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On homomorphisms of partially ordered semigroups

By R. McFADDEN in Belfast (N. Ireland)

In attempting to describe the order preserving homomorphisms of a partially ordered semigroup G onto a partially ordered semigroup G', it has proved necessary to impose conditions on G, on G', on the congruence ϱ determined by the homomorphism, or on a combination of these [7], [3], [1]. The approach used here is to assume that G/ϱ is residuated in such a way that each element of G/ϱ is both a left and a right residual of itself, and that the ϱ-class of each element of G contains a maximum element. Without assuming that G is residuated, as in [4], or even generalized residuated, [3], it is shown that if t is maximum in its ϱ-class, the residuals $t\,.\!\cdot a$ and $t\cdot\!.\,a$ exist for any $a \in G$, and that ϱ is determined by a subset of all such residuals.

When G/ϱ is a group the form of ϱ has been determined by Mme. DUBREIL-JACOTIN [1]; since a group is residuated, her result may be deduced from those described here. As an extension of this, the condition that G/ϱ be a group is replaced by the condition that G/ϱ be an integrally closed semigroup, and the structure of ϱ is then determined.

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Let G be a partially ordered set. That is, a set in which is defined a relation $\leqq$, which is reflexive, antisymmetric and transitive. For $x, y \in G$, the *greatest lower bound* of x and y, if it exists, is denoted by $x \wedge y$, and the *least upper bound*, if it exists, is denoted by $x \vee y$. An equivalence relation ϱ on G is called an *m-equivalence* if ϱ satisfies the following conditions:

(i) *for any $x \in G$, the ϱ-class $x\varrho$ of x contains a maximum element t_x,*

(ii) *for $x, y \in G$, $x \leqq y$ implies $t_x \leqq t_y$.*

The following notation will be used:

$$T(\varrho) = \{t \in G \mid t \text{ is maximum in its } \varrho\text{-class}\}.$$

For an equivalence relation ϱ satisfying (i), it is easily seen that (ii) is equivalent to:

(ii)′ *$x, y \in G$, $x < y$, $x \not\equiv y(\varrho)$, $x'\varrho x$ imply that there exists in G an element y' such that $y'\ \varrho\ y$ and $x' < y'$.* (Condition (ii)′ is the property (S) discussed in (2, 5).)

When ϱ is an m-equivalence, the set $G/\varrho = \{x\varrho \mid x \in G\}$ may be partially ordered by:

$$x\varrho \leqq y\varrho \quad \text{in} \quad G/\varrho \quad \text{if and only if} \quad t_x \leqq t_y \quad \text{in} \quad G.$$

We use the same notation for the partial orders in G and G/ϱ; it is clear that G/ϱ

is the order homomorphic image of G, in the sense that $x \leqq y$ in G implies $x\varrho \leqq y\varrho$ in G/ϱ. Note that $x\varrho \leqq y\varrho$ in G/ϱ if and only if there exist $x' \in x\varrho$ and $y' \in y\varrho$ such that $x' \leqq y'$. Further, the ϱ-classes are convex, for if $x, y, z \in G$ with $x \leqq y \leqq z$ and $x\varrho z$, then $t_x \leqq t_y \leqq t_z = t_x$ implies $t_x = t_y$, or $x\varrho y$.

A *partially ordered groupoid* is a partially ordered set G on which is defined a binary operation, which will be written multiplicatively, such that for $a, b, x \in G$, $a \leqq b$ implies $ax \leqq bx$ and $xa \leqq xb$. If multiplication is associative, G is called a *partially ordered semigroup*. If for $a, b \in G$ the set of all $x \in G$ such that $ax \leqq b$ ($xa \leqq b$) is non-empty and contains a maximum element, this element is called the *right (left) residual of b by a*, and is written $b \,.\!\cdot\, a$ ($b \cdot\!.\, a$). If $b \,.\!\cdot\, a$ ($b \cdot\!.\, a$) exists for all $a, b \in G$, then G is called *right (left) residuated*, and if G is both right and left residuated, it is said to be *residuated.*

A *congruence relation* on a partially ordered groupoid G is an equivalence relation ϱ on G which satisfies:

(iii) *for* $x, y, z \in G$, $x \varrho y$ *implies* $xz \varrho yz$ *and* $zx \varrho zy$.

An *m-congruence on* G is an m-equivalence on G which satisfies (iii).

When ϱ is an m-congruence, G/ϱ is a groupoid, and a homomorphic image of G, if multiplication in G/ϱ is defined by $a\varrho \cdot b\varrho = (ab)\varrho$. Further, G/ϱ is a partially ordered groupoid with the partial order defined above, for if $x\varrho, y\varrho, z\varrho \in G/\varrho$ with $x\varrho \leqq y\varrho$, then $t_x \leqq t_y$ implies $t_z t_x \leqq t_z t_y$, whence by (i) and (iii), $t_{zx} \leqq t_{zy}$; similarly for multiplication on the right.

Lemma 1. *Let ϱ be an m-congruence on a partially ordered groupoid G. Then G/ϱ is right residuated if and only if $t \,.\!\cdot\, a$ exists for every $t \in T(\varrho)$ and for every $a \in G$. In this case $t \,.\!\cdot\, a \in T(\varrho)$ $(t \,.\!\cdot\, a)\varrho = t\varrho \,.\!\cdot\, a\varrho$, and $a' \varrho\, a$ implies $t \,.\!\cdot\, a = t \,.\!\cdot\, a'$.*

Proof. *Sufficiency:* Let $a\varrho, b\varrho \in G/\varrho$, and let $a \in a\varrho$. Since $a(t_b \,.\!\cdot\, a) \leqq t_b$, it follows that $a\varrho(t_b \,.\!\cdot\, a)\varrho \leqq b\varrho$; on the other hand, if $a\varrho\, x\varrho \leqq b\varrho$ then $at_x \leqq t_b$, $t_x \leqq t_b \,.\!\cdot\, a$, $x\varrho \leqq (t_b \,.\!\cdot\, a)\varrho$. Hence $b\varrho \,.\!\cdot\, a\varrho$ exists, equal to $(t_b \,.\!\cdot\, a)\varrho$.

Necessity: Let $a \in G$, $t \in T(\varrho)$. Consider $t\varrho \,.\!\cdot\, a\varrho$ in G/ϱ, and let u be the maximum element in the class $t\varrho \,.\!\cdot\, a\varrho$. Then $a\varrho(t\varrho \,.\!\cdot\, a\varrho) \leqq t\varrho$ implies $au \leqq t$; but if $ax \leqq t$ then $a\varrho\, x\varrho \leqq t\varrho$, $x\varrho \leqq t\varrho \,.\!\cdot\, a\varrho$, $x \leqq t_x \leqq u$. Hence $t \,.\!\cdot\, a$ exists, equal to u.

Since $t \,.\!\cdot\, a$ is the maximum element in $t\varrho \,.\!\cdot\, a\varrho$, it follows that if $a' \varrho\, a$ then $t \,.\!\cdot\, a' = t \,.\!\cdot\, a = u$.

Lemma 1 will be used as stated, but it may be noted that the following holds: the residual $b\varrho \,.\!\cdot\, a\varrho$ exists in G/ϱ if and only if $t_b \,.\!\cdot\, a$ exists for some (and hence all) $a \in a\varrho$.

It follows from Lemma 1 that if G/ϱ is right residuated, then for any $x \in G$, $t \,.\!\cdot\, x$ exists for any right residual $t [= t_b \,.\!\cdot\, a]$ of any element of $T(\varrho)$; for $t \in T(\varrho)$.

Residuals obey the following rules, quoted here without proof (see [2]); it is not necessary to assume that the groupoid G concerned is residuated, but only that the residuals concerned exist.

1. $b \leqq a \,.\!\cdot\, (a \cdot\!.\, b)$, *with equality if and only if* $b = a \,.\!\cdot\, x$ *for some* $x \in G$.
2. If G *is a semigroup,* $a \,.\!\cdot\, bc = (a \,.\!\cdot\, b) \,.\!\cdot\, c$ *and* $a \cdot\!.\, bc = (a \cdot\!.\, c) \cdot\!.\, b$.
3. $a \leqq b$ *implies* $a \,.\!\cdot\, c \leqq b \,.\!\cdot\, c$ *and* $c \,.\!\cdot\, b \leqq c \,.\!\cdot\, a$.

The existence of an identity element e in G, together with Rule 1, implies that $a = a \cdot\!.\, (a \,.\!\cdot\, a) = a \,.\!\cdot\, (a \cdot\!.\, a)$ for any $a \in G$, since $a = a \cdot\!.\, e = a \,.\!\cdot\, e$. However, even if G does not have an identity, it may still be true that $a=a^{\cdot}.(a.^{\cdot}a)=a.^{\cdot}(a^{\cdot}.a)$ for any $a\in G$; this is the case if and only if every element of G is both a left and a right residual of itself, in the sense that for every $a\in G$ there exists $x\in G$ such that $a=a^{\cdot}.x(a=a.^{\cdot}x)$. We shall call *self-residuated* a groupoid having this property (5).

Theorem 1. *Let G be a partially ordered semigroup, and let T be a non-empty subset of G satisfying, the following conditions:*

(α) *For any $t\in T$ and for any $a, x\in G$ there exist:*

$$t.^{\cdot}a,\quad t^{\cdot}.a,\quad \bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a),\quad \Big(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\Big).^{\cdot}x,\qquad \Big(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\Big)^{\cdot}.x.$$

(β) *For any $t\in T$ and for any $a\in G$, $t.^{\cdot}a\in T$.*

(γ) *Each $t\in T$, and each $\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)$, for $a\in G$, is both a left and a right residual of itself.*

Define the relation ϱ_T on G by:

$$a\,\varrho_T\,b \quad \text{if and only if} \quad t.^{\cdot}a=t.^{\cdot}b \quad \text{for every} \quad t\in T.$$

Then ϱ_T is an m-congruence on G, and G/ϱ_T is residuated and self-residuated.

Conversely, if ϱ is an m-congruence on G such that G/ϱ is residuated and self-residuated, then $T(\varrho)$ satisfies (α), (β) and (γ), and $\varrho=\varrho_{T(\varrho)}$.

Proof. Clearly ϱ_T is an equivalence relation. For $a, x\in G$ and $t\in T$, $t.^{\cdot}a\in T$ (by (β)), and this implies, by (α), that $(t.^{\cdot}a).^{\cdot}x$ exists; by (α) again, $t.^{\cdot}ax$ exists, and then $(t.^{\cdot}a).^{\cdot}x=t.^{\cdot}ax$, each being the maximum $z\in G$ such that $axz\leqq t$. (It is here that we use the fact that G is a semigroup). Hence if $b\in G$ and $a\,\varrho_T\,b$, then

$$t.^{\cdot}ax=(t.^{\cdot}a).^{\cdot}x=(t.^{\cdot}b).^{\cdot}x=t.^{\cdot}bx,$$

$$t.^{\cdot}xa=(t.^{\cdot}x).^{\cdot}a=t'.^{\cdot}a=t'.^{\cdot}b=(t.^{\cdot}x).^{\cdot}b=t.^{\cdot}xb,$$

where $t'=t.^{\cdot}x\in T$, by (β). Thus ϱ_T is a congruence relation. To see that ϱ_T is an m-congruence, we note that by Rule 1, $a\leqq t^{\cdot}.(t.^{\cdot}a)$; this implies that

$$a\leqq \bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\leqq t^{\cdot}.(t.^{\cdot}a) \quad \text{for any} \quad t\in T,$$

and since (by Rule 1 again), $t.^{\cdot}(t^{\cdot}.(t.^{\cdot}a))=t.^{\cdot}a$ it follows from Rule 2 that $t.^{\cdot}a=t.^{\cdot}\big(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\big)$. That is, $a\,\varrho_T\big(\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\big)$. Clearly $\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)$ is maximum in $a\varrho_T$. Using Rule 3 twice, we see that $a\leqq c$ in G implies that $t^{\cdot}.(t.^{\cdot}a)\leqq t^{\cdot}.(t.^{\cdot}c)$ for any $t\in T$, and hence that $\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)\leqq \bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}c)$, so that ϱ_T is indeed an m-congruence.

By (α), ϱ_T satisfies the conditions of Lemma 1, and so G/ϱ_T is residuated. For $a\varrho_T\in G/\varrho_T$, write $t_a=\bigwedge_{t\in T} t^{\cdot}.(t.^{\cdot}a)$; then

$$a\varrho_T{}^{\cdot}.(a\varrho_T.^{\cdot}a\varrho_T)=t_a\varrho_T{}^{\cdot}.(t_a\varrho_T.^{\cdot}t_a\varrho_T)=$$

$$=(t_a{}^{\cdot}.(t_a.^{\cdot}t_a))\varrho_T=t_a\varrho_T \quad (\text{using } (\gamma))=a\varrho_T,$$

and similarly $a\varrho_T=a\varrho_T.^{\cdot}(a\varrho_T{}^{\cdot}.a\varrho_T)$, so that G/ϱ_T is self-residuated.

Conversely, let G and ϱ be as stated; write $T=T(\varrho)$. Lemma 1 implies the existence of $t.\cdot a$ and $t\cdot.a$, and that $t.\cdot a\in T$; for the rest, it is enough to show that each $t\in T$ is both a left and a right residual of itself, and that $\bigwedge_{t\in T} t\cdot.(t.\cdot a)=t_a$.

Let $a\varrho\in G/\varrho$; since G/ϱ is self-residuated, we have from Lemma 1 that

$$a\varrho=t_a\varrho=t_a\varrho\cdot.(t_a\varrho.\cdot t_a\varrho)=t_a\varrho\cdot.(t_a.\cdot t_a)\varrho=(t_a\cdot.(t_a.\cdot t_a))\varrho.$$

Since $t_a\cdot.(t_a.\cdot t_a)$ is maximum in its ϱ-class (Lemma 1 again), we deduce $t_a=t_a\cdot.\cdot.(t_a.\cdot t_a)$; similarly $t_a=t_a.\cdot(t_a\cdot.t_a)$. By Rule 2, $t_a\leqq t\cdot.(t.\cdot t_a)$ for any $t\in T$; in particular, $t_a=t_a\cdot.(t_a.\cdot t_a)$. Hence $t_a=\bigwedge_{t\in T} t\cdot.(t.\cdot t_a)$, and since, by Lemma 1 again, $t.\cdot t_a=t.\cdot a$, it follows that $t_a=\bigwedge_{t\in T} t\cdot.(t.\cdot a)$.

By the first part of the Theorem, we now have that ϱ_T is an m-congruence on G such that G/ϱ_T is residuated and self-residuated, and it only remains to show that $\varrho=\varrho_T$. By Lemma 1, ϱ is finer than ϱ_T. Let $a\equiv b(\varrho_T)$; then with t_a maximum in $a\varrho$, $t_a.\cdot t_a=t_a.\cdot a=t_a.\cdot b=t_a.\cdot t_b$, (using Lemma 1), implies $t_b\,(t_a.\cdot t_a)\leqq t_a$, which in turn implies $t_b\leqq t_a\cdot.(t_a.\cdot t_a)=t_a$, using ($\gamma$) for the last equality. Similarly $t_a\leqq t_b$, whence $t_a=t_b$; that is, $a\,\varrho\,b$. We conclude that $\varrho=\varrho_T$, and the theorem is proved.

Corollary 1.1. *If ϱ and σ are m-congruences on a partially ordered semigroup G such that G/ϱ and G/σ are residuated and self-residuated, then ϱ is finer than σ if and only if $T(\sigma)\subseteqq T(\varrho)$.*

Proof. If ϱ is finer than σ, any element of G maximum in its σ-class must be maximum in its ϱ-class. Conversely, if $T(\sigma)\subseteqq T(\varrho)$, then $a\,\varrho\,b$ implies $t.\cdot a=t.\cdot b$ for any $t\in T(\varrho)$, and therefore for any $t\in T(\sigma)$; by Theorem 1, $a\,\sigma\,b$.

Definition. An element a of a partially ordered groupoid G is called *equiresidual* if whenever one of $a.\cdot x$, $a\cdot.x$ exists for $x\in G$, so does the other, and $a.\cdot x=a\cdot.x$. We shall denote their common value by $a{:}x$.

Corollary 1.2. *Let G, ϱ be as in Theorem 1. Then G/ϱ is commutative if and only if each $t\in T(\varrho)$ is equiresidual.*

Proof. Let $a\varrho, b\varrho\in G/\varrho$, and suppose that each $t\in T(\varrho)$ is equiresidual. Then $b\varrho.\cdot a\varrho=(t_b.\cdot a)\varrho=(t_b\cdot.a)\varrho=b\varrho\cdot.a\varrho$, by Lemma 1. Since G/ϱ is residuated and self-residuated, $a\varrho\, b\varrho.\cdot a\varrho\, b\varrho=a\varrho\, b\varrho\cdot.a\varrho\, b\varrho=(a\varrho\, b\varrho\cdot.b\varrho)\cdot.a\varrho= =(a\varrho\, b\varrho.\cdot b\varrho).\cdot a\varrho=a\varrho\, b\varrho.\cdot b\varrho\, a\varrho$, using Rule 2, and so $b\varrho\, a\varrho\,(a\varrho\, b\varrho.\cdot a\varrho\, b\varrho)\leqq \leqq a\varrho\, b\varrho$. Hence $b\varrho\, a\varrho\leqq a\varrho\, b\varrho\cdot.(a\varrho\, b\varrho.\cdot a\varrho\, b\varrho)=a\varrho\, b\varrho$. Similarly $a\varrho\, b\varrho\leqq b\varrho\, a\varrho$, $\leqq b\varrho\, a\varrho$, whence equality.

Conversely, if G/ϱ is commutative, $(t_b.\cdot a)\varrho=b\varrho.\cdot a\varrho=b\varrho\cdot.a\varrho=(t_b\cdot.a)\varrho$. Since each of $t_b.\cdot a$, $t_b\cdot.a$ is maximum in its ϱ-class, equality follows.

It follows from the proof of Corollary 1.2 that a residuated, self-residuated semigroup G is commutative if and only if every element of G is equiresidual.

Note 1. If each $t\in T$ is equiresidual, and if G is residuated, (α) and (γ) are enough to ensure that ϱ_T in Theorem 1 is an m-congruence. Condition (β) was used only to show that ϱ_T is regular on the left with respect to multiplication; but for $a, b, x\in G$ with $a\equiv b(\varrho_T)$, we now have

$$t.\cdot xa=t\cdot.xa=(t\cdot.a)\cdot.x=(t\cdot.b)\cdot.x=t\cdot.xb=t.\cdot xb.$$

For a discussion of the case where T consists of a single equiresidual element in a residuated semigroup with identity, and where G/ϱ_T is a group, see MAURY [6].

Note 2. Condition (γ) is not necessary if G has an identity element.

Thus in a commutative, residuated semigroup G with identity, any non-empty subset T of G defines an m-congruence ϱ_T as in Theorem 1, provided only that for any $a \in G$, $\bigwedge_{t \in T} t^{\cdot}.(t{:}a)$ exists. Then G/ϱ_T is a residuated semigroup with identity; the maximum element in the ϱ_T class of a is $\bigwedge_{t \in T} t{:}(t{:}a)$. In particular, if x is a fixed element of G, let $T = \{x\}$, and write $\varrho_x = \varrho_{\{x\}}$. Then

$$a\varrho_x b \quad \text{if and only if} \quad x{:}a = x{:}b,$$

and $T(\varrho_x) = \{\bigwedge_{x \in T} x : (x : a),\ a \in G\} = \{x : (x : a),\ a \in G\}$, so that ϱ_x is MOLINARO's congruence relation A_x (7) (see below).

Note 3. There is a difference between the two parts of Theorem 1. Given that ϱ is an m-congruence such that G/ϱ is residuated and self-residuated, it follows that $\varrho = \varrho_{T(\varrho)}$, and that for any $a \in G$, $\bigwedge_{t \in T(\varrho)} t^{\cdot}.(t.^{\cdot}a) \in T(\varrho)$. Yet given $T \subseteq G$ satisfying (α), (β) and (γ), to establish that ϱ_T is an m-congruence and that G/ϱ_T is residuated and self-residuated, it is not necessary to assume that $t_a = \bigwedge_{t \in T} t^{\cdot}.(t.^{\cdot}a)$ is in T for every $a \in G$, but only that t_a, $t_a{}^{\cdot}.x$ and $t_a.^{\cdot}x$ exist, for any $x \in G$. Then the set of elements maximum in their ϱ_T-classes is $T(\varrho_T) = \{\bigwedge_{t \in T} t^{\cdot}.(t.^{\cdot}a),\ \text{for } a \in G\}$, of which T is a subset, in general a proper subset. Even the fact that $t.^{\cdot}a \in T$ does not force the equality of T and $T(\varrho_T)$; in the semigroup $G = \{e, a, b, c, z\}$, with $e > a > c > z$, $e > b > c > z$ and $xy = x \wedge y$ for all $x, y \in G$, let $T = \{e, a, b\}$. Then G is a residuated, commutative semigroup with identity e; T satisfies (α), (β) and (γ), so Theorem 1 holds. Yet $T(\varrho_T) = \{e, a, b, c\}$, which properly contains T. Hence in general the representation of ϱ described in Theorem 1 is not unique.

Note 4. Although T is closed under residuation, in the sense that (β) holds, in general T is not closed under multiplication. In the example above, $a, b \in T$ but $ab = c \notin T$.

Note 5. Given ϱ satisfying the conditions of Theorem 1, it follows by symmetry that $T(\varrho) = T$ satisfies:

(α)′ *For any $t \in T$ and for any $a, x \in G$ there exist:*

$$t^{\cdot}.a,\quad t.^{\cdot}a,\quad \bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a),\quad \Big(\bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a)\Big)^{\cdot}.x \quad\text{and}\quad \Big(\bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a)\Big).^{\cdot}x.$$

(β)′ *For any $t \in T$ and for any $a \in G$, $t^{\cdot}.a \in T$.*

(γ)′ *Each $t \in T$, and each $\bigwedge_{t \in T} t.^{\cdot}(t^{\cdot}.a)$, for $a \in G$, is both a left and a right residual of itself.*

Although this argument applies to $T(\varrho)$, it does not apply to any T satisfying (α), (β) and (γ); for example, T may satisfy (α) without satisfying (α)′. If T satisfies (α), (β) and (γ) as well as (α)′, (β)′ and (γ)′, then

$$a\,\varrho\,b,\ t\,.\!\cdot a=t\,.\!\cdot b \quad \text{for all} \quad t\in T,\ t\cdot\!.\,a=t\cdot\!.\,b \quad \text{for all} \quad t\in T,$$

are all equivalent.

Note 6. In [7], I. Molinaro considered equivalence relations on a residuated semigroup S. He showed that for $t\in S$, the relation A_t $({}_tA)$ defined by $a\equiv b(A_t)$ $(a\equiv b({}_tA))$ if and only if $t\,.\!\cdot a=t\,.\!\cdot b$ $(t\cdot\!.\,a=t\cdot\!.\,b)$ is an m-equivalence, regular on the right (left) with respect to multiplication.

If a subset T of a partially ordered semigroup G satisfies (α), (β) and (γ), one may still define the relation A_t as above, since, by (α), the residuals concerned exist; obviously A_t is an equivalence relation. Further, as in the proof of Theorem 1, A_t is regular on the right with respect to multiplication. Finally, $t\,.\!\cdot x=t\,.\!\cdot\{t\cdot\!.\,(t\,.\!\cdot x)\}$ implies that A_t is an m-equivalence, the maximum element in the class of $x\in G$ being $t\cdot\!.\,(t\,.\!\cdot x)$. By (γ), the maximum element in the class containing t is t itself. Thus an m-congruence ϱ on G, which satisfies the conditions of Theorem 1, may be expressed as the intersection of the m-equivalences A_t $({}_tA)$ for $t\in T$, each A_t $({}_tA)$ being regular on the right (left) with respect to multiplication.

When t is equiresidual A_t is a congruence relation, and several papers (cf. [4], [6], [7]) have been written about the situation where A_t is defined on residuated gerbiers, where by definition a gerbier is a semigroup G in which every two elements x, y have a least upper bound $x\vee y$ satisfying $a(x\vee y)=ax\vee ay$, $(x\vee y)a=xa\vee ya$ for all $x, y\in G$. If in addition every pair $x, y\in G$ have a greatest lower bound $x\wedge y$, G is called a lattice semigroup.

Note 7. If x and y are equiresidual elements of a residuated lattice semigroup G, then $\varrho=A_x\cap A_y$ is an m-congruence on G, with

$$T(\varrho)=\{t_a=x:(x:a)\wedge y:(y:a),\ \text{ for } a\in G\}.$$

For ϱ is certainly a congruence on G, while

$$a\leqq x:(x:a)\wedge y:(y:a)\leqq x:(x:a)$$

implies $a\equiv t_a(A_x)$, by convexity, and similarly $a\equiv t_a(A_y)$, so $a\,\varrho\,t_a$. Clearly $a\leqq t_a$, t_a is maximum in its ϱ-class, and $a\leqq b$ implies $t_a\leqq t_b$.

