *ordered* field). If bla denotes the operation of forming the blank hull, then $\operatorname{bla} X=\bigcup_{n \in N} \operatorname{bla}_n X$, where $\operatorname{bla}_n X$, denotes the set of all blank combinations of n (or fewer) points of X.

The individual operations bla_n are also of interest. When $\Phi=R$, $\operatorname{aff}_2 X$ is (X together with) the union of all lines determined by two points of X, $\operatorname{aff}_3 X$ is ($\operatorname{aff}_2 X$ together with) the union of all planes determined by three points of X, etc. It is easily verifield that

$$\operatorname{bla}_m(\operatorname{bla}_n X)=\operatorname{bla}_{mn} X \quad \text{for} \quad \operatorname{bla} \neq \operatorname{aff}. \quad \text{(See 1. 2.)}$$

The present paper is motivated by the fact that while

$$\operatorname{aff}_m(\operatorname{aff}_n X) \subset \operatorname{aff}_{mn} X,$$

the two sets need not be equal. For example, if the affinely independent set $Z \subset E$ consists of four points $z_1, \ldots, z_4$, then

$$\operatorname{aff}_4 Z \sim \operatorname{aff}_2(\operatorname{aff}_2 Z)=\left\{\frac{1}{2}\left(\sum_1^4 z_i\right)-z_j:\ 1 \leqq j \leqq 4\right\}.$$

To describe the same example more geometrically, let a, b, c, and d be the vertices of a tetrahedron in R^3. For each permutation u, v, w, x of these four vertices, let $\Pi(u, v; w, x)$ denote the plane which contains the line $\overline{wx}$ and is parallel to the

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line $\overline{uv}$. Then

$$\Pi(a, b; c, d)\cap\Pi(a, c; b, d)\cap\Pi(a, d; b, c) = \left\{\frac{1}{2}b+\frac{1}{2}c+\frac{1}{2}d-\frac{1}{2}a\right\},$$

and the set $\mathrm{aff}_4\{a, b, c, d\}\sim\mathrm{aff}_2(\mathrm{aff}_2\{a, b, c, d\})$ consists of this point together with three others which are similarly situated.

We study here sets of the form $\mathrm{aff}_{n_1}(\mathrm{aff}_{n_2}(\ldots(\mathrm{aff}_{n_k}X)\ldots))$ and others which are formed in a similar way. Since many of the results are rather technical in nature, the reader is referred to the text for full statements. However, the general spirit of our results is indicated by the following corollaries (3. 3 and 3. 4):

For all X, $\mathrm{aff}_{mn-1}X\subset\mathrm{aff}_m(\mathrm{aff}_nX)$; and if $m\neq n$, $\mathrm{aff}_{mn}X=\mathrm{aff}_m(\mathrm{aff}_nX)\cup\mathrm{aff}_n(\mathrm{aff}_mX)$.

1. Results on $\mathrm{bla}_m(\mathrm{bla}_nX)$. Let us begin by extending the definition of bla_nX. For $n_1, \ldots, n_m\in N$, $\mathrm{bla}_{(n_1,\ldots,n_m)}X$ will denote the set of all points of the form $\sum_1^m\alpha_iy_i$ for $y_i\in\mathrm{bla}_{n_i}X$ and $(\alpha_1, \ldots, \alpha_m)\in B_m$, where for bla = lin the last condition imposes no restriction, for bla = aff it means that $\sum_1^m\alpha_i=1$, for bla = pos it means that $\alpha_i\geqq 0$, and for bla = con it means that $\alpha_i\geqq 0$ and $\sum_1^m\alpha_i=1$. Thus, in particular, $\mathrm{bla}_mX=\mathrm{bla}_{(1,\ldots,1)}X$ with m 1's and $\mathrm{bla}_m(\mathrm{bla}_nX)=\mathrm{bla}_{(n,\ldots,n)}X$ with m n's.

1.1. Proposition. *For all four types of operation (and for all X), $\mathrm{bla}_{(n_1,\ldots,n_m)}X\subset\mathrm{bla}_{\sum_1^m n_i}X$; in particular, $\mathrm{bla}_m(\mathrm{bla}_nX)\subset\mathrm{bla}_{mn}X$.*

Proof. For $p\in\mathrm{bla}_{(n_1,\ldots,n_m)}X$, let $p=\sum_1^m\alpha_iy_i$ with $y_i\in\mathrm{bla}_{n_i}X$ and $(\alpha_1,\ldots,\alpha_m)\in B_m$. For each i, y_i can be expressed in the form $\sum_{j=1}^{n_i}\beta_{ij}x_{ij}$ with $x_{ij}\in X$ and $(\beta_{i1}, \ldots, \beta_{in_i})\in B_{n_i}$. But then of course

$$p=\sum_{i=1}^m\sum_{j=1}^{n_i}\alpha_i\beta_{ij}x_{ij},$$

where it is easily verified that

$$(\alpha_1\beta_{11}, \ldots, \alpha_1\beta_{1n_1}, \ldots, \alpha_m\beta_{m1}, \ldots, \alpha_m\beta_{mn_m})\in B_{\sum_1^m n_i}. \blacksquare$$

The next observation is due jointly to W. E. Bonnice and the author.[1])

1.2. Proposition. *For bla $\neq$ aff, $\mathrm{bla}_{(n_1,\ldots,n_m)}X=\mathrm{bla}_{\sum_1^m n_i}X$; in particular, $\mathrm{bla}_m(\mathrm{bla}_nX)=\mathrm{bla}_{mn}X$.*

Proof. Since this is obvious for linear or positive combinations, we discuss only the case of convex combinations. Let $k_0=0$ and $k_i=\sum_{r=1}^i n_i$ for $1\leqq i\leqq m$.

[1]) See also William Bonnice and Victor Klee, The generalisation of convex hulls, *Math. Annalen*, **149** (1963), to appear.

Consider a point $p \in \mathrm{con}_{k_m} X$ — say $p = \sum_1^{k_m} \alpha_j x_j$ with $x_j \in X$, $a_j \geqq 0$, and $\sum_1^{k_m} \alpha_j = 1$. For $1 \leqq i \leqq m$, let $\sigma_i = \sum_{k_{i-1}+1}^{k_i} \alpha_j$. For $k_{i-1}+1 \leqq j \leqq k_i$, define

$$\beta_j = \alpha_j / \sigma_i \text{ when } \sigma_i \neq 0, \quad \beta_j = \alpha_j \text{ when } \sigma_i = 0.$$

Then $p = \sum_{i=1}^{m} \left(\sigma_i \sum_{j=k_{i-1}+1}^{k_i} \beta_j x_j \right)$, where all coefficients are $\geqq 0$, $\sum_1^m \sigma_i = 1$, and

$$\sum_{j=k_{i-1}+1}^{k_i} \beta_i = 1 \text{ when } \sigma_i \neq 0,$$

$$\sum_{j=k_{i-1}+1}^{k_i} \beta_j x_j = 0 \text{ when } \sigma_i = 0.$$

Consequently $p \in \mathrm{con}_{(n_1, \ldots, n_m)} X$. ∎

The proof of 1.2 depends on partitioning the α_j's into m groups such that there are n_i of them in the i^{th} group and such that those in each group are all zero or have nonzero sum. The same problem arises in connection with affine combinations, but there the desired partition may not exist. In order to discuss the situation efficiently, we shall introduce the notion of a weighted set and shall study partitions of such sets.

2. Partitions of weighted sets. Here and subsequently, Γ denotes a fixed (but arbitrary) ordered abelian group, while $<$, $+$, and $-$ are used for the ordering, addition, and subtraction in both Γ and N. A *weighted point* is an ordered pair $w = (\overline{w}, w')$ for which $w' \in \Gamma$ ($\overline{w}$ arbitrary); w' is called the *weight* of w. A *weighted set* is a finite set W of weighted points such that for $u, v \in W$, $\overline{u} = \overline{v} \Rightarrow u' = v'$. The *weight* $\mu(W)$ of a weighted set W is the sum of the weights of its points $\left(\mu(W) = \sum_{w \in W} w'\right)$; $(W, \mu(W))$ is a weighted point. A weighted set will be called *good* unless its weight is zero while at least one of its points has nonzero weight, where zero is the neutral element of Γ.

A *partition* of a set S is a finite family of pairwise disjoint subsets of S whose union is S. For $n \in N$, an *n-partition* is one in which each member consists of n points. An $(n_1, \ldots, n_m)$*-partition* is one consisting of m sets which can be ordered in such a way that (for $1 \leqq i \leqq m$) the i^{th} set is of cardinality n_i. A partition $\mathscr{P}$ of a weighted set will be called *nice* iff each of its members is good; thus $\mathscr{P}$ is nice unless there exist $P \in \mathscr{P}$ and $w \in P$ such that $w' \neq 0 = \mu(P)$.

2.1. Theorem. *Suppose W is a weighted set and $n_1, \ldots, n_m \in N$ with $\sum_1^m n_i = \operatorname{card} W$. Then W admits a nice $(n_1, \ldots, n_m)$-partition if and only if the following three statement are all false:*

(S_1) *$m = 1$ and W is not good;*

(S_2) *$n_i = 2$ for all i; W is the union of two sets of odd cardinality such that all points of one set have the same nonzero weight α and all points of the other set have weight $-\alpha$;*

(S_3) *there exists $n \geqq 3$ such that $n_i = n$ for all i, all but one point of W have the same nonzero weight α, and the exceptional point has weight $(1-n)\alpha$.*

Proof. It is easily verified that if S_1, S_2, or S_3 is true, W does not admit a nice $(n_1, \ldots, n_m)$-partition. We now assume that W does not admit such a partition, and wish to show that S_1, S_2, or S_3 is true. If $m=1$, it is evident that W is not good and S_1 holds, so we assume $m>1$. Let $k=\text{card } W$ and consider an enumeration of the points of W in order of increasing weight:

$$w_1' \leqq w_2' \leqq \ldots \leqq w_k'.$$

Let P_1 be (the set consisting of) the first n_1 of the w_j's, P_2 the next n_2 of them, ..., P_m the last n_m of them. Then some set P_r fails to be good, and from the method of construction it is clear that $w'<0$ for all $w \in \bigcup_1^{r-1} P_i$, while $w'>0$ for all $w \in \bigcup_{r+1}^{m} P_i$. Since $m>1$, there are three cases to be considered:

$$1 = r<m;\ 1<r<m;\ 1<r = m.$$

However, the first case is treated like the third, so it suffices to consider the second and third cases. We assume, then, that $1<r$.

Note that if $u \in P_r$ with $u' \leqq 0$ and $w \in P_i$ for $i<r$, then $w'=u'$, for otherwise a nice $(n_1, \ldots, n_m)$-partition of W results from the partition $\{P_1, \ldots, P_m\}$ upon interchanging u and w. Since $1<r$, this implies the existence of $\alpha>0$ such that $w' = -\alpha$ whenever $w \in W$ with $w' \leqq 0$. Further, if $i<r$ and $v \in P_r$ with $v'>0$, then $\mu(P_i) = -n_i\alpha$ and hence $v' = (n_i-1)\alpha$, for otherwise a nice $(n_1, \ldots, n_m)$-partition of W results from interchanging v with a point of P_i. Thus there exists $n \geqq 2$ such that $n_i=n$ for all $i<r$, and $v' = (n-1)\alpha$ whenever $v \in P_r$ with $v'>0$.

We wish next to show that $n_r=n$, and for this purpose will consider another $(n_1, \ldots, n_m)$-partition of W. Let Q_r be the first n_r of the w_j's (in terms of the given ordering), Q_1 the next n_1 of them, Q_2 the next n_2 of them, ..., Q_{r-1} the next n_{r-1} of them, Q_{r+1} the next n_{r+1} of them, ..., Q_m the last n_m of them. Then $Q_i=P_i$ for $i>r$ and (since $1<r$) there exists $s<r$ such that the weighted set Q_s is not good. If $s>1$ it follows from reasoning in the preceding paragraph that $\text{card } Q_r=\text{card } Q_1$, whence $n_r=n_1=n$. If $s=1$, let R_1 be the first n_1 of the w_j's, R_r the next n_r of them, and $R_i=Q_i$ for $i \notin \{1, r\}$. By hypothesis, some R_i fails to be good, and clearly it is R_r. But then from the method of construction (using the ordering of the w_j's) we conclude that $R_r=Q_1$, whence $n_r=n_1=n$.

We have now established that $n_i = n \geqq 2$ for $1 \leqq i \leqq r$, that $w' = -\alpha$ for all $w \in W$ with $w' \leqq 0$, and that $v' = (n-1)\alpha$ for all $v \in P_r$ with $v'>0$. Let l denote the number of points of P_r which are of positive weight. Then

$$0 = \mu(P_r) = l(n-1)\alpha = (n-l)\alpha = (l-1)n\alpha,$$

whence $l=1$. Thus S_3 holds (with its α the negative of our present α) if $r=m$ and $n \geqq 3$, while S_2 holds if $r=m$ and $n=2$. If $r<m$, the reasoning of the above paragraphs shows that $n_i=n$ for *all* i and that $\alpha=(n-1)\beta$ whenever $\beta=w'>0$ for some $w \in W$. But then $\alpha = (n-1)^2\alpha$, whence $n=2$ and S_2 holds. ∎

3. The basic theorem on $\mathrm{aff}_{(n_\alpha, \ldots, n_m)}X$.

3. 1. Theorem. *Let E be a vector space over a field Φ of characteristic zero, $n_1, \ldots, n_m \in N$, and $p \in \mathrm{aff}_{\sum_1^m n_i} X$. Then $p \in \mathrm{aff}_{(n_1, \ldots, n_m)}X$ unless one of the following statements is true:*

(i) $m>1$; $n_i=2$ *for all* i; *for each expression of* p *in the form* $\sum_1^{2m} \alpha_i x_i$ *with* $x_i \in X$, $\alpha_i \in \Phi$, *and* $\sum_1^{2m} \alpha_i = 1$, *there is an odd number* $l \in N$ *such that* $m<l<2m$, l *of the* α_i's *are equal to* $\frac{1}{2(l-m)} \in \Phi$, *and the remaining* $2m-l$ α_i's *are equal to* $-\frac{1}{2(l-m)} \in \Phi$;

(ii) $m>1$; *all* n_i's *have the same value* $n \geqq 3$; *for each expression of* p *in the form* $\sum_1^{mn} \alpha_i x_i$ *with* $x_i \in X$, $\alpha_i \in \Phi$, *and* $\sum_1^{mn} \alpha_i = 1$, *one* α_i *is equal to* $\frac{1-n}{(m-1)n} \in \Phi$ *and the others are all equal to* $\frac{1}{(m-1)n} \in \Phi$.

(If 1 is the unit element of Φ and $a, b \in N$, the point $\underbrace{(1+1+\ldots+1)}_{a \text{ terms}} \div \underbrace{(1+1+\ldots+1)}_{b \text{ terms}} \in \Phi$ is denoted simply by $\frac{a}{b}$.

P r o o f. Let us suppose first that $p \in \operatorname{aff}_{(n_1, \ldots, n_m)} X$, whence there exist $y_i \in \operatorname{aff}_{n_i} X$ and $\beta_i \in \Phi$ such that $\sum_1^m \beta_i = 1$ and $\sum_1^m \beta_i y_i = p$. For each i, there exist points $z_1^i, \ldots, z_{n_i}^i$ (not necessarily distinct) of X and numbers $\gamma_1^i, \ldots, \gamma_{n_i}^i \in \Phi$ such that $\sum_{j=1}^{n_i} \gamma_j^i = 1$ and $\sum_{j=1}^{n_i} \gamma_j^i z_j^i = y_i$. Now with $s = \sum_1^m n_i$,

$$(x_1, \ldots, x_s) = (z_1^1, \ldots, z_{n_1}^1, \ldots, z_1^m, \ldots, z_{n_m}^m),$$

and

$$(\alpha_1, \ldots, \alpha_s) = (\beta_1 \gamma_1^1, \ldots, \beta_1 \gamma_{n_1}^1, \ldots, \beta_m \gamma_1^m, \ldots, \beta_m \gamma_{n_m}^m),$$

we have

$$p = \sum_1^s \alpha_r x_r, \ x_r \in X, \text{ and } \sum_1^s \alpha_r = 1.$$

Further, in their natural ordering the indices $1, \ldots, s$ are partitioned into m sets such that there are n_i indices in the i^{th} set, and the sum $\beta_i \left(= \sum_{r=1}^{n_i} \beta_i \gamma_r^i \right)$ of the α_r's corresponding to the i^{th} set of incides is different from zero unless all of these α_r's are zero. From this it is easily verified that (i) and (ii) are both false.

Conversely, we assume (i) and (ii) to be false and want to prove that $p \in \operatorname{aff}_{(n_1, \ldots, n_m)} X$. Since the field Φ is of characteristic zero, its additive group Γ is isomorphic with a subgroup of a direct sum of a (possibly infinite) number of copies of the additive group of rational numbers. Since this direct sum is an ordered group under the lexicographic ordering based on the natural ordering of rational numbers, we may assume without loss of generality that Γ is an ordered group (not implying, of course, that Φ is an ordered field).

Now taking Γ as an ordered group, we see from 2. 1 (and the assumption that (i) and (ii) are false) that p admits an expression in the form $p = \sum_1^s \alpha_r x_r$ with $x_r \in X$, $\alpha_r \in \Phi$, $\sum_1^s \alpha_r = 1$, and such that the weighted set $\{(r, \alpha_r) : 1 \leqq r \leqq s\}$ admits a nice $(n_1, \ldots, n_m)$-partition. We may assume without loss of generality that the members of the partition are the sets $\{(r, \alpha_r): s_{i-1} < r \leqq s_i\}$ for $1 \leqq i \leqq m$, where $s_0 = 0$ and $s_i = \sum_1^i n_r$. Defining $\sigma_i = \sum_{s_i-1}^{s_i} \alpha_r$, we see that either $\sigma_i \neq 0$ or $\alpha_r = 0$ for all r with $s_{i-1} < r \leqq s_i$. It then follows as in the proof of 1.2 that $p \in \mathrm{aff}_{(n_1, \ldots, n_m)} X$. ∎

3. 2. Corollary. *If the numbers* $n_1, \ldots, n_m (\in N)$ *are not all the same,* $\mathrm{aff}_{(n_1, \ldots, n_m)} X = \mathrm{aff}_{\sum_1^m n_i} X$.

3. 3. Corollary. *For all* X *and all* $n_1, \ldots, n_m \in N$, $\mathrm{aff}_{\left(\sum_1^m n_i\right)-1} X \subset \mathrm{aff}_{(n_1, \ldots, n_m)} X \subset \subset \mathrm{aff}_{\sum_1^m n_i} X$; *in particular,* $\mathrm{aff}_{mn-1} X \subset \mathrm{aff}_m (\mathrm{aff}_n X) \subset \mathrm{aff}_{mn} X$.

3. 4. Corollary. *For* $m \neq n$, $\mathrm{aff}_{mn} X = \mathrm{aff}_m (\mathrm{aff}_n X) \cup \mathrm{aff}_n (\mathrm{aff}_m X)$.

3. 5. Corollary. *If* X *is affinely independent and consists of* k *points, and* $m \geqq 2$, *the cardinality of the set*

$$X' = \mathrm{aff}_{mn} X \sim \mathrm{aff}_m (\mathrm{aff}_n X)$$

is equal to
$$\begin{cases} \binom{k}{mn} mn & \textit{when } n \geqq 3, \\ \binom{k}{2m} 2^{2m-2} & \textit{when } n = 2 \textit{ and } m \textit{ is even,} \\ \binom{k}{2m} \left(2^{2m-2} - \frac{1}{2}\binom{2m}{m}\right) & \textit{when } n = 2 \textit{ and } m \textit{ is odd.} \end{cases}$$

3. 6. Corollary. *If* X *is finite, so is* X'. *If* $\dim(\mathrm{aff}\, X) < mn - 1$, X' *is empty. If* $\dim(\mathrm{aff}\, X) = mn - 1$, $\mathrm{card}\, X' \leqq c(m, n)$, *where*

$$c(m, n) = \begin{cases} mn & \textit{when } n \geqq 3, \\ 2^{2m-2} & \textit{when } n = 2 \textit{ and } m \textit{ is even,} \\ 2^{2m-2} - \frac{1}{2}\binom{2m}{m} & \textit{when } n = 2 \textit{ and } m \textit{ is odd.} \end{cases}$$

If $\dim E \geqq mn$ *and* $m \geqq 2 \leqq n$, *then* E *containts a set* X *for which* X' *consists of* $c(m, n)$ *distinct parallel „lines" (genuine lines when* $\Phi = R$*).*

Proofs. The Corollaries 3. 2, 3. 3, and 3. 4 follow immediately from 3. 1. For the first part of 3. 5, apply 3. 1 (ii) to show that $\mathrm{card}\, X' = \binom{k}{mn}\binom{mn}{1}$. For the second and third parts of 3. 5, apply 3. 1 (i) to show that card X' is equal to $\binom{k}{mn}$ times the number of sets $Y \subset \{1, \ldots, mn\}$ for which card Y is odd and $\mathrm{card}\, Y < mn - - \mathrm{card}\, Y$. The first three assertions of 3. 6 follow from 3. 1, 3. 3, and 3. 5 respectively. For the fourth, let F be an $(mn-1)$-dimensional linear subspace of E, Y an affinely independent set in F with card $Y=mn$, $z\in E\sim F$, and $X = Y+\Phi z$. It is easily verified that

$$\mathrm{aff}_{mn}X\sim\mathrm{aff}_m(\mathrm{aff}_n X) = (\mathrm{aff}_{mn}Y\sim\mathrm{aff}_m(\mathrm{aff}_n Y))+\Phi z,$$

whence the desired conclusion follows from 3. 5. ∎

4. A qualitative approach. For $Y\subset E$ and $n_i^j\in N$, consider the set

$$\mathrm{bla}_{(n_1^1,\ldots,n_{m(1)}^1)}(\mathrm{bla}_{(n_1^2,\ldots,n_{m(2)}^2)}(\ldots(\mathrm{bla}_{(n_1^k,\ldots,n_{m(k)}^k)}Y)\ldots)).$$

With $n=\prod_{j=1}^{k}\left(\sum_{i=1}^{m(j)}n_i^j\right)$, it follows from 1. 2 that this set is equal to $\mathrm{bla}_n Y$ when bla $\neq$ $\neq$ aff. For bla = aff, the situation is much more complex and a full analysis would probably cost more than it is worth. In any case, the problem of describing the above set reduces to one concerning the interaction of operations aff_n for various values of n, since (by 3. 2) $\mathrm{aff}_{(n_1,\ldots,n_m)}X=\mathrm{aff}_{\sum_1^m n_i}X$ for all X if the n_i's assume at least two different values, while of course $\mathrm{aff}_{(n_1,\ldots,n_m)}X=\mathrm{aff}_m(\mathrm{aff}_n X)$ if all the n_i's have the same value n.

From 3. 3 it follows that always

$$\mathrm{aff}_l(\mathrm{aff}_m(\mathrm{aff}_n X))\subset\mathrm{aff}_{lmn-l-1}X\cup\mathrm{aff}_{lmn-n-1}X.$$

However, this is a crude approach and becomes cruder as the number of operations increases. The present section shows by means of a qualitative approach that always

$$\mathrm{aff}_{n_1}(\mathrm{aff}_{n_2}(\ldots(\mathrm{aff}_{n_k}X)\ldots))\subset\mathrm{aff}_{(n_1n_2\ldots n_k)-1}X, \tag{1}$$

and that if X is finite, so is the set

$$\mathrm{aff}_{n_1n_2\ldots n_k}X\sim\mathrm{aff}_{n_1}(\mathrm{aff}_{n_2}(\ldots(\mathrm{aff}_{n_k}X)\ldots)). \tag{2}$$

Section 5 contains a more quantitative analysis, leading to a description of sets of the form (2) for $k=3$ which is similar in completeness to that of Section 3 for the case $k=2$ (cf. 6. 6).

A basic tool is the notion of a weighted partition. When $\mathscr{P}$ is a partition of a weighted set, the corresponding *weighted partition* is the weighted set $\mathscr{P}^* = \{(P,\mu(P)):P\in\mathscr{P}\}$. To illustrate the combinatorial problem which is involved in the study of sets of the form (2), let us consider a weighted set W consisting of twelve points, ten of weight 1/6 and two of weight $-2/6$. Though W admits a nice 3-partition $\mathscr{P}$, the weighted partition $\mathscr{P}^*$ must consist of four „points" (the sets $P\in\mathscr{P}$), two of weight 1/2 and one of weight $-1/2$, whence $\mathscr{P}^*$ does not admit a nice 2-partition. This corresponds to the fact that if an affinely independent set $X\subset E$ consists of twelve distinct points $x_1,\ldots,x_{12}$, and if $p=\left(\sum_1^{10}\frac{1}{6}x_i\right)-\frac{1}{3}x_{11}-\frac{1}{3}x_{12}$, then $p\notin\mathrm{aff}_2(\mathrm{aff}_2(\mathrm{aff}_3X))$, even though $p\in\mathrm{aff}_m(\mathrm{aff}_n X)$ whenever $mn=12$. Thus in studying sets of the form (2), trouble is caused (speaking roughly) not only by weighted sets which admit no nice partitions but also by those whose nice partitions admit no nice partitions, and so on down the line. To establish (1) we must show that if W is a weighted set of cardinality $n_1 n_2 \dots n_k$ with at least one point of zero weight, then W admits a nice n_k-partition $\mathscr{P}_{(k)}$ such that $\mathscr{P}^*_{(k)}$ admits a nice n_{k-1}-partition $\mathscr{P}_{(k-1)}$ such that $\mathscr{P}^*_{(k-1)} \dots$ such that $\mathscr{P}^*_{(3)}$ admits a nice n_2-partition $\mathscr{P}_{(2)}$.

The basic lemma is easy to prove, but its statement requires some additional notation. Let T be a finite set, $\mathfrak{S}$ the class of all nonempty subsets of T, and Ξ the class of all functions on T to Γ. For $S \in \mathfrak{S}$ and $\xi \in \Xi$, let S_ξ denote the weighted set $\{(s, \xi s): s \in S\}$ and let $\mu_\xi S$ denote its weight $((\mu_\xi S = \mu(S_\xi) = \sum_{s \in S} \xi s))$. Let $n_1, \dots, n_m \in N$ with $m \geqq 2$ and $\sum_1^m n_i = \text{card}\, T$, and let $\mathfrak{P}$ denote the class of all $(n_1, \dots, n_m)$-partitions of T. For each $\mathscr{P} \in \mathfrak{P}$ and $\xi \in \Xi$, let $\mathscr{P}_{(\xi)}$ denote the corresponding partition of the weighted set T_ξ; that is, $P_{(\xi)} = \{S_\xi : S \in \mathscr{P}\}$.

4.1. Lemma. *Suppose Δ is a finite subset of Γ and H is the set of all $\xi \in \Xi$ such that T_ξ admits at least one nice $(n_1, \dots, n_m)$-partition $\mathscr{P}_{(\xi)}$, with $\mathscr{P}^{*\prime}_{(\xi)} \subset \Delta$ for all such $\mathscr{P}_{(\xi)}$. Then the set H is finite.*

Proof. Let I be the class of all ordered triples $(\mathfrak{Q}, f, g)$ for which $\mathfrak{Q}$ is a nonempty subset of $\mathfrak{P}$, f is a function whose domain is $\mathfrak{P} \sim \mathfrak{Q}$, and the following conditions are satisfied:

for each $\mathscr{Q} \in \mathfrak{Q}$, $f_{\mathscr{Q}}$ is a function on $\mathscr{Q}$ to Δ;

for each $\mathscr{P} \in \mathfrak{P} \sim \mathfrak{Q}$, $g\mathscr{P}$ is a nonempty subset of $\mathscr{P}$.

For each $\iota = (\mathfrak{Q}, f, g) \in I$, let H_ι denote the set of all $\eta \in H$ which have the following two properties:

$\mathfrak{Q} = \{\mathscr{Q} \in \mathfrak{P} : \mathscr{Q}_{(\eta)}$ is nice$\}$; whenever $S \in \mathscr{Q} \in \mathfrak{Q}$, then $\mu_\eta S = f_{\mathscr{Q}} S$;

for each $\mathscr{P} \in \mathfrak{P} \sim \mathfrak{Q}$, $g\mathscr{P} = \{S \in \mathscr{P} : \mu_\eta S = 0\}$.

It is evident that $H = \bigcup_{\iota \in I} H_\iota$ and that I is finite. To complete the proof it suffices to show (for $\iota \in I$) that the difference of any two functions in H is constant on X, for then it is apparent that each set H_ι has at most one member.

Let $\iota = (\mathfrak{Q}, f, g) \in I$ and consider two functions $\xi, \eta \in H_\iota$. Choose $\mathscr{Q} \in \mathfrak{Q}$. To show that $\xi - \eta$ is constant it suffices to show that whenever u_1 and u_2 are points of T which lie in different members U_1 and U_2 of $\mathscr{Q}$, then $\xi u_1 - \eta u_1 = \xi u_2 - \eta u_2$. For such U_i it follows from the definition of H_ι that $\mu_\xi U_i = \mu_\eta U_i$ $(i = 1, 2)$. Let $\mathscr{P}$ denote the partition of T which is obtained from $\mathscr{Q}$ by interchanging u_1 and u_2. Then

$$\mathscr{P} = (\mathscr{Q} \sim \{U_1, U_2\}) \cup \{V_1, V_2\},$$

where

$$V_i = (U_i \sim \{u_i\}) \cup \{u_j\} \qquad (i \neq j).$$

Clearly

$$\mu_\xi V_i = \mu_\xi U_i - \xi u_i + \xi u_j$$

and

$$\mu_\eta V_i = \mu_\eta U_i - \eta u_i + \eta u_j.$$

If $\mathscr{P} \in \mathfrak{Q}$, then (for $i = 1$ *and* $i = 2$) $\mu_\xi V_i = f_P V_i = \mu_\eta V_i$, and (recalling that $\mu_\xi U_i = \mu_\eta U_i$) we conclude that $\xi u_j - \eta u_j = \xi u_i - \eta u_i$. Suppose, on the other hand, that $\mathscr{P} \in \mathfrak{P} \sim \mathfrak{Q}$. Then by the definition of H_ι, neither $\mathscr{P}_{(\xi)}$ nor $\mathscr{P}_{(\eta)}$ is nice. Since $\mathscr{Q}$ was nice it follows that $\mu_\xi V_i = 0$ for $i = 1$ *or* $i = 2$ (but not necessarily both), whence $V_i \in g\mathscr{P}$ and $\mu_\eta V_i = 0$. Then, as before, $\xi u_j - \eta u_j = \xi u_i - \eta u_i$. ∎

For a finite set T and for $\gamma \in \Gamma$, let $A_\gamma(T)$ denote the set of all functions ξ on T to Γ such that $\sum_{t \in T} \xi t = \gamma$. For $n_1, \ldots, n_k \in N$ with $\prod_{i=1}^{k} n_i = \text{card } T$, let $A_\gamma(T; n_1, \ldots, n_k)$ denote the set of all $\xi \in A_\gamma(T)$ for which there exist weighted sets $T_\xi = W_{k+1}, W_k, \ldots, W_2$ with $W_i = \mathscr{P}_i^*$ for some nice n_i-partition $\mathscr{P}_i$ of $W_{i+1} (2 \leqq i \leqq k)$.

4. 2. Theorem. *Suppose T is a finite set, $\gamma \in \Gamma$, and $n_1, \ldots, n_k \in N$ with $\prod_{i=1}^{k} n_i =$ card T. Then the set $A_\gamma(T) \sim A_\gamma(T; n_1, \ldots, n_k)$ is finite.*

Proof. When $k=2$, the assertion follows from 2. 1. Suppose it is known for $k = j-1 \geqq 2$ and consider the case $k=j$. Let S be a set of cardinality $\prod_{i=1}^{j-1} n_i$ and let $B = A_\gamma(S) \sim A_\gamma(S; n_1, \ldots, n_{j-1})$. Then B is finite by the inductive hypothesis, so the set $\Delta = \bigcup_{\eta \in B} \eta S$ is also finite. Now with card $T = \prod_{i=1}^{j} n_i$, let G denote the set of all $\xi \in A_\gamma(T)$ such that T_ξ admits no nice n_j-partition. The set G is finite by 2. 1. Let H denote the set of all $\xi \in A_\gamma(T)$ such that T_ξ admits at least one nice n_j-partition $\mathscr{P}_{(\xi)}$, but $\mathscr{P}_\xi^{*\prime} \subset \Delta$ for all such $\mathscr{P}_{(\xi)}$. Then H is finite by 4. 1, and it is easily verified that

$$A_\gamma(T) \sim A_\gamma(T; n_1, \ldots, n_k) \subset G \cup H. \blacksquare$$

4. 3. Theorem. *For each set $X \subset E$,*

$$\text{aff}_{(n_1 n_2 \ldots n_k)-1} X \subset \text{aff}_{n_1}(\text{aff}_{n_2}(\ldots(\text{aff}_{n_k} X)\ldots)).$$

If X is finite, so is the set

$$\text{aff}_{n_1 n_2 \ldots n_k} X \sim \text{aff}_{n_1}(\text{aff}_{n_2}(\ldots(\text{aff}_{n_k} X)\ldots)).$$

Proof. Let $r = \prod_{i=1}^{k} n_i$ and let $T = \{1, \ldots, r\}$. As in the proof of 3. 1, we see that if $\xi \in A_1\ (T; n_1, \ldots, n_k)$ and if $x_1, \ldots, x_r$ are (not necessarily distinct) points of X, then

$$\sum_1^r \xi(i) x_i \in \text{aff}_{n_1}(\text{aff}_{n_2}(\ldots(\text{aff}_{n_k} X)\ldots)).$$

The second statement of 4. 3 follows at once from this fact in conjunction with 4. 2. For the first part of 4. 3, consider an arbitrary point $p = \sum_1^{r-1} \alpha_i x_i$ with $x_i \in X$, $\alpha_i \in \Gamma$, and $\sum_1^{r-1} \alpha_i = 1$. For each $\beta \in \Gamma$, let the function $\xi\beta \in A_1(T)$ be defined as follows:

$$\xi_\beta(i) = \alpha_i \quad \text{for} \quad 1 \leqq i \leqq r-2; \quad \xi_\beta(r-1) = \alpha_{r-1} - \beta; \ \xi_\beta(r) = \beta.$$

Since Γ is infinite, 4. 2 implies the existence of $\beta \in \Gamma$ for which $\xi_\beta \in A_1\ (T; n_1 \ldots, n_k)$. With $x_r = x_{r-1}$, we have

$$p = \sum_{i=1}^{r} \xi_\beta(i) x_i \in \text{aff}_{n_1}(\text{aff}_{n_2}(\ldots(\text{aff}_{n_k} X)\ldots)). \blacksquare$$

5. Troublesome sets: Lemmas. For a weighted set W, W' will denote the set $\{w': w \in W\} \subset \Gamma$. W will be said to *have the form* $(\gamma_1)^{a_1} \dots (\gamma_k)^{a_k}$ iff $a_i \in \Gamma$, $a_i \in N$, $\sum_1^k a_i = \text{card } W$, and $a_i = \text{card } \{w \in W: w' = \gamma_i\}$ for $1 \leqq i \leqq k$; and *W has the crude form* $(\gamma_1)^{a_1} \dots (\gamma_k)^{a_k}$ iff $\gamma_i \in \Gamma$, $a_i \in N \cup \{0\}$, $\sum_1^k a_i = \text{card } W$, and W admits a partition into pairwise disjoint sets $P_1, \dots, P_k$ such that card $P_i = a_i$ and $\subset \{\gamma_i\}$ for $1 \leqq i \leqq k$. With $a_i > 0$, the first condition requires that $W' = \{\gamma_i: 1 \leqq i \leqq k\}$ and the γ_i's are distinct; the second condition requires that $W' \subset \{\gamma_i: 1 \leqq i \leqq k\}$ but permits $a_i = 0$ (with of course $P_i = \emptyset$) and $\gamma_i = \gamma_j$, for $i \neq j$.

A weighted set W will be called *troublesome* iff W has the form

$$\text{(T)} \qquad (\alpha)^r (\beta_1)^{r_1} \dots (\beta_s)^{r_s} \text{ with } r \geqq 3, s \geqq 1,$$

and

$$0 < \alpha \leqq \min \{-\beta_i: 1 \leqq i \leqq s\} \quad \textit{or} \quad 0 > \alpha \geqq \max \{-\beta_i: 1 \leqq i \leqq s\}.$$

We shall often refer to the expression (T), using its notation without further explanation.

A weighted set W will be called *positively* ⟨resp. *negatively*⟩ *troublesome* iff W has the form (T) with $\alpha > 0$ ⟨resp. $\alpha < 0$⟩, *doubly troublesome* iff it has the form (T) with $s = 1$ and $\beta_1 = -\alpha$, *singly troublesome* iff it has the form (T) with $s = 1 = r_1$, and *t-singly troublesome* (for $t \in N$) iff it has the form (T) with $s = 1 = r_1$ and $\beta_1 = -t\alpha$. In connection with 2. 1 and with the principal goal of this section, the doubly and t-singly troublesome sets are of special interest; unification in the treatment of these two special types is achieved through the more general notion. Note that a set which is both positively and negatively troublesome must be doubly troublesome, but not conversely, and that a troublesome set may be both doubly and singly troublesome but need not be either.

A partition $\mathcal{P}$ of a weighted set will be called *troublesome* ⟨resp. *doubly troublesome* etc.⟩ iff the weighetd set $(\mathcal{P}^*)'$ is troublesome ⟨resp. doubly troublesome, etc.⟩. When $\mathcal{P}$ is a partition of W and $\gamma \in \Gamma$, we define $\mathcal{P}_\gamma = \{P \in \mathcal{P}: \mu(P) = \gamma\}$, $\mathcal{P}_- = \{P \in \mathcal{P}: \mu(P) < 0\}$, and $\mathcal{P}_+ = \{P \in \mathcal{P}: \mu(P) > 0\}$. For any family $\mathcal{F}$ of sets, $\amalg\mathcal{F}$ will denote the union of all members of $\mathcal{F}$. Thus (for example) $\amalg(\mathcal{P}_-)$ is the union of all members of $\mathcal{P}$ which have negative weight, while $((\mathcal{P}_-)^*)'$ is the set of all negative weights attained by members of $\mathcal{P}$. Since the danger of confusion is slight, we shall usually omit the parentheses in expressions such as these.

When $\mathcal{P}$ is a partition of W and x and y are points of W, $\mathcal{P}(x, y)$ will denote the partition which results from $\mathcal{P}$ upon interchanging x and y. Thus for $x \in X \in \mathcal{P}$ and $y \in Y \in \mathcal{P}$,

$$\mathcal{P}(x, y) = (\mathcal{P} \sim \{X, Y\}) \cup \{(X \sim \{x\}) \cup \{y\}, (Y \sim \{y\}) \cup \{x\}\}.$$

When more complicated interchanges are required, they will be described explicitly.

For the remainder of this section, we make the

STANDING HYPOTHESES: *W is a weighted set and* $n_1, \dots, n_m \in N$, *with* $m \geqq 4$ *and* $\sum_1^m n_i = \text{card } W$. *W admits a nice* $(n_1, \dots, n_m)$*-partition, but all such partitions are troublesome.*

Partition *will mean an* $(n_1, \ldots, n_m)$*-partition of* W. *A partition* $\mathscr{P}$ *will be called an* α-partition *iff* $\mathscr{P}^*$ *has the form* (T) *and in addition*

($\dagger$) $x'-y' \in \{-\alpha, 0, \alpha\}$ *whenever* x *and* y *are points of distinct members of* $\mathscr{P}_\alpha$. *An* α*-partition* $\mathscr{P}$ *will be called a* minimal α-partition *iff there is no* α*-partition* $\mathscr{Q}$ *for which* $\mathscr{Q}_\alpha$ *is a proper subset of* $\mathscr{P}_\alpha$.

* * *

The first lemma is

5. 1. *If* $\mathscr{P}$ *is a partition and* $\mathscr{P}^*$ *has the form* (T) *(but requiring only* $r \geqq 2$*), then* $\max \cup\mathscr{P}' \leqq \min \cup\mathscr{P}'_+$.

Proof. It suffices to consider the case $\alpha > 0$. If $u \in \cup\mathscr{P}_{\beta_j} \subset \cup\mathscr{P}_-$ and $x \in \cup\mathscr{P}_\alpha = \cup\mathscr{P}_+$, then $\mathscr{P}(u, x)$ is a partition for which

$$\mathscr{P}(u, x)^{*\prime} = B \cup \{\beta_i : i \neq j\} \cup \{\beta_j - (u'-x'), \alpha + (u'-x'), \alpha\},$$

where $B \subset \{\beta_j\}$. If $u'-x' > 0$, the partition $\mathscr{P}(u, x)$ is nice but cannot be troublesome, for $\beta_j - (u'-x') < \beta_j \leqq -\alpha < 0 < \alpha < \alpha + (u'-x')$. The contradiction shows that $u'-x' \leqq 0$ and yields the desired conclusion. ∎

5. 2. *If* $\mathscr{P}$ *is a partition and* $\mathscr{P}^*$ *has the form* (T) *with* $|\alpha| < \max\{|\beta_i| : 1 \leqq i \leqq s\}$, *then* $\mathscr{P}$ *is an* α*-partition.*

Proof. We assume without loss of generality that $\alpha > 0$. If x and y lie in different members of $\mathscr{P}_\alpha$, and $x' > y'$, then

$$\mathscr{P}(x, y)^{*\prime} = \{\beta_i : 1 \leqq i \leqq s\} \cup \{\alpha - (x'-y'), \alpha + (x'-y'), \alpha\},$$

and $\mathscr{P}(x, y)$ is not positively troublesome since $0 < \alpha < \alpha + (x'-y')$. If $\mathscr{P}(x, y)$ is negatively troublesome, then $s = 1$ and $0 > \beta_1 \geqq -\alpha$. Since we knew already that $\alpha \leqq -\beta_1$, it follows that

$$|\alpha| = |\beta_1| = \max\{|\beta_i| : 1 \leqq i \leqq s\},$$

contradicting the hypothesis of 5. 2. Thus $\mathscr{P}(x, y)$ must fail to be nice, whence $\alpha - (x'-y') = 0$. ∎

5. 3. *For some* α, W *admits an* α*-partition.*

Proof. Let $\mathscr{P}$ be a nice partition, whence $\mathscr{P}^*$ has the form (T). Suppose $\mathscr{P}$ is not an α-partition, whence there exists points x and y in different members of $\mathscr{P}_\alpha$ such that $x'-y' \notin \{-\alpha, 0, \alpha\}$. The partition $\mathscr{P}(x, y)$ is nice and hence troublesome. We assume without loss of generality that $\alpha > 0$ and $x'-y' > 0$, whence $\mathscr{P}(x, y)^{*\prime}$ contains at least two positive weights and $\mathscr{P}(x, y)$ must be negatively troublesome; this implies $s = 1$ and $\beta_1 \geqq -\alpha$, whence $\beta_1 = -\alpha$. With $0 \neq \alpha - (x'-y') < \alpha$, it follows that $\alpha - (x'-y') = -\alpha$, whence $x'-y' = 2\alpha$ and $\mathscr{P}(x, y)^*$ has the form $(-\alpha)^{r_1+1}(\alpha)^{r-2}(3\alpha)^1$. But then $r_1 \geqq 2$ and $\mathscr{P}(x, y)$ is a $(-\alpha)$-partition by 5. 2. ∎

5. 4. *For each* α*-partition* $\mathscr{P}$ *there is a minimal* α*-partition* $\mathscr{Q}$ *with* $\mathscr{Q}_\alpha \subset \mathscr{P}_\alpha$.

Now we add to the

STANDING HYPOTHESES: $\mathscr{Q}$ *is a minimal* α*-partition of* W, *with* $\alpha > 0$; $z \in Z \in \mathscr{Q}_\alpha$ *with* $z' = \min \cup\mathscr{Q}'_\alpha$; $\gamma = z'$. (The assumption $\alpha > 0$ is only for convenience, since the case $\alpha < 0$ can be treated in the same way.)

From (†) there follows

5. 5. *Either* (i) $\cup\mathcal{Q}'_\alpha=\{\gamma, \gamma+\alpha\}$

or (ii) $\{\gamma, \gamma+2\alpha\}\subset Z'\subset\{\gamma, \gamma+\alpha, \gamma+2\alpha\}$ *and* $\cup(\mathcal{Q}_\alpha\sim\{Z\})'=\{\gamma+\alpha\}$.

The discussion is now divided into three cases, as follows:

$$\text{(A) } \gamma\geqq 0; \qquad \text{(B) } \gamma<0;\ \gamma+2\alpha\in\cup\mathcal{Q}'_\alpha; \qquad \text{(C) } \gamma<0;\ \gamma+2\alpha\notin\cup\mathcal{Q}'_\alpha.$$

By adding the appropriate letter to the number of a lemma, we indicate the addition of one of these three conditions to the standing hypotheses.

* * *

5. 6_A. *There exists $n\in N$ such that $\alpha=n\gamma$ and each member of $\mathcal{Q}_\alpha$ has the form $(\gamma)^n$. In particular, $\gamma>0$.*

Proof. Since $\gamma\geqq 0$ by condition (A), it follows from the definition of γ that all points of $\cup\mathcal{Q}_\alpha$ have non-negative weight. Consider two points x and y lying in different members of $\mathcal{Q}_\alpha$. The partition $\mathcal{Q}(x, y)$ is nice but is not troublesome if $x'\neq y'$, for then $\mathcal{Q}(x, y)^{*\prime}$ contains β_1 as well as three different non-negative weights, and one of the latter is $<\alpha\leqq-\beta_1$. This shows that $x'=y'$ and consequently $\cup\mathcal{Q}'_\alpha=\{\gamma\}$. The desired conclusions follow. ∎

5. 7_A. *If $Q\in\mathcal{Q}_{\beta_j}$, then $Q'\subset\{\gamma, \gamma-\alpha, \gamma-2\alpha, \gamma+\beta_j, \gamma+\beta_j-\alpha\}$.*

Proof. By 5. 1, $\max\cup\mathcal{Q}'_-\leqq\gamma$. Let $U_0=\{u\in Q: u'<\gamma\}$, and define the subsets U_i of U_0 by saying that if $u\in U_0$, then
$u\in U_1$ iff $\mathcal{Q}(u, z)$ is not nice;
$u\in U_2$ iff $\mathcal{Q}(u, z)$ is positively troublesome;
$u\in U_3$ iff $\mathcal{Q}(u, z)$ is negatively troublesome.
Obviously $U_0=U_1\cup U_2\cup U_3$. For $u\in U_0$, we have

$$\mathcal{Q}(u, z)^{*\prime} = B\cup\{\beta_i: i\neq j\}\cup\{\beta_j-u'+\gamma, \alpha-\gamma+u', \alpha\},$$

with $B\subset\{\beta_j\}$. Clearly $u\in U_1$ implies $u' = \gamma+\beta_j$ or $u' = \gamma-\alpha$. If $u\in U_2\cup U_3$, then $\alpha-\gamma+u'<0$, for otherwise $\mathcal{Q}(u, z)^{*\prime}$ contains the positive weights α and $\alpha-\gamma+u'$ with

$$\alpha-\gamma+u'<\alpha\leqq\min\{-\beta_i: 1\leqq i\leqq s\},$$

and $\mathcal{Q}(u, z)$ is not troublesome. If $u\in U_2$, then (since $\mathcal{Q}$ is a *minimal* α-partition) $\beta_j-u'+\gamma>0$, whence $\beta_j-u'+\gamma = \alpha$ and $u' = \gamma+\beta_j-\alpha$. If $u\in U_3$, then $\alpha-\gamma+u'= =\beta_i = -\alpha$ (for $i\neq j$, where in fact this situation entails $s=2$ and $r_j=1$). We have now proved that $U'_0\subset\{\gamma-\alpha, \gamma-2\alpha, \gamma+\beta_j, \gamma+\beta_j-\alpha\}$. ∎

5. 8_A. *W is troublesome when $n\geqq 2$.*

Proof. Use 5. 6, 5. 7, and the fact that

$$\max\{\gamma-\alpha, \gamma-2\alpha, \gamma+\beta\quad \gamma+\beta_j-\alpha\} = \gamma-\alpha = (1-n)\gamma. ∎$$

5. 9_A. *With $Q\in\mathcal{Q}_{\beta_j}$, let a, b, and c denote the number of points of Q which are of weight γ, $\gamma-\alpha$, and $\gamma-2\alpha$ respectively. Let $d=0$ if $\beta_j\in\{-\alpha, -2\alpha\}$ and otherwise $d=$*card *$\{u\in Q: u' = \gamma+\beta_j\}$. Let $e=0$ if $\beta_j = -\alpha$ and otherwise $e=$*card *$\{u\in Q: u' = = \gamma+\beta_j-\alpha\}$. Then one of the following statements is true:*

(i) $d=1, c=e=0;\ a+1 = (n-1)b;$
(ii) $e=1, c=d=0;\ a+1 = (n-1)b+n;$
(iii) $d=e=0;\ (a-(n-1)b-(2n-1)c) = \beta_j.$

Proof. Clearly

$$\gamma(a+b+c+d+e)-\alpha(b+2c+e)+\beta_j(d+e) = \mu(Q) = \beta_j,$$

and since $\alpha=n\gamma$ it follows that

$$(a+(1-n)b+(1-2n)c+d+(1-n)e)\gamma = (1-d-e)\beta_j.$$

To gain more information about the numbers $a, \ldots, e$, we consider the partition $\mathcal{Q}_{(u,v)}$ which is obtained from $\mathcal{Q}$ by interchanging two points u and v of Q with two points which lie in different members of $\mathcal{Q}_\alpha$. Then

$$\mathcal{Q}^{*\prime}_{(u,v)} = B_j \cup A(u,v) \cup \{\alpha\},$$

where

$$\{\beta_i : i \neq j\} \subset B_j \subset \{\beta_i : 1 \leqq i \leqq s\}$$

and

$$A(u,v) = \{\beta_j - u' - v' + 2\gamma, \alpha-\gamma+u', \alpha-\gamma+v'\}.$$

The possibilities of special interest are described in the following table:

	u'	v'	$A(u,v)$
$(d\geqq 2)$	$\gamma+\beta_j$	$\gamma+\beta_j$	$\{-\beta_j, \alpha+\beta_j\}$
$(e\geqq 2)$	$\gamma+\beta_j-\alpha$	$\gamma+\beta_j-\alpha$	$\{2\alpha-\beta_j, \beta_j\}$
$(d\geqq 1\leqq e)$	$\gamma+\beta_j$	$\gamma+\beta_j-\alpha$	$\{\alpha-\beta_j, \alpha+\beta_j, \beta_j\}$.
$(c\geqq 1\leqq d)$	$\gamma-2\alpha$	$\gamma+\beta_j$	$\{2\alpha, -\alpha, \alpha+\beta_j\}$
$(c\geqq 1\leqq e)$	$\gamma-2\alpha$	$\gamma+\beta_j-\alpha$	$\{3\alpha, -\alpha, \beta_j\}$.

Recalling that $d\neq 0$ implies $\beta_j \notin \{-\alpha, -2\alpha\}$, we see that $\mathcal{Q}_{(u,v)}$ is nice in each case and hence must be troublesome. In the first case, $\mathcal{Q}_{(u,v)}$ cannot be positively troublesome since $-\beta_j \neq \alpha$ and cannot be negatively troublesome since $0 > \alpha+\beta_j \neq -\alpha$. In the second case, $Q_{(u,v)}$ cannor be positively troublesome since $0<\alpha<2\alpha-\beta_j$ and cannot be negatively troublesome since (with $e\neq 0$) $\beta_j < -\alpha$. Similar contradictions ensue in the other three cases. It follows that $d+e\leqq 1$, and $d+e=1$ implies $c=0$, whence the remaining possibilities for $a, \ldots, e$ are exactly as described in 5. 9. ∎

5. 10_A. *If $n=1$, each member of $\mathcal{Q}_{\beta_j}$ has the form $(\beta_j)^1$ or the crude form $(\gamma)^a(-\gamma)^c$. Thus W is troublesome.*

Proof. With $n=1$, 5. 9 (i) is impossible, 5. 9 (ii) implies $a=0$, and 5. 9 (iii) becomes $(a-c)\gamma=\beta_j$. The corresponding possibilities for the crude form of $Q\in\mathcal{Q}_{\beta_j}$ are $(0)^b\ (\beta_j)^1$ and $(\gamma)^a(0)^b(-\gamma)^c$; to establish 5. 10 we must prove $b=0$. Suppose $b>0$ and let $u\in Q$ with $u'=0$. Then $\mathcal{Q}(u,z)^{*\prime} = B_j \cup \{\beta_j+\gamma, 0, \gamma\}$, so $\mathcal{Q}(u,z)$ is not troublesome and hence not nice. Since $u\in\{u\}\in\mathcal{Q}(u,z)$, the fact that $u'=0$ does not account for $\mathcal{Q}(u,z)$'s lack of niceness, and it follows that $\beta_j = -\gamma$. Thus $\mu(Q) = -\gamma$, Q contains a point v with $v' = -\gamma$, and $\mathcal{Q}^{*\prime}_{(u,v)} = B_j \cup \{2\gamma, 0, -\gamma, \gamma\}$. Since 0 appears only as the weight of a onepointed member of $\mathcal{Q}_{(u,v)}$, $\mathcal{Q}_{(u,v)}$ is nice but not troublesome. The contradiction implies $b=0$. ∎

5.11$_A$. *Suppose $n=1$ and there exists $Q\in\mathcal{Q}_-$ with card $Q>1$. Then Q has the crude form $(\gamma)^a(-\gamma)^c$ for $c\in\{a+1, a+2, a+3\}$ and each member of $\mathcal{Q}_-\sim\{Q\}$ has the form $(-\gamma)^1$.*

Proof. Clearly $c\geqq a+1$, for $(c-a)\gamma = \mu(Q)<0$. Suppose $\mu(Q)=\beta_j$ and let $v_1\in Q$ with $v_1' = -\gamma$. Then $\mathcal{Q}(v_1, z)^{*\prime} = B_j\cup\{(a-c+2)\gamma, -\gamma, \gamma\}$. If $\mathcal{Q}(v_1, z)$ is not nice, $a-c+2 = 0$. If $\mathcal{Q}(v_1, z)$ is positively troublesome, then (since $\mathcal{Q}$ is a minimal α-partition) $a-c+2 = 1$. If $\mathcal{Q}(v_1, z)$ is negatively troublesome, then $a-c+2 = -1$ when $a-c+2<0$ and $-1\geqq -a+c-2$ when $a-c+2>0$. It follows that $c-a\in\in\{1, 2, 3\}$, with $c = a+3$ only when $\mathcal{Q}(v_1, z)$ is negatively troublesome.