Example. Let $S=\{x|x$ is a real number and $x\leqq -2\}\cup\{-1\}\cup\{0\}$, with the usual ordering. If $x\leqq -2$, define $xy=yx=-2$ for any $y\in S$, and for $x, y>-2$, define $xy=yx=\min\{x, y\}$. Then S is a partially ordered semigroup without identity element. Let Z denote the integers under addition, with the usual ordering; as an ordered group, Z is residuated, with $i:j=i-j$ for $i, j\in Z$. Let $G=S\times Z$, with co-ordinatewise multiplication and ordering. For $a, b\in S$ and $a<-2$, there is no $x\in S$ such that $bx\leqq a$, so S is not residuated. It is easy to see that the direct product of residuated semigroups is residuated if and only if each factor is residuated, so G is not residuated. Yet for $x\in S$, $0:x=0$, $-1:0=-1$, $-1:x=0$ if $x\leqq -1$, $-2:0=-2$, $-2:-1=-2$, $-2:x=0$ if $x\leqq -2$. It follows that

$$T=\{(n, i)|\, n=-2,\ -1 \ \text{ or } \ 0,\ i\in Z\}$$

is a subset of G satisfying the first two parts of (α), (and satisfying (β)). Consider $a=(x,j)\in G$, for $x\leqq -2$. For $t=(n,i)\in T$, $t:(t:a)=(n,j)$ and so $\bigwedge_{t\in T} t:(t:a)=$ $=(-2,j)\in T$. For $a=(-1,j)$ and $t=(0,i)$ or $t=(-2,i)$, $t:(t:a)=(0,j)$, while for $t=(-1,i)$, $t:(t:a)=(-1,j)$. Thus $\bigwedge_{t\in T} t:(t:a)=(-1,j)\in T$. Similarly, for $a=(0,j)$, $\bigwedge_{t\in T} t:(t:a)=(0,j)\in T$. Hence T satisfies (α). Finally, for $t=(n,i)\in T$, $t:(t:t)=(n,i):(0,0)=(n,i)=t$, so T satisfies (γ). By Theorem 1, G/ϱ_T is a residuated, self-residuated semigroup. The formulae above show that the ϱ_T-classes consist of the points $\{(0,i)\}$ and $\{(-1,i)\}$, $i\in Z$, and the lines $\{(x,i)|x\leqq -2, i\in Z\}$. If $U=\{-2,-1,0\}$, with the usual ordering and $xy=yx=\min\{x,y\}$ for $x,y\in U$, then G/ϱ_T is isomorphic to the residuated, self-residuated semigroup $U\times Z$.

The formulae above also show that $A_{(n,i)}=A_{(n,j)}$ for any $i,j\in Z$, so that $\varrho_T=A_{(0,0)}\cap A_{(-1,0)}\cap A_{(-2,0)}$; since $A_{(-1,0)}\leqq A_{(0,0)}$ and $A_{(-2,0)}\leqq A_{(0,0)}$, in fact $\varrho_T=A_{(-2,0)}\cap A_{(-1,0)}$, though ϱ_T is not a congruence of the A type. Thus the representation of an m-congruence is not in general unique.

The situation illustrated in this example is typical of that in general. One may show that if $\varrho=\varrho_T$ and $T'\subseteqq T$ satisfies

$$\bigwedge_{t'\in T'} t'^{\cdot}.(t'.^{\cdot}a)=\bigwedge_{t\in T}(t^{\cdot}.(t.^{\cdot}a) \quad \text{for any } a\in G,$$

then $\varrho=\varrho_{T'}$, using the fact that $\varrho_{T'}$ is an m-congruence such that $T(\varrho_{T'})=T(\varrho)$.

II

A residuated semigroup G, with identity e, for which $a.^{\cdot}a=a^{\cdot}.a=e$ for every non-zero $a\in G$, is called *integrally closed.* We now investigate under what conditions a partially ordered semigroup G has an integrally closed homomorphic image, under the hypothesis that each congruence class contains a maximum element.

Let ϱ be an m-congruence on a partially ordered semigroup G such that G/ϱ is integrally closed. Then G/ϱ has an identity element $f\varrho$; let f be the element maximum in this class. Since G/ϱ is then self-residuated, Theorem 1 and its dual hold, and we have

(a) *$f.^{\cdot}a$ and $f^{\cdot}.a$ exist for any $a\in G$.*

Further, f satisfies the following conditions:

(b) *f is equiresidual.*

(c) *f is a residual of itself.*

(d) *$(f:a).^{\cdot}(f:a)=f=(f:a)^{\cdot}.(f:a)$ for any $a\in G$. In particular, $f=f:f$.*

For (b), $(f.^{\cdot}a)\varrho=f\varrho.^{\cdot}a\varrho$ (by Lemma 1) $=(a\varrho^{\cdot}.a\varrho).^{\cdot}a\varrho$ (since G/ϱ is integrally closed) $=(a\varrho.^{\cdot}a\varrho)^{\cdot}.a\varrho=f\varrho^{\cdot}.a\varrho=(f^{\cdot}.a)\varrho$; by Lemma 1, $f^{\cdot}.a=f.^{\cdot}a$, each being maximum in its class. The third condition follows at once from Theorem 1. Finally, for any $x\varrho\in G/\varrho$, $x\varrho.^{\cdot}x\varrho=f\varrho=x\varrho^{\cdot}.x\varrho$ implies that $t_x.^{\cdot}t_x=f=t_x^{\cdot}.t_x$; then $f:a\in T(\varrho)$ implies (d). In particular, $f=(f:f).^{\cdot}(f:f)=(f:(f:f))^{\cdot}.f=f:f$.

We have now proved the first part of the following Theorem.

Theorem 2. *A necessary and sufficient condition that there exist an m-con-*

gruence ϱ on a partially ordered semigroup G such that G/ϱ is integrally closed, is that G contain an element f satisfying (a), (b), (c) *and* (d).

For the second part, we require the following Lemma.

Lemma 2. *Let G, ϱ, $T(\varrho)$ be as in Theorem 1. Then G/ϱ has an identity element if and only if there exists $f\in G$ such that $t.\cdot f=t=t^{\cdot}.f$ for every $t\in T(\varrho)$.*

Proof. The necessity is obvious, so let $f\in G$ be such that $t.\cdot f=t=t^{\cdot}.f$ for all $t\in T(\varrho)$. Then for $a\varrho\in G/\varrho$, $a\varrho.\cdot f\varrho=(t_a.\cdot f)\varrho=t_a\varrho=a\varrho$ implies $f\varrho\, a\varrho\leqq a\varrho$. On the other hand, $a\varrho\leqq(f\varrho\, a\varrho).\cdot f\varrho=f\varrho\, a\varrho$, so that $a\varrho=f\varrho\, a\varrho$. Similarly $f\varrho$ is a right identity for G/ϱ.

Proof of sufficiency of Theorem 2. Let $f\in G$ satisfy (a), (b), (c) and (d), and consider $T=\{t=f:a|\, a\in G\}$; we show that T satisfies the conditions of Theorem 1. First, for any $x, a\in G$, $f:a$ and $f:ax$ exist, and so therefore does $(f:a).\cdot x=t.\cdot x$; similarly $t^{\cdot}.x$ exists. Both these residuals are elements of T, so (β) is satisfied. Next, for any $y\in G$,

$$(f:y)^{\cdot}.((f:y).\cdot a)=(f:y)^{\cdot}.(f:ya)=$$
$$=(f:y)^{\cdot}.((f:a)^{\cdot}.y)=f:\{(f:a)^{\cdot}.y\}y\geqq f:(f:a).$$

But $f=(f:f).\cdot(f:f)=(f:(f:f))^{\cdot}.f=f:f$, so $(f:f)^{\cdot}.\{(f:f).\cdot a\}=f:(f:a)$, whence $f:(f:a)=\bigwedge_{t\in T}t^{\cdot}.(t.\cdot a)\in T$. Condition ($\alpha$) follows at once. For ($\gamma$), we use Rule 2 and the fact that $f=f:f$ is equiresidual to obtain

$$t=f:a=(f:f):a=f:fa=(f:a)^{\cdot}.f=f:af=(f:a).\cdot f.$$

By Theorem 1, $\varrho=\varrho_T$ is an m-congruence on G such that G/ϱ is residuated and self-residuated. To show that G/ϱ is integrally closed, we prove that in fact G/ϱ is a group. Since $a\,\varrho\, b$ if and only if $t_a=t_b$, and since $t_a=f:(f:a)$ (see above), we may use Rule 1 to obtain; $a\,\varrho\, b$ if and only if $f:a=f:b$. Then for any $t\in T$, $t.\cdot f=(f:a).\cdot f=(f:f)^{\cdot}.a=f:a=t=t^{\cdot}.f$, and Lemma 2 shows that $f\varrho$ is the identity element of G/ϱ. Finally, $a(f:a)\equiv f(\varrho)$ for any $a\in G$, since $f:a(f:a)=(f:a).\cdot(f:a)=f:f$, so G/ϱ is a group, and is a fortiori integrally closed. The Theorem is proved.

Since we do not require that G is a residuated semigroup, Theorem 2 generalizes the result of Maury [6]. One may deduce from Theorem 2 the result ([1], p. 107), of Mme. Dubreil-Jacotin, that any m-congruence ϱ on G resulting in a group image G/ϱ is necessarily defined by: $a\,\varrho\, b$ if and only if $\langle f:a\rangle=\langle f:b\rangle=\{x\in G|\, ax\leqq f\}$, where f is the maximum element in the identity class of G/ϱ. See also L. Fuchs [3].

Note 8. It is not necessary to assume that f is idempotent. It does follow from $f=f:f$ that $f^2\leqq f$, but it may happen that $f^2<f$. Nevertheless f is the maximum element satisfying $x^2\leqq x$ in G, for if $x^2\leqq x$ then

$$f:x\leqq f:x^2=(f:x).\cdot x \text{ implies } (f:x)x\leqq f:x,$$

so $x\leqq(f:x)^{\cdot}.(f:x)=f$.

If G has an identity element e, then $e\,\varrho\, f$ and $e\leqq f$. We note also (cf. [1], Theorem 5), that f is the maximum element of the form $x.^{\cdot}x$ or $x^{\cdot}.x$ in G, for $f=f:f$, while $f=(f:x).^{\cdot}(f:x)=(f:(f:x))^{\cdot}.x\geqq x^{\cdot}.x$.

In the example above, the element $f=(0,0)$ satisfies (a), (b), (c) and (d), and G/ϱ is isomorphic to Z. Here $f^2=f$, though G has no identity element.

Theorem 2 makes use of the fact that if a partially ordered semigroup G has an integrally closed image by means of an m-congruence ϱ, then G has a group image by means of an m-congruence. However, G may have an integrally closed image G/ϱ which is not a group. An additional condition on $T(\varrho)$ necessary (and sufficient) for G/ϱ to be integrally closed is described in the following Theorem.

Theorem 3. *Let G, ϱ, $T=T(\varrho)$ be as in Theorem* 1. *Then G/ϱ is integrally closed if and only if*

(δ) *there exists $f\in T$ such that $t.^{\cdot}t=t^{\cdot}.t=f$ for any $t\in T$.*

Proof. Suppose G/ϱ integrally closed, and let f be the maximum element in the identity class of G/ϱ. Then $f\varrho=t\varrho.^{\cdot}t\varrho=(t.^{\cdot}t)\varrho$ implies $f=t.^{\cdot}t$; similarly $f=t^{\cdot}.t$. Conversely, let T satisfy (δ). Then

$$t\varrho.^{\cdot}t\varrho=(t.^{\cdot}t)\varrho=f\varrho=t\varrho^{\cdot}.t\varrho.$$

Since G/ϱ is self-residuated,

$$t\varrho=t\varrho.^{\cdot}(t\varrho^{\cdot}.t\varrho)=t\varrho.^{\cdot}f\varrho=t\varrho^{\cdot}.(t\varrho.^{\cdot}t\varrho)=t\varrho^{\cdot}.f\varrho.$$

By Lemma 2, $f\varrho$ is the identity element of G/ϱ, whence G/ϱ is integrally closed.

In the example above, T satisfies (δ), for $f=(0,0)\in T$ is such that $t:t=f$ for any $t\in T$. The semigroup $G/\varrho=U\times Z$ is integrally closed, but is not a group.

*

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Homomorphisms of partially ordered semigroups onto groups

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In a recent paper [1] L. FUCHS considered the order preserving homomorphisms of a partially ordered semigroup S with identity e onto a partially ordered group G. Assuming that the partial order in G is determined in a natural way by that in S, that the congruence classes are convex, and that e is greater than or equal to any element of S whose class is less than or equal to that of e, FUCHS determined all such homomorphisms. He showed that whenever S is generalized residuated (see below), the solution is a generalization of ARTIN's equivalence, which provided the answer for a commutative, residuated, semilattice semigroup with identity. (See [2], [3], [4]). The purpose of the present paper is to show that similar results may be obtained even if S has no identity, and even if S is not generalized residuated.

The results described here were presented in Dr. R. J. KOCH's seminar at Louisiana State University, and I should like to thank the members of the seminar for their comments.

Let S be a *partially ordered semigroup*. That is, S is a semigroup on which defined a partial order $\leqq$ with the property that for all $a, b, c \in S$, $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$. For $a, b \in S$ define the *generalized left residual* of a by b to be the set $\langle a^{\cdot} . b\rangle = [x \in S \mid xb \leqq a]$, and the *generalized right residual* of a by b to be the set $\langle a . ^{\cdot} b\rangle = [x \in S \mid bx \leqq a]$. Using $\emptyset$ to denote the empty set, call S *generalized left (right) residuated* if $\langle a^{\cdot} . b\rangle \neq \emptyset$ $(\langle a . ^{\cdot} b\rangle \neq \emptyset)$ for all $a, b \in S$. If S is both right and left generalized residuated, call S generalized residuated.

If for $a, b \in S$ the set $\langle a^{\cdot} . b\rangle$ $(\langle a . ^{\cdot} b\rangle)$ is not empty and contains a maximum element, this element is called the *left (right) residual of a by b*, and is written $a^{\cdot} . b$ $(a . ^{\cdot} b)$; if $a^{\cdot} . b$ $(a . ^{\cdot} b)$ exists for all $a, b \in S$ then S is called *left (right) residuated.*

Define a *congruence relation* θ on S to be an equivalence relation satisfying, for all $a, b \in S$:

1. *$a \equiv b(\theta)$ implies $ac \equiv bc(\theta)$ and $ca \equiv cb(\theta)$.*

The set of congruence classes $\theta(a)$, $a \in S$, forms a semigroup S/θ if we define $\theta(a)\theta(b) = \theta(ab)$, but if we define an order relation $\leqq'$ on S/θ by setting:

$$\theta(a) \leqq' \theta(b) \quad \textit{if and only if there exist} \quad a' \in \theta(a), \quad b' \in \theta(b) \quad \textit{such that} \quad a' \leqq b',$$

then in general $\leqq'$ is not a partial order on S/θ. However, $\leqq'$ is compatible with the multiplication defined above, in the sense that for any $\theta(a)$, $\theta(b)$, $\theta(c) \in S/\theta$, $\theta(a) \leqq' \theta(b)$ implies $\theta(a)\theta(c) \leqq' \theta(b)\theta(c)$ and $\theta(c)\theta(a) \leqq' \theta(c)\theta(b)$; for if $a \in \theta(a)$, $b \in \theta(b)$,

$c\in\theta(c)$ with $a\leqq b$, then $ac\leqq bc$ implies $\theta(a)\theta(c)\leqq'\theta(b)\theta(c)$, and similarly for multiplication on the left.

If θ is a congruence relation on S which satisfies

2. $a\leqq c\leqq b$ and $a\equiv b(\theta)$ *imply* $a\equiv c(\theta)$,

we shall call θ a convex congruence relation on S.

Whenever θ is a convex congruence relation on S, with the properties that S/θ is a group and the identity class of S/θ contains an element f such that θ satisfies also:

3. $\theta(a)\leqq'\theta(f)$ *implies* $a\leqq f$,

it is true that $\leqq'$ is a partial order (so that S/θ is then a partially ordered group). For if now $\theta(a)\leqq'\theta(b)\leqq'\theta(a)$, choose $a'\in S$ such that $aa'\equiv f(\theta)$. Then since $\leqq'$ is compatible with multiplication in S/θ, $\theta(f)=\theta(aa')\leqq'\theta(ba')\leqq'\theta(aa')$, and so there exist $f'\in\theta(f)$ and $ba'\in\theta(ba')$ such that $f'\leqq ba'\leqq f$. By 2. it follows that $ba'\equiv f$, whence $\theta(b)=\theta(a)$. Thus $\leqq'$ is antisymmetric, and, similarly, transitive.

Note that we do not require that f be the identity of S, nor even that S have identity; in fact, f may not be idempotent.

Lemma 1. *The identity class of the group S/θ contains an element f satisfying* 3. *if and only if S contains an element $f=f^{\cdot}.f=f.{}^{\cdot}f$ satisfying* 3.

Proof. If $\theta(f)$ is the identity of S/θ, then $f^2\in\theta(f)\theta(f)=\theta(f)$ implies $f^2\leqq f$, while if $fx\leqq f$ then $\theta(fx)=\theta(f)\theta(x)=\theta(x)\leqq'\theta(f)$ implies that $x\leqq f$, so $f=f.{}^{\cdot}f$; similarly, $f=f^{\cdot}.f$. Conversely, if $f=f.{}^{\cdot}f=f^{\cdot}.f\in S$ satisfies 3., let $\theta(a)$ be the identity of S/θ. Then $\theta(f)=\theta(a)\theta(f)=\theta(af)$ implies $af\leqq f$, using 3. once more, and so $a\leqq f^{\cdot}.f=f$. Hence $\theta(a)\leqq'\theta(f)$. Now $f=f.{}^{\cdot}f$ satisfies $f^2\leqq f$, whence $\theta(f)\theta(f)\leqq'\theta(f)$ and $\theta(f)\leqq'\theta(a)$ in the group S/θ. As above, it follows that $\theta(a)=\theta(f)$.

If now $a\in S$ and $\langle f.{}^{\cdot}a\rangle\neq\emptyset$ ($\langle f^{\cdot}.a\rangle\neq\emptyset$), call $a'\in\langle f.{}^{\cdot}a\rangle$ ($\langle f^{\cdot}.a\rangle$) right (left) multiplicatively maximal in $\langle f.{}^{\cdot}a\rangle$ ($\langle f^{\cdot}.a\rangle$) if $x\in S$ and $a'x\in\langle f.{}^{\cdot}a\rangle$ ($xa'\in\langle f^{\cdot}.a\rangle$) imply $x\leqq f$. (See [1].)

Theorem 1. *Let S be a partially ordered semigroup containing an element $f=f.{}^{\cdot}f=f^{\cdot}.f$, and let θ be an equivalence relation on S satisfying* 1. *and* 2. *If S/θ is a group and θ satisfies* 3. *then:*

(i) *for any* $a\in S$, $\langle f.{}^{\cdot}a\rangle\neq\emptyset$, $\langle f^{\cdot}.a\rangle\neq\emptyset$, $\langle f.{}^{\cdot}a\rangle=\langle f^{\cdot}.a\rangle$,

(ii) *for any* $a\in S$, $\langle f^{\cdot}.a\rangle$ *contains left multiplicatively maximal elements.*

(iii) $\langle f^{\cdot}.a\rangle=\langle f^{\cdot}.b\rangle$ *if and only if* $a\equiv b(\theta)$.

Using Lemma 1, the proof is almost the same as that of Theorem 1 of [1], and we omit it. See also [5], p. 107.

Under the present hypotheses one cannot prove that S is generalized residuated; in general there may exist $a,b\in S$ such that $\langle a.{}^{\cdot}b\rangle=\emptyset$.

Theorem 2. *For any $a\in S$, $a.{}^{\cdot}a=f$ if and only if $af\leqq a$.*

Proof. For $a\in S$, the elements $a'\in S$ such that $aa'\equiv f(\theta)$ are right multiplicatively maximal in $\langle f.{}^{\cdot}a\rangle$, because $aa'\leqq f$ by 3., while if $a'x\in\langle f.{}^{\cdot}a\rangle$ for some $x\in S$ then $aa'x\leqq f$ implies $\theta(x)=\theta(aa')\theta(x)\leqq'\theta(f)$, so by 3. again, $x\leqq f$. Since $aa'\equiv f(\theta)$ is equivalent to $a'a\equiv f(\theta)$, and since $\langle f.{}^{\cdot}a\rangle=\langle f^{\cdot}.a\rangle$, a' is also left multiplicatively maximal in $\langle f.{}^{\cdot}a\rangle$. Now $a\in\langle f.{}^{\cdot}a'\rangle$, and if $ax\in\langle f.{}^{\cdot}a'\rangle=\langle f^{\cdot}.a'\rangle$ then $axa'\leqq f$, so $xa'\in\langle f.{}^{\cdot}a\rangle=\langle f^{\cdot}.a\rangle$. Since a' is left multiplicatively maximal in $\langle f^{\cdot}.a\rangle$

it follows that $x \leqq f$, and that a is then right, and similarly left, multiplicatively maximal in $\langle f . \cdot a' \rangle$.

Now suppose that $ax \leqq a$. For a' as above, $axa' \leqq aa' \leqq f$, so $xa' \in \langle f . \cdot a \rangle$; but a' is left multiplicatively maximal in this set, and therefore $x \leqq f$. Hence if $af \leqq a$ then the residual $a . \cdot a$ exists, and $a . \cdot a = f$; conversely, if $a . \cdot a = f$ then $af \leqq a$. It may happen that the only $a \in S$ for which $af \leqq a$ is $a = f$.

For the case considered in [1], where f is the identity of S, $a^{\cdot} . a = a . \cdot a = f$ for every $a \in S$; I am indebted to Mr. J. E. L'HEUREUX for this remark. Mme DUBREIL-JACOTIN points out [5], Lemma 5, that since $aff \leqq af$, one has $af . \cdot af = f$ for every $a \in S$.

Theorem 3. *Let S be a partially ordered semigroup containing an element $f = f^{\cdot} . f = f . \cdot f$, and let* (i) *and* (ii) *hold. Define a relation θ on S by* (iii). *Then θ satisfies* 1., 2. *and* 3., *and S/θ is a group.*

Proof. From the obvious properties of generalized residuals, θ satisfies 1 and 2. For 3., let $\theta(a) \leqq' \theta(f)$; there exist $a' \in \theta(a), f' \in \theta(f)$ such that $a' \leqq f'$, and then $f \in \langle f^{\cdot} . f \rangle = \langle f^{\cdot} . f' \rangle \subseteqq \langle f^{\cdot} . a' \rangle = \langle f^{\cdot} . a \rangle$, so $fa \leqq f$, $a \leqq f . \cdot f = f$. To show that S/θ is a group, note first that $\theta(f)$ is the identity of S/θ, for the following are equivalent: $x \in \langle f^{\cdot} . a \rangle$, $xa \leqq f$, $xaf \leqq f$, $x \in \langle f^{\cdot} . af \rangle$; that is, $\theta(a) = \theta(af) = \theta(a)\theta(f)$. Using $\langle f . \cdot a \rangle = \langle f^{\cdot} . a \rangle$, $\theta(f)$ is also a left identity for S/θ. Now let $a' \in S$ be left multiplicatively maximal in $\langle f^{\cdot} . a \rangle$, and let $x \in \langle f^{\cdot} . a'a \rangle$. Then $xa'a \leqq f$ implies $xa' \in \langle f^{\cdot} . a \rangle$, so $x \leqq f$ and $xf \leqq f^2 \leqq f$; that is, $x \in \langle f^{\cdot} . f \rangle$. Conversely, if $x \in \langle f^{\cdot} . f \rangle$ then $xf \leqq f$, $x \leqq f^{\cdot} . f = f$, $xa'a \leqq f^2 \leqq f$, $x \in \langle f^{\cdot} . a'a \rangle$. Hence $a'a \equiv f(\theta)$ and S/θ is a group, completing the proof.

A subset X of a semigroup S is said to be *reflective* if $ab \in X$ implies $ba \in X$. When S is partially ordered, an element $x \in S$ is called *reflective* if $ab \leqq x$ implies $ba \leqq x$. Mme. DUBREIL-JACOTIN proves [5], Theorem 7, that under the present hypotheses, f is reflective, and conversely, [5], Lemma 8, that if f is reflective, then $\langle f . \cdot a \rangle = \langle f^{\cdot} . a \rangle$ for any $a \in S$.

Let $H = [x \in S \mid x \leqq f]$. Clearly H is a subsemigroup of S, $x \in H$ and $y \leqq x$ imply $y \in H$, H is reflective, and for any $a \in S$ there exists $a' \in S$ such that $aa'x \in H$ implies $x \in H$. Thus H satisfies the conditions of Theorem 1 of [6], and so $\theta(a) = \theta(b)$ if and only if $H : a = H : b$, where $H : a = [x \in S \mid ax \in H] = [x \in S \mid xa \in H]$.

Recalling that the identity of a partially ordered group G is the maximum $x \in G$ satisfying $x^2 \leqq x$ in G, we note that with the present hypotheses the element f above is the maximum element of S which satisfies $x^2 \leqq x$ in S. For since $f = f . \cdot f$, certainly $f^2 \leqq f$; while if $x^2 \leqq x$ in S then $\theta(x)\theta(x) \leqq' \theta(x)$ in the group S/θ, so $\theta(x) \leqq' \theta(f)$, and by 3., $x \leqq f$. If S has identity e, then of course $e \equiv f(\theta)$, and also $e \leqq f = f^2$. Mme. DUBREIL-JACOTIN notes [5], Theorem 5, that f is maximum in each of the sets $U\langle a . \cdot a \rangle$, $U\langle a^{\cdot} . a \rangle$, where the unions are over all those $a \in S$ for which $\langle a . \cdot a \rangle \neq \emptyset$, $\langle a^{\cdot} . a \rangle \neq \emptyset$.

The following example may illustrate the situation. Let T be the set of points (p, i) in the plane, where $-\infty < p < 0$ and i is an integer, together with the point $(0, 0)$. Write a_{pi} for the point (p, i). Partially order T by setting $a_{pi} \leqq a_{qj}$ if and only if $p \leqq q$ and $i \leqq j$.