Now suppose $c = a+3$ and let v_2 and v_3 be distinct points of $Q\sim\{v_1\}$ such that $v_2'= v_3' = -\gamma$. Suppose some member P of $\mathcal{Q}_-\sim\{Q\}$ has form other than $(-\gamma)^1$. Since $\mathcal{Q}(v_1, z)$ is negatively troublesome, it is evident that $\mu(P) = -\gamma$ and hence (using 5.10) if P does not have the form $(-\gamma)^1$ there exists $w\in P$ with $w'=\gamma$. But then

$$\mathcal{Q}_{(v_1, v_2)}(v_3; w)^{*\prime} = B\cup\{-3\gamma, 3\gamma, -\gamma, \gamma\}$$

with $B\subset\{\beta_i: 1\leqq i\leqq 1\}$, whence the partition $\mathcal{Q}_{(v_1, v_2)}(v_3, w)$ is nice but not troublesome and the contradiction shows that P has the form $(-\gamma)^1$.

Alternatively, suppose $c-a\in\{1, 2\}$ and note that $c\geqq 2$ since card $Q>1$. Suppose some member P of $\mathcal{Q}_-\sim\{Q\}$ has form other than $(-\gamma)$.^{1} With $v_1, v_2\in Q$ and $v_1'=v_2' = -\gamma$, we have

$$\mathcal{Q}^{*\prime}_{(v_1, v_2)} = B\cup\{\mu(P), (a-c+4)\gamma, -\gamma, \gamma\},$$

so $\mathcal{Q}_{(v_1, v_2)}$ is nice and the fact that it is troublesome implies $\mu(P) = -\gamma$. Thus there exists $w\in P$ with $w'=\gamma$ and we have

$$\mathcal{Q}(v_1, z)(v_2, w)^{*\prime} = B\cup\{-3\gamma, (a-c+4)\gamma, -\gamma, \gamma\},$$

a contradiction which yields the desired conclusion. ∎

5.12$_A$. *If all the n_i's have the same value $n\geqq 2$, each member of $\mathcal{Q}_{\beta_j}$ has one of the following crude forms:*

$(\beta_j\notin\{-\alpha, -2\alpha\})$	(i) $(\gamma)^{n-2}(\gamma-\alpha)^1(\gamma+\beta_j)^1$;	
$(\beta_j\neq -\alpha)$	(ii) $(\gamma)^{n-1}(\gamma+\beta_j-\alpha)^1$;	
$(\beta_j=-\alpha)$	(iii) $(\gamma)^{n-2}(\gamma-\alpha)^2$;	(iv) $(\gamma)^{n-1}(\gamma-2\alpha)^1$;
$(\beta_j=-2\alpha)$	(v) $(\gamma)^{n-3}(\gamma-\alpha)^3$;	(vi) $(\gamma)^{n-2}(\gamma-\alpha)^1(\gamma-2\alpha)^1$;
$(\beta_j=-3\alpha)$	(vii) $(\gamma)^{n-4}(\gamma-\alpha)^4$;	(viii) $(\gamma)^{n-3}(\gamma-\alpha)^2(\gamma-2\alpha)^1$;
	(ix) $(\gamma)^{n-2}(\gamma-2\alpha)^2$.	

If some member Q of $\mathcal{Q}_-$ has the crude form (vii), (viii), *or* (ix), *then $\mathcal{Q}_-\sim\{Q\}$ is nonempty and all its members have the crude form* (iii) *or* (iv).

Proof. Here 5.9 (i) becomes

$$n = a+b+d = (n-1)b-1+b+1 = nb,$$

whence $b=1$, $a=n-2$, and Q has the crude form (i) above. And 5. 9 (ii) becomes

$$n = a+b+e = (n-1)b+n-1+b+1 = nb+n,$$

whence $b=0$, $a=n-1$, and Q has the form (ii) above.

For 5. 9 (iii) we have $n = a+b+c$ (and of course $\alpha=n\gamma$), so $\beta_j = (1-b-2c)\alpha$. Now with $g\leqq b$, $h\leqq c$, and $g+2h>1$, let $\mathscr{R}_{(g,h)}$ denote a partition which is obtained from $\mathcal{Q}$ by interchanging g points of weight $\gamma-\alpha$ in Q and h points of weight $\gamma-2\alpha$ in Q with $g+h$ points of weight γ in a single member of $\mathcal{Q}_\alpha$. (When all n_i's have the same value, such an interchange is possible.) Then

$$\mathscr{R}^{*\prime}_{(g,h)} = B_j\cup\{\beta_j+(g+2h)\alpha, (1-g-2h)\alpha, \alpha\}.$$

Note that $1-g-2h<0$. Thus if $\mathscr{R}_{(g,h)}$ is not nice, $\beta_j = -(g+2h)\alpha$ and $g+2h= =b+2c-1$. If $\beta_j+(g+2h)\alpha<0$, then (since $\mathcal{Q}$ is a minimal α-partition) $\mathscr{R}_{(g,h)}$ is negatively troublesome and

$$\beta_j+(g+2h)\alpha = (1-g-2h)\alpha = -\alpha,$$

whence $g+2h = 2$, $\beta_j = -3\alpha$, and $b+2c = 4$. If $\beta_j+(g+2h)\alpha>0$, then $g+2h = =b+2c$ when $\mathscr{R}_{(g,h)}$ is positively troublesome, while negative troublesomeness of $\mathscr{R}_{(g,h)}$ implies

$$-\beta_j-(g+2h)\alpha\leqq(1-g-2h)\alpha = -\alpha,$$

whence $g+2h = 2$ and $-\beta_j\leqq\alpha$. But then $\beta_j = -\alpha$ and $b+2c = 2$.

The preceding paragraph shows that if $Q(\in\mathcal{Q}_-)$ has the crude form $(\gamma)^{n-b-c}\cdot(\gamma-\alpha)^b(\gamma-2\alpha)^c$, the pair (b, c) must be such that $b+2c>1$, and such that whenever $g\leqq b$, $h\leqq c$, and $1<g+2h<b+2c$, then $g+2h = b+2c-1$, or $g+2h = 2$ and $b+2c = 4$. It is obvious that $b\leqq 4$ and $c\leqq 2$, and a closer examination shows that

$$(b, c)\in\{(0, 1), (0, 2), (1, 1), (2, 0), (2, 1), (3, 0), (4, 0)\},$$

whence Q has one of the crude forms (iii)—(ix). Note also that if $(b, c)\in\{(0, 2), (2, 1), (4, 0)\}$, then $\mu(Q) = -3\alpha$ and there exist g and h as described for which $\mu(Q)+(g+2h)\alpha<0$. But then $\mathscr{R}_{(g,h)}$ is negatively troublesome, whence $\mathcal{Q}_-\sim\{Q\}$ is nonempty and all members of $\mathcal{Q}_-\sim\{Q\}$ have weight $-\alpha$. ∎

* * *

We consider now the case in which condition (B) is satisfied. In this case, $\gamma<0$ and a single member Z of $\mathcal{Q}_\alpha$ contains points of weights γ and $\gamma+2\alpha$ (perhaps also $\gamma+\alpha$), while the other members of $\mathcal{Q}_\alpha$ consist exclusively of points of weight $\gamma+\alpha$.

5. 13_{B}. $\alpha = -2\gamma$. *Z has the form $(\gamma)^1(-3\gamma)^1$, while all other members of $\mathcal{Q}_\alpha$ have the form $(-\gamma)^2$. All members of $\mathcal{Q}_-$ have the form (γ) with at most one exception, and the exceptional member Q (if there is one) has the form $(\gamma)^1(3\gamma)^1$ or the form $(\gamma)^4$. If there is such a Q, then $\mathcal{Q}_-\sim\{Q\}$ is nonempty.*

Proof. Note the existence of $n\in N$ such that each member of $\mathcal{Q}_\alpha\sim\{Z\}$ has the form $(\gamma+\alpha)^n$; with $\alpha=n(\gamma+\alpha)$, we have $(1-n)\alpha=n\gamma$ and thus $n\geqq 2$.

Let $x, z\in Z$ and $y\in\mathfrak{u}\mathcal{Q}_\alpha\sim Z$ with $x' = \gamma+2\alpha$, $y'=\gamma+\alpha$, and $z'=\gamma$. For each $u\in\mathfrak{u}\mathcal{Q}_{\beta_j}\subset\mathfrak{u}\mathcal{Q}$, let $\mathcal{Q}_{(u)}$ denote the partition which results from $\mathcal{Q}$ under cyclic permutation of u, x, and y (replacing x by u, y by x, and u by y). Then

$$\mathcal{Q}^{*\prime}_{(u)} = B\cup\{\beta_i : i\neq j\}\cup\{\beta_j-u'+\gamma+\alpha, -\gamma-\alpha+u', 2\alpha, \alpha\}$$

with $B\subset\{\beta_j\}$, so $\mathcal{Q}_{(u)}$ is not positively troublesome. Thus $\mathrm{u}\mathcal{Q}'_- = \bigcup_{1\leqq j\leqq s}(U_j^1\cup U_j^2)$, where U_j^1 is the set of all $u\in\mathrm{u}\mathcal{Q}_{\beta_j}$ for which $\mathcal{Q}_{(u)}$ is negatively troublesome and U_j^2 is the set of all $u\in\mathrm{u}\mathcal{Q}_{\beta_j}$ for which $\mathcal{Q}_{(u)}$ is not nice.

Now we claim that for $1\leqq j\leqq s$,

(1) $$(U_j^2)'=\{\gamma+\beta_j+\alpha\}$$

and

(2) $$(U_j^1)'=\{\gamma\}.$$

The statement (1) is immediate from the definition of U_j^2 and the form of $\mathcal{Q}^*_{(u)}$, since (by 5. 1) $-\gamma-\alpha+u'\leqq-\alpha$ for all $u\in\mathrm{u}\mathcal{Q}_-$. Now suppose $u\in U_j^1$. Since $\alpha\in\mathcal{Q}^{*\prime}_{(u)}$, it follows that $-\gamma-\alpha+u'=-\alpha$, whence $u'=\gamma$ and $\beta_j+\alpha\in\mathcal{Q}^{*\prime}_{(u)}$. This establishes (2) and (continuing with the assumption that $u\in U_j^1$) since $\beta_j+\alpha\leqq 0$ we see that $\beta_j+\alpha=-\alpha$, whence it follows that $\beta_j=-2\alpha$, $r_j=1$, $\beta_i=-\alpha$ for $i\neq j$, and $s=2$.

If $u\in Q\cap U_j^1$ (where $Q\in\mathcal{Q}_{\beta_j}$), the above reasoning shows that $u'=\gamma$ and there exists $k\in N$ such that each member of $\mathcal{Q}_-\sim\{Q\}$ has the form $(\gamma)^k$, with $k\gamma=-\alpha$. Recalling that $n\gamma=(1-n)\alpha$, we see that $k=n/(n-1)$, whence $n=2=k$ and $\alpha=-2\gamma$. Thus each member of $\mathcal{Q}_\alpha\sim\{Z\}$ has the form $(-\gamma)^2$ while each member of $\mathcal{Q}_-\sim\{Q\}$ has the form $(\gamma)^2$. We want to show that Z has the form $(\gamma)^1(-3\gamma)^2$ while Q has the form $(\gamma)^1(3\gamma)^1$ or the form $(\gamma)^4$.

From (1) and (2) we know that Q consists of a points of weight γ and b of weight 3γ, with $a\gamma+3b\gamma=\mu(Q)=-2\alpha=4\gamma$. Hence $a=1$ and $b=1$ or $a=4$ and $b=0$; in either case, Q has the desired form. A simple interchange shows that if some such Q occurs with $\mathcal{Q}_-=\{Q\}$, then W admits a partition $\mathcal{S}$ for which $\mathcal{S}^*$ has the form $(2\gamma)^2(-2\gamma)^c(-4\gamma)$, an impossibility since $\mathcal{S}$ is nice but not troublesome.

We know that Z consists of c points of weight γ, d of weight $-\gamma$, and e of weight -3γ, with $c\geqq 1\leqq e$. Now suppose $e\geqq 2$ $\langle$resp. $d\geqq 1\rangle$ and let $\mathcal{R}$ denote the partition which is obtained from $\mathcal{Q}$ by interchanging two points of weight -3γ $\langle$resp. one of weight -3γ and one of weight $-\gamma\rangle$ from Z with the two points of weight γ from a single member of $\mathcal{Q}_-\sim\{Q\}$. Then $\mathcal{R}^{*\prime}=\{4\gamma,2\gamma,-2\gamma\}\cup A$, where $A=\{-6\gamma,6\gamma\}$ $\langle$resp. $A=\{-4\gamma,4\gamma\}\rangle$. Thus $\mathcal{R}$ is nice but not troublesome, and the contradiction implies that $d=0$ and $e=1$, whence $c=1$ and Z has the desired form.

For the proof of 5. 13, it remains to consider the case in which $\bigcup_{1\leqq j\leqq s} U_j^2=\mathrm{u}\mathcal{Q}'_-$, whence (for all j) each member of $\mathcal{Q}_{\beta_j}$ has the form $(\gamma+\beta_j+\alpha)^{k_j}$ for some $k_j\in N$. With $(1-k_j)\beta_j=k_j(\gamma+\alpha)$ and $\alpha=n(\gamma+\alpha)$, we have $n(1-k_j)\beta_j=k_j\alpha$. Now for $u\in\mathrm{u}\mathcal{Q}_{\beta_j}$, note that

$$\mathcal{Q}(u,z)^{*\prime}=B\cup\{\beta_i:i\neq j\}\cup\{-\alpha,\beta_j+2\alpha,\alpha\}$$

with $B\subset\{\beta_j\}$. If $\mathcal{Q}(u,z)$ is not nice, then $\beta_j=-2\alpha$ and $2n=k_j/(k_j-1)$. This implies $k_j=2$ and $n=1$, whence $\gamma=0$ in contradiction of our basic assumption that $\gamma<0$. If $\beta_j+2\alpha<0$, then (since $\mathcal{Q}$ is a minimal α-partition) $Q(u,z)$ must be negatively troublesome, whence $\beta_j+2\alpha=-\alpha$ and $3n=k_j/(k_j-1)$, an impossibility. Suppose, finally, that $\beta_j+2\alpha>0$. If $\mathcal{Q}(u,z)$ is positively troublesome, then $\beta_j+2\alpha=\alpha$, while negative troublesomeness of $\mathcal{Q}(u,z)$ implies $-\alpha\geqq-\beta_j-2\alpha$. But we know already that $\beta_j\leqq-\alpha$, so both possibilities imply $\beta_j=-\alpha$. From this it follows that $n=k_j/(k_j-1)$, whence $k_j=2$, $n=2$, and $\mathrm{u}\mathcal{Q}'_-=\{\gamma\}$. Thus each member of $\mathcal{Q}_-$ has the form $(\gamma)^2$ while each member of $\mathcal{Q}_\alpha\sim\{Z\}$ has the form

$(-\gamma)^2$. The argument of the preceding paragraph shows that Z has the form $(\gamma)^1(-3\gamma)^1$, and this completes the proof of 5. 13. ▮

* * *

We turn finally to the case in which condition C is satisfied.

5. 14_C. *Each set $X\in Q_\alpha$ has the crude form $(\gamma)^{a(X)}(\gamma+\alpha)^{b(X)}$ with $(a(X)+b(X))\gamma = (1-b(X))\alpha$ and $b(X)\geqq 2$. Of course $a(Z)\geqq 1$.*

Proof. With $a\gamma+b(\gamma+\alpha) = \mu(X) = \alpha$, the equality $(a+b)\gamma = (1-b)\gamma$ is immediate; $b\geqq 2$ because $\gamma<0<\alpha$. Further, $z\in Z$ with $z'=\gamma$ (as part of the standing hypotheses). ▮

5. 15_C. *Suppose $a(Z_i)\geqq 1$ for at least two different members Z_1 and Z_2 of $\mathcal{Q}_\alpha$, or $a(Z_3)\geqq 2$ for some $Z_3\in\mathcal{Q}_\alpha$. Then $\mathcal{Q}$ is doubly troublesome,* card $\mathcal{Q}_-\geqq 2$, $\amalg\mathcal{Q}_-=\{\gamma\}$, *and W admits a nice $(n_1, \ldots, n_m)$-partition which is neither doubly nor singly troublesome.*

Proof. In the first instance, let $z_1\in Z_1$ and $z_2\in Z_2$, and in the second $z_1, z_2\in Z_3$, with $z_1'=\gamma=z_2'$ in each case. Let Y be a member of $\mathcal{Q}_\alpha$ different from the Z_i's, and $y_1, y_2\in Y$ with $y_1' = \gamma+\alpha \doteq y_2'$.

Let $\mathcal{R}=\mathcal{Q}(y_1, z_1)(y_2, z_2)$, whence

$$\mathcal{R}^{*\prime} = \{\beta_i: 1\leqq i\leqq s\}\cup\{-\alpha\}\cup A$$

with $\{2\alpha\}\subset A\subset\{\alpha, 2\alpha\}$ or $A=\{\alpha, 3\alpha\}$. In the first case, 2α appears as the weight of two different members of $\mathcal{R}$, so in neither case is $\mathcal{R}$ doubly or singly troublesome. On the other hand, $\mathcal{R}$ is nice and hence troublesome, which can happen only if $\beta_i = -\alpha$ for all i (whence $\mathcal{Q}$ is doubly troublesome) and card $\mathcal{Q}_-\geqq 2$.

Now suppose $u\in\amalg\mathcal{Q}_-$ with $u'\neq\gamma$. Then $u'<\gamma$ by 5. 1, whence $\alpha-\gamma+u'<\alpha$. Since

$$\mathcal{Q}(u, z_1) = \{-\alpha, -\alpha-u'+\gamma, \alpha-\gamma+u', \alpha\},$$

it follows easily that $\alpha-\gamma+u' = 0$ or $\alpha-\gamma+u' = -\alpha$. Now if a member of $\mathcal{Q}_{-\alpha}$ contains a point of weight $<\gamma-\alpha$, or two points of weight $\leqq\gamma-\alpha$, then by interchanging these with points of weight $\gamma+\alpha$ in a single member of $\mathcal{Q}_\alpha$ we obtain a nice partition whose members have weights $<-\alpha$, $=-\alpha$, $=\alpha$, and $>\alpha$. Since this is impossible, we conclude that each member of $\mathcal{Q}_{-\alpha}$ has the crude form $(\gamma-\alpha)^c(\gamma)^d$ with $c\in\{0, 1\}$. But then $(c+d)\gamma-c\alpha = -\alpha$, so $c=1$ implies $\gamma=0$. This contradiction completes the proof. ▮

5. 16_C. *W is troublesome.*

Proof. By 5. 14, W contains at least six points of weight $\gamma+\alpha$. By 5. 1, $w'\leqq\gamma$ whenever $w\in W$ with $w'<\gamma+\alpha$. Thus W is surely troublesome if $\gamma+\alpha\leqq-\gamma$. Suppose, on the other hand, that $\alpha>2(-\gamma)$. Then for each $X\in\mathcal{Q}_\alpha$, $(a(X)+b(X))> >2(b(X)-1)$, whence $a(X)\geqq 1$. It then follows from 5. 15 that W has the form $(\gamma)^e(\gamma+\alpha)^f$ with $\gamma<0<\gamma+\alpha$ and $e\geqq 3\leqq f$, so of course W is troublesome. ▮

5. 17_C. *If all n_i' s have the same value n, then $n\geqq 3$; each member of $\mathcal{Q}_-$ has the form $(\gamma)^n$ while each member of $\mathcal{Q}_\alpha$* has the form $(\gamma)^{n-2}(\gamma+\alpha)^2$.

Proof. For each $X\in\mathcal{Q}_\alpha$ we have $a(X)+b(X) = n$, whence (by 5. 14) $(1-b(X))\,\alpha=n\gamma$ and $b(X)$ has the same value for all $X\in\mathcal{Q}_\alpha$. Thus the same is also true of $a(X)$, and 5. 15 applies to show that $-\alpha = n\gamma$. But then $1-b(X) = -1$ and the desired conclusions follow. ∎

6. Troublesome sets: Theorems. The results of this section are based on the lemmas of Section 5.

6. 1. Theorem. *Suppose W is a weighted set and $n_1, \ldots, n_m \in N$ with $m \geqq 4$ and $\sum_1^m n_i = \text{card } W$. If all nice $(n_1, \ldots, n_m)$-partitions of W are troublesome, then either W itself is troublesome or all n_i's are equal to 2 and W has the form $(-3\gamma)^1(-\gamma)^{4+2a}(\gamma)^{4+2b}(3\gamma)^1$ for some $\gamma \in \Gamma \sim \{0\}$ and $a, b \in N \cup \{0\}$.*

Proof. If W admits no nice $(n_1, \ldots, n_m)$-partition, 2. 1 implies that W is troublesome. Suppose, then, that W admits a nice $(n_1, \ldots, n_m)$-partition, and let $\mathcal{Q}$ be a minimal α-partition of W as described in Section 5. Referring to 5. 8, 5. 10, 5. 13, and 5. 16, we see that W can fail to be troublesome only if $\mathcal{Q}$ satisfies the condition (B). By 5. 13, the only non-troublesome possibility for this case is that described above. ∎

It would be interesting to have an intrinsic *characterization* of those weighted sets W and m-tuples $(n_1, \ldots, n_m$ such that all nice $(n_1, \ldots, n_m)$-partitions of W are troublesome. (This is not provided by 6. 1, for a troublesome set may admit nice partitions which are not troublesome). Relevant information is supplied by 5. 6_A, 5. 7_A, 5. 9_A, 5. 11_A, 5. 12_A, 5. 13_B, 5. 15_C and 5. 17_C. The picture is complete for condition (B) and could probably be completed without difficulty for (C), but the case of (A) seems more complicated. We have a complete solution only when all the n_i's have the same value. For $m \geqq 4$ and $n \geqq 2$, let $\mathfrak{T}(m, n)$ denote the class of all weighted sets W of cardinality mn such that all nice n-partitions of W are troublesome. Let $\mathfrak{T}_N(m, n)$ denote the class of all $W \in \mathfrak{T}(m, n)$ such that W admits no nice n-partition, and for $D \in \{A, B, C\}$ let $\mathfrak{T}_D(m, n)$ denote the class of all $W \in \mathfrak{T}(m, n)$ such that for some $\alpha \in \Gamma \sim \{0\}$, W admits a minimal α-partition which satisfies condition (D). Then

$$\mathfrak{T}(m, n) = \mathfrak{T}_N(m, n) \cup \mathfrak{T}_A(m, n) \cup \mathfrak{T}_B(m, n) \cup \mathfrak{T}_C(m, n).$$

The class $\mathfrak{T}_N(m, n)$ is completely described in 2. 1, and the other classes are described in the following result.

6. 2. Theorem. *Suppose $m \geqq 4$, $n \geqq 2$, and W is a weighted set of cardinality mn. Then*

(a) $W \in \mathfrak{T}_A(m, n)$ iff W has the crude form

$$(\gamma)^{kn+a(n-2)+b(n-1)+c(n-2)+d(n-1)+e(n-3)+f(n-2)+g(n-4)+h(n-3)+i(n-2)}\ldots$$

$$\ldots(\gamma-\alpha)^{a+2c+3e+f+4g+2h}(\gamma-2\alpha)^{d+f+h+2i}(\gamma+\beta_0)^a(\gamma+\beta_1-\alpha)^1\ldots(\gamma+\beta_b-\alpha)^1$$

for some $\gamma \in \Gamma \sim \{0\}$, $\alpha = n_\gamma$, β_j of opposite sign from α but of greater absolute value $(0 \leqq j \leqq b)$, $\beta_0 \neq -2\alpha$, $3 \leqq k < m$, and $a, b, \ldots h, i \in N \cup \{0\}$ with $e > 0 \Rightarrow n \geqq 3$, $g > 0 \Rightarrow n \geqq 4$, $h > 0 \Rightarrow n \geqq 3$, and one of the following four conditions satisfied:

(a_1) $0 = e = f = h = i$, $a = 1$;
(a_2) $0 = a = g = h = i$, $b \geqq 1$, $e + f \leqq 1$;
(a_3) $0 = a = b = g = h = i$, $c + d \geqq 1$, $e + f \leqq 2$;
(a_4) $0 = a = b = e = f = 0$, $c + d \geqq 1$, $g + h + i = 1$;

(b) $W \in \mathfrak{T}_B(m, n)$ *iff* $n=2$ *and* W *has the form* $(-3\gamma)^1(-\gamma)^{4+2a}(\gamma)^{3+2b}$ *or the form* $(-3\gamma)^1(-\gamma)^{4+2a}(\gamma)^{4+2b}(3\gamma)^1$ *for some* $\gamma \in \Gamma \sim \{0\}$ *and* $a, b \in N \cup \{0\}$.

(c) $W \in \mathfrak{T}_C(m, n)$ *iff* $n \geqq 3$ *and* W *has the form* $(\gamma-\alpha)^{6+2a}(\gamma)^{5n-6+a(n-2)+bn}$ *for some* $\gamma \in \Gamma \sim \{0\}$, $\alpha = n\gamma$, *and* $a, b \in N \cup \{0\}$.

Proof. It is tedious but not difficult to verify that if W has one of the stated forms, then W is a member of the appropriate class $\mathfrak{T}_D(m, n)$. This task is left to the reader. That the members of $\mathfrak{T}_B(m, n)$ and $\mathfrak{T}_C(m, n)$ must have the indicated forms is an almost immediate consequence of 5. 13 and 5. 17 respectively, with a slight change of notation in the latter case and use of 5. 15 to show that card $\mathfrak{Q}_- \geqq 2$ when $\alpha > 0$. This takes care of (*b*) and (*c*). For (*a*) we use 5. 12, but some additional argument is necessary.

Let Q be as in 5. 12, whence $\mathfrak{Q}_\alpha$ consists of k sets of the form $(\gamma)^n$, a sets of the crude form (i) (for various $\beta_j \notin \{-\alpha, -2\alpha\}$), b sets of the form (ii) (for various $\beta_j \neq -\alpha$), c sets of the crude form (iii), ..., i sets of the crude form (ix), where $3 \leqq k < m$ and the designations (i)...(ix) refer to the statement of 5. 12. From 5. 12 it follows that if $g+h+i \geqq 1$, then $g+h+i = 1$, $a=b=e=f=0$, and $c+d \geqq 1$. And $e+f \leqq 2$ in any case, for if $e+f \geqq 3$ a simple interchange leads from $\mathfrak{Q}$ to another minimal α-partition of W for which $g+h+i \geqq 1$ and $e+f \geqq 1$, in contradiction of 5. 12. Note also that if $a \geqq 1$, then $a=1$ and $e=f=g=h=i=0$, for otherwise a simple interchange leads from $\mathfrak{Q}$ to another minimal α-partition one of whose members has a crude form other than those indicated in 5. 12. We now see further that if $b \geqq 1$, then $e+f \leqq 1$, for otherwise an interchange leads from $\mathfrak{Q}$ to another minimal α-partition for which $a \geqq 1$ and $e+f \geqq 1$. A review of the assembled facts shows that one of the four conditions $(a_1)-(a_4)$ must be satisfied. ∎

We next discuss weighted sets all of whose nice n-partitions are doubly or singly troublesome. While the discussion could be based on 6. 2, it will be simpler to apply the relevant lemmas.

6. 3. Theorem. *Suppose* $m \geqq 4$, $n \geqq 2$, *and* W *is a weighted set of cardinality* mn *which admits a nice n-partition. Then all nice n-partitions of W are doubly troublesome iff* W *has the crude form* $(\gamma)^{an+b(n-2)+c(n-1)}(\gamma-\alpha)^{2b}(\gamma-2\alpha)^c$ *for some* $\gamma \in \Gamma \sim \{0\}$, $\alpha = n\gamma$, *and* $a, b, c \in N \cup \{0\}$ *such that* $a+b+c = m$ *and one of the following additional restrictions is satisfied:*

$n=2$; $3 \leqq a < m$ *and* $c \leqq 1$ *or* $m \in \{4, 5\}$, $a = m-3$,; $b=2$ *and* $c=1$

$n=3$; $3 \leqq a < m$; $b=0$ *and* $c=1$ *or* $b \in \{1, 2\}$ *and* $c=0$;

$n \geqq 4$; $a = m-1$; $b=0$ *and* $c=1$ *or* $b=1$ *and* $c=0$.

Proof. The stated crude form for W is equivalent to W's being the union of a sets of the form $(\gamma)^n$, b of the crude form $(\gamma)^{n-2}(\gamma-\alpha)^2$, and c of the form $(\gamma)^{n-1}(\gamma-2\alpha)^1$. If a, b, and c are subject to the restrictions given above, it can be verified that all nice n-partitions of W are doubly troublesome.

Now suppose conversely that all nice n-partitions of W are doubly troublesome, and let $\mathfrak{Q}$ be a minimal α-partition of W. From 5. 15 it follows that $\mathfrak{Q}$ satisfies condition (*A*) or condition (*B*) of Section 5, whence 5. 12_A and 5. 13_B will apply. Since $\mathfrak{Q}$ is doubly troublesome, the form $(\gamma)^1(3\gamma)^1$ (for a member of $\mathfrak{Q}$) of 5. 13

is eliminated, as are all the forms mentioned in 5. 12 except for (iii) and (iv). Thus if 5. 12 holds, W clearly has the desired form with $3 \leqq a < m$ (but ignoring, for the moment, the restrictions on b and c). And with the aid of a simple substitution (the $-\gamma$ of 5. 13 being the γ of 6. 3), W as described under 5. 13 is seen to have one of the two forms listed above for $n=2$. It remains only to justify the restrictions on b and c.

If $c=2$, we may interchange two points of weight $\gamma-2\alpha$ in ${}_{\text{II}}\mathcal{Q}_{-\alpha}$ with two points of weight γ in a single member of $\mathcal{Q}_\alpha$ to obtain from $\mathcal{Q}$ a nice n-partition $\mathcal{R}$ of W for which $\mathcal{R}^{*\prime} \supset \{-3\alpha, \alpha\}$, contradicting the assumption that all nice n-partitions of W are doubly troublesome. If $n \geqq 3$ and $c \geqq 1 \leqq b$, a similar contradiction arises from an interchange involving one point of weight $\gamma-2\alpha$, two of weight $\gamma-\alpha$, and three of weight γ. If $n \geqq 3$ and $b \geqq 3$, then interchanging the two points of weight $\gamma-\alpha$ in one member of $\mathcal{Q}_{-\alpha}$ with points of weight γ in two other members of $\mathcal{Q}_{-\alpha}$ leads to a nice n-partition $\mathcal{S}$ with $\mathcal{S}^{*\prime} \supset \{-2\alpha, \alpha\}$, again an impossibility. Finally, if $n \geqq 3$ and $b \geqq 2$, a contradictory partition is obtained in a similar way by choosing the two points of weight γ from a single member of $\mathcal{Q}_{-\alpha}$. The stated restrictions have now been justified. ∎

Note that if $n \geqq 4$ and all nice n-partitions of W are doubly troublesome, then all are singly troublesome.

6. 4. Theorem. *Suppose $m \geqq 4$, $n \geqq 2$, and W is a weighted set of cardinality mn which admits a nice n-partition. Then all nice n-partitions of W are singly troublesome iff W has one of the following forms for some $\gamma \in \Gamma \sim \{0\}$, $\alpha = n\gamma$, δ and ε of opposite sign from α but $|\delta| > |\alpha|$ and $|\varepsilon| \geqq 2|\alpha|$:*

$$(\gamma)^{mn-1}(\gamma-\varepsilon)^1; \quad (\gamma)^{mn-2}(\gamma-\alpha)^1(\gamma-\delta)^1; \quad (\gamma)^{mn-2}(\gamma-\alpha)^2;$$

$$(\gamma)^{mn-3}(\gamma-\alpha)^3 \qquad \textit{(only for } n \geqq 3);$$

$$(3\gamma)^1(\gamma)^{mn-4}(-\gamma)^3 \qquad \textit{(only for } n=2).$$

Proof. Again, case (C) is eliminated by 5. 15. Under 5. 12_{A}, the forms (vii), (viii) and (ix) are eliminated by the fact that card $\mathcal{Q}_- = 1$ (since $\mathcal{Q}$ is singly troublesome). Combining the representations of (i) and (vi) and of (ii) and (iv), we see that W has one of the first four forms listed above.

Under 5. 13_{B}, W is seen to have the last form listed. Finally, it can be verified that if W has one of the five stated forms, then all nice n-partitions of W are singly troublesome. ∎

6. 5. Corollary. *Suppose $m \geqq 4$, $n \geqq 2$, and W is a weighted set of cardinality mn which admits a nice n-partition. Then all nice n-partitions of W are t-singly troublesome (for $t \in N$) iff W has one of the following forms for some $\gamma \in \Gamma \sim \{0\}$:*

$$(\gamma)^{mn-1}(\gamma-(1+t)n\gamma)^1; \quad (\gamma)^{mn-2}(\gamma-n\gamma)^1(\gamma-tn\gamma)^1;$$

$$(\gamma)^{mn-3}(\gamma-n\gamma)^3 \qquad \textit{(only for } n \geqq 3,\ t=2);$$

$$(3\gamma)^1(\gamma)^{mn-4}(-\gamma)^3 \qquad \textit{(only for } n=2,\ t=1).$$

With the aid of 2. 1, 6. 3, and 6. 5, it is possible to give a detailed description of sets of the form $\operatorname{aff}_l(\operatorname{aff}_m(\operatorname{aff}_n X))$. By way of illustration, we prove

6. 6. Theorem. *Suppose X is an affinely independent subset of E and* card $X=lmn$, *where $l, m, n \in N\sim\{1\}$. Then the cardinality of the set* $\mathrm{aff}_{lmn}X\sim\mathrm{aff}_l(\mathrm{aff}_m(\mathrm{aff}_n X))$ *is equal to $c(l, m, n)$ as given by the following formulae:*

when $n\geqq 3$ *and* $m\geqq 4$, $c(l, m, n)=lmn(lmn+1)$;

when $n\geqq 3$ *and* $m=3$, $c(l, 3, n) =\frac{1}{2} ln(9(ln)^2+9ln+8)$;

when $n\geqq 3$ *and* $m=2$, $c(l, 2n) =ln(2ln+3)$;

when $n=2$ *and* $m\geqq 3$, $c(l, m, 2)=2^{2lm-2}+(2lm)^2-f$, *where* $f=0$ *when* lm *is even and* $f=\frac{1}{2}\binom{2lm}{lm}$ *when* lm *is odd;*

when $n=2$ *and* $m=2$, $c(l, 2, 2)=2^{4l-2}+\sum_{i=1}^{l}\left(\binom{4l}{4i-2}+4l\binom{4l-1}{4i-1}\right)-g$, *where* $g=-280$ *when* $l=2$, $g=0$ *when* l *is even but* >2, *and* $g=\binom{4l}{2l}+4l\binom{4l-1}{2l-2}$ *when* l *is odd.*

Proof. Let A denote the set of all functions ξ on X to Φ such that $\sum_{x\in X}\xi x=1$, and for each $\xi\in A$ let X_ξ denote the weighted set $\{(x, \xi x): x\in X\}$. Then $c(l, m, n)$ is equal to card $B+$card C, where B is the set of all $\xi\in A$ such that X_ξ admits no nice n-partition and C is the set of all $\xi\in A\sim B$ such that for each nice n-partition $\mathscr{P}$ of X_ξ the weighted set $\mathscr{P}^*$ admits no nice m-partition. From 2. 1 and 3. 5 it follows that

$$\text{card } B=\begin{Bmatrix} lmn \\ 2^{2lm-2} \\ 2^{2lm-2}-\frac{1}{2}\binom{2lm}{lm}\end{Bmatrix} \text{ when } \begin{cases} n\geqq 3 \\ n=2 \text{ and } lm \text{ is even} \\ n=2 \text{ and } lm \text{ is odd.}\end{cases}$$

When $m\geqq 3$, the set C is determined by 6. 5 (with $t=m-1$) in conjunction with 2. 1, whence it is seen that

$$\text{card } C=\begin{Bmatrix} lmn+lmn(lmn-1 \\ lmn+lmn(lmn-1)+\binom{lmn}{3}\end{Bmatrix} \text{ when } \begin{cases} m\geqq 4 \text{ or } n=2 \\ m=3 \text{ and } n\geqq 3.\end{cases}$$

When $m=2$, the set C is determined by 6. 3 in conjunction with 2. 1. For $n\geqq 3$, we see that $\xi\in C$ iff X_ξ has the form $(\gamma)^{2ln-2}((1-n)\gamma)^2$, or the form $(\gamma)^{2ln-1}((1-2n)\gamma)^1$ (where $\gamma=1/(2ln-2)$), and it follows that

$$\text{card } C=\binom{2ln}{2}+\binom{2ln}{1}.$$

When $m=2=n$, the above considerations show that $\xi\in C$ iff X_ξ has the form $(\gamma)^{4l-2b}(-\gamma)^{2b}$ with b odd, $b\neq l$, and $1\leqq b\leqq 2l-1$, or the crude form $(\gamma)^{4-2b-1}$

$(-\gamma)^{2b}(-3\gamma)^1$ with b even, $b \neq l-1$, and $0 \leqq b \leqq 2l-2$ or (only when also $l=2$) the form $(\gamma)^3(-\gamma)^4(-3\gamma)^1$, where in each case the value of γ is determined by the fact that $\mu(X_\xi)=1$. Thus for $l \geqq 3$,

$$\text{card } C = \Sigma' \binom{4l}{2b} + 4l\Sigma'' \binom{4l-1}{2b},$$

where $'$ and $''$ indicate the appropriate range and restrictions for b, while for $l=2$, there must be added a term equal to $8\binom{7}{3}=280$. It can be verified that

$$\text{card } C = \sum_{i=1}^{l}\left(\binom{4l}{4i-2} + 4l\binom{4l-1}{4i-4}\right) + g,$$

where g is as described in the statement of 6. 6.

A review of the assembled facts shows that the value of $c(l, m, n)$ is indeed given by the stated formulae. ∎

We conclude with the following table:

l	m	n	$c(l, m, n)$	l	m	n	$c(l, m, n)$
2	2	2	688	3	2	2	3148
2	2	3	90	3	2	3	189
2	3	2	1168	3	3	2	41550
2	3	3	1158	3	3	3	3681

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Semi-Carleman operators*

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1. VON NEUMANN [1] showed in 1935 that a self-adjoint operator on Hilbert space may be represented on $L_2(-\infty, \infty; dx)$ as an integral operator of Carleman type if and only if 0 is a limit point of its spectrum. In this note we show that this result survives in the non-self-adjoint case. In so doing we are lead to the consideration of what we shall call *semi-Carleman* integral operators. They are operators T on $L_2(-\infty, \infty; dx)$, given by a kernel $K(x, y)$ by the relation

$$Tf(x) = \int_{-\infty}^{\infty} K(x, y) f(y)\, dy,$$

such that

$$\int_{-\infty}^{\infty} |K(x, y)|^2\, dx \equiv M^2(y) < \infty, \quad \text{a. e. in } y. \tag{1}$$

According to standard usage (see [2, p. 397]) Carleman integral operators have symmetric kernels $(K(y, x) = \overline{K(x, y)})$. We drop the requirement of symmetry. There is a natural choice of domain for such an operator making it closed and densely defined. We shall prove that such operators always have the point 0 as a limit point of their spectra, extending (and simplifying the proof of) [1, Theorem IV], and we shall obtain also a converse to this statement.

We are indebted to Dr. L. GROSS for a number of interesting and helpful conversations on this subject.

2. We shall say that a complex number λ is a limit point of the spectrum of an operator T if there exist unit vectors x_n $(n = 1, 2, \ldots)$ which converge weakly to 0 and such that

$$(T - \lambda) x_n \to 0.$$

(*Cf.* [5, n° 133].) Suppose that T is closed and densely defined, and that 0 is a limit point of its spectrum (which we are implicitly assuming is not empty). We know that we may express T in the form $T = U(T^*T)^{1/2}$ where $(T^*T)^{1/2}$ is self-adjoint and U is a partial isometry whose initial domain is the closure of the range of $(T^*T)^{1/2}$ [3, p. 53], and we claim that 0 is a limit point of $(T^*T)^{1/2}$. For if we have unit vectors x_n tending weakly to 0 such that $Tx_n \to 0$ then $\|(T^*T)^{1/2}x_n\| = \|U(T^*T)^{1/2}x_n\| = \|Tx_n\| \to 0$, so that $(T^*T)^{1/2}x_n \to 0$, as required. Hence by the

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von Neumann theorem [1, Theorem III] $(T^*T)^{1/2}$ is unitarily equivalent to an operator H of Carleman type on $L_2(-\infty, \infty; dx)$, and therefore T may be represented as a partial isometry times a Carleman operator. We can in fact say more, by following VON NEUMANN's method adapted to the present (non-self-adjoint) circumstances. By [1, Theorem I] there exists a self-adjoint operator X of arbitrarily small Hilbert—Schmidt norm such that $A = (T^*T)^{1/2}+X$ is a *pure point operator* (in the sense that it has a complete orthonormal set of eigenvectors) having 0 as a limit point of its spectrum. Thus there exists a complete orthonormal set $\{\varphi_n\}_{n=1}^{\infty}$ of vectors and real numbers $\{\lambda_n\}_{n=1}^{\infty}$ (not necessarily distinct) such that $A\varphi_n=\lambda_n\varphi_n$ $(n=1, 2, \ldots)$. We know that 0 is a limit point of $\{\lambda_n\}_{n=1}^{\infty}$.

If it happens that $\sum_{n=1}^{\infty}|\lambda_n|^2<\infty$, then, writing $\psi_n=U\varphi_n$, we may choose a basis $\{\tilde{\varphi}_n\}_{n=1}^{\infty}$ of $L_2(-\infty, \infty; dx)$ and, defining the unitary operator W by $W\varphi_n=\tilde{\varphi}_n$, we write $\tilde{\psi}_n=W\psi_n$, and finally we define $K(x, y)=\sum_n \lambda_n\tilde{\psi}_n(x)\bar{\tilde{\varphi}}_n(y)$. Since U is not unitary, but only the partial isometry from [Range $(T^*T)^{1/2}$] to [Range (T)], we cannot conclude that the family $\{\psi_n\}$ (and the same applies to $\{\tilde{\psi}_n\}$) is orthonormal. All we know is $\|\tilde{\psi}_n\|=\|\psi_n\|=\|U\varphi_n\|\leqq 1$. Nevertheless $\iint\tilde{\psi}_n(x)\bar{\tilde{\varphi}}_n(y)\bar{\tilde{\psi}}_m(x)\tilde{\varphi}_m(y)\,dxdy= =\delta_{nm}\|\tilde{\psi}_n\|^2\leqq\delta_{nm}$, which is to say that the functions $F_n(x, y)=\tilde{\psi}_n(x)\bar{\tilde{\varphi}}_n(y)$ are orthogonal and of norm $\leqq 1$ on the plane. Hence the series defining K is L_2 convergent on the plane and $\iint|K(x, y)|^2dxdy=(K, K)=\left(\sum\lambda_nF_n, \sum\lambda_mF_m\right)=\sum|\lambda_n|^2\|F_n\|^2\leqq \leqq\sum|\lambda_n|^2$. Hence K is a Hilbert—Schmidt kernel, and the operator B it determines has the property $B\varphi_n=\lambda_n\tilde{\psi}_n$ That is, $B=W\{UA\}W^{-1}$. (This argument, proving that UA has a representation on $L_2(-\infty, \infty)$ as a Hilbert—Schmidt integral operator, is slightly different from the usual argument (see [4, p. 35]) because of the perturbation X, so that U is not necessarily isometric on the range of $A=(T^*T)^{1/2}+X$ Note that the argument shows that *such a representation is achieved no matter what basis* $\{\tilde{\varphi}_n\}$ *is chosen in* $L_2(-\infty, \infty)$.) Hence $WTW^{-1}=B-WUXW^{-1}$. Now UX is of Hilbert—Schmidt type since X is (see [5, p. 157]), so that, as pointed out above, $WUXW^{-1}$ is an integral operator with a Hilbert—Schmidt kernel L. Hence WTW^{-1} is an integral operator of Hilbert—Schmidt type with (nonsymmetric) kernel $K-L$.

If $\sum|\lambda_n|^2=\infty$, and we know only that $\{\lambda_n\}_{n=1}^{\infty}$ has 0 as a limit point, then we employ the following rearrangement of $\{\lambda_n\}$ (in which we are following VON NEUMANN exactly). Let $|\lambda_{m_\nu}|\leqq\frac{1}{\nu}$ $(\nu=1, 2, \ldots)$, and let $\{\lambda_{n_\nu}\}$ be the remaining members of $\{\lambda_n\}_{n=1}^{\infty}$. Set $l(\nu, k)=m_{2^{k-2}(2\nu-1)}$ for $k=2, 3, \ldots$, and $l(\nu, 1)=n_\nu$. Then $|\lambda_{l(\nu,k)}|\leqq\frac{1}{2^{k-2}(2\nu-1)}\leqq\frac{1}{2^{k-2}}$ $(k=2, 3, \ldots)$, so that $\sum_k|\lambda_{l(\nu,k)}|^2<\infty$ for all $\nu=1, 2, \ldots$. Renumber the system so that $\nu=0, \pm1, \pm2, \ldots$. Define $U\varphi_n=\psi_n$. Choose a basis of uniformly bounded functions $\{\tilde{\varphi}_n\}_{n=1}^{\infty}$ of $L_2(0, 1; dx)$ and define a unitary operator W_1 by $W_1\varphi_n=\tilde{\varphi}_n$, and write $\tilde{\psi}_n=W_1\psi_n$. Now define, for $\nu=0, \pm1, \pm2, \ldots$,

$$\Phi_{\nu,n}(t)=\begin{cases}\tilde{\varphi}_n(t-\nu), & \nu\leqq t\leqq\nu+1\\ 0 & , \text{otherwise.}\end{cases}$$

Then $\{\Phi_{\nu,n}\}$ is a complete uniformly bounded orthonormal system in $L_2(-\infty, \infty; dx)$ and the map V: $L_2(0, 1; dx) \to L_2(-\infty, \infty; dx)$ defined as

$$V\tilde{\varphi}_{l(\nu,k)} = \Phi_{\nu,k}$$

is unitary. Let $\Psi_{\nu,k} = V\tilde{\psi}_{l(\nu,k)}$, and define kernels K_ν by the relation

$$K_\nu(x, y) = \sum_k \lambda_{l(\nu,k)} \Psi_{\nu,k}(x)\overline{\Phi_{\nu,k}(y)}. \tag{2}$$

Now K_ν is square integrable on the strip $(\nu \leqq y \leqq \nu+1, -\infty < x < \infty)$ because for each fixed ν, $\sum |\lambda_{l(\nu,k)}|^2 < \infty$ (see above), so K_ν defines a Hilbert—Schmidt integral operator T_ν from $L_2(\nu, \nu+1; dx)$ to $L_2(-\infty, \infty; dx)$, with the property that $T_\nu \Phi_{\nu,k} = \lambda_{l(\nu,k)} \Psi_{\nu,k}$. If we now write

$$K(x, y) = \sum_\nu K_\nu(x, y) \tag{3}$$

as we may since the summands are supported on disjoint strips, we have a kernel defined on the whole plane which defines an operator S such that $S\Phi_{\nu,k} = \lambda_{l(\nu,k)}\Psi_{\nu,k}$ for all ν, k. S is densely defined, since $\mathfrak{D}_S$ contains the linear span of the basis $\{\Phi_{\nu,k}\} = \{VW_1\varphi_{l(\nu,k)}\}$, UA is defined on the linear span of the basis $\{\varphi_{l(\nu,k)}\} = \{\varphi_n\}$, and we have clearly

$$VW_1[UA](VW_1)^{-1} = S$$

on these dense sets. Now $|K(x, y)|^2 = \sum_\nu |K_\nu(x, y)|^2$, $\int |K_\nu(x, y)|^2\, dx = \sum_k |\lambda_{l(\nu,k)}|^2 |\Phi_{\nu,k}(y)|^2 < \infty$ for a. e. y, and for each fixed y, $\sum_\nu \int |K_\nu(x, y)|^2\, dx = \int |K_{\nu_0}(x, y)|^2\, dx$ where $\nu_0 \leqq y \leqq \nu_0 + 1$, so that $\int |K(x, y)|^2\, dx < \infty$ for a. e. y. Thus UA has a representation on $L_2(-\infty, \infty; dx)$ as a semi-Carleman operator.

To summarize, we have $T = U[(T^*T)^{1/2} + X] - UX = UA - UX$, where X is self-adjoint of Hilbert—Schmidt type, and UA is representable on $L_2(-\infty, \infty; dx)$ as the semi-Carleman operator S above. Now, just as before, UX goes over by the same representation on $L_2(-\infty, \infty; dx)$ (i. e., VW_1) into an integral operator of Hilbert—Schmidt type. Hence, upon adding the kernels, we arrive at the following

Theorem 1. *If T is closed, $\mathfrak{D}_T$ is dense, and* 0 *is a limit point of the spectrum of T, then T may be represented on $L_2(-\infty, \infty; dx)$ by a semi-Carleman integral operator.*

We have noted above that operators of Hilbert—Schmidt type have kernels no matter what representation on L_2 is chosen, and this is true even if l_2 is chosen as the representation space (here the kernel is the matrix). We do not assert this invariance of representation for the more general operators considered in Theorem 1. Indeed, every bounded operator A has a representation on l_2 as a semi-Carleman operator, where the kernel is the matrix. For, denoting by $\{x_n\}_{n=-\infty}^{\infty}$ the usual basis in l_2, T has the matrix representation $((Tx_n, x_m))$, and $\sum_n |(Tx_n, x_m)|^2 = \sum_n |(x_n, T^*x_m)|^2 = \|T^*x_m\|^2 \leqq \|T\|^2$. But it is not true that every bounded operator has a semi-Carleman representation on L_2. The identity operator may be offered as a counterexample (as may be verified just as for Hilbert—Schmidt operators, but we shall not do it that way since the same conclusion will follow from our Theorem 2 below). Thus it is essential that we employ a non-atomic measure space in Theorem 1.

3. Suppose we are given a measurable function $K(x, y)$ defined on the whole plane and satisfying the semi-Carleman condition (1). Let us write (with M as defined in (1))

$$\mathfrak{D}=\left\{f\in L_2(-\infty, \infty; dx)\,\middle|\,\int M(x)\,|f(x)|\,dx<\infty\right\}.$$

Let $\sigma_n=\{x|M(x)\leqq n\}$ $(n=1, 2, \dots)$ and let $\alpha\subset\sigma_n$ be an arbitrary measurable set of finite positive measure. Then the characteristic function of α is in $\mathfrak{D}$. Suppose $\int g\bar{f}dx=0$ for all $f\in\mathfrak{D}$. Then $\int_\alpha g\,dx=0$, so that $g(x)=0$ for a. e. $x\in\sigma_n$. But the complement of $\bigcup_n\sigma_n$ has measure 0, whence $g(x)=0$ for a. e. x. Hence $\mathfrak{D}$ is dense in $L_2(-\infty, \infty; dx)$. (This is essentially the argument used in [2, p. 398] for Carleman operators, and we have included it for the sake of completeness.) If $f\in\mathfrak{D}$ then

$$\int\left|\int K(x, y)f(y)\,dy\right|^2 dx\leqq\iint dydz\,|f(y)|\,|f(z)|\int dx\,|K(x, y)|\,|K(x, z)|\leqq$$
$$\leqq\iint dydz\,|f(y)|\,|f(z)|\left(\int|K(x, y)|^2\,dx\right)^{1/2}\left(\int|K(x, z)|^2\,dx\right)^{1/2}=$$
$$=\iint dydz\,|f(y)|\,|f(z)|\,M(y)M(z)=\left[\int|f(u)|\,M(u)\,du\right]^2<\infty.$$

Hence the operator T given by $(Tf)(x)=\int K(x, y)f(y)\,dy$ for $f\in\mathfrak{D}$ is a densely defined semi-Carleman operator. Let $\sigma_n=\{x|0\leqq x\leqq 1,\ M(x)\leqq n\}$. By (1) the measure of $\bigcup_n\sigma_n$ is 1, so there exists n_0 such that $0<$ measure of $\sigma_{n_0}\leqq 1$. Then

$$\int_{\sigma_{n_0}}\int_{-\infty}^{\infty}|K(x, y)|^2\,dxdy=\int_{\sigma_{n_0}}dy\int_{-\infty}^{\infty}|K(x, y)|^2\,dx=\int_{\sigma_{n_0}}M(y)^2\,dy\leqq n_0^2,$$

so that we may regard K as an element of $L_2((-\infty, \infty)\times\sigma_{n_0})$ and $K^*(x, y)=\overline{K(y, x)}$ as an element of $L_2(\sigma_{n_0}\times(-\infty, \infty))$. As such, K and K^* define operators $S\colon L_2(\sigma_{n_0})\to L_2(-\infty, \infty)$ and $S^*\colon L_2(-\infty, \infty)\to L_2(\sigma_{n_0})$, respectively, of Hilbert—Schmidt type, with $N(S)\leqq n_0$, $N(S^*)\leqq n_0$, where N denotes the Hilbert—Schmidt norm. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a basis in $L_2(\sigma_{n_0})$. Then

$$\{N((S^*S)^{1/2})\}^2=\sum_n\|(S^*S)^{1/2}\varphi_n\|_{\sigma_{n_0}}^2=$$
$$=\sum_n(S^*S\varphi_n, \varphi_n)_{\sigma_{n_0}}=\sum\|S\varphi_n\|_{(-\infty,\infty)}^2=N(S)^2\leqq n_0^2,$$

where the subscripts indicate the norm employed. Hence $(S^*S)^{1/2}\colon L_2(\sigma_{n_0})\to L_2(\sigma_{n_0})$ is a self-adjoint Hilbert—Schmidt operator. Since Hilbert—Schmidt operators are completely continuous (see [4, p. 32]) we know there exists a set $\{\psi_n\}_{n=1}^{\infty}$ of unit vectors in $L_2(\sigma_{n_0})$, which are orthogonal (because $(S^*S)^{1/2}$ is self-adjoint) such that $(S^*S)^{1/2}\psi_n \to 0$ as $n\to\infty$. Let U: [Range $(S^*S)^{1/2}$] $\to$ [Range S] be the partial isometry in the polar decomposition of S: $S=U(S^*S)^{1/2}$. Then $\|S\psi_n\|_{(-\infty,\infty)}=$ $=\|U(S^*S)^{1/2}\psi_n\|_{(-\infty,\infty)}=\|(S^*S)^{1/2}\psi_n\|_{\sigma_{n_0}}\to 0$, so $S\psi_n\to 0$ in $L_2(-\infty,\infty)$. Define functions θ_n by

$$\theta_n(x)=\begin{cases}\psi_n(x), & x\in\sigma_{n_0},\\ 0, & x\notin\sigma_{n_0}.\end{cases}$$

Then θ_n is an orthonormal system in $L_2(-\infty,\infty)$. We have $\theta_n\in\mathfrak{D}$ for all n, for

$$\int_{-\infty}^{\infty} M(x)|\theta_n(x)|\,dx=\int_{\sigma_{n_0}} M(x)|\psi_n(x)|\,dx\leqq n_0\int_{\sigma_{n_0}}|\psi_n(x)|\,dx\leqq n_0\|\psi_n\|_{\sigma_{n_0}}=n_0<\infty.$$

Further, we have

$$T\theta_n(x)=\int K(x,y)\theta_n(y)\,dy=\int_{\sigma_{n_0}} K(x,y)\psi_n(y)\,dy=[S\psi_n](x),$$

so $T\theta_n = S\psi_n\to 0$ in $L_2(-\infty,\infty)$. Since θ_n converges weakly to 0, so we have proved

Theorem 2. *An integral operator of semi-Carleman type has* 0 *as a limit point of its spectrum (which is thereby, in particular, non-empty).*

From this it follows, as we mentioned earlier, that the identity operator cannot be represented as a semi-Carleman operator.

4. To complete the circle and achieve a characterization of operators of this type we have to show that semi-Carleman operators are closed. Let T be a semi-Carleman operator with kernel K acting on the domain $\mathfrak{D}$ defined above, and write $\mathcal{E}=\left\{f\,\middle|\,\int K(x,y)f(y)\,dy\in L_2(-\infty,\infty)\right\}$. We have seen above that $\mathcal{E}\supset\mathfrak{D}$. One may verify that T^* is determined by the kernel $K^*(x,y)=\overline{K(y,x)}$ acting on $\mathcal{E}^*=$ $=\left\{f\,\middle|\,\int K^*(x,y)f(y)\,dy\in L_2\right\}$ and that T^{**} is determined by K acting on $\mathcal{E}$ (the steps in the verification are the same, *mutatis mutandis*, as in [2, Theorem 10. 1, p. 398] and we omit them). Hence T has the closed extension T^{**}, and if we adopt $\mathcal{E}$ for the domain of K at the outset then the semi-Carleman operator it determines is already closed. With this understanding, we have now shown that *an operator T is representable on $L_2(-\infty,\infty)$ as a semi-Carleman operator if and only if T is closed, densely defined, and has* 0 *as a limit point of its spectrum.*

(Any partial isometry or projection with infinite-dimensional null space satisfies the above criterion, and it is easy to see what the representation is for such operators. For a partial isometry U, we have formulae (2) and (3) above, where $\lambda_{l(\nu,k)}=0$, $k=2,3,\ldots,\lambda_{l(\nu,1)}=1$, and $\Phi_{\nu,1}$, $\Psi_{\nu,1}$ correspond to bases for the initial and final spaces of U. Thus $K(x,y)=\sum_\nu \Psi_{\nu,1}(y)\overline{\Phi_{\nu,1}(x)}$, with the ν^{th} summand supported (and square integrable) on $(\nu\leqq y\leqq\nu+1,\ -\infty<x<\infty)$, $\nu=0,\pm1,\pm2,\ldots$. For a projection P the representation is even simpler because then $\Psi_{\nu,1}=\Phi_{\nu,1}$, so P is represented as the direct sum (on $\oplus L_2(\nu\leqq x\leqq\nu+1)$) of operators of rank 1).

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On a pair of commutative contractions

By T. ANDÔ in Bloomington (Indiana, U. S. A.)

1. Introduction

Let T be a contraction on a Hilbert space $\mathfrak{H}$, i. e. $\|T\| \leqq 1$. A unitary (resp. isometric) operator $\mathbf{U}$ is called a *unitary* (resp. *isometric*) *dilation* of T if $\mathbf{U}$ acts on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace, and

(1) $$T^n f = \mathbf{P}\mathbf{U}^n f \quad (f \in \mathfrak{H}) \qquad n = 1, 2, \ldots$$

where $\mathbf{P}$ is the orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$. SZ.-NAGY [3, 4] proved the existence of a unitary dilation of any contraction. In this paper we shall concern ourselves with a pair of commutative contractions and prove the following theorem.

Theorem. *Let T_1, T_2 be a pair of commutative contractions. Then there exists a pair of commutative unitary operators $\mathbf{U}_1, \mathbf{U}_2$ on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace such that*

(2) $$T_1^{n_1} T_2^{n_2} f = \mathbf{P}\mathbf{U}_1^{n_1}\mathbf{U}_2^{n_2} f \qquad (f \in \mathfrak{H};\ n_1, n_2 = 1. 2. \ldots),$$

where $\mathbf{P}$ is the orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$.

This gives a partial answer to a problem raised by SZ.-NAGY [5] in which a finite number of commutative contractions comes into question.

The author would like to thank Professor SZ.-NAGY for his valuable suggestions.

2. Reduction of the problem

First of all, if the theorem is proved, replacing the word "*unitary*" by "*isometric*", the unitary operators in question can be readily obtained, because a pair of commutative isometries can be extended to a pair of commutative unitary operators on a larger Hilbert space by ITO's theorem [2] (see also BREHMER [1]). Secondly, if $\mathbf{U}_1, \mathbf{U}_2$ are isometries on $\mathfrak{K} \supset \mathfrak{H}$ such that

(3) $$T_i f = \mathbf{P}\mathbf{U}_i f \qquad (f \in \mathfrak{H};\ i = 1, 2)$$

and

(4) $$\mathbf{U}_i(\mathfrak{K} \ominus \mathfrak{H}) \subset \mathfrak{K} \ominus \mathfrak{H} \qquad (i = 1, 2)$$

then the condition (2) is necessarily satisfied. Thus it suffices to prove the following proposition instead of the theorem.

For any pair of commutative contractions T_1, T_2 there exists a pair of commutative isometries $\mathbf{U}_1$, $\mathbf{U}_2$ with the properties (3) *and* (4).

3. Proof

For the purpose, SCHÄFFER's construction [6] is used in the following modified form; $\mathfrak{K}$ is the orthogonal sum of countably many copies of $\mathfrak{H}$, indexed by all non-negative integers: the elements of $\mathfrak{K}$ are the sequences $\varphi=\{f_n\}_0^\infty$ of elements $f_n\in\mathfrak{H}$ with norm $\|\varphi\|^2=\sum_{n=0}^{\infty}\|f_n\|^2$. $\mathfrak{H}$ is embedded in $\mathfrak{K}$ by identifying $f\in\mathfrak{H}$ with the sequence $\{f_n\}$ where $f_0=f$ and $f_n=0$ for $n>0$. Then operators $\mathbf{V}_i$ $(i=1, 2)$ are defined as follows: $\{g_n\}=\mathbf{V}_i\{f_n\}$ if and only if $g_0=T_if_0$, $g_1=Z_if_0$, $g_2=0$ and $g_n=f_{n-2}$ for $n>2$ where $Z_i=(I-T_i^*T_i)^{1/2}$. Since

$$\|Z_if\|^2=\|f\|^2-\|T_if\|^2 \qquad (f\in\mathfrak{H};\ i=1, 2) \tag{5}$$

from the definitions of $\mathbf{V}_1$, $\mathbf{V}_2$ it is readily seen that they are isometries with the properties (3) and (4) for $\mathbf{V}_i$ instead of $\mathbf{U}_i$. Moreover from (5) it follows that

$$\|Z_2T_1f\|^2+\|Z_1f\|^2 = \|T_1f\|^2-\|T_2T_1f\|^2+\|f\|^2-\|T_1f\|^2 = \|f\|^2-\|T_2T_1f\|^2$$

and similarly

$$\|Z_1T_2f\|^2+\|Z_2f\|^2 = \|f\|^2-\|T_1T_2f\|^2;$$

hence the commutativity of T_1 with T_2 implies that

$$\|Z_2T_1f\|^2+\|Z_1f\|^2 = \|Z_1T_2f\|^2+\|Z_2f\|^2. \tag{6}$$

Now consider the orthogonal sum $\mathfrak{G}$ of four copies of $\mathfrak{H}$, i. e. $\mathfrak{G} = \mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H}\oplus\oplus\mathfrak{H}$ and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the subspace consisting of all the elements of the form

$$\{Z_2T_1f, 0, Z_1f, 0\} \qquad (f\in\mathfrak{H})$$

and

$$\{Z_1T_2f, 0, Z_2f, 0\} \qquad (f\in\mathfrak{H}),$$

respectively. From the relation (6) it follows that there exists an isometry W with domain $\mathfrak{M}_2$ and range $\mathfrak{M}_1$ wich assigns $\{Z_1T_2f, 0, Z_2f, 0\}$ to $\{Z_2T_1f, 0, Z_1f, 0\}$ $(f\in\mathfrak{H})$. If $\dim(\mathfrak{G}\ominus\mathfrak{M}_2) = \dim(\mathfrak{G}\ominus\mathfrak{M}_1)$, W can be extended to a unitary operator on $\mathfrak{G}$. This restriction on dimensions is actually guaranteed; in fact, in case $\mathfrak{H}$ is finite dimensional, it follows from the fact $\dim(\mathfrak{M}_1)=\dim(\mathfrak{M}_2)$, and in the contrary case, $\dim(\mathfrak{H})=\dim(\mathfrak{G})\geqq\dim(\mathfrak{G}\ominus\mathfrak{M}_i)\geqq\dim(\mathfrak{H})$ $(i=1, 2)$, because each $\mathfrak{G}\ominus\mathfrak{M}_i$ contains the subspace, isomorphic to $\mathfrak{H}$, consisting of all the elements of the form $\{0, f, 0, 0\}$ $(f\in\mathfrak{H})$. The unitary operator obtained is denoted by the same symbol W.

Now $\mathfrak{K}$ can be identified with the orthogonal sum

$$\mathfrak{H}\oplus\sum_{n=1}^{\infty}\oplus\mathfrak{G}_n,$$

where each $\mathfrak{G}_n$ is a copy of $\mathfrak{G}$, under the correspondence

$$\{f_0, f_1, f_2, \ldots, f_n, \ldots\} \leftrightarrow \{f_0, \{f_1, f_2, f_3, f_4\}, \ldots, \{f_{4n-3}, f_{4n-2}, f_{4n-1}, f_{4n}\}, \ldots\}.$$

In the sequel, this identification will always be in mind.

Let $\mathbf{W}$ be the operator on $\mathfrak{K}$ defined as follows: $\{g_n\} = \mathbf{W}\{f_n\}$ if and only if $g_0 = f_0$ and $\{g_{4n-3}, g_{4n-2}, g_{4n-1}, g_{4n}\} = W\{f_{4n-3}, f_{4n-2}, f_{4n-1}, f_{4n}\}$ $(n > 0)$. Then the unitarity of $\mathbf{W}$ follows from the unitarity of W on G, and both $\mathbf{W}$ and $\mathbf{W}^*$ have the property (4). Finally the isometries $\mathbf{U}_1, \mathbf{U}_2$ in question are defined by

$$\mathbf{U}_1 = \mathbf{W}\mathbf{V}_1 \quad \text{and} \quad \mathbf{U}_2 = \mathbf{V}_2\mathbf{W}^* \tag{7}$$

Since all $\mathbf{W}$, $\mathbf{W}^*$, $\mathbf{V}_1$ and $\mathbf{V}_2$ are isometries with the property (4), $\mathbf{U}_1, \mathbf{U}_2$ are isometries with the property (4). Obviously each $\mathbf{U}_i$ has the property (3). It remains only to prove the commutativity of $\mathbf{U}_1$ with $\mathbf{U}_2$. For any $\{f_n\} \in \mathfrak{K}$ putting

$$\{g_n\} \equiv \mathbf{U}_1\mathbf{U}_2\{f_n\} \equiv \mathbf{W}\mathbf{V}_1\mathbf{V}_2\mathbf{W}^*\{f_n\}$$

and

$$\{h_n\} \equiv \mathbf{U}_2\mathbf{U}_1\{f_n\} = \mathbf{V}_2\mathbf{W}^*\mathbf{W}\mathbf{V}_1\{f_n\} = \mathbf{V}_2\mathbf{V}_1\{f_n\},$$

simple calculations using the definitions of $\mathbf{W}$ and $\mathbf{U}_i$'s show that

$$\begin{aligned} g_0 &= T_1T_2f_0 \\ \{g_1, g_2, g_3, g_4\} &= W\{Z_1T_2f_0, 0, Z_2f_0, 0\} \\ g_n &= f_{n-4} \qquad (n > 4), \end{aligned}$$

and

$$\begin{aligned} h_0 &= T_2T_1f_0 \\ \{h_1, h_2, h_3, h_4\} &= \{Z_2T_1f_0, 0, Z_1f_0, 0\} \\ h_n &= f_{n-4} \qquad (n > 4). \end{aligned}$$

Since $T_1T_2 = T_2T_1$ and

$$W\{Z_1T_2f_0, 0, Z_2f_0, 0\} = \{Z_2T_1f_0, 0, Z_1f_0, 0\}$$

by the definition of W, it follows that $\mathbf{U}_1\mathbf{U}_2\{f_n\} = \mathbf{U}_2\mathbf{U}_1\{f_n\}$. Thus $\mathbf{U}_1$ commutes with $\mathbf{U}_2$.

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Matrices of normal extensions of subnormal operators

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1. A (bounded) operator T on a Hilbert space $\mathfrak{H}$ is called *subnormal* in case there exists a normal operator $\mathbf{N}$, called a *normal extension* of T, acting on a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace such that

$$\mathbf{N}f = Tf \qquad (f \in \mathfrak{H}). \tag{1}$$

A characterization of subnormality in terms of T has been obtained by HALMOS [2] and BRAM [1]; T is subnormal if and only if

$$\sum_{i,j} (T^i f_j, T^j f_i) \geqq 0 \tag{2}$$

for every finite sequence (f_i) in $\mathfrak{H}$. Their construction of the space $\mathfrak{K}$, however, depends heavily on T. It seems natural to raise a problem whether $\mathfrak{K}$ can be taken to be a fixed Hilbert space, independent of T as in SCHÄFFER's construction [4] for a unitary dilation of a contraction, and whether $\mathbf{N}$ can be constructed on $\mathfrak{K}$ along a definite line from T. In this paper this problem will be settled (Theorem 1), producing another characterization of subnormality (Theorem 2). At the same time a discussion concerning a commutative family of a subnormal operators will be made (Theorem 3).

Introduction of some notations will simplify later discussions. For any positive integer n, $\mathfrak{H}^n$ stands for the orthogonal sum of n copies of $\mathfrak{H}$, indexed by $0, 1, 2, \ldots, n-1$. In other words, the elements of $\mathfrak{H}^n$ are the n-sequences $\varphi = \{f_0, f_1, \ldots, f_{n-1}\}$ of elements $f_i \in \mathfrak{H}$ with norm $\|\varphi\|^2 = \sum_{i=0}^{n-1} \|f_i\|^2$. $\mathfrak{H}^\infty$ is similarly defined. In case $n > m$, $\mathfrak{H}^m$ is embedded into $\mathfrak{H}^n$ by identifying $\{f_0, f_1, \ldots, f_{m-1}\} \in \mathfrak{H}^m$ with $\{f_0, f_1, \ldots, f_{m-1}, 0, \ldots, 0, 0\} \in \mathfrak{H}^n$. $\mathfrak{H}$ is always identified with $\mathfrak{H}^1$. An operator $\mathbf{M}$ on $\mathfrak{H}^n$ $(1 \leqq n \leqq \infty)$ can be associated with a square n-rowed matrix each of whose entries is an operator on $\mathfrak{H}$. More precisely, if $M(i,j)$ stands for the (i,j)-th entry of $\mathbf{M}$, $\{g_i\} = \mathbf{M}\{f_i\}$ means that

$$g_i = \sum_{j=0}^{n-1} M(i,j) f_j \qquad (0 \leqq i \leqq n-1).$$

The requirement that $\mathfrak{H}$ is invariant under $\mathbf{M}$ and the restriction of $\mathbf{M}$ to $\mathfrak{H}$ coincides with T can be expressed by the requirement that $M(0,0) = T$ and $M(i,0) = 0$ for all $i > 0$. Finally we shall formulate a simple Lemma.

Lemma 1. *If T is subnormal and V is an operator from $\mathfrak{H}$ into another Hilbert space $\mathfrak{M}$ such that $V^* V T = T$, then VTV^* is subnormal on $\mathfrak{M}$.*

In fact, since $(VTV^*)^k = VT^kV^*$ $(k=1, 2, \ldots)$ by assumption, for every finite sequence (φ_i) in $\mathfrak{M}$

$$\sum_{i,j} ((VTV^*)^i \varphi_j, (VTV^*)^j \varphi_i) = \sum_{i,j} (VT^iV^* \varphi_j, VT^jV^* \varphi_i) =$$
$$= \sum_{i,j} (V^*VT^iV^* \varphi_j, T^jV^* \varphi_i) = \sum_{i,j} (T^iV^* \varphi_j, T^jV^* \varphi_i) \geqq 0$$

(the last inequality follows from (2)), hence the criterion (2) yields the subnormality of VTV^*.

2. First of all, if $\mathbf{N}$ is a normal extension of T, from (1) and the normality of $\mathbf{N}$ it follows that

$$\mathbf{N}\mathbf{N}^*f = \mathbf{N}^*\mathbf{N}f = \mathbf{N}^*Tf, \tag{3}$$

$$(\mathbf{N}^*f, g) = (T^*f, g) \qquad (f, g \in \mathfrak{H}), \tag{4}$$

$$\|Tf\| = \|\mathbf{N}f\| = \|\mathbf{N}^*f\| \geqq \|T^*f\|, \tag{5}$$

and moreover on account of BRAM's theorem [1] the norm $\|\mathbf{N}\|$ may be assumed to be equal to $\|T\|$.

(5) is equivalent to the positive definiteness of $T^*T - TT^*$. Let $S = (T^*T - TT^*)^{\frac{1}{2}}$, then

$$\|(\mathbf{N}^* - T^*)f\| = \|Sf\| \qquad (f \in \mathfrak{H}), \tag{6}$$

because by (4) and (5)

$$\|(\mathbf{N}^* - T^*)f\|^2 = \|\mathbf{N}^*f\|^2 - 2\,\mathrm{Re}(\mathbf{N}^*f, T^*f) + \|T^*f\|^2 = \|Tf\|^2 - \|T^*f\|^2 = \|Sf\|.$$

From this it follows that $Sf = 0$ is equivalent to $\mathbf{N}^*f = T^*f$, and the latter, in turn, is equivalent to the fact that $\mathbf{N}^*f$ is contained in $\mathfrak{H}$. Now each element φ in $\mathfrak{H} + \mathbf{N}^*\mathfrak{H}$ can be written in the form

$$\varphi = f + (\mathbf{N}^* - T^*)g \qquad \text{with} \qquad f, g \in \mathfrak{H}$$

and this decomposition is unique, because of the orthogonality of $\mathfrak{H}$ with $(\mathbf{N}^* - T^*)\mathfrak{H}$ by (4), consequently

$$\|\varphi\|^2 = \|f\|^2 + \|(\mathbf{N}^* - T^*)g\|^2. \tag{7}$$

Combining (7) with (6), it follows that the operator $\mathbf{V}$ which assigns $\{f, Sg\}$ to φ maps isometrically $\mathfrak{H} + \mathbf{N}^*\mathfrak{H}$ into $\mathfrak{H}^2$, and can be extended isometrically on the closure $\mathfrak{L}$ of $\mathfrak{H} + \mathbf{N}^*\mathfrak{H}$. On the other hand, $\mathfrak{L}$ is invariant under $\mathbf{N}$, because by (2)

$$\mathbf{N}(\mathfrak{H} + \mathbf{N}^*\mathfrak{H}) \subset T\mathfrak{H} + \mathbf{N}^*T\mathfrak{H} \subset \mathfrak{H} + \mathbf{N}^*\mathfrak{H}.$$

Therefore the restriction $\mathbf{M}$ of the normal operator $\mathbf{N}$ to the invariant subspace $\mathfrak{L}$ is subnormal with norm equal to $\|T\|$ by the definition of subnormality. Since clearly $\mathbf{V}^*\mathbf{V}\mathbf{M} = \mathbf{M}$, Lemma 1 yields the subnormality of $\mathbf{T} = \mathbf{V}\mathbf{M}\mathbf{V}^*$ and the norm $\|\mathbf{T}\|$ is equal to $\|T\|$.

In order to obtain the matrix of $\mathbf{T}$ on $\mathfrak{H}^2$ it suffices to calculate $\mathbf{T}\{f, Sg\}$ $(f, g \in \mathfrak{H})$, because $\mathbf{V}^*\{0, h\} = 0$ whenever $S^*h (= Sh) = 0$ and the orthogonal complement of the null space of S coincides with the closure of the range of S. To this effect, consider the densely defined operator S^{-1}, called the *partial inverse* of S, such that $S^{-1}S=P$ and $S^{-1}(I-P)=0$ where P denotes the orthogonal projection from $\mathfrak{H}$ onto the closure of the range of S. From (3) and the definition of $\mathbf{V}$ it follows that

$$\mathbf{T}\{f, Sg\} = \mathbf{VN}(f+(N^*-T^*)g) =$$
$$= \mathbf{V}(Tf+(T^*T-TT^*)g+(\mathbf{N}^*-T^*)Tg) = \{Tf+S^2g, STg\}$$

and this, in turn, means that the matrix in question is given by $\begin{pmatrix} T & S \\ 0 & STS^{-1} \end{pmatrix}$, *a fortiori* STS^{-1} is bounded. The bounded extension of STS^{-1} on $\mathfrak{H}$ will be denoted by the same symbol. Moreover, since $\mathbf{N}^*f\in\mathfrak{H}$ implies $\mathbf{N}^*Tf = \mathbf{NN}^*f\in\mathfrak{H}$ by (3), it follows that $Sf=0$ implies $STf=0$, i. e. $ST=STP=STS^{-1}\cdot S$.

Summing up, if T is subnormal, then T^*T-TT^* is positive definite, STS^{-1} is bounded and $ST=STS^{-1}\cdot S$, and the operator $\begin{pmatrix} T & S \\ 0 & STS^{-1} \end{pmatrix}$ on $\mathfrak{H}^2$ is subnormal with norm equal to $\|T\|$. This can be further generalized as follows:

Lemma 2. *Let T be subnormal and let R_n, S_n and T_n be defined by the following recurrent formulas:*

$$R_0=S_0=0,\ T_0=T,$$

$$R_n=S_{n-1}^2+T_{n-1}^*T_{n-1}-T_{n-1}T_{n-1}^*,\ S_n=R_n^{\frac{1}{2}},\ T_n=S_nT_{n-1}S_n^{-1} \qquad (n=1, 2, \dots)$$

Then, in each step, R_n is positive definite, T_n is bounded and $S_nT_{n-1}=T_nS_n$, and the operator $\mathbf{N}_n$ on $\mathfrak{H}^n$ with the entries $N_n(i,i)=T_i$ $(0\leqq i\leqq n-1)$, $N_n(i,i+1)=S_{i+1}$ $(0\leqq i\leqq n-2)$, $N_n(i,j)=0$ (for all other indices), is subnormal with norm equal to $\|T\|$.

Proof by induction. The assertions for $n=1$ have been just proved above. Suppose that the assertions on R_i, S_i and T_i $(0\leqq i\leqq n-1)$ and on $\mathbf{N}_n$ have been proved. On account of the arguments preceding this lemma, $\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*$ is positive definite, $\mathbf{WN}_n\mathbf{W}^{-1}$ is bounded, where $\mathbf{W}=(\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*)^{\frac{1}{2}}$ and $\mathbf{W}^{-1}$ is its partial inverse, and $\mathbf{WN}_n=\mathbf{WN}_n\mathbf{W}^{-1}\mathbf{W}$ and the operator $\begin{pmatrix} \mathbf{N}_n & \mathbf{W} \\ 0 & \mathbf{WN}_n\mathbf{W}^{-1} \end{pmatrix}$ on the orthogonal sum $\mathfrak{H}^n\oplus\mathfrak{H}^n$ is subnormal with norm equal to $\|\mathbf{N}_n\|=\|T\|$. Putting $\mathbf{N}_n^*\mathbf{N}_n^*=\mathbf{A}$ and $\mathbf{N}_n\mathbf{N}_n^*=\mathbf{B}$, simple calculations show that

$$\begin{aligned}
&A(i,i-1)=S_iT_{i-1} && (1\leqq i\leqq n-1),\\
&A(i,i)\quad\ =S_i^2+T_i^*T_i && (0\leqq i\leqq n-1),\\
&A(i,i+1)=T_i^*S_{i+1} && (0\leqq i\leqq n-2),\\
&A(i,j)\quad\ =0 && \text{(for all other indices)},
\end{aligned}$$

and similarly

$$\begin{aligned}
&B(i,i-1) &&=T_iS_i && (1\leqq i\leqq n-1),\\
&B(i,i) &&=T_iT_i^*+S_{i+1}^2 && (0\leqq i\leqq n-2),\\
&B(i,i+1) &&=S_{i+1}T_{i+1}^* && (0\leqq i\leqq n-2),\\
&B(n-1,n-1) &&=T_{n-1}T_{n-1}^* && \\
&B(i,j) &&=0 && \text{(for all other indices)}.
\end{aligned}$$

Since, by assumption,

$$S_iT_{i-1}=T_iS_i \qquad (1\leqq i\leqq n-1),$$

$$S_i^2+T_i^*T_i=T_iT_i^*+S_{i+1}^2 \qquad (0\leqq i\leqq n-2),$$

all the entries of $\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*$ are equal to 0 except the $(n-1, n-1)$th, which is equal to $S_{n-1}^2+T_{n-1}^*T_{n-1}-T_{n-1}T_{n-1}^* = R_n$ by definition. Hence the positive definiteness of $\mathbf{N}_n^*\mathbf{N}_n-\mathbf{N}_n\mathbf{N}_n^*$ implies the positive definiteness of R_n. Similarly all the entries of $\mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}$ are equal to 0 except the $(n-1, n-1)$ th which is equal to $S_nT_{n-1}S_n^{-1}=T_n$ by definition and is bounded. Moreover $\mathbf{W}\mathbf{N}_n=\mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\cdot\mathbf{W}$ implies $S_nT_{n-1}=T_nS_n$. Finally considering the operator $\mathbf{V}$, with norm one, from $\mathfrak{H}^n\oplus\mathfrak{H}^n$ into $\mathfrak{H}^{n+1}$ defined by $\mathbf{V}\{\{f_0, f_1, \dots, f_{n-1}\}, \{g_0, g_1, \dots, g_{n-1}\}\} = \{f_0, f_1, \dots \dots, f_{n-1}, g_{n-1}\}$,

$$\mathbf{V}^*\mathbf{V}\begin{pmatrix}\mathbf{N}_n & \mathbf{W}\\ 0 & \mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\end{pmatrix}=\begin{pmatrix}\mathbf{N}_n & \mathbf{W}\\ 0 & \mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\end{pmatrix} \text{ and } \mathbf{N}_{n+1}=\mathbf{V}\begin{pmatrix}\mathbf{N}_n & \mathbf{W}\\ 0 & \mathbf{W}\mathbf{N}_n\mathbf{W}^{-1}\end{pmatrix}\mathbf{V}^*$$

hence by Lemma 1 $\mathbf{N}_{n+1}$ is also subnormal with norm equal to $\|\mathbf{N}_{n+1}\|=\|T\|$. Thus induction is complete.

Inspecting the above proof, from the definitions of R_n, S_n and T_n, and of $\mathbf{N}_n$ and from the relations $S_nT_{n-1}=T_nS_n$ $(n=1, 2, \dots)$, it follows

$$\|\mathbf{N}_{n+1}^*\varphi\|=\|\mathbf{N}_n\varphi\| \qquad (\varphi\in\mathfrak{H}^n) \tag{8}$$

where, on the right side, φ is considered as an element of $\mathfrak{H}^{n+1}$.

Now the matrix representation of a normal extension of T is near at hand, using R_n, S_n and T_n in Lemma 2.

Theorem 1. *If T is subnormal, the operator $\mathbf{N}$ on $\mathfrak{H}^\infty$ with the entries $N(i, i)=T_i$ $(i\geqq 0)$, $N(i, i+1)=S_{i+1}$ $(i\geqq 0)$, $N(i, j)=0$ (for all other indices), is a normal extension with norm equal to $\|T\|$.*

In fact, in view of Lemma 2, all $\mathbf{P}_n\mathbf{N}\mathbf{P}_n$ are bounded with norm equal to $\|T\|$ $n=0, 1, 2, \dots$, where each $\mathbf{P}_n$ is the orthogonal projection from $\mathfrak{H}^\infty$ onto $\mathfrak{H}^n$, consequently, as readily seen, $\mathbf{N}$ itself is bounded with norm equal to $\|T\|$, and is an extension of T. Moreover from (8) it follows that

$$\|\mathbf{P}_{n+1}\mathbf{N}^*\mathbf{P}_n\varphi\|=\|\mathbf{P}_n\mathbf{N}\mathbf{P}_n\varphi\| \qquad (\varphi\in\mathfrak{H}^\infty)\ (n=0, 1, 2, \dots)$$

hence

$$\|\mathbf{N}\varphi\|=\lim_{n\to\infty}\|\mathbf{P}_n\mathbf{N}\mathbf{P}_n\varphi\|=\lim_{n\to\infty}\|\mathbf{P}_{n+1}\mathbf{N}^*\mathbf{P}_n\varphi\|=\|\mathbf{N}^*\varphi\|.$$

This shows the normality of $\mathbf{N}$.

Lemma 2 also produces a characterization of subnormality in terms of R_n, S_n, and T_n in it.

Theorem 2. *If, for an operator T, each R_n is positive definite, each T_n is bounded and $S_nT_{n-1}=T_nS_n$ $(n=0, 1, 2, \dots)$, then T is subnormal.*

In fact, the operator $\mathbf{N}$ on $\mathfrak{H}^\infty$ in Theorem 1 can be defined on the linear sum $\mathfrak{M}$ of all $\mathfrak{H}^n$'s, and is an extension of T. Moreover by (8)

$$\|\mathbf{N}\varphi\|=\|\mathbf{N}^*\varphi\| \qquad (\varphi\in\mathfrak{M}).$$

Since $\mathfrak{M}$ is dense in $\mathfrak{H}^\infty$, it follows that $\mathbf{N}^*\mathbf{N}\varphi=\mathbf{N}\mathbf{N}^*\varphi$ ($\varphi\in\mathfrak{M}$), in particular $\mathbf{N}^{*j}\mathbf{N}^i f=\mathbf{N}^i\mathbf{N}^{*j}f$ ($f\in\mathfrak{H}$) ($i,j=0,1,2,\ldots$). Therefore, for every finite sequence (f_i) in $\mathfrak{H}$,

$$\sum_{i,j}(T^j f_j, T^i f_i)=\sum_{i,j}(\mathbf{N}^{*j}\mathbf{N}^i f_j, f_i)=\sum_{i,j}(\mathbf{N}^{*j}f_j, \mathbf{N}^{*i}f_i)=\|\sum_k \mathbf{N}^{*k}f_k\|^2\geqq 0,$$

and the criterion (2) can be applied.

3. Itô [3] answered to the question when a commutative family of subnormal operators admits simultaneous commutative normal extensions. At this moment, it seems, however, difficult for us to construct matrices for these simultaneous commutative extensions along the line as that developed in § 2. Here we shall confine ourselves to a special case, namely, a doubly commutative family of subnormal operators.

Let $(T_\omega)_{\omega\in\Omega}$ be a *doubly commutative* family of subnormal operators, that is, each T_ω commutes with both T_γ and T_γ^* whenever $\omega\neq\gamma$. Let Δ denote the space of all generalized sequences $\{i_\omega\}$ such that all i_ω are non-negative integers and $\sum_{\omega\in\Omega} i_\omega<\infty$. θ denotes the element of Δ whose terms are all equal to 0. For any $\omega\in\Omega$ and $\Gamma\in\Delta$, ω_Γ is the ω-th term of Γ and $\Gamma+\omega$ stands for the element Λ such that $\omega_\Lambda=\omega_\Gamma+1$ and $\gamma_\Lambda=\gamma_\Gamma$ for all $\gamma\neq\omega$. $\mathfrak{H}^\Delta$ is the orthogonal sum of copies of $\mathfrak{H}$ indexed by all the elements in Δ; the elements of $\mathfrak{H}^\Delta$ are the generalized sequences $\varphi=\{f_\Gamma\}$ whose terms are in $\mathfrak{H}$ with norm $\|\varphi\|^2=\sum_{\Gamma\in\Delta}\|f_\Gamma\|^2$. $\mathfrak{H}$ is embedded in $\mathfrak{H}^\Delta$ by identifying $f\in\mathfrak{H}$ with $\{f_\Gamma\}$ where $f_\theta=f$ and $f_\Gamma=0$ ($\Gamma\neq\theta$). In Theorem 3 below, $S_{\omega,n}$ and $T_{\omega,n}$ correspond to S_n and T_n respectively in Lemma 2, starting from T_ω instead of T.

Theorem 3. *A doubly commutative family of subnormal operators $(T_\omega)_{\omega\in\Omega}$ has simultaneous commutative normal extensions $(\mathbf{N}_\omega)_{\omega\in\Omega}$ on $\mathfrak{H}^\Delta$ with the entries: $N_\omega(\Gamma,\Gamma)=T_{\omega,\omega_\Gamma}$, $N_\omega(\Gamma,\Gamma+\omega)=S_{\omega,\omega_\Gamma+1}$, $N_\omega(\Gamma,\Lambda)=0$ for all other indices.*

Proof. Just as in Theorem 1, each $\mathbf{N}_\omega$ is a normal extension of T_ω ($\omega\in\Omega$). For $\omega\neq\gamma$, putting $\mathbf{N}_\omega\mathbf{N}_\gamma=\mathbf{A}$ and $\mathbf{N}_\gamma\mathbf{N}_\omega=\mathbf{B}$, simple calculations based on the definitions of $\mathbf{N}_\omega$'s show that

$$\begin{aligned}
A(\Gamma,\Gamma)&=T_{\omega,\omega_\Gamma}T_{\gamma,\gamma_\Gamma}, & B(\Gamma,\Gamma)&=T_{\gamma,\gamma_\Gamma}T_{\omega,\omega_\Gamma},\\
A(\Gamma,\Gamma+\omega)&=S_{\omega,\omega_\Gamma+1}T_{\gamma,\gamma_\Gamma}, & B(\Gamma,\Gamma+\omega)&=T_{\gamma,\gamma_\Gamma}\cdot S_{\omega,\omega_\Gamma+1},\\
A(\Gamma,\Gamma+\gamma)&=T_{\omega,\omega_\Gamma}S_{\gamma,\gamma_\Gamma+1}, & B(\Gamma,\Gamma+\gamma)&=S_{\gamma,\gamma_\Gamma+1}T_{\omega,\omega_\Gamma},\\
A(\Gamma,\Gamma+\omega+\gamma)&=S_{\omega,\omega_\Gamma+1}S_{\gamma,\gamma_\Gamma+1}, & B(\Gamma,\Gamma+\omega+\gamma)&=S_{\gamma,\gamma_\Gamma+1}S_{\omega,\omega_\Gamma+1},
\end{aligned}$$

and all other entries of $\mathbf{A}$ and $\mathbf{B}$ are equal to 0. Therefore the commutativity of $\mathbf{N}_\omega$ with $\mathbf{N}_\gamma$ will follows from the commutativity of the family $\{S_{\omega,i}, T_{\omega,i}\}_{i=0}^\infty$ with the family $\{S_{\gamma,i}, T_{\gamma,i}\}_{i=0}^\infty$. In order to prove the latter commutativity, we shall show, by induction, that $T_\omega=T_{\omega,0}$ is doubly commutative with all $S_{\gamma,n}$ and $T_{\gamma,n}$, $n=0,1,2,\ldots$ The assertion for $n=0$ follows directly from the assumption. Suppose that the assertion for n is proved, then T_ω commutes with $S_{\gamma,n+1}$ because, as in [2], the latter is uniformly approximated by polynomials of $S_{\gamma,n}^2+T_{\gamma,n}^*T_{\gamma,n}-T_{\gamma,n}T_{\gamma,n}^*$.

which commutes with T_ω. This, in turn, implies the commutativity of T_ω with $S_{\gamma,n+1}^{-1}$, hence with $T_{\gamma,n+1}$. Similarly T_ω commutes with $T_{\gamma,n+1}^{*}$. In quite a similar way it is proved that the family $\{S_{\omega,i}, T_{\omega,i}\}_{i=0}^{\infty}$ commutes with the family $\{S_{\gamma,i}, T_{\gamma,i}\}_{i=0}^{\infty}$.

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Über die Weylsche Vertauschungsrelation

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Herrn Professor Béla Sz.-Nagy zum 50. Geburtstag gewidmet

Einführung

Es seien P und Q zwei selbstadjungierte Operatoren, für die die quantenmechanische Vertauschungsrelation

$$PQ - QP = -iI \tag{1}$$

erfüllt ist. Wie H. WEYL [1] zuerst bemerkt hat, geht (1) durch eine formelle Rechnung in die Relation (die sog. Weylsche Vertauschungsrelation):

$$e^{itP}e^{isQ} = e^{its}e^{isQ}e^{itP} \qquad (-\infty < t, s < +\infty) \tag{2}$$

über, wobei $\{e^{itP}\}_{-\infty<t<+\infty}$ und $\{e^{isQ}\}_{-\infty<s<+\infty}$ die durch die infinitesimalen Generatoren iP bzw. iQ erzeugten einparametrigen starkstetigen Gruppen von unitären Operatoren sind.

In der Arbeit [2] wurden Bedingungen angegeben, unter denen (1) und (2) streng äquivalent sind. Sogar wurde das allgemeinere Problem betrachtet, wobei statt Gruppen von unitären Operatoren, Halbgruppen von Kontraktionen auftreten.

Dieser allgemeinere Fall kann in gewißem Sinne auf den ursprünglichen Fall unitärer Gruppen zurückgeführt werden. In dieser Arbeit werden wir nämlich den folgenden Satz beweisen.

Satz. *Es seien $\{T_t\}_{t\geqq 0}$ und $\{S_s\}_{s\geqq 0}$ zwei einparametrige starkstetige Halbgruppen von Kontraktionen in einem Hilbertschen Raum H, für die die Weylsche Vertauschungsrelation*

$$T(t)S(s) = e^{its}S(s)T(t) \qquad (-\infty < t, s < +\infty) \tag{3}$$

erfüllt ist, wobei

$$T(t) = \begin{cases} T_t & \text{für } t \geqq 0 \\ T^*_{-t} & \text{für } t < 0 \end{cases} \quad \text{und} \quad S(s) = \begin{cases} S_s & \text{für } s \geqq 0 \\ S^*_{-s} & \text{für } s < 0 \end{cases}$$

gesetzt wird. Dann gibt es in einem geigneten Erweiterungsraum $\mathbf{H}$ zwei einparametrige starkstetige Gruppen von unitären Operatoren, $\{\mathbf{U}(t)\}_{-\infty<t<+\infty}$ und $\{\mathbf{V}(s)\}_{-\infty<s<+\infty}$, für die die Weylsche Relation

$$\mathbf{U}(t)\mathbf{V}(s) = e^{its}\mathbf{V}(s)\mathbf{U}(t) \qquad (-\infty < t, s < +\infty) \tag{4}$$

erfüllt ist und für die

$$T(t)\,S(s) = \mathrm{pr}\,\mathbf{U}(t)\mathbf{V}(s) \qquad (-\infty < t, s < +\infty) \tag{5}$$

gilt[1]). *Der Raum* $\boldsymbol{H}$ *kann in dem Sinne minimal gewählt werden, daß er von den Elementen* $\mathbf{U}(t)\mathbf{V}(s)h$ $(h \in H;\ -\infty < t, s < +\infty)$ *aufgespannt wird. Dann ist die Struktur* $\{\boldsymbol{H}, \mathbf{U}(t), \mathbf{V}(s), H\}$ *bis auf Isomorphie bestimmt.*

Die gewünschten unitären Dilatationen $\{\mathbf{U}(t)\}$ und $\{\mathbf{V}(s)\}$ werden wir in zwei Schritten konstruieren. Im ersten Schritt konstruieren wir in einem Erweiterungsraum $\boldsymbol{H}^\circ$ von H eine einparametrige starkstetige Gruppe $\{\mathbf{U}^\circ(t)\}$ von unitären Operatoren und eine einparametrige starkstetige Halbgruppe $\{\mathbf{V}^\circ_s\}_{s \geqq 0}$ von Kontraktionen für die die Weylsche Relation

$$\mathbf{U}^\circ(t)\mathbf{V}^\circ(s) = e^{its}\mathbf{V}^\circ(s)\mathbf{U}^\circ(t) \qquad (-\infty < t, s < +\infty)$$

erfüllt ist und

$$T(t)\,S(s) = \mathrm{pr}\,\mathbf{U}^\circ(t)\mathbf{V}^\circ(s) \qquad (-\infty < t, s < +\infty)$$

gilt, wobei $\mathbf{V}^\circ(s) = \begin{cases} \mathbf{V}^\circ_s & \text{für } s \geqq 0 \\ \mathbf{V}^{\circ *}_{-s} & \text{für } s < 0 \end{cases}$ gesetzt wird. Im zweiten Schritt konstruieren wir in einem Erweiterungsraum $\boldsymbol{H}$ von $\boldsymbol{H}^\circ$ unitäre Dilatationen $\{\mathbf{U}(t)\}_{-\infty < t < +\infty}$ und $\{\mathbf{V}(s)\}_{-\infty < s < +\infty}$ von $\{\mathbf{U}^\circ(t)\}$ bzw. von $\{\mathbf{V}^\circ(s)\}$, die die gewünschten Eigenschaften besitzen.

Zum Beweis benützen wir den folgenden Satz von B. Sz.-Nagy (siehe [3]):

Es sei $\{T_t\}_{t \geqq 0}$ *eine einparametrige starkstetige Halbgruppe von Kontraktionen in einem Hilbertschen Raum* H. *Dann gibt es in einem geeigneten Erweiterungsraum* $\boldsymbol{H}$ *eine einparametrige starktetige Gruppe* $\{\mathbf{U}(t)\}_{-\infty < t < +\infty}$ *von unitären Operatoren, für die*

$$T(t) = \mathrm{pr}\,\mathbf{U}(t) \qquad (-\infty < t < +\infty)$$

gilt, wobei

$$T(t) = \begin{cases} T_t & \text{für } t \geqq 0 \\ T^*_{-t} & \text{für } t < 0 \end{cases}$$

gesetzt wird. Der Raum $\boldsymbol{H}$ *kann in dem Sinne minimal gewählt werden, daß er von den Elementen von der Form* $\mathbf{U}(t)h$ $(h \in H;\ -\infty < t < +\infty)$ *aufgespannt wird. Dann ist die Struktur* $\{\boldsymbol{H}, \mathbf{U}(t), H\}$ *bis auf Isomorphie bestimmt.*

Beweis des Satzes

Es sei $\{\mathbf{U}^\circ(t)\}_{-\infty < t < +\infty}$ die *minimale unitäre Dilatation* von $\{T(t)\}$; der entsprechende Erweiterungsraum $\boldsymbol{H}^\circ$ ist von den Elementen von der Form $\mathbf{U}^\circ(t)x$ $(x \in H;\ -\infty < t < +\infty)$ aufgespannt, also ist die durch die endlichen Linearkombinationen

$$\xi = \sum_j \mathbf{U}^\circ(t_j)x_j \tag{6}$$

[1]) Sind A und $\mathbf{B}$ beschränkte lineare Operatoren in H, bzw. in einem Erweiterungsraum $\boldsymbol{H}$ von H, so bedeutet $A = \mathrm{pr}\mathbf{B}$, daß für jedes Element $h \in H$, $Ah = \mathbf{PB}h$ gilt, wobei $\mathbf{P}$ die orthogonale Projektion von $\boldsymbol{H}$ auf H ist. $\mathbf{B}$ heißt eine *Dilatation* von A.

gebildete lineare Mannigfaltigkeit $\boldsymbol{L}^\circ$ in $\boldsymbol{H}^\circ$ dicht. Ist ξ durch (6) gegeben, so setzen wir

(7) $$\eta(s) = \sum_j e^{-ist_j} \mathbf{U}^\circ(t_j) S(s) x_j \qquad (-\infty < s < +\infty).$$

Diese Zuordnung $\xi \to \eta(s)$ ist *eindeutig*, d. h. von der speziellen Wahl der Darstellung (6) des Elementes ξ unabhängig. Offenbar genügt es hierzu zu zeigen, daß aus $\xi = 0$ folgt $\eta(s) = 0$ $(-\infty < s < +\infty)$.

Aus
$$\xi = \sum_j \mathbf{U}^\circ(t_j) x_j = 0$$
folgt für jede reelle Zahl t
$$\sum_j T(t+t_j) x_j = \mathbf{P}^\circ \sum_j \mathbf{U}^\circ(t+t_j) x_j = \mathbf{P}^\circ \mathbf{U}^\circ(t) \sum_j \mathbf{U}^\circ(t_j) x_j = \mathbf{P}^\circ \mathbf{U}^\circ(t) \xi = 0,$$
wobei $\mathbf{P}^\circ$ die orthogonale Projektion von $\boldsymbol{H}^\circ$ auf H bedeutet. Daraus folgt weiter
$$\mathbf{P}^\circ \mathbf{U}^\circ(t) \eta(s) = \mathbf{P}^\circ \sum_j e^{-ist_j} \mathbf{U}^\circ(t+t_j) S(s) x_j =$$
$$= e^{its} \sum_j e^{-is(t+t_j)} T(t+t_j) S(s) x_j = e^{its} S(s) \sum_j T(t+t_j) x_j = 0$$
$(-\infty < t < +\infty)$, und daher
$$(\eta(s), \mathbf{U}^\circ(t) x) = (\mathbf{U}^\circ(-t)\eta(s), x) = (\mathbf{P}^\circ \mathbf{U}^\circ(-t)\eta(s), x) = 0$$
für jedes $x \in H$. Der Raum $\boldsymbol{H}^\circ$ ist aber von den Elementen $\mathbf{U}^\circ(t)x$ $(x \in H;\ -\infty < t < +\infty)$ aufgespannt, also ist $\eta(s) = 0$, w. z. b. w.

Aus (6), (7) und aus der soeben bewiesenen Eindeutigkeit der Zuordnung $\xi \to \eta(s)$ folgt, daß der durch $\eta(s) = \mathbf{V}^\circ(s)\xi$ definierte Operator $\mathbf{V}^\circ(s)$ die Linearmannigfaltigkeit $\boldsymbol{L}^\circ$ eindeutig und linear in sich überführt.

a) Von (7) kann man unmittelbar die folgenden Eigenschaften von $\mathbf{V}^\circ(s)$ ablesen:

1° $\mathbf{V}^\circ(s)$ ist starkstetig in s, d. h. von $s_n \to s$ folgt $\mathbf{V}^\circ(s_n)\xi \to \mathbf{V}^\circ(s)\xi$ für jedes $\xi \in \boldsymbol{L}^\circ$.