Define a multiplication $(\cdot)$ on T by setting

$$a_{pi} \cdot a_{qj} = a_{qj} \cdot a_{pi} = a_{\min(p,q), i+j} .$$

Then $S=T(\cdot)$ is a commutative generalized residuated semigroup with identity $f=a_{00}$, where

$$\langle a_{pi} \cdot . a_{qj}\rangle = [a_{rk}\in S \mid k \leqq i-j \text{ and (i) } -\infty < r \leqq 0 \text{ if } q \leqq p, \text{ (ii) } r \leqq p \text{ if } q > p].$$

For $i \neq j$ and $q \leqq p$, $\langle a_{pi} \cdot . a_{qj}\rangle$ does not contain a maximum element, so S is not residuated; yet for any $a_{pi}\in S$, $a_{pi} \cdot . a_{pi} = a_{pi} . \cdot a_{pi} = f$. If we define θ on S by (iii), then $a_{pi}\theta a_{qj}$ if and only if $i=j$, and θ satisfies 1., 2. and 3. The θ-classes are lines parallel to the x-axis, and S/θ is isomorphic to the additive, linearly ordered group of the integers. The multiplicatively maximal elements in $\langle a_{00} \cdot . a_{pi}\rangle$ are the elements $a_{q,-i}$, where $-\infty < q \leqq 0$, and these are exactly the elements $a'\in S$ such that $a' a_{pi} \equiv f(\theta)$. Except when $i=0$, the set $[a_{q,-i} \mid -\infty < q \leqq 0]$ has no maximum element.

Now define $(*)$ on T by: $a_{pi} * a_{qj} = a_{qj} * a_{pi} = a_{-1,i+j}$.

Then $S' = T(*)$ is a commutative partially ordered semigroup without identity. Write $f=a_{00}$. We have

$$\langle a_{pi} \cdot . a_{qj}\rangle = \emptyset \quad \text{if} \quad p < -1 =$$
$$= [a_{rk}\in S \mid k \leqq i-j, \; -\infty < r \leqq 0 \quad \text{if} \quad p \geqq -1].$$

Clearly S' is not generalized residuated, but $f = f \cdot . f = f . \cdot f \in S'$ and $\langle f \cdot . a_{pi}\rangle$ is non-empty for $a_{pi}\in S'$. If θ is defined by (iii), conditions 1., 2. and 3. are satisfied, the θ-classes are the same as those in S, and S'/θ is isomorphic to the integers. The multiplicatively maximal elements in $\langle a_{00} \cdot . a_{pi}\rangle$ are as in S. For $p \geqq -1$, $f a_{pi} \leqq a_{pi}$ and $a_{pi} \cdot . a_{pi} = f$, but for $p < -1$, $\langle a_{pi} \cdot . a_{pi}\rangle = \emptyset$. Finally, $f^2 = a_{-1,0} < f$.

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Über lokal linear kompakte Ringe

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1. Einleitung. Die Struktur der linear kompakten bzw. lokal kompakten Ringe ist schon von mehreren Verfassern weitgehend untersucht worden. Es stellt sich die Frage, was sich über die Struktur der lokal linear kompakten Ringe sagen läßt. Für Vektormoduln wurde dieser Begriff schon im Buch [3] von LEFSCHETZ (S. 79) definiert, doch wurden meines Wissens bis jetzt keine Ergebnisse über lokal linear kompakte Ringe veröffentlicht. In dieser Note machen wir einige Schritte in der Untersuchung der lokal linear kompakten Ringen, und zwar beweisen wir, daß jeder halbeinfache lokal linear kompakte Ring ein minimales Linksideal hat. Davon können wir leicht einige Behauptungen über primitive bzw. einfache lokal linear kompakte Ringe ableiten. Ein topologisch einfacher lokal linear kompakter Ring erweist sich als direkte Summe eines linear kompakten Linksideals und eines diskreten Linksideals; ferner beweisen wir, daß ein topologisch einfacher lokal linear kompakter Ring mit gröbster Topologie, oder mit Rechtseinselement stets linear kompakt ist.

2. Vorbereitungen. In dieser Arbeit betrachten wir nur *Hausdorffsche Topologien.* Die Terminologie von LEPTIN [4] folgend, bezeichnen wir als *Filter* ein System $\boldsymbol{F}$ von Mengen F_μ mit der Eigenschaft, daß zu F_μ, F_ν ein $F_\lambda \in \boldsymbol{F}$ mit $F_\lambda \subset F_\mu \cap F_\nu$ existiert. Ein *Basisfilter* eines topologischen Moduls ist ein Filter, dessen Elemente ein Fundamentalsystem für die Umgebungen von 0 bilden. Ist $\boldsymbol{F}$ ein Filter und $a \in \bigcap_{F \in \boldsymbol{F}} \bar{F} = \downarrow\boldsymbol{F}$, dann sagen wir, a sei ein *Berührungspunkt* von $\boldsymbol{F}$. Ein Filter $\boldsymbol{F}$ *konvergiert* gegen a, falls jede Umgebung von a ein $F \in \boldsymbol{F}$ enthält. Aus $\lim \boldsymbol{F} = a$ folgt immer $\downarrow\boldsymbol{F} = a$.

Ein topologischer Linksmodul M über einem Ring R heißt *linear topologisch,* falls M ein Basisfilter aus Untermoduln besitzt. Ein Filter in einem linear topologischen Modul heißt *Cauchyfilter*, wenn er aus Restklassen von Linksidealen eines Basisfilters besteht. Ein linear topologischer Modul wird *vollständig* genannt, falls für jeden Cauchyfilter $\boldsymbol{C}$ $\downarrow\boldsymbol{C}$ nicht die leere Menge ist.

Ein topologischer R-Modul M heißt *linear kompakt,* falls M linear topologisch ist und jedes Filter von Restklassen nach abgeschlossenen Untermoduln einen nicht leeren Durchschnitt hat. Ein topologischer Ring R ist linear kompakt, wenn er als R-Linksmodul linear kompakt ist. Wir nennen einen topologischen Ring R *lokal linear kompakt,* falls R ein von 0 verschiedenes offenes Linksideal besitzt, welches ein linear kompakter R-Modul ist. In dieser Definition haben wir also den trivialen Fall ausgeschlossen.

Sei L ein Linksideal eines topologischen Ringes R. Es zeigt sich unmittelbar, daß die Menge

$$A = \{r \in R \mid rL = 0\}$$

ein abgeschlossenes Ideal bildet; dieses Ideal A wird *Linksannihilatorideal* von L genannt. Bezeichne Q den Faktorring R/A. Man kann jedes Element $q = r + A \in Q$ als einen stetigen Endomorphismus der additiven Gruppe von L auffassen, welche durch

$$q(x) = rx \qquad (x \in L)$$

definiert wird.

Wie üblich, wird ein Ring *halbeinfach* genannt, falls sein Jacobsonsches Radikal verschwindet. Ein Ring R, der einen treuen [1]) irreduziblen R-Linksmodul besitzt, heißt *primitiv*. Wir nennen einen Ring *einfach*, wenn R kein Radikalring ist und kein echtes abgeschlossenes Ideal enthält. Gemäß dieser Definition ist ein einfacher Ring stets primitiv und ein primitiver Ring ist halbeinfach. Ist R radikalfrei, und besitzt kein echtes abgeschlossenes Ideal, dann wird R *topologisch einfach* genannt.

Bezüglich des Jacobsonschen Radikals und weiterer algebraischer Begriffe verweisen wir auf [2] und [7].

3. Ergebnisse. Das Hauptresultat dieser Arbeit ist der folgende

Satz 1. *Ist R ein halbeinfacher lokal linear kompakter Ring, so enthält R ein minimales Linksideal.*

Beweis. Sei $L \neq 0$ ein offenes linear kompaktes Linksideal in R und bezeichne A das Linksannihilatorideal von L. Da der Durchschnitt $A \cap L$ ein Linksideal ist mit $(A \cap L)^2 = 0$, ist $A \cap L$ im Radikal von R enthalten. Infolge der Halbeinfachheit muß $A \cap L = 0$ bestehen. Bezeichne φ den natürlichen Homomorphismus von R auf $Q = R/A$, und τ die induzierte Topologie in Q. Wegen $A \cap L = 0$ ist $L' = \varphi(L)$ als Q-Modul zu L im algebraischen und topologischen Sinne isomorph. Folglich ist L' ein linear kompakter Q-Modul.

Wir nehmen an, daß die Elemente von Q Endomorphismen der additiven Gruppe von L sind. Seien $x_1, \ldots, x_n \in L$ endlich viele Elemente und $U \subset L$ ein offenes Linksideal. Die Menge

$$U^*(x_1, \ldots, x_n; U) = \{q \in Q \mid qx_1, \ldots, qx_n \in U\}$$

ist offenbar ein Linksideal von Q und sämtliche $U^*(x_1, \ldots, x_n; U)$ bilden ein Filter $\mathfrak{U}^*$. Wir wählen $\mathfrak{U}^*$ als Basisfilter einer neuen Topologie τ^* in Q. Es zeigt sich, daß die Topologie τ^* größer als τ ist. Sei nämlich $U^* = U^*(x_1, \ldots, x_n; U)$ ein Linksideal aus $\mathfrak{U}^*$, und V ein offenes Linksideal in R mit $V \cup Vx_1 \cup \ldots \cup Vx_n \subset U$. V bestimmt das offene Linksideal $V' = \varphi(V)$ in Q, und für V' gilt $V' \subset U^*$. Daraus folgt $\tau \leqq \tau^*$.

Bezeichne nun $\tilde{Q}$ die Komplettierung von Q bezüglich der Topologie τ^*. Wir beweisen, daß $\tilde{Q}$ linear kompakt ist. Nach ZELINSKY [8], Theorem 3', läßt sich $\tilde{Q}$ als inverser Limes der diskreten Faktormoduln Q/U^* ($U^* \in \mathfrak{U}^*$) darstellen. Für die lineare Kompaktheit von $\tilde{Q}$ genügt es zu beweisen, daß jeder diskrete Faktormodul Q/U^* linear kompakt ist. Sei $U^* = U^*(x_1, \ldots, x_n; U)$ ein τ^*-offenes

[1]) Bekanntlich nennt man einen R-Modul L treu, falls das Linksannihilatorideal von L Null ist.

Linksideal und bezeichne M die direkte Summe von n Exemplaren des Faktormoduls L/U. M ist offenbar diskret und linear kompakt. Die Abbildung

$$\varphi(q) = (qx_1 + U, \ldots, qx_n + U) \qquad (q \in Q)$$

ist ein Homomorphismus des Q-Moduls Q in M. Da $\varphi(q) = 0$ gleichbedeutend ist mit $qx_1, \ldots, qx_n \in U$, ist $\operatorname{Ker} \varphi = U^*(x_1, \ldots, x_n; U)$. Folglich ist Q/U^* als Q-Modul zu einem Untermodul M' von M isomorph, ferner sind Q/U^* und M' diskret und M' linear kompakt.

Es zeigt sich, daß man auch $\tilde{Q}$ als ein Endomorphismenring von L auffassen kann. Ist nämlich $\tilde{q} \in \tilde{Q}$, so ist $\tilde{q} = \lim \boldsymbol{C}$, wo $\boldsymbol{C} = \{q + U^*(x_1, \ldots, x_n; U)\}$ ein Cauchyfilter von Q in der Topologie τ^* ist. $\boldsymbol{C}_z = \{q + U^*(z, x_1, \ldots, x_n; U)\}$ bildet auch ein Cauchyfilter, und es gilt $\tilde{q} = \lim \boldsymbol{C}_z$. Zu jedem $q + U^*(z, x_1, \ldots, x_n; U)$ gehört ein Cauchyfilter $\boldsymbol{D}_z = \{qz + U\}$ von L. Da L linear kompakt ist, ist $\downarrow \boldsymbol{D}_z$ nicht leer. Sei $\tilde{q}(z) = \downarrow \boldsymbol{D}_z$: damit haben wir eine Abbildung von L in sich definiert. Wir zeigen, daß diese Abbildung ein Endomorphismus ist. Da

$$U^*(x, y; U) \subset U^*(x; U) \cap U^*(y; U) \subset U^*(x+y; U)$$

gültig ist, so gilt auch für die entsprechenden Restklassen aus $\boldsymbol{C}$:

$$q + U^*(x, y; U) \subset (q_1 + U^*(x; U)) \cap (q_2 + U^*(y; U)) \subset q_3 + U^*(x+y; U).$$

Daraus folgt

$$qx + U = q_1 x + U, \quad qy + U = q_2 y + U, \quad q(x+y) + U = q_3(x+y) + U,$$

und so ergibt sich

$$q_3(x+y) + U = q(x+y) + U = qx + qy + U = q_1 x + U + q_2 y + U.$$

Dementsprechend bekommen wir

$$\tilde{q}(x+y) = \downarrow\{q(x+y) + U\} \subset \downarrow\{qx + U\} + \downarrow\{qy + U\} = \tilde{q}(x) + \tilde{q}(y),$$

$\tilde{q}$ ist also ein Endomorphismus der additiven Gruppe von L.

Ist B ein abgeschlossener R-Untermodul von L, so enthält jede Umgebung von $\tilde{q}(b)$ ($\tilde{q} \in Q$, $b \in B$) ein Produkt qb ($q \in Q$), ferner ist $qb \in B$. Da B abgeschlossen ist, so muß $\tilde{q}(b) \in B$ bestehen. Das bedeutet, daß jeder abgeschlossene R-Untermodul von L zugleich ein $\tilde{Q}$-Untermodul ist.

Es zeigt sich, daß $L' = \varphi(L)$ ein abgeschlossenes Linksideal von $\tilde{Q}$ ist. Bezeichne $\bar{L}'$ die abgeschlossene Hülle von L' in $\tilde{Q}$ und sei $l \in \bar{L}'$, $l = \lim \boldsymbol{C}$, wo $\boldsymbol{C} = \{l_\alpha + U_\alpha^*\}$ ($l_\alpha \in L'$) ein Cauchyfilter von Q in der Topologie τ^* ist. $\boldsymbol{C}_0 = \{(l_\alpha + U_\alpha^*) \cap L'\} = \{l_\alpha + U_\alpha^* \cap L'\}$ bildet offenbar auch ein Filter, und zwar gilt $l = \lim \boldsymbol{C}_0$. Wegen $\tau \leqq \tau^*$ besteht $\boldsymbol{C}_0$ aus Restklassen nach τ-abgeschlossenen Q-Untermoduln von L'. Da L' als Q-Modul linear kompakt ist, gilt $l = \lim \boldsymbol{C}_0 = \downarrow \boldsymbol{C}_0 \in L'$. L' ist also abgeschlossen.

Ist $q \in \tilde{Q}$ und $l \in L'$, so gibt es zu jeder Umgebung U_{ql}^* von ql eine Umgebung V_q^* von q mit $V_q^* l \subset U_{ql}^*$. Da Q in $\tilde{Q}$ dicht ist, enthält V_q^* ein Element $r \in Q$. Folglich gilt $rl \in U_{ql}^* \cap L'$. Das bedeutet eben $ql \in \bar{L}' = L$. Damit ist bewiesen, daß L' ein abgeschlossenes Linksideal von $\tilde{Q}$ ist.

Nun beweisen wir, daß L und L' als $\tilde{Q}$-Moduln operatorisomorph sind. Dazu müssen wir die Gültigkeit von

$$\varphi(ql)=q\varphi(l) \qquad (q\in\tilde{Q},\ l\in L)$$

zeigen. Sei

$$U^*(x;\ U_\alpha)=\{q\in\tilde{Q}\mid qx\in U_\alpha\} \qquad (x\in L,\ U_\alpha\subset L)$$

ein offenes Linksideal von $\tilde{Q}$. Zu $U^*(x;\ U_\alpha)$ wählen wir ein offenes Linksideal $V_{x,\alpha}$ von R mit $V_{x,\alpha}x\subset U_\alpha$. Zur Umgebung $ql+V_{x,\alpha}$ von ql gibt es eine Umgebung $q+W^*_{x,\alpha}$ von q so, daß $W^*_{x,\alpha}l\subset V_{x,\alpha}$ und $W^*_{x,\alpha}\varphi(l)\subset U^*(x;\ U_\alpha)$ erfüllt ist. Da Q in $\tilde{Q}$ dicht ist, existiert ein Element $r_{x,\alpha}\in(q+W^*_{x,\alpha})\cap Q$. So ergibt sich

$$q+W^*_{x,\alpha}=r_{x,\alpha}+W^*_{x,\alpha} \quad \text{und} \quad ql+V_{x,\alpha}=r_{x,\alpha}l+W^*_{x,\alpha}.$$

Wegen $ql=\bigcap\limits_{x,\alpha}(r_{x,\alpha}l+V_{x,\alpha})$ ergibt sich

$$\varphi(ql)=\bigcap_{x,\alpha}(\varphi(r_{x,\alpha}l)+\varphi(V_{x,\alpha}))\subset\bigcap_{x,\alpha}(r_{x,\alpha}\varphi(l)+U^*(x;\ U_\alpha)).$$

Infolge $q=\bigcap\limits_{x,\alpha}(r_{x,\alpha}+W^*_{x,\alpha})$ gilt andererseits

$$q\varphi(l)=\bigcap_{x,\alpha}(r_{x,\alpha}\varphi(l)+W^*_{x,\alpha}\varphi(l))\subset\bigcap_{x,\alpha}(r_{x,\alpha}\varphi(l)+U^*(x;\ U_\alpha)).$$

Daraus folgt $\varphi(ql)=q\varphi(l)$. L und L' sind also als $\tilde{Q}$-Moduln operatorisomorph.

Jetzt zeigen wir, daß $\tilde{Q}$ halbeinfach ist. Bezeichne J das Radikal von $\tilde{Q}$. Da L und L' operatorisomorph sind, so ist $\varphi^{-1}(JL')$ ein quasireguläres Linksideal von R in L. Infolge der Halbeinfachheit von R muß aber $JL'=0$ bestehen. Wegen der Operatorisomorphie von L und L' ergibt sich

$$JL\cdot L=\varphi(JL)L=JL'\cdot L=0$$

folglich besteht $JL\subset A\cap L=0$. Da die Elemente von J Endomorphismen von L sind, ist $J=0$.

Nach Leptin [4], Satz 13, enthält der linear kompakte halbeinfache Ring $\tilde{Q}$ ein Einselement. Da L und L' als $\tilde{Q}$-Moduln operatorisomorph sind, erweist sich L als ein unitärer $\tilde{Q}$-Modul, d.h. das Einselement $e\in\tilde{Q}$ erfüllt die Bedingung $ex=x$ $(x\in L)$. Da ein Endomorphismus $q(\in\tilde{Q})$ jeden abgeschlossenes Untermodul von L in sich abbildet, und L bezüglich Q linear kompakt ist, deshalb ist L auch als $\tilde{Q}$-Modul linear kompakt. Nach Leptin [5], Satz 2, ist L eine direkte Summe minimaler $\tilde{Q}$-Untermoduln. Bezeichne K einen minimalen $\tilde{Q}$-Untermodul von L, und sei $K_1\neq 0$ ein Linksideal von R in K. Wäre $L'K_1=0$, so wäre auch

$$K_1^2\subset LK=L'K=0,$$

und das Radikal von R enthielte $K_1\neq 0$, was unmöglich ist. Es ist also $L'K_1\neq 0$, ferner gilt

$$K=QL'K_1\subset L'K_1=LK_1\subset K_1.$$

K ist also ein minimales Linksideal von R. Damit ist der Beweis vollendet.

Korollar 1. *Jeder primitive lokal linear kompakte Ring besitzt ein minimales Linksideal.*

Da die primitiven Ringe stets halbeinfach sind, folgt die Behauptung unmittelbar aus Satz 1. Korollar 1 zeigt, daß die primitiven lokal linear kompakten Ringe genau die primitiven Ringe mit minimalen Linksidealen sind. Diese Ringe sind in dem Buch [2] von JACOBSON weitgehend untersucht (Kapitel IV, vgl. insbesondere: Structure Theorem auf Seite 75).

Die einfachen lokal linear kompakten Ringe sind durch den Litoffschen Satz gekennzeichnet, es gilt nämlich

Korollar 2. *Jeder einfache lokal linear kompakte Ring R ist lokal ein Matrixring über einem Schiefkörper S, d.h. jede endliche Teilmenge von R läßt sich in einen Unterring M so einbetten, daß M zu einem vollen Matrixring über S isomorph ist.*

Nach Satz 1 hat R ein minimales Linksideal, die Behauptung folgt also unmittelbar aus dem Litoffschen Satz (vgl. JACOBSON [2], S. 90, oder FAITH—UTUMI [1]).

Korollar 3. *Ist R ein einfacher lokal linear kompakter Ring mit Rechtseinselement, dann ist R ein Matrixring endlichen Grades über einem Schiefkörper.*

Aus Satz 1 folgt, daß R ein minimales Linksideal enthält. Bekanntlich ist der durch alle minimalen Linksideale erzeugte Unterring, der sogenannte *Sockel*, ein zweiseitiges Ideal in R. Wegen der Einfachheit ist R durch minimale Linksideale erzeugt. Ist e das Rechtseinselement von R, dann ist $e=e_1+\ldots+e_n$, wo die Komponenten e_i in minimalen Linksidealen L_i liegen. Daraus folgt, daß R durch endlich viele minimale Linksideale erzeugt ist, also erweist sich als direkte Summe endlich vieler minimaler Linksideale. Der wohlbekannte Satz von E. NOETHER über halbeinfache Ringe bestätigt unsere Behauptung.

Es ist merkwürdig, wie einfach die Struktur der einfachen lokal linear kompakten Ringe ist. Dagegen sind die Verhältnisse unter lokal kompakten Ringen ganz anders. Neulich hat SKORNJAKOV in seiner Arbeit [6] einfache, nicht diskrete, lokal kompakte Ringe mit Einselement konstruiert, die keine Matrixringe sind.

Der folgende Satz beschreibt die Struktur der topologisch einfachen lokal linear kompakten Ringe.

Satz 2. *Ist R ein topologisch einfacher lokal linear kompakter Ring und $L\neq 0$ ein linear kompaktes offenes Linksideal von R, dann ist L ein direkter Summand von R im algebraischen und topologischen Sinne. In der Zerlegung $R=L\oplus K$ ist K ein durch minimale Linksideale erzeugtes diskretes Linksideal.*

Beweis. Nach Satz 1 enthält R ein minimales Linksideal, folglich ist der Sockel B in R von Null verschieden, ferner ist B in R dicht.

Ist $L=R$, dann ist die Behauptung trivial. Im Fall $L\neq R$ betrachten wir die Menge sämtlicher Linksideale $K_1, \ldots, K_\alpha, \ldots$ die durch minimale Linksideale erzeugt sind, und $K_\alpha \cap L=0$ genügen. Wegen $L\neq R$ und $\bar{B}=R$ ist diese Menge nicht leer. Ist $K_{\alpha_1}\subset K_{\alpha_2}\subset\ldots$ eine aufsteigende Kette solcher Linksideale und $K_0=\bigcup_{\alpha_i} K_{\alpha_i}$, dann ist K_0 ein Linksideal. Da aus $a\in K_0$ folgt $a\in K_{\alpha_i}$ für einen Index α_i, deshalb ist a in der Summe endlich vieler minimaler Linksideale enthalten. Folglich ist K_0 durch minimale Linksideale erzeugt. Ist $b\in K_0\cap L$, dann gilt für irgendeinen Index α_j: $b\in K_{\alpha_j}\cap L=0$, d.h. $K_0\cap L=0$. Wegen des Kuratowski—Zornschen Lemmas gibt es ein durch minimale Linksideale erzeugtes Linksideal K, welches maximal bezüglich der Bedingung $K\cap L=0$ ist.

Die Summe $L+K$ enthält den Sockel von R. Sonst gibt es nämlich ein minimales Linksideal N mit $N \cap (L+K)=0$; wegen der Maximalität von K ist $(N+K) \cap \cap L \neq 0$, also gilt eine Gleichung $l=n+k \neq 0$ $(l \in L,\ n \in N,\ k \in K)$, d.h. $n=l-k \neq 0$. Folglich ist $N \subset L+K$ gültig, was ein Wiederspruch ist.

Deshalb ergibt sich

$$R=\bar{B} \subset \overline{L+K}=L+K$$

R ist also die algebraische direkte Summe von L und K. Da L in R offen ist, so ist diese direkte Summe auch topologisch.

Korollar 4. *Läßt sich in den topologisch einfachen lokal linear kompakten Ring R keine gröbere Hausdorffsche Topologie einführen, dann ist R in dieser Topologie linear kompakt.*

Nach Satz 2 gibt es eine Zerlegung $R=L \oplus K$. Da in R und so auch in K die Topologie die gröbste ist, muß K die Summe endlich vieler Linksideale sein. Somit ist K und auch R linear kompakt.

Korollar 5. *Besitzt der topologisch einfache lokal linear kompakte Ring R ein Rechtseinselement, dann ist R linear kompakt.*

Wegen $R=L \oplus K$ gilt für das Rechtseinselement $e \in R$ eine Zerlegung $e=l+ +e_1+\ldots+e_n$, wo $l \in L$ ist und die Komponenten $e_1, \ldots, e_n$ in den minimalen Linksidealen $K_1, \ldots, K_n$ liegen. $e_1+\ldots+e_n$ ist ein Rechtseinselement von K, folglich ist K durch $K_1, \ldots, K_n$ erzeugt. Damit ist K und auch R linear kompakt.

Nach LEPTIN [4], Satz 12, ist ein topologisch einfacher linear kompakter Ring voller Endomorphismenring eines Vektormoduls über einem Schiefkörper. Deshalb lassen sich die letzten zweien Ergebnisse folgenderweise fassen:

Satz 3. *Ist in dem topologisch einfachen lokal linear kompakten Ring R die Topologie die gröbste, oder besitzt R ein Rechtseinselement, dann ist R zu einem vollen Endomorphismenring eines Vektormoduls über einem Schiefkörper isomorph.*

MATHEMATISCHES FORSCHUNGSINSTITUT,
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A note on completely regular semigroups

By SÁNDOR LAJOS in Budapest

Let S be a semigroup. It is well known,[1]) that S is regular if and only if the relation

$$R \cap L = RL \tag{1}$$

holds for every left ideal L and for every right ideal R of S. It is natural to ask the following question: In what semigroups does a similar relation hold only for left or right ideals, respectively?