2° Für jedes $\xi \in \boldsymbol{L}^\circ$ und für beliebige $s, s' \geqq 0$ gilt $\mathbf{V}^\circ(s+s')\xi = \mathbf{V}^\circ(s)\mathbf{V}^\circ(s')\xi$, ferner ist $\mathbf{V}^\circ(0)\xi = \xi$.

b) Für je zwei Elemente $\xi', \xi'' \in \boldsymbol{L}^\circ$ ist
$$(\mathbf{V}^\circ(s)\xi', \xi'') = (\xi', \mathbf{V}^\circ(-s)\xi'') \qquad (-\infty < s < +\infty).$$
Es ist nämlich
$$(\mathbf{V}^\circ(s)\xi', \xi'') = \Big(\sum_j e^{-ist'_j} \mathbf{U}^\circ(t_j) S(s) x'_j,\ \sum_k \mathbf{U}^\circ(t''_k) x''_k\Big) =$$
$$= \sum_j \sum_k e^{-ist'_j} \big(\mathbf{U}^\circ(t'_j) S(s) x'_j,\ \mathbf{U}^\circ(t''_k) x''_k\big) =$$
$$= \sum_j \sum_k e^{-ist'_j} \big(S(s) x'_j,\ \mathbf{U}^\circ(t''_k - t'_j) x''_k\big) =$$
$$= \sum_j \sum_k e^{-ist'_j} \big(x'_j,\ S(-s) T(t''_k - t'_j) x''_k\big) =$$
$$= \sum_j \sum_k e^{-ist'_j} \big(x'_j,\ e^{-i(-s)(t''_k - t'_j)} T(t''_k - t'_j) S(-s) x''_k\big) =$$
$$= \sum_j \sum_k e^{-ist''_k} \big(x'_j,\ \mathbf{U}^\circ(t''_k - t'_j) S(-s) x''_k\big) =$$
$$= \sum_j \sum_k e^{-ist''_k} \big(\mathbf{U}^\circ(t'_j) x'_j,\ \mathbf{U}^\circ(t''_k) S(-s) x''_k\big) = (\xi', \mathbf{V}^\circ(-s)\xi'').$$

c) Für jedes $\xi \in \boldsymbol{L}^\circ$ und reelle s gilt $\|\mathbf{V}^\circ(s)\xi\| \leqq \|\xi\|$. Ist nämlich

$$\xi = \sum_j \mathbf{U}^\circ(t_j)x_j,$$

so gilt

$$\mathbf{V}^\circ(-s)\mathbf{V}^\circ(s)\xi = \sum_j \mathbf{U}^\circ(t_j)S(-s)S(s)x_j,$$

also

$$\begin{aligned}\|\mathbf{V}^\circ(s)\xi\|^2 &= (\mathbf{V}^\circ(s)\xi, \mathbf{V}^\circ(s)\xi) = (\mathbf{V}^\circ(-s)\mathbf{V}^\circ(s)\xi, \xi) = \\ &= \sum_j \sum_k (\mathbf{U}^\circ(t_j)S^*(s)S(s)x_j, \mathbf{U}^\circ(t_k)x_k) = \\ &= \sum_j \sum_k (S^*(s)S(s)x_j, \mathbf{U}^\circ(t_k-t_j)x_k) = \\ &= \sum_j \sum_k (S^*(s)S(s)x_j, T(t_k-t_j)x_k) = \\ &= \sum_j \sum_k (T(t_j-t_k)S^*(s)S(s)x_j, x_k);\end{aligned}$$

andererseits ist

$$\begin{aligned}\|\xi\|^2 &= \sum_j \sum_k (\mathbf{U}^\circ(t_j)x_j, \mathbf{U}^\circ(t_k)x_k) = \\ &= \sum_j \sum_k (\mathbf{U}^\circ(t_j-t_k)x_j, x_k) = \sum_j \sum_k (T(t_j-t_k)x_j, x_k).\end{aligned}$$

Wir haben also zu beweisen, daß

$$\|\xi\|^2 - \|\mathbf{V}^\circ(s)\xi\|^2 = \sum_j \sum_k (T(t_j-t_k)(I-S^*(s)S(s))x_j, x_k) \geqq 0.$$

Wegen $\|S(s)\| \leqq 1$ ist $I - S^*(s)S(s) \geqq 0$; man setze $Q(s) = [I - S^*(s)S(s)]^{\frac{1}{2}}$. Aus

$$\begin{aligned}S^*(s)S(s)T(t) &= S(-s)S(s)T(t) = S(-s)e^{-its}T(t)S(s) = \\ &= T(t)S(-s)S(s) = T(t)S^*(s)S(s)\end{aligned}$$

folgt, daß $Q(s)$ mit $T(t)$ vertauschbar ist $(-\infty < t, s < +\infty)$; folglich gilt

$$\|\xi\|^2 - \|\mathbf{V}^\circ(s)\xi\|^2 = \sum_j \sum_k (T(t_j-t_k)Q(s)x_j, Q(s)x_k) =$$

$$= \sum_j \sum_k (\mathbf{U}^\circ(t_j-t_k)Q(s)x_j, Q(s)x_k) = \sum_j \sum_k (\mathbf{U}^\circ(t_j)Q(s)x_j, \mathbf{U}^\circ(t_k)Q(s)x_k) =$$

$$= \Big\|\sum_j \mathbf{U}^\circ(t_j)Q(s)x_j\Big\|^2 \geqq 0, \qquad \text{w. z. b. w.}$$

d) In $\boldsymbol{L}^\circ$ erfüllen $\mathbf{U}^\circ(t)$ und $\mathbf{V}^\circ(s)$ die Weylsche Vertauschungsrelation. Für beliebige reelle t und s gilt nämlich

$$\begin{aligned}\mathbf{U}^\circ(t)\mathbf{V}^\circ(s)\xi &= \mathbf{U}^\circ(t) \sum_j e^{-ist_j}\mathbf{U}^\circ(t_j)S(s)x_j = \\ &= e^{its} \sum_j e^{-is(t+t_j)}\mathbf{U}^\circ(t+t_j)S(s)x_j = \\ &= e^{its}\mathbf{V}^\circ(s) \sum_j \mathbf{U}^\circ(t+t_j)x_j = e^{its}\mathbf{V}^\circ(s)\mathbf{U}^\circ(t)\xi.\end{aligned}$$

e) Endlich gilt für $\xi = x \in H$, und für beliebige t und s: $\xi = \mathbf{U}^\circ(0)x$, $\mathbf{V}^\circ(s)\xi = S(s)x$, $\mathbf{U}^\circ(t)\mathbf{V}^\circ(t)\xi = \mathbf{U}^\circ(t)\,S(s)\xi$, also

$$\mathbf{P}^\circ\mathbf{U}^\circ(t)\mathbf{V}^\circ(s)x = T(t)\,S(s)x.$$

Wegen c) kann man die Definition von $\mathbf{V}^\circ(s)$ auf den ganzen Raum $\boldsymbol{H}^\circ$ durch Stetigkeit erweitern: $\{\mathbf{V}^\circ(s)\}_{s\geqq 0}$ wird eine einparametrige starkstetige Kontraktions-halbgruppe in $\boldsymbol{H}^\circ$, $\{\mathbf{V}^\circ(s)\}_{-\infty<s<+\infty}$ und $\{\mathbf{U}^\circ(t)\}_{-\infty<t<+\infty}$ werden die Weylsche Vertauschungsrelation im ganzen Raum $\boldsymbol{H}^\circ$ erfüllen, ferner gilt

$$T(t)\,S(s) = \operatorname{pr} \mathbf{U}^\circ(t)\mathbf{V}^\circ(s) \qquad (-\infty<t,\, s<+\infty).$$

Es sei jetzt $\{\mathbf{V}(s)\}_{-\infty<s<+\infty}$ die minimale unitäre Dilatation von $\{\mathbf{V}^\circ(s)\}_{-\infty<s<+\infty}$ in einem entsprechend gewählten minimalen Erweiterungsraum $\boldsymbol{H}$ von $\boldsymbol{H}^\circ$. Die lineare Mannigfaltigkeit $\boldsymbol{L}$ der Elemente von der Form

$$(9) \qquad \vartheta = \sum_j \mathbf{V}(s_j)\boldsymbol{x}_j \qquad (\boldsymbol{x}_j \in \boldsymbol{H}^\circ)$$

ist in $\boldsymbol{H}$ dicht. Für so ein ϑ setzen wir

$$(10) \qquad \sigma(t) = \sum_j e^{its_j}\mathbf{V}(s_j)\mathbf{U}^\circ(t)\boldsymbol{x}_j \qquad (-\infty<t<+\infty).$$

Die Zuordnung $\vartheta \to \sigma(t)$ ist eindeutig, d. h. von der speziellen Wahl der Darstellung (9) des Elementes ϑ unabhängig. Definiert man den linearen Operator $\mathbf{U}(t)$ $(-\infty<t<+\infty)$ in $\boldsymbol{L}$ mit $\mathbf{U}(t)\vartheta = \sigma(t)$, so kann man mit Wiederholung der Rechnungen im ersten Schritte leicht sehen, daß die Definition von $\mathbf{U}(t)$ $(-\infty<t<+\infty)$ zum ganzen Raum $\boldsymbol{H}$ fortgesetzt werden kann, derart, daß die folgenden Beziehungen erfüllt werden:

1) $\mathbf{U}(-t) = \mathbf{U}^*(t)$ $(-\infty<t<+\infty)$,
2) $\{\mathbf{U}(t)\}_{t\geqq 0}$ ist eine einparametrige starkstetige Kontraktionshalbgruppe,
3) $\mathbf{U}(t)\,\mathbf{V}(s) = e^{its}\mathbf{V}(s)\mathbf{U}(t)$ $(-\infty<t,\, s<+\infty)$,
4) $T(t)\,S(s) = \operatorname{pr} \mathbf{U}(t)\,\mathbf{V}(s)$ $(-\infty<t,\, s<+\infty)$.

Wir werden beweisen, daß $\mathbf{U}(t)$ sogar unitär ist.

Wiederholt man die Rechnung c) aus dem ersten Schritte, so sieht man, daß $\mathbf{U}(t)$ $(-\infty<t<+\infty)$ auf $\boldsymbol{L}$ isometrisch ist. $\mathbf{U}(t)$ $(-\infty<t<+\infty)$ bildet aber die lineare Mannigfaltigkeit $\boldsymbol{L}$ ein-eindeutig auf sich ab, wie man von (9) und (10) unmittelbar ablesen kann. $\boldsymbol{L}$ ist in $\boldsymbol{H}$ dicht, voraus die Behauptung folgt.

Also besitzen $\{\mathbf{U}(t)\}$ und $\{\mathbf{V}(s)\}$ die gewünschten Eigenschaften, w. z. b. w.

Bemerkung. Wir führen die folgenden Bezeichnungen ein:

$$\mathbf{X}(t, s) = e^{-\frac{i}{2}ts}\mathbf{U}(t)\mathbf{V}(s) = e^{\frac{i}{2}ts}\mathbf{V}(s)\mathbf{U}(t),$$

$$Y(t, s) = e^{-\frac{i}{2}ts}T(t)S(s) = e^{\frac{i}{2}ts}S(s)T(t).$$

Man kann leicht nachrechnen, daß die Beziehungen

$$(11) \qquad e^{\frac{i}{2}(t_1 s_2 - s_1 t_2)}\mathbf{X}(t_1+t_2, s_1+s_2) = \mathbf{X}(t_1, s_1)\mathbf{X}(t_2, s_2)$$

$(-\infty<t_1, t_2, s_1, s_2<+\infty)$, $\mathbf{X}^*(t, s) = \mathbf{X}(-t, -s)$ und $\mathbf{X}(0, 0) = \mathbf{I}$ gelten.

Es sei $\{x_n\}$ ein beliebiges endliches System der Elementen von H, und seien $\{s_n\}$, $\{t_n\}$ zwei entsprechende Systeme von reellen Zahlen. Dann folgt aus (5) und (9):

$$\sum_j \sum_k e^{-\frac{i}{2}(s_k t_j - t_k s_j)} (Y(t_j - t_k, s_j - s_k)x_k, x_j) =$$

$$= \sum_j \sum_k e^{-\frac{i}{2}(s_k t_j - t_k s_j)} (\mathrm{X}(t_j - t_k, s_j - s_k)x_k, x_j) = \| \sum_k \mathrm{X}^*(t_k, s_k)x_k \|^2,$$

und folglich

$$\sum_j \sum_k e^{-\frac{i}{2}(s_k t_j - t_k s_j)} (Y(t_j - t_k, s_j - s_k)x_k, x_j) \geqq 0. \tag{12}$$

Man kann beweisen, daß das Bestehen der Ungleichung (12) für jedes endliche System $\{x_n\}$ von Elementen von H und für entsprechende Systeme $\{s_n\}$, $\{t_n\}$ von reellen Zahlen, auch *hinreichend* dafür ist, daß unitäre Dilatationen $\{\mathrm{U}(t)\}_{-\infty<t<+\infty}$ und $\{\mathrm{V}(s)\}_{-\infty<s<+\infty}$ mit den gewünschten Eigenschaften existieren. (Diese Konstruktion ist analog einer Konstruktion in [4].)

Könnten wir also die Ungleichung (12) unmittelbar beweisen, so würden wir einen neuen Beweis des Satzes dieser Arbeit bekommen. Wir haben jedoch (12) bisher nur in dem Falle *unmittelbar* beweisen können, daß mindestens eine der Halbgruppen $\{S_s\}_{s\geqq 0}$, $\{T_t\}_{t\geqq 0}$ aus lauter normalen Kontraktionen besteht.

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Ergodic theorems for gages

By I. KOVÁCS in Szeged

To Professor Béla Szőkefalvi-Nagy on his 50th birthday

Introduction

The theory of "non-commutative integration" which summarizes various analogies between the theory of measures and the theory of von Neumann algebras has been investigated by several authors in the last decade (especially *cf.* [3], [8] and [10]).

The purpose of the present work is to extend some of the notions and results of ergodic theory to the case of non-commutative integration.

§ 1 is devoted to general preliminaries. In § 2 a special case of the Riesz convexity theorem is extended to non-commutative L^p-spaces. This result is applied in § 3 where a non-commutative analogue of the concept of measurable transformation is introduced and a non-commutative extension of the von Neumann—Dunford—Miller mean ergodic theorem is given. In § 4 an ergodicity concept for "gages" on a von Neumann algebra A with respect to a group of *-automorphisms of A is introduced, and it is shown that the extreme points of the convex set formed by the probability gages on a von Neumann algebra A, which are invariant under a group of *-automorphisms of A, are precisely the ergodic ones.

The proofs are modelled on the corresponding proofs in the ordinary integration theory supplemented by some devices necessitated by the non-commutative character of the situation. The key role in the course of our proofs is played by a method of J. Dixmier used in § 3 of [3].

The results of this paper were announced in [6].

§ 1. Definitions and preliminaries

1. Let $\mathfrak{H}$ be a complex Hilbert space. A *von Neumann algebra*[1]) on $\mathfrak{H}$ will mean a self-adjoint algebra of bounded, every-where defined linear operators on $\mathfrak{H}$, which is closed in the weak (or strong) operator topology, and contains the identity operator $I_{\mathfrak{H}}$ of $\mathfrak{H}$[2]). In what follows, A_P will denote the set of the projections of the von Neumann algebra A.

[1]) For the theory of von Neumann algebras *cf.* [4], chap. I, §§ 1—6. Reference to this book in each particular case will be omitted.

[2]) For any Hilbert space $\mathfrak{H}$, $I_{\mathfrak{H}}$ will denote its identity operator.

Let A be a von Neumann algebra. A non-negative valued function φ on A^+ [3]) is called a *trace* on A^+, if it has the following properties:

(i) if $S, T \in \mathsf{A}^+$ and $\lambda, \mu \geqq 0$, then $\varphi(\lambda S + \mu T) = \lambda\varphi(S) + \mu\varphi(T)$;

(ii) for every $T \in \mathsf{A}^+$ and for every unitary operator U in A: $\varphi(UTU^{-1}) = \varphi(T)$.

A trace φ on A^+ is said to be *a) faithful* if the conditions $T \in \mathsf{A}^+$, $\varphi(T) = 0$ imply $T = 0$; *b) normal* if, for every increasing directed set $\mathsf{F} \in \mathsf{A}^+$ with $\sup_{S \in \mathsf{F}} S = T \in \mathsf{A}^+$, we have $\varphi(T) = \sup_{S \in \mathsf{F}} \varphi(S)$; *c) finite* if $\varphi(T) < +\infty$ for every $T \in \mathsf{A}^+$; *d) semi-finite* if, for every $T \in \mathsf{A}^+$, $T \neq 0$ there exists $S \in \mathsf{A}^+$, $S \neq 0$ such that $S \leqq T$ and $\varphi(S) < +\infty$.

Let A be a von Neumann algebra, and let φ be a trace on A^+. The set of elements T of A^+ for which $\varphi(T) < +\infty$, is the positive portion of a two-sided ideal $\mathfrak{m}_\varphi$, called the *two-sided ideal associated with* φ. φ can be uniquely extended to a positive linear form $\dot\varphi$ on $\mathfrak{m}_\varphi$, and for every $S \in \mathfrak{m}_\varphi$, $T \in \mathsf{A}$, we have $\dot\varphi(ST) = \dot\varphi(TS)$. If φ is normal, then for every $S \in \mathfrak{m}_\varphi$ the linear form $T \to \varphi(ST)$ $(T \in \mathsf{A})$ is strongly continuous on the unit sphere of A. If φ is finite, evidently we have $\mathfrak{m}_\varphi = \mathsf{A}$ (in this case $\dot\varphi$ is a positive linear form on A).

Let now φ be a semi-finite faithful normal trace on A^+. For any $S, T \in \mathfrak{m}_\varphi^{\frac{1}{2}}$ [4]), we define $\langle S|T\rangle_{\dot\varphi} = \dot\varphi(T^* S)$. Then $\mathfrak{m}_\varphi^{\frac{1}{2}}$ becomes a unitary algebra [5]) with inner product $\langle S|T\rangle_{\dot\varphi}$. Let $\mathfrak{H}_{\mathfrak{m}_\varphi^{\frac{1}{2}}}$ be the completion of the pre-Hilbert space $\mathfrak{m}_\varphi^{\frac{1}{2}}$. For any $R \in \mathfrak{m}_\varphi^{\frac{1}{2}}$, the mapping $S \to RS$ (resp. $S \to SR$) can be uniquely extended to a bounded linear operator $\Phi(R)$ [resp. $\Psi(R)$] on $\mathfrak{H}_{\mathfrak{m}_\varphi^{\frac{1}{2}}}$. Φ (resp. Ψ) is a *-isomorphism [6]) (resp. *-antiisomorphism), called *canonical *-isomorphism* (resp. **-antiisomorphism*) between A and the left ring $\mathbf{R}^g$ (resp. right ring $\mathbf{R}^d$) of $\mathfrak{m}_\varphi^{\frac{1}{2}}$.

2. Under a *non-commutative measurable space* we shall mean a system $(\mathfrak{H}, \mathsf{A})$ composed of a complex Hilbert space $\mathfrak{H}$ and a von Neumann algebra A on $\mathfrak{H}$. A *gage-space* $(\mathfrak{H}, \mathsf{A}, m)$ is a non-commutative measurable space $(\mathfrak{H}, \mathsf{A})$ with a non-negative valued function m on A_P which is completely additive, unitarily invariant and such that every projection in A is the supremum of the projections on which m is finite. (We say that m is *completely additive*, if $m(P) = \sum_{\iota \in I} m(P_\iota)$ for any set $(P_\iota)_{\iota \in I}$ of mutually orthogonal projections in A with $\sum_{\iota \in I} P_\iota = P$, and we say that m is *unitarily invariant* if for every unitary operator $U \in \mathsf{A}$ and projection $P \in \mathsf{A}_P$, we have $m(UPU^{-1}) = m(P)$.) The function m is called a "gage" (a "non-commutative

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[3]) For any set M of linear operators in a Hilbert space $\mathfrak{H}$, M^+ denotes the *positive portion* of M, i. e. the set of all non-negative symmetric elements of M.

[4]) Let $\mathfrak{m}$ be a two-sided ideal in a von Neumann algebra A. If T runs over $\mathfrak{m}^+$ then T^α $(0 < \alpha < +\infty)$ runs over the positive portion of a uniquely determined two-sided ideal of A: it will be denoted by $\mathfrak{m}^\alpha$ (*cf.* [2]).

[5]) A *unitary algebra* $\mathbf{R}$ is an algebra over the complex numbers, on which an involutive anti-automorphism $x \to x^*$ and an inner product $\langle x|y\rangle$ are defined, such that $\mathbf{R}$ becomes a pre-Hilbert space satisfying the following axioms: *(i)* $\langle x|y\rangle = \langle y^*|x^*\rangle$; *(ii)* $\langle xy|z\rangle = \langle y|x^*z\rangle$; *(iii)* the mapping $x \to xy$ with fixed y is continuous; *(iv)* the set of elements of the form xy is dense in $\mathbf{R}$ (x, y, z arbitrary in $\mathbf{R}$). Let $\mathfrak{H}_\mathbf{R}$ be the completion of the pre-Hilbert space $\mathbf{R}$. For every $x \in \mathbf{R}$ there exists a bounded operator U_x (resp. V_x) on $\mathfrak{H}_\mathbf{R}$ satisfying $U_x y = xy$ (resp. $V_x y = yx$) for every $y \in \mathbf{R}$. The weak (or strong) closure of the operators U_x (resp. V_x) is a von Neumann algebra $\mathbf{R}^g$ (resp. $\mathbf{R}^d$), called the *left* (resp. *right*) *ring* of $\mathbf{R}$. The commutant $(\mathbf{R}^g)'$ of $\mathbf{R}^g$ is identical with $\mathbf{R}^d$ (*cf.* [4], chap. I, § 5).

[6]) A **-isomorphism* is an isomorphism (in algebraical sense) preserving the adjunction.

measure") of A. It is evident that the restriction on A_P of a semi-finite normal trace on A^+ is a gage of A. Conversely, one can show (cf. [1]) that every gage of A can be uniquely extended to a semi-finite normal trace on A^+. For any gage m, φ_m will denote this extension.

A gage space $(\mathfrak{H}, A, m)$ is said to be *finite* (resp. *regular*) if φ_m is finite (resp. faithful).

In any gage space $(\mathfrak{H}, A, m)$ there exists, by virtue of the complete additivity of m, a maximal among those projections of A on which m vanishes; let it be denoted by F_m. It belongs to the centre of A. $I_{\mathfrak{H}} - F_m$ is called the *support* of m. In the following it will be denoted by E_m. Then for every $P \in A_P$ we have $m(E_m P) = m(P)$. $(\mathfrak{H}, A, m)$ is regular if and only if $E_m = I_{\mathfrak{H}}$.

Let $(\mathfrak{H}, A)$ be a non-commutative measurable space. A closed linear operator T on $\mathfrak{H}$ is said to be "measurable" with respect to A if:

(i) T is affiliated[7]) with A;

(ii) there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections of A such that, for every n, $P_n\mathfrak{H} \subset \mathfrak{D}_T$ ($\mathfrak{D}_T$ denotes the domain of T), $I_{\mathfrak{H}} - P_n$ is algebraically finite[8]), and $I_{\mathfrak{H}} - P_n \downarrow 0$ strongly ($n \to \infty$). It is evident that $A \subset \mathfrak{B}(A)$. Defining the "strong sum" and "strong product" of any two $S, T \in \mathfrak{B}(A)$ by the closure of their usual sum and product, respectively, $\mathfrak{B}(A)$ is a selfadjoint algebra relative to the strong sum and product, the usual operation of multiplication by scalars, and the adjunction. In what follows, when sum or product of measurable operators occurs, always the strong sum or strong product is understood, respectively.

Let $(\mathfrak{H}, A, m)$ be a gage space. For every $T \in \mathfrak{B}(A)^+$, we put

$$m(T) = \sup_{S \in \mathfrak{m}_{\varphi_m}^+,\, S \leqq T} \varphi_m(S).$$

Then m can be uniquely extended to a complex (possibly infinite) valued linear form on $\mathfrak{B}(A)$ (identical with φ_m on $\mathfrak{m}_{\varphi_m}$), designated by the same letter m. An element $T \in \mathfrak{B}(A)$ is said to be *integrable* (with respect to m) if $m(|T|) < +\infty$[9]). An element $T \in \mathfrak{B}(A)$ is said to be p^{th} *power integrable* if $|T|^p$ is integrable. Let $\mathbf{L}^p(m)$ $(1 \leqq p < +\infty)$ denote the set of all p^{th} power integrable operators of $\mathfrak{B}(A)$. The $\mathbf{L}^p$-norm of $T \in \mathbf{L}^p(m)$ is defined as $[m(|T|^p)]^{1/p}$, and denoted by $\|T\|_p$.

Let $(\mathfrak{H}, A, m)$ be a regular gage space. Then, for every $1 \leqq p < +\infty$, $\mathbf{L}^p(m)$ is a Banach space with the $\mathbf{L}^p$-norm defined above. Further we have:

(i) $\mathfrak{m}_{\varphi_m}^{1/p}$ is dense in $\mathbf{L}^p(m)$ $(1 \leqq p < +\infty)$;

(ii) if $1 < p < +\infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, there is an isometric isomorphism between the dual space $[\mathbf{L}^p(m)]^*$ of $\mathbf{L}^p(m)$ and $\mathbf{L}^q(m)$ in which corresponding elements

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[7]) A closed, densely defined linear oparator T in a Hilbert space $\mathfrak{H}$ is said to be *affiliated* with a von Neumann algebra A on $\mathfrak{H}$ (in sign $T\eta A$) if it commutes with every operator of A'.

[8]) A projection $P \in A$ is called *algebraically finite* if there exists no partially isometric operator $V \in A$ such that $V^*V = P$, $VV^* = Q < P$.

[9]) Every closed densely defined operator T in a Hilbert space can be uniquely written as a product of a partially isometric operator with the closure of the range of $|T| = (T^*T)^{\frac{1}{2}}$ as initial domain and the closure of the range of T as final domain. The decomposition $T = W|T|$ is called the *polar decomposition* of T. If $T\eta A$ (A being a von Neumann algebra), then $W \in A$, $|T|\eta A$. Hence, if $T \in \mathfrak{B}(A)$, we have $|T| \in \mathfrak{B}(A)$.

$F \in [\mathbf{L}^p(m)]^*$ and $S_F \in \mathbf{L}^q(m)$ are related by the identity

$$F(T) = m(TS_F), \quad T \in \mathbf{L}^p(m).$$

The dual space $\mathbf{L}^\infty(m)$ of $\mathbf{L}^1(m)$ is identical with the Banach space A considered with the usual operator norm;

(iii) if $\frac{1}{p} + \frac{1}{q} = 1$, where $1 \leqq p$, $q \leqq +\infty$, then $m(ST) = m(TS)$ for $S \in \mathbf{L}^p(m)$, $T \in \mathbf{L}^q(m)$;

(iv) $|m(T_1T_2 \dots T_n)| \leqq m(|T_1T_2 \dots T_n|) \leqq \|T_1\|_{p_1} \|T_2\|_{p_2} \dots \|T_n\|_{p_n}$, $T_i \in \mathbf{L}^{p_i}(m)$ with $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i \geqq 1$ $(i = 1, 2, \dots, n)$.

For the enumerated facts concerning the theory of the non-commutative integration, we refer the reader to [3], [8] and [10].

Let $(\mathfrak{H}, \mathrm{A}, m)$ be a gage space. An element T of A is said to be *quasi-simple* if it has the form $T = VT_0$ where T_0 is a finite linear combination of mutually orthogonal projections in $\mathfrak{m}_{\varphi_m}$: $T_0 = \sum_{j=1}^{n} \lambda_j P_j$, $P_i P_k = 0$ $(i \neq k)$, $P_j \in \mathfrak{m}_{\varphi_m}$, and V is a partially isometric operator in A whose initial domain contains the subspace $(P_1 + \dots + P_n)\mathfrak{H}$. It is easy to see that for a quasi-simple element $T = VT_0 = V \sum_{j=1}^{n} \lambda_j P_j$ we have

$$|T| = \sum_{j=1}^{n} |\lambda_j| P_j;$$

if $1 \leqq p < +\infty$

$$|T|^p = \sum_{j=1}^{n} |\lambda_j|^p P_j \quad \text{and} \quad \|T\|_p = \left[\sum_{j=1}^{n} |\lambda_j|^p m(P_j) \right]^{\frac{1}{p}};$$

further

$$\|T\|_\infty = \|T\| = \sup(|\lambda_1|, \dots, |\lambda_n|).$$

In what follows the terms and symbols introduced here will be used without further references.

§ 2. A convexity theorem for finite regular gage spaces

The following lemma which will be often applied throughout this paper is due to J. Dixmier (*cf.* [3], § 3). For the convenience of the reader we recall its proof.

Lemma 2. 1. *Let $(\mathfrak{H}, \mathrm{A}, m)$ be a regular gage space. Then the set of the quasi-simple elements of* A *is dense in $\mathbf{L}^p(m)$ for $1 \leqq p < +\infty$.*

Proof. As $\mathfrak{m}_{\varphi_m}$ is dense in $\mathbf{L}^p(m)$ for $1 \leqq p < +\infty$ (*cf.* § 1.), it is enough to show that every element of $\mathfrak{m}_{\varphi_m}$ is the limit in $\mathbf{L}^p$-norm of a sequence of quasi-simple elements of A.

Let T be an arbitrary element of $\mathfrak{m}_{\varphi_m}$. Let $T = W|T|$ be the polar decomposition of T. Using the spectral representation of $|T|$, we can determine a sequence $\{T_n\}_{n=1}^{\infty}$

of elements of A^+ commuting with T such that: 1) $0 \leqq T_n \leqq I_{\mathfrak{H}}$; 2) TT_n is quasi-simple for every $n=1, 2, \dots$; 3) $T_n \uparrow I_{\mathfrak{H}}$ strongly as $n \to \infty$.

By the uniqueness of the polar decomposition, we can see that $|T-TT_n| = |T(I_{\mathfrak{H}}-T_n)| = |T|(I_{\mathfrak{H}}-T_n)$. Therefore $\|T-TT_n\|_p^p = m(|T|^p(I_{\mathfrak{H}}-T_n)^p) = \varphi_m(|T|^p(I_{\mathfrak{H}}-T_n)^p)$. As $|T|^p \in \mathfrak{m}_{\varphi_m}$, $0 \leqq (I_{\mathfrak{H}}-T_n)^p \leqq I_{\mathfrak{H}}$ and $(I_{\mathfrak{H}}-T_n) \downarrow 0$ strongly, we have $\|T-TT_n\|_p \to 0$ as $n \to \infty$ (*cf.* § 1.)

To facilitate the statement of the next lemma which is a companion result to Lemma 7 of VI. 10 of [5], it will be convenient to introduce the following notations.

Definition 2. 1. Let $(\mathfrak{H}, \mathsf{A}, m)$ be a finite regular gage space. If $a \in R^1$ [10]) and $a > 0$, we define $\mathsf{A}(a)$ to be the set of quasi-simple elements T of A for which

$$(*) \qquad m(|T|^{\frac{1}{a}}) \leqq 1.$$

If $a=0$, the condition $(*)$ is replaced by

$$(**) \qquad \|T\| \leqq 1.$$

Definition 2. 2. Let $(\mathfrak{H}^{(j)}, \mathsf{A}^{(j)}, m^{(j)})$ be a finite regular gage space for each $j=1, 2$. Let $\mathscr{A}$ be the product of $\mathsf{A}^{(1)}$ and $\mathsf{A}^{(2)}$: $\mathscr{A} = \mathsf{A}^{(1)} \times \mathsf{A}^{(2)}$. If $\mathbf{a}=(a_1, a_2) \in R^2$ with $a_1 \geqq 0$, $a_2 \geqq 0$, we define $\mathscr{A}(\mathbf{a})$ to be the set of all elements $\mathbf{T}=(T_1, T_2)$ of $\mathscr{A}$ with $T_j \in \mathsf{A}^{(j)}(a_j)$.

Lemma. 2. 2. *With the notations of the preceding definitions, let F be a complex valued bilinear form on $\mathscr{A} = \mathsf{A}^{(1)} \times \mathsf{A}^{(2)}$ and let*

$$(1) \qquad M(\mathbf{a}) = \sup_{\mathbf{S} \in \mathscr{A}_1, \mathbf{T} \in \mathscr{A}(\mathbf{a})} |F(\mathbf{ST})|. \text{ [11])}$$

Then $\log M(\mathbf{a})$ *is a convex function*[12]) *of* $\mathbf{a}=(a_1, a_2)$ *for* $0 \leqq a_1 \leqq 1$, $0 \leqq a_2 \leqq 1$.

Proof. Let $\mathscr{A}^+(\mathbf{a})$ denote the totality of all $\mathbf{T}=(T_1, T_2)$ in $\mathscr{A}(\mathbf{a})$ for which $T_1 \geqq 0$, $T_2 \geqq 0$. First we prove that

$$(2) \qquad M(\mathbf{a}) = \sup_{\mathbf{S} \in \mathscr{A}_1, \mathbf{T} \in \mathscr{A}^+(\mathbf{1})} |F(\mathbf{ST}^{\mathbf{a}})|,$$

where $\mathbf{T}^{\mathbf{a}} = (T_1^{a_1}, T_2^{a_2})$ and $\mathscr{A}(\mathbf{1}) = \mathscr{A}(1, 1)$. To see this we have to show that the sets $\mathfrak{M} = \{\mathbf{ST}\}_{\mathbf{S} \in \mathscr{A}_1, \mathbf{T} \in \mathscr{A}(\mathbf{a})}$ and $\mathfrak{N} = \{\mathbf{ST}^{\mathbf{a}}\}_{\mathbf{S} \in \mathscr{A}_1, \mathbf{T} \in \mathscr{A}(\mathbf{1})}$ are identical. Let $\mathbf{T} = (T_1, T_2) = \left(\sum_{j=1}^{n_1} \lambda_j^{(1)} P_j^{(1)}, \sum_{j=1}^{n_2} \lambda_j^{(2)} P_j^{(2)} \right)$ be an arbitrary element of $\mathscr{A}^+(\mathbf{1})$. Then, for every $\mathbf{a}=(a_1, a_2)$ with $a_1 \geqq 0, a_2 \geqq 0$, $\mathbf{T}^{\mathbf{a}} = (T_1^{a_1}, T_2^{a_2}) = \left(\sum_{j=1}^{n_1} (\lambda_j^{(1)})^{a_1} P_j^{(1)}, \sum_{j=1}^{n_2} (\lambda_j^{(2)})^{a_2} P_j^{(2)} \right)$ is an element of $\mathscr{A}^+(\mathbf{a}) \subset \mathscr{A}(\mathbf{a})$, and it follows that $\mathfrak{N} \subset \mathfrak{M}$. Let now $\mathbf{T}' = (T_1', T_2') =$

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[10]) R^k $(k=1, 2)$ denotes the k-dimensional real Euclidean space.

[11]) For a von Neumann algebra A, A_1 denotes its unit sphere.

[12]) Let C be a convex subset of R^2, and let M be a function defined on C having values which are either real or $+\infty$. M is said to be convex if for any $u, v \in C$

$$M(\alpha u + (1-\alpha)v) \leqq \alpha M(u) + (1-\alpha) M(v)$$

whenever $0 \leqq \alpha \leqq 1$.

$=\left(V_1 \sum_{j=1}^{n_1} \lambda_j^{(1)} P_j^{(1)}, V_2 \sum_{j=1}^{n_2} \lambda_j^{(2)} P_j^{(2)}\right)$ be arbitrary in $\mathscr{A}(\mathbf{a})$. First suppose that $a_1>0, a_2>0$. It is evident that

$$T_1'=V_1\left(\sum_{j=1}^{n_1} e^{i\arg\lambda_j^{(1)}} P_j^{(1)}\right)|T_1'|,\quad T_2'=V_2\left(\sum_{j=1}^{n_2} e^{i\arg\lambda_j^{(2)}} P_j^{(2)}\right)|T_2'|.$$

Putting

$$\mathbf{S}=(S_1, S_2)=\left(V_1 \sum_{j=1}^{n_1} e^{i\arg\lambda_j^{(1)}} P_j^{(1)}, V_2 \sum_{j=1}^{n_2} e^{i\arg\lambda_j^{(2)}} P_j^{(2)}\right),$$

$$\mathbf{T}=(T_1, T_2)=(|T_1'|^{\frac{1}{a_1}}, |T_2'|^{\frac{1}{a_2}}),$$

we have $\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})$ and $\mathbf{T}'=\mathbf{S}\mathbf{T}^{\mathbf{a}}$. If $a_1=a_2=0$, then we have $\mathbf{T}'\in\mathscr{A}_1$ and $\mathbf{T}'=\mathbf{T}'\mathbf{T}^0$ for every $\mathbf{T}\in\mathscr{A}^+(\mathbf{1})\,(\mathbf{0}=(0,0))$. As $\mathscr{A}_1=\mathscr{A}_1\mathscr{A}_1$, it follows that $\mathfrak{M}\subset\mathfrak{N}$. The cases when either $a_1>0, a_2=0$ or $a_1=0, a_2>0$, can be treated by a similar way. Hence $\mathfrak{M}=\mathfrak{N}$ which proves (2).

Let now $\mathbf{b}=(b_1, b_2)\in R^2$ and $\mathbf{T}=(T_1, T_2)\in\mathscr{A}^+(\mathbf{1})$ be arbitrary. Put $\mathbf{T}^{i\mathbf{b}}=(T_1^{ib_1}, T_2^{ib_2})$. Then for every $\mathbf{a}=(a_1, a_2)\in R^2$ with $a_1\geqq 0, a_2\geqq 0$, we have

$$\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})| = \sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{i\mathbf{b}}\mathbf{T}^{\mathbf{a}})| \leqq M(\mathbf{a}).$$

Therefore,

$$\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}\leqq M(\mathbf{a}).$$

On the other hand, it is clear that

$$\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}\leqq M(\mathbf{a}).$$

Hence

$$\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})} |F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}= M(\mathbf{a}).$$

Let now $\mathbf{T}=(T_1, T_2)\in\mathscr{A}^+(\mathbf{1})$ be arbitrary. Then, for every $\mathbf{b}=(b_1, b_2)\in R^2$ and $\mathbf{a}=(a_1, a_2)\in R^2$ with $a_1\geqq 0, a_2\geqq 0$, $\mathbf{T}^{\mathbf{a}+i\mathbf{b}}=(T_1^{a_1+ib_1}, T_2^{a_2+ib_2})$ belongs to $\mathscr{A}(\mathbf{a})$. In fact, if $T_k=\sum_{j=1}^{n_k}\lambda_j^{(k)}P_j^{(k)}$ $(k=1,2)$, then $T_k^{a_k+ib_k}=\sum_{j=1}^{n_k}(\lambda_j^{(k)})^{a_k+ib_k}P_j^{(k)}$ which means that $T_k^{a_k+ib_k}$ is a quasi-simple element of $\mathrm{A}^{(k)}$. Further, $m^{(k)}(|T_k^{a_k+ib_k}|^{\frac{1}{a_k}})\leqq\leqq\|e^{ib_k\log T_k}\|\, m^{(k)}(|T_k|)\leqq 1$ if $a_k>0$, and $\|T_k^{a_k+ib_k}\|=\|e^{ib_k\log T_k}\|=1$ if $a_k=0$ [13]). Consequently,

$$|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\leqq\sup_{\mathbf{b}\in R^2}|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\leqq M(\mathbf{a}).$$

It follows that

$$M(\mathbf{a})=\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})}|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}\leqq$$

$$\leqq\sup_{\mathbf{b}\in R^2}\{\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})}[\sup_{\mathbf{b}\in R^2}|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|]\}=$$

$$=\sup_{\mathbf{S}\in\mathscr{A}_1, \mathbf{T}\in\mathscr{A}^+(\mathbf{1})}\{\sup_{\mathbf{b}\in R^2}|F(\mathbf{S}\mathbf{T}^{\mathbf{a}+i\mathbf{b}})|\}\leqq M(\mathbf{a}).$$

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[13]) We may suppose that $T_k>0$ for $k=1,2$.

Hence we have

$$M(\mathbf{a})=\sup_{\mathbf{S}\in\mathscr{A}_1,\mathbf{T}\in\mathscr{A}^+(1)}\{\sup_{\mathbf{b}\in R^2}|F(\mathbf{ST}^{\mathbf{a}+i\mathbf{b}})|\}. \tag{3}$$

Let $\mathbf{z}=(z_1, z_2)=(a_1+ib_1, a_2+ib_2)$ and let $\mathbf{T}=\left(\sum_{j=1}^{n_1}\lambda_j^{(1)}P_j^{(1)}, \sum_{j=1}^{n_2}\lambda_j^{(2)}P_j^{(2)}\right)\in$ $\in\mathscr{A}^+(\mathbf{1})$. We may suppose that every $\lambda_j^{(k)}>0$ $(j=1, \dots, n_k;\ k=1, 2)$. Then, for every $S=(S_1, S_2)\in\mathscr{A}_1$, we have

$$\mathbf{ST}^{\mathbf{z}}=\left(\sum_{j=1}^{n_1}(\lambda_j^{(1)})^{z_1}S_1P_j^{(1)}, \sum_{j=1}^{n_2}(\lambda_j^{(2)})^{z_2}S_2P_j^{(2)}\right)=$$

$$=\left(\sum_{j=1}^{n_1}e^{z_1\log\lambda_j^{(1)}}S_1P_j^{(1)}, \sum_{j=1}^{n_2}e^{z_2\log\lambda_j^{(2)}}S_2P_j^{(2)}\right).$$

As F is bilinear, $F(\mathbf{ST}^{\mathbf{z}})$ can be written as a finite sum $\sum_n f_n(z_1, z_2)F(\mathbf{B}_n)$ with $\mathbf{B}_n\in\mathscr{A}$, where $f_n(z_1, z_2)$ is an analytic[14]) function of the complex variables (z_1, z_2) and is bounded on the strip $0\leqq a_j\leqq 1$ $(i=1, 2)$. Hence, by VI. 10. 4 and VI. 10. 2 of [5], and the increasing nature of the logarithm, we obtain that

$$\log M(\mathbf{a})=\sup_{\mathbf{S}\in\mathscr{A}_1,\mathbf{T}\in\mathscr{A}^+(1)}\log\{\sup_{\mathbf{b}\in R^2}|F(\mathbf{ST}^{\mathbf{a}+i\mathbf{b}})|\}$$

is a convex function of $\mathbf{a}=(a_1, a_2)$ for $0\leqq a_1\leqq 1$, $0\leqq a_2\leqq 1$. Hence Lemma 2. 2 is proved.

The next theorem can be considered as a non-commutative extension of a special case of the Riesz convexity theorem (*cf.* [5], VI. 10. 11).

Theorem 2. 1. *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* Φ *be a linear mapping of* A *into itself. If for a given* p $(1\leqq p\leqq+\infty)$ Φ *has an extension to a bounded linear mapping of the Banach space* $\mathbf{L}^p(m)$ *into itself, let* $\|\Phi\|_p$ *denote the norm of this extension; if no such extension exists, let* $\|\Phi\|_p=+\infty$. *Then* $\log\|\Phi\|_{1/a}$ *is a convex function of* a *for* $0\leqq a\leqq 1$.

Proof. It is evident that

$$F(\mathbf{T})=F(T_1, T_2)=m(\Phi(T_1)T_2)$$

is a complex valued bilinear form on $\mathscr{B}=\mathsf{A}\times\mathsf{A}$. Let $a=\dfrac{1}{p}$ [$(1\leqq p\leqq+\infty$; $a=0$ if $p=+\infty$), and let]

$$M(a, 1-a)=\sup_{\mathbf{S}\in\mathscr{B}_1,\ \mathbf{T}\in\mathscr{B}(a,1-a)}|F(\mathbf{ST})|.$$

Now we are going to show that $\|\Phi\|_{1/a}\equiv M(a, 1-a)$. If both of $M(a, 1-a)$ and $\|\Phi\|_{1/a}$ are identically infinite, our assertion is trivial. Therefore, we show that

[14]) Let G be an open set in the space of the complex variables (z_1, z_2). A complex valued function f defined on G is said to be analytic in G if f is continuous and the first partial derivates $\partial f/\partial z_i$ $(i=1, 2)$ exist at each point of G.

$M(a, 1-a)$ is finite if and only if $\|\Phi\|_1$ is finite and in this case we have $M(a, 1-a) = \|\Phi\|_{1/a}$.

For any $a \in [0, 1]$ we have

$$M(a, 1-a) = \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} |F(\mathbf{ST})| = \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} |m(\Phi(S_1 T_1) S_2 T_2)| \leqq$$

$$\leqq \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} \|S_2\| \, \|T_2\|_{p'} \, \|\Phi(S_1 T_1)\|_p \leqq$$

$$\leqq \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} \|\Phi(S_1 T_1)\|_p \qquad \left(p = \frac{1}{a}\right).$$

As $\|S_1 T_1\|_p \leqq \|S_1\| \|T_1\|_p \leqq 1$ and A_1 contains the identity operator, by virtue of Lemma 2. 1 we have

$$\|\Phi\|_p = \sup_{T_1 \in \mathscr{A}(a)} \|\Phi(T_1)\|_p \leqq \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} \|\Phi(S_1 T_1)\|_p \leqq$$

$$\leqq \sup_{T \in \mathscr{A},\, \|T\|_p \leqq 1} \|\Phi(T)\|_p = \|\Phi\|_p,$$

which implies $$\sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} \|\Phi(S_1 T_1)\|_p = \|\Phi\|_p.$$

Hence $$M(a, 1-a) \leqq \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} \|\Phi(S_1 T_1)\| = \|\Phi\|_p.$$

It follows that if $\|\Phi\|_{1/a}$ is finite for a given $a \in [0, 1]$, then $M(a, 1-a)$ is finite and $M(a, 1-a) \leqq \|\Phi\|_{1/a}$. Conversely, suppose that $M(a, 1-a)$ is finite for some a in $[0, 1]$. Let $p = \frac{1}{a}$ ($p = +\infty$ if $a = 0$), and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for every quasi-simple element T_1' of A, the linear form

$$H_{T_1'}(R) = m(\Phi(T_1') R) \quad (R \in \mathsf{A})$$

is bounded in $\mathbf{L}^{p'}$-norm on a dense subset of $\mathbf{L}^{p'}(m)$, namely on $\mathsf{A} \subset \mathbf{L}^{p'}(m)$. In fact,

$$\|H_{T_1'}\|_{p'} = \sup_{R \in \mathscr{A}(1-a)} |m(\Phi(T_1') R) = \sup_{R \in \mathscr{A}(1-a)} \|T_1'\|_p \left| m\left(\Phi\left(\frac{T_1'}{\|T_1'\|_p}\right) R\right)\right| \leqq$$

$$\leqq \|T_1'\|_p \sup_{\mathbf{S} \in \mathscr{B}_1,\, \mathbf{T} \in \mathscr{B}(a, 1-a)} |m(\Phi(S_1 T_1) S_2 T_2)| = \|T_1'\|_p M(a, 1-a)$$

($\mathbf{S} = (S_1, S_2)$, $\mathbf{T} = (T_1, T_2)$). Consequently, $H_{T_1'}$ can be uniquely extended to all $\mathbf{L}^{p'}(m)$, i. e. $H_{T_1'} \in (\mathbf{L}^{p'}(m))^*$. Hence there exists an element $Q \in \mathbf{L}^p(m)$ with $\|Q\|_p \leqq M(a, 1-a) \|T_1'\|_p$ (*cf.* § 1) such that

$$H_{T_1'}(R) = m(\Phi(T_1') R) = m(QR)$$

for all $R \in \mathbf{L}^{p'}(m)$. It follows (*cf.* § 1) that $Q = \Phi(T_1)$. Hence

$$\|\Phi(T_1')\|_p \leqq M(a, 1-a) \|T_1'\|_p$$

for all quasi-simple elements T_1' in A. Hence

$$\|\Phi\|_p \leqq M(a, 1-a)$$

and we can conclude that $\|\Phi\|_{1/a} \equiv M(a, 1-a)$ for $0 \leqq a \leqq 1$. By Lemma 2. 2,

$$\log M(\mathbf{a}) = \log [\sup_{S \in \mathscr{B}_1,\, T \in \mathscr{B}(\mathbf{a})} |F(ST)|]$$

is a convex function of $\mathbf{a} = (a_1, a_2)$ for $0 \leqq a_1 \leqq 1$, $0 \leqq a_2 \leqq 1$, therefore $\log \|\Phi\|_{1/a}$ is also convex for $0 \leqq a \leqq 1$, and the proof is completed.

§ 3. The non-commutative mean-ergodic theorem

We begin this section by giving a non-commutative analogue of the concept of measurable transformation.

Let (X, S) be a "measurable space", i. e. a set X and a σ-algebra S of subsets of X. Denote by $\mathfrak{B}(X)$ the algebra of all complex valued functions $f(x)$ defined on X which are measurable with respect to S. Let T be a measurable transformation of (X, S), i. e. a mapping of X into itself such that the inverse image of every element of S by T belongs to S. By $f(x) \to \theta(f(x)) = f(Tx)$, T defines an endomorphism θ of $\mathfrak{B}(X)$. By the nature of the theory of gage spaces as a non-commutative extension of the classical theory of integration over an abstract measure space, it will be natural to define a non-commutative measurable trasformation as a mapping of the set of all measurable operators into itself with analogous algebraical and topological properties as θ.

Definition 3. 1. Let $(\mathfrak{H}, A)$ be a non-commutative measurable space. A *measurable transformation* of $(\mathfrak{H}, A)$ is a *-endomorphism (homomorphism into itself which preserves the adjunction) θ of $\mathfrak{B}(A)$ with the following properties:

(i) $\theta(I_{\mathfrak{H}}) = I_{\mathfrak{H}}$;

(ii) the restriction of θ to A is a normal[15]) *-endomorphism of A sending the set of all algebraically finite projections of A into itself. An invertible measurable transformation of $(\mathfrak{H}, A)$ is a *-automorphism of $\mathfrak{B}(A)$, whose restriction to A is a *-automorphism of A.

It follows immediately from the preceding definition the

Proposition 3. 1. *Let $(\mathfrak{H}, A)$ be a non-commutative measurable space and let θ be a measurable transformation of $(\mathfrak{H}, A)$. If a sequence $\{T_n\}_{n=1}^{\infty}$ of elements of $\mathfrak{B}(A)$ converges nearly everywhere[16]) (relative to A) to an element T of $\mathfrak{B}(A)$ then $\{\theta(T_n)\}_{n=1}^{\infty}$ converges nearly everyhere to $\theta(T)$.*

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[15]) A *-endomorphism θ of A is said to be *normal* if it has the following property: if $T \in A^+$ is the supremum of an increasing directed set F of elements in A^+, then we have $\theta(T) = \sup_{S \in F} S$.

[16]) A sequence $\{T_n\}_{n=1}^{\infty}$ of elements of $\mathfrak{B}(A)$ is said to be convergent nearly everywhere (relative to A) to an element T of $\mathfrak{B}(A)$) if for every $\varepsilon > 0$ there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections in A such that $P_n \uparrow I_{\mathfrak{H}}$ as $n \to \infty$, $\|(T - T_n)P_n\| < \varepsilon$ $(n = 1, 2, \ldots)$ and $I_{\mathfrak{H}} - P_n$ is algebraically finite for every $n = 1, 2, \ldots$ (*cf.* [9], def. 23).

The next proposition can be proved by the same way as Theorem 1 in [8], hence the details are omitted.

Proposition 3. 2. *Let* $(\mathfrak{H}, \mathsf{A})$ *be a non-commutative measurable space, and let* θ_0 *be a normal* $*$*-endomorphism of* A *with the following properties:* (i) $\theta_0(I_{\mathfrak{H}})=I_{\mathfrak{H}}$; (ii) θ_0 *sends the set of all algebraically finite projections of* A *into itself.*

Then θ_0 *can be uniquely extended to a measurable transformation* θ *of* $(\mathfrak{H}, \mathsf{A})$.

The preceding propositions imply

Proposition 3. 3. *Let* θ *be a measurable transformation of the non-commutative measurable space* $(\mathfrak{H}, \mathsf{A})$. *Then* θ *is uniquely determined by its restriction to* A.

Now we formulate our main result which can be considered as a non-commutative extension of the von Neumann—Dunford—Miller mean ergodic theorem (*cf.* [5], VIII. 5. 9.).

Theorem 3. 1. *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* θ *be a measurable transformation of* $(\mathfrak{H}, \mathsf{A})$. *Suppose that, for every projection* $P\in\mathsf{A}_P$ *and for every* $n=1, 2, \ldots$, θ *satisfies the inequality*

$$\frac{1}{n}\sum_{j=0}^{n-1} m(\theta^j(P)) \leqq M\cdot m(P) \tag{3.1}$$

with a constant M *independent of* P *and* n. *Then, for every* p *with* $1\leqq p<+\infty$, $T\to\theta(T)$ *is a continuous linear mapping of* $\mathbf{L}^p(m)$ *into itself and the sequence of operators* $\left\{\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(\cdot)\right\}_{n=1}^{\infty}$ *is strongly convergent in the Banach space* $\mathbf{L}^p(m)$.

The following lemmas are required for the proof.

Lemma 3. 1. (*cf.* [5], VIII. 5. 3). *Let* T *be a bounded operator in an arbitrary complex Banach space* $\mathfrak{X}$. *If the sequence* $\left\{\frac{1}{n}\sum_{j=0}^{n-1}T^j\right\}_{n=1}^{\infty}$ *is bounded (in norm), then it converges strongly in* $\mathfrak{X}$ *if and only if* $\frac{1}{n}T^n x\to 0$ *as* $n\to\infty$ *for* x *in a fundamental set*[17]) *in* $\mathfrak{X}$ *and the sequence* $\left\{\frac{1}{n}\sum_{j=0}^{n-1}T^j x\right\}_{n=1}^{\infty}$ *is weakly*[18]) *sequentially compact*[19]) *for* x *in a fundamental set in* $\mathfrak{X}$.

Lemma 3. 2 (*cf.* [12], th. 3). *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* $\mathbf{K}$ *be a bounded subset of* $\mathbf{L}^1(m)$. *If, for any sequence of projections* $\{P_n\}_{n=1}^{\infty}$ *in* A *with* $P_n\downarrow 0$ $(n\to\infty)$, $m(TP_n)$ *converges to zero uniformly with respect to* $T\in\mathbf{K}$, *then* $\mathbf{K}$ *is weakly sequentially compact.*

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[17]) A subset $\mathfrak{E}$ of a Banach space $\mathfrak{X}$ is said to be fundamental in $\mathfrak{X}$ if the linear subspace spanned by $\mathfrak{E}$ is equal to $\mathfrak{X}$.

[18]) By the weak topology of $\mathfrak{X}$ is understood the weak topology induced by the dual of $\mathfrak{X}$.

[19]) A subset $\mathfrak{K}$ of $\mathfrak{X}$ is said to be *sequentially compact* if every sequence of points in $\mathfrak{K}$ has a subsequence converging to a point of $\mathfrak{X}$.

Lemma 3. 3. *Let* $(\mathfrak{H}, \mathsf{A}, m)$ *be a finite regular gage space, and let* θ *be a measurable transformation of* (H, A). *Suppose that, for every* $P \in \mathsf{A}_p$, m *satisfies the inequality*

$$m(\theta(P)) \leqq Km(P) \tag{3. 2}$$

with a constant K *independent of* P. *Then for every* p *with* $1 \leqq p < +\infty$, $T \to \theta(T)$ *is a continuous linear mapping of* $\mathbf{L}^p(m)$ *into itself.*

Proof. For the sake of brevity, denote by A_0 the set of all quasi-simple elements of A. It is not hard to see that θ maps A_0 into itself. Further, for every $T \in \mathsf{A}_0$ with $T = V \sum_{j=1}^{n} \lambda_j P_j$ we have

$$|\theta(T)|^p = \theta(|T|^p) = \sum_{j=1}^{n} |\lambda_j|^p \theta(P_j),$$

and

$$\|\theta(T)\|_p = \left[\sum_{j=1}^{n} |\lambda_j|^p m(\theta(P_j))\right]^{\frac{1}{p}}.$$

Hence, by (3. 2), we have

$$\|\theta(T)\|_p \leqq K^{\frac{1}{p}} \left[\sum_{j=1}^{n} |\lambda_j|^p m(P_j)\right]^{\frac{1}{p}} = K^{\frac{1}{p}} \|T\|_p \qquad (1 \leqq p < +\infty). \tag{3. 3}$$

Let now $T \in \mathsf{A}$ be arbitrary. As in the proof of Lemma 2. 1, we can determine a sequence $\{T_n\}_{n=1}^{\infty}$ of elements of A^+ commuting with $|T|$ such that: 1) $0 \leqq T_n \leqq I_{\mathfrak{H}}$; 2) $TT_n \in \mathsf{A}_0$; 3) $T_n \uparrow I_{\mathfrak{H}}$ strongly as $n \to \infty$. It follows that $|TT_n|^p = |T|^p T_n \uparrow |T|^p$ strongly as $n \to \infty$. As φ_m is normal, we have $\|TT_n\|_p = m(|TT_n|^p)^{\frac{1}{p}} = \varphi_m(|TT_n|^p)^{\frac{1}{p}} \to \varphi_m(|T|^p)^{\frac{1}{p}} = m(|T|^p)^{\frac{1}{p}} = \|T\|_p$ $(n \to \infty)$. Further, $|\theta(TT_n)|^p = \theta(|TT_n|^p)$ for every $1 \leqq p < +\infty$. Since θ is normal on A (*cf.* Def. 3. 1), $|\theta(TT_n)|^p \uparrow |\theta(T)|^p$ as $n \to \infty$ and thus $\lim_{n\to\infty} \|\theta(TT_n)\|_p = \lim_{n\to\infty} m(|\theta(TT_n)|^p)^{\frac{1}{p}} = m(|\theta(T)|^p)^{\frac{1}{p}} = \|\theta(T)\|_p$. Since $TT_n \in \mathsf{A}_0$, by (3. 3) we have

$$\|\theta(TT_n)\|_p \leqq K^{\frac{1}{p}} \|TT_n\|_p.$$

Thus, we obtain

$$\|\theta(T)\|_p = \lim_{n\to\infty} \|\theta(TT_n)\|_p \leqq K^{\frac{1}{p}} \lim_{n\to\infty} \|TT_n\|_p = K^{\frac{1}{p}} \|T\|_p,$$

i. e.

$$\|\theta(T)\|_p \leqq K^{\frac{1}{p}} \|T\|_p \qquad (1 \leqq p < +\infty) \tag{3. 4}$$

for every $T \in \mathsf{A}$. The inequality (3. 4) shows that the restriction of θ to A, denoted by θ_0, is a continuous linear mapping of A into itself with respect to the $\mathbf{L}^p$-norm. Since A is dense in $\mathbf{L}^p(m)$, θ_0 can be uniquely extended to a continuous linear mapping θ_1 of $\mathbf{L}^p(m)$ into itself. Now, using the fact that every sequence $\{T_n\}_{n=1}^{\infty}$ of elements of A which converges in $\mathbf{L}^p$-norm to a measurable operator T contains a subsequence converging nearly everywhere to T (*cf.* [8] and [10]), it can be seen as in the classical case that $\theta_1(T)=\theta(T)$ for every $T\in\mathbf{L}^p(m)$. Thus Lemma 3. 2 is proved.

Proof of Theorem 3. 1. If $n=2$, the inequality (3. 2) gives $m(\theta(P))\leqq\leqq(2M-1)m(P)$ for any projection P in A. Hence, by Lemma 3. 3, θ is a bounded linear operator in $\mathbf{L}^p(m)$ $(1\leqq p<+\infty)$. To complete the proof of Theorem 3. 1, we have only to show the following (*cf.* Lemma 3. 1):

a) for every $T\in\mathbf{L}^p(m)$, $\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|_p\leqq M^{\frac{1}{p}}\|T\|_p$ $(n=1,2,\ldots)$;

b) $\frac{1}{n}\theta^n(T)$ converges strongly to zero as $n\to\infty$ for T in a fundamental set in $\mathbf{L}^p(m)$;

c) the sequence $\left\{\frac{1}{n}\sum_{j=1}^{n-1}\theta^j(T)\right\}_{n=1}^{\infty}$ of elements of $\mathbf{L}^p(m)$ is weakly sequentially compact for T in a fundamental set in $\mathbf{L}^p(m)$.

Let us prove a). First we show that

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|_1\leqq M\|T\|_1 \qquad (n=1,2,\ldots), \tag{3. 5}$$

for every $T\in$ A. The reasoning in the proof of Lemma 3. 3 shows that it is enough to prove (3. 5) for the quasi-simple elements of A. Let $T=VT_0$ be an arbitrary element of A_0 with $T_0=\sum_{i=1}^{N}\lambda_iP_i$. Then we have

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|_1=\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(V)\theta^j(T_0)\right\|_1\leqq\frac{1}{n}\sum_{j=0}^{n-1}\|\theta^j(V)\|\,\|\theta^j(T_0)\|_1\leqq$$

$$\leqq\frac{1}{n}\sum_{j=1}^{n-1}\sum_{i=1}^{N}|\lambda_i|m(\theta^j(P_i))=\sum_{i=1}^{N}\left[|\lambda_i|\frac{1}{n}\sum_{j=1}^{n-1}m(\theta^j(P_i))\right]\leqq M\|T\|_1,$$

which proves (3. 5). Further, for every $T\in A$ we have

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(T)\right\|\leqq\frac{1}{n}\sum_{j=0}^{n-1}\|\theta^j(T)\|\leqq\frac{1}{n}\sum_{j=0}^{n-1}\|T\|=\|T\|. \tag{3. 6}$$

Putting $\Phi_n(\cdot)=\frac{1}{n}\sum_{j=0}^{n-1}\theta^j(\cdot)$ for every $n=1,2,\ldots$, we have obtained

$$\|\Phi_n\|_1\leqq M,\quad \|\Phi_n\|_\infty\leqq 1.$$

As $\Phi_n(\cdot)$ is a linear mapping of A into itself, Theorem 2. 1 now gives

$$\log\|\Phi_n\|_p\leqq\left(1-\frac{1}{p}\right)\log\|\Phi_n\|_\infty+\frac{1}{p}\log\|\Phi_n\|_1\leqq\log\|\Phi_n\|_1^{\frac{1}{p}}\leqq\log M^{\frac{1}{p}},$$

and so

$$\|\Phi_n\|_p=M^{\frac{1}{p}} \qquad (n=1,2,\ldots),$$

which gives a).

To prove b), we note that the set A_P is fundamental in every $\mathbf{L}^p(m)$ for $1 \leqq p < +\infty$ (*cf.* [9], [11]). Now, if $P \in A_P$, we have

$$\left\| \frac{1}{n} \theta^n(P) \right\|_p = \frac{1}{n} [m(|\theta^n(P)|^p)]^{\frac{1}{p}} = \frac{1}{n} [m(\theta^n(P))]^{\frac{1}{p}} \leqq \frac{1}{n} [m(I_{\mathfrak{H}})]^{\frac{1}{p}} \to 0 \text{ as } n \to \infty,$$

whence b).

Finally, c) follows from Lemma 3. 2. Indeed, let $P_n \in A_P$ such that $P_n \downarrow 0$ strongly as $n \to \infty$. Then, for every $Q \in A_P$,

$$\left| m\left(P_n \frac{1}{k} \sum_{j=0}^{k-1} \theta^j(Q)\right) \right| \leqq \left(\frac{1}{k} \sum_{j=0}^{k-1} \|\theta^j(Q)\| \right) \|P_n\|_1 \leqq \|P_n\|_1 = \varphi_m(P_n) \to 0 \text{ as } n \to \infty$$
$$(k = 1, 2, \ldots)$$

independently from Q, and this completes the proof of Theorem 3. 1.

§ 4. An ergodicity concept for gages

In his paper [11], H. UMEGAKI introduced a concept of ergodicity for "traces" of a D*-algebra R (a normed *-algebra over the complex number, with an approximative identity) which are "stationary" (i. e. invariant) concerning a group of *-automorphisms **G** of A. He called a stationary trace of R *ergodic* if it is not a linear combination with positive coefficients of two other linearly independent stationary traces of R, and he characterized the ergodic traces with the aid of the two-sided representations corresponding to them. The ergodicity concept for gages introduced by us is analogous to that for measures in the ordinary integration theory[20]). We shall show the relation between our definition of ergodicity and UMEGAKI's, the latter definition being considered in the case when R is supposed to be a von Neumann algebra.

Let $(\mathfrak{H}, A, m)$ be a gage space, **G** a group of invertible measurable transformations of $(\mathfrak{H}, A)$ (*cf.* Def. 3. 1). In what follows, an element $T \in \mathfrak{B}(A)$ is said to be $(m, \mathbf{G})$-*invariant* if for every $\theta \in \mathbf{G}$ we have $E_m \theta(T) = E_m T$ (E_m is the support of m). T is said simply to be **G**-*invariant* if for every $\theta \in \mathbf{G}$ we have $\theta(T) = T$.

Our ergodicity concept for gages is given by the following

Definition 4. 1. Let $(\mathfrak{H}, A, m)$ be a gage space, **G** a group of invertible measurable transformations of $(\mathfrak{H}, A)$. m is called **G**-*ergodic* if for every $(m, \mathbf{G})$-invariant projection P of $A \cap A'$ we have either $m(P) = 0$ or $m(I_{\mathfrak{H}} - P) = 0$.

Analogously to the classical case we have

Theorem 4. 1. *Let $(\mathfrak{H}, A, m)$ be a gage space,* **G** *a group of invertible measurable transformations of $(\mathfrak{H}, A)$. In order that to m be* **G**-*ergodic it is necessary and sufficient*

20) Let (X, S, μ) be a measure space, and let **G** be a group of one-to-one mappings of X onto itself and which at the same time is a group of automorphisms of S. We recall that μ is said to be **G**-ergodic if for any $E \in S$ such that $[E \cup \theta(E)] - [E \cap \theta(E)]$ has μ-measure 0 for every $\theta \in \mathbf{G}$, we have either $\mu(E) = 0$ or $\mu(X - E) = 0$. μ is **G**-ergodic if and only if every S-measurable function $f(x)$ such that, for every $\theta \in \mathbf{G}$, $f(\theta(x)) = f(x)$ almost everywhere, is equal to a constant almost everywhere.

that every $(m, \mathbf{G})$-invariant element T of $\mathfrak{B}(\mathsf{A})$ affiliated with $\mathsf{A} \cap \mathsf{A}'$ be a scalar multiple of E_m.

Proof. If the condition of Theorem 4 is fulfilled, then every $(m, \mathbf{G})$-invariant projection P in $\mathsf{A} \cap \mathsf{A}'$ satisfies the equality $E_m P = \lambda E_m$ with some scalar λ. Since $E_m \in (\mathsf{A} \cap \mathsf{A}')$, $E_m P$ is a projection, so we have either $\lambda = 1$ or $\lambda = 0$. Hence either $m(P) = m(E_m P) = 0$ or $m(I_{\mathfrak{H}} - P) = m(E_m(I_{\mathfrak{H}} - P)) = m(E_m) - m(E_m P) = 0$. This means that m is $\mathbf{G}$-ergodic. Conversely, suppose that m is $\mathbf{G}$-ergodic. Let $T \in (\mathsf{A} \cap \mathsf{A}')$ be a self-adjoint $(m, \mathbf{G})$-invariant operator with $T = \int \lambda dE_\lambda$. Since $E_m \in (\mathsf{A} \cap \mathsf{A}')$, we have $E_m T = \int \lambda d(E_m E_\lambda)$, and $E_m \theta(T) = \int \lambda d(E_m \theta(E_\lambda))$ for every $\theta \in \mathbf{G}$. As $E_m(\theta(T)) = E_m T$, it follows from the uniqueness of the spectral representation that $E_m \theta(E_\lambda) = E_m E_\lambda$ for every λ and $\theta \in \mathbf{G}$. Since $E_\lambda \in (A \cap A')$, we obtain, by the $\mathbf{G}$-ergodicity of m, that for every λ either $m(E_\lambda) = 0$ or $m(I_{\mathfrak{H}} - E_\lambda) = 0$, i. e. either $E_m E_\lambda = 0$ or $E_m E_\lambda = E_m$. This means that the spectral family of $E_m T$ contains only two projections, namely 0 and E_m. Hence we have $E_m T = \lambda_0 E_m$. Let now $T \in (\mathsf{A} \cap \mathsf{A}')$ be an arbitrary $(m, \mathbf{G})$-invariant operator. It is easy to see that T can be written as a linear combination of two self-adjoint $(m, \mathbf{G})$-invariant operators in $\mathsf{A} \cap \mathsf{A}'$. Hence T is also a scalar multiple of E_m. Finally, let T be an arbitrary $(m, \mathbf{G})$-invariant operator in $\mathfrak{B}(\mathsf{A})$ affiliated with $\mathsf{A} \cap \mathsf{A}'$. Let $T = W|T|$ be the polar decomposition of T with $T = \int \lambda dE_\lambda$. It is known that $W \in (\mathsf{A} \cap \mathsf{A}')$, and $E_\lambda \in (\mathsf{A} \cap \mathsf{A}')$ for every λ. Further, as $E_m \theta(T) = E_m \theta(W) \theta(|T|) = (E_m \theta(W))(E_m \theta(|T|) = E_m T = E_m W|T| = (E_m W)(E_m |T|)$, it follows from the uniqueness of the polar decomposition that $E_m \theta(W) = E_m W$ and $E_m \theta(|T|) = E_m |T|$ for every $\theta \in G$. Since $W \in (\mathsf{A} \cap \mathsf{A}')$, we have $E_m W = \alpha E_m$. Since $\theta(|T|) = \int \lambda d\theta(E_\lambda)$ (*cf.* [9]), $E_m \theta(|T|) = \int \lambda d(E_m \theta(E_\lambda))$, similarly as above, it may be seen that the spectral family of $E_m |T|$ contains only two projections: 0 and E_m. Thus we obtain $E_m |T| = \beta E_m$, which proves Theorem 4. 1.

Definition 4. 2. Let $(\mathfrak{H}, \mathsf{A}, m)$ be a gage space, $\mathbf{G}$ a group of invertible measurable transformations of $(\mathfrak{H}, \mathsf{A})$. m is said to be $\mathbf{G}$-invariant if for every projection P of A and for every $\theta \in \mathbf{G}$ we have $m(\theta(P)) = m(P)$.

Let now A be a von Neumann algebra, $\mathbf{G}$ a group of $*$-automorphisms of A. Let $\mathsf{P}^{\mathbf{G}}$ denote the set of all $\mathbf{G}$-invariant probability[21]) gages on $(\mathfrak{H}, \mathsf{A})$, and $\dot{\mathsf{P}}^{\mathbf{G}} = \{\dot{\varphi}_m : m \in \mathsf{P}^{\mathbf{G}}\}$. It is evident that $\dot{\mathsf{P}}^{\mathbf{G}}$ is a convex subset of A^*[22]). The next theorem characterizes the $\mathbf{G}$-ergodic elements of $\mathsf{P}^{\mathbf{G}}$ as follows

Theorem 4. 2. *$m \in \mathsf{P}^{\mathbf{G}}$ is $\mathbf{G}$-ergodic if and only if $\dot{\varphi}_m$ is an extreme[23]) point of $\dot{\mathsf{P}}^{\mathbf{G}}$.*

Proof. First we note that if $m \in \mathsf{P}^{\mathbf{G}}$ then E_m is $\mathbf{G}$-invariant. Indeed, for every $\theta \in \mathbf{G}$ we have $m(I_{\mathfrak{H}} - \theta(E_m)) = m(I_{\mathfrak{H}}) - m(\theta(E_m)) = m(E_m) - m(E_m) = 0$. This means that $I_{\mathfrak{H}} - \theta(E_m) \leqq I_{\mathfrak{H}} - E_m$ $(\theta \in \mathbf{G})$. It follows that $I_{\mathfrak{H}} - E_m \leqq I_{\mathfrak{H}} - \theta^{-1}(E_m)$ for every

[21]) A gage m of A is said to be a *probability gage* if $m(I_{\mathfrak{H}}) = 1$.

[22]) For a von Neumann algebra A, A^* denotes the dual space of A when A is considered as a Banach space with $\|T\|$ as its norm.

[23]) $\dot{\varphi}_m$ is an *extreme* point of $\dot{\mathsf{P}}^{\mathbf{G}}$ if it is not a middle point of any segment belonging to $\dot{\mathsf{P}}^{\mathbf{G}}$.

$\theta\in\mathbf{G}$. As the mapping $\theta\to\theta^{-1}$ of $\mathbf{G}$ onto itself is one-two-one, we have $I_{\mathfrak{H}}-E_m\leqq\leqq I_{\mathfrak{H}}-\theta(E_m)$ $(\theta\in\mathbf{G})$. Thus we have $E_m=\theta(E_m)$ for every $\theta\in\mathbf{G}$.

Further, if $m\in\mathsf{P}^{\mathbf{G}}$ then for every $T\in\mathsf{A}$ and $\theta\in\mathbf{G}$ we have $\dot{\varphi}_m(\theta(T))=\dot{\varphi}_m(T)$. In fact, let T be an arbitrary element of A^+. As in the proof of Lemma 2. 1, we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ of elements of A^+ such that: 1) $0\leqq T_n\leqq I_{\mathfrak{H}}$; 2) $T_n\uparrow I_{\mathfrak{H}}$ strongly; 3) TT_n is a finite linear combination of elements of A_P with positive coefficients. The $\mathbf{G}$-invariance of m implies that $\varphi_m(\theta(TT_n))=\varphi_m(TT_n)$. As θ is a *-automorphism, it follows that $\theta(T_n)\uparrow I_{\mathfrak{H}}$. Thus $\theta(TT_n)=\theta(T)\theta(T_n)\uparrow\theta(T)$. By the normality of φ_m, we have $\varphi_m(\theta(T))=\lim\limits_{n\to\infty}\varphi_m(\theta(TT_n))=\lim\limits_{n\to\infty}\varphi_m(TT_n)=\varphi_m(T)$. Since every element of A can be written as a finite linear combination of elements in A^+, our assertion follows.

For $m\in\mathsf{P}^{\mathbf{G}}$, consider the von Neumann sub-algebra $\mathsf{A}_{E_m}=\{T\in\mathsf{A}: TE_m=T\}$ of A. We note that the restriction of φ_m to $\mathsf{A}_{E_m}^+$, denoted by the same letter φ_m, is a finite faithful normal trace on $\mathsf{A}_{E_m}^+$. Let $\mathbf{R}_m$ be the unitary algebra associated with $\dot{\varphi}_m$, and let Φ_m be the canonical *-isomorphism between A_{E_m} and the left ring $\mathbf{R}_m^g$ of $\mathbf{R}_m$. Since $\theta(E_m)=E_m$ for every $\theta\in\mathbf{G}$, it is easy to see that the mapping $T\to\theta'(T)==\Phi_m[\theta(\Phi_m^{-1}(T))]$ defines a *-automorphism θ' of $\mathbf{R}_m^g$ for every θ, and so $\mathbf{G}$ induces through Φ_m a group of *-automorphisms $\mathbf{G}'$ of $\mathbf{R}_m^g$. Further, it is not hard to see that an element $T\in\mathbf{R}_m^g$ is $\mathbf{G}'$-invariant if and only if $\Phi_m^{-1}(T)$ is $\mathbf{G}$-invariant.

Let now $\mathbf{V}_m$ be the set of all bounded linear operators V on $\mathfrak{H}_{\mathbf{R}_m}$ for which

$$\langle V\theta(S)\,|\,\theta(T)\rangle_{\dot{\varphi}_m}=\langle VS\,|\,T\rangle_{\dot{\varphi}_m}$$

for all $S, T\in\mathbf{R}_m$, $\theta\in\mathbf{G}$. It is easy to see that $\mathbf{V}_m$ is a von Neumann algebra on $\mathfrak{H}_{\mathbf{R}_m}$. By a theorem of H. UMEGAKI (*cf.* [11], Th. 5), $\dot{\varphi}_m$ is an extreme point of $\dot{\mathsf{P}}^{\mathbf{G}}$ if and only if $(\mathbf{V}_m\cap\mathbf{R}_m^g\cap\mathbf{R}_m^d)=\{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$. Hence we have to prove that $m\in\mathsf{P}^{\mathbf{G}}$ is $\mathbf{G}$-ergodic if and only if $(\mathbf{V}_m\cap\mathbf{R}_m^g\cap\mathbf{R}_m^d)=\{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$.

First we show that for an element $T\in(\mathbf{R}_m^g\cap\mathbf{R}_m^d)$ we have $T\in\mathbf{V}_m$ if and only if T is $\mathbf{G}'$-invariant. Suppose that $T\in(\mathbf{R}_m^g\cap\mathbf{R}_m^d)$ is $\mathbf{G}'$-invariant. Then $\Phi_m^{-1}(T)$ is $\mathbf{G}$-invariant. Thus, for every $\theta\in\mathbf{G}$ and $R, S\in\mathbf{R}_m$ we have

$$\langle T\theta(R)\,|\,\theta(S)\rangle_{\dot{\varphi}_m}=\dot{\varphi}_m(\theta(S^*)\Phi_m^{-1}(T)\theta(R))=\dot{\varphi}_m(\theta(S^*\Phi_m^{-1}(T)R))=$$
$$=\dot{\varphi}_m(S^*\Phi_m^{-1}(T)R)=\langle TR\,|\,S\rangle_{\dot{\varphi}_m},$$

which gives that $T\in\mathbf{V}_m$. Conversely, suppose that $T\in(\mathbf{V}_m\cap\mathbf{R}_m^g\cap\mathbf{R}_m^d)$. Then for every $\theta\in\mathbf{G}$ and $R, S\in\mathbf{R}_m$, we have

$$\dot{\varphi}_m(\Phi_m^{-1}(T)\theta(R)\theta(S^*))=\dot{\varphi}_m(\theta(S^*)\Phi_m^{-1}(T)\theta(R))=\langle T\theta(R)\,|\,\theta(S)\rangle_{\dot{\varphi}_m}=\langle TR\,|\,S\rangle_{\dot{\varphi}_m}=$$
$$=\dot{\varphi}_m(S^*\Phi_m^{-1}(T)R)=\dot{\varphi}_m(\Phi_m^{-1}(T)RS^*)=\dot{\varphi}(\theta(\Phi_m^{-1}(T))\theta(R)\theta(S^*)).$$

In particular, for $S=E_m\in\mathbf{R}_m$ we have

$$\dot{\varphi}_m(\theta(\Phi_m^{-1}(T))\theta(R))=\dot{\varphi}_m(\Phi_m^{-1}(T)\theta(R)).$$

Thus for every $\theta\in\mathbf{G}$ and $R\in\mathbf{R}_m$,

$$\dot{\varphi}_m([\Phi_m^{-1}(T)-\theta(\Phi_m^{-1}(T)]\theta(R))=0.$$

It follows that $\Phi_m^{-1}(T)=\theta(\Phi_m^{-1}(T))$ for every $\theta\in\mathbf{G}$, which gives that T is $\mathbf{G}'$-invariant.

Suppose now that $m \in \mathsf{P}^{\mathbf{G}}$ is $\mathbf{G}$-ergodic, and let $T \in (\mathbf{V}_m \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d)$ be arbitrary. Then $\Phi_m^{-1}(T)$ as an element of $\mathsf{A} \cap \mathsf{A}'$ is $(m, \mathbf{G})$-invariant. By Theorem 4. 1, $\Phi_m^{-1}(T) = \alpha E_m$. Hence $T = \alpha \Phi_m(E_m) = \alpha I_{\mathfrak{H}_{\mathbf{R}_m}}$, which gives that $(\mathbf{V}_m \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d) = \{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$. Conversely, suppose that $(\mathbf{V}_m' \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d) = \{\alpha I_{\mathfrak{H}_{\mathbf{R}_m}}\}$, and let $T \in (\mathsf{A} \cap \mathsf{A}')$ be $(m, \mathbf{G})$-invariant. Then TE_m is a $\mathbf{G}$-invariant element of $\mathsf{A}_{E_m} \cap \mathsf{A}'_{E_m}$. It follows that $\Phi_m(TE_m) \in (\mathbf{V}_m \cap \mathbf{R}_m^g \cap \mathbf{R}_m^d)$, therefore $\Phi_m(TE_m) = \alpha I_{\mathfrak{H}_{\mathbf{R}_m}} = \alpha \Phi_m(E_m)$. Thus $TE_m = \alpha E_m$, which completes the proof of Theorem 4. 2.

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Über projektive Veränderung der Übertragung in Linienelementmannigfaltigkeiten

Von ARTHUR MOÓR in Szeged

Herrn Professor Béla Sz.-Nagy zum 50 Geburtstage gewidmet

§ 1. Einleitung

Das Differentialgleichungssystem

$$\frac{d^2x^i}{ds^2}+2G^i\left(x, \frac{dx}{ds}\right)=0 \qquad (i=1, 2, \ldots, n), \tag{1.1}$$

wo die Größen $G^i(x, x')$ in den $x'^i\equiv\frac{dx^i}{ds}$ homogen von zweiter Dimension sind, bestimmt eine Geometrie der Bahnen, deren projektive und affine Eigenschaften schon vielfaltig untersucht wurden; wir erwähnen nur die grundlegende Arbeit [1] von L. BERWALD, wo man auch weitere Literaturangaben findet.

Die Geometrie $\mathfrak{G}_n$ der durch (1. 1) definierten Bahnen ist durch die Grundgrößen $G^i(x, x')$ bestimmt; man kann aber $\mathfrak{G}_n$ auch mit Finslerräumen $\mathfrak{F}_n$ in Zusammenhang bringen, falls man bedingt, daß die Gleichung (1. 1) eben die Extremalen einer Finslerschen Geometrie $\mathfrak{F}_n$ bestimmt.

Eine projektive Veränderung der Grundgrößen ist durch die Formeln

$$\hat{G}^i(x, x')=G^i(x, x')+\psi(x, x')x'^i, \qquad x'^i\equiv\frac{dx^i}{ds} \tag{1.2}$$

angegeben[1]), wo $\psi(x, x')$ eine in den x'^i von erster Dimension homogene skalare Funktion der Veränderlichen (x^i, x'^i) ist. Im folgenden wollen wir nun für die skalare Funktion $\psi(x, x')$ diejenigen Bedingungen bestimmen, die notwendig und hinreichend dafür sind, daß nach der projektiven Veränderung (1. 2) der Grundgrößen G^i, die Krümmungstensoren des Raumes unverändert bleiben, bzw. daß der Basisraum $\mathfrak{G}_n$ in einen sog. affin-skalaren Raum übergehe.

Der affin-skalare Raum ist ein spezieller Typ der metrisch-affinen Räume. Ein *metrisch-affiner Raum* ist ein n-dimensionaler Linienelementraum bezogen auf ein lokales Koordinatensystem, in dem die Metrik durch einen metrischen Grundtensor $g_{ik}(x, \dot{x})$ festgelegt ist und in dem eine kovariante Ableitung der Vektoren

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[1]) In der Arbeit [1] ist $\psi(x, x')$ durch $-P(x, x')$ bezeichnet.

definiert ist. Der metrisch-affine Raum ist ein *affin-skalarer Raum,* falls sein Krümmungstensor eine gewisse spezielle Form hat (vgl. unsere Formeln (3. 1)—(3. 3)), die in die charakteristische Form der Finslerräume von skalarer Krümmung übergeht, falls die Übertragungsparameter mit den Cartanschen Übertragungsparametern eines Finslerraumes von skalarer Krümmung übereinstimmen. *Diese Räume sind also Verallgemeinerungen der Finslerräume von skalarer Krümmung* (vgl. [4], Definition 2 auf Seite 159 und § 4); bezüglich weiterer geometrischer Eigenschaften dieser Räume verweisen wir auf die Sätze 5—7 von [4].

§ 2. Projektive Veränderung der Übertragung

Wir führen die folgenden — auch von L. Berwald benützten — Bezeichnungen ein (vgl. [1] Formeln (1. 4)):

$$G^i_j \stackrel{\text{def}}{=} \frac{\partial G^i(x,\dot{x})}{\partial \dot{x}^j}, \quad G^i_{jk} \stackrel{\text{def}}{=} \frac{\partial^2 G^i(x,\dot{x})}{\partial \dot{x}^j \partial \dot{x}^k}, \quad G^i_{jkl} \stackrel{\text{def}}{=} \frac{\partial^3 G^i(x,\dot{x})}{\partial \dot{x}^j \partial \dot{x}^k \partial \dot{x}^l}.$$

Diese Größen sind Funktionen des Linienelementes $(x, \dot{x})$. Die Ableitung $\frac{dx^i}{ds}$ nach dem affinen Parameter s wird mit x'^i bezeichnet. Die Größen G^i sind in den $\dot{x}^i$ homogen von zweiter Dimension; dementsprechend sind G^i_j, G^i_{jk} bzw. G^i_{jkl} in den $\dot{x}^i$ homogen von erster, nullter, bzw. (-1)-ter Dimension.

Die wichstigsten Krümmungstensoren des Raumes $\mathfrak{G}_n$ sind die folgenden (vgl. [1] § 2):

$$K^i_{\ j} \stackrel{\text{def}}{=} 2\frac{\partial G^i}{\partial x^j} - \frac{\partial G^i_j}{\partial x^r}\dot{x}^r + 2G^i_{jr}G^r - G^i_r G^r_j \tag{2. 1}$$

ist der *affine Abweichungstensor,*

$$K^i_{\ jk} \stackrel{\text{def}}{=} \frac{1}{3}\left(\frac{\partial K^i_{\ k}}{\partial \dot{x}^j} - \frac{\partial K^i_{\ j}}{\partial \dot{x}^k}\right) \equiv \frac{\partial G^i_j}{\partial x^k} - \frac{\partial G^i_k}{\partial x^j} + G^r_j G^i_{rk} - G^r_k G^i_{rj} \tag{2. 2}$$

ist der *Grundtensor der affinen Krümmung* und

$$K^i_{hjk} \stackrel{\text{def}}{=} \frac{\partial K^i_{\ jk}}{\partial \dot{x}^h} \tag{2. 3}$$

ist der *affine Krümmungstensor.*

Die Transformationsformel von G^i_{jk} stimmt mit der Transformationsformel der gewöhnlichen affinen Übertragungsparameter überein, somit ist die von L. Berwald eigeführte Operation:

$$\xi^i_{\ ,k} \stackrel{\text{def}}{=} \frac{\partial \xi^i}{\partial x^k} - \frac{\partial \xi^i}{\partial \dot{x}^r} G^r_k + G^i_{\ rk}\xi^r$$

eine kovariante Ableitung des kontravarianten Vektors ξ^i; diese Operation kann in der gewöhnlichen Weise auf beliebige Tensoren erweitert werden (vgl. [1] § 3)[2]).

[2]) In [1] ist diese Operation durch ein Semikolon bezeichnet.

Für einen Skalar $S(x, \dot{x})$ lautet die Berwaldsche kovariante Ableitung:

$$S_{,k}=\frac{\partial S}{\partial x^k}-\frac{\partial S}{\partial \dot{x}^r}G_k^r.$$

Die Krümmungstensoren $K^i{}_j$ und $K^i{}_{jk}$ verändern sich nach einer projektiven Veränderung (1. 2) der Grundgrößen G^i gemäß den Formeln:

$$\hat{K}^i{}_j=K^i{}_j+2\psi_{,j}\dot{x}^i-\psi_{j,t}\dot{x}^t\dot{x}^i-\psi\psi_j\dot{x}^i+\delta^i_j(\psi^2-\psi_{,t}\dot{x}^t) \tag{2. 4}$$

und

$$\hat{K}^i{}_{jk}=K^i{}_{jk}-\{\delta^i_k(\psi_{,j}-\psi\psi_j)-\delta^i_j(\psi_{;k}-\psi\psi_k)+(\psi_{k,j}-\psi_{j,k})\dot{x}^i\}, \tag{2. 5}$$

wo $$\psi_j\overset{\text{def}}{=}\frac{\partial\psi}{\partial\dot{x}^j},\quad \psi_{j,k}\overset{\text{def}}{=}\frac{\partial\psi_j}{\partial x^k}-\frac{\partial\psi_j}{\partial\dot{x}^r}G_k^r-G^r_{jk}\psi_r$$

bedeuten. Die Formeln (2. 4) und (2. 5) erhält man leicht aus (2. 1) und (2. 2), wenn man diese Formeln statt G^i mit den durch (1. 2) angegebenen $\hat{G}^i$ aufschreibt (vgl. [1] Gleichungen (5. 3), (7. 3) und (7. 4)).

Nach diesen Vorbereitungen beweisen wir den folgenden

Satz 1. *Für die Relationen*

$$\hat{K}^i{}_j=K^i{}_j \tag{2. 6}$$

bzw.

$$\hat{K}^i{}_{jk}=K^i{}_{jk}\qquad (n>2) \tag{2. 7}$$

ist die Relation

$$\psi_{,j}-\psi\psi_j=0\qquad\left(\psi_j\overset{\text{def}}{=}\frac{\partial\psi}{\partial\dot{x}^j}\right) \tag{2. 8}$$

notwendig und hinreichend.

Beweis. Wir beweisen zuerst, daß aus der Relation (2. 6) die Relation (2. 8) folgt. Die Relation (2. 6) ist nach der Formel (2. 4) mit

$$(2\psi_{,j}-\psi_{j,t}\dot{x}^t-\psi\psi_j)\dot{x}^i+\delta^i_j(\psi^2-\psi_{,t}\dot{x}^t)=0 \tag{2. 9}$$

gleichwertig. Eine Verjüngung über die Indizes i, j gibt wegen der aus den Homogenitätseigenschaften von ψ und G^i folgenden Identität

$$\psi_{j,k}\dot{x}^j\dot{x}^k=\psi_{,k}\dot{x}^k$$

und $\psi_t\dot{x}^t=\psi$ unter Beachtung der Ungleichung $n\neq 1$

$$\psi^2-\psi_{,t}\dot{x}^t=0. \tag{2. 10}$$

Nun ist

$$\psi_{k,j}\equiv\frac{\partial\psi_{,j}}{\partial\dot{x}^k} \tag{2. 11}$$

eine Identität, wie das durch eine unmittelbare Rechnung sofort bestätigt werden kann. Eine partielle Ableitung von (2. 10) nach $\dot{x}^j$ gibt wegen der Identität (2. 11):

$$2\psi\psi_j - \psi_{,j} - \psi_{j,t}\dot{x}^t = 0. \tag{2. 12}$$

Auf Grund von (2. 10) und (2. 12) bekommt man aus (2. 9) durch Elimination von $\psi_{j,t}\dot{x}^t$ die Formel $\dot{x}^i(\psi_{,j} - \psi\psi_j) = 0$, woraus wegen $\dot{x}^i \neq 0$ die zu beweisende Relation (2. 8) unmittelbar folgt.

Die Relation (2. 8) ist also für die Relation (2. 6) notwendig. Wir beweisen jetzt, daß sie auch hinreichend ist. Nehmen wir also an, daß die Relation (2. 8) gilt. Wir zeigen, daß dann auch (2. 6) gelten wird.

Aus (2. 8) folgt offenbar nach einer Überschiebung mit $\dot{x}^j$ wegen der Homogenität von erster Dimension von ψ in den $\dot{x}^i$ die Relation (2. 10). Aus (2. 10) erhält man ebenso wie vorher, durch partielle Ableitung nach $\dot{x}^j$, die Relation (2. 12).

Auf Grund der Formeln (2. 8), (2. 10) und (2. 12) folgt aber, daß die Relation (2. 9) gültig ist, man muß nur aus (2. 9) die Größen $\psi_{j,t}\dot{x}^t$ mit Hilfe von (2. 12) und dann $\psi_{,j}$ mit Hilfe von (2. 8) eliminieren. (2. 9) ist aber schon auf Grund von (2. 4) mit der zu beweisenden Relation (2. 6) gleichwertig.

Wir gehen jetzt zum Beweis des zweiten Teiles des Satzes über und zeigen, daß (2. 8) auch für das Bestehen von (2. 7) notwendig und hinreichend ist.

Aus (2. 5) und (2. 7) folgt:

$$\delta^i_k(\psi_{,j} - \psi\psi_j) - \delta^i_j(\psi_{,k} - \psi\psi_k) + (\psi_{k,j} - \psi_{j,k})\dot{x}^i = 0. \tag{2. 13}$$

Nach einer Verjüngung über i, k wird:

$$(n-1)(\psi_{,j} - \psi\psi_j) + (\psi_{t,j} - \psi_{j,t})\dot{x}^t = 0 \tag{2. 14}$$

und nach einer neuen Überschiebung mit $\dot{x}^j$:

$$(\psi_{,t} - \psi\psi_t)\dot{x}^t = 0. \tag{2. 15}$$

Da $\dot{x}^i \neq 0$ für einen beliebigen Index i bedingt werden kann, erhält man aus der Gleichung (2. 13) nach einer Überschiebung mit $\dot{x}^k$ in Hinsicht auf die Relation (2. 15):

$$\psi_{,j} - \psi\psi_j + (\psi_{t,j} - \psi_{j,t})\dot{x}^t = 0. \tag{2. 16}$$

Eliminiert man nun aus dieser Gleichung den Ausdruck $(\psi_{t,j} - \psi_{j,t})\dot{x}^t$ mit Hilfe der Formel (2. 14), so erhält man wegen der Bedingung $n > 2$ (vgl. unsere Formel (2. 7)) die zu beweisende Relation (2. 8).

Wir müssen noch zeigen, daß aus (2. 8) die Relation (2. 7) folgt. Substituieren wir $\psi_{,j}$ aus (2. 8) in die Identität (2. 11), so wird aus der Formel (2. 11):

$$\psi_{k,j} = \psi_j\psi_k + \psi\psi_{jk}, \qquad \psi_{jk} \stackrel{\text{def}}{=} \frac{\partial^2\psi}{\partial\dot{x}^j\partial\dot{x}^k}. \tag{2. 17}$$

Aus (2. 8) und (2. 17) folgt schon, daß die Relation (2. 13) gültig ist; aus (2. 13) und (2. 5) folgt aber die Relation (2. 7). Der Satz 1 ist also vollständig bewiesen.

Die Relation (2. 8) verallgemeinert den Fall der affinzusammenhängenden Punkträume (vgl. [3] § 32). Ist der Skalar ψ in den $\dot{x}^i$ linear, d. h. ist ψ von der Form

$$\psi(x, \dot{x}) = \varphi_t(x)\dot{x}^t \tag{2. 18}$$

und sind die Übertragungsparameter G^j_{ik} nur von x^k abhängig, von $\dot{x}^i$ aber unabhängig, so bekommt man aus (2. 8) wegen (2. 18) und wegen $G^s_{tj}\dot{x}^t \equiv G^s_j$:

$$(\varphi_{t,j} - \varphi_t\varphi_j)\dot{x}^t = 0; \tag{2. 19}$$

da aber nach den gestellten Bedingungen φ_j und $\varphi_{t,j}$ von den $\dot{x}^k$ unabhängig sind und (2. 19) für jede Richtung $\dot{x}^i$ besteht, folgt aus (2. 19) die Relation

$$\varphi_{t,j} - \varphi_t\varphi_j = 0,$$

und das stimmt mit dem Fall der Punkträume überein (vgl. [3], Formel (32. 17)).

§ 3. Affin-skalare Räume

Wir nehmen jetzt an, daß unser Raum $\mathfrak{G}_n$ ein Finslerraum $\mathfrak{F}_n$ ist. Das bedeutet, daß im Raume eine metrische Fundamentalfunktion $F(x, \dot{x})$ existiert[3]), die durch die Formeln

$$g_{ij} \overset{\text{def}}{=} \frac{1}{2}\frac{\partial^2 F^2}{\partial\dot{x}^i\partial\dot{x}^j}$$

einen metrischen Grundtensor definiert, und daß die Grundgrößen durch die Formeln

$$G^i = \frac{1}{2}\Gamma^{*i}_{j\ k}\dot{x}^j\dot{x}^k$$

bestimmt sind, wo $\Gamma^{*i}_{j\ k}(x, \dot{x})$ die wohlbekannten Cartanschen Übertragungsparameter des Finslerraumes $\mathfrak{F}_n$ bedeuten (vgl. [2]).

Nach der projektiven Veränderung (1. 2) von G^i bekommt man einen metrisch-affinen Raum $\hat{\mathfrak{B}}_n$, der ein affin-skalarer Raum von erster, zweiter, dritter Art ist, je nach dem sein Krümmungstensor die Form

$$\hat{K}^i{}_j = KF^2(\hat{\gamma}^i{}_j\hat{\gamma}_{oo} - \hat{\gamma}^i{}_o\hat{\gamma}_{oj}), \tag{3. 1}$$

$$\text{(a)}\quad \hat{K}^i{}_j = KF^2(\hat{\gamma}^i{}_j - \hat{\gamma}^i{}_o l_j), \qquad \text{(b)}\quad \hat{K}^i{}_j = KF^2(\delta^i_j\hat{\gamma}_{oo} - l^i\hat{\gamma}_{oj}), \tag{3. 2}$$

bzw.

$$\hat{K}^i{}_j = KF^2(\delta^i_j - l^i l_j) \tag{3. 3}$$

hat, wo K einen Skalar — möglicherweise ist $K \equiv 1$ — und $\hat{\gamma}^i{}_j$ einen von ψ und g_{ij}, l^i bzw. von den kovarianten Ableitungen dieser Größen gebildeten gemischten Tensor bedeuten. Die Abhängigkeit von $\hat{\gamma}^i{}_j$ von den erwähnten Grundgrößen des Raumes ist in den einzelnen Fällen verschieden, die Formeln (3. 1)—(3. 3) drücken aber aus, daß der Tensor $\hat{K}^i{}_j$ der affin-skalaren Räume eine sehr spezielle Form haben muß, die ermöglicht, daß mehrere geometrischen Eigenschaften der Finslerräume skalarer Krümmung auch in diesen Räumen gültig seien (vgl. [4] § 4). Der

[3]) Die gewöhnlichen Eigenschaften von $F(x, \dot{x})$ sind B. z. in der Einleitung von [4] angegeben.

Index: o bedeutet — wie gewöhnlich — die Überschiebung mit dem Einheitsvektor

$$l^i \overset{\text{def}}{=} \frac{\dot{x}^i}{F}, \quad \text{bzw.} \quad l_i \overset{\text{def}}{=} \frac{\partial F}{\partial \dot{x}^i}$$

des Finslerraumes.

Wir werden jetzt notwendige und hinreichende Bedingungen bestimmen dafür, daß der $\hat{\mathfrak{B}}_n$-Raum ein affin-skalarer Raum sei:

Satz 2. *Damit $\hat{K}^i{}_j$ die Form* (3. 2) (*b*) *hat, ist notwendig und hinreichend, daß $K^i{}_j$ ebenfalls die Form* (3. 2) (*b*) *habe.*

Beweis. Die Formel (2. 4) schreiben wir in der Form:

$$\hat{K}^i{}_j = K^i{}_j + F^2\{\delta^i_j(\psi_t\psi_r - \psi_{t,r})l^t l^r - l^i(-2\psi_{t,j} + \psi_{j,t} + \psi_t\psi_j)l^t\}. \tag{3. 4}$$

Offenbar ist (3. 4) mit (2. 4) vollständig identisch, da wegen der Homogenität von erster Dimension: $\psi_t \dot{x}^t = \psi$ besteht und $l^i{}_{,k} = 0$ ist. Mit der Bezeichnung

$$\gamma^*_{tj} \overset{\text{def}}{=} -2\psi_{t,j} + \psi_{j,t} + \psi_t\psi_j \tag{3. 5}$$

geht die Formel (3. 4) in

$$\hat{K}^i{}_j = K^i{}_j + F^2(\delta^i_j \gamma^*_{oo} - l^i \gamma^*_{oj}) \tag{3. 6}$$

über. Hat nun $\hat{K}^i{}_j$ die Form (3. 2) (*b*), so besteht für $K^i{}_j$ nach (3. 6) die Formel:

$$K^i{}_j = F^2(\delta^i_j \gamma_{oo} - l^i \gamma_{oj}) \quad \text{mit} \quad \gamma_{oj} \overset{\text{def}}{=} K\hat{\gamma}_{oj} - \gamma^*_{oj}.$$

Damit haben wir die Notwendigkeit bewiesen.

Wir nehmen jetzt an, daß $K^i{}_j$ die Form

$$K^i{}_j = K^* F^2(\delta^i_j \gamma_{oo} - l^i \gamma_{oj})$$

hat. Offenbar wird dann $\hat{K}^i{}_j$ nach (3. 6) die Form (3. 2) (*b*) haben, wo jetzt

$$\hat{\gamma}_{km} = K^* \gamma_{km} + \gamma^*_{km} \qquad (K \equiv 1)$$

bedeuten wird. Damit haben wir also den Satz 2 vollständig bewiesen.

Mit Hilfe der Formeln (3. 5) und (3. 6) kann auch der folgende Satz leicht bewiesen werden:

Satz 3. *Notwendig und hinreichend dafür, daß $\hat{K}^i{}_j$ die Form* (3. 3) *habe, sind die folgenden Relationen:*

$$\gamma^*_{oj} = l_j\left(K - \frac{B}{F^2}\right), \tag{3. 7a}$$

$$K^i{}_j = B(\delta^i_j - l^i l_j), \tag{3. 7b}$$

wo B den Skalar

$$B \overset{\text{def}}{=} \frac{1}{n-1} K^t{}_t \tag{3. 7c}$$

bedeutet und γ^*_{ij} durch (3. 5) *definiert ist.*

Beweis. Substituiert man γ^*_{oj} und $K^i{}_j$ aus den Formeln (3. 7*a*) und (3. 7*b*) in die Formel (3. 6), so bekommt man für $\hat{K}^i{}_j$ eben die Form (3. 3). Die Bedingungen (3. 7*a*) und (3. 7*b*) sind also hinreichend.

Um die Notwendigkeit der Formeln (3. 7*a*) und (3. 7*b*) zu beweisen, nehmen wir jetzt an, daß $\hat{K}^i{}_j$ die Form (3. 3) hat. Aus (3. 6) und (3. 3) folgt dann:

$$K^i{}_j + F^2(\delta^i_j \gamma^*_{oo} - l^i \gamma^*_{oj}) = KF^2(\delta^i_j - l^i l_j). \tag{3. 8}$$

In diesem Paragraphen haben wir vorausgesetzt, daß der Basisraum ein Finslerraum ist. Im Finslerschen Fall ist aber

$$K^i{}_j \equiv F^2 R_o{}^i{}_{oj},$$

wo $R_o{}^i{}_{oj}$ der kontrahierte Krümmungstensor des Finslerraumes ist (vgl. [2], § 38). In diesem Falle ist aber wegen $R_{(ki)mj} = 0$ (vgl. [2], § 38) auch

$$K_{oj} \equiv F^2 R_{oooj} = 0 \text{ [4]}.$$

Eine Überschiebung der Relation (3. 8) mit l_i ergibt somit:

$$l_j \gamma^*_{oo} - \gamma^*_{oj} = 0. \tag{3. 9}$$

Eine Verjüngung von (3. 8) über den Indexen i, j ergibt:

$$K^t{}_t + F^2(n-1)\gamma^*_{oo} = KF^2(n-1).$$

Nach der Bezeichnung (3. 7*c*) bekommt man aus unserer letzten Gleichung:

$$\gamma^*_{oo} = K - \frac{B}{F^2}.$$

Setzen wir nun diesen Wert von γ^*_{oo} in die Gleichung (3. 9) ein, lösen wir dann (3. 9) bezüglich γ^*_{oj}, so erhalten wir eben die Formel (3. 7*a*). Eliminieren wir nun γ^*_{oj} und γ^*_{oo} mit Hilfe von (3. 7*a*) aus der Gleichung (3. 8), so bekommen wir für $K^i{}_j$ die Formel (3. 7*b*). Damit ist der Satz 3 vollständig bewiesen.

Die projektive Veränderung (1. 2) verändert auch die Berwaldschen Übertragungsparameter G^i_{jk}. Es wird:

$$\hat{G}^i_{jk} = G^i_{jk} + \psi_{jk}\dot{x}^i + \psi_j \delta^i_k + \psi_k \delta^i_j, \quad \psi_{jk} \overset{\text{def}}{=} \frac{\partial \psi_j}{\partial \dot{x}^k} \equiv \frac{\partial^2 \psi}{\partial \dot{x}^j \partial \dot{x}^k}. \tag{3. 10}$$

Definieren wir nun ein invariantes Differential mit den Übertragungsparametern $\hat{G}^i_{jk}$ durch die Formeln

$$\hat{D}\xi^i = d\xi^i + \hat{G}^i_{jk}\xi^j dx^k + A_j{}^i{}_k \xi^j \hat{\omega}^k(d),$$

wo $A_j{}^i{}_k$ den Torsionstensor des Finslerraumes $\mathfrak{F}_n$ und

$$\hat{\omega}^k(d) \overset{\text{def}}{=} dl^k + \hat{G}^k_{ot} dx^t \equiv \hat{D}l^k$$

bedeuten, so bekommen wir einen von uns in der Arbeit [4[als affinen Finslerraum

[4]) Offenbar ist $K^o{}_j \equiv K_{ij} g^{ir} l_r \equiv K_{ij} l^i \equiv K_{oj}$.

bezeichneten Raum: $\hat{\mathfrak{A}}_n$ (vgl. [4] Definition 1 auf S. 159; statt „*" bezeichnen wir jetzt und im folgenden die Größen unseres Raumes mit „∧").

Die Hyperflächen des Raumes $\hat{\mathfrak{A}}_n$ betrachten wir als die Mannigfaltigkeit der tangentialen Linienelemente, d. h. die Grundgleichungen einer Hyperfläche $\mathfrak{H}_{n-1}$ sind durch

$$x^i = x^i(u^1, u^2, \ldots, u^{n-1}), \quad \dot{x}^i = \frac{\partial x^i}{\partial u^\alpha}\dot{u}^\alpha .$$

angegeben, wo über α von 1 bis $(n-1)$ summiert werden soll und $(u^\alpha, \dot{u}^\alpha)$ ein Linienelement der Hyperfläche bedeutet. Das invariante Differential $\hat{D}$ induziert durch Projektion ein invariantes Differential für die Hyperfläche $\mathfrak{H}_{n-1}$. Die zum invarianten Differential $\hat{D}$ gehörigen autoparallelen Kurven sind im allgemeinen von den autoparallelen Linien des induzierten invarianten Differentials von $\mathfrak{H}_{n-1}$ verschieden. Sind aber die autoparallelen Linien von $\mathfrak{H}_{n-1}$ gleichzeitig autoparallele Kurven des Raumes, so nennt man $\mathfrak{H}_{n-1}$ eine *autoparallele Hyperfläche erster Art* (vgl. [4] § 7, insbesondere Definition 5). Mit Hilfe der autoparallelen Hyperflächen erster Art kann man die $\hat{\mathfrak{A}}_n$-Räume von skalarer Krümmung vierter Art definieren. Nach der Definition ist der Raum $\hat{\mathfrak{A}}_n$ ein Raum von *skalarer Krümmung vierter Art,* falls in jedem Punkte eine autoparallele Hyperfläche erster Art A_{n-1} existiert, und für den Normalenvektor von A_{n-1} jede Richtung möglich ist (vgl. [4] § 8, insbesondere die Definition 6 auf S. 183).

Die Forderung, daß der Raum ein $\hat{\mathfrak{A}}_n$-Raum von skalarer Krümmung vierter Art sei, führt zu einer Reihe der Bedingungsgleichungen für die Krümmungstensoren. Sie bestimmen die analytischen Bedingungen dafür, daß der $\mathfrak{A}_n$-Raum ein Raum von skalarer Krümmung vierter Art sei (vgl. [4] Formeln (8. 3)). Wir wollen im folgenden die mögliche Form der Krümmungstensoren $\hat{K}^i{}_j$ und $\hat{G}^i_{jkl}$ der $\hat{\mathfrak{A}}_n$-Räume von skalarer Krümmung vierter Art zu bestimmen.

Im Beweis des Satzes 9 unserer Arbeit [4] bemerkten wir, daß der Tensor $\hat{G}^i{}_{jkl}$ ähnliche Eigenschaften hat, wie der Tensor $G^\alpha_{\beta\gamma\delta}$ in den Untersuchungen von A. Rapcsák (vgl. [5], Hauptsatz I auf S. 12). Die Ursache hiervon ist die folgende: Ist der Raum $\mathfrak{A}_n$ ein Finslerraum $\mathfrak{F}_n$, so stimmen die autoparallelen und die geodätischen Kurven des Raumes überein, und die Fläche A_{n-1} geht in eine Hyperebene erster Art über. Die Finslerräume, in denen A_{n-1}-Flächen in jedem Punkt und zu jeder Richtung existieren, sind von skalarer Krümmung vierter Art. In unseren Räumen $\hat{\mathfrak{A}}_n$ hat nur $\hat{K}_o{}^i{}_{ok}$ einen anderen Charakter als in den Räumen $\mathfrak{F}_n$. Es muß also, falls unser Raum $\hat{\mathfrak{A}}_n$ ein Raum von skalarer Krümmung vierter Art ist, $\hat{G}^i{}_{jkl}$ die Form:

$$\hat{G}^i_{jkl} = \dot{x}^i \varphi_{jkl} + \varphi_{jk}\delta^i_l + \varphi_{jl}\delta^i_k + \varphi_{kl}\delta^i_l \tag{3.11}$$

und $\hat{K}_o{}^i{}_{ok}$ nach der Formel (8. 6) von [4] die Form:

$$\hat{K}_o{}^i{}_{ok} = A_k l^i + \delta^i_k B \tag{3.12}$$

haben, wobei φ_{ij}, φ_{ijk} solche symmetrische Tensoren bedeuten, die von den Grundgrößen des Raumes gebildet sind, während A_k und B einen Vektor bzw. einen Skalar bedeuten. Es ist dabei $\hat{K}_h{}^i{}_{jk}$ das Analogon des Tensors (2. 3), d. h.

$$\hat{K}_h{}^i{}_{jk} \overset{\text{def}}{=} \frac{\partial \hat{K}^i{}_{jk}}{\partial \dot{x}^h}, \tag{3. 13}$$

wo $\hat{K}^i{}_{jk}$ durch (2. 5) angegeben ist.