We shall prove in this short note, that a semigroup S satisfying the relation

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for each pair of left ideals in S, is a left regular semigroup, that is $a \in Sa^2$ for all a in S. Analogously, if a semigroup S satisfies the relation

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

for every pair of right ideals of S, then S is a right regular semigroup, that is $a \in a^2 S$, for each a in S. It will be also proved that the semigroup S is a union of disjoint groups provided it satisfies both (2) and (3) for left and right ideals, respectively. Finally, we give a characterization of semigroups having either property (2) for left ideals or property (3) for right ideals.

First we prove the following simple

Lemma. *In an arbitrary semigroup S*

$$(a)_L^2 = (a)a \quad \textit{and} \quad (a)_R^2 = a(a), \tag{4}$$

where $a \in S$ and $(a)_L$ [$(a)_R$, (a)] denotes the principal left [right, two-sided] ideal of S generated by a.

Proof. We have

$$(a)a = a^2 \cup aSa \cup Sa^2 \cup SaSa,$$

and

$$(a)_L^2 = (a \cup Sa)(a \cup Sa) = a^2 \cup a\,Sa \cup Sa^2 \cup SaSa$$

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[1]) See the references [1], [2] or [3]. Concerning the definitions of the fundamental notions in the algebraic theory of semigroups, we refer to the books [1] and [3].

because $(a)_L = a \cup Sa$ and $(a) = a \cup aS \cup Sa \cup SaS$. Therefore $(a)_L^2 = (a)a$, as we stated. The second statement is the left-right dual of the first one.

Theorem 1. *Let S be a semigroup satisfying the relation*

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for any two left ideals L_1, L_2 of S. Then every left ideal of S is a two-sided ideal of S.

Proof. Let $L_2 = S$, then relation (2) implies $L_1 = L_1 S$, therefore the left ideal L_1 is also a right ideal of S, which proves the theorem.

It is also true the following left-right dual of Theorem 1.

Theorem 2. *Let S be a semigroup satisfying the relation*

$$R_1 \cap R_2 = R_1 R_2 \tag{3}$$

for any two right ideals R_1, R_2 of S. Then every right ideal of S is a two-sided ideal of S.

Theorem 3. *Let S be a semigroup satisfying relation* (2) *for any two left ideals. Then for any element a in S there exists at least one element x in S such that $a = xa^2$, i.e. S is a left regular semigroup.*

Proof. Let $L_1 = L_2 = (a)_L$. Then by (2) we have

$$(a)_L = (a)_L^2. \tag{5}$$

Applying the Lemma it follows that

$$(a)_L = (a)_L a, \tag{6}$$

because $(a)_L = (a)$ in view of Theorem 1. (6) implies

$$a \in (a)_L = (a \cup Sa)a = a^2 \cup Sa^2. \tag{7}$$

Thus we obtain either $a = a^2$ or $a = xa^2$, where $x \in S$. This means that $a \in Sa^2$ for any element a in S, i. e. the semigroup S is left regular. Theorem 3 is proved.

The dual statement reads as follows.

Theorem 4. *Let S be a semigroup satisfying relation* (3) *for any two right ideals of S. Then for each element a in S, there exists at least one element x in S so that $a = a^2x$, i.e. S is a right regular semigroup.*

Remark. The following example shows that converse of Theorem 3 is not true, i. e. relation (2) does not characterize the class of left regular semigroups:

	a	b	c	d
a	a	a	c	c
b	a	b	c	d
c	a	a	c	c
d	a	b	c	d

The semigroup $S = \{a, b, c, d\}$ with the above multiplication table is left regular, because S is an idempotent semigroup, but the relation (2) does not holds for any two left ideals of S. If

$$L_1=\{a, b, c\}$$

and

$$L_2=\{a, c, d\}$$

then

$$L_1 \cap L_2=\{a, c\} \neq L_2=L_1 L_2.$$

Now we prove the main result of this note.

Theorem 5. *If S is a semigroup having the property*

(2) $$L_1 \cap L_2=L_1 L_2$$

for any two left ideals L_1, L_2 of S, and

(3) $$R_1 \cap R_2=R_1 R_2$$

for any two right ideals R_1, R_2 of S, then the semigroup S is a union of disjoint groups.[2])

Proof. Let S be a semigroup satisfying both the relations (2) and (3) for left and right ideals, respectively. Let a be an arbitrary element of S. Then by Theorem 1

(8) $$(a)_L=(a)=(a)_R.$$

On the other hand $a=xa^2=a^2y$, where $x, y \in S$, by Theorems 3 and 4. Hence it follows that

(9) $$Sa=(a)=aS.$$

We define an $\mathscr{L}$-equivalence in S as follows:

(10) $$a\mathscr{L}b \quad \text{if and only if} \quad aS=bS.$$

It is easy to see that the relation $\mathscr{L}$ is reflexive, symmetric and transitive, that is, $\mathscr{L}$ is indeed an equivalence relation.[3]) The relation $\mathscr{L}$ determines a decomposition of S into disjoint classes. Denote by L_a the $\mathscr{L}$-class containing the element a in S. We show that L_a is a group.

To show that L_a is a subsemigroup of S, consider the elements a, b in L_a. Then by (9) and (3) we have

$$abS=abS^2=a(bS)S=a(Sb)S=aS \cap bS=aS=bS,$$

that is, $a\mathscr{L}ab$, and we conclude $ab \in L_a$. Thus L_a is a subsemigroup of S.

Next we show that L_a is a left simple semigroup, i. e. if $b \in L_a$, then $b=ca$,

---

[2]) It is known that a semigroup S is a disjoint union of groups if and only if it is completely regular (see [3]). Therefore our Theorem 5 implies that *the semigroup S having properties* (2) *and* (3) *concerning left and right ideals respectively, is a completely regular semigroup.*

[3]) See [4]. The relation $\mathscr{L}$ is a two-sidedly stable equivalence relation, that is, $\mathscr{L}$ is a congruence relation on S.

where $c \in L_a$. We know that $ba \in L_a$. Hence $b = xba$, with $x \in S$. Let $xb = c$. To show that $c \in L_a$, let y be an element of S such that $x = yx^2$. Thus

$$b = xba = yx^2ba = (yx)(xba) = yxb = yc.$$

The equations $b = yc$ and $c = xb$ imply $b\mathscr{L}c$, therefore $c \in L_b = L_a$.

Analogously we can prove the right simplicity of the semigroup L_a. Thus the semigroup L_a is both left and right simple, which implies that L_a is a group (see [1]). Hence S is a union of the disjoint classes any of which is a group.

The proof of Theorem 5 is complete.

In what follows we characterize the class of semigroups satisfying either (2) for left ideals or (3) for right ideals.

Theorem 6. *A semigroup S has the property*

$$L_1 \cap L_2 = L_1 L_2 \tag{2}$$

for any two left ideals L_1, L_2 of S if and only if S is left regular and any two left ideals of S commute, i.e. $L_1 L_2 = L_2 L_1$.

Proof. If the semigroup S has the property (2) for left ideals, then by Theorem 3 it is left regular and it follows from (2) that $L_1 L_2 = L_2 L_1$ holds for any left ideals L_1, L_2 of S.

Conversely, suppose that S is a left regular semigroup any two left ideals of which commute. Then $L_1 L_2 \subseteq L_2$ and $L_1 L_2 = L_2 L_1 \subseteq L_1$, whence it follows that

$$L_1 L_2 \subseteq L_1 \cap L_2.$$

To show the converse inclusion, consider an element a of $L_1 \cap L_2$. Then by the left regularity of S we have

$$a = xa^2 = (xa)a \in L_1 L_2.$$

Thus we obtain that

$$L_1 \cap L_2 = L_1 L_2,$$

for any two left ideals of S, which completes the proof.

Similarly, the following result also can be proved.

Theorem 7. *A semigroup S has the property*

$$L_1 \cap L_2 = L_1 L_2$$

for any two left ideals L_1, L_2 of S if and only if S is left regular and each left ideal L of S is a two-sided ideal of S.

Theorems 6 and 7 imply

Theorem 8. *For any semigroup S the following conditions are equivalent:*

(i) *$L_1 \cap L_2 = L_1 L_2$ for any two left ideals L_1, L_2 of S;*

(ii) *S is left regular and $L_1 L_2 = L_2 L_1$ for any two left ideals of S;*

(iii) *S is left regular and each left ideal of S is at the same time a two-sided ideal of S.*

The Theorems 6, 7 and 8 also have a left-right dual. We formulate only the dual of Theorem 8.

Theorem 9. *For any semigroup S the following conditions are equivalent:*

(i) $R_1 \cap R_2 = R_1 R_2$ *for any two right ideals* R_1, R_2 *of* S;

(ii) S *is right regular and* $R_1 R_2 = R_2 R_1$ *for any two right ideals of* S;

(iii) S *is right regular and every right ideal of* S *is a two-sided ideal of* S.

Remark (added in proof). The following example shows that the converse of Theorem 5 is not true:

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	a	b	c	d
d	a	a	d	c

The semigroup S of the elements a, b, c, d with the above multiplication table is a union of the disjoint subgroups $G_1=\{a, b\}$ and $G_2=\{c, d\}$, but relation (2) does not hold in S, because G_1 and G_2 are left ideals of S and

$$\emptyset = G_1 \cap G_2 \neq G_1 G_2 = G_2.$$

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Transitivity of implication in orthomodular lattices

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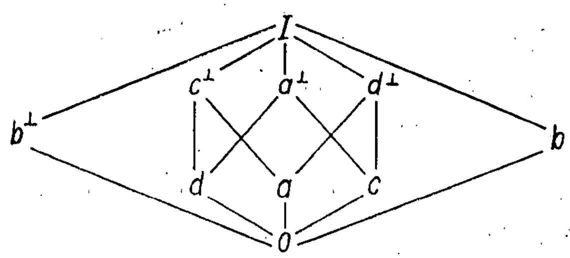
Extensive investigations in lattice theory [2—5] and logic [6—8] regarding the quantum logic of G. BIRKHOFF and J. VON NEUMANN [1] have induced many authors to call orthomodular lattices "generalized logic" [9]. KUNSEMÜLLER [6] pointed out that in quantum logic the relation of implication defined by

$$a^{\perp} \cup b = I$$

is not transitive ($a^{\perp}$ = orthocomplement of a, I = greatest element). The same holds also for orthomodular lattices in general. For example, the lattice with the diagram



is orthomodular and $a^{\perp} \cup b = I$, $b^{\perp} \cup c = I$, but $a^{\perp} \cup c \neq I$ in it.

KOTAS [7] has analysed the relations of implications defined on orthocomplemented modular lattices and characterized quantum logic on this basis by logical postulates. From the point of view of quantum logic the transitivity of the above established "classical" relation of implication is an interesting question. We have noticed that this property is characteristic to classical logic. Obviously it is the least we may demand of a logic L, and this we must demand [10], that L should be a lattice with unique orthocomplements. ROSE [11] has proved that such lattices coincide with orthomodular lattices.

We will prove that a lattice with unique orthocomplements is a Boolean algebra (i. e. a generalized logic is classical) if and only if the classical relation of implication defined in it is transitive.

Definition 1. A complemented lattice L is called *orthocomplemented* if the complementation is an involutory dualautomorphism in L, i. e. if to every $x\in L$ there exists an $x^\perp\in L$, such that

$$x\cap x^\perp=0,\quad x\cup x^\perp=I,\quad x\leqq y\Rightarrow y^\perp\leqq x^\perp,\quad x^{\perp\perp}=x.$$

Definition 2. An element x of an orthocomplemented lattice L is said to be *orthogonal* to $y\in L$, in symbols $x\perp y$, if $x\leqq y^\perp$.

The relation of orthogonality is evidently symmetrical.

Definition 3. An orthocomplemented lattice L is called *uniquely orthocomplemented* if each $x\in L$ has at most one complement orthogonal to x.

Lemma 1. *The De Morgan laws hold in every orthocomplemented lattice L, i.e.*

$$(x\cap y)^\perp=x^\perp\cup y^\perp \quad \textit{and} \quad (x\cup y)^\perp=x^\perp\cap y^\perp$$

for each $x, y\in L$.

Proof. $x,y\leqq x\cup y\Rightarrow x^\perp,\quad y^\perp\geqq(x\cup y)^\perp\Rightarrow x^\perp\cap y^\perp\geqq(x\cup y)^\perp$

and

$$x^\perp, y^\perp\geqq x^\perp\cap y^\perp\Rightarrow x^{\perp\perp}, y^{\perp\perp}\leqq(x^\perp\cap y^\perp)^\perp\Rightarrow x^{\perp\perp}\cup y^{\perp\perp}\leqq(x^\perp\cap y^\perp)^\perp\Rightarrow$$

$$\Rightarrow x\cup y\leqq(x^\perp\cap y^\perp)^\perp\Rightarrow(x^\perp\cap y^\perp)^{\perp\perp}\leqq(x\cup y)^\perp\Rightarrow x^\perp\cap y^\perp\leqq(x\cup y)^\perp,$$

whence $(x\cap y)^\perp=x^\perp\cup y^\perp$. The other statement follows by duality.

Lemma 2. *In every lattice L with unique orthocomplements we have*

$$a\leqq b\Rightarrow a=b\cap(b^\perp\cup a),\quad a, b\in L.$$

Proof. Let $a\leqq b$. Then $b^\perp\leqq a^\perp$ and according to Lemma 1 we get

$$a\cap[b\cap(b^\perp\cup a)]^\perp=a\cap[b^\perp\cup(b\cap a^\perp)]\leqq a\cap[a^\perp\cup(b\cap a^\perp)]=a\cap a^\perp=0,$$

$$a\cup[b\cap(b^\perp\cup a)]^\perp=a\cup[b^\perp\cup(b\cap a^\perp)]=(a\cup b^\perp)\cup(a^\perp\cap b)$$

$$=(a^\perp\cap b)^\perp\cup(a^\perp\cap b)=I,$$

$$[b\cap(b^\perp\cup a)]^\perp=b^\perp\cup(b\cap a^\perp)\leqq a^\perp\cup(b\cap a^\perp)=a^\perp.$$

Hence $[b\cap(b^\perp\cup a)]^\perp$ is a complement of a and also orthogonal to a, and so by the assumption is equal to $a^\perp$.

Lemma 3. *If L is a lattice with unique orthocomplements, then*

$$x\cap y=y\cap[(x\cup y)^\perp\cup x]$$

for each $x, y\in L$.

Proof. Applying Lemma 2 for $a=x$, $b=x\cup y$, we get

$$y\cap[(x\cup y)^\perp\cup x]=[y\cap(x\cup y)]\cap[(x\cup y)^\perp\cup x]=$$

$$=y\cap\{(x\cup y)\cap[(x\cup y)^\perp\cup x]\}=y\cap x.$$

Theorem. *Let L be a lattice with unique orthocomplements. If*

(1) $$(x^{\perp} \cup y = I, y^{\perp} \cup z = I) \Rightarrow x^{\perp} \cup z = I,$$

for each $x, y, z \in L$, *then L is distributive (and thus is a Boolean algebra).*

Proof. First we prove that (1) implies

(2) $$x^{\perp} \cup y = I \Rightarrow x \leqq y \quad \text{for each} \quad x, y \in L.$$

Let $x^{\perp} \cup y = I$, and

$$z = (x \cap y) \cup [x^{\perp} \cap (x \cup y)].$$

Then

$$y^{\perp} \cup z = y^{\perp} \cup \{(x \cap y) \cup [x^{\perp} \cap (x \cup y)]\} =$$
$$= (x \cap y) \cup \{y^{\perp} \cup [x^{\perp} \cap (x \cup y)]\} =$$
$$= (x \cap y) \cup \{y \cap [x \cup (x \cup y)^{\perp}]\}^{\perp},$$

by Lemma 1, hence, by Lemma 3,

$$y^{\perp} \cup z = (x \cap y) \cup (x \cap y)^{\perp} = I.$$

Thus from (1) we obtain $x^{\perp} \cup z = I$, i. e.

$$I = x^{\perp} \cup z = x^{\perp} \cup \{(x \cap y) \cup [x^{\perp} \cap (x \cup y)]\} =$$
$$= (x \cap y) \cup \{x^{\perp} \cup [x^{\perp} \cap (x \cup y)]\} = (x \cap y) \cup x^{\perp}.$$

But

$$(x \cap y) \cap x^{\perp} = (x \cap x^{\perp}) \cap y = 0 \cap y = 0,$$

and

$$(x \cap y) \leqq x = x^{\perp\perp}.$$

Consequently, $x \cap y$ is a complement of $x^{\perp}$ and is orthogonal to it, which means that

$$x \cap y = x^{\perp\perp} = x,$$

i. e.

$$x \leqq y.$$

Thus (2) is proved.

Let us take now $a, b \in L$ arbitrarily and let $x = b$, $y = a^{\perp} \cup (a \cap b)$. Then

$$x^{\perp} \cup y = I,$$

since

$$x^{\perp} \cup y = b^{\perp} \cup [a^{\perp} \cup (a \cap b)] = (b^{\perp} \cup a^{\perp}) \cup (a \cap b) = (a \cap b)^{\perp} \cup (a \cap b) = I.$$

Thus, according to (2), $x \leqq y$ which means

(3) $$b \leqq a^{\perp} \cup (a \cap b)$$

for each $a, b \in L$. Similarly, if $x = c$, $y = a^{\perp} \cup (a \cap c)$, where $a, c \in L$ are taken arbitrarily, then

(4) $$c \leqq a^{\perp} \cup (a \cap c).$$

By making use of (3) and (4), we get

$$a \cap (b \cup c) \leqq a \cap \{[a^{\perp} \cup (a \cap b)] \cup [a^{\perp} \cup (a \cap c)]\} =$$
$$= a \cap \{(a^{\perp} \cup a^{\perp}) \cup [(a \cap b) \cup (a \cap c)]\} =$$
$$= a \cap \{a^{\perp} \cup [(a \cap b) \cup (a \cap c)]\}.$$

Here, the last term is equal to $(a \cap b) \cup (a \cap c)$ by Lemma 2 (since $a \leqq (a \cap b) \cup (a \cap c)$). Therefore $a \cap (b \cup c) \leqq (a \cap b) \cup (a \cap c)$.

The opposite inequality, and the converse of the Theorem is well known. Cf. for instance [12].

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Über äquivalente Variationsprobleme von mehreren Veränderlichen

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§ 1. Einleitung

Im n-dimensionalen Raum X_n $(n \geqq 2)$ sei ein $(n-1)$-parametriges Integral von der Form

$$\mathscr{I}(F) \stackrel{\text{def}}{=} \int_{\mathscr{A}} \cdots \int F\left(x^i(u^\alpha), \frac{\partial x^i}{\partial u^\alpha}\right) du^1 \ldots du^{n-1} \qquad \begin{pmatrix} i = 1, 2, \ldots, n \\ \alpha = 1, 2, \ldots, n-1 \end{pmatrix}$$

angegeben [1]), wo $\mathscr{A}$ ein $(n-1)$-dimensionales Bereich von X_n bedeutet. Wir nehmen an, daß das zu $\mathscr{I}(F)$ gehörige Variationsproblem regulär ist. Die Extremalen $x^i(u^\alpha)$ sind in diesem Falle durch das partielle Differentialgleichungssystem

$$\text{(1.1)} \qquad \mathscr{E}_i(F) \stackrel{\text{def}}{=} \frac{\partial F}{\partial x^i} - \frac{\partial}{\partial u^\alpha} \frac{\partial F}{\partial x^i_\alpha} = 0, \qquad x^i_\alpha \stackrel{\text{def}}{=} \frac{\partial x^i}{\partial u^\alpha}$$

angegeben. Jetzt und im folgenden soll die Einsteinsche Summationskonvention gelten, d. h. auf doppelt vorkommende Indizes soll es immer summiert werden.

Es sei nun auch ein zweites Integral $\mathscr{I}(F^*)$ mit der Grundfunktion F^* angegeben, zu dem auch ein reguläres Variationsproblem gehört. Die beiden Variationsprobleme nennen wir äquivalent, falls die zu ihnen gehörigen Scharen der Extremalen übereinstimmen. Als ein spezielles Problem in dieser Richtung wollen wir den Zusammenhang von F^* und F bestimmen, falls

$$\text{(1.2)} \qquad \mathscr{E}_i(F^*) \equiv \lambda \mathscr{E}_i(F) \qquad (\lambda \not\equiv 0)$$

besteht, wo der Operator $\mathscr{E}_i(F)$ durch (1. 1) angegeben ist, und λ eine Funktion von $x^i(u^\alpha)$ ist. Ein bemerkenswertes Resultat ist, daß (1. 2) für die mögliche Form der Funktion F eine wesentliche Einschränkung gibt, wenn λ nicht eine Konstante ist. Ist aber λ eine Konstante, für die dann wegen

$$\lambda \mathscr{E}_i(F) = \mathscr{E}_i(\lambda F), \qquad \lambda = \text{Konst.}$$

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[1]) Lateinische bzw. griechische Indizes sollen immer die Zahlen $1, 2, \ldots, n$, bzw. $1, 2, \ldots, (n-1)$ durchlaufen.

$\lambda=1$ gesetzt werden kann, so ist F beliebig wählbar, nur die Form von F^* ist jetzt durch (1. 2) bestimmt. Unsere diesbezüglichen Resultate sind im Satz 1 formuliert. Für den Fall, in dem die Grundintegrale $\mathcal{I}(F)$ und $\mathcal{I}(F^*)$ nur von einer Veränderlichen abhängig sind, verweisen wir auf den Aufsatz [4].

In unseren weiteren Untersuchungen werden wir werschiedene Verallgemeinerungen untersuchen. Erstens werden wir den Fall betrachten, in dem die Grundfunktionen F und F^* auch von $\dfrac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$ abhängig sind. In diesem Falle hat der Euler—Lagrangesche Operator $\mathcal{E}_i(F)$ die Form: [2])

$$\mathcal{E}_i(F) \overset{\text{def}}{=} \frac{\partial F}{\partial x^i} - \frac{\partial}{\partial u^\alpha}\frac{\partial F}{\partial x^i_\alpha} + \chi \frac{\partial^2}{\partial u^\alpha \partial u^\beta}\frac{\partial F}{\partial x^i_{\alpha\beta}}, \tag{1. 3}$$

$$x^i_\alpha \overset{\text{def}}{=} \frac{\partial x^i}{\partial u^\alpha}, \quad x^i_{\alpha\beta} \overset{\text{def}}{=} \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad \chi \overset{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{für} \quad \alpha \neq \beta \\ 1 & \text{für} \quad \alpha = \beta. \end{cases}$$

Diesen Fall wollen wir nur kurz behandeln, da — wie wir es sehen werden — in diesem Falle die Relation (1. 2) für F^* nicht so strenge Bedingungen stellt; die Form von F^* wäre nur mit weiteren Bedingungen vollständig bestimmbar. Zweitens betrachten wir solche Differentialoperatoren $\mathcal{E}_i^*(F)$, die nicht unbedingt Euler—Lagrangesche Operatoren einer Funktion F sind, und die Form

$$\mathcal{E}_i^*(F) = a_i^k(x)\mathcal{E}_k(F) \tag{1. 4}$$

haben, wo die $a_i^k(x)$ einen Tensor bilden. Es wird sich zeigen, daß derjenige Fall neue Erweiterungen gibt, in dem der Rang des Matrix (a_i^k) kleiner als n ist.

§ 2. Der Fall $\lambda=\lambda(x)$

Wir nehmen an, daß für zwei Variationsprobleme mit den Grundfunktionen $F(x, x_\alpha)$ und $F^*(x, x_\alpha)$ die Relationen (1. 2) gelten, wo $\lambda=\lambda(x)$ eine allein von x^i abhängige Skalare Funktion ist. Hier und im folgenden setzt man $x=(x^1, x^2, \ldots, x^n)$ bzw. $x_\alpha=(x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)$. Bezüglich der Funktionen F und F^* stellen wir die Bedingung, daß sie nach ihren Veränderlichen mindestens zweimal stetig differenzierbar sind. Die Relation (1. 2) bestimmen wir in der Form:

$$\mathcal{E}_i(F^*) - \lambda(x)\mathcal{E}_i(F) \equiv 0 \tag{2. 1}$$

die auf Grund von (1. 1) offenbar mit

$$\frac{\partial F^*}{\partial x^i} - \lambda\frac{\partial F}{\partial x^i} - x^j_\alpha\left(\frac{\partial^2 F^*}{\partial x^j \partial x^i_\alpha} - \lambda\frac{\partial^2 F}{\partial x^j \partial x^i_\alpha}\right) - x^j_{\alpha\beta}\frac{\partial^2}{\partial x^i_{(\alpha}\partial x^j_{\beta)}}(F^* - \lambda F) \equiv 0 \tag{2. 2}$$

identisch ist. Wie gewöhnlich, bedeuten in dieser Formel die Klammern bei α und β den in α, β symmetrischen Teil des entsprechenden Ausdrucks.

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[2]) Vgl. [1] S. 28.

Da (2. 2) in $x^i, x^i_\alpha, x^i_{\alpha\beta}$ eine Identität ist, muß der Koeffizient von $x^i_{\alpha\beta}$ verschwinden, d. h. es gilt

(2. 3)
$$\frac{\partial^2}{\partial x^i_{(\alpha}\partial x^j_{\beta)}}(F^*-\lambda(x)F)\equiv 0.$$

Wir stellen nun die Forderung:
Es soll die Relation

(2. 4)
$$\frac{\partial^2}{\partial x^i_{[\alpha}\partial x^j_{\beta]}}(F^*-\lambda(x)F)\equiv 0$$

gelten. (Die eckigen Klammern bedeuten den in α, β schiefsymmetrischen Teil.)