Bemerkung. Die Tensoren $\hat{K}^i{}_j$, $\hat{K}^i{}_{jk}$ und $\hat{K}_h{}^i{}_{jk}$ erhält man, wenn man in die Formeln (2. 1)—(2. 3) statt G^i die Größen $\hat{G}^i$ schreibt.—

Aus (3. 13) folgt wegen der Homogenitätseigenschaften von $\hat{K}^i{}_j$ und $\hat{K}^i{}_{jk}$, daß

$$\hat{K}_o{}^i{}_{ok} = \frac{1}{F^2}\hat{K}^i{}_k \tag{3. 14}$$

ist. Aus (3. 12) und (3. 14) folgt somit, daß in einem Raum von skalarer Krümmung vierter Art $\hat{K}^i{}_k$ die Form

$$\hat{K}^i{}_k = \delta^i_k \hat{Q} + l^i \hat{P}_k \tag{3. 15}$$

haben muß, wobei $\hat{Q}$ einen Skalar und $\hat{P}_k$ einen kovarianten Vektor bedeuten: $\hat{Q} = F^2 B$, $\hat{P}_k = F\, A_k$.

Satz 4. *$\hat{K}^i{}_j$ und $\hat{G}^i_{hjk}$ bestimmen dann und nur dann einen affinen Finslerraum von skalarer Krümmung vierter Art, falls auch $K^i{}_j$ und G^i_{hjk} einen solchen Raum bestimmen.*

Beweis. Aus (3. 10) folgt:

$$\hat{G}^i_{jkl} = G^i_{jkl} + \dot{x}^i \psi_{jkl} + \psi_{jk}\delta^i_l + \psi_{jl}\delta^i_k + \psi_{kl}\delta^i_j, \tag{3. 16}$$

mit

$$\psi_{jkl} \overset{\text{def}}{=} \frac{\partial^3 \psi}{\partial \dot{x}^j \partial \dot{x}^k \partial \dot{x}^l}, \quad \psi_{jk} \overset{\text{def}}{=} \frac{\partial^2 \psi}{\partial \dot{x}^j \partial \dot{x}^k}.$$

Vergleicht man (3. 11) und (3. 16), so sieht man unmittelbar, daß G^i_{jkl} die Form

$$G^i_{jkl} = \dot{x}^i \Phi_{jkl} + \Phi_{jk}\delta^i_l + \Phi_{jl}\delta^i_k + \Phi_{kl}\delta^i_l \tag{3. 17}$$

haben muß, wo Φ_{jkl} und Φ_{jk} symmetrische Tensoren bedeuten und umgekehrt: aus (3. 17) folgt (3. 11). Es ist

$$\Phi_{jk} \overset{\text{def}}{=} \varphi_{jk} - \psi_{jk}, \quad \Phi_{jkl} \overset{\text{def}}{=} \varphi_{jkl} - \psi_{jkl}.$$

Ebenso folgt aus (3. 6) und (3. 15), daß $K^i{}_j$ die Form

$$K^i{}_j = \delta^i_j Q + l^i P_j \tag{3. 18}$$

hat, wo Q einen Skalar und P_j einen kovarianten Vektor bedeuten, und offenbar folgt aus (3. 18) und (3. 6) für $\hat{K}^i{}_j$ eine Form von (3. 15). Damit haben wir den Satz 4 vollständig bewiesen: es ist $Q = \hat{Q} - F^2\gamma^*_{oo}$, $P_j = \hat{P}_j - F^2\gamma^*_{oj}$.

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Über die Rieszschen Mittel allgemeiner Orthogonalreihen

Von L. LEINDLER in Szeged

Herrn Professor Béla Szőkefalvi-Nagy zum 50. Geburtstag gewidmet

J. MEDER [4] und K. TANDORI [6] haben Approximationssätze für die $(C, 1)$-Mittel allgemeiner Orthogonalentwicklungen mit den üblichen Methoden der Theorie der allgemeinen Orthogonalreihen bewiesen. G. ALEXITS und D. KRÁLIK [3] haben neuerlich den Satz von K. TANDORI, welcher den von J. MEDER erweitert, mit *reihentheoretischen* Methoden noch verallgemeinert. Ihr Satz lautet folgenderweise:

Sei $\{\varphi_n(x)\}$ ein im Intervall (a, b) definiertes, beliebiges Orthonormalsystem, $\{c_n\} \in l^2$ [1]) eine reelle Zahlenfolge und

$$(1) \qquad f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x) \qquad \text{(Konvergenz in } L^2(a, b)\text{)}.$$

Wir nehmen an, daß die Bedingung

$$\sum_{n=0}^{\infty} c_n^2 \vartheta^2(n) < \infty$$

die $(C, 1)$-Summierbarkeit der Reihe (1) auf einer Menge $E \subset (a, b)$ sichert, wobei $\vartheta(x)$ eine positive, monoton gegen $+\infty$ wachsende Funktion ist. Bedeutet $\bar{l}(x)$ eine positive, von unten konkave, monoton gegen $+\infty$ strebende Funktion, für welche $x^\gamma/\bar{l}(x)$ mit festem $0 < \gamma < 1$ bei genügend großem x monoton nicht abnimmt, so folgt aus dem Erfülltsein der Bedingung

$$\sum_{n=0}^{\infty} c_n^2 \bar{l}^2(n) \vartheta^2(n) < \infty,$$

daß die $(C, 1)$-Mittel $\sigma_n(x)$ der Orthogonalreihe (1) die Funktion $f(x)$ auf E mit dem Annäherungsgrad

$$(2) \qquad |\sigma_n(x) - f(x)| = o_x\left(\frac{1}{\bar{l}(n)}\right)$$

approximieren.

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[1]) D. h. $\sum_{n=0}^{\infty} c_n^2 < \infty$.

In § 2 beweisen wir wieder mit den klassischen Methoden der Theorie der allgemeinen Orthogonalreihen drei mit Rieszscher Summation verknüpfte Sätze, aus denen sich als Spezialfälle der Satz von G. ALEXITS und D. KRÁLIK und das Ergebnis, daß im Falle $\bar{l}(x)=x^\gamma$ $(0<\gamma<1)$ schon die Bedingung $\sum c_n^2\bar{l}^2(n)\equiv\sum c_n^2 n^{2\gamma}<\infty$ für (2) *hinreichend* ist, ergeben. Bevor wir unsere Sätze formulieren, führen wir einige Begriffe und Bezeichnungen ein, die wir im folgenden immer in demselben Sinne verwenden.

Sei $\lambda(\omega)$ $(\omega\geqq 0)$ eine positive, im strengen Sinne wachsende Funktion mit $\lambda(0)=0$ und $\lambda(n)\to\infty$ und $\Lambda(\omega)$ die eindeutig bestimmte inverse Funktion von $\lambda(\omega)$. Die Orthogonalreihe (1) heißt $(R, \lambda(n), 1)$-summierbar zur im Falle $\{c_n\}\in l^2$ durch den Riesz—Fischerschen Satz bis auf eine Nullmenge eindeutig bestimmten Funktion $f(x)$, wenn für $\omega=n$ mit $n\to\infty$

$$R_\omega(x)=\frac{1}{\lambda(\omega)}\sum_{k\leqq\omega}(\lambda(\omega)-\lambda(k))c_k\varphi_k(x)\to f(x)$$

fast überall gilt. Wir setzen

$$s_\omega(x)=\sum_{k\leqq\omega}c_k\varphi_k(x),\quad m_n=\Lambda(2^n)\quad\text{und}\quad \bar{m}_n=[m_n];\,{}^{2)}$$

hier sind ω und m_n nicht notwendigerweise ganze Zahlen. Es seien $\varrho(\omega)$ $(\omega\geqq 0)$ eine positive monoton gegen Unendlich wachsende, $\mu(\omega)$ $(\omega\geqq 0)$ eine positive, monoton nichtabnehmende Funktion und $\gamma(\omega)$ $(\omega\geqq 0)$ eine positive, monotone Funktion, für welche die Funktion $\gamma(\omega)/\varrho(\omega)$ monoton gegen Null konvergiert.

Satz I. *Es sei $l(\omega)$ eine positive monoton nichtabnehmende Funktion mit $l(\bar{m}_n+1)=l(\bar{m}_n+2)=\ldots=l(\bar{m}_{n+1})$ $(n=1,2,\ldots)$ und*

$$l(m_{n+1})\leqq Kl(m_n)\qquad(n=1,2,\ldots;\ 0<K<2).\tag{3}$$

Ferner sei $\{c_n\}$ eine Zahlenfolge mit der Eigenschaft

$$\sum_{n=0}^{\infty}c_n^2l^2(n)<\infty.\tag{4}$$

Ist die Reihe

$$\sum_{n=0}^{\infty}c_nl(n)\varphi_n(x)\tag{5}$$

auf einer Menge $E\subset(a,b)$ $(R,\lambda(n),1)$-summierbar, so approximieren die $(R,\lambda(n),1)$-Mittel $R_n(x)$ der Orthogonalreihe (1) die Funktion $f(x)$ auf E fast überall mit dem Annäherungsgrad

$$|R_n(x)-f(x)|=o_x\left(\frac{1}{l(n)}\right).\tag{6}$$

In diesem Satz können die „guten" Eigenschaften des Systems $\{\varphi_n(x)\}$ ausgenutzt werden. Im Spezialfall $\lambda(n)=n$ ist es bekannt, daß die Reihe (5) im Fall (4) z. B. für das trigonometrische System $(R, n, 1)$- (d. h. $(C, 1)$-) summierbar ist; dasselbe gilt für ein beliebiges Orthogonalsystem $\{\varphi_n(x)\}$, dessen 2^n-ten Lebesgue-

[2]) $[a]$ bezeichnet den ganzen Teil von a.

schen Funktionen ($n=1, 2, \ldots$) auf der Menge E gleichmäßig beschränkt sind (siehe z. B. ALEXITS [2]). Die Ersatzbedingungen für $l(\omega)$ sind leider nötig.

Aus dem Satz I ergibt sich der folgende

Satz II. *Sichert die Bedingung*

$$\sum_{n=0}^{\infty} c_n^2 \varrho^2(n) < \infty \tag{7}$$

die $(R, \lambda(n), 1)$*-Summierbarkeit der Reihe* (1) *für jede die Bedingung* (7) *erfüllenden Koeffizientenfolgen* $\{c_n\}$ *auf einer Menge* $E \subset (a, b)$ *fast überall, so folgt aus dem Erfülltsein der Bedingungen*

$$\sum_{n=0}^{\infty} c_n^2 \varrho^2(n) \mu^2(n) < \infty \tag{8}$$

und

$$\mu(m_{n+1}) \leqq K\mu(m_n) \qquad (n=1, 2, \ldots;\ 0<K<2), \tag{9}$$

daß die $(R, \lambda(n), 1)$*-Mittel* $R_n(x)$ *der Orthogonalreihe* (1) *die Funktion* $f(x)$ *auf* E *fast überall mit dem Annäherungsgrad*

$$|R_n(x) - f(x)| = o_x\left(\frac{1}{\mu(n)}\right) \tag{10}$$

approximieren.

Der folgende Satz besagt, daß für die $(R, \lambda(n), 1)$-Summierbarkeit die Heranziehung der Folge $\{\varrho(n)\}$ in gewißen Fällen unnötig ist.

Satz III. *Gilt*

$$K_1\mu(m_n) \leqq \mu(m_{n+1}) \leqq K_2\mu(m_n) \quad \textit{mit} \quad 1 < K_1 \leqq K_2 < 2, \tag{11}$$

so folgt schon aus der Bedingung

$$\sum_{n=0}^{\infty} c_n^2 \mu^2(n) < \infty, \tag{12}$$

daß die $(R, \lambda(n), 1)$*-Mittel* $R_n(x)$ *der Orthogonalreihe* (1) *die Funktion* $f(x)$ *in* (a, b) *fast überall mit dem Annäherungsgrad* (10) *approximieren.*

Es ist bekannt, daß im Falle des $(C, 1)$-Verfahrens $\lambda(n) = \Lambda(n) = n$ ist. So sind die Bedingung (9) mit $\mu(2^{n+1}) \leqq K\mu(2^n)$ $(0<K<2)$ und die Bedingung (11) mit $\mu(n) = \alpha n^\gamma$ $(\alpha > 0, 0 < \gamma < 1)$ erfüllt. Daraus ergeben sich unsere vorerwähnten Behauptungen.

Der folgende Satz ergibt sich unmittelbar aus dem Hilfssatz I, der in den Approximationssätzen eine analoge Rolle spielt, wie das Kroneckersche Lemma in den Größenordnungssätzen:

Satz IV. *Es sei* $\{p_n\}$ *eine im strengen Sinne wachsende Indexfolge und* $\{w_n\}$ *eine positive, monoton nichtabnehmende Zahlenfolge mit* $w_{p_n+1} = w_{p_n+2} = \ldots = w_{p_{n+1}}$.

$(n=1, 2, \ldots)$. *Konvergieren die Partialsummen* $\tilde{s}_{p_n}(x)$ *der Orthogonalreihe*

$$\sum_{n=1}^{\infty} c_n w_n \varphi_n(x)$$

auf einer Menge $E \subset (a, b)$, *so approximieren die Partialsummen* $s_{p_n}(x)$ *der Reihe* (1) *auf* E *die Funktion* $f(x)$ *mit dem Annäherungsgrad*

$$|s_{p_n}(x) - f(x)| = o_x\left(\frac{1}{w_n}\right).$$

Dieser Satz ist eine Verallgemeinerung eines Satzes von K. Tandori [6].

In § 3 geben wir Sätze für die Größenordnung der $(R, \lambda(n), 1)$-Mittel der Orthogonalreihe (1).

Satz V. *Es sei* $l(\omega)$ *eine positive, monoton nichtabnehmende Funktion mit* $l(\overline{m}_n+1) = l(\overline{m}_n+2) = \ldots = l(\overline{m}_{n+1})$ $(n=1, 2, \ldots)$ und $l(n) \to \infty$. *Ferner sei* $\{c_n\}$ *eine Koeffizienten folge mit der Eigenschaft*

$$\sum_{n=1}^{\infty} c_n^2 [l(n)]^{-2} < \infty. \tag{13}$$

Ist die Reihe

$$\sum_{n=1}^{\infty} c_n [l(n)]^{-1} \varphi_n(x) \tag{14}$$

auf einer Menge $E \subset (a, b)$ $(R, \lambda(n), 1)$-*summierbar, so gilt die Abschätzung*

$$R_n(x) = o_x(l(n)) \tag{15}$$

für die $(R, \lambda(n), 1)$-*Mittel* $R_n(x)$ *der Orthogonalreihe* (1) *auf* E *fast überall.*

Die „guten" Eigenschaften des Systems $\{\varphi_n(x)\}$ können auch im Satz V ausgenutzt werden.

Aus dem Satz V folgt der folgende:

Satz VI. *Sichert die Bedingung* (7) *die* $(R, \lambda(n), 1)$-*Summierbarkeit der Reihe* (1) *für jede, die Bedingung* (7) *erfüllende Koeffizientenfolge* $\{c_n\}$ *auf einer Menge* $E \subset (a, b)$, *so folgt aus der Bedingung*

$$\sum_{n=1}^{\infty} c_n^2 \gamma^2(n) < \infty$$

für die $(R, \lambda(n), 1)$-*Mittel* $R_n(x)$ *der Orthogonalreihe* (1) *auf* E *die Abschätzung*

$$R_n(x) = o_x\left(\frac{\varrho(n)}{\gamma(n)}\right). \tag{16}$$

Für $\gamma(n) \equiv 1$ wurde dieser Satz im Spezialfall $\varrho(n) = \log\log n$, $\lambda(n) = n$ von K. Tandori [7] und im Fall $\varrho(n) = \log\log\log n$, $\lambda(n) = \log n$ von J. Meder [5] bewiesen.

§ 1. Hilfssätze

Zum Beweis unserer Sätze werden wir einige Hilfssätze vorausschicken.

Hilfssatz I. *Es sei* $\{p_m\}$ *eine Indexfolge:* $(1 \leqq) p_1 < p_2 < \ldots < p_m < \ldots$. *Sind* $\{u_n\}$ *eine beliebige, und* $\{\lambda_n\}$ *eine positive, monoton nichtabnehmende Zahlenfolge mit* $\lambda_{p_m+1} = \ldots = \lambda_{p_{m+1}}$ $(m = 1, 2, \ldots)$, *so folgt aus der Konvergenz der* p_m*-ten Partialsummen* $s^*_{p_m}$ *der Reihe* $\sum_{n=1}^{\infty} u_n \lambda_n$ *die Konvergenz der* p_m*-ten Partialsummen* s_{p_m} *der Reihe* $\sum_{n=1}^{\infty} u_n$ *und es gilt für* $s = \lim s_{p_m}$ *die Beziehung*

$$|s_{p_m} - s| = o\left(\frac{1}{\lambda_{p_{m+1}}}\right). \tag{1. 1}$$

Die Notwendigkeit der Ersatzbedingung für $\{\lambda_n\}$ kann mit einem Gegenbeispiel leicht bewiesen werden.

Beweis von Hilfssatz I. Durch Abelsche Umformung ergibt sich für

$$s_{p_{m+k}} - s_{p_m} = \sum_{\nu = p_m + 1}^{p_{m+k}} \frac{1}{\lambda_\nu} \lambda_\nu u_\nu$$

die Abschätzung

$$|s_{p_{m+k}} - s_{p_m}| = \left| \sum_{\nu = p_m+1}^{p_{m+k}-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right)(s^*_\nu - s^*_{p_m}) + \frac{1}{\lambda_{p_{m+k}}}(s^*_{p_{m+k}} - s^*_{p_m}) \right| =$$

$$= \left| \sum_{i=m+1}^{m+k-1} \left(\frac{1}{\lambda_{p_i}} - \frac{1}{\lambda_{p_{i+1}}}\right)(s^*_{p_i} - s^*_{p_m}) + \frac{1}{\lambda_{p_{m+k}}}(s^*_{p_{m+k}} - s^*_{p_m}) \right| =$$

$$= o\left(\frac{1}{\lambda_{p_{m+1}}} - \frac{1}{\lambda_{p_{m+k}}}\right) + o\left(\frac{1}{\lambda_{p_{m+k}}}\right) = o\left(\frac{1}{\lambda_{p_{m+1}}}\right).$$

Daraus folgt im Falle $k \to \infty$ die Beziehung (1. 1).

Damit haben wir Hilfssatz I bewiesen.

Hilfssatz II. *Gilt*

$$\frac{\mu(\nu_{n+1})}{\mu(\nu_n)} \geqq K > 1, \tag{1. 2}$$

so folgt aus der Bedingung (12), *daß die Partialsummen* $s_{\nu_n}(x)$ *der Reihe* (1) *in* (a, b) *die Funktion* $f(x)$ *fast überall mit dem Annäherungsgrad*

$$|s_{\nu_n}(x) - f(x)| = o_x\left(\frac{1}{\mu(\nu_n)}\right) \tag{1. 3}$$

approximieren. (Hier sind ν_n nicht notwendigerweise ganze Zahlen.)

Beweis von Hilfssatz II. Dann ist nach (1. 2) und (12)

$$\sum_{n=1}^{\infty} \mu^2(\nu_n) \int_a^b (s_{\nu_n}(x)-f(x))^2\,dx = \sum_{n=1}^{\infty} \mu^2(\nu_n) \sum_{k>\nu_n} c_k^2 =$$

$$= \sum_{n=1}^{\infty} \mu^2(\nu_n) \sum_{m=n}^{\infty} \Big(\sum_{\nu_m<k\leqq\nu_{m+1}} c_k^2 \Big) = O(1) \sum_{m=1}^{\infty} \Big(\sum_{\nu_m<k\leqq\nu_{m+1}} c_k^2 \Big) \sum_{n=1}^{m} \mu^2(\nu_n) =$$

$$= O(1) \sum_{m=1}^{\infty} \Big(\sum_{\nu_m<k\leqq\nu_{m+1}} c_k^2 \Big) \mu^2(\nu_m) = O(1) \sum_{n=1}^{\infty} c_n^2 \mu^2(n) < \infty.$$

Nach dem Satz von B. Levi konvergiert die Reihe

$$\sum_{n=1}^{\infty} \mu^2(\nu_n) (s_{\nu_n}(x)-f(x))^2$$

fast überall und so gilt (1. 3) in (a, b) fast überall, womit der Hilfssatz II bewiesen ist.

Hilfssatz III. *Unter den Voraussetzungen von Hilfssatz I und der ergänzenden Voraussetzung $\lambda_n \to \infty$ folgt aus der Konvergenz der p_m-ten Partialsummen $\tilde{s}_{p_m}$ der Reihe $\sum_{n=1}^{\infty} u_n \lambda_n^{-1}$ die Abschätzung*

$$s_{p_m} = o(\lambda_{p_m}). \tag{1. 4}$$

Beweis von Hilfssatz III. Die Behauptung folgt unmittelbar aus dem Kroneckerschen Lemma (siehe z. B. Alexits [1], S. 68), wenn wir dieses auf die Zahlenfolge $\{\lambda_{p_m}\}$ und die Reihe $\sum_{m=0}^{\infty} \Big(\sum_{\nu=p_m+1}^{p_{m+1}} u_\nu \Big)$ anwenden.

Hilfssatz IV. *Es sei $\tilde{l}(\omega)$ eine positive, monotone Funktion mit $\tilde{l}(\overline{m}_n+1) = \tilde{l}(\overline{m}_n+2) = \ldots = \tilde{l}(\overline{m}_{n+1})$ und*

$$\tilde{l}(m_{n+1}) \leqq K\tilde{l}(m_n) \qquad (n=1, 2, \ldots;\ 0<K<2). \tag{1. 5}$$

Dann folgt aus der Bedingung

$$\sum_{n=1}^{\infty} c_n^2 \tilde{l}^2(n) < \infty \tag{1. 6}$$

die Abschätzung

$$|s_{\overline{m}_n}(x) - R_k(x)| = o_x\left(\frac{1}{\tilde{l}(k)}\right) \tag{1. 7}$$

für jede n und k mit $\overline{m}_n < k \leqq \overline{m}_{n+1}$, fast überall in (a, b).

Beweis von Hilfssatz IV. Wir werden die Gleider auf der rechten Seite der Ungleichung

$$|s_{\overline{m}_n}(x) - R_k(x)| \leqq |s_{\overline{m}_n}(x) - R_{m_n}(x)| + |R_{m_n}(x) - R_{\overline{m}_n+1}(x)| + |R_{\overline{m}_n+1}(x) - R_k(x)| \tag{1. 8}$$

einzeln abschätzen. Mit einfacher Rechnung ergibt sich nach (1. 5) und (1. 6)

$$\sum_{n=1}^{\infty} \tilde{l}^2(m_n) \int_a^b (s_{m_n}(x) - R_{m_n}(x))^2 \, dx = O(1) \sum_{n=1}^{\infty} \tilde{l}^2(m_n) \frac{1}{2^{2n}} \sum_{k=1}^{\overline{m}_n} \lambda^2(k) c_k^2 =$$

$$= O(1) \sum_{n=1}^{\infty} \tilde{l}^2(m_n) \frac{1}{2^{2n}} \sum_{\nu=0}^{n-1} \sum_{k=\overline{m}_\nu+1}^{\overline{m}_{\nu+1}} \lambda^2(k) c_k^2 =$$

$$= O(1) \sum_{\nu=0}^{\infty} \sum_{k=\overline{m}_\nu+1}^{\overline{m}_{\nu+1}} \lambda^2(k) c_k^2 \sum_{n=\nu+1}^{\infty} \frac{\tilde{l}^2(m_n)}{2^{2n}} =$$

$$= O(1) \sum_{\nu=0}^{\infty} \sum_{k=\overline{m}_\nu+1}^{\overline{m}_{\nu+1}} \lambda^2(k) c_k^2 \frac{\tilde{l}^2(m_{\nu+1})}{2^{2\nu}} = O(1) \sum_{n=1}^{\infty} c_n^2 \tilde{l}^2(n) < \infty.$$

Daraus ergibt sich, daß die Reihe

$$\sum_{n=1}^{\infty} \tilde{l}^2(m_n)(s_{m_n}(x) - R_{m_n}(x))^2$$

fast überall konvergiert; also gilt fast überall

$$s_{\overline{m}_n}(x) - R_{m_n}(x) = o_x\left(\frac{1}{\tilde{l}(m_n)}\right). \tag{1.9}$$

Weiterhin gilt die Beziehung

$$|R_{\overline{m}_n+1}(x) - R_{m_n}(x)| = o_x\left(\frac{1}{\tilde{l}(m_n)}\right) \qquad (m_n < \overline{m}_{n+1}) \tag{1.10}$$

fast überall in (a, b). Dies folgt mit Anwendung des Satzes von B. LEVI daraus, daß

$$\sum_{n=1}^{\infty}{}' \tilde{l}^2(m_n) \int_a^b (R_{\overline{m}_n+1}(x) - R_{m_n}(x))^2 \, dx = O(1) \sum_{n=1}^{\infty}{}' \tilde{l}^2(m_n) \frac{1}{\lambda^2(m_n)} \sum_{k=1}^{\overline{m}_n} \lambda^2(k) c_k^2 =$$

$$= O(1) \sum_{\nu=0}^{\infty} \sum_{k=\overline{m}_\nu+1}^{\overline{m}_{\nu+1}} \lambda^2(k) c_k^2 \sum_{n=\nu+1}^{\infty} \frac{\tilde{l}^2(m_n)}{2^{2n}} = O(1) \sum_{n=1}^{\infty} c_n^2 \tilde{l}^2(n) < \infty. \text{ [3]}$$

Nach (1. 6) gilt auch die Ungleichung

$$\sum_{i=1}^{\infty} \frac{\lambda(i) \tilde{l}^2(i)}{\lambda(i+1) - \lambda(i)} \int_a^b (R_i(x) - R_{i-1}(x))^2 \, dx =$$

$$= O(1) \sum_{i=1}^{\infty} \frac{(\lambda(i+1) - \lambda(i)) \tilde{l}^2(i)}{\lambda(i) \lambda^2(i+1)} \sum_{k=1}^{i} \lambda^2(k) c_k^2 =$$

[3] Σ' bedeutet, daß man nur für die n mit $m_n < \overline{m}_{n+1}$ summiert.

$$= O(1) \sum_{k=1}^{\infty} \lambda^2(k) c_k^2 \sum_{i=k}^{\infty} \frac{\tilde{l}^2(i)(\lambda(i+1)-\lambda(i))}{\lambda(i)\lambda^2(i+1)} =$$

$$= O(1) \sum_{k=1}^{\infty} \lambda^2(k) c_k^2 \sum_{i=k}^{\infty} \tilde{l}^2(i) \left(\frac{1}{\lambda^2(i)} - \frac{1}{\lambda^2(i+1)}\right) =$$

$$= O(1) \sum_{k=1}^{\infty} \lambda^2(k) c_k^2 \frac{\tilde{l}^2(k)}{\lambda^2(k)} = O(1) \sum_{k=1}^{\infty} c_k^2 \tilde{l}^2(k) < \infty.$$

Nach dem Obigen gilt

$$\sum_{i=\bar{m}_n+1}^{\bar{m}_{n+1}} \frac{\lambda(i)\,\tilde{l}^2(i)}{\lambda(i+1)-\lambda(i)} (R_i(x)-R_{i-1}(x))^2 = o_x(1).$$

Offensichtlich ist

$$\sum_{i=\bar{m}_n+1}^{\bar{m}_{n+1}} \frac{\lambda(i+1)-\lambda(i)}{\lambda(i)} \leqq \frac{\lambda(\bar{m}_{n+1}+1)}{\lambda(\bar{m}_n+1)} = O(1),$$

also ergibt sich durch Anwendung der Cauchyschen Ungleichung:

$$|R_{\bar{m}_n+1}(x) - R_k(x)| \leqq \left\{\sum_{i=\bar{m}_n+2}^{k} \frac{\lambda(i)\,\tilde{l}^2(i)}{\lambda(i+1)-\lambda(i)} (R_i(x)-R_{i-1}(x))^2\right\}^{1/2} \cdot$$

$$\cdot \left\{\sum_{i=\bar{m}_n+2}^{k} \frac{\lambda(i+1)-\lambda(i)}{\lambda(i)\,\tilde{l}^2(i)}\right\}^{1/2} = o_x\left(\frac{1}{\tilde{l}(k)}\right).$$

Daraus und aus (1. 9) und (1. 10) folgt die Abschätzung (1. 7) nach (1. 8).

Damit haben wir den Hilfssatz IV bewiesen.

Hilfssatz V. *Damit die Orthogonalreihe* (1) *mit* $\{c_n\} \in l^2$ *auf einer Menge* E $(R, \lambda(n), 1)$*-summierbar ist, ist notwendig und hinreichend, daß die Partialsummen* $s_{\bar{m}_n}(x)$ *auf* E *fast überall konvergieren.*

Der Hilfssatz V wurde von A. Zygmund [8] bewiesen.

§ 2. Approximationen

Beweis von Satz I. Wir nehmen an, daß die Koeffizientenfolge $\{c_n\}$ und die Funktion $l(\omega)$ die Bedingungen des Satzes I erfüllen. Dann können wir den Hilfssatz V mit der Koeffizientenfolge $\{c_n l(n)\}$ anwenden, und so ergibt sich die Konvergenz der Partialsummen $\tilde{s}_{\bar{m}_n}(x)$ der Reihe (5) auf E fast überall. Also können wir den Hilfssatz I für die Reihe (5) mit $p_n = \bar{m}_n$ anwenden, und so ergibt sich auf E fast überall der Annäherungsgrad

$$|s_{\bar{m}_n}(x) - f(x)| = o_x\left(\frac{1}{l(\bar{m}_n)}\right). \tag{2. 1}$$

Nun wenden wir den Hilfssatz IV mit $\tilde{l}(\omega) = l(\omega)$ an; dies ist möglich nach den

Bedingungen (3) und (4). So ergibt sich die Abschätzung

$$|s_{\overline{m}_n}(x)-R_k(x)| = o_x\left(\frac{1}{l(k)}\right) \tag{2. 2}$$

für jede n und k mit $\overline{m}_n<k\leqq\overline{m}_{n+1}$, fast überall in $(a; b)$. Aus (2. 1) und (2. 2) folgt die Behauptung (6).

Damit haben wir den Satz I bewiesen.

Beweis von Satz II. Es sei

$$\tilde{\mu}(\omega)=\begin{cases}\mu(\overline{m}_{n+1}) & \text{für} \quad \overline{m}_n+1\leqq\omega\leqq\overline{m}_{n+1},\\ \mu(\omega) & \text{sonst.}\end{cases}$$

Nach den Bedingungen des Satzes II ist die Reihe $\sum c_n\tilde{\mu}(n)\varphi_n(x)$ fast überall auf E $(R, \lambda(n), 1)$-summierbar, weiter bestehen die Bedingungen (3) und (4) des Satzes I mit $l(\omega)=\tilde{\mu}(\omega)$. Darum können wir Satz I anwenden: so folgt nach der Definition von $\tilde{\mu}(\omega)$ der Annäherungsgrad (10) fast überall auf E.

Damit haben wir den Satz II vollständig bewiesen.

Beweis von Satz III. Wir wenden den Hilfssatz II mit $v_n=m_n$ an, so bekommen wir die Beziehung

$$|s_{m_n}(x)-f(x)| = o_x\left(\frac{1}{\mu(m_n)}\right)$$

in (a, b) fast überall. Auf Grund von (11) und (12) können wir Hilfssatz IV mit $\tilde{l}(\omega)=\mu(\omega)$ anwenden und so ergibt sich nach (1. 7) die Abschätzung

$$|s_{m_n}(x)-R_k(x)| = o_x\left(\frac{1}{\mu(k)}\right)$$

für jede n und k mit $m_n<k\leqq m_{n+1}$, fast überall in (a, b). Aus dem Obigen und (11) folgt die Behauptung des Satzes III.

§ 3. Abschätzungen

Beweis von Satz V. Wir nehmen an, daß die Bedingungen des Satzes V erfüllt sind. Wir wenden zuerst Hilfssatz V mit der Koeffizientenfolge $\{c_n l^{-1}(n)\}$ an. So ergibt sich die Konvergenz der Partialsummen $\tilde{s}_{\overline{m}_n}(x)$ der Reihe (14) auf E fast überall. Also können wir Hilfssatz III für die Reihe (14) mit $p_n=\overline{m}_n$ anwenden: so ergibt sich nach (1. 4) fast überall auf E die Abschätzung

$$s_{\overline{m}_n}(x) = o_x(l(\overline{m}_n)).$$

Hiernach wenden wir Hilfssatz IV mit $\tilde{l}(\omega)=l^{-1}(\omega)$ an. So bekommen wir die Abschätzung

$$|s_{\overline{m}_n}(x)-R_k(x)| = o_x(l(k))$$

für jede n und k mit $m_n<k\leqq m_{n+1}$, fast überall in (a, b). Aus dem Obigen ergibt sich die Behauptung (15).

Damit haben wir den Satz V vollständig bewiesen.

Beweis von Satz VI. Es sei

$$\tilde{\gamma}(\omega)=\begin{cases}\dfrac{\gamma(\overline{m}_{n+1})}{\varrho(\overline{m}_{n+1})} & \text{für} \quad \overline{m}_n+1\leqq\omega\leqq\overline{m}_{n+1},\\[2ex] \dfrac{\gamma(\omega)}{\varrho(\omega)} & \text{sonst.}\end{cases}$$

Nach den Bedingungen des Satzes VI ist die Reihe

$$\sum_{n=1}^{\infty} c_n\tilde{\gamma}(n)\varphi_n(x)$$

auf E fast überall $(R, \lambda(n), 1)$-summierbar. Mit $[l(\omega)]^{-1}=\tilde{\gamma}(\omega)$ erfüllen sich auch die weiteren Bedingungen des Satzes V, also können wir den Satz V anwenden. So ergibt sich die Abschätzung (16) fast übrall auf E.

Damit ist der Satz VI bewiesen.

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Über die Konvergenz der Orthogonalreihen

Von KÁROLY TANDORI in Szeged

Herrn Professor Béla Szőkefalvi-Nagy zum 50. Geburtstag gewidmet

Einleitung

Für jede endliche Folge $c_1, \dots, c_N$ von reellen Zahlen setzen wir

$$I(c_1, \dots, c_N) = \sup \int_0^1 \Big(\max_{1 \leqq i \leqq j \leqq N} |c_i \varphi_i(x) + \dots + c_j \varphi_j(x)|\Big)^2 \, dx,$$

wobei das Supremum für alle im Intervall [0, 1] orthonormierten Funktionensysteme $\{\varphi_n(x)\}$ $(n=1, \dots, N)$ gebildet wird. Offensichtlich hängt $I(c_1, \dots, c_N)$ nur von den von Null verschiedenen Gliedern der Folge $\{c_n\}$ ab, und zwar stetig.

Satz I. *Sei $\{a_n\}_1^\infty$ eine gegebene Folge von reellen Zahlen. Gilt*

$$\sum_{k=0}^{\infty} I(a_{n_k+1}, \dots, a_{n_{k+1}}) < \infty \tag{1}$$

für jede Indexfolge $(0=)n_0 < \dots < n_k < \dots$, so konvergiert die Reihe

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{2}$$

für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall. Gilt aber (1) für eine Indexfolge nicht, so gibt es ein orthonormiertes System $\{\varphi_n(x)\}$, für welches die Reihe (2) sogar fast überall divergiert. Insbesondere ist also das Bestehen von (1) für jede Indexfolge notwendig und hinreichend dafür, daß die Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall konvergiert.

Für die Funktion I gibt es keine explizite Darstellung, wohl aber verschiedene Abschätzungen. Aus dem klassischen Resultat von MENCHOFF und RADEMACHER[1])

[1]) D. MENCHOFF, Sur les séries de fonctions orthogonales (Première partie), *Fundamenta Math.*, 4 (1923), 82–105; H. RADEMACHER, Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen, *Math. Annalen*, 87 (1922), 112–138.

folgt leicht die obere Abschätzung:

$$I(c_1, \ldots, c_N) = O(1) \sum_{n=1}^{N} c_n^2 \log^2 (n+1),$$

woraus man, auf Grund von Satz I, den klassischen Satz von MENCHOFF und RADEMACHER[2]) bekommt, daß im Falle $\sum a_n^2 \log^2 n < \infty$ die Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall konvergiert.

Da

$$|c_1 \varphi_1(x) + \ldots + c_N \varphi_N(x)| \leqq \max_{1 \leqq i \leqq j \leqq N} |c_i \varphi_i(x) + \ldots + c_j \varphi_j(x)|$$

gilt, besteht die untere Abschätzung

$$c_1^2 + \ldots + c_N^2 \leqq I(c_1, \ldots, c_N).$$

Verfasser[3]) hat eine feinere untere Abschätzung angegeben:

$$I(c_1, \ldots, c_N) \geqq \varrho \sum_{n=1}^{N} c_n^2 \lambda_n(c_1, \ldots, c_N) \qquad (\varrho > 0)$$

mit

$$\lambda_i(c_1, \ldots, c_N) = \begin{cases} (\log (c_1^2 + \ldots + c_N^2) - \log c_i^2)^2, & \text{wenn} \quad c_1^2 + \ldots + c_N^2 \geqq 8 c_i^2 > 0, \\ 1 & \text{sonst.} \end{cases}$$

Auf Grund dieser Abschätzung folgt aus dem Satz I leicht, daß $\sum a_n^2 \log^2 1/a_n^2 < \infty$ mit $a_n \neq 0$ und $a_n \to 0$ notwendig dafür ist, damit die Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall konvergiert[4]).

Es sei $(0=)m_0 < \ldots < m_k < \ldots$ eine gegebene Indexfolge. Wir setzen

$$A_k = \{a_{m_k+1}^2 + \ldots + a_{m_{k+1}}^2\}^{1/2} \qquad (k=0, 1, \ldots).$$

Satz II. *Das Bestehen der Bedingung*

$$\sum_{k=0}^{\infty} I(A_{n_k+1}, \ldots, A_{n_{k+1}}) < \infty \tag{3}$$

für jede Indexfolge $0 = n_0 < \ldots < n_k < \ldots$ ist hinreichend dafür, daß die Folge der m_k-ten Partialsummen von (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überal konvergiert. Gilt aber (3) für eine Indexfolge nicht, so gibt es ein orthonormiertes System $\{\varphi_n(x)\}$, für welches die Folge der m_k-ten Partialsummen der Reihe (2) sogar fast überall divergiert. Insbesondere ist also das Bestehen von (3) für jede Indexfolge $\{n_k\}$ notwendig und hinreichend dafür, daß die m_k-ten Partialsummen der Reihe (2) für jedes Orthonormalsystem fast überall konvergieren.

Es sei $T = \|c_{i,j}\|$ eine Matrix mit

$$\lim_{i\to\infty} c_{i,j} = 0 \quad (j=1, 2, \ldots) \quad \text{und} \quad \lim_{i\to\infty} \sum_{j=1}^{\infty} c_{i,j} = 1,$$

weiterhin nehmen wir an, daß die T-Summierbarkeit der Orthogonalreihe (2) für

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[2]) Siehe loc. cit. 1).

[3]) K. TANDORI, Über die Divergenz der Orthogonalreihen, *Publicationes Math. Debrecen*, **8** (1961), 291–307.

[4]) Siehe loc. cit. 3).

eine beliebige Koeffizientenfolge $\{a_n\}$ $(\sum a_n^2 < \infty)$ und für ein beliebiges orthonormiertes System $\{\varphi_n(x)\}$ mit der Konvergenz der Folge der m_k-ten Partialsummen fast überall äquivalent ist. Es gilt dann die folgende Behauptung:

Damit die Reihe (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}$ *fast überall T-summierbar ist, ist es notwending und hinreichend, daß die Bedingung* (3) *für jede Indexfolge* $\{n_k\}$ *gilt.*

Ist nämlich (3) für jede Indexfolge $\{n_k\}$ erfüllt, so konvergiert die Folge der m_k-ten Partialsummen der Reihe (2) fast überall. Aus (3) folgt also $\sum a_n^2 < \infty$ und auf Grund unserer Annahme über das Summationsverfahren T ergibt sich, daß die Reihe (2) fast überall T-summierbar ist. Ist aber (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall T-summierbar, so ist $\sum a_n^2 < \infty$. Im Falle $\sum a_n^2 = \infty$ ist nämlich die Rademachersche Reihe $\sum a_n r_n(x)$ fast überall nicht T-summierbar[5]). So folgt aus unserer Annahme über das Verfahren T, daß die Folge der m_k-ten Partialsummen der Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall konvergiert und so besteht (3) auf Grund des Satzes II.

Nach den Sätzen von KOLMOGOROFF[6]) und KACZMARZ[7]) ist die $(C, 1)$-Summierbarkeit der Reihe (2) für eine beliebige Koeffizientenfolge $\{a_n\}$ $(\sum a_n^2 < \infty)$ und für jedes orthonormierte System $\{\varphi_n(x)\}$ mit der Konvergenz der Folge der 2^m-ten Partialsummen fast überall äquivalent. Also gilt die folgende Behauptung:

Damit die Reihe (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}$ *fast überall* $(C, 1)$-*summierbar ist, ist es notwendig und hinreichend, daß die Bedingung*

$$\sum_{k=0}^{\infty} I(\bar{A}_{n_k+1}, \ldots, \bar{A}_{n_{k+1}}) < \infty \tag{4}$$

für jede Indexfolge $(0=)n_0 < \ldots < n_k < \ldots$ *erfüllt wird, wobei*

$$\bar{A}_m = \{a_{2^m+1}^2 + \ldots + a_{2^{m+1}}^2\}^{1/2} \qquad (m=0, 1, \ldots)$$

gesetzt wird.

Offensichtlich können ähnliche Sätze auch für andere Summationsverfahren, z. B. für die Rieszsche Summation bewiesen werden. Mit Anwendung der erwähnten Abschätzungen folgen aus Satz II einige bekannte Sätze über die $(C, 1)$-Summierbarkeit der Orthogonalreihen.[8])

Da aus Bedingung (1) insbesondere

$$\sum a_n^2 < \infty$$

folgt, so konvergiert die Reihe (2) jedenfalls im quadratischen Mittel gegen eine Funktion $f(x)$. Wir wählen eine Indexfolge derart, daß

$$\sum_{k=0}^{\infty} \sum_{n=n_k+1}^{\infty} a_n^2 < \infty$$

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[5]) A. ZYGMUND, On the convergence of lacunary trigonometric series, *Fundamenta Math.*, **16** (1930), 90—107.

[6]) A. N. KOLMOGOROFF, Une contribution à l'étude de la convergence des séries de Fourier, *Fundamenta Math.*, **5** (1924), 96—97.

[7]) S. KACZMARZ, Über die Summierbarkeit der Orthogonalreihen, *Math. Zeitschrift*, **26** (1927), 99—105.

[8]) Siehe z. B. D. MENCHOFF, Sur les séries de fonctions orthogonales. Deuxième partie, *Fundamenta Math.*, **8** (1926), 56—108.; S. KACZMARZ, loc. cit. 7); K. TANDORI, loc. cit. 3).

besteht. Dann ist

$$\sum_{k=0}^{\infty}\int_0^1 (f(t)-s_{n_k}(t))^2\,dt<\infty,$$

wobei $s_n(x)$ die n-te Partialsumme der Reihe (2) bezeichnet. Hieraus, auf Grund des Satzes von B. LEVI folgt, daß die Reihe

$$\sum_{k=0}^{\infty}(f(x)-s_{n_k}(x))^2$$

fast überall konvergiert. Die positive Quadratwurzel der Summe dieser Reihe bezeichnen wir mit $F(x)$. Also ist $F(x)$ quadratisch integrierbar; ihr Quadratintegral hängt offenbar nur von den Koeffizienten a_n ab. Aus (1) folgt, daß die Funktion

$$G(x)=\left\{\sum_{k=0}^{\infty}\big(\max_{n_k<i\leqq j\leqq n_{k+1}}|a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)|\big)^2\right\}^{1/2}$$

quadratisch integrierbar ist; ihr Quadratintegral hängt nur von den Koeffizienten a_n ab. Es sei n ein beliebiger Index und $n_k<n\leqq n_{k+1}$. Da offenbar

$$|s_n(x)|\leqq|f(x)|+|s_{n_k}(x)-f(x)|+|s_n(x)-s_{n_k}(x)|\leqq|f(x)|+F(x)+G(x)$$

gilt, so ergibt sich auf Grund des Satzes I Folgendes: ist die Bedingung (1) erfüllt, so konvergiert die Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ „beschränkt", d. h. konvergiert fast überall und die Partialsummen bleiben im absoluten Betrag unterhalb einer nur von dem System $\{\varphi_n(x)\}$ abhängigen, quadratisch integrierbaren Funktion; das Quadratintegral dieser Funktion bleibt unterhalb einer nur von den Koeffizienten a_n abhängigen Konstante.

Also gilt auch die folgende Behauptung:

Satz III. *Die Konvergenz fast überall der Orthogonalreihe* (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}$ *ist mit ihrer beschränkten Konvergenz für jedes orthonormierte System* $\{\varphi_n(x)\}$ *gleichwertig.*

Nach der Definition von $I(c_1,\ldots,c_N)$ gilt $I(\pm c_1,\ldots,\pm c_N)=I(c_1,\ldots,c_N)$ und $I(c_1+d_1,\ldots,c_N+d_N)\leqq 2I(c_1,\ldots,c_N)+2I(d_1,\ldots,d_N)$. Es sei $c_1',\ldots,c_N'$ eine Folge, die sich aus $c_1,\ldots,c_N$ so ergibt, daß einige der c_i gleich 0 gesetzt werden. Auf Grund der obigen Bemerkungen folgt die Ungleichung $I(c_1',\ldots,c_N')\leqq 4I(c_1,\ldots,c_N)$. Daraus, auf Grund des Satzes I folgt

Satz IV. *Ist die Reihe* (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}$ *fast überall konvergent, so ist jede Teilreihe von* (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}$ *fast überall konvergent.*

Es soll aber bemerkt werden, daß aus der Konvergenz von (2) fast überall für jedes orthonormierte System $\{\varphi_n(x)\}$ die unbedingte Konvergenz von (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ nicht folgt. Es kann nämlich eine Koeffizientenfolge $\{a_n\}$ mit $a_n^2\geqq a_{n+1}^2$

$$\sum a_n^2\log^2 n<\infty\quad\text{und}\quad\sum_{\nu=0}^{\infty}\left[\sum_{n=2^{2^\nu}+1}^{2^{2^{\nu+1}}}a_n^2\log^2 n\right]^{1/2}=\infty$$

angegeben werden. Auf Grund der zweiten Bedingung existiert ein orthonormiertes System $\{\varphi_n(x)\}$ derart, daß die Reihe (2) in gewisser Anordnung ihrer Glieder fast überall divergiert[9]), und wegen der ersten Bedingung konvergiert der Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ fast überall[10]).

Die n-te $(C, 1)$-Mittel der Orthogonalreihe (2) bezeichnen wir mit $\sigma_n(x)$, d. h.

$$\sigma_n(x) = \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) a_k \varphi_k(x).$$

Offensichtlich ist die Konvergenz bzw. die beschränkte Konvergenz der Folge der 2^m-ten Partialsummen der Orthogonalreihe (2) mit der Konvergenz bzw. mit der beschränkten Konvergenz der Reihe

$$\sum_{m=0}^{\infty} \bar{A}_m \Phi_m(x) \tag{5}$$

äquivalent, wobei

$$\Phi_m(x) = \bar{A}_m^{-1}\left(a_{2^m+1}\varphi_{2^m+1}(x) + \ldots + a_{2^{m+1}}\varphi_{2^{m+1}}(x)\right) \qquad (m=0, 1, \ldots)$$

gesetzt wird. (Ist $\bar{A}_m = 0$, so soll man statt $\bar{A}_m^{-1}$ z. B. 1 setzen.) Ist (4) erfült, so erhalten wir — wie oben — daß die Partialsummen der Reihe (5), also die 2^m-ten Partialsummen der Reihe (2), beschränkt konvergieren; also gibt es eine quadratisch integrierbare Funktion $H(x)$ mit

$$|s_{2^m}(x)| \leqq H(x) \qquad (m=0, 1, \ldots).$$

Wegen

$$\sum_{m=0}^{\infty} \int_0^1 (s_{2^m}(x) - \sigma_{2^m}(x))^2\, dx \leqq \sum_{m=0}^{\infty} \frac{1}{2^{2m}} \sum_{k=1}^{2^m} a_k^2 k^2 = O(1) \sum_{k=1}^{\infty} a_k^2 < \infty$$

ist

$$M(x) = \left\{\sum_{m=0}^{\infty} (s_{2^m}(x) - \sigma_{2^m}(x))^2\right\}^{1/2} \in L^2.$$

Durch einfache Rechnung erhalten wir weiterhin:

$$\sum_{n=1}^{\infty} n \int_0^1 (\sigma_{n+1}(x) - \sigma_n(x))^2\, dx \leqq \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=1}^{n} a_k^2 k^2 = O(1) \sum_{k=1}^{\infty} a_k^2 < \infty;$$

es besteht also auch

$$N(x) = \left\{\sum_{n=1}^{\infty} n\left(\sigma_{n+1}(x) - \sigma_n(x)\right)^2\right\}^{1/2} \in L^2.$$

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[9]) K. Tandori, Über die orthogonalen Funktionen. X (Unbedingte Konvergenz), *Acta Sci. Math.*, **23** (1962), 185—221.

[10]) D. Menchoff, loc. cit. 1); H. Rademacher, loc. cit. 2).

Es sei n ein beliebiger Index und $2^m < n < 2^{m+1}$. Dann ist

$$|\sigma_n(x)-\sigma_{2^m}(x)| = \left|\sum_{k=2^m}^{n-1}(\sigma_{k+1}(x)-\sigma_k(x))\right| \leqq$$

$$\leqq \left\{\sum_{k=2^m}^{2^{m+1}-1}\frac{1}{k}\right\}^{1/2}\left\{\sum_{k=2^m}^{2^{m+1}-1}k(\sigma_{k+1}(x)-\sigma_k(x))^2\right\}^{1/2} \leqq \sqrt{2}N(x).$$

Nach den Obigen gilt

$$|\sigma_n(x)| \leqq |s_{2^m}(x)| + |\sigma_{2^m}(x)-s_{2^m}(x)| + |\sigma_{2^m}(x)-\sigma_n(x)| \leqq$$

$$\leqq H(x)+M(x)+\sqrt{2}N(x) \in L^2.$$

Ist also die Bedingung (4) erfüllt, so ist die Orthogonalreihe (2) beschränkt $(C, 1)$-summierbar. Damit haben wir auch die folgende Behauptung bewiesen:

Satz V. *Die $(C, 1)$-Summierbarkeit fast überall der Reihe (2) für jedes orthonormierte System $\{\varphi_n(x)\}$ ist mit ihrer beschränkten $(C, 1)$-Summierbarkeit für jedes orthonormierte System $\{\varphi_n(x)\}$ äquivalent.*

Offensichtlich kann ein ähnlicher Satz auch für andere Summationsverfahren, z. B. für die Rieszsche Summation bewiesen werden.

§ 1. Beweis des Satzes I

Hinlänglichkeit. Wie schon bemerkt wurde, aus (1) folgt, mit Anwendung des Satzes von Riesz—Fischer, daß es eine Indexfolge $\{n_k\}$ derart gibt, daß die Folge $\{s_{n_k}(x)\}$ fast überall konvergiert. Aus (1) folgt weiterhin

$$\delta_k(x) = \max_{n_k < i \leqq j \leqq n_{k+1}} |a_i\varphi_i(x)+\ldots+a_j\varphi_j(x)| \to 0 \qquad (k\to\infty)$$

fast überall. Es sei $n_k < n < n_{k+1}$. Dann ist also

$$|s_n(x)-s_{n_k}(x)| \leqq \delta_k(x) \to 0 \qquad (n\to\infty)$$

fast überall. Damit haben wir die Hinlänglichkeit der Bedingung (1) bewiesen.

Mit derselben Methode kann auch die folgende Behauptung bewiesen werden: *Ist für eine Indexfolge $\mu_0 < \ldots < \mu_k < \ldots$ die Folge $\{s_{\mu_k}(x)\}$ fast überall konvergent und besteht*

$$\sum_{k=0}^{\infty} I(a_{\mu_k+1}, \ldots, a_{\mu_{k+1}}) < \infty,$$

so ist die Reihe (2) fast überall konvergent.

Notwendigkeit. Zum Beweis benötigen wir den folgenden Hilfssatz.