Diese Forderung ermöglicht schon F^* durch F auszudrücken. Aus (2. 3) und (2. 4) folgt nämlich

$$\frac{\partial^2}{\partial x^i_\alpha\partial x^j_\beta}(F^*-\lambda(x)F)\equiv 0,$$

also ist $(F^*-\lambda F)$ in x^j_γ linear. Es gilt somit

(2. 5)
$$F^*(x, x_\alpha)-\lambda(x)F(x, x_\alpha) = S^\gamma_j(x)x^j_\gamma+\varphi(x).$$

Aus (2. 1) wird somit nach (2. 5)

(2. 6)
$$\frac{\partial\varphi}{\partial x^i}+\frac{\partial\lambda}{\partial x^i}F-\frac{\partial\lambda}{\partial u^\alpha}\frac{\partial F}{\partial x^i_\alpha}+\left(\frac{\partial S^\alpha_j}{\partial x^i}-\frac{\partial S^\alpha_i}{\partial x^j}\right)x^j_\alpha\equiv 0.$$

Nach partieller Ableitung nach x^k_β wird wegen

$$\frac{\partial\lambda}{\partial u^\alpha}=\frac{\partial\lambda}{\partial x^j}x^j_\alpha$$

die Relation

(2. 7)
$$\frac{\partial\lambda}{\partial x^{[i}}\frac{\partial F}{\partial x^{k]}_\beta}-\frac{1}{2}\frac{\partial\lambda}{\partial u^\alpha}\frac{\partial^2F}{\partial x^i_\alpha\partial x^k_\beta}+\frac{\partial}{\partial x^{[i}}S^\beta_{k]}\equiv 0$$

gelten.

Bilden wir nun den in i, k symmetrischen bzw. schiefsymmetrischen Teil von (2. 7), so erhält man die folgenden beiden Identitäten:

(2. 8)
$$\frac{\partial\lambda}{\partial u^\alpha}\frac{\partial^2F}{\partial x^{(i}_\alpha\partial x^{k)}_\beta}\equiv 0,$$

(2. 9)
$$\frac{\partial\lambda}{\partial x^{[i}}\frac{\partial F}{\partial x^{k]}_\beta}-\frac{1}{2}\frac{\partial\lambda}{\partial u^\alpha}\frac{\partial^2F}{\partial x^{[i}_\alpha\partial x^{k]}_\beta}+\frac{\partial}{\partial x^{[i}}S^\beta_{k]}\equiv 0.$$

Die Gleichungen (2. 6)—(2. 9) *bestimmen also die Form von* F, λ, S_j^γ *und* φ, *wenn nur diese Differentialgleichungen bezüglich diese Größen überhaupt lösbar sind.*

Als Beispiel nehmen wir an, daß F die Form

(2. 10)
$$F(x, x_\alpha)=a^\gamma_j(x)x^j_\gamma+A(x)$$

hat, d. h. F ist in x_α^i linear. Offensichtlich ist dann (2. 8) erfüllt. Die Identität (2. 9) geht in

$$\frac{\partial}{\partial x^{[i}} S^\beta_{k]} = \frac{\partial \lambda}{\partial x^{[k}} a^\beta_{i]} \tag{2. 11}$$

über. Da

$$\frac{\partial S^\beta_k}{\partial x^i} = \frac{\partial}{\partial x^{[i}} S^\beta_{k]} + \frac{\partial}{\partial x^{(i}} S^\beta_{k)}$$

besteht, wird nach (2. 11)

$$\frac{\partial S^\beta_k}{\partial x^i} = \frac{\partial \lambda}{\partial x^{[k}} a^\beta_{i]} + \varphi^\beta_{ik}(x), \tag{2. 12}$$

wo

$$\varphi^\beta_{ki} = \varphi^\beta_{ik} \overset{\text{def}}{=} \frac{\partial}{\partial x^{(i}} S^\beta_{k)}$$

beliebig gewählt werden kann, da wegen der Symmetrie in i, k diese Größen in (2. 6)—(2. 9) nicht vorkommen werden. (2. 12) ist nun ein partielles Differentialgleichungssystem für S^β_k, a^β_k und λ, wo die in i, k symmetrische $\varphi^\beta_{ik}(x)$ noch beliebig gewählt werden können. Aus (2. 6) wird auf Grund von (2. 10) und (2. 11)

$$\frac{\partial \varphi}{\partial x^i} + \frac{\partial \lambda}{\partial x^i} A(x) = 0.$$

Diese Gleichung ist wegen der Willkürlichkeit von $A(x)$ offenbar immer lösbar.

§ 3. Der Fall $\lambda = 1$

Ist in unserem Fundamentalproblem (1. 2) λ eine Konstante, so kann — wie wir das schon in der Einleitung bemerkt haben — $\lambda = 1$ gesetzt werden.

Ist aber $\lambda = 1$, so entsteht ein Spezial-Fall des im vorigen Paragraphen behandelten Typs. Die Gleichung (2. 8) ist offenbar identisch erfüllt. Aus (2. 9) bekommt man, daß die S^β_k $(n-1)$ Gradientvektoren bestimmen; es ist:

$$S^\beta_k(x) = \frac{\partial S^\beta(x)}{\partial x^k} \tag{3. 1}$$

und aus (2. 6) bekommt man, daß $\varphi =$ Konstante ist. (2. 5) bestimmt jetzt die Fundamentalfunktion F^* in Hinsicht auf (3. 1) in der Form

$$F^*(x, x_\alpha) = F(x, x_\alpha) + \frac{\partial S^\gamma(x(u))}{\partial u^\gamma} + \varphi \tag{3. 2}$$

wo φ eine Konstante ist. Damit haben wir den Zusammenhang von F und F^* bestimmt, da falls (2. 7) gültig ist, dann auch (1. 2) erfüllt ist.

Wir wollen noch darauf hinweisen, daß falls (3. 2) besteht, so ist die Übereinstimmung der Variationsprobleme

$$\delta \int_{\mathscr{A}} \dots \int F^* \, du = 0 \quad \text{und} \quad \delta \int_{\mathscr{A}} \dots \int F \, du = 0, \quad du \overset{\text{def}}{=} du^1 \dots du^{n-1}$$

trivial, da jetzt nach (2. 7) mittels des Stokesschen Satzes

$$\int_{\mathscr{A}} \dots \int F^* \, du^1 \dots du^{n-1} = \int_{\mathscr{A}} \dots \int F \, du^1 \dots du^{n-1} +$$

$$+ \int_{\partial \mathscr{A}} \dots \int \sum_{\beta=1}^{n-1} (-1)^{\beta-1} S^\beta \, du^1 \dots du^{\beta-1} \, du^{\beta+1} \dots du^{n-1} + \text{Konst.}$$

besteht, wo die Konstante aus $\int \dots \int \varphi \, du$ entstanden ist und $\partial \mathscr{A}$ die Grenze von $\mathscr{A}$ bedeutet. (Für den Stokesschen Satz vgl. z. B. [2], S. 64—66.)

Unsere bisherigen Resultate fassen wir im folgenden Satz zusammen.

Satz 1. *Gelten die Relationen* (1. 2) *und* (2. 4), *so ist der Zusammenhang von F und F^* durch* (2. 5) *bestimmt, wo die Funktionen λ, S_j^γ, F und φ den Relationen* (2. 6), (2. 8) *und* (2. 9) *genügen müssen. Ist in* (1. 2) $\lambda = 1$, *so ist der Zusammenhang von F und F^* durch* (3. 2) *bestimmt, wo $S^\gamma(x)$ beliebige Funktionen von x^i sind und φ eine Konstante bedeutet.*

§ 4. Verallgemeinerungen

Eine der einfachsten Verallgemeinerungen ist die Annahme, daß die Grundfunktionen auch von den $x^i_{\alpha\beta}$ abhängig sind. In diesem Falle ist der Euler—Lagrangesche Operator durch (1. 3) festgelegt. Ist λ eine Konstante, so kann (1. 2) in der Form

$$(4.\,1) \qquad \frac{\partial}{\partial x^i}(F^* - \lambda F) - \frac{\partial}{\partial u^\alpha}(F^*_{x^i_\alpha} - \lambda F_{x^i_\alpha}) + \chi \frac{\partial^2}{\partial u^\alpha \partial u^\beta}(F^*_{x^i_{\alpha\beta}} - \lambda F_{x^i_{\alpha\beta}}) \equiv 0$$

geschrieben werden. Diesen Fall wollen wir nur nur skizzieren, da — wie wir es sehen werden — die Relation (4. 1) die Form von F^*, wegen der Symmetrie von $x^i_{\alpha\beta\gamma\delta}$ und $x^i_{\alpha\beta\gamma}$ in den $\alpha, \beta, \gamma, \delta$, nicht bestimmt. Berechnen wir in (4. 1) die partiellen Ableitungen nach u^α und u^β, so zeigt sich, daß die linke Seite von (4. 1) ein Polynom von $x^i_{\alpha\beta\gamma\delta}$ und $x^i_{\alpha\beta\gamma}$ ist. Offensichtlich ist dieses Polynom in $x^i_{\alpha\beta\gamma}$ von zweitem Grade.

Das Verschwinden der Koeffizienten von $x^i_{\alpha\beta\gamma\delta}$ bzw. $x^i_{\alpha\beta\gamma} x^j_{\delta\varepsilon\eta}$ gibt zwei Relationen, in denen die homogenen linearen Ausdrücke der Größen von der Form

$$(4.\,2) \qquad \frac{\partial^2 (F^* - \lambda F)}{\partial x^i_{\alpha\beta} \, \partial x^j_{\gamma\delta}}$$

bzw.

$$(4.\,3) \qquad \frac{\partial^3 (F^* - \lambda F)}{\partial x^i_{\alpha\beta} \, \partial x^j_{\gamma\delta} \, \partial x^k_{\varepsilon\eta}}$$

vorkommen. Diese Relationen sind sicher erfüllt, falls z.B. $(F^* - \lambda F)$ in $x^i_{\alpha\beta}$ linear ist, d.h. F^* hat die Form:

$$F^*(x, x_\alpha) = \lambda F(x, x_\alpha) + \chi S^{\alpha\beta}_j(x, x_\gamma) x^j_{\alpha\beta} + \varphi(x, x_\alpha), \tag{4. 4}$$

$$S^{\alpha\beta}_j(x, x_\gamma) \equiv S^{\beta\alpha}_j(x, x_\gamma).$$

$S^{\alpha\beta}_j$ und φ sind nicht beliebig angebbar, da (4. 1) identisch erfüllt sein muß. Es wird somit

$$\chi \frac{\partial S^{\alpha\beta}_j}{\partial x^i} x^j_{\alpha\beta} + \frac{\partial \varphi}{\partial x^i} - \frac{\partial}{\partial u^\gamma}\left(\chi \frac{\partial S^{\alpha\beta}_j}{\partial x^i_\gamma} x^j_{\alpha\beta} + \frac{\partial \varphi}{\partial x^i_\gamma}\right) + \chi \frac{\partial^2 S^{\alpha\beta}_i}{\partial u^\alpha \partial u^\beta} \equiv 0. \tag{4. 5}$$

Da der Koeffizient von $x^j_{\alpha\beta\gamma}$ verschwinden muß, bekommt man die Relation

$$\chi\left(\frac{\partial S^{\alpha\beta}_i(x, x_\delta)}{\partial x^j_\gamma} - \frac{\partial S^{\alpha\beta}_j(x, x_\delta)}{\partial x^i_\gamma}\right) x^j_{\alpha\beta\gamma} \equiv 0. \tag{4. 6}$$

Diese Relation ist erfüllt, falls z.B. $S^{\alpha\beta}_i$ die Form:

$$S^{\alpha\beta}_i(x, x_\delta) = S^{\alpha\beta\varrho}_{ik}(x) x^k_\varrho + \psi^{\alpha\beta}_i(x)$$

hat, und die $S^{\alpha\beta\varrho}_{ik}$ in i, k und in α, β symmetrisch sind. Die linke Seite von (4. 5) wird somit ein Polynom von zweitem Grade in $x^i_{\alpha\beta}$, dessen Koeffizienten aber selbstverständlich auch verschwinden müssen. Das wird für $S^{\alpha\beta}_j$ bzw. φ noch mehrere Bedingungen geben, die wir aber nicht explizit berechnen wollen.

Kurz zusammenfassend können wir behaupten, daß (4. 1) *erfüllt ist, falls F^* die Form* (4. 4) *hat, und* (4. 5) *besteht.*

Aus den bisherigen Untersuchungen ist ersichtlich, daß die von uns verwandte Methode im Wesentlichen nicht für äquivalente Variationsprobleme, sondern für äquivalente partielle Differentialgleichungssysteme benützt wurde. Deshalb wollen wir im folgenden solche äquivalente partielle Differentialgleichungssysteme untersuchen, die nicht die Euler—Lagrangeschen Differentialgleichungssysteme eines Variationsproblems sind.

Wir nehmen an, daß $\mathscr{E}^*_k(F)$ den Differentialoperator

$$\mathscr{E}^*_k(F) \overset{\text{def}}{=} a^i_k(x) F_{x^i} - \sum_{\varrho=1}^{n-1} \underset{(\varrho)}{b^i_k}(x) \frac{\partial}{\partial u^\varrho} F_{x^i_\varrho} \tag{4. 7}$$

bedeutet. Im folgenden wollen wir untersuchen, was für eine Form F^* hat, falls

$$\mathscr{E}^*_k(F^*) \equiv \lambda\, \mathscr{E}^*_k(F), \quad \lambda = \text{Konstante} \tag{4. 8}$$

besteht. Offenbar sind auf Grund von (4. 8) die Lösungshyperflächen der partiellen Differentialgleichungssysteme

$$\mathscr{E}^*_k(F) = 0 \quad \text{und} \quad \mathscr{E}^*_k(F^*) = 0$$

identisch. Wir beweisen das folgende

Lemma. *Ist $\mathscr{E}_k^*(F)$ für jeden Skalar F ein kovarianter Vektor, so gilt*

(4. 9) $$a_k^i = \underset{(\varrho)}{b_k^i} \qquad (\varrho = 1, \ldots, n-1),$$

und $\mathscr{E}_k^*(F)$ hat somit die Form:

(4. 10) $$\mathscr{E}_k^*(F) = a_k^i(x)\mathscr{E}_i(F).$$

Beweis. Es sei $\bar{x}^i = \bar{x}^i(x)$ eine mindestens zweimal stetig differenzierbare Koordinatentransformation mit von Null verschiedener Jacobischer Determinante. Da nach unserer Annahme $\mathscr{E}_k^*(F)$ ein kovarianter Vektor ist, gilt

(4. 11) $$\overline{\mathscr{E}_k^*}(\bar{F}) = \frac{\partial x^i}{\partial \bar{x}^k}\mathscr{E}_i^*(F),$$

wo

(4. 11a) $$\overline{\mathscr{E}_k^*}(\bar{F}) = \bar{a}_k^i \frac{\partial \bar{F}}{\partial \bar{x}^i} - \sum_{\varrho=1}^{n-1} \underset{(\varrho)}{\bar{b}_k^i} \frac{\partial}{\partial u^\varrho}\frac{\partial \bar{F}}{\partial \bar{x}_\varrho^i}$$

bedeutet. Nun ist F ein Skalar, d. h.

$$\bar{F}(\bar{x}, \bar{x}_\alpha) = F(x, x_\alpha),$$

woraus die Transformationsformeln

$$\frac{\partial \bar{F}}{\partial \bar{x}^i} = \frac{\partial F}{\partial x^l}\frac{\partial x^l}{\partial \bar{x}^i} + \frac{\partial F}{\partial x_\varrho^l}\frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^i}\bar{x}_\varrho^j,$$

$$\frac{\partial}{\partial u^\varrho}\frac{\partial \bar{F}}{\partial \bar{x}_\varrho^i} = \left(\frac{\partial}{\partial u^\varrho}\frac{\partial F}{\partial x_\varrho^l}\right)\frac{\partial x^l}{\partial \bar{x}^i} + \frac{\partial F}{\partial x_\varrho^l}\frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^i}\bar{x}_\varrho^j$$

folgen. — Bei der Herleitung dieser Formeln haben wir auch die Transformationsformeln

$$x_\beta^t = \frac{\partial x^t}{\partial \bar{x}^m}\bar{x}_\beta^m$$

benützt.

Substituieren wir nun diese Größen in (4. 11a), und dann $\overline{\mathscr{E}_k^*}(\bar{F})$ in (4. 11), so muß der Koeffizient von $\partial^2_{\bar{x}^j\bar{x}^i} x^l$ wegen der Willkürlichkeit der Transformation $\bar{x}^i = \bar{x}^i(x)$ verschwinden. Das gibt

$$\sum_{\varrho=1}^{n-1} (\bar{a}_k^i - \underset{(\varrho)}{\bar{b}_k^i})\frac{\partial F}{\partial x_\varrho^l}\bar{x}_\varrho^j = 0.$$

Es war aber F auch beliebig, somit muß $\bar{a}_k^i = \underset{(\varrho)}{\bar{b}_k^i}$ bestehen. Offenbar muß diese letzte Relation in jedem Koordinatensystem gelten, da $\bar{x}^i$ ein beliebiges Koordinatensystem bestimmt, woraus die Relation (4. 9) folgt. Aus (4. 7) wird man aber nach (4. 9) im Hinblick auf (1. 1) unmittelbar (4. 10) bekommen, w.z.b.w.

Im folgenden können wir uns also auf den Typ (4. 10) beschränken. Aus (4. 8) bekommt man das Differentialglechungssystem

(4. 12) $$a_k^i(x)\big(\mathscr{E}_i(F^*) - \lambda\mathscr{E}_i(F)\big) \equiv 0.$$

Ist der Rang der Determinante $|a_k^i|$ gleich n, so ist (4. 12) und somit auch (4. 8) mit (1. 2) identisch. Nehmen wir also an, daß der Rang r von $|a_k^i|$ kleiner als n ist. Dann kann (4. 12) in der Form

$$\mathscr{E}_i(F^*)-\lambda\mathscr{E}_i(F)=\sum_{t=r+1}^{n} A_{it}B_t \qquad (i=1,2,\ldots,r) \tag{4. 13}$$

geschrieben werden, wobei

$$A_{it}\overset{\text{def}}{=}\begin{vmatrix} a_1^1\ldots a_1^{i-1}\,a_1^t\,a_1^{i+1}\ldots a_1^r \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_r^1\ldots a_r^{i-1}\,a_r^t\,a_r^{i+1}\ldots a_r^r \end{vmatrix}, \qquad \begin{pmatrix} i=1,\ldots,r \\ t=r+1,\ldots,n \end{pmatrix}$$

bedeutet, und

$$B_t(x,x_\alpha,x_{\alpha\beta})\overset{\text{def}}{=}\mathscr{E}_t(F^*)-\lambda\mathscr{E}_t(F), \qquad (t=r+1,\ldots,n)$$

eine beliebige Funktion von x^i, x_α^i und ein linearer Ausdruck von $x_{\alpha\beta}^i$ ist. Beachten wir noch, daß λ eine Konstante ist, so ist (4. 13) mit

$$\mathscr{E}_i(F^*-\lambda F)=\sum_{t=r+1}^{n} A_{it}B_t, \qquad (i=1,\ldots,r) \tag{4. 14}$$

identisch.

Ist nun

$$\Phi=\Phi(x,x_\alpha)$$

die allgemeine Lösung von

$$\mathscr{E}_i(\Phi)\equiv\sum_{t=r+1}^{n} A_{it}B_t, \qquad (i=1,\ldots,r), \tag{4. 15}$$

wo die B_t beliebige vorgegebene, in $x_{\alpha\beta}^i$ lineare Funktionen bedeuten, so ist nach (4. 14)

$$F^*-\lambda F=\Phi.$$

Wir können somit den folgenden Satz behaupten:

Satz 2. *Ist der Rang r von $|a_k^i|$ kleiner als n, und gilt* (4. 8), *wo $\mathscr{E}_k^*$ durch* (4. 10) *festgelegt ist, so hat F^* die Form*

$$F^*(x,x_\alpha)=\lambda F(x,x_\alpha)+\Phi(x,x_\alpha),$$

wobei Φ dem Differentialgleichungssystem (4. 15) *genügt.*

Bemerkung. Da die B_t $(t=r+1,\ldots,n)$ auch von $x_{\alpha\beta}^i$ abhängig sind, müssen die Koeffizienten von $x_{\alpha\beta}^j$ in (4. 15) verschwinden. Die Funktion Φ ist von $x_{\alpha\beta}^j$ unabhängig, und $\mathscr{E}_i(\Phi)$ ist in $x_{\alpha\beta}^j$ linear.

Zum Schluß wollen wir einen sehr wichtigen Satz beweisen, welcher zeigt, daß wie stark die Relation (1. 2) die Funktionen F und F^* miteinander verbindet.

Satz 3. *Ist in* (1. 2) $\lambda=1$, *ist* (2. 4) *gültig und ist die Dimensionszahl des Grundraumes: $n>2$, ferner sind die Grundintegrale $\mathscr{I}(F)$ und $\mathscr{I}(F^*)$ parameterinvariant, so folgt aus* (1. 2) $F^*(x,x_\alpha)\equiv F(x,x_\alpha)$.

Beweis. Aus dem Parameterinvarianz der Grundintegrale $\mathscr{I}(F)$ und $\mathscr{I}(F^*)$ folgt (vgl. [5], Gleichung (1. 8)):

$$\frac{\partial F}{\partial x^i_\alpha} x^i_\beta = \delta^\alpha_\beta F, \quad \frac{\partial F^*}{\partial x^i_\alpha} x^i_\beta = \delta^\alpha_\beta F^*.$$

Nach einer Verjüngung bezüglich α und β folgt:

$$\frac{\partial F}{\partial x^i_\alpha} x^i_\alpha = (n-1)F, \quad \frac{\partial F^*}{\partial x^i_\alpha} x^i_\alpha = (n-1)F^*. \tag{4. 16}$$

Beachten wir jetzt, daß nach unserem Satz 1 F^* die Form (3. 2) hat, so folgt aus (4. 16) $\varphi=0$, und wegen

$$\frac{\partial S^\gamma(x(u))}{\partial u^\gamma} \equiv \frac{\partial S^\gamma(x)}{\partial x^j} x^j_\gamma$$

gilt die Relation:

$$\frac{\partial S^\alpha(x)}{\partial x^i} x^i_\alpha = (n-1)\frac{\partial S^\alpha(x)}{\partial x^j} x^j_\alpha,$$

woraus, in Hinsicht auf $n>2$, wird:

$$\frac{\partial S^\alpha}{\partial u^\alpha} \equiv \frac{\partial S^\alpha(x)}{\partial x^i} x^i_\alpha = 0.$$

Aus (3. 2) folgt dann die Behauptung des Satzes 3.

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Bemerkung zur Divergenz der Fourierreihen

Von KÁROLY TANDORI in Szeged

Die n-te Partialsumme der Fourierreihe einer Funktion $f(x) \in L(0, 2\pi)$ setzen wir in der Form $s_n = s_n' + s_n''$ an, mit

$$s_n'(f; x) = \frac{1}{\pi} \int_0^\pi f(x+t) \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt, \quad s_n''(f; x) = \frac{1}{\pi} \int_0^\pi f(x-t) \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt.$$

Wir beweisen den folgenden

Satz. *Es sei* $\{\lambda_n\}$ *eine positive Zahlenfolge mit* $\lambda_n \to \infty$ *und* $\lambda_n = o(\log n)$. *Dann gibt es eine stetige, nach* 2π *periodische Funktion* $f(x)$ *derart, daß*

$$\overline{\lim_{n\to\infty}}\, \lambda_n^{-1} |s_n'(f; x)| > 0 \quad \textit{und} \quad \overline{\lim_{n\to\infty}}\, \lambda_n^{-1} |s_n''(f; x)| > 0 \tag{1}$$

fast überall bestehen.

Beweis. Nach dem Satz von L. CARLESON [1]) konvergiert die Folge $\{s_n(f; x)\}$ im Falle $f(x) \in L^2(0, 2\pi)$ fast überall; es ist also nur die erste der Ungleichungen (1) für eine stetige, nach 2π periodische Funktion $f(x)$ zu beweisen. Weiterhin, nach dem Riemann—Lebesgueschen Lemma gilt

$$s_n'(f; x) = \frac{1}{\pi} \int_0^\pi f(x+t) \frac{\sin nt}{t}\,dt + o(1) = s_n^*(f; x) + o(1)$$

für jedes x, darum ist es genügend eine stetige, nach 2π periodische Funktion $f(x)$ anzugeben, für die

$$\overline{\lim_{n\to\infty}}\, \lambda_n^{-1} |s_n^*(f; x)| > 0 \tag{2}$$

fast überall gilt.

Aus der Relation

$$\lim_{\alpha\to\infty} \int_0^\delta \frac{\sin \alpha t}{t}\,dt = \frac{\pi}{2} \quad (\delta > 0)\,^{2)} \tag{3}$$

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[1]) L. CARLESON, On convergence and growth of partial sums of Fourier series, *Acta Math.*, **166** (1966), 135—157.

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und aus dem Riemann—Lebesgueschen Lemma erhalten wir:

$$\lim_{\alpha\to\infty}\int_0^\pi \frac{\cos nt}{t}\sin \alpha t\,dt=\frac{\pi}{2},\quad \lim_{\alpha\to\infty}\int_0^\pi \frac{\sin nt}{t}\sin \alpha t\,dt=0 \qquad (n=1,2,\ldots).$$

Auf Grund dieser Relationen und der Annahme $\lambda_n=o(\log n)$ können wir eine Indexfolge $(1\leqq)\,n(1)<\ldots<n(k)<\ldots$ angeben, für die

$$\text{(4)}\qquad \left|\int_0^\pi \frac{\cos n(k)t}{t}\sin n(l)t\,dt\right|\leqq 2,\quad \left|\int_0^\pi \frac{\sin n(k)t}{t}\sin n(l)t\,dt\right|\leqq 2 \qquad (k\neq l),$$

$$\text{(5)}\qquad \frac{\lambda_{n(k)}}{\log n(k)}\leqq\frac{1}{k^2}\qquad (k=1,2,\ldots),$$

$$\text{(6)}\qquad n(k)|n(l)\qquad (k<l)$$

gelten. Wir setzen:

$$f(x)=\sum_{k=1}^\infty \frac{\sin n(k)x}{k^2}.$$

Offensichtlich ist $f(x)$ eine stetige, nach 2π periodische Funktion und gilt

$$s^*_{n(k_0)}(f;x)=\frac{1}{\pi}\sum_{k=1}^\infty\frac{1}{k^2}\int_0^\pi\frac{\sin n(k)(x+t)}{t}\sin n(k_0)t\,dt.$$

Durch eine einfache Rechnung ergibt sich

$$s^*_{n(k_0)}(f;x)=\frac{1}{\pi}\sum_{k=1}^{k_0-1}\frac{1}{k^2}\cdot$$

$$\cdot\left(\sin n(k)x\int_0^\pi\frac{\cos n(k)t}{t}\sin n(k_0)t\,dt+\cos n(k)x\int_0^\pi\frac{\sin n(k)t}{t}\sin n(k_0)t\,dt\right)+$$

$$+\frac{1}{\pi k_0^2}\left(\frac{\sin n(k_0)x}{2}\int_0^\pi\frac{\sin 2n(k_0)t}{t}\,dt+\cos n(k_0)x\int_0^\pi\frac{\sin^2 n(k_0)t}{t}\,dt\right)+\frac{1}{\pi}\sum_{k=k_0+1}^\infty\frac{1}{k^2}\cdot$$

$$\cdot\left(\sin n(k)x\int_0^\pi\frac{\cos n(k)t}{t}\sin n(k_0)t\,dt+\cos n(k)x\int_0^\pi\frac{\sin n(k)t}{t}\sin n(k_0)t\,dt\right).$$

Daraus, auf Grund von (3) und (4) folgt

$$\text{(7)}\qquad |s^*_{n(k_0)}(f;x)|\geqq\frac{1}{\pi k_0^2}|\cos n(k_0)x|\int_0^\pi\frac{\sin^2 n(k_0)t}{t}\,dt-\frac{4}{\pi}\sum_{k=1}^\infty\frac{1}{k^2}.$$

Da

$$\int_0^\pi\frac{\sin^2 n(k_0)t}{t}\,dt\geqq a\log n(k_0)$$

mit einer positiven absoluten Konstante a gilt, erhalten wir aus (5) und (7)

$$\lambda_{n(k_0)}^{-1} |s_{n(k_0)}^*(f; x)| \geqq b \qquad \left(x \in \left(l\frac{\pi}{n(k_0)} - \frac{\pi}{2n(k_0)}, l\frac{\pi}{n(k_0)} + \frac{\pi}{2n(k_0)}\right); 0 \leqq l < 2n(k_0)\right)$$

für genügend großes k_0, wobei b eine positive, absolute Konstante bezeichnet. Es sei

$$E(k) = \bigcup_{l=0}^{2n(k)-1} \left(l\frac{\pi}{n(k)} - \frac{\pi}{2n(k)}, l\frac{\pi}{n(k)} + \frac{\pi}{2n(k)}\right).$$

Da $\operatorname{mes}(E(k)) = \pi$ $(k = 1, 2, \ldots)$ ist, und die Mengen $E(k)$ wegen (6) stochastisch unabhängig sind, ergibt sich durch Anwendung des zweiten Borel—Cantellischen Lemmas $\operatorname{mes}(\overline{\lim}_{k\to\infty} E(k)) = 2\pi$. Also gilt

$$\lambda_{n(k)}^{-1} |s_{n(k)}^*(f; x)| > b$$

für unendlich viele k fast überall.

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Sums of operators with square zero*)

By P. A. FILLMORE in Bloomington (Indiana, U.S.A.)

Let $\mathfrak{H}$ be a separable infinite-dimensional complex Hilbert space. There are a number of results concerning generators in various senses for certain spaces of operators (bounded linear transformations) on $\mathfrak{H}$. For example, a von Neumann algebra is the linear span of its unitary elements [2, p. 4], the algebra of all operators on $\mathfrak{H}$ is generated as an algebra by its elements of square zero [3], and a von Neumann algebra with no abelian summand is generated as an algebra by its projections [5]. One of the most striking is the result of STAMPFLI [7] asserting that every operator on $\mathfrak{H}$ is the sum of eight idempotents. The purpose of this note is to show that STAMPFLI's theorem implies (in an elementary fashion) the following:

Theorem 1. *Every operator on $\mathfrak{H}$ is a sum of* 64 *operators with square zero.*

Theorem 2. *Every operator on $\mathfrak{H}$ is a linear combination of* 257 *projections.*

The author wishes to acknowledge many helpful and stimulating conversations with DAVID TOPPING, who also provided the proof of Lemma 3.

Several preliminary lemmas are necessary. An operator A is an *idempotent* if $A^2=A$, an (orthogonal) *projection* if $A^2=A$ and $A^*=A$, and an *involution* if $A^2=I$. We recall that P is an idempotent if and only if $2P-I$ is an involution. For any operator A with null-space $\mathfrak{N}=\text{null } A$ we write $\nu(A)=\dim \mathfrak{N}$. If the range $\mathfrak{R}=\text{ran } A$ is closed we write $\varrho(A)=\dim \mathfrak{R}$.

Lemma 1. *If P is an idempotent with $\nu(P)=\varrho(P)$ then the corresponding involution $S=2P-I$ is the sum of two operators with square zero.*

Proof. Since every idempotent is similar to a projection, the hypothesis implies easily that if $\mathfrak{K}$ is a separable infinite-dimensional Hilbert space, P is similar to the operator $\begin{pmatrix} I & O \\ O & O \end{pmatrix}$ on the space $\mathfrak{K}\oplus\mathfrak{K}$. Since $U=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is unitary, P is also similar to $U^*\begin{pmatrix} I & O \\ O & O \end{pmatrix}U=\frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$. Consequently, S is similar to $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$. But $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$ is the sum of $\begin{pmatrix} O & I \\ O & O \end{pmatrix}$ and $\begin{pmatrix} O & O \\ I & O \end{pmatrix}$, each having square zero, and the lemma follows.

*) Research supported in part by a grant from the National Science Foundation.

Lemma 2. *An idempotent P is either the sum or the difference of idempotents Q_1 and Q_2 such that* $\nu(Q_i)=\varrho(Q_i)$ $(i=1,2)$.

Proof. Suppose first that P is a projection. If $\varrho(P)=\infty$, then P is the sum of orthogonal projections Q_1 and Q_2 with $\varrho(Q_i)=\infty$; clearly then $\nu(Q_i)=\infty$. If $\varrho(P)<\infty$, then $\varrho(I-P)=\nu(P)=\infty$, so $I-P=Q_1+Q_2$ as above, and $P=(P+Q_1)-Q_1$ meets our requirements. Since any idempotent is similar to a projection, the lemma follows.

Proof of Theorem 1. Let A be an operator. By Stampfli's theorem and Lemma 2 we have $\frac{1}{2}A=\sum_{i=1}^{16}\pm P_i$, where each P_i is an idempotent with $\nu(P_i)=\varrho(P_i)$. But $P_i=\frac{1}{2}(S_i+I)$, where the involution S_i is the sum of two operators with square zero (by Lemma 1). Hence A is the sum of 32 operators with square zero and an integer multiple of I. Temporarily taking $A=\frac{1}{2}I$, we find that I is itself the sum of 32 operators with square zero. Consequently A is the sum of 64 operators with square zero.

Corollary 1. *Any operator is a sum of commutators.*

Proof.
$$\begin{pmatrix} O & A \\ O & O \end{pmatrix}=\begin{pmatrix} O & I \\ O & O \end{pmatrix}\begin{pmatrix} O & O \\ O & A \end{pmatrix}-\begin{pmatrix} O & O \\ O & A \end{pmatrix}\begin{pmatrix} O & I \\ O & O \end{pmatrix}.$$

This result was obtained by Halmos [6], and also follows from the recent description of commutators by Brown and Pearcy [1].

Corollary 2. *If K and L are any operators, there exist decompositions*

$$K=K_1K_1'+\dots+K_nK_n' \quad \textit{and} \quad L=L_1L_1'+\dots+L_nL_n' \qquad (n\leqq 64)$$

such that

$$K_i'K_i=L_i'L_i \qquad (i=1,\dots,n).$$

Proof. To begin with, any operator on $\mathfrak{H}\oplus\mathfrak{H}$ of the form $\begin{pmatrix} AXE & AXF \\ CXE & CXF \end{pmatrix}$ has square zero, provided $EA+FC=O$. Conversely, any operator on $\mathfrak{H}\oplus\mathfrak{H}$ with square zero is similar to an operator $\begin{pmatrix} O & X \\ O & O \end{pmatrix}$, and consequently has the above form $\left(\text{multiply on the left by} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and on the right by its inverse } \begin{pmatrix} G & H \\ E & F \end{pmatrix}\right)$.

Applying Theorem 1 to the operator $\begin{pmatrix} K & O \\ O & -L \end{pmatrix}$, we find that $K=A_1X_1E_1+\dots$ $\dots+A_nX_nE_n$ and $L=-C_1X_1F_1-\dots-C_nX_nF_n$ $(n\leqq 64)$ with $E_iA_i+F_iC_i=O$. Taking $L_i=A_i$, $L_i'=X_iE_i$, $K_i=-C_i$, and $K_i'=X_iF_i$, we have $L_i'L_i=X_iE_iA_i=$ $=-X_iF_iC_i=K_i'K_i$.

Lemma 3. *Any operator of the form* $\begin{pmatrix} O & A \\ O & O \end{pmatrix}$ *on* $\mathfrak{K}\oplus\mathfrak{K}$ *is a linear combination of 8 projections and I.*

Proof. The operator A is a linear combination of two self-adjoint contractions, each of which is a linear combination of two unitary operators [2, p. 4]. But

$$\begin{pmatrix} O & U \\ O & O \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I & U \\ U^* & I \end{pmatrix} + \frac{i}{2}\begin{pmatrix} I & -iU \\ iU^* & I \end{pmatrix} - \frac{1}{2}(1+i)I,$$

and the matrices on the right are easily seen to be multiples of projections when U is unitary.

Proof of Theorem 2. Let A be an operator with square zero. To apply Lemma 3 we need to know that A is unitarily equivalent to an operator of the form $\begin{pmatrix} O & B \\ O & O \end{pmatrix}$ on a space $\mathfrak{K} \oplus \mathfrak{K}$. The hypothesis implies that ran $A \subset$ null A and that $\nu(A) = \infty$. Therefore there exists a closed subspace $\mathfrak{K}$ between ran A and null A such that $\dim \mathfrak{K} = \dim \mathfrak{K}^\perp$. Let U be a unitary operator from $\mathfrak{K}^\perp$ onto $\mathfrak{K}$, and define W from $\mathfrak{H}$ onto $\mathfrak{K} \oplus \mathfrak{K}$ by $Wx = y \oplus Uz$, where $x = y + z$ with $y \in \mathfrak{K}$ and $z \in \mathfrak{K}^\perp$. It is easy to see that W is unitary and that $WAW^* = \begin{pmatrix} O & B \\ O & O \end{pmatrix}$, where $B = AU^{-1}|\mathfrak{K}$. Lemma 3 now implies that any operator of square zero is a linear combination of 8 projections and I. But the proof of Theorem 1 shows that any operator is a sum of 32 operators of square zero and a multiple of I. Combining these statements completes the proof.

Remarks. 1. The numbers mentioned in the theorems are undoubtedly not the best possible, but we have not resolved this question.

2. Theorem 1 and both Corollaries fail if $\dim \mathfrak{H} < \infty$, but Theorem 2 remains true. If, on the other hand, STAMPFLI's theorem is valid for nonseparable spaces (as seems likely), then so are the above results.

3. If either A or B is invertible, then AB and BA are similar. Thus the relation of Corollary 2 is an attenuated form of similarity.

4. The above results probably persist relative to a von Neumann algebra with no summand of finite type.

5. It is easy to see that a self-adjoint operator is a real linear combination of projections. However, it is not true that every positive operator is a linear combination of projections with positive coefficients. In fact, let A be positive and compact, and suppose $A = \lambda_1 P_1 + \ldots + \lambda_n P_n$ with the $\lambda_i > 0$ and the P_i projections. Then $\lambda_i P_i \leqq A$ for each i, so that rng $P_i \subset$ rng $\sqrt{A}$ by [4]. Since $\sqrt{A}$ is compact, this implies that each P_i is finite-dimensional, and consequently so is A.

6. The real Banach space $\mathfrak{S}$ of all self-adjoint operators has the following curious property: it is the linear span of the extreme points of its unit ball $\mathfrak{U}$, but $\mathfrak{U}$ is not the convex hull of its extreme points. This is because $\mathfrak{U}$ is affinely equivalent to its positive part $\mathfrak{P}$ (by $U \to \frac{1}{2}(U+I)$), and the preceding remark shows that $\mathfrak{S}$ is the linear span of the extreme points of $\mathfrak{P}$, but that $\mathfrak{P}$ is not the convex hull of its extreme points.

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Positive definite contraction valued functions on locally compact abelian groups *)

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Introduction

In this paper we will discuss two separate yet related problems. In § 1 we ask under what conditions would the minimal unitary dilations of a positive definite contraction valued function on the LCA groups be unitary equivalent to the sum of many copies of the regular representation of the group. In § 2 we study the relationships between positive definiteness, von Neumann condition and Heinz condition for certain contraction valued functions in case of ordered groups. We are only interested in complex Hilbert spaces. We denote them by H, K, etc.; $B(H)$ (or $B(K)$) will be the algebra of bounded linear operators on H (or K). All topological spaces will be Hausdorff, and the notation $L^p(X, \Omega, \mu, C)$ for the set X, field Ω of subsets of X, measure μ and the complex field C will be as on p. 121 of [2].

§ 1

To study the first problem we make use of a new construction of the minimal unitary dilation of positive definite contraction valued function on LCA groups. For this purpose we need the following notations.

1. 1. Definition. Let $E(\cdot)$ be a bounded additive positive $B(H)$-valued set function defined on a field of subsets of a set X. If

$$f(x) = \sum_{l=1}^{r} \alpha_l \chi_{A_l}, \quad \Phi(x) = \sum_{i=1}^{m} h_i \chi_{D_i} \quad \text{and} \quad \Phi'(x) = \sum_{j=1}^{n} h'_j \chi_{D'_j}$$

are simple functions where α_l's are complex numbers, h_i, h'_j are in H and χ_ω denotes the characteristic function of the set ω in Ω, and A_l, D_i, D'_j are in Ω, then we define

$$\int_\omega f(x)(E(dx)\Phi(x), \Phi'(x)) = \sum_{i,j,l} \alpha_l (E(\omega \cap A_l \cap D_i \cap D'_j) h_i, h'_j)$$

whenever ω is in Ω. It is easily verified that this is independent of the representations

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of f, Φ, Φ'. Now suppose f is a bounded measurable complex valued function and Φ, Φ' are bounded measurable functions on X with values in a finite-dimensional subspace H_1 of H, choose a sequence f_n of simple complex valued functions converging to f uniformly on X, and sequences Φ_n, Φ'_n of simple measurable H_1-valued functions converging uniformly on X to Φ and Φ' respectively. For ω in Ω we then define

$$\int_\omega f(x)(E(dx)\Phi(x), \Phi'(x)) = \lim_{n\to\infty} \int_\omega f_n(x)(E(dx)\Phi_n(x), \Phi'_n(x)).$$

With standard argument it can be shown that this is independent of the choices of $H_1, f_n, \Phi_n, \Phi'_n$. Furthermore,

$$\int_\omega f(x)(E(dx)\Phi(x), \Phi'(x)) = \int_\omega (E(dx)f(x)\Phi(x), \Phi'(x)) =$$

$$= \int_\omega (E(dx)\Phi(x), \overline{f(x)}\,\Phi'(x)).$$

If we denote $\int_X (E(dx)\Phi(x), \Phi'(x))$ by $\langle \Phi, \Phi' \rangle$ then $\langle \Phi, \Phi' \rangle$ is a positive definite Hermitian form.

Using the above notations we now give a new proof for LCA groups, to a theorem of Sz.-Nagy which we need later.

1.2. Theorem. *Every weakly continuous positive definite $B(H)$-valued function $\{T_\gamma\}$ on a LCA group Γ with $T_0 = I$ (0 is the identity of Γ) has a minimal unitary dilation $\{U_\gamma, K\}$.*

Proof. Let G be the dual group of Γ and $E(\cdot)$ the $B(H)$-valued set function on the Borel sets of G such that $T_\gamma = \int_G (x, \gamma)E(dx)$ [9]. Let D be the set of all H-valued bounded measurable functions with finite-dimensional range. If Φ, Ψ are in D, define $\langle \Phi, \Psi \rangle$ to be $\int_G (E(dx)\Phi(x), \Psi(x))$. Thus D is a linear manifold with $\langle \Phi, \Psi \rangle$ as a positive definite scalar product (see Definition 1. 1). Denote by N the linear subspace of D consisting of those Φ for which $\langle \Phi, \Phi \rangle = 0$. Denote D/N by K_0 and the coset $\Phi + N$ in K_0 by $[\Phi]$. Then $\langle [\Phi], [\Psi] \rangle = \langle \Phi, \Psi \rangle$ is well-defined on K_0 so that K_0 is an inner product space and its completion K is a Hilbert space. Define U_γ on K_0 by $U_\gamma[\Phi] = [\Psi]$ where $\Psi(x) = (x, \gamma)\Phi(x)$. It is easily verified that the map is independent of the choice of coset representatives and is in fact an isometry of K_0 onto itself. Thus U_γ extends by continuity to a unitary transformation of K (which we also denote by U_γ). Evidently $\{U_\gamma\}$ is a unitary representation of Γ. Given any two elements Φ, Ψ in D let H_1 be the finite-dimensional subspace of H generated by the ranges of Φ and Ψ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of H_1. Since $(E(\cdot)e_i, e_j)$ is a finite regular Borel measure, so we have

$$\int_G (x, \gamma)(E(dx)\Phi(x), \Psi(x)) = \sum_{i,j=1}^{n} \int_G (x, \gamma)(\Phi(x), e_i)\overline{(\Psi(x), e_j)}(E(dx)e_i, e_j)$$

which is a continuous function of γ for $[\Phi]$, $[\Psi]$ in K_0. By the uniform boundedness of $\{U_\gamma\}$, $\{U_\gamma\}$ is weakly continuous on Γ. Embed H in K by mapping h in H to $[\Phi_h]$ in K_0 where $\Phi_h(x)=h$ for all x in G. Obviously this is a linear and isometric embedding. For arbitrary h, h' in H we have

$$(U_\gamma[\Phi_h], [\Phi_{h'}]) = \int_G (x, \gamma)(E(dx)h, h') = (T_\gamma h, h')$$

so $PU_\gamma P = T_\gamma$ where P is the projection from K onto H. Finally a standard argument would show that the elements of the form $U_\gamma[\Phi_h]$ where h in H generate K, so $\{U_\gamma, K\}$ is a minimal unitary dilation.

For the remainder of this section we shall assume H is separable. If Γ is σ-compact (in particular if Γ is the integer group or the group of real numbers) this is only a slight restriction since in this case H is an orthogonal direct sum of separable subspaces each of which is invariant under T_γ for all γ in Γ as we shall see later (Remark 1. 8).

Suppose that H is separable. Denote the dual group of Γ by G and the Haar measure of G by σ. Suppose there exists a positive $B(H)$-valued function $M(\cdot)$ on G such that for any Borel set ω of G and any h, h' in H we have

$$(E(\omega)h, h') = \int_\omega (M(x)h, h')\sigma(dx).$$

Set $H(x)=\overline{M(x)H}=\overline{M(x)^{1/2}H}$. Then $x\to H(x)$ is a field of Hilbert spaces on G. Define the unitary operator S_γ on the direct integral space

$$\mathbf{H} = \int^{\oplus} H(x)\sigma(dx)$$

by $(S_\gamma\xi)(x)=(x, \gamma)\xi(x)$ where ξ is in $\mathbf{H}$. We first establish

1. 3. Theorem. $\{\mathbf{H}, S_\gamma\}$ *is unitarily equivalent to the minimal unitary dilation* $\{U_\gamma, K\}$ *of* $\{T_\gamma, H\}$.

Proof. We use the same notations as in the proof of theorem 1. 2. Define W_0 on D into $\mathbf{H}$ by $(W_0\Phi)(x)=M(x)^{1/2}\Phi(x)$ for Φ in D. It is easy to verify that

$$\int_G \|M(x)^{1/2}\Phi(x)\|^2\sigma(dx) = \langle\Phi, \Phi\rangle$$

so W_0 is a linear isometry map from D into $\mathbf{H}$. We claim that the range of W_0 is dense in $\mathbf{H}$. Let $\{g_i(\cdot)|\, i=1, 2, \dots\}$ be a measurable field of orthonormal bases, so

$$g_i(x) = \sum_{j=1}^{m_i} c_j^i(x)\big(M(x)^{1/2}e_j\big)$$

where $c_j^i(x)$ are complex valued measurable functions. Suppose ξ in $\mathbf{H}$ and $\varepsilon>0$ are given. Then $\xi(x)=\sum_{i=1}^{k}\alpha_i(x)g_i(x)$ where $\alpha_i(x)=(\xi(x), g_i(x))$ is measurable.

By the monotone convergence theorem we can find an integer k such that if we define $\xi_1(x)=\sum_{i=1}^{\infty}\alpha_i(x)g_i(x)$ then

$$\int_G \|\xi(x)-\xi_1(x)\|^2\sigma(dx) = \int_G\left(\sum_{i=k+1}^{\infty}|\alpha_i(x)|^2\right)\sigma(dx) < \varepsilon^2/4$$

or $\|\xi-\xi_1\|<\varepsilon/2$. A similar argument shows that there is a positive constant C such that if we define $\xi_2(x)=\xi_1(x)$ whenever $|c_j^i(x)|\leqq C$ and $|\alpha_i(x)|\leqq C$ for $j\leqq m_i$ and $i\leqq k$ and we define $\xi_2(x)=0$ otherwise, then $\|\xi_2-\xi_1\|<\varepsilon/2$ so $\|\xi_2-\xi\|<\varepsilon$. Now define $\eta(x)=\sum_{i=1}^{k}\alpha_i(x)\cdot\sum_{j=1}^{m_i}c_j^i(x)e_i$ if $\xi_2(x)\neq 0$ and $\eta(x)=0$ if $\xi_2(x)=0$. It follows easily that η is in D and $W_0\eta=\xi_2$ so W_0D is dense in $\mathbf{H}$. Now W_0 induces an isometry of K_0 onto a dense subspace of $\mathbf{H}$ which extends by continuity to a unitary map W of K onto $\mathbf{H}$ and clearly $WU_\gamma=S_\gamma W$ so W is a unitary equivalence between two representations of Γ.

Now we are going to answer partly the first problem mentioned earlier.

1.4. Theorem. *Under the hypotheses of* 1.3, *the minimal unitary dilation of* $\{T_\gamma, H\}$ *is unitarily equivalent to the sum of* d_0 *copies of the regular representation of* Γ *iff* $\dim(H(x))=d_0$ *for almost all* x *with respect to* σ.

Proof. Assume that $\dim(H(x))=d_0$ for almost all x. Without loss of generality we may assume $\dim(H(x))=d_0$ for all x. Let $\{g_i(\cdot)|\, i=1,2,\ldots\}$ be a measurable field of orthonormal bases for $H(x)$. We map ξ in $\mathbf{H}$ to the element $(\alpha_1,\alpha_2,\ldots)$ in the direct sum of d_0 copies of $L^2(G,\Omega,\sigma,C)$ such that $\alpha_i(x)=(\xi(x),g_i(x))$. It can be verified that this gives a unitary equivalence between $\{\mathbf{H}, S_\gamma\}$ and the sum of d_0 copies of regular representation of Γ. By 1.3 it now follows that the minimal unitary dilation of $\{T_\gamma, H\}$ is unitarily equivalent to the sum of d_0 copies of regular representation of Γ. For the converse now assume that the minimal unitary dilation of $\{T_\gamma, H\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of Γ. Since H is assumed to be separable, if $\{U_\gamma, K\}$ denotes the minimal unitary dilation it follows that K is countably generated. That is there is a countable subset of K such that the closed subspace invariant under all U_γ generated by this countable set is the whole K. Therefore the regular representation of Γ is countably generated and thus G is σ-compact and its Haar measure σ is σ-finite. For a measurable set ω in G define $A(\omega)$ on $\mathbf{H}$ by $A(\omega)\xi(x)=\chi_\omega(x)\xi(x)$. ($\chi_\omega$ is the characteristic function of ω.) It is readily seen that $A(\cdot)$ is the spectral family given by STONE's theorem for the representation $\{S_\gamma\}$ of Γ, so $A(\omega)$ is in the weakly closed algebra of operators generated by the $\{S_\gamma\}$. Now G may be decomposed into disjoint measurable sets ω_p for $p=1,2,\ldots$ such that $\dim(H(x))=p$ if x is in ω_p. Denote $A(\omega_p)$ by A_p. Define the representation $\{V_{\gamma,p}\}$ as follows: $V_{\gamma,p}$ acts in the space $L^2(\omega_p,\Omega_p,\sigma,C)$ (where Ω_p is the σ-field of Borel subsets of ω_p) and is such that $(V_{\gamma,p}\zeta)(x)=(x,\gamma)\zeta(x)$. The argument used in the first part of the proof will also show that the representation $\{A_pS_\gamma A_p\}$ on the range of A_p is unitarily equivalent to the sum of p copies of the regular representation $\{V_{\gamma,p}\}$. Now since G is σ-finite, so is ω_p, hence there exists a nowhere vanishing L^2 function on ω_p and we conclude that the representation $\{V_{\gamma,p}\}$ is cyclic, and thus locally simple (see p. 42 of [3]). Thus corresponding to the decomposition

$$I = A_\infty + A_1 + A_2 + \dots$$

we obtain a decomposition

$$\{S_\gamma\} = \infty\{V_{\gamma,\infty}\} + 1\{V_{\gamma,1}\} + 2\{V_{\gamma,2}\} + \dots .$$

Since the A_i's are in the weak closure of the $\{S_\gamma\}$, they are in the center of the algebra of intertwining operators, so this decomposition is the unique decomposition of $\{S_\gamma\}$ (see p. 40 of [3]). It follows that $\{S_\gamma\}$ is unitarily equivalent to the sum of d_0 copies of a locally simple representation only if the only non-trivial term in this decomposition is the one for $p = d_0$. Thus $\dim (H(x)) = d_0$ for almost all x.