Hilfssatz. *Für jede Folge $c_1, \ldots, c_N$ mit $c_1^2+\ldots+c_N^2>0$ gibt es ein in $[0, 1]$ orthonormiertes und von der Folge $\{c_n\}$ abhängiges System von Treppenfunktionen $\psi_1(x), \ldots, \psi_N(x)$ derart, daß*

$$\max_{1\leqq i\leqq j\leqq N} |c_i\psi_i(x)+\ldots+c_j\psi_j(x)| \geqq 1$$

in einem Intervall $E(\subseteq[0, 1])$ *mit*

$$\operatorname{mes}(E) \geqq \alpha I(c_1, \dots, c_N)$$

gilt, wobei α *eine positive, absolute Konstante ist.*

Beweis des Hilfssatzes. Nach der Definition von $I(c_1, \dots, c_N)$ gibt es ein orthonormiertes System $\varphi_1(x), \dots, \varphi_N(x)$, für welches

$$\int_0^1 \big(\max_{1\leqq i\leqq j\leqq N} |c_i\varphi_i(x)+\dots+c_j\varphi_j(x)|\big)^2\,dx \geqq \frac{1}{2} I(c_1, \dots, c_N) \tag{6}$$

gilt. Zu einem beliebigen $\varepsilon>0$ gibt es Treppenfunktionen $\chi_1(x), \dots, \chi_N(x)$ mit

$$\int_0^1 (\varphi_i(x)-\chi_i(x))^2\,dx \leqq \varepsilon \qquad (i=1, \dots, N).$$

Wir setzen

$$\alpha_{i,j} = \int_0^1 \chi_i(x)\chi_j(x)\,dx \qquad (i, j=1, \dots, N)$$

und

$$\sum_{l=1}^{i-1} |\alpha_{l,i}| + \sum_{l=i+1}^{N} |\alpha_{l,i}| = \eta_i \qquad (i=1, \dots, N).$$

Bei genügend kleinem ε gelten nach (6) offenbar:

$$\int_0^1 \big(\max_{1\leqq i\leqq j\leqq N} |c_i\chi_i(x)+\dots+c_j\chi_j(x)|\big)^2\,dx \geqq \frac{1}{8} I(c_1, \dots, c_N), \tag{7}$$

$$\int_0^1 \left(\max_{1\leqq i\leqq j\leqq N} \left| c_i\left(1-\frac{1}{\sqrt{\alpha_{i,i}+\eta_i}}\right)\chi_i(x)+\dots+c_j\left(1-\frac{1}{\sqrt{\alpha_{j,j}+\eta_j}}\right)\chi_j(x)\right|\right)^2 dx \leqq \tag{8}$$
$$\leqq \frac{1}{16} I(c_1, \dots, c_N).$$

Wir definieren ein in $[0, 2]$ orthogonales System von Treppenfunktionen $\bar\chi_1(x), \dots$ $\dots, \bar\chi_N(x)$ folgenderweise. Wir teilen das Intervall $(1, 2]$ in $N(N-1)$ Teilintervalle $I_{i,j}$ $(i, j=1, \dots, N;\ i\neq j)$ von gleicher Länge ein und setzen:

$$\bar\chi_i(x) = \begin{cases} \chi_i(x), & x\in[0, 1], \\ [2^{-1}N(N-1)|\alpha_{l,i}|]^{1/2}, & x\in I_{i,l} \quad (l=1, \dots, N;\ l\neq i), \\ -[2^{-1}N(N-1)|\alpha_{l,i}|]^{1/2}\operatorname{sign}\alpha_{l,i} & x\in I_{l,i} \quad (l=1, \dots, N;\ l\neq i) \end{cases}$$

$(i=1, \dots, N)$. Wegen

$$\int_0^2 \bar\chi_i^2(x)\,dx = \int_0^1 \chi^2(x)\,dx + \sum_{l=1}^{i-1}|\alpha_{l,i}| + \sum_{l=i+1}^{N}|\alpha_{l,i}| = \alpha_{i,i}+\eta_i$$

bilden die Treppenfunktionen

$$\bar{\varphi}_i(x) = \frac{1}{\sqrt{\alpha_{i,i}+\eta_i}}\bar{\chi}_i(x) \qquad (i=1, \ldots, N)$$

ein orthonormiertes System in [0, 2], für welches wegen (7) und (8) gilt:

$$(9) \qquad \int_0^2 \big(\max_{1\le i\le j\le N} |c_i\bar{\varphi}_i(x)+\ldots+c_j\bar{\varphi}_j(x)|\big)^2 dx \geqq \frac{1}{16} I(c_1, \ldots, c_N).$$

$F(x) = \max_{1\le i\le j\le N} |c_i\bar{\varphi}_i(x)+\ldots+c_j\bar{\varphi}_j(x)|$ ist eine Treppenfunktion: sie nimmt auf den nacheinander folgenden Intervallen $I_1, \ldots, I_\varrho$ konstante Werte, etwa $w_1, \ldots, w_\varrho$ an. Es sei

$$\sum_{r=1}^{\varrho} w_r^2 \operatorname{mes}(I_r) = A;$$

ohne Beschränkung der Allgemeinheit können wir $A \leqq 2$ annehmen. Wir setzen

$$u_0 = 0, \; u_r = \frac{1}{2}\sum_{s=1}^{r} w_s^2 \operatorname{mes}(I_s) \qquad (r=1, \ldots, \varrho)$$

und

$$\psi_i(x) = \begin{cases} \dfrac{\sqrt{2}}{w_{r+1}}\bar{\varphi}_i\left(\dfrac{2}{w_{r+1}^2}(x-u_r)+\sum_{s=1}^{r}\operatorname{mes}(I_s)\right), & x\in[u_r, u_{r+1}),\ w_r\neq 0,\ r=0, 1, \ldots, \varrho-1, \\ 0 \text{ sonst in } [0, 1] \end{cases}$$

$(i=1, \ldots, N)$. Offensichtlich genügen diese Funktionen $\psi_i(x)$ und das Intervall $E=[0, u_\varrho]$ allen den gestellten Bedingungen.

Damit haben wir den Hilfssatz bewiesen.

Ist die Bedingung (1) nicht erfüllt, so gibt es eine Indexfolge $(0=)n_0<\ldots$ $\ldots<n_k<\ldots$, für die

$$(10) \qquad \sum_{k=0}^{\infty} I(a_{n_k+1}, \ldots, a_{n_{k+1}}) = \infty$$

ist. Ohne Beschränkung der Allgemeinheit kann $I(a_{n_k+1}, \ldots, a_{n_{k+1}})>0$, d. h. $a_{n_k+1}^2+\ldots+a_{n_{k+1}}^2>0$ angenommen werden.

Durch Induktion werden wir ein in [0, 1] orthonormiertes System von Treppenfunktionen $\Phi_n(x)$ und eine Folge von einfachen (d. h. als Vereinigung endlich vieler Intervalle entstehenden) Mengen $E_k(\subseteqq[0, 1])$ definieren mit den folgenden Eigenschaften:

a) die Mengen E_k sind stochastisch unabhängig und für jedes k gilt

$$(11) \qquad \operatorname{mes}(E_k) \geqq \alpha I(a_{n_k+1}, \ldots, a_{n_{k+1}});$$

b) für jedes k gibt es von x abhängige Indizes $\nu_k=\nu_k(x)$, $\mu_k=\mu_k(x)$ $(n_k<\nu_k\leqq \mu_k\leqq n_{k+1})$ derart, daß für $x\in E_k$

$$(12) \qquad |a_{\nu_k}\Phi_{\nu_k}(x)+\ldots+a_{\mu_k}\Phi_{\mu_k}(x)| \geqq 1$$

besteht.

Wir wenden den Hilfssatz in Falle $c_i = a_i$ $(i=1, \ldots, n_1)$ an, die entsprechenden Funktionen und das entsprechende Intervall bezeichnen wir mit $\Phi_1(x), \ldots, \Phi_{n_1}(x)$ bzw. mit E_1.

Es sei $\varkappa(>1)$ eine beliebige natürliche Zahl. Wir nehmen an, daß die Treppenfunktionen $\Phi_n(x)$ $(1 \leqq n \leqq n_{k-1})$ und die einfachen Mengen E_k $(1 \leqq k \leqq \varkappa - 1)$ schon derart definiert sind, daß diese Funktionen orthonormiert und diese Mengen stochastisch unabhängig sind, und (11), (12) für $k = 1, \ldots, \varkappa - 1$ bestehen.

Dann können wir das Intervall [0, 1] in endlich viele Teilentervalle $J_q = (\alpha_q, \beta_q)$ $(q = 1, \ldots, Q)$ einteilen, derart, daß die Funktionen $\Phi_n(x)$ in jedem J_q konstant sind und jede Menge E_k die Vereinigung gewisser J_q ist.

Wir wenden den Hilfssatz im Falle $c_i = a_i (i = n_{\varkappa-1} + 1, \ldots, n_\varkappa)$ an; die entsprechenden Funktionen und das entsprechende Intervall bezeichnen wir mit $\varphi_n(x)$ $(n_{\varkappa-1} + 1 \leqq n \leqq n_\varkappa)$ bzw. mit E. Es sei

$$\Phi_n(x) = \sum_{q=1}^{Q} \varphi_n(J_q; x) \qquad (n_{\varkappa-1} + 1 \leqq n \leqq n_\varkappa)$$

und

$$E_\varkappa = \bigcup_{q=1}^{Q} E(J_q),$$

wobei für ein beliebiges Intervall $I = [u, v]$

$$f(I; x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right), & u < x < v, \\ 0 & \text{sonst} \end{cases}$$

gesetzt wird und $G(I)$ das mit der linearen Transformation $x = (v-u)y + u$ erhaltene Bild in $[u, v]$ der Menge $G(\subseteqq [0, 1])$ bedeutet.

Offensichtlich bilden die Treppenfunktionen $\Phi_n(x)$ $(1 \leqq n \leqq n_\varkappa)$ ein orthonormiertes System, die einfachen Mengen E_k $(1 \leqq k \leqq \varkappa)$ sind stochastisch unabhängig, und (11), (12) sind auch im Falle $k = \varkappa$ erfüllt.

Das Funktionensystem $\{\Phi_n(x)\}$ und die Mengenfolge $\{E_k\}$ mit den geforderten Eigenschaften erhalten wir also mit Induktion.

Es sei $H = \overline{\lim}\, E_k$ $(k \to \infty)$. Auf Grund von (10) und (11), mit Anwendung des zweiten Borel-Cantellischen Lemmas folgt

$$\operatorname{mes}(H) = 1.$$

Für $x \in H$ ist die Orthogonalreihe $\sum a_n \Phi_n(x)$ wegen (12) divergent.

Damit haben wir den Satz I vollständig bewiesen.

§ 2. Beweis des Satzes II

Hinlänglichkeit. Diese folgt leicht, mit Anwendung des Satzes I, aus der Bedingung (3), da die Konvergenz fast überall der Folge der m_k-ten Partialsummen der Reihe (2) äquivalent mit der Konvergenz der Orthogonalreihe

$$\sum A_k \Psi_k(x) \tag{13}$$

ist, wobei

$$\Psi_k(x) = A_k^{-1}(a_{m_k+1}\varphi_{m_k+1}(x)+\ldots+a_{m_{k+1}}\varphi_{m_{k+1}}(x)) \qquad (k=0, 1, \ldots)$$

gesetzt wird. (Für $A_k=0$ soll man hier anstatt A_k^{-1} etwa 1 schreiben.)

Notwendigkeit. Wir werden Folgendes beweisen: Ist die Bedingung (3) nicht erfüllt, so gibt es ein orthonormiertes System $\{\Phi_n(x)\}$, für welches die Folge der m_k-ten Partialsummen der Reihe $\sum a_n\Phi_n(x)$ fast überall divergiert.

Ist die Bedingung (3) nicht erfüllt, so gibt es nämlich eine Indexfolge $0=n_0<\ldots<n_k<\ldots$, für die

$$\sum I(A_{n_k+1}, \ldots, A_{n_{k+1}})=\infty$$

gilt. Ohne Beschränkung der Allgemeinheit kann $I(A_{n_k+1}, \ldots, A_{n_{k+1}})>0$, d. h. $A^2_{n_k+1}+\ldots+A^2_{n_{k+1}}>0$ angenommen werden.

Wir wählen eine Folge $\{b_n\}$ von rationalen Zahlen derart, daß

$$(14) \qquad \sum |b_n-a_n|<\infty \quad \text{und} \quad \sum I(B_{n_k+1}, \ldots, B_{n_{k+1}})=\infty$$

gilt, wobei

$$B_k=\{b^2_{m_k+1}+\ldots+b^2_{m_{k+1}}\}^{1/2} \qquad (k=0, 1, \ldots)$$

gesetzt wird. Dies ist offenbar möglich.

Mit der im § 1 angewandten Methode kann man ein in [0, 1] orthonormiertes System von Treppenfunktionen $\psi_n(x)$ und eine stochastisch unabhängige Folge von einfachen Mengen $E_k(\subseteqq[0, 1])$ derart angeben, daß

$$(15) \qquad \sum \operatorname{mes}(E_k)=\infty$$

besteht und es für jedes $x\in E_k$ von x abhängige Indizes $\nu_k=\nu_k(x)$, $\mu_k=\mu_k(x)$ $(n_k<\nu_k\leqq\mu_k\leqq n_{k+1})$ gibt, so daß

$$(16) \qquad |B_{\nu_k}\psi_{\nu_k}(x)+\ldots+B_{\mu_k}\psi_{\mu_k}(x)|\geqq 1$$

besteht.

Durch Induktion werden wir ein orthonormiertes System von Treppenfunktionen $\Phi_n(x)$ und eine Folge von einfachen Mengen F_k definieren mit den folgenden Eigenschaften:

a) die Mengen F_k sind stochastisch unabhängig und für jedes k besteht

$$(17) \qquad \operatorname{mes}(F_k)=\operatorname{mes}(E_k);$$

b) zu jedem $x\in F_k$ gibt es Indizes $\nu_k=\nu_k(x)$, $\mu_k=\mu_k(x)$ $(n_k<\nu_k\leqq\mu_k\leqq n_{k+1})$ derart, daß die Ungleichung

$$(18) \qquad |b_{m_{\nu_k}+1}\Phi_{m_{\nu_k}+1}(x)+\ldots+b_{m_{\mu_k+1}}\Phi_{m_{\mu_k+1}}(x)|\geqq 1$$

besteht.

Wir schreiben die rationalen Zahlen b_n^2/B_k^2 $(m_k<n\leqq m_{k+1}; k=0, \ldots, n_1; m_0=0)$ als Brüche von natürlichen Zahlen mit gemeinsamem Nenner auf:

$$b_n^2/B_k^2=p_n^{(0)}/q_0.$$

Wir teilen das Intervall [0, 1] in q_0 Teilintervalle gleicher Länge $I_\nu=[u_\nu, v_\nu]$ $(1\leqq\nu\leqq q_0)$ ein. Es sei für $m_k<n\leqq m_{k+1}$ $(k=0, \dots, n_1)$

$$\Phi_n(x)=\frac{B_k}{b_n}\sum_{\nu=p^{(0)}_{m_k+1}+\dots+p^{(0)}_{n-1}+1}^{p^{(0)}_{m_k+1}+\dots+p^{(0)}_n}\psi_k(I_\nu; x) \quad \text{und} \quad F_0=\bigcup_{\nu=1}^{q_0}E_0(I_\nu).$$

Offensichtlich bilden die Treppenfunktionen $\Phi_n(x)$ $(1\leqq n\leqq m_{n_1+1})$ ein orthonormiertes System, die Menge F_0 ist einfach und besteht (17) für $k=0$. Jeder Punkt $x\in F_0$ ist in einem $E_0(I_\nu)$ enthalten $(1\leqq\nu\leqq q_0)$. Dann ist $y=\dfrac{x-u_\nu}{v_\nu-u_\nu}\in E_0$ und so gibt es auf Grund von (16) Indizes $\nu_0=\nu_0(x)$, $\mu_0=\mu_0(x)$ $(0<\nu_0\leqq\mu_0\leqq n_1)$ derart, daß

$$|B_{\nu_0}\psi_{\nu_0}(y)+\dots+B_{\mu_0}\psi_{\mu_0}(y)|\geqq 1$$

gilt. Also gilt auch

$$|B_{\nu_0}\psi_{\nu_0}(I_\nu; x)+\dots+B_{\mu_0}\psi_{\mu_0}(I_\nu; x)|\geqq 1.$$

Nach der Definition von $\Phi_n(x)$ ist aber

$$b_{m_{\nu_0}+1}\Phi_{m_{\nu_0}+1}(x)+\dots+b_{m_{\mu_0}+1}\Phi_{m_{\mu_0}+1}(x)=B_{\nu_0}\psi_{\nu_0}(I_\nu; x)+\dots+B_{\mu_0}\psi_{\mu_0}(I_\nu; x),$$

also ist (18) für $k=0$ erfüllt.

Es sei $\varkappa(\geqq 1)$ eine beliebige natürliche Zahl. Wir nehmen an, daß die Treppenfunktionen $\Phi_n(x)$ $(1\leqq n\leqq m_{n_\varkappa+1})$ und die einfachen Mengen F_k $(0\leqq k\leqq\varkappa-1)$ schon derart definiert sind, daß die Funktionen $\Phi_n(x)$ ein orthonormiertes System bilden, die Mengen F_k stochastisch unabhängig sind, und (17), (18) für $k=0, \dots, \varkappa-1$ erfüllt sind.

Dann kann das Intervall [0, 1] in endlich viele Teilintervalle J_s $(1\leqq s\leqq\sigma)$ derart zerlegt werden, daß in jedem J_s die Funktionen $\Phi_n(x)$ $(1\leqq n\leqq m_{n_\varkappa+1})$ konstant sind und jede Menge F_k $(0\leqq k\leqq\varkappa-1)$ die Vereinigung gewisser J_s ist. Wir schreiben die endlich vielen rationalen Zahlen b_n^2/B_k^2 $(m_k<n\leqq m_{k+1}; k=n_\varkappa+1, \dots n_{\varkappa+1})$ mit gemeinsamem Nenner auf:

$$b_n^2/B_k^2=p_n^{(\varkappa)}/q_\varkappa.$$

Wir teilen jedes J_s in $q_\varkappa$ Teilintervalle gleicher Länge $J_{s,\varrho}=[u_{s,\varrho}, v_{s,\varrho}]$ $(1\leqq s\leqq\sigma,$ $1\leqq\varrho\leqq q_\varkappa)$ ein und wir setzen für $m_k<n\leqq m_{k+1}$ $(k=n_\varkappa+1, \dots, n_{\varkappa+1})$

$$\Phi_n(x)=\frac{B_k}{b_n}\sum_{s=1}^{\sigma}\sum_{\varrho=p^{(\varkappa)}_{m_k+1}+\dots+p^{(\varkappa)}_{n-1}+1}^{p^{(\varkappa)}_{m_k+1}+\dots+p^{(\varkappa)}_n}\psi_k(J_{s,\varrho}; x)$$

und

$$F_\varkappa=\bigcup_{s=1}^{\sigma}\bigcup_{\varrho=1}^{q_\varkappa}E_\varkappa(J_{s,\varrho}).$$

Offensichtlich bilden die Treppenfunktionen $\Phi_n(x)$ $(1\leqq n\leqq m_{n_{\varkappa+1}+1})$ ein orthonormiertes System in [0, 1]. Die Menge $F_\varkappa$ ist einfach, die Mengen F_k $(0\leqq k\leqq\varkappa)$ sind stochastisch unabhängig und (17) ist auch für $k=\varkappa$ erfüllt. Es sei $x\in F_\varkappa$. Dann ist $x \in E_\varkappa(J_{s,\varrho})$ für gewisse s und ϱ $(1 \leqq s \leqq \sigma,\ 1 \leqq \varrho \leqq q_\varkappa)$. Also ist

$$y = \frac{x - u_{s,\varrho}}{v_{s,\varrho} - u_{s,\varrho}} \in E_\varkappa$$

und so gibt es auf Grund von (16) von x abhängige Indizes $\nu_\varkappa = \nu_\varkappa(x)$, $\mu_\varkappa = \mu_\varkappa(x)$ $(n_\varkappa < \nu_\varkappa \leqq \mu_\varkappa \leqq n_{\varkappa+1})$, für die

$$|B_{\nu_\varkappa}\psi_{\nu_\varkappa}(y) + \ldots + B_{\mu_\varkappa}\psi_{\mu_\varkappa}(y)| \geqq 1$$

besteht. Daraus folgt

$$|B_{\nu_\varkappa}\psi_{\nu_\varkappa}(J_{s,\varrho};\, x) + \ldots + B_{\mu_\varkappa}\psi_{\mu_\varkappa}(J_{s,\varrho};\, x)| \geqq 1.$$

Nach der Definition von $\Phi_n(x)$ ist aber

$$b_{m_{\nu_\varkappa}+1}\Phi_{m_{\nu_\varkappa}+1}(x) + \ldots + b_{m_{\mu_\varkappa}+1}\Phi_{m_{\mu_\varkappa}+1}(x) = B_{\nu_\varkappa}\psi_{\nu_\varkappa}(J_{s,\varrho};\, x) + \ldots + B_{\mu_\varkappa}\psi_{\mu_\varkappa}(J_{s,\varrho};\, x),$$

also ist (18) auch für $k = \varkappa$ erfüllt.

Durch Induktion ergibt sich also das Funktionensystem $\{\Phi_n(x)\}$ und die Mengenfolge $\{F_k\}$ mit den erwähnten Eigenschaften.

Wegen (15) und (17) folgt

$$\sum \operatorname{mes}(F_k) = \infty.$$

Daraus und aus der stochastischen Unabhängigkeit der Mengen F_k erhalten wir mit Anwendung des zweiten Borel-Cantellischen Lemmas:

$$\operatorname{mes}(\overline{\lim_{k\to\infty}} F_k) = 1.$$

Für $x \in \overline{\lim}\, F_k$ divergiert aber die Folge der m_k-ten Partialsummen der Reihe

$$\sum b_n \Phi_n(x)$$

wegen (18) fast überall.

Es ist leicht einzusehen, daß wegen (14) die Orthogonalreihe

$$\sum (b_n - a_n)\Phi_n(x)$$

fast überall konvergiert. Daraus folgt endlich, daß die Folge der m_k-ten Partialsummen der Reihe $\sum a_n \Phi_n(x)$ fast überall divergiert.

Damit haben wir den Satz II vollständig bewiesen.

Für die Folge der m_k-ten Partialsummen der Orthogonalreihe (2) können wir auch das Analogon der Bemerkung in § 2 beweisen.

Für eine Indexfolge $(0=)\, m_0 < \ldots < m_k < \ldots$ betrachten wir die Mittel

$$\frac{1}{k}\,[s_{m_1}(x) + \ldots + s_{m_k}(x)] \qquad (k = 1, 2, \ldots), \tag{19}$$

wo $s_n(x)$ die n-te Partialsumme der Orthogonalreihe (2) bezeichnet. Offensichtlich ist die Konvergenz der Mittel (19) mit der $(C, 1)$-Summierkeit der Orthogonalreihe (13) äquivalent. Ist $\sum A_k^2 < \infty$, d. h. $\sum a_n^2 < \infty$, so gibt es nach dem Satz von Riesz—Fischer eine quadratisch integrierbare Funktion $f(x)$, für die

$$\int_0^1 (S_k(x)-f(x))^2\,dx = \int_0^1 (s_{m_{k+1}}(x)-f(x))^2\,dx \to 0$$

gilt, wobei

$$S_k(x) = \sum_{l=0}^{k} A_l \Psi_l(x) = \sum_{n=1}^{m_{k+1}} a_n \varphi_n(x)$$

bedeutet. Nach einem Satz von ZYGMUND[11]) hat man

$$\frac{1}{N}\sum_{k=0}^{N-1} [S_k(x)-f(x)]^2 = \frac{1}{N}\sum_{k=1}^{N} [s_{m_k}(x)-f(x)]^2 \to 0 \tag{20}$$

fast überall. Da aus (20) die Konvergenz fast überall der Mittel (19) folgt, ist die Konvergenz von (19) mit (20) fast überall äquivalent.

Die Orthogonalreihe (2) heißt *sehr stark* summierbar, wenn die Relation (20) für jede Indexfolge $\{m_k\}$ fast überall erfüllt wird. Aus dem Satz II und aus den erwähnten Sätzen von KOLMOGOROFF und KACZMARZ ergibt sich also der folgende Satz:

Damit die Orthogonalreihe (2) *für jedes orthonormierte System* $\{\varphi_n(x)\}$ *sehr stark summierbar ist, ist es notwendig und hinreichend, daß*

$$\sum I(\bar{A}_{n_l+1}(\{m_k\}), \ldots, \bar{A}_{n_{l+1}}(\{m_k\})) < \infty$$

für alle Indexfolgen $\{m_k\}$ *und* $\{n_k\}$ *gilt, wobei*

$$\bar{A}_l(\{m_k\}) = \{A^2_{2^l+1} + \ldots + A^2_{2^{l+1}}\}^{1/2} = \{a^2_{m_{2^l+1}+1} + \ldots + a^2_{m_{2^{l+1}+1}}\}^{1/2}$$

bedeutet.

(Eingegangen am 26. Oktober 1962)

[11]) A. ZYGMUND, Sur l'application de la première moyenne arithmétique dans la théorie des séries de fonctions orthogonales, *Fundamenta Math.*, **10** (1927), 356—362.

Equivalence of a problem of set theory to a hypothesis concerning the powers of cardinal numbers

By G. FODOR in Szeged

To Professor Béla Szőkefalvi-Nagy on his 50th birthday

Let E be an arbitrary set of power $\aleph_\alpha$ and suppose that with every element x of E is associated a non empty set $f(x)$ such that for any $x\in E$ the power of the set $f(x)$ is smaller than a given cardinal number $\aleph_\beta$ which is smaller than $\aleph_\alpha$ and $f(x)\neq f(y)$ $(x\neq y)$. We say that the subset Γ of E has the property $T(\mathfrak{q}, \mathfrak{p})$, where $\mathfrak{q}$ and $\mathfrak{p}$ are two cardinal numbers such that $\mathfrak{p}\leqq\mathfrak{q}\leqq\aleph_\alpha$, if

$$\overline{\overline{\bigcup_{x\in\Gamma} f(x)}} = \mathfrak{q} \quad \text{and} \quad \overline{\overline{\bigcup_{\substack{x,y\in\Gamma\\ x\neq y}} (f(x)\cap f(y))}} < \mathfrak{p}.$$

We define the ordinal number β_0 as follows:

Let β_0 be the smallest ordinal number $\varrho<\beta$ such that the set $E^{(\varrho)}$ of the elements $x\in E$ for which $\overline{\overline{f(x)}}<\aleph_\varrho$ has the power $\aleph_\alpha$.

Consider now the following propositions.

(I) *Under the above conditions E has a subset Γ with the property $T(\aleph_\alpha, \aleph_\alpha)$.*

(II) *For every ordinal number γ, $\beta<\gamma<\alpha$, the inequality*

$$(\aleph_\gamma^{\underline{\aleph_{\beta_0}}})^{\underline{\aleph_{\beta_0}}} < \aleph_\alpha$$

holds, where $\aleph_\gamma^{\underline{\aleph_{\beta_0}}} = \sum_{\varrho<\beta_0} \aleph_\gamma^{\aleph_\varrho}$.

We shall prove in this paper the following

Theorem. *The propositions* (I) *and* (II) *are equivalent.*

We shall use the following notations. For any subset Γ of E let

$$\Pi_\Gamma = \bigcup_{\substack{x,y\in\Gamma\\ x\neq y}} (f(x)\cap f(y)).$$

For any cardinal number $\mathfrak{r}$ we denote by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$. The symbols $\overline{\overline{S}}$ and $\overline{\gamma}$ denote the cardinal numbers of the set S and of the ordinal number γ, respectively. For every ordinal number τ, $\aleph_{\mathrm{cf}(\tau)}$ denotes the least cardinal number $\mathfrak{n}$ such that $\aleph_\tau$ can be expressed as the sum of $\mathfrak{n}$ cardinal numbers each $<\aleph_\tau$. If $\mathfrak{m}$ and $\mathfrak{n}$ cardinal numbers, then we define $\mathfrak{m}^{\underline{\mathfrak{n}}}=\sum_{\mathfrak{r}<\mathfrak{n}}\mathfrak{m}^{\mathfrak{r}}$. Put, for every ordinal number γ, $W(\gamma)=\{\xi:\xi<\gamma\}$.

In the proof of the theorem we shall use the following theorems:

Theorem 1. *If $\aleph_\alpha$ is regular and $\bigcup_{x\in E} f(x)$ has the power $\aleph_\alpha$, then E has a subset with the property $T(\aleph_\alpha, \aleph_\alpha)$.* (See [1], theorem 1.)

Theorem 2. *Let $\aleph_\alpha$ be a singular cardinal number, $\mathfrak{r}_0$ a cardinal number which is smaller then $\aleph_\alpha$ and $\{\aleph_\xi\}_{\xi<\omega_{cf(\alpha)}}$ a sequence of regular cardinal numbers such that $\aleph_\sigma > \aleph_\tau$ $(\sigma > \tau)$; $\max\{\aleph_{\mathrm{cf}(\alpha)}, \aleph_\beta, \mathfrak{r}_0\} < \aleph_\xi < \aleph_\alpha$ and $\aleph_\alpha = \sum_{\xi<\omega_{cf(\alpha)}} \aleph_\xi$. If for every $\xi < \omega_{\mathrm{cf}(\alpha)}$, E_ξ is a subset of power $\geqq \aleph_\xi$ of E such that E_ξ has a subset E'_ξ with the property $T(\aleph_\xi, \mathfrak{r}_0)$, then E has a subset with the property $T(\aleph_\alpha, [\aleph_{\mathrm{cf}(\alpha)}\mathfrak{r}_0]^+)$.* (See [1], theorem 4.)

Theorem 3. *If M is an infinite set of power $\mathfrak{m}$, and if $\mathfrak{n} \leqq \mathfrak{m}$, then the set S of subsets $X \subset M$ with $\overline{\overline{X}} < \mathfrak{n}$ has the power $\overline{\overline{S}} = \sum_{\mathfrak{r}<\mathfrak{n}} \mathfrak{m}^{\mathfrak{r}}$.* (See for example the theorem 3 of § 34 in [2].)

Theorem 4.

$$(\mathfrak{m}^{\aleph_\varrho})^{\aleph_\mu} = \begin{cases} \mathfrak{m}^{\aleph_\varrho} & \text{for } \mu \leqq \mathrm{cf}(\varrho), \\ \mathfrak{m}^{\aleph_\varrho} & \text{for } \mathrm{cf}(\varrho) < \mu \leqq \varrho+1, \\ \mathfrak{m}^{\aleph_\mu} & \text{for } \mu > \varrho. \end{cases}$$

(See theorem 7 of § 34 in [2].)

Theorem 5. *Let $\aleph_\alpha$ be a singular cardinal number and η an ordinal number smaller than ω_α. If to every element γ of $W(\omega_\alpha)$ there corresponds an ordinal number $h(\gamma) < \eta$, then there exists a subset M of power $\aleph_\alpha$ of $W(\omega_\alpha)$ such that*

$$\overline{\overline{h[M]}} \leqq \aleph_{\mathrm{cf}(\alpha)}.$$

Proof. Let $\{\alpha_\xi\}_{\xi<\omega_{cf(\alpha)}}$ be an increasing sequence of ordinal numbers such that $\lim_{\xi<\omega_{cf(\alpha)}} \alpha_\xi = \alpha$ for every $\xi < \omega_{\mathrm{cf}(\alpha)}$, $\omega_{\alpha_\xi} > \eta$ and ω_{α_ξ} is regular. It is clear that

$$W(\omega_\alpha) = \bigcup_{\xi<\omega_{cf(\alpha)}} W(\omega_{\alpha_\xi}).$$

Let us define $g_\xi(\gamma)$ on $W(\omega_{\alpha_\xi})$ as follows:

$$g_\xi(\gamma) = h(\gamma) \quad (\gamma \in W(\omega_{\alpha_\xi})).$$

Since ω_{α_ξ} is regular and $\omega_{\alpha_\xi} > \eta$, there exists an ordinal number $\pi_\xi \in W(\eta)$ and a subset M_ξ of power $\aleph_{\alpha_\xi}$ of $W(\omega_{\alpha_\xi})$ such that

$$g_\xi[M_\xi] = \{\pi_\xi\}.$$

Let
$$M = \bigcup_{\xi<\omega_{cf(\alpha)}} M_\xi.$$

Clearly the power of M is $\aleph_\alpha$. Let further N be the set of all distinct elements of the sequence $\{\pi_\xi\}_{\xi<\omega_{cf(\alpha)}}$. It is clear that

$$h[M] = N.$$

Since $\overline{\overline{N}} \leqq \aleph_{\mathrm{cf}(\alpha)}$, theorem 5 is proved.

Corollary. *If η is an ordinal number of the second kind and* $\mathrm{cf}(\eta) \neq \mathrm{cf}(\alpha)$, *then there exists a subset M' of power $\aleph_\alpha$ of M and an ordinal number $\eta' < \eta$ such that*

$$h[M'] \subseteqq W(\eta').$$

Proof. (i) If $\overline{\overline{N}} < \aleph_{\mathrm{cf}(\alpha)}$, then it follows from the regularity of $\omega_{\mathrm{cf}(\alpha)}$ that there exists an increasing sequence $\{\xi_\nu\}_{\nu < \omega_{cf(\alpha)}}$ of the type $\omega_{\mathrm{cf}(\alpha)}$ of ordinal numbers smaller than $\omega_{\mathrm{cf}(\alpha)}$ such that

$$\pi_{\xi_0} = \pi_{\xi_1} = \ldots = \pi_{\xi_\nu} = \ldots \quad (\nu < \omega_{\mathrm{cf}(\alpha)}).$$

But then
$$\overline{\overline{\{\gamma \in M : h(\gamma) = \pi_{\xi_0}\}}} = \sum_{\xi_\nu < \omega_{cf(\alpha)}} \aleph_{\xi_\nu} = \aleph_\alpha.$$

and
$$h[M'] = h[\{\gamma \in M : h(\gamma) = \pi_{\xi_0}\}] \subseteqq W(\pi_{\xi_0} + 1).$$

(j) If $\overline{\overline{N}} = \aleph_{\mathrm{cf}(\alpha)}$, then let $\{\eta_\nu\}_{\nu < \omega_{cf(\eta)}}$ be an increasing sequence of ordinal numbers for which $\lim\limits_{\nu < \omega_{cf(\eta)}} \eta_\nu = \eta$.

(j_1) If $\mathrm{cf}(\alpha) < \mathrm{cf}(\eta)$, then it follows from the inequality $N \subset W(\eta)$ that there exists an ordinal number $\nu_0 < \omega_{\mathrm{cf}(\eta)}$, for which

$$N \subseteqq W(\eta_{\nu_0}) \subset W(\eta).$$

(j_2) If $\mathrm{cf}(\alpha) > \mathrm{cf}(\eta)$, then let $N_\nu = N \cap W(\eta_\nu)$. It is clear that

$$\bigcup_{\nu < \omega_{cf(\eta)}} N_\nu = N.$$

Since $\omega_{\mathrm{cf}(\alpha)}$ is regular, there exists an ordinal number $\nu_0 < \omega_{\mathrm{cf}(\eta)}$ such that

$$\overline{\overline{N}}_{\nu_0} = \aleph_{\mathrm{cf}(\alpha)}.$$

It follows that there exists an increasing sequence $\{\xi_\varrho\}_{\varrho < \omega_{cf(\alpha)}}$ of the type $\omega_{\mathrm{cf}(\alpha)}$ such that

$$N_{\nu_0} = \{\pi_{\xi_\varrho}\}_{\varrho < \omega_{cf(\alpha)}}.$$

Thus we get from the definition of $\{\pi_\xi\}_{\xi < \omega_{cf(\alpha)}}$ that $M' = \bigcup\limits_{\nu < \omega_{cf(\alpha)}} M_{\xi_\nu}$ has the power $\aleph_\alpha = \sum \aleph_{\alpha_{\xi_\nu}}$ and

$$h[M'] \subset W(\eta_{\nu_0}).$$

Theorem 6. *Let $\aleph_\alpha$ be a singular cardinal number and η an ordinal number smaller than ω_α. If to every element γ of $W(\omega_\alpha)$ there corresponds an ordinal number $h(\gamma) < \eta$, then the smallest ordinal number η_0, for which there exists a subset M of power $\aleph_\alpha$ of $W(\omega_\alpha)$ such that*

$$h[M] \subset W(\eta_0) \subseteqq W(\eta),$$

is either of the first kind, i. e. $\eta_0 = \tau_0 + 1$ or of the second kind with $\mathrm{cf}(\eta_0) = \mathrm{cf}(\alpha)$.

Proof. (i) $W(\eta_0)$ has a greatest element. In this case the power of the set M', for which $h[M'] = \{\pi_0\}$, is $\aleph_\alpha$ and the power of the set M'', for which

$$h[M''] \subseteqq W(\tau_0),$$

is smaller than $\aleph_\alpha$. Thus $\eta_0 = \tau_0 + 1$.

(ii) $W(\eta_0)$ does not contain a greatest element. Then η_0 is of the second kind. It follows from the definition of η_0 and the corollary of theorem 5 that $\mathrm{cf}(\eta_0) = \mathrm{cf}(\alpha)$. Theorem 6 is proved. With the aid of theorem 6 we get

Theorem 7. *The ordinal number β_0 is either of the first kind or of the second kind with* $\mathrm{cf}(\beta_0)=\mathrm{cf}(\alpha)$.

Proof of the theorem. (A) First we prove that (I) follows from (II). Suppose also that the proposition (II) holds. Put

$$(\aleph_{\gamma}^{\underset{\smile}{\aleph_{\beta_0}}})^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta_0(\gamma)}.$$

It follows from theorem 4, that

$$\aleph_{\beta_0(\gamma)} = \begin{cases} \aleph_{\gamma}^{\underset{\smile}{\aleph_{\beta_0}}} & \text{for} \quad \mathrm{cf}(\beta_0)=\beta_0 . \\ \aleph_{\gamma}^{\aleph_{\beta_0}} & \text{for} \quad \mathrm{cf}(\beta_0)<\beta_0. \end{cases}$$

This implies that

$$\aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = (\aleph_{\gamma}^{\underset{\smile}{\aleph_{\beta_0}}})^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\gamma}^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta_0(\gamma)}$$

for $\mathrm{cf}(\beta_0)=\beta_0$ and

$$\aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = \sum_{\varrho<\beta_0} \aleph_{\beta_0(\gamma)}^{\aleph_{\varrho}} = \sum_{\varrho<\beta_0} (\aleph_{\gamma}^{\aleph_{\beta_0}})^{\aleph_{\varrho}} = \sum_{\varrho<\beta_0} \aleph_{\gamma}^{\aleph_{\beta_0}\cdot\aleph_{\varrho}} = \sum_{\varrho<\beta_0} \aleph_{\gamma}^{\aleph_{\beta_0}} = \aleph_{\gamma}^{\aleph_{\beta_0}}\cdot\aleph_{\beta_0} = \aleph_{\beta_0(\gamma)}$$

for $\mathrm{cf}(\beta_0)<\beta_0$, i. e. in both cases $\aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta(\gamma_0)}$ holds. As the sets $f(x)$ are distinct it follows from this that the set $\bigcup_{x\in E} f(x)$ has the power $\aleph_\alpha$. Thus, if $\aleph_\alpha$ is regular, we get by theorem 1, that E has a subset with the property $T(\aleph_\alpha, \aleph_\alpha)$. Suppose now that $\aleph_\alpha$ is singular. Then $E^{(\beta_0)}$ has for every γ, $\beta<\gamma<\alpha$, a subset E_γ with the property $T(\aleph_{\beta_0(\gamma)+1}, \aleph_{\beta_0(\gamma)+1})$, i. e.

$$\overline{\overline{\Pi}}_{E_\gamma} \leqq \aleph_{\beta_0(\gamma)} < \aleph_{\beta_0(\gamma)+1}.$$

Let $S(\gamma)$ be the set of subsets $X\subset \Pi_{E_\gamma}$ with $\overline{\overline{X}}<\aleph_{\beta_0}$. It follows from theorem 3 that $\overline{\overline{S(\gamma)}} \leqq \aleph_{\beta_0(\gamma)}^{\underset{\smile}{\aleph_{\beta_0}}} = \aleph_{\beta_0(\gamma)}$. Hence, since for given γ the sets $f^{(\gamma)}(x)=f(x)-\Pi_{E_\gamma}$ $(x\in E_\gamma)$ are mutually disjoint, we obtain that there exists an element X_0 of $S(\gamma)$ and to this a subset E'_γ of power $\aleph_{\beta_0(\gamma)+1}$ of E_γ such that $f^{\gamma}(x)\neq 0$ and

$$f(x)=f^{(\gamma)}(x)\cup X_0$$

for every $x\in E'_\gamma$, i. e. E'_γ has the property $T(\aleph_{\beta_0(\gamma)+1}, \aleph_{\beta_0})$. It follows from theorem 2 that E has a subset with the property $T(\aleph_\alpha, \aleph_{\beta_0}\aleph_{\mathrm{cf}(\alpha)+1})$.

(B) We prove now that from the proposition (I) follows the proposition (II). Suppose therefore that (II) does not hold. Then we prove that the proposition (I) is false.

Let β_0 is an ordinal number of the first kind, i. e. $\beta_0=\tau_0+1$. If (II) does not hold, then there exists an ordinal number γ_0, $\beta<\gamma_0<\alpha$ for which

$$\aleph_{\gamma_0}^{\aleph_{\tau_0}} \geqq \aleph_\alpha.$$

Let E_1 be a subset of power $\aleph_{\gamma_0}$ of E and T_1 a set of power $\aleph_\alpha$ of subsets of power $\aleph_{\tau_0}$ of E_1. Let further $f(x)$ be a one-to-one mapping of E into T_1. It follows that if Γ is a subset of E with the property $T(\mathfrak{q}, \mathfrak{p})$ then $\mathfrak{q} \leqq \aleph_{\gamma_0}$, because the sets

$$f'(x) = f(x) - \Pi_\Gamma \subset E_1$$

must be not empty and mutually disjoint for $\mathfrak{q}$ elements x of Γ.

Let β_0 be an ordinal number of the second kind. Then $\mathrm{cf}(\beta_0) = \mathrm{cf}(\alpha)$ by the theorem 7. Let $\{\alpha_\eta\}_{\eta < \omega_{cf(\alpha)}}$ and $\{\beta_\eta\}_{\eta < \omega_{cf(\alpha)}}$ be two increasing sequences of ordinal numbers such that $\lim\limits_{\eta < \omega_{cf(\alpha)}} \alpha_\eta = \alpha$ and $\lim\limits_{\eta < \omega_{cf(\alpha)}} \beta_\eta = \beta_0$. We have two cases:

(i) there exists a smallest ordinal number $\eta_0 < \omega_{\mathrm{cf}(\alpha)}$ and an ordinal number γ_0, $\beta < \gamma_0 < \alpha$, such that $\aleph_{\gamma_0}^{\aleph_{\beta_{\eta_0}}} \geqq \aleph_\alpha$;

(ii) for every $\varrho < \beta_0$ there exists an $\varrho' < \beta_0$ such that $\aleph_{\gamma_0}^{\aleph_{\varrho'}} > \aleph_{\gamma_0}^{\aleph_\varrho}$. In this case we assume that, for every $\eta < \omega_{\mathrm{cf}(\alpha)}$, β_η is the smallest ordinal number such that

$$\aleph_{\gamma_0}^{\aleph_{\beta_\eta}} \geqq \aleph_{\alpha_\eta}.$$

Let T_η be in both cases (but in the case (i) we assume that $\eta_0 \leqq \eta < \beta_0$ holds) a set of power $\aleph_{\alpha_\eta}$ of subsets of power $\aleph_{\beta_\eta}$ of E_1, where $\overline{\overline{E}}_1 = \aleph_{\gamma_0}$. It is clear that the set

$$T = \bigcup_{\eta < \omega_{cf(\alpha)}} T_\eta$$

has the power $\aleph_\alpha$. Let $f(x)$ be a one-to-one mapping of E into T. If Γ is a subset of E with the property $T(\mathfrak{q}, \mathfrak{p})$, then $\mathfrak{q} \leqq \aleph_{\gamma_0}$, because the sets $f'(x) = f(x) - \Pi_\Gamma \subset E_1$ must be non empty and mutually disjoint for $\mathfrak{q}$ elements x of Γ. The theorem is proved.

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ПРИМИТИВНЫЕ КЛАССЫ АЛГЕБР, ЭКВИВАЛЕНТНЫЕ КЛАССАМ ПОЛУМОДУЛЕЙ И МОДУЛЕЙ

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Целью настоящей статьи является продолжение исследований, проведенных в работе автора [3], в частности, усиление некоторых результатов, там изложенных. Ввиду этого предполагается знакомство читателя с определениями, обозначениями и результатами упомянутой работы.

§ 1

В [3] сформулированы следующие условия, которыми может обладать некоторый примитивный класс $\mathfrak{A}$:

I. В $\mathfrak{A}$ существует нульместная операция, отмеченный которой элемент образует подалгебру в любой алгебре класса $\mathfrak{A}$.

II′. В любой алгебре из $\mathfrak{A}$ каждая конгруэнция однозначно определяется своим классом, являющимся нормальной подалгеброй.

III. В любой алгебре из $\mathfrak{A}$ каждая подалгебра нормальна.

IV. Класс $\mathfrak{A}$ нормальный.

Кроме этих условий, нам понадобится и следующее:

V. В классе $\mathfrak{A}$ прямое и свободное произведения двух алгебр совпадают. Иными словами, между прямым и $\mathfrak{A}$-свободным произведениями алгебр A и B класса $\mathfrak{A}$ существует такой изоморфизм, при котором элементы A и B соответствуют самим себе.

Условие V исследовал А. А. Терехов [2]. Им отмечено, что V имеет смысл лишь при наличии I, а также дана характеристика квазипримитивных классов, удовлетворяющих условиям I, V. Напомним, что существование свободного произведения в примитивном классе установлено Сикорским [4].

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Алгебру R с основными операциями $+, \cdot$, и с нулевым элементом $\mathbf{0}$, для которых выполняются аксиомы

$$r_1+r_2=r_2+r_1,\ (r_1+r_2')+r_3=r_1+(r_2+r_3),$$
$$(r_1+r_2)r_3=r_1r_3+r_2r_3,\ r_3(r_1+r_2)=r_3r_1+r_3r_2,$$
$$(r_1r_2)r_3=r_1(r_2r_3),$$
$$r_1+\mathbf{0}=r_1,\ r_1\mathbf{0}=\mathbf{0}\ r_1=\mathbf{0},$$

где $r_1, r_2, r_3 \in R$, мы будем называть (ассоциативным) полукольцом.

Аддитивную полугруппу A с единичным элементом 0, в которой определено операторное произведение $a\varrho$, где $a \in A$, ϱ — элемент некоторого полукольца с единицей R, подчиненное условиям

$$a(\varrho_1+\varrho_2)=a\varrho_1+a\varrho_2,\ (a+b)\varrho_1=a\varrho_1+b\varrho_1,$$
$$a(\varrho_1\varrho_2)=(a\varrho_1)\varrho_2,\ a\mathbf{0}=0,\ a1=a,\ 0\varrho_1=0,$$

где $a, b \in A$, $\varrho_1, \varrho_2 \in R$, $\mathbf{0}$ — нулевой элемент, а 1 — единица в R, мы назовем правым унитарным R-полумодулем.

Легко видеть, что если R фиксированное полукольцо с единицей, то все правые унитарные R-полумодули образуют примитивный класс.

§ 2

Сейчас мы охарактеризуем с точностью до эквивалентности примитивные классы адгебр с условиями I, V. Подготовкой служит следующая

Лемма. Если эквивалентность примитивных классов $\mathfrak{A}$ и $\mathfrak{B}$ определяется отображением операций φ и алгебрам A, A' из $\mathfrak{A}$ соответствуют алгебры B, B' из $\mathfrak{B}$, то алгебре $A \times A'$ соответствует $B \times B'$, а $\mathfrak{A}$-свободному произведению $A * A'$ — $\mathfrak{B}$-свободное произведение $B * B'$.

Доказательство. Пусть θ — отображение элементов A на B при рассматриваемой эквивалентности, а θ — отображение элементов A' на B'. Построим отображение η элементов $A \times A'$ на $B \times B'$ так:

$$(a, a')\eta=(a\theta, a'\theta'),\quad (a \in A, a' \in A').$$

η взаимно однозначно и если ν произвольная n-местная операция класса $\mathfrak{A}$, то для любых $a_i \in A$, $a_i' \in A'$ $(i=1, \ldots, n)$

$$((a_1, a_1')\ldots(a_n, a_n')\nu)\eta=(a_1\ldots a_n\nu, a_1'\ldots a_n'\nu)\eta=$$
$$=((a_1\ldots a_n\nu)\theta, (a_1'\ldots a_n'\nu)\theta')=((a_1\theta)\ldots(a_n\theta)(\nu\varphi), (a_1'\theta')\ldots(a_n'\theta')(\nu\varphi))=$$
$$=(a_1\theta, a_1'\theta')\ldots(a_n\theta, a_n'\theta')(\nu\varphi)=((a_1, a_1')\eta\ldots(a_n, a_n')\eta)(\nu\varphi).$$

Значит, $A\times A'$ эквивалентна $B\times B'$, а поэтому $B\times B'$ изоморфна той алгебре класса $\mathfrak{B}$, которая соответствует алгебре $A\times A'$ при рассматриваемой эквивалентности классов $\mathfrak{A}$ и $\mathfrak{B}$.

Перейдем к свободным произведениям. При определении в $B*B'$ операций класса $\mathfrak{A}$ посредством

(1) $$b_1\ldots b_r\varrho=b_1\ldots b_r(\varrho\varphi),\quad (b_i\in B*B',\ i=1,\ldots,r),$$

$B*B'$ превращается в алгебру класса $\mathfrak{A}$ $\overline{B*B'}$ (см. лемму 1 из [3] и ее доказательство). При этом θ, θ' изоморфно отображают A, A' на подалгебры B, B' алгебры $\overline{B*B'}$. Поскольку $A*A'$ — $\mathfrak{A}$-свободное произведение, θ и θ' можно продолжить до гомоморфного отображения η $A*A'$ в $\overline{B*B'}$. Гомоморфизм η на самом деле есть отображение на $\overline{B*B'}$. Действительно, если $x\in B*B'$, то существует запись вида $x=b_1\ldots b_nb_1'\ldots b_m'\sigma$, где σ — главная производная операция класса $\mathfrak{A}$, $b_i\in B$ $(i=1,\ldots,n)$, $b_j'\in B'$ $(j=1,\ldots,m)$. Однако $\sigma=\varrho\varphi$, где ϱ — некоторая операция класса $\mathfrak{A}$, а поэтому, ввиду (1), в алгебре $\overline{B*B'}$

$$x=b_1\ldots b_nb_1'\ldots b_m'\varrho.$$

Определяя в $A*A'$ операции класса $\mathfrak{B}$ путем

$$a_1\ldots a_r(\varrho\varphi)=a_1\ldots a_r\varrho,\quad (a_i\in A*A',\ i=1,\ldots,r),$$

мы превратим $A*A'$ в алгебру класса $\mathfrak{B}$ $\overline{A*A'}$, притом θ^{-1}, θ'^{-1} изоморфно отображают B, B' в подалгебры A, A' алгебры $\overline{A*A'}$. Как и выше, θ^{-1} и θ'^{-1} можно продолжить до гомоморфного отображения χ $B*B'$ на $\overline{A*A'}$.

$\eta\chi$ является отображением $A*A'$ на себя, тождественным для элементов A и A'. Имеет место

$$(a_1\ldots a_na_1'\ldots a_m'\varrho)(\eta\chi)=\big((a_1\eta)\ldots(a_n\eta)(a_1'\eta)\ldots(a_m'\eta)(\varrho\varphi)\big)\chi=$$
$$=\big((a_1\theta)\ldots(a_n\theta)(a_1'\theta')\ldots(a_m'\theta')(\varrho\varphi)\big)\chi=a_1\ldots a_na_1'\ldots a_m'\varrho,$$

т. е. $\eta\chi$ — тождественное отображение. Итак, η — взаимно однозначно, т. е. оно есть изоморфизм $A*A'$ на $\overline{B*B'}$. Ввиду (1), $A*A'$ и $B*B'$ эквивалентны при отображении операций φ. Этим лемма доказана.

Еще раз отметим, что отображение операций φ, определяющее эквивалентность классов $\mathfrak{A}$ и $\mathfrak{B}$, тем самым однозначно определяет и соответствие между алгебрами классов $\mathfrak{A}$ и $\mathfrak{B}$.

Теорема 1. *Примитивный класс алгебр $\mathfrak{A}$ тогда и только тогда эквивалентен примитивному классу всех правых унитарных полумодулей над некоторым ассоциативным полукольцом с единицей, если $\mathfrak{A}$ удовлетворяет условиям I и V.*

Доказательство. Если в $\mathfrak{A}$ выполняются условия I, V, то свободную алгебру класса $\mathfrak{A}$ с двумя свободными образующими можно отождествить с прямым произведением двух свободных алгебр класса $\mathfrak{A}$ с одним свободным образующим каждая. Из этого факта, как и при доказательстве теоремы 1 в [3], вытекает, что в $\mathfrak{A}$ существует такая двуместная ассоциативная операция $+$ с нулем, что для любой n-местной операции ω класса $\mathfrak{A}$ тождественно

$$(x_1+y_1)\dots(x_n+y_n)\omega = x_1\dots x_n\omega + y_1\dots y_n\omega. \tag{2}$$

Отсюда, в частности, получаем коммутативность операции $+$.

Множество M, состоящее из всех одноместных операций и из нульместной операции $\mathbf{0}$ класса $\mathfrak{A}$, можно превратить в полукольцо с единицей, если в нем определить операции полукольца следующим образом:

$$x(\mu_1+\mu_2)=x\mu_1+x\mu_2, \quad x\mathbf{0}=0, \quad x(\mu_1\mu_2)=(x\mu_1)\mu_2.$$

Полученное полукольцо обозначим через R. Сейчас мы можем доказать, что примитивный класс $\mathfrak{R}$ всех правых унитарных R-полумодулей эквивалентен классу $\mathfrak{A}$. Доказательство состоит в дословном повторении доказательства теоремы 2 из [3]. Отметим, что при этом нам приходится использовать разложимость операций класса $\mathfrak{R}$ в сумму одноместных операций, которая вместо ссылки на свойства абелевых Ω-групп доказывается индукцией по степени слова, определяющего операцию, над системой операций, состоящей из сложения и умножений на элементы полукольца R.