The following results are the applications of this theorem to some special cases. The first is a result of Sz.-Nagy and Foiaş (see p. 125 of [8]) but we give a different proof here.

1. 5. Theorem. *Let T be a completely non-unitary contraction of H and suppose the intersection of the spectrum of T with the unit circle is a set of measure zero. Define $T_n = T^n$ for $n \geq 0$ and $T_n = T^{*(-n)}$ for $n < 0$ and let $\{U_n\}$ be its minimal unitary dilation. Then $\{U_n\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of the integer group where $d_0 = \dim \overline{(I - T^*T)H} = \dim \overline{(I - TT^*)H}$.*

Proof. Denote the intersection of the spectrum of T with the unit circle by ω_0. Since ω_0 has Lebesgue measure zero, by Theorem 2 of [7] we have $E(\omega_0) = O$ where $E(\cdot)$ is the $B(H)$-valued set function on the circle such that

$$T_n = \int_0^{2\pi} e^{in\theta} E(d\theta).$$

For $z = re^{i\theta}$ $(r < 1)$ define

$$M(z) = \sum_{m=-\infty}^{\infty} r^{|m|} e^{-im\theta} T_m = \operatorname{Re} [(I + \bar{z}T)(I - \bar{z}T)^{-1}].$$

It is obvious that $(M(z)h, h')$ is the Poisson integral of the measure $(E(\cdot)h, h')$ for h, h' in H. If F is any closed interval on the unit circle which does not intersect ω_0, we notice that $M(z)$ has a harmonic extension $\operatorname{Re} [(I + \bar{z}T)(I - \bar{z}T)^{-1}]$ to a neighborhood of F, denote this extension by $\tilde{M}(z)$. Using Fatou's theorem it then follows that if ω is any measurable subset of F,

$$(E(\omega)h, h') = \int_\omega (\tilde{M}(z)h, h')\sigma(dz) \quad \text{for } h, h' \text{ in } H.$$

If we extend $\tilde{M}(z)$ to the whole circle by defining $\tilde{M}(z) = O$ if z is in ω_0, then from the fact that $E(\omega_0) = O$ it follows that $(E(\omega)h, h') = \int_\omega (M(z)h, h')\alpha(dz)$ for any measurable set ω on the circle. Now for z on the circle but not in ω_0, we have

$$\tilde{M}(z) = (I - zT^*)^{-1}(I - T^*T)(I - \bar{z}T)^{-1} = (I - \bar{z}T)^{-1}(I - TT^*)(I - zT^*)^{-1}.$$

Thus $\dim \overline{\tilde{M}(z)H} = \dim \overline{(I - T^*T)H} = \dim \overline{(I - TT^*)H} = d_0$ for almost all z in the circle group. Using theorem 1. 4 we then have the desired conclusion.

1. 6. Theorem. *Let Γ be the group of real numbers and $\{T_t\}$ a weakly continuous positive definite $B(H)$-valued function on Γ with $T_0 = I$. Assume that $\limsup_{t\to\infty} \|T_t\|^{1/t} < 1$. Then the minimal unitary dilation $\{U_t\}$ of $\{T_t\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of Γ where $d_0 \leqq \dim H$.*

Proof. Define $M(z)$ by

$$\big(M(z)h, h'\big) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (e^{-izt} T_t h, h')\, dt.$$

Then for

$$|\mathrm{Im}\, z| < -\log\big(\limsup_{t\to+\infty} \|T_t\|^{1/t}\big)$$

$M(z)$ exists and is a bounded operator, a positive operator if z is a real number and $\big(M(z)h, h'\big)$ is an analytic function for every h, h' in H. If G is the dual group of Γ (that is G is the group of real numbers) and if $E(\cdot)$ is the $B(H)$-valued set function on G such that $T_t = \int_{-\infty}^{\infty} e^{ist} E(ds)$, from the inversion theorem on Fourier transform we have $\big(E(\omega)h, h'\big) = \int_{\omega} \frac{1}{\sqrt{2\pi}} \big(M(s)h, h'\big) ds$ for any Borel set ω of G and h, h' in H. We claim that $\dim \overline{M(s)H}$ is a constant for almost all real numbers with respect to Lebesgue measure. In fact let s_0 be a real number and n an integer such that $\dim \overline{M(s_0)H} \geqq n$. Select elements $h_1, h_2, \ldots, h_n$ in H so that $M(s_0)h_1$, $M(s_0)h_2, \ldots, M(s_0)h_n$ are linearly independent vectors. Consider the Gram determinant of $M(s)h_1$, $M(s)h_2, \ldots, M(s)h_n$. This is a real analytic function of s which does not vanish at s_0, hence it vanishes for at most a countable set. Thus $M(s)h_1, \ldots$ $\ldots, M(s)h_n$ are linearly independent except for at most countably many values of s. From the separability of H it follows that $\dim \overline{M(s)H}$ is constant almost everywhere, and the conclusion follows from Theorem 1. 4.

Instead of analytic functions on the real line we may consider analytic functions on the circle. So we get

1. 7. Theorem. *Let Γ be the group of integers and $\{T_n\}$ a positive definite $B(H)$-valued function on Γ such that $T_0 = I$. Assume that $\limsup \|T_n\|^{1/n} < 1$. Then the unitary dilation of $\{T_n\}$ is unitarily equivalent to the sum of d_0 copies of the regular representation of the integer group.*

1. 8. Remark. If Γ is σ-compact, many of above discussions can be applied for non-separable H. If $\{T_\gamma\}$ is positive definite with $\{U_\gamma\}$ its minimal unitary dilation, observe that since $\{U_\gamma\}$ is strongly continuous, so is $\{T_\gamma\}$. Thus if h is an element of H, the set of $\{T_\gamma h \mid \gamma \in \Gamma\}$ is a σ-compact subset of the metric space H, hence separable and so generate a separable subspace. From this we conclude by a standard argument that the smallest subspace containing h and invariant under all T_γ is separable. Using Zorn's Lemma and standard argument we can show that H may be written as a direct sum of subspaces H_α each of which is separable and invariant under all T_γ. If $T_{\gamma,\alpha}$ denotes the restriction of T_γ to H_α and if $\{U_{\gamma,\alpha}\}$ is the minimal unitary dilation of $\{T_{\gamma,\alpha}\}$, it is easily verified that the minimal uni-

tary dilation of $\{T_\gamma\}$ is the direct sum of $\{U_{\gamma,\alpha}\}$. Thus the results obtained in separable case are also true for general H provided that Γ is σ-compact. In particular, 1. 5, 1. 6, 1. 7 can be extended to the non-separable case in this manner.

§ 2

In 1951 VON NEUMANN proved that if T is a contraction, and for every analytic function $f(z)=\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} |a_n|<\infty$ and $|f(z)|\leqq 1$ for $|z|\leqq 1$, we define $f(T)$ to be $\sum_{n=0}^{\infty} a_n T^n$, then we have $\|f(T)\|\leqq 1$. Later E. HEINZ proved that if instead of $|f(z)|\leqq 1$, we require that $\operatorname{Re} f(z)\geqq 0$ for $|z|\leqq 1$, then for any h in H we have $\operatorname{Re}(f(T)h, h)\geqq 0$. From this result HEINZ could give an easy proof of the von Neumann theorem. In [6] SZ.-NAGY showed that the von Neumann and Heinz theorems follow easily from the existence of unitary dilations. Here we shall exhibit the relationship between the positive definiteness, the von Neumann theorem and the Heinz theorem in a more general setting. As both theorems depend on positive elements of the integer group, we shall only consider ordered LCA groups from now on. If Γ is an ordered group, Γ^+ will denote the set of all non-negative elements of Γ, Σ the σ-field of Borel sets of Γ and ϱ the Haar measure of Γ.

2. 1. Definition. Let A_1 denote the set of all functions ξ in $L^1(\Gamma, \Sigma, \varrho, C)$ such that $\xi(\gamma)=0$ for $\gamma<0$. With the L^1-norm and the convolution as multiplication A_1 is a Banach algebra. (See p. 380 of [1].) Define a norm $|||\ |||$ on A_1 by $|||\xi|||=\|\hat{\xi}\|_\infty$ where $\hat{\xi}$ is the Fourier transform of ξ in A_1 and $\|\ \|_\infty$ is the sup norm. Denote the completion of A_1 in the norm $|||\ |||$ by A_0.

2. 2. Definition. Let $\mathcal{T}=\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. For ξ in A_1 define $\xi(\{T_\gamma\})$ to be $\int_\Gamma \xi(\gamma)T_\gamma\,\varrho(d\gamma)$ where the integral is taken in the weak sense. It is clear that $\|\xi(\{T_\gamma\})\|\leqq\|\xi\|_1$.

2. 3. Remark. Let $\{T_\gamma\}$ be the same as in 2. 2 and in addition positive definite. Let $E(\cdot)$ be the $B(H)$-valued set function defined on the dual group G of Γ such that $T_\gamma=\int_G (x, \gamma)E(dx)$. For any bounded Borel measurable complex valued function φ on G, $\int_G \varphi(x)E(dx)$ defines an element of $B(H)$ which will be denoted by $\varphi(\mathcal{T})$. If $\{U_\gamma, K\}$ is the minimal unitary dilation of $\{T_\gamma, H\}$ and $F(\cdot)$ is the spectral measure of $\{U_\gamma\}$, then the map from φ to $\varphi(\mathcal{U})=\int_G \varphi(x)F(dx)$ is a homomorphism of the B^*-algebra of bounded Borel functions on G (under pointwise multiplication) into the B^*-algebra $B(K)$. If P is the projection from K onto H, so that $PU_\gamma P=T_\gamma$ and $PF(\cdot)P=E(\cdot)$, then $\varphi(\mathcal{T})=P\varphi(\mathcal{U})P$. If ξ is in A_1, $\hat{\xi}$ is complex valued bounded measurable (in fact continuous) so $\hat{\xi}(\mathcal{T})=\int_G \hat{\xi}(x)E(dx)$ is defined. From FUBINI's theorem it follows that $\hat{\xi}(\mathcal{T})=\xi(\{T_\gamma\})$.

2. 4. Definition. Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. We say that $\{T_\gamma\}$ satisfies the von Neumann condition if ξ in A_1 implies $\|\xi(\{T_\gamma\})\|\leqq|||\xi|||=\|\hat{\xi}\|_\infty$. We say that $\{T_\gamma\}$ satisfies the Heinz condition if ξ in A_1 and $\operatorname{Re}\hat{\xi}(x)\geqq 0$ for all x in G imply that $\operatorname{Re}(\xi(\{T_\gamma\})h, h)\geqq 0$ for all h in H.

2. 5. Proposition. *If $\{T_\gamma\}$ is a $B(H)$-valued weakly continuous positive definite function on Γ with $T_0=I$, then $\{T_\gamma\}$ satisfies both the von Neumann condition and the Heinz condition.*

Proof. Let $E(\cdot)$ be the $B(H)$-valued set function on the dual group G such that $T_\gamma=\int_G (x, \gamma)E(dx)$. Suppose $\{U_\gamma, K\}$ is the minimal unitary dilation of $\{T_\gamma, H\}$ and $F(\cdot)$ is the spectral measure of $\{U_\gamma\}$. For an arbitrary element ξ in A_1, we have $\left\|\int_G \hat{\xi}(x)F(dx)\right\|\leqq\|\hat{\xi}\|_\infty$ (see p. 900 of [2]). Thus

$$\left\|\int_G \hat{\xi}(x)E(dx)\right\|\leqq\|\hat{\xi}\|_\infty, \quad \text{so} \quad \|\xi(\{T_\gamma\})\|\leqq|||\xi|||$$

(see 2. 3). Hence $\{T_\gamma\}$ satisfies the von Neumann condition. Next suppose that $\operatorname{Re}\hat{\xi}(x)\geqq 0$ for all x in G. For h in H we have

$$\operatorname{Re}(\xi(\{T_\gamma\})h, h)=\operatorname{Re}\int_G \hat{\xi}(x)(E(dx)h, h)=\int_G \operatorname{Re}\hat{\xi}(x)(E(dx)h, h)$$

which is non-negative because $\operatorname{Re}\hat{\xi}(x)\geqq 0$ and $E(\omega)$ is a positive operator for every ω in Ω (σ-field of Borel subsets of G). Thus $\{T_\gamma\}$ satisfies the Heinz condition.

2. 6. Proposition. *Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. If $\{T_\gamma\}$ satifies the Heinz condition, $\{T_\gamma\}$ is positive definite.*

Proof. For ζ in $L^1(\Gamma, \Sigma, \varrho, C)$ define $\tilde{\zeta}$ by $\tilde{\zeta}(\gamma)=\overline{\zeta(-\gamma)}$ for γ in Γ. It is easily verified that $(\zeta*\tilde{\zeta})(\gamma)=\overline{(\zeta*\tilde{\zeta})(-\gamma)}$. Define ξ by

$\xi(\gamma)=2(\zeta*\tilde{\zeta})(\gamma)$ if $\gamma>0$,

$\xi(\gamma)=0$ if $\gamma<0$, and

$\xi(0)=(\zeta*\tilde{\zeta})(0)$.

It is readily seen that ξ is in A_1 and for x in G, $\operatorname{Re}\hat{\xi}(x)=\hat{\zeta}(x)\hat{\tilde{\zeta}}(x)=|\hat{\zeta}(x)|^2\geqq 0$. Using Heinz condition we get $\operatorname{Re}(\xi(\{T_\gamma\})h, h)\geqq 0$ for every h in H. Thus $(\operatorname{Re}\xi(\{T_\gamma\})h, h)\geqq 0$ where

$$\operatorname{Re}\xi(\{T_\gamma\})=1/2[\xi(\{T_\gamma\})+\xi(\{T_\gamma\})^*]=$$

$$=1/2\int_\Gamma [\overline{\xi(-\gamma)}+\xi(\gamma)]T_\gamma\varrho(d\gamma)=\int_\Gamma (\zeta*\tilde{\zeta})(\gamma)T_\gamma\varrho(d\gamma).$$

(All integrals are taken in the weak sense.) Therefore $\int_\Gamma (\zeta*\tilde{\zeta})(\gamma)(T_\gamma h, h)\varrho(d\gamma)\geqq 0$, that is the bounded continuous function $(T_\gamma h, h)$ on Γ is an integral positive definite function and so equal locally almost everywhere to a continuous positive definite function (see p. 397 of [4]). Thus from the continuity of $(T_\gamma h, h)$ it follows that it is positive definite [9]. Hence $\{T_\gamma\}$ is positive definite.

2. 7. Proposition. *Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ with $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. If $\{T_\gamma\}$ satisfies the von Neumann condition, $\{T_\gamma\}$ is positive definite.*

Proof. Denote by α the map from ξ in A_1 to $\xi(\{T_\gamma\})$. Clearly α is linear and norm decreasing. Since $\{T_\gamma\}$ satisfies the von Neumann condition $\|\alpha(\xi)\| \leqq |||\xi||| = \|\hat{\xi}\|_\infty$. Thus α extends by continuity to A_0. For h in H we define the linear functional $\tilde{h}$ on A_0 by $\tilde{h}(\eta)=(\alpha(\eta)h, h)$ for η in A_0. Since α is norm decreasing, $\|\tilde{h}\| \leqq \|h\|^2$. Thus by the Hahn—Banach and Riesz representation theorems there is a measure μ_h on the Borel sets of G such that for ξ in A_1, $\tilde{h}(\xi)=\int_G \hat{\xi}(x)\mu_h(dx)$ and μ_h has total variation at most $\|h\|^2$. We claim that $\mu_h(G)=\|h\|^2$ from which it will follow that μ_h is a positive measure. Since Γ is ordered, it is metric (see p. 196 of [5]) and thus satisfies the first axiom of countability. Choose a decreasing sequence $\{N_k\}$ of compact neighborhoods of 0 in Γ which form a neighborhoods base at 0. Define φ_k by

$$\varphi_k(\gamma)=1/\varrho(N_k\cap\Gamma^+) \quad \text{if} \quad \gamma\in N_k\cap\Gamma^+,$$

$$\varphi_k(\gamma)=0 \quad \text{otherwise.}$$

Clearly $\varphi_k\in A_1$ and $\|\varphi_k\|=1$. Furthermore it is readily verified that $\varphi_k(x)\to 1$ for every x in G as $k\to\infty$. Thus by the dominated convergence theorem,

$$\lim_{k\to\infty}\int_G \hat{\varphi}_k(x)\mu_h(dx) = \mu_h(G).$$

On the other hand,

$$\lim_{k\to\infty}\int_G \hat{\varphi}_k(x)\mu_h(dx) = \lim_{k\to\infty}\int_\Gamma \varphi_k(\gamma)(T_\gamma h, h)\varrho(d\gamma) = \|h\|^2.$$

Thus μ_h is a positive measure. Since it is clear that for ξ in A_1,

$$\int_\Gamma \xi(\gamma)(T_\gamma h, h)\varrho(d\gamma) = \int_\Gamma \xi(\gamma)\hat{\mu}_h(\gamma)\varrho(d\gamma)$$

so we conclude from the continuity of $(T_\gamma h, h)$ and $\hat{\mu}_h(\gamma)$ that they are equal for γ in Γ^+. Since $T_{-\gamma}=T_\gamma^*$ and $\hat{\mu}_h(-\gamma)=\bar{\hat{\mu}}_h(\gamma)$ it follows that $(T_\gamma h, h)=\hat{\mu}_h(\gamma)$ for all γ in Γ. Thus $(T_\gamma h, h)$ is the Fourier—Stieltjes transform of the positive measure μ_h, so is positive definite, that is, $\{T_\gamma\}$ itself is positive definite.

2. 8. Theorem. *Let $\{T_\gamma\}$ be a weakly continuous contraction valued function on Γ such that $T_0=I$ and $T_{-\gamma}=T_\gamma^*$. Then the following three statements are equivalent:* (1) *$\{T_\gamma\}$ is positive definite.* (2) *$\{T_\gamma\}$ satisfies the von Neumann condition.* (3) *$\{T_\gamma\}$ satisfies the Heinz condition.*

Proof. Propositions 2. 5, 2. 6, 2. 7.

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On the spectrum of unitary ϱ-dilations

By E. DURSZT in Szeged

We shall consider operators T in a Hilbert space $\mathfrak{H}$, which admit *unitary ϱ-dilations* ($\varrho > 0$), i. e. unitary operators U defined on some Hilbert space $\mathfrak{K}(\supset \mathfrak{H})$, such that

(1) $$T^n \varphi = \varrho P U^n \varphi \qquad (\varphi \in \mathfrak{H};\ n = 1, 2, \ldots),$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Obviously, (1) implies

(1*) $$T^{*n} \varphi = \varrho P U^{-n} \varphi.$$

B. SZ.-NAGY and C. FOIAŞ have characterized the operators T which have unitary ϱ-dilations; see [4].

We shall denote by $\sigma(U)$ the spectrum of U. In the case $\varrho = 1$, $\sigma(U)$ has been studied extensively, cf. in particular [1], [2], [3]. The purpose of this paper is to study the spectral properties of U for arbitrary $\varrho > 0$.

First we recall some definitions, familiar in the case $\varrho = 1$, but which extend immediately to the general case too.

The unitary ϱ-dilation U of T is *minimal*, if

(2) $$\mathfrak{K} = \bigvee_{n=-\infty}^{\infty} U^n \mathfrak{H}.$$

In this case U is uniquely determined up to isomorphism. (The proof is similar to that given in the special case $\varrho = 1$, cf. [$*$].)

Let E_ϑ ($0 \leqq \vartheta \leqq 2\pi$) be the spectral function of U. We say that the spectral measure of U is *absolutely continuous* if, for every vectors $\varphi, \psi \in \mathfrak{K}$, the function $(E_\vartheta \varphi, \psi)$ of ϑ is absolutely continuous on $0 \leqq \vartheta \leqq 2\pi$, i. e., if there exists a function $f_{\varphi,\psi}(\vartheta) \in L(0, 2\pi)$ such that

(3) $$(E_\vartheta \varphi, \psi) = \int_0^\vartheta f_{\varphi,\psi}(\tau)\, d\tau.$$

T is called completely non-unitary, if there exists no vector $\varphi \in \mathfrak{H}$, $\varphi \neq 0$, for which

$$\ldots = \|T^{*2}\varphi\| = \|T^*\varphi\| = \|\varphi\| = \|T\varphi\| = \|T^2\varphi\| = \ldots$$

We shall use the following notations:

4) $$\mathfrak{L}_n = U^n \overline{(U-T)\mathfrak{H}}, \quad \mathfrak{L}_n^* = U^{*n} \overline{(U^*-T^*)\mathfrak{H}} \qquad (n = 0, \pm 1, \ldots),$$

(5) $$\mathfrak{L} = \bigvee_{n=-\infty}^{\infty} \mathfrak{L}_n, \quad \mathfrak{L}^* = \bigvee_{n=-\infty}^{\infty} \mathfrak{L}_n^*.$$

We denote by Q the orthogonal projection of $\mathfrak{L}$ onto $\bigvee_{n=1}^{\infty} \mathfrak{L}_n$, and by Q' the orthogonal projection of $\mathfrak{L}^*$ onto $\bigvee_{n=1}^{\infty} \mathfrak{L}_n^*$. Further, we set

$$\mathfrak{V} = \overline{Q\mathfrak{L}_0}, \quad \mathfrak{V}^* = \overline{Q'\mathfrak{L}_0^*}, \tag{6}$$

$$\mathfrak{W} = \overline{(I-Q)\mathfrak{L}_0}, \quad \mathfrak{W}^* = \overline{(I-Q')\mathfrak{L}_0^*}. \tag{7}$$

Lemma 1. *$\mathfrak{L}_n \perp \mathfrak{L}_k$ and $\mathfrak{L}_n^* \perp \mathfrak{L}_k^*$ if $n-k \geqq 2$.*

Proof. (4) shows that it sufficies to prove

$$\left.\begin{array}{ll} (8) & (U^m(U-T)\varphi, (U-T)\psi) = 0 \\ (8^*) & (U^{*m}(U^*-T^*)\varphi, (U^*-T^*)\psi) = 0 \end{array}\right\} \quad \text{if} \quad m = 2, 3, \ldots,$$

where φ and ψ are arbitrary vectors in $\mathfrak{H}$.

In order to prove (8), we use (1):

$$(U^m(U-T)\varphi, (U-T)\psi) = (U^m\varphi, \psi) - (U^{m-1}T\varphi, \psi) - (U^{m+1}\varphi, T\psi) + (U^mT\varphi, T\psi) =$$

$$= \left(\frac{1}{\varrho}T^m\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{m-1}T\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{m+1}\varphi, T\psi\right) + \left(\frac{1}{\varrho}T^mT\varphi, T\psi\right) = 0.$$

Similarly, using (1*) we get (8*) as follows:

$$(U^{*m}(U^*-T^*)\varphi, (U^*-T^*)\psi) =$$

$$= (U^{-m}\varphi, \psi) - (U^{-m+1}T^*\varphi, \psi) - (U^{-m-1}\varphi, T^*\psi) + (U^{-m}T^*\varphi, T^*\psi) =$$

$$= \left(\frac{1}{\varrho}T^{*m}\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{*m-1}T^*\varphi, \psi\right) - \left(\frac{1}{\varrho}T^{*m+1}\varphi, T^*\psi\right) + \left(\frac{1}{\varrho}T^{*m}T^*\varphi, T^*\psi\right) = 0.$$

Lemma 2. *$\mathfrak{V}$, $\mathfrak{V}^*$, $\mathfrak{W}$ and $\mathfrak{W}^*$ are wandering subspaces for U. I. e., if $n \neq k$ then*

$$U^n\mathfrak{V} \perp U^k\mathfrak{V}, \quad U^n\mathfrak{V}^* \perp U^k\mathfrak{V}^*,$$

$$U^n\mathfrak{W} \perp U^k\mathfrak{W}, \quad U^n\mathfrak{W}^* \perp U^k\mathfrak{W}^*.$$

Proof. It suffices to prove that

$$U^mQ\mathfrak{L}_0 \perp Q\mathfrak{L}_0, \quad U^mQ'\mathfrak{L}_0^* \perp Q'\mathfrak{L}_0^*, \tag{9}$$

$$U^m(I-Q)\mathfrak{L}_0 \perp (I-Q)\mathfrak{L}_0, \quad U^m(I-Q')\mathfrak{L}_0^* \perp (I-Q')\mathfrak{L}_0^*. \tag{10}$$

for $m = 1, 2, \ldots$

In order to prove (9) choose arbitrary vectors $\varphi = Q\varphi'$ and $\psi = Q\psi'$ ($\varphi', \psi' \in \mathfrak{L}_0$). We have

$$(U^m\varphi, \psi) = (U^mQ\varphi', Q\psi') = (QU^mQ\varphi', \psi').$$

Now, $Q\varphi'$ is an element of $\bigvee_{n=1}^{\infty} \mathfrak{L}_n$, hence $QU^mQ\varphi'$ is an element of $\bigvee_{n=m+1}^{\infty} \mathfrak{L}_n$. Thus, by Lemma 1, the last inner product equals 0. This implies the first part of (9), and we can prove its second part in a similar way.