С другой стороны, пусть примитивный класс $\mathfrak{A}$ эквивалентен примитивному классу $\mathfrak{R}$ всех правых унитарных полумодулей над некоторым ассоциативным полукольцом с единицей R. Выполнение I в $\mathfrak{A}$ очевидно. Рассмотрим в $\mathfrak{A}$ прямое произведение $A\times A'$ ($A, A'\in\mathfrak{A}$) и возьмем прямое произведение полумодулей P, P', соответствующих в $\mathfrak{R}$ алгебрам A, A'. Согласно лемме, $P\times P'$ соответствует алгебре $A\times A'$. Убедимся, что $P\times P'$ — свободное произведение полумодулей P и P' в классе $\mathfrak{R}$.[1]) В самом деле, $P\times P'$ порождается своими подмодулями P и P'. Далее, если θ, θ' — гомоморфизмы P, соответственно P' в некоторый R-полумодуль Q, то пусть отображение η $P\times P'$ в Q определяется так:

$$(p,p')\eta=(p,0)\theta+(0,p')\theta' \qquad (p\in P, p'\in P').$$

Для элементов из P η совпадает с θ, а для элементов из P' — с θ'; кроме того, оно является гомоморфизмом $P\times P'$ в Q, ибо если $p_1, p_2\in P$,

[1]) Доказательство этого факта по существу идет от А. А. Терехова [2] и мы включаем его лишь для удобства читателя.

$p_1', p_2' \in P'$, то

$$[(p_1, p_1') + (p_2, p_2')]\eta = (p_1 + p_2, p_1' + p_2')\eta =$$
$$= (p_1 + p_2, 0)\theta + (0, p_1' + p_2')\theta' =$$
$$= (p_1, 0)\theta + (p_2, 0)\theta + (0, p_1')\theta' + (0, p_2')\theta' = (p_1, p_1')\eta + (p_2, p_2')\eta$$

и для любого $\varrho \in R$

$$[(p_1, p_1')\varrho]\eta = (p_1\varrho, p_1'\varrho)\eta = (p_1\varrho, 0)\theta + (0, p_1'\varrho)\theta' =$$
$$= [(p_1, 0)\theta]\varrho + [(0, p_1')\theta']\varrho = [(p_1, 0)\theta + (0, p_1')\theta']\varrho = [(p_1, p_1')\eta]\varrho.$$

Из доказанного на основании леммы следует, что $A \times A'$ есть $\mathfrak{A}$-свободное произведеиие своих подалгебр A и A'. Этим показано выполнение условия V в $\mathfrak{A}$, что завершает доказательство теоремы 1.

§ 3

Теперь мы будем рассматривать, как и в работе [3], класс унитарных модулей над кольцом.

Теорема 2. *Для примитивного класса $\mathfrak{A}$ следующие четыре утверждения равносильны:*

(A). *$\mathfrak{A}$ эквивалентен примитивному классу всех правых унитарных модулей над некоторым ассоциативным кольцом с единицей.*

(B). *$\mathfrak{A}$ удовлетворяет условиям* I, II′, III.

(C). *$\mathfrak{A}$ удовлетворяет условиям* I, II′, V.

(D). *$\mathfrak{A}$ удовлетворяет условиям* I, IV, V.

Доказательство. (A) влечет (B). В самом деле, всякий прмитивный класс правых унитарных модулей над ассоциативном кольцом с единицей удовлетворяет условиям I, II′, III. Поэтому нам достаточно заметить, что если примитивные классы $\mathfrak{A}$ и $\mathfrak{B}$ эквивалентны, то отображение элементов, устанавливающее эквивалентность соответствующих алгебр A из $\mathfrak{A}$ и B из $\mathfrak{B}$, переводит подалгебры A в подалгебры B, а также конгруэнции A в конгруэнции B.

(B) влечет (C). В классе $\mathfrak{A}$ со свойством (B) свободное произведение двух алгебр согласно лемме 2 из [3] совпадает с их прямым произведением.

(C) влечет (D). Класс $\mathfrak{A}$ со свойством (C) по теореме 1 эквивалентен примитивному классу $\mathfrak{R}$ всех правых унитарных полумодулей над некоторым ассоциативным полукольцом с единицей R. Рассмотрим в классе $\mathfrak{R}$ свободный полумодуль F с двумя свободными образующими x_1, x_2. Элементы F представимы словами класса $\mathfrak{R}$ с переменными x_1, x_2. Индукцией по степени слов над системой операций, состоящей из сложения и умножений на элементы из R, получим, что всякий элемент из F имеет вид $x_1\varrho_1+x_2\varrho_2$, $\varrho_1, \varrho_2\in R$. Такое представление является единственным, ибо если $x_1\varrho_1+x_2\varrho_2=x_1\varrho_1'+x_2\varrho_2'$, $\varrho_1', \varrho_2'\in R$, то, так как F свободен в классе $\mathfrak{R}$, это равенство выполняется в $\mathfrak{R}$ тождественно, и, подставляя $x_1=x, x_2=0$ и $x_1=0$, $x_2=x$, мы получаем, что $\varrho_1=\varrho_1'$, $\varrho_2=\varrho_2'$.

Введем в F следующее бинарное отношение: $x_1\sigma_1+x_2\sigma_2\equiv x_1\tau_1+x_2\tau_2$ тогда и только тогда, если $\sigma_1+\sigma_2=\tau_1+\tau_2$. Из аксиом полумодуля вытекает, что это отношение определяет конгруэнцию $\varkappa$ в F. В классе $\mathfrak{R}$, эквивалентном классу $\mathfrak{A}$, выполняется условие II′. Принимая во внимание, что $x_1\equiv x_2$, ибо $x_1=x_1 1+x_2\mathbf{0}$, $x_2=x_1\mathbf{0}+x_2 1$, мы видим, что нулевой класс конгруэнции $\varkappa$ содержит ненулевой элемент, в противном случае $\varkappa$ оказалась бы тривиальной в силу II′. Поэтому существует такой элемент $x_1\tau_1+x_2\tau_2\in F$, что $x_1\tau_1+x_2\tau_2\equiv 0$, но $x_1\tau_1+x_2\tau_2\neq 0$. Таким образом, существуют такие $\tau_1, \tau_2\in R$, что $\tau_1+\tau_2=0$, притом они отличны от нуля, поскольку $\tau_1=0$ влечет за собой $\tau_2=0$, откуда $x_1\tau_1+x_2\tau_2=0$ вопреки предположению. Следовательно, в полукольце R существуют ненулевые элементы, обладающие противоположными элементами относительно сложения. Совокупность всех таких элементов образует подкольцо R_1 в R, являющееся в нем даже (двусторонним) идеалом. Покажем, что $R_1=R$.

Допустим, что это не так, т. е. $R_1\subset R$ и элементы из R, не принадлежащие к R_1, не имеют противоположного элемента. Рассмотрим в R смежные классы $R_1+\varrho(\varrho\in R)$. Если $\varrho_1+\varrho=\varrho_1'+\varrho'$, $(\varrho_1, \varrho_1'\in R_1, \ \varrho, \varrho'\in R)$, то $\varrho=(-\varrho_1+\varrho_1')+\varrho'$, а поэтому $R_1+\varrho=R_1+(-\varrho_1+\varrho_1')+\varrho'=R_1+\varrho'$, откуда следует, что если два смежных класса имеют общий элемент, то они совпадают. Итак, эти классы осуществляют разбиение полукольца R, которое, очевидно, является и конгруэнцией в R. Поскольку нулевой класс этой конгруэнции есть R_1, то полученное факторполукольцо мы можем обозначать через R/R_1; его единицей служит R_1+1.

Рассмотрим примитивный класс всех правых унитарных R/R_1-полумодулей. Каждый R/R_1-полумодуль естественным образом можно рассматривать как R-полумодуль и при этом конгруэнции в нем остаются те же самые. Поэтому примитивный класс всех правых унитарных R/R_1-полумодулей удовлетворяет условию II′, справедливость же условий I и V вытекает из теоремы 1. Повторением предыдущего процесса мы получим, что R/R_1 содержит ненулевое подкольцо R_2/R_1, где R_2 — соответствующее подполукольцо в R. Пусть $\varrho_2\in R_2$, $\varrho_2\notin R_1$. Тогда существует $\varrho_2'\in R_2$, для которого $\varrho_2+\varrho_2'=\varrho_1\in R_1$. Мы видим, что $\varrho_2+(\varrho_2'-\varrho_1)=0$, т. е. ϱ_2 обладает противоположным элементом, что противоречит предположению.

Мы видим, что R является кольцом, а поэтому единица из R имеет противоположный элемент. Рассмотрим в $\mathfrak{R}$ операцию $xyz\omega = x + y(-1) + z$. Тождественно имеет место $xxz\omega = zxx\omega = z$. Берем операцию ω' из $\mathfrak{A}$, соответствующую ω при отображении операций, определяющем эквивалентность между $\mathfrak{R}$ и $\mathfrak{A}$. Согласно лемме 1 из [3], в $\mathfrak{A}$ тождественно $xxz\omega' = zxx\omega' = z$. По теореме 4 из [1] это равносильно тому, что в $\mathfrak{A}$ выполняется IV.

(D) влечет (A). Класс $\mathfrak{A}$ со свойством (D) эквивалентен примитивному классу $\mathfrak{R}$ всех правых унитарных полумодулей над некоторым полукольцом с единицей R. Из выполнения IV следует, по [1], существование в $\mathfrak{A}$, а также в $\mathfrak{R}$, тернарных операций ω', соответственно ω с тождеством $xxz\omega = zxx\omega = z$. Как упомянуто при доказательстве теоремы 1, ω разлагается следующим образом: $xyz\omega = x\omega_1 + y\omega_2 + z\omega_3$. Здесь ω_i ($i = 1, 2, 3$) — элементы из R, что показывается таким же путем, как и в доказательстве теоремы 2 из [3]. Тогда, пользуясь тождеством (2) и тем, что R само является правым унитарным R-полумодулем, получим:

$$1 + \omega_2 = 1 + 1\omega_2 = 100\omega + 010\omega = (1+0)(0+1)(0+0)\omega = 110\omega = 0.$$

Значит 1, а вместе с ней и каждый элемент из R, имеет противоположный элемент, т. е. R — кольцо. Учитывая, что система аксиом унитарного полумодуля содержит систему аксиом унитарного модуля, мы видим, что $\mathfrak{R}$ есть примитивный класс всех правых унитарных R-модулей. Теорема доказана.

Поскольку эквивалентность примитивных классов является транзитивной, из теоремы 2 и из теоремы 2 работы [3] вытекает

Следствие. Примитивный класс $\mathfrak{A}$ тогда и только тогда эквивалентен некоторому примитивному классу абелевых Ω-групп, если для $\mathfrak{A}$ верно какое-либо из утверждений (A), (B), (C) и (D).

Отметим, что системы условий в утверждениях (B), (C), (D) являются минимальными в том смысле, что ни в одной из них нельзя вычеркнуть ни одного из условий II′, III, IV, V. Для доказательства этого факта достаточно заметить, что в примитивном классе всех групп выполняются I, II′, IV, но не выполняются III, V; в примитивном классе всех коммутативных полугрупп с единицей выполняются I, V, но не выполняются II′, IV; наконец, в примитивном классе всех полугрупп с тождественным соотношением $x_1x_2 = x_3x_4$ выполняется I, III, но не выполняется II′.

Отметим также, что в результате Шоды [4], цитированном и в § 5 работы [3], условие нормальности рассматриваемого примитивного класса оказывается излишним, так как оно является следствием условий I, II′, III.

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О НЕКОТОРЫХ КЛАССАХ ПОЛУМОДУЛЕЙ И МОДУЛЕЙ

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Введение

Известно (см. [2]), что тем примитивным классам универсальных алгебр, в которых прямые и свободные произведения совпадают, можно поставить в соответствие некоторое полукольцо с единицей, однозначно определенное с точностью до изоморфизма, которое может рассматриваться, как область операторов данного класса. Аналогично, любому примитивному классу абелевых Ω-групп соответствует некоторое вполне определенное кольцо с единицей [1]. В настоящий работе мы выясним строение колец, соответствующих некоторым конкретным примитивным классам указанных видов. В заключительной части даются признаки относительно эквациональной полноты исследованных классов.

§ 1

Будем предполагать, что для читателя знакомы обозначения, определения и результаты работ [1] и [2]. Сейчас для удобства мы напомним некоторые из них, а также приводим дальнейшие обозначения и определения.

Подкласс $\mathfrak{B}$ примитивного класса $\mathfrak{A}$, образующий примитивный класс относительно операций того же класса, называется примитивным подклассом класса $\mathfrak{A}$. Множество всех тождеств примитивного класса $\mathfrak{A}$ мы обозначим через $\Lambda(\mathfrak{A})$.

Пусть R — произвольное кольцо. Примитивный класс всех правых модулей над R обозначается через $\mathfrak{M}^{\mathsf{R}}$, а при наличии единичного элемента в R для примитивного класса всех правых унитарных R-модулей мы будем пользоватся обозначением $\mathfrak{M}_1^{\mathsf{R}}$. Если M является модулем над кольцами R и L, то оно называется бимодулем над R и L. Примитивный класс всех M,

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являющихся одновременно левым R- и правым L-модулем, обозначается через ${}^{R}\mathfrak{M}^{L}$, а в случае унитарности операторных областей R и L — через ${}^{R}\mathfrak{M}^{L}_{1}$.

Очевидно, $\mathfrak{M}^{R}$, $\mathfrak{M}^{R}_{1}$, ${}^{R}\mathfrak{M}^{L}$ и ${}^{R}\mathfrak{M}^{L}_{1}$ — примитивные классы абелевых Ω-групп (см. [5], [1]).

Согласно теореме 2 из [1] каждому примитивному классу $\mathfrak{A}$ абелевых Ω-групп соответствует кольцо R с единицей, определенное с точностью до изоморфизма, с таким свойством, что $\mathfrak{M}^{R}_{1}$ эквивалентен классу $\mathfrak{A}$. В дальнейшем R называется кольцом, соответствующим классу $\mathfrak{A}$.

Под тензорным произведением колец R и L мы будем понимать кольцо $R\otimes L$, состоящее из элементов $\sum_{i=1}^{k} r_i\otimes s_i$ $(k=1, 2, \ldots;\ r_i\in R,\ s_i\in L)$, где кольцевые операции производятся следующим образом: $\sum_{i=1}^{k} r_i\otimes s_i+\sum_{l=k+1}^{k+m=n} r_l\otimes s_l=$ $=\sum_{j=1}^{n} r_j\otimes s_j$; $\left(\sum_i r_i\otimes s_i\right)\left(\sum_j r'_j\otimes s'_j\right)=\sum_{i,j} r_ir'_j\otimes s_is'_j$ и выполняются следующие тождественные определяющие соотношения:

$$(\alpha)\qquad (r_1+r_2)\otimes s=r_1\otimes s+r_2\otimes s \qquad (r_1, r_2\in R;\ s\in L),$$

$$(\beta)\qquad r\otimes(s_1+s_2)=r\otimes s_1+r\otimes s_2 \qquad (r\in R;\ s_1, s_2\in L),$$

$$(\gamma)\qquad 0\otimes s=r\otimes 0.$$

Если P и T — полукольца [2], то $\mathfrak{N}^{P^*}$ означает примитивный класс всех P-полумодулей; а в унитарном случае применяется обозначение $\mathfrak{N}^{P}$. Естественным путем определяются биполумодули, для которых мы вводим обозначение ${}^{P}\mathfrak{N}^{T}$, соотв. ${}^{P}\mathfrak{N}^{T}_{1}$. Методом А. А. Терехова ([3], [1]) можно показать, что для всех этих классов выполняются условия:

I. Существование нульместной операции, отмечанный которой элемент образует подалгебру во всех алгебрах данного класса.

II. Совпадание прямого и свободного произведений двух произвольных алгебр в данном классе.

По теореме 1 из [2] примитивный класс $\mathfrak{B}$, удовлетворяющий условиям I и II, эквивалентен примитивному классу всех правых унитарных полумодулей над некоторым вполне определенным полукольцом P с единицей, которое мы будем называть полукольцом, соответствующим классу $\mathfrak{B}$.

Тензорное произведение полуколец определяется аналогично случаю колец.

I означает кольцо целых рациональных чисел, а I^* — полукольцо с нулем натуральных чисел. Через $(I+R)$ мы обозначим известное расширение Доро кольца R при помощи I (см. [8], [4]). Вполне аналогично определяется (I^*+R).

Сопоставим пару $[m, k]$ той конгруэнции I^*, в которой m — наименьшее число конгруэнтное отличному от него числу, а $m+k$ — наименьшее среди чисел, конгруэнтных m, но отличных от него ($m=0, 1, 2, \ldots$; $k=1, 2, \ldots$). Легко убедиться, что это соответствие взаимно однозначно.

В дальнейшем буквы R и L означают кольца, или же полукольца в соответствие тому, являются-ли они областями операторов модулей, или же полумодулей.

Примитивный класс $\mathfrak{A}$ алгебр называется эквационально полным [6], если прибавляя к множеству всех его тождеств хоть одно новое тождество, полученная система тождеств выполняется лишь в тривиальном (состоящем из единственной одноэлементной алгебры) примитивном подклассе класса $\mathfrak{A}$.

§2

Пусть $\mathfrak{F}$ — некоторый примитивный класс полумодулей, а R — соответствующее ему полукольцо. Имеет место

Лемма 2.1. *Примитивный класс $\mathfrak{F}$ имеет нетривиальный примитивный подкласс тогда и только тогда, если R обладает нетривиальной конгруенцией. Между примитивными подклассами $\mathfrak{F}'$ класса $\mathfrak{F}$ и конгруэнциями $\mathfrak{C}'$ полукольца R можно установить взаимно однозначное соответствие ($\mathfrak{F}' \to \mathfrak{C}'$) так, что примитивный класс всех правых унитарных полумодулей над фактор-полукольцом по $\mathfrak{C}'$ эквивалентен классу $\mathfrak{F}'$.*

Доказательство. Пусть $\mathfrak{F}'$ — примитивный подкласс класса $\mathfrak{F}$. В случае $\mathfrak{F}'=\mathfrak{F}$ классу полумодулей $\mathfrak{F}'$ ставится в соответствие конгруэнция $\mathfrak{C}_0$ полукольна R, классы которой суть отдельные элементы. Если $\mathfrak{F}'=0$, то классу $\mathfrak{F}'$ соответствует конгруэнция $\mathfrak{C}_1$ полукольца R, единственным классом которой является само R. В оставшемся случае, если $\mathfrak{F}' \subset \mathfrak{F}$, то $\Lambda(\mathfrak{F}) \subset \Lambda(\mathfrak{F})'$, т. е. в подклассе $\mathfrak{F}'$ имеет место некоторое тождество $x_1 \ldots x_m \mu = x_1 \ldots x_m \nu$, не принадлежащее $\Lambda(\mathfrak{F})$. Имеют место разложения $x_1 \ldots x_m \mu = x_1 \mu_1 + \ldots + x_m \mu_m$ и $x_1 \ldots x_m \nu = x_1 \nu_1 + \ldots + x_m \nu_m$. Подставляя $x_i = x$, $x_j = 0$ ($i \neq j$; $i, j = 1, 2, \ldots, m$) мы получим $x\mu_i = x\nu_i$. Пусть теперь φ — взаимно однозначное отображение операций класса $\mathfrak{F}$ на операции класса $\mathfrak{N}_1^{\mathsf{R}}$, при котором тождества класса $\mathfrak{F}$ и только они переидут в тождества класса $\mathfrak{N}_1^{\mathsf{R}}$. Рассмотрим следующее разбиение $\mathfrak{C}'$ полукольца R: $r_1, r_2 (\in \mathsf{R})$ содержатся в одном и том же классе тогда и только тогда, если $x(r_1\varphi^{-1}) = x(r_2\varphi^{-1})$ — тождество в $\mathfrak{F}$. Очевидно, $\mathfrak{C}'$ является конгруэнцией.

С другой стороны, пусть $\mathfrak{C}'$ — некоторая нетривиальная конгруэнция полукольца R, и пусть $\mathfrak{F}'$ состоит из всех тех алгебр класса $\mathfrak{F}$, в которых тождества $x\mu_i = x\nu_i$ выполняются каждый раз, когда $\mu_i\varphi$ и $\nu_i\varphi$ содержатся в одном и том же классе конгруэнции $\mathfrak{C}'$. $\mathfrak{C}' \to \mathfrak{F}'$ и есть требуемое взаимно однозначное соответствие между конгруэнциями R и примитивными подклассами $\mathfrak{F}$, а примитивный класс всех правых унитарных полумодулей над фактор-полукольцом по $\mathfrak{C}'$ эквивалентен классу $\mathfrak{F}'$.

Обозначим через $\mathfrak{M}$ примитивный класс абелевых Ω-групп, а R — соответствующее ему кольцо. Такими же рассуждениями получается

Лемма 2.2. *Класс $\mathfrak{M}$ имеет нетривиальный примитивный подкласс тогда и только тогда, если кольцо R обладает нетривиальным идеалом. Можно установить взаимно однозначное соответствие между подклассами $\mathfrak{M}'$ класса $\mathfrak{M}$ и идеалами N кольца R так, что примитивный класс всех правых унитарных модулей над фактор-кольцом по N эквивалентен классу $\mathfrak{M}'$.*

С помощью лемм 2.1 и 2.2 можно определить полукольца, соответствующие примитивным подклассам класса $\mathfrak{F}$ всех коммутативных полугрупп с единицей и кольца, соответствующие примитивным подклассам класса $\mathfrak{A}$ всех абелевых групп.

Класс $\mathfrak{F}$ есть не что иное, как примитивный класс $\mathfrak{N}^0$ всех правых полумодулей над полукольцом I^*. Результаты следующего параграфа показывают (но и непосредственно легко видеть), что $\mathfrak{N}^0$ эквивалентен примитивному классу $\mathfrak{N}_1^{I^*}$. Конгруэнции полукольца I^* определяются парами $[m, k]$ (см. § 1). Так как $\mathfrak{N}^0 \sim \mathfrak{N}_1^{I^*}$, то по лемме 2.1 для произвольного примитивного подкласса $\mathfrak{N}'$ класса $\mathfrak{N}^0$ существуют такие числа m и k ($m=0, 1, 2, \ldots$; $k=1, 2, \ldots$), что $\mathfrak{N}'$ является классом всех коммутативных полугрупп с единицей, удовлетворяющих тождеству $x^m = x^{m+k}$.

Класс $\mathfrak{A}$ является примитивным классом $\mathfrak{M}^0$ всех правых модулей над кольцом I. Легко видеть, что $\mathfrak{M}^0 \sim \mathfrak{M}_1^I$. Так как все фактор-кольца кольца I имеют вид $I/\{k\}$, всякий примитивный подкласс класса $\mathfrak{A}$ является классом всех абелевых групп экспонента k, где k — подходящее натуральное число.

Мы видим, что в обоих случаях определены и полукольца (соотв. кольца), соответствующие примитивным подклассам.

§ 3

Теперь посмотрим примитивные классы $\mathfrak{N}^R$ и $\mathfrak{M}^R$, где R — произвольное. Покажем, что имеет место следующая

Теорема 3.1. *Примитивный класс $\mathfrak{N}^R$ эквивалентен примитивному классу $\mathfrak{N}_1^{(I^*+R)}$ ($\mathfrak{N}^R \sim \mathfrak{N}_1^{(I^*+R)}$).*

Доказательство. Пусть μ — m-местная операция ($m>0$) в классе $\mathfrak{N}^R$. μ разлагается в одноместные операции: $x_1 \ldots x_m \mu = x_1 \mu_1 + \ldots + x_m \mu_m$. Методом индукции можно показать (см. [1]), что произвольная одноместная операция μ_k в классе $\mathfrak{N}^{\mathsf{R}}$ имеет вид: $x_k\mu_k = x_k(r_k + n_k)$, где $r_k \in \mathsf{R}$, $n_k \in I^*$. Так, $x_1 \ldots x_m \mu = x_1(r_1 + n_1) + \ldots + x_m(r_m + n_m)$. Обший вид операций в классе $\mathfrak{N}_1^{(I^*+\mathsf{R})}$ следующий: $x_1 \ldots x_m \mu' = x_1\langle n_1, r_1\rangle + \ldots + x_m\langle n_m, r_m\rangle$ ($n_i \in I^*$, $r_i \in \mathsf{R}$; $i = 1, 2, \ldots, m$). Возьмем следующее отображение φ множества $O(\mathfrak{N}^{\mathsf{R}})$ на $O(\mathfrak{N}_1^{(I^*+\mathsf{R})})$: $\mu\varphi = \mu'$ и $0\varphi = \mathbf{0}$. Покажем, что φ — взаимно однозначно. Если $x_1 \ldots x_m \mu = x_1 \ldots x_m \nu$ то $x_1(r_1 + n_1) + \ldots + x_m(r_m + n_m) = x_1(r_1' + n_1') + \ldots + x_m(r_m' + n_m')$ и, подставляя $x_i = x$, $x_j = 0$ ($i \neq j$; $i, j = 1, 2, \ldots, m$), мы получим $x(r_i + n_i) = x(r_i' + n_i')$. Покажем, что $\langle n_i, r_i\rangle = \langle n_i', r_i'\rangle$, т. е. $n_i = n_i'$ и $r_i = r_i'$. Действительно, само полукольцо R является правосторонней областью операторов полугруппы $(I^* + \mathsf{R})^+$ полукольца $(I^* + \mathsf{R})$:

$$\langle n, r\rangle r_1 = \langle n, r\rangle\langle 0, r_1\rangle \quad (\langle n, r\rangle \in (I^* + \mathsf{R});\ r_1 \in \mathsf{R}).$$

В этом случае $\langle n, r\rangle(r_1 + n_1) = \langle n, r\rangle\langle n_1, r_1\rangle$. Тождество $x(r_i + n_i) = x(r_i' + n_i')$ в случае $x = \langle 1, 0\rangle$ дает искомый результат. Этим однозначность φ показана.

С другой стороны, если $\mu\varphi = \nu\varphi$, то, учитывая, что φ переводит m-местную операцию в m-местную же, получим:

$$x_1\langle n_1, r_1\rangle + \ldots + x_m\langle n_m, r_m\rangle = x_1\langle n_1', r_1'\rangle + \ldots + x_m\langle n_m', r_m'\rangle,$$

откуда, подставляя $x_i = x$, $x_j = 0$ ($i \neq j$; $i, j = 1, 2, \ldots, m$): $\langle n_i, r_i\rangle = \langle n_i', r_i'\rangle$, значит $n_i = n_i'$, $r_i = r_i'$.

При отображении φ тождества класса $\mathfrak{N}^{\mathsf{R}}$ и только они переходят в тождества класса $\mathfrak{N}_1^{(I^*+\mathsf{R})}$. Это показывается таким же путем, как это было сделано при доказательстве эквивалентности абелевых Ω-групп и правых унитарных модулей (см. [1] теорему 1). Этим утверждение доказано.

Для модулей над кольцом аналогично получается

Теорема 3.2. Примитивный класс $\mathfrak{M}^{\mathsf{R}}$ эквивалентен примитивному классу $\mathfrak{M}_1^{(I+\mathsf{R})}$ ($\mathfrak{M}^{\mathsf{R}} \sim \mathfrak{M}_1^{(I+\mathsf{R})}$).

Следствие 3.1. Пусть R — произвольное полукольцо с единицей. Существует фактор-полукольцо полукольца $(I^* + \mathsf{R})$, изоморфное полукольцу R.

Доказательство. Примитивный класс $\mathfrak{N}_1^{\mathsf{R}}$ является примитивным подклассом класса $\mathfrak{N}^{\mathsf{R}}$. Однако $\mathfrak{N}^{\mathsf{R}} \sim \mathfrak{N}_1^{(I^*+\mathsf{R})}$, так что по лемме 2.1 существует фактор-полукольцо полукольца $(I^* + \mathsf{R})$, примитивный класс полумодулей над которым эквивалентен примитивному классу $\mathfrak{N}_1^{\mathsf{R}}$. По теореме 1 из [2] это полукольцо с единицей является единственным с точностью до изоморфизма, что завершает доказательство.

По теореме 3.2 аналогично получим

Следствие 3.2. Пусть R — произвольное кольцо с единицей. Тогда существует фактор-кольцо кольца $(I + \mathsf{R})$, изоморфное кольцу R. (Это фактор-кольцо является фактор-кольцом по идеалу, порождаемому элементом $\langle 1, -\varepsilon\rangle$, где ε — единица кольца R).

Заметим, что А. Кертес в работе [7] показал возможность рассмотрения любого модуля над произвольным кольцом в качестве унитарного модуля над кольцом с единицей. Теорема 3.2 является естественным обобщением этого результата для примитивных классов. Далее, результаты следующего параграфа тоже являются обобщениями некоторых результатов Кертеса.

§ 4

Легко убедиться, что, если полукольцо R служит правой областью операторов для полумодуля F, то его антиизоморфный образ $\bar{\mathsf{R}}$ можно рассматривать левой областью операторов F так, чтобы при этом результаты применений образов и прообразов совпадали. Соответствующее утверждение имеет место также для модулей над кольцом. Эти замечания дают нам возможность ограничиться рассмотрением случаев ${}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}$, соотв. ${}^{\mathsf{R}}\mathfrak{M}^{\mathsf{L}}$.

Имеет место следующая

Теорема 4.1. *Примитивный класс* ${}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}$ *эквивалентен примитивному классу* $\mathfrak{N}_1^{(I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L})}$ $({}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}\sim\mathfrak{N}^{(I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L})})$.

Доказательство. Если μ — m-местная операция $(m>0)$ в классе ${}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}$, то она разлагается в одноместные операции: $x_1\ldots x_m\mu=x_1\mu_1+\ldots+x_m\mu_m$. Произвольная одноместная операция μ_i в классе ${}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}$ имеет вид:

$$x_i\mu_i=(r_{i,1}+n_{i,1})x_i(s_{i,1}+n'_{i,1})+\ldots+(r_{i,k_i}+n_{i,k_i})x_i(s_{i,k_i}+n'_{i,k_i}).$$

В классе $\mathfrak{N}_1^{(I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L})}$ общий вид операций следующий:

$$\begin{aligned}x_1\ldots x_m\mu'=&\,x_1\left(\langle n_{1,1},\bar r_{1,1}\rangle\otimes\langle n'_{1,1},s_{1,1}\rangle+\ldots+\langle n_{1,k_1},\bar r_{1,k_1}\rangle\otimes\langle n'_{1,k_1},s_{1,k_1}\rangle\right)+\ldots\\ &\vdots\\ &+x_m\left(\langle n_{m,1},\bar r_{m,1}\rangle\otimes\langle n'_{m,1},s_{m,1}\rangle+\ldots+\langle n_{m,k_m},\bar r_{m,k_m}\rangle\otimes\langle n'_{m,k_m},s_{m,k_m}\rangle\right).\end{aligned}$$

Возьмем следующее отображение φ множества операторов $O({}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}})$ на $O(\mathfrak{N}_1^{(I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L})})$: $\mu\varphi=\mu'$ и $0\varphi=\mathbf{0}$. Покажем, что φ — взаимно однозначно. Если $x_1\ldots x_m\mu=x_1\ldots x_m\nu$, где

$$\begin{aligned}x_1\ldots x_m\nu=&\,(r^*_{1,1}+n^*_{1,1})x_1(s^*_{1,1}+n^{*\prime}_{1,1})+\ldots+(r^*_{1,k_1}+n^*_{1,k_1})x_1(s^*_{1,k_1}+n^{*\prime}_{1,k_1})+\ldots\\ &\vdots\\ &+(r^*_{m,1}+n^*_{m,1})x_m(s^*_{m,1}+n^{*\prime}_{m,1})+\ldots+(r^*_{m,k_m}+n^*_{m,k_m})x_m(s^*_{m,k_m}+n^{*\prime}_{m,k_m}),\end{aligned}$$

то подставляя $x_i=x$, $x_j=0$ $(i\neq j;\ i,j=1,2,\ldots m)$, получим

$$\begin{aligned}(1)\qquad &(r_{i,1}+n_{i,1})x(s_{i,1}+n'_{i,1})+\ldots+(r_{i,k_i}+n_{i,k_i})x(s_{i,k_i}+n'_{i,k_i})=\\ &=(r^*_{i,1}+n^*_{i,1})x(s^*_{i,1}+n^{*\prime}_{i,1})+\ldots+(r^*_{i,k_i}+n^*_{i,k_i})x(s^*_{i,k_i}+n'_{i,k_i}).\end{aligned}$$

Покажем, что

$$(2)\qquad \langle n_{i,1}, \bar{r}_{i,1}\rangle\otimes\langle n'_{i,1}, s_{i,1}\rangle+\ldots+\langle n_{i,k_i}, \bar{r}_{i,k_i}\rangle\otimes\langle n'_{i,k_i}, s_{i,k_i}\rangle=$$
$$=\langle n^*_{i,1}, \bar{r}^*_{i,1}\rangle\otimes\langle n^{*\prime}_{i,1}, s^*_{i,1}\rangle+\ldots+\langle n^*_{i,k_i}, \bar{r}^*_{i,k_i}\rangle\otimes\langle n^{*\prime}_{i,k_i}, s^*_{i,k_i}\rangle.$$

Действительно, $(I^*+\mathsf{R})$ и $(I^*+\mathsf{L})$ является лево- и правосторонней унитарной областью полумодуля $((I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L}))^+$:

$$\langle n, r\rangle\Big(\sum_j\langle n_j, \bar{r}_j\rangle\otimes\langle n'_j, s_j\rangle\Big)\langle n', s\rangle=\sum_j\langle n_j, \bar{r}_j\rangle\langle n, \bar{r}\rangle\otimes\langle n_j, s_j\rangle\langle n', s\rangle$$

Так как тождество (1) выполняется и в полумодуле $((I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L}))^+$, в случае $x=\langle 1, 0\rangle\otimes\langle 1, 0\rangle$ получим равенсво (2).

Обратно, если

$$(3)\qquad x(\langle n_{i,1}, \bar{r}_{i,1}\rangle\otimes\langle n'_{i,1}, s_{i,1}\rangle+\ldots+\langle n_{i,k_i}, \bar{r}_{i,k_i}\rangle\otimes\langle n'_{i,k_i}, s_{i,k_i}\rangle)=$$
$$=x(\langle n^*_{i,1}, \bar{r}^*_{i,1}\rangle\otimes\langle n^{*\prime}_{i,1}, s^*_{i,1}\rangle+\ldots+\langle n^*_{i,k_i}, \bar{r}^*_{i,k_i}\rangle\otimes\langle n^{*\prime}_{i,k_i}, s^*_{i,k_i}\rangle),$$

тогда

$$(4)\qquad (r_{i,1}+n_{i,1})x(s_{i,1}+n'_{i,1})+\ldots+(r_{i,k_i}+n_{i,k_i})x(s_{i,k_i}+n'_{i,k_i})=$$
$$=(r^*_{i,1}+n^*_{i,1})x(s^*_{i,1}+n^{*\prime}_{i,1})+\ldots+(r^*_{i,k_i}+n^*_{i,k_i})x(s^*_{i,k_i}+n^{*\prime}_{i,k_i}).$$

Действительно, если полугруппы всех полумодулей класса ${}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}$ снабдить естественным образом областью правых унитарных операторов $(I^*+\bar{\mathsf{R}})\otimes\otimes(I^*+\mathsf{L})$, то из тождества (3) получим (4).

При отображении φ тождества класса ${}^{\mathsf{R}}\mathfrak{N}^{\mathsf{L}}$ и только они переходят в тождества класса $\mathfrak{N}_1^{(I^*+\bar{\mathsf{R}})\otimes(I^*+\mathsf{L})}$. Это показывается опять таким же образом, как при доказательстве в теореме 1 работы [1].

В случае бимодулей аналогично получится

Теорема 4.2. *Примитивный класс ${}^{\mathsf{R}}\mathfrak{M}^{\mathsf{L}}$ эквивалентен примитивному классу $\mathfrak{M}_1^{(I+\bar{\mathsf{R}})\otimes(I+\mathsf{L})}$ (${}^{\mathsf{R}}\mathfrak{M}^{\mathsf{L}}\sim\mathfrak{M}_1^{(I+\bar{\mathsf{R}})\otimes(I+\mathsf{L})}$).*

§ 5

Обозначим через $\mathfrak{N}$ примитивный класс алгебр, удовлетворяющий условиям I и II.

Теорема 5.1. *Примитивный класс $\mathfrak{N}$ является эквационально полным тогда и только тогда, если он эквивалентен примитивному классу всех унитарных полумодулей над полукольцом R с единицей, обладающим лишь тривиальными конгруэнциями.*

Доказательство. По лемме 2.1 примитивные подклассы класса $\mathfrak{N}$ можно поставить в взаимно однозначное соответствие конгруэнциям полукольца R. При этом $\mathfrak{N}$ соответствует конгруэнции $\mathfrak{C}_0$, а примитивный подкласс 0, содержащий лишь одноэлементную алгебру, — конгруенции $\mathfrak{C}_1$.

Добавим к множеству $\Lambda(\mathfrak{N})$ произвольное новое тождество. Полученное множество тождеств определит некоторый примитивный подкласс класса $\mathfrak{N}$. Конгруэнция полукольца R, соответствующая этому подклассу, отлична от $\mathfrak{C}_0$ и по условию совпадает с $\mathfrak{C}_1$. Отсюда видно, что единственным примитивным подклассом класса $\mathfrak{N}$ является 0. Необходимость же условия теоремы очевидно.

Аналогично доказывается

Теорема 5.2. *Примитивный класс $\mathfrak{A}$ абелевых Ω-групп является эквационально полным тогда и только тогда, если он еквивалентен примитивному классу всех правых унитарных модулей над некоторым простым кольцом R с единицей.*

Мы видели, что $\mathfrak{M}^{\mathsf{R}} \sim \mathfrak{M}_1^{(I+\mathsf{R})}$. Но кольцо $(I+\mathsf{R})$ никогда не является простым, так что имеет место

Следствие 5.1. *Примитивный класс всех правых модулей над кольцом R не может быть эквационально полным.*

Наконец, из простоты полей вытекает

Следствие 5.2. *Всякий примитивный класс векторных пространств является эквационально полным.*

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Bibliographie

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In der Physik, Biologie usw. kommt das folgende Problem oft hervor. Nach gewissen Experimenten ergibt sich eine Kurve, worüber man annehmen kann, daß sie die Superposition von Verteilungsfunktionen bzw. Dichtefunktionen von gewissem Typ ist; aus der Kurve sollen diese Komponenten bestimmt werden.

Das Buch gibt eine systematische, mathematische Behandlung solcher Probleme und umfaßt neben den vorherigen Resultaten auch neue, vom Verfasser selbst erzielte Ergebnisse.

In § I wird das Grundproblem in der folgenden Form formuliert. Es sei $F(x; \alpha, \beta)$ eine nichtentartete und von den Parametern α, β abhängige Verteilungsfunktion. Es sei weiterhin

$$\mathfrak{G}(x)=\sum_{k=1}^{N} p_k F(x; \alpha_k, \beta_k),$$

wo die Parameter $N, p_k, \alpha_k, \beta_k$ $(k=1, ..., N)$ unbekannt sind. Auf Grund der Kenntnis von $G(x)$ sollen diese Unbekannten bestimmt werden. Es wird an einigen Beispielen gezeigt, wie dieses Grundproblem in verschiedenen Wissenschaften aufgeworfen wird. Z. B., in der Spektroskopie soll die Intensitätskurve $g(x)$ in normalen Komponenten zerlegt werden, d. h. in der Formel

$$g(x)=\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{N}\frac{p_k}{\beta_k}\exp\left[-\frac{(x-\alpha_k)^2}{2\beta_k^2}\right]$$

die Parameter $N, p_k, \alpha_k, \beta_k (k-1, ..., N)$ bestimmt werden. In § II werden allgemeine Lösungsmethoden für das Grundproblem behandelt. In den weiteren Paragraphen werden die allgemeinen Lösungsmethoden in speziellen Fällen, z. B. im Fall von normalen Komponenten, angewendet, die Wirksamkeit der allgemeinen Lösungsmethoden diskutiert und weitere Lösungsmethoden behandelt.

Das Buch wurde in erster Reihe für diejenigen Fachleute geschrieben, die die Lösung von derartigen Problemen in ihrer Praxis benötigen. Darum sind die gebrauchten, weniger einfachen mathematischen Instrumente nicht im Text ausgearbeitet, sondern werden sie am Ende des Buches in 10 Anhängen zusammengefaßt.

Mit diesem Buch bekommen Fachleute von verschiedenen Wissenschaften ein gewiß nützliches Hilfsmittel.

K. Tandori (Szeged)

Lajos Takács, Introduction to the theory of queues (University Texts in the Mathematical Sciences), X+268 pages, New York, Oxford University Press, 1962.

The theory of queues is the fashionable term for the mathematical study of service systems in which either the demand for service or the services given, or both, have a stochastic or probabilistic nature. In telephony, where the subject began at about the turn of the century, it is clear that the times at which calls are made and the lengths of such calls (the holding times of the circuits transmitting the calls) are of such nature, and a great deal of the development of the subject has appeared in this context, as attested by the following roll call: Engset, Erlang, Fry, Kosten, Molina, Palm, Pollaczek, Vaulot (and indeed the present author). Lately, similar problems have appeared elsewhere: in air traffic control at airports, in automobile road traffic, in docking of ships, in hospital clinics, in the filling and emptying of water reservoirs, and in a variety of other situations.

The mathematical description of a class of such systems may be made in three parts. First, the traffic input is given by a sequence $\{t_1, t_2, \ldots\}$ of time epochs where service demands arise. Next, the service times for each demand and each server are given by distribution functions (giving the probability that the service time in question is at most t, say, for every t). Finally, the service arrangements, the number of servers, the assignment of servers to customers, the provision for waiting or not, the order of serving waiting customers, and so on, appear in almost infinite variety. Curiously he "busy signal" systems of telephony, in which demands arising when all servers are busy are dismissed because there is no provision for waiting and hence no queue, are still regarded as belonging to the theory of queues! The simplest traffic input has single demands at demand epochs, which are such that the differences $t_i - t_{i-1}$ have a common exponential distribution; this is called Poisson input, because the number of demands in a finite time interval has the Poisson distribution. When the common distribution is arbitrary, the input is called "recurrent", which, of course, includes the Poisson. Usually all distributions of service time are taken to be alike, and to be independent of each other, as they are in the book under review.

This introduction, by a distinguished Hungarian mathematician, is addressed to mathematical analysts. It is written in clear, simple style with many repetitions of the mathemaical specification, and with main emphasis on the transient (that is, complete) behavior. Of course, the limits (the steady state results) are also noticed. The development for the most part is the author's own and throughout meets his high standard of mathematical elegance; indeed the tyro may despair over the prospect of similar attainment. Over half of the book is devoted to single servers (the very rare case in telephony) but also there are chapters on the many server case, with and without waiting lines (the author calls the latter telephone traffic processes, although, of course, there are many delay systems in telephony), the infinite server case, the machine-repair problem, and (electronic) particle counters. Finally an appendix collects statements of a number of auxiliary theorems. Each chapter is followed by a bibliography, usually very extensive.

While this reviewer would have preferred a somewhat less general treatment, at least occasionally, in favor of greater intuitive simplicity, there is no doubt that the reader in search of mathematical rigor will find his answer in this book. As a combinatorialist, the reviewer must make one small cavil; at the top of page 30 the author says that to obtain a certain formula "we have to use Lagrange's expansion of $[g(w)]^k$." Actualy the definition of $g(w)$ relates $g^2(w)$ and $g(w)$ and may be used to give a recurrence relation for the coefficients of powers of w in the expansion of powers of $g(w)$, which is an easy alternate to Lagrange's expansion.

John Riordan (Murray Hill, N. J., USA)

O. Ore, Theory of graphs (American Mathematical Society, Colloquium Publications, Vol. XXXVIII), IX+270 pages, Providence, R. I., American Mathematical Society, 1962.

The present monograph deals primarily with such branches of the graph theory which had not yet been explained in any book. The author turns his interest with preference for graph-theoretical aspects of notions originating from set theory and algebra, further for questions concerning extreme subgraphs and numerical extreme values. The book has a rich content. However, to the reviewer's opinion, the importance of the selected material is not quite homogeneous and the presentation is sometimes not the most felicitous.

The book consists of 15 chapters, there are more than 200 bibliographical references grouped according to the chapters. It is promised a second volume, containing chiefly practical applications and questions related to topology.

In Chapters 1—2 (Fundamental concepts; Connectedness) the most important initial concepts are introduced. Chapter 3 (Path problems) is devoted to studying Euler paths, Hamilton lines and labyrinth questions. Chapter 4 (Trees) deals not only with trees but also with questions involving general graphs, namely with circuit rank, structural investigation of certain directed graphs, special one-to-one correspondences between vertices and edges. Chapter 5 (Leaves and lobes) discusses some very natural homomorphism concepts for non-directed graphs. In Chapter 6 (The axiom of choice) some maximal principles of set theory are investigated and applied for proving the existence of certain maximal subgraphs of infinite graphs. Chapter 7 (Matching theorems) presents a detailed discussion of maximal subgraphs with degree 1 in bipartite graphs. Chapter 8 (Directed graphs) deals with homomorphisms, embeddability in order relations and basis graphs. Chapter 9 (Acyclic graphs) gives an analogon of the Jordan—Hölder theorem (arising in abstract algebra) and discusses the possibility of certain bipartition of the vertices in a directed graph. Chapters 10—11 (Partial order; Binary relations and Galois correspondences) study some notions, being well-known in other branches of mathematics, in terms of graph theory. The main assertion of Chapter 12 (Connecting paths) is MENGER's theorem on minimal separating vertex sets and maximal families of disjoint connecting arcs. Chapter 13 (Dominating sets, covering sets and independent sets) investigates subgraphs with certain extremal properties. Chapter 14 (Chromatic graphs) is devoted to studying the chromatic number of graphs. In the final chapter (Groups and graphs) it is proved that any finite group appears as the automorphism group of a suitable graph, there is studied how edge isomorphism and circuit isomorphism are related to the isomorphism in customary sense.

A. Ádám (Szeged)

J. Favard, Cours d'Analyse de l'Ecole Polytechnique, tome I, VIII+675 pages; tome II, 578 pages; tome III, fasc. I, 294 pages; fasc. II, 542 pages (Cahiers Scientifiques publiés sous la direction de Gaston Julia), Paris, Gauthier—Villars, 1960—62.

Les trois volumes traitent d'une matière étendue aussi bien en largeur qu'en profondeur; pour s'en rendre compte, il est indiqué, d'abord, de jeter un coup d'oeil sur le contenu.

Tome I. (Introduction. Opération.) — Ensembles, éléments d'algèbre et de topologie, introduction à la théorie des espaces fonctionnels, séries et produits infinis, fonctions à variation bornée, fonctions convexes. Dérivées, différentielles, fonctions implicites, déterminants fonctionnels, éléments de géométrie différentielle, points singuliers. Mesure de Jordan, intégrale de Cauchy—Riemann, quadrature mécanique, intégrales curvilignes, intégrale de Stieltjes, fonctionnelles, analyse vectorielle, intégrale de Lebesgue et ses extensions, dérivabilité et recherche des primitives, théorème de Riesz—Fischer, représentation des fonctionnelles linéaires, convergence faible et forte, espace produit, théorème de Fubini. — La notion de fonction y est introduite comme application d'un espace métrique sur un autre espace métrique. Les espaces vectoriels normés jouent, naturellement, un rôle important dans cette manière d'exposer la matière introductoire. Or ce point de vue général est conservé dans l'ouvrage entier, ce qui permet au lecteur de se familiariser avec les notions modernes, souvent nécessaires même pour ceux qui s'intéressent aux mathématiques du point de vue des applications. Le point de vue général n'est quitté que lorsque la nature du sujet traité l'exige nécessairement (p. ex. au cas des fonctions monotones ou des théorèmes spéciaux de dérivabilité, etc.)

Tome II. (Représentations. Fonctions analytiques.) — Fonction Γ, principes de convergence dans les espaces de Banach, théorème d'approximation de Weierstrass, représentations dans L^2. Séries trigonométriques, convergence, sommation, intégrale de Fourier, polynômes orthogonaux, interpolation, éléments de la théorie des distributions. Fonctions à une variable complexe, fonctions monogènes, transformations, théorème de Cauchy, théorèmes d'unicité, théorème de Liouville, points singuliers isolés, fonctions méromorphes, zéros et pôles. Transformations conformes, lemme de Schwarz, théorème de Bloch, espaces de fonctions holomorphes et méromorphes, familles normales, représentations conformes par fonctions univalentes, théorème de Picard. Séries entières, séries de Laurent, théorème de Mittag—Leffler, produits infinis, fonctions entières, transformations de Laplace, représentations diverses. Fonctions elliptiques. Prolongement analytique, surfaces de Riemann, théorème de monodromie, théorème de Weyl sur le caractère topologique des surfaces de Riemann. Fonctions analytiques à plusieures variables. Fonctions algébriques, théorème de Nöther, intégrales abéliennes, théorème de Riemann—Roch et d'Abel, fonctions thétas. Généralisation de la notion de fonction holomorphe, fonctions vectorielles analytiques d'une variable scalaire et vectorielle.

Tome III. (Théorie des équations.) — Equations différentielles dans le champ réel, théorèmes d'existence et d'unicité, méthode de Cauchy, théorème de Peano, stabilité et instabilité, systèmes d'équations différentielles, théorème de Frobenius, équation de Sturm—Liouville, systèmes stationnaires dans l'espace euclidien, systèmes définis sur les variétés. Equations différentielles dans le champ analytique, existence et unicité, singularités, théorème de Painlevé, équations de Fuchs, équations de fonctions hypergéométriques, fonctions de Legendre et de Bessel, développements asymptotiques, méthode de Laplace. — Equations aux dérivées partielles, problème de Cauchy, systèmes d'équations, théorème d'existence de Cauchy—Kovalevskaya, problèmes d'unicité et de stabilité, équations du second ordre, équations hyperboliques, équations des télégraphistes, des ondes, de la corde et des plaques vibrantes, opérateurs de Heaviside, méthodes numériques d'intégration approchée, équations elliptiques, fonctions harmoniques et sous-harmoniques, problème de Dirichlet, potentiel de volume, méthode numérique pour la solution approchée du problème de Dirichlet, équations paraboliques, fonctions caloriques et sous-caloriques, problèmes aux limites. — Equations intégrales, équations de Fredholm et de Volterra, étude des types de noyaux, applications aux équations différentielles. — Calcul des variations, fonctionnelles semi-continues, équation d'Euler, condition de Weierstrass, Legendre et de Jacobi, existence de l'extremum, méthode directe et solution du problème de Dirichlet, problème de Plateau.

Cette matière vaste est encore complétée et approfondie par de nombreux exercises et compléments à la fin de chaque chapitre. Les compléments conduisent souvent jusqu'à des problèmes profonds. Pour en donner une idée relevons, à titre d'exemple, le théorème de compactification d'Alexandroff, l'homéomorphie d'un complexe $K^{(n)}$ avec un polyèdre du $R^{(2n+1)}$, le théorème de point fixe de Brouwer, dans le chapitre introductoire de topologie; ou bien, dans la partie traitant des fonctions analytiques, le théorème de Phragmén—Lindelöf, quelques théorèmes de Stoïlow concernant les transformations internes, recherche de la périodicité des intégrales hyperelliptiques, éléments de la fonction $\zeta(s)$ de Riemann, etc. Ainsi, il est visible que ces trois volumes contiennent, outre la théorie classique, une grande quantité de méthodes et de résultats modernes.

On peut se demander si, par l'agglomération de tant de faits, on ne risque pas de composer une sorte d'encyclopédie qui, évidemment, ne peut pas être assez profonde pour le spécialiste, mais qui est trop large pour un technicien, même créateur? Oui, ce problème subsiste, mais il n'est pas le problème de cet ouvrage, mais celui de notre temps comme conséquence du fait que les idées mathématiques utilisées par le technicien créateur deviennent de jour en jour plus abstraites et plus compliquées. Les temps sont passés, où on a pu se contenter d'une suite de recettes, comme la „Cuisine de Tante Marie"; aujourd'hui, il faut initier les futures cadres supérieurs de la technique à la pensée mathématique moderne, puisque les jeunes élèves de l'Ecole Polytechnique d'aujourd'hui seront les techniciens dirigeants à la veille du XXI. siècle, quand le constructeur technique sera, probablement, comblé de problèmes mathématiques, s'il veut tenir au courant du développement de sa spécialité.

L'auteur a rendu un grand service à l'enseignement supérieur des mathématiques — d'ailleurs toujours en crise — en composant les trois volumes de son Cours d'Analyse, car il n'a pas perdu de vue ni l'exigence moderne de généralité ni celle du technicien, car il a voulu que la matière présentée soit applicable directement aux problèmes de physique et de technique. C'est un ouvrage qu'on peut consulter avec fruit; il est profitable tant pour le technicien avancé que pour le jeune mathématicien cherchant une initiation à l'Analyse moderne. La présentation des idées est toujours aussi simple que le sujet traité le permet, et le texte peut être bien suivi par un lecteur intéressé. Nous croyons que ce Cours d'Analyse restera pour longtemps un ouvrage recherché par tous ceux qui désirent approfondir leurs connaissances antérieures et se faire une image des méthodes de l'Analyse moderne.

G. Alexits (Budapest)

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