Now, every vector in $(I-Q)\mathfrak{L}_0$ has the form $\varphi=\varphi'-\varphi''$, where $\varphi'\in\mathfrak{L}_0$ and $\varphi''\in\bigvee_{n=1}^{\infty}\mathfrak{L}_n$, thus we have $U^m\varphi\in\bigvee_{n=1}^{\infty}\mathfrak{L}_n$. This implies $U^m\varphi\perp(I-Q)\mathfrak{L}_0$. So we get the first part of (10); the second part can be proved similarly.

Lemma 3. *If* $\mathfrak{W}=\mathfrak{W}^*=\{0\}$, *then* T *is a unitary operator and* $\varrho=1$.

Proof. Let $\mathfrak{W}=\{0\}$. In this case (7) implies $\varphi=Q\varphi$ for every $\varphi=\mathfrak{L}_0$, consequently $\mathfrak{L}_0\subset\bigvee_{n=1}^{\infty}\mathfrak{L}_n$. Now we have $\mathfrak{L}_1=U\mathfrak{L}_0\subset U\bigvee_{n=1}^{\infty}\mathfrak{L}_n=\bigvee_{n=2}^{\infty}\mathfrak{L}_n$, consequently $\mathfrak{L}_0\subset\bigvee_{n=2}^{\infty}\mathfrak{L}_n$ holds too. On the other hand, Lemma 1 shows that $\mathfrak{L}_0\perp\bigvee_{n=2}^{\infty}\mathfrak{L}_n$. So we get $\mathfrak{L}_0\equiv\{0\}$. Hence, $U\varphi=T\varphi$ for $\varphi\in\mathfrak{H}$. We can similarly prove that $\mathfrak{W}^*=\{0\}$ implies $U^*\varphi=T^*\varphi$ for $\varphi\in\mathfrak{H}$. Moreover, $U\varphi=T\varphi$ implies $\|\varphi\|=\|U\varphi\|=\|T\varphi\|=$ $=\|\varrho PU\varphi\|=\varrho\|\varphi\|$ for every $\varphi\in\mathfrak{H}$, consequently $\varrho=1$ and we have finished the proof.

Theorem 1. *If* T *is non-unitary, or if* $\varrho\neq1$, *then* $\sigma(U)$ *is the whole unit circle of the complex plane.*

Proof. Since U is unitary, $\sigma(U)$ is situated on the unite circle. On the other hand, by Lemma 3, there exists an element $\varphi\neq0$ in $\mathfrak{W}$ or $\mathfrak{W}^*$. By Lemma 2, U is a "bilateral shift" on $\bigvee_{n=-\infty}^{\infty}U^n\varphi$. Since the spectrum of the bilateral shift coincides with the unit circle C so we have a fortiori $\sigma(U)=C$.

A direct proof of the last statement can be given as follows: Suppose the converse case, i. e. that there exists ε such that $|\varepsilon|=1$ and $\varepsilon\notin\sigma(U)$. In this case $(I-\varepsilon U)^{-1}$ is bounded, and using the notation $S_n=I+\varepsilon U+\ldots+(\varepsilon U)^n$, we have $S_n(I-\varepsilon U)=$ $=I-(\varepsilon U)^{n+1}$. Hence

$$\|S_n\|=\|[I-(\varepsilon U)^{n+1}](I-\varepsilon U)^{-1}\|\leqq2\|(I-\varepsilon U)^{-1}\|.$$

Thus $\|S_n\|\leqq K$ with K independent of n. Now choosing φ as above,

$$\|S_n\varphi\|^2=\left\|\sum_{k=0}^{n}(\varepsilon U)^k\varphi\right\|^2=\sum_{k=0}^{n}\|(\varepsilon U)^k\varphi\|^2=(n+1)\|\varphi\|^2,$$

and this contradicts $\|S_n\|\leqq K$.

Lemma 4. *If* T *is completely non-unitary then* $\mathfrak{H}$ *is a subspace of* $\mathfrak{L}\vee\mathfrak{L}^*$.

Proof. If an element φ of $\mathfrak{H}$ can be written as $\varphi=(I-T^{*n}T^n)\psi$ for some n, then we have:

$$\varphi=U^{-1}(U-T)\psi+U^{-2}(U-T)T\psi+\ldots+U^{-n}(U-T)T^{n-1}\psi+$$
$$+U^{-n+1}(U^*-T^*)T^n\psi+U^{-n+2}(U^*-T^*)T^*T^n\psi+\ldots+(U^*-T^*)T^{*n-1}T^n\psi,$$

and, by (5) and (4) this means that $\varphi\in\mathfrak{L}\vee\mathfrak{L}^*$. In case $\varphi=(I-T^nT^{*n})\psi$ for some n, we get the same result by changing the roles of T and T^*, U and U^*.

$\mathfrak{L}\vee\mathfrak{L}^*$ is closed, consequently it contains the space spanned by the ranges of $(I-T^{*n}T^n)$ and $(I-T^nT^{*n})$ for all positive integer n. Thus we only have to prove that if, for some $\varphi\in\mathfrak{H}$,

(11) $$\varphi\perp(I-T^{*n}T^n)\mathfrak{H} \quad\text{and}\quad \varphi\perp(I-T^nT^{*n})\mathfrak{H} \qquad (n=1,2,\ldots)$$

then $\varphi=0$. (11) implies

$$(I-T^{*n}T^n)\varphi=0 \quad\text{and}\quad (I-T^nT^{*n})\varphi=0.$$

So we have $T^{*n}T^n\varphi=T^nT^{*n}\varphi=\varphi$, hence $\|T^n\varphi\|^2=\|T^{*n}\varphi\|^2=\|\varphi\|^2$ for $n=1,2,\ldots$. This implies that $\varphi=0$, because T is completely non-unitary.

Lemma 5. *If U is the minimal unitary ϱ-dilation of a completely non-unitary operator T, then $\mathfrak{K}=\mathfrak{L}\vee\mathfrak{L}^*$.*

Proof. Using (2), it suffices to prove that

(12) $$\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}=\mathfrak{L}\vee\mathfrak{L}^*.$$

By Lemma 4, $\mathfrak{H}$ is a subspace of $\mathfrak{L}\vee\mathfrak{L}^*$. (4) and (5) imply, that both $\mathfrak{L}$ and $\mathfrak{L}^*$ reduce U, consequently $U^n\mathfrak{H}$ is a subspace of $\mathfrak{L}\vee\mathfrak{L}^*$ for $n=0,\ \pm1,\ldots$. So we have

(12′) $$\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}\subset\mathfrak{L}\vee\mathfrak{L}^*.$$

On the other hand, (4) implies that both $\mathfrak{L}_k$ and $\mathfrak{L}_k^*$ are contained in $\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}$, for $k=0,\ \pm1,\ldots$. Now (5) shows that

$$\bigvee_{n=-\infty}^{\infty} U^n\mathfrak{H}\supset\mathfrak{L}\vee\mathfrak{L}^*.$$

This relation and (12′) prove (12).

Lemma 6. $\mathfrak{L}\vee\mathfrak{L}^*=\bigvee_{n=-\infty}^{\infty} U^n(\mathfrak{V}\vee\mathfrak{W}\vee\mathfrak{V}^*\vee\mathfrak{W}^*)$.

Proof. (6) and (7) show that

$$\mathfrak{V}\vee\mathfrak{W}\supset\mathfrak{L}_0 \quad\text{and}\quad \mathfrak{V}^*\vee\mathfrak{W}^*\supset\mathfrak{L}_0^*.$$

So, by (5) and (4)

(13) $$\bigvee_{n=-\infty}^{\infty} U^n(\mathfrak{V}\vee\mathfrak{W}\vee\mathfrak{V}^*\vee\mathfrak{W}^*)\supset\mathfrak{L}\vee\mathfrak{L}^*.$$

On the other hand, (4), (5), (6) and (7) show that $\mathfrak{V}$ and $\mathfrak{W}$ are contained in $\mathfrak{L}$; similarly $\mathfrak{V}^*$ and $\mathfrak{W}^*$ are contained in $\mathfrak{L}^*$. Since both $\mathfrak{L}$ and $\mathfrak{L}^*$ reduce U, this implies

$$\bigvee_{n=-\infty}^{\infty} U^n(\mathfrak{V}\vee\mathfrak{W}\vee\mathfrak{V}^*\vee\mathfrak{W}^*)\subset\mathfrak{L}\vee\mathfrak{L}^*.$$

This relation and its converse (13) prove the lemma.

Now, by Lemma 3 of [3], the following is true: If U is a unitary operator on

$\mathfrak{K}$ and $\mathfrak{A}_1, \ldots, \mathfrak{A}_N$ are wandering subspaces of U, and the set of the finite linear combinations $\sum_{n,k}' \varphi_{n,k}$ $(\varphi_{n,k} \in U^n \mathfrak{A}_k)$ is dense in $\mathfrak{K}$, then U has absolutely continuous spectral measure.

Thus, Lemma 6 implies

Lemma 7. *The restriction of U to the reducing subspace $\mathfrak{L} \vee \mathfrak{L}^*$ has absolutely continuous spectral measure.*

Combining this fact with Lemma 5 we get

Theorem 2. *If U is the minimal unitary ϱ-dilation of a completely non-unitary operator T, then the spectral measure of U is absolutely continuous.*

We shall use the following obvious

Lemma 8. *If T has some unitary ϱ-dilation U with absolutely continuous spectral measure, then T^n converges weakly to O as $n \to \infty$.*

Indeed, using (3) and the Riemann—Lebesgue lemma, we get for $\varphi, \psi \in \mathfrak{H}$

$$(T^n \varphi, \psi) = \varrho (U^n \varphi, \psi) = \varrho \int_0^{2\pi} e^{in\vartheta} \, d(E_\vartheta \varphi, \psi) = \varrho \int_0^{2\pi} e^{in\vartheta} f_{\varphi,\psi}(\vartheta) \, d\vartheta \to 0.$$

Thus Theorem 2 has the following

Corollary. *If T is completely non-unitary and has some unitary ϱ-dilation, then T^n converges weakly to O as $n \to \infty$.*

The next theorem gives a decomposition for T.

Theorem 3. *If T has some unitary ϱ-dilation U, then $\mathfrak{H}$ can be decomposed as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, in such a way that:*

(i) *$\mathfrak{H}_1$ and $\mathfrak{H}_2$ reduce T,*

(ii) *$T_1 = T|\mathfrak{H}_1$ has a unitary ϱ-dilation with absolutely continuous spectral measure,*

(iii) *$T_2 = T|\mathfrak{H}_2$ is unitary.*

Proof. Set $\mathfrak{H}_1 = \mathfrak{H} \cap (\mathfrak{L} \vee \mathfrak{L}^*)$. If $\varphi \in \mathfrak{H}_1$, then

$$T\varphi = (T-U)\varphi + U\varphi \in \mathfrak{L}_0 \vee U(\mathfrak{L} \vee \mathfrak{L}^*) \subset \mathfrak{L} \vee \mathfrak{L}^*$$

thus $T\mathfrak{H}_1 \subset \mathfrak{H}_1$. Similarly, $T^*\mathfrak{H}_1 \subset \mathfrak{H}_1$, so $\mathfrak{H}_1$ reduces T.

Since $\mathfrak{L} \vee \mathfrak{L}^*$ reduces U, the part U_1 of U in $\mathfrak{L} \vee \mathfrak{L}^*$ will be a unitary ϱ-dilation of $T_1 = T|\mathfrak{H}_1$. Now, by Lemma 7, $U_1 = U|(\mathfrak{L} \vee \mathfrak{L}^*)$ has absolutely continuous spectral measure.

It remains to show that if $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$, then $T_2 = T|\mathfrak{H}_2$ is unitary. Now, the relations

$$(I - T^*T)\varphi = U^{-1}(U - T)\varphi + (U^* - T^*)T\varphi,$$

$$(I - TT^*)\varphi = U(U^* - T^*)\varphi + (U - T)T^*\varphi$$

($\varphi \in \mathfrak{H}$) show that $\mathfrak{H}_1$ contains the ranges of both $I - T^*T$ and $I - TT^*$. Thus $\psi \in \mathfrak{H}_2$ implies $\psi \perp (I - T^*T)\mathfrak{H}$ and $\psi \perp (I - TT^*)\mathfrak{H}$, hence $T^*T\psi = \psi$ and $TT^*\psi = \psi$. This means that T is unitary on $\mathfrak{H}_2$, and so we have finished the proof.

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Sur les transformations de classe $\mathscr{C}_\varrho$ dans l'espace de Hilbert

Par A. RÁCZ à Timişoara (R. S. Roumanie)

1. Le but de la présente Note est d'étendre certains faits connus pour les contractions de l'espace de Hilbert au cas des opérateurs de classe $\mathscr{C}_\varrho$ et cela en adaptant les démonstrations correspondantes dans le cas des contractions de [1], [2], [3].

Rappelons (voir [4]) qu'un opérateur linéaire borné T de l'espace de Hilbert $\mathfrak{H}$ est de classe $\mathscr{C}_\varrho$ $(\varrho > 0)$ s'il existe un espace $\mathfrak{K} \supset \mathfrak{H}$ et un opérateur unitaire U dans $\mathfrak{K}$ tel qu'on a

$$T^n = \varrho \cdot \mathrm{pr}\, U^n \qquad (n = 1, 2, \ldots). \tag{1}$$

On peut exiger aussi que $\mathfrak{K}$ soit sous-tendu par les éléments de la forme $U^n h$ $(h \in \mathfrak{H};$ $n = 0, \pm 1, \pm 2, \ldots)$; U est alors déterminé à isomorphie près et s'appelle la ϱ-dilatation unitaire minimum de T.

Désignons par C le cercle unité dans le plan des nombres complexes, par D le disque unité ouvert et par $\bar{D}$ le disque unité fermé.

2. Soit A_0 la classe des fonctions analytiques dans D de la forme

$$u(\lambda) = \sum_{n=1}^{\infty} c_n \lambda^n \quad \text{avec} \quad \sum_{n=1}^{\infty} |c_n| < \infty; \tag{2}$$

A_0 est une algèbre (sans unité) par rapport aux opérations usuelles. Soit $T \in \mathscr{C}_\varrho$ et soit U la ϱ-dilatation unitaire minimum de T. Puisque (1) entraîne $\|T^n\| \leqq \varrho$ $(n \geqq 1)$, on peut définir l'opérateur $u(T)$ pour $u \in A_0$ par la série, convergente en norme,

$$u(T) = \sum_{n=1}^{\infty} c_n T^n. \tag{3}$$

Notons que $u \to u(T)$ est un homomorphisme d'algèbre de A_0 dans l'algèbre opérateurs linéaires bornés dans $\mathfrak{H}$. De plus, (1) entraîne

$$u(T) = \varrho \cdot \mathrm{pr}\, u(U), \tag{4}$$

où

$$u(U) = \sum_{n=1}^{\infty} c_n U^n = \int_0^{2\pi} u(e^{it})\, dE_t, \tag{5}$$

$\{E_t\}$ étant la famille spectrale de l'opérateur unitaire U.

3. Désignons par $H_{0,T}$ la classe des fonctions $u(\lambda)$, analytiques et bornés dans D, s'annulant à l'origine et telles que la limite radiale $u(e^{it}) = \lim_{r\to 1} u(re^{it})$ existe en tout point $e^{it}\in C$ sauf peut-être les points d'un ensemble C_u de mesure O par rapport à la mesure spectrale $E(\cdot)$ engendrée par $\{E_t\}$. Il est manifeste que cette classe est une algèbre et que $u_r(\lambda)=u(r\lambda)\in A_0$ pour tout $u\in H_{0,T}$ et $0\leqq r<1$. Ainsi $u_r(T)$ a un sens pour $T\in\mathscr{C}_\varrho$. Des relations (4) et (5), appliquées à u_r, et de la relation

$$\|[u(U)-u_r(U)]h\|^2 = \int_0^{2\pi} |u(e^{it})-u_r(e^{it})|^2\, d(E_t h, h)\to 0 \qquad (r\to 1),$$

valable pour tout $h\in\mathfrak{K}$, il s'ensuit que la limite

$$(6) \qquad u(T) = \lim_{r\to 1} u_r(T)$$

existe au sens fort. Pour $u\in A_0$, la définition (6) de $u(T)$ est évidemment cohérente à celle donnée dans la section 2. De plus, l'application $u\to u(T)$ est un homomorphisme d'algèbre de $H_{0,T}$ dans l'algèbre des opérateurs linéaires bornés de H.

Remarquons aussi que (4) s'étend à $H_{0,T}$, $u(U)$ étant défini par l'intégrale spectrale figurant au dernier membre de (5).

De (4) on déduit que si $u_n\in H_{0,T}$, $|u_n(\lambda)|\leqq M$ $(\lambda\in D;\ n=1, 2, \ldots)$ et $u_n(e^{it})\to 0$ $(n\to\infty)$ pp. par rapport à la mesure spectrale $E(\cdot)$ de U, alors $u_n(T)\to 0$ (fortement.)

Proposition 1. *Soit $T\in\mathscr{C}_\varrho$ et soit U la ϱ-dilatation unitaire minimum de T. Si $\varrho\neq 1$, le spectre de U recouvre le cercle unité C. Il en est de même dans le cas $\varrho=1$ si T n'est pas unitaire.*

Démonstration. Supposons que le spectre de U est situé dans un arc fermé α de C, $\alpha\neq C$. Comme α est alors dans l'intérieur d'un domaine simplement connexe du plan complexe, ne contenant pas le point $\lambda=0$, il s'ensuit du théorème de Runge qu'il existe une suite de polynômes $q_n(\lambda)$ tendant vers $1/\lambda^2$ uniformément sur α et par conséquent $p_n(\lambda)=\lambda q_n(\lambda)$ tendant vers $1/\lambda$ uniformément sur α. En vertu du calcul fonctionnel pour U on a

$$U^\nu\cdot p_n(U)\to U^{\nu-1} \qquad (\nu = 0, 1, \ldots;\ n\to\infty).$$

En prenant les projections sur H et eu égard à ce que $p_n(0)=0$, on obtient (avec $\delta=1/\varrho$):

$$(\nu=0) \qquad \delta p_n(T)\to \mathrm{pr}\ U^*=(\mathrm{pr}\ U)^*=\delta T^*,$$

$$(\nu=1) \qquad \delta T p_n(T)=\delta p_n(T)T\to I,$$

$$(\nu=2) \qquad \delta T^2 p_n(T)\to\delta T.$$

En comparant ces résultats on déduit

$$\delta TT^*=\delta T^*T=I, \quad \delta T=T.$$

Dans le cas $\delta=1/\varrho\neq 1$ la seconde relation donne $T=O$, ce qui est en contradiction avec la première relation. Donc $\varrho=1$ et la première des relations exprime alors que T est unitaire.

4. Nous disons que T est complètement non-unitaire si aucun sous-espace $\neq\{0\}$ ne réduit T à un opérateur unitaire. Il est manifeste que si $T\in\mathscr{C}_\varrho$ $(\varrho<1)$, T est complètement non-unitaire.

Proposition 2. *Soit $T\in\mathscr{C}_\varrho$, complètement non-unitaire, et soit U sa ϱ-dilatation unitaire minimum. La mesure spectrale $E(\cdot)$ de U est alors absolument continue (par rapport à la mesure de Lebesgue m).*

Démonstration. Il suffit de démontrer que si un ensemble *fermé* $\sigma(\subset C)$ est de mesure 0 par rapport à la mesure de Lebesgue, on a aussi $E(\sigma)=O$.

A cet effet envisageons une fonction $u(\lambda)$, continue dans $\bar{D}$, holomorphe dans D et telle que

$$u(\lambda)=1 \quad \text{pour} \quad \lambda\in\sigma, \quad |u(\lambda)|<1 \quad \text{pour} \quad \lambda\in\bar{D}\setminus\sigma$$

(cf. [1], p. 253). Soit $z\to l(z)$ l'homographie de $\bar{D}$ sur $\bar{D}$ telle que $u(0)\to 0$ et $1\to 1$. La fonction $v(\lambda)=l(u(\lambda))$ jouit alors des mêmes propriétés que $u(\lambda)$, de plus on a $v(0)=0$. Il est manifeste que les opérateurs $v(T)$ et $v(U)$ existent et qu'on a

$$v(T)=\varrho\cdot\mathrm{pr}\; v(U).$$

On a évidemment

$$[v(U)]^n\to E(\sigma) \qquad (n\to\infty)$$

d'où il dérive

$$[v(T)]^n\to\varrho\cdot\mathrm{pr}\; E(\sigma)=B(\sigma).$$

De cette représentation on déduit que $B(\sigma)$ est une projection orthogonale, permutant à T.

Soit σ_t la partie de σ située dans l'arc fermé $[1, e^{it}]$ de C. Comme σ_t est aussi fermé et de mesure 0, on peut affirmer que

$$B(\sigma_t)=\varrho\cdot\mathrm{pr}\; E(\sigma_t) \qquad (0\leqq t\leqq 2\pi)$$

est une projection orthogonale permutant à T; de plus on a évidemment $B(\sigma_t)\leqq \leqq B(\sigma_{t'})$ pour $t\leqq t'$, donc, en posant $\mathfrak{H}(\sigma)=B(\sigma)\mathfrak{H}$, $B(\sigma_t)|\mathfrak{H}(\sigma)$ forme une famille spectrale dans l'espace $\mathfrak{H}(\sigma)$. Comme on a

$$T|\mathfrak{H}(\sigma)=\varrho\cdot\mathrm{pr}\; U|\mathfrak{H}(\sigma)=\int_0^{2\pi} e^{it}\, d[\varrho\cdot\mathrm{pr}\; E_t|\mathfrak{H}(\sigma)]=\int_0^{2\pi} e^{it}\, d[B(\sigma_t)|\mathfrak{H}(\sigma)],$$

$T|\mathfrak{H}(\sigma)$ est unitaire. Vu que T était supposé complètement non-unitaire, cela entraîne $\mathfrak{H}(\sigma)=\{0\}$, donc $B(\sigma)=0$. En répétant le raisonnement de [1] on conclut que $E(\sigma)=O$.

5. Nous terminons cette Note par la

Proposition 3. *Dans les conditions de la proposition 2, soit σ un sous-ensemble borélien de C tel que $m(\sigma)>0$. On a alors $E(\sigma)h\neq 0$ pour $h\in\mathfrak{H}$, $h\neq 0$.*

Démonstration. Le cas $\varrho=1$ étant envisagé dans [1], nous supposons désormais que $\varrho\neq 1$.

Soit σ un ensemble borélien tel que $m(\sigma)>0$ et soit $h\in H$ tel que $E(\sigma)h=0$. Nous allons démontrer que ces hypothèses entraînent $h=0$.

Notons d'abord que, en vertu de la proposition 2, on a

$$p_f(t) = \frac{d}{dt}(E_t h, f)\in L^1(0, 2\pi)$$

pour tout f de l'espace de dilatation U. De plus, $p_f(t)$ s'annule pp. dans l'ensemble de mesure positive $(\sigma) = \{t: e^{it}\in\sigma\}$ et

$$(E(\omega)h, f) = \int_{(\omega)} p_f(t)\,dt \quad \text{où} \quad (\omega) = \{t:\ e^{it}\in\omega\}.$$

Choisissons d'abord $f=(U-T)g$, avec $g\in\mathfrak{H}$, arbitraire. On a alors

$$c_k = \int_0^{2\pi} e^{-ikt}p_f(t)\,dt = \int_0^{2\pi} e^{-ikt}\,d(E_t h, f) =$$
$$= (h, U^k f) = (h, U^{k+1}g - U^k Tg) \qquad (k = 0, \pm 1, \ldots).$$

Pour $k\geqq 1$ cela donne $c_k=0$. Vu que $p_f(t)$ s'annule dans un ensemble de mesure positive, cela entraîne $p_f(t)=0$ pp. Donc on doit avoir $c_k=0$ aussi pour $k<1$. Or, on a

$$c_0 = (h, Ug-Tg) = \left(\frac{1}{\varrho}-1\right)(h, Tg) = \left(\frac{1}{\varrho}-1\right)(T^*h, g).$$

On en déduit (vu que $1/\varrho-1\neq 0$ et que g est arbitraire)

(7) $$T^*h=0.$$

Choisissons maintenant $f=(U^*-T^*)g$, $g\in\mathfrak{H}$. On a

$$d_k = \int_0^{2\pi} e^{ikt}p_f(t)\,dt = \int_0^{2\pi} e^{ikt}\,d(E_t h, f) = (h, U^{*k}f) =$$
$$= (h, U^{*k+1}g - U^{*k}T^*g) \qquad (k = 0, \pm 1, \ldots).$$

Pour $k\geqq 1$ cela donne $d_k=0$, d'où il s'ensuit $d_k=0$ aussi pour $k<1$, en particulier $d_{-1}=0$. Or,

$$d_{-1} = (h, g-UT^*g) = \left(h, g-\frac{1}{\varrho}TT^*g\right) = \left(\left(I-\frac{1}{\varrho}TT^*\right)h, g\right),$$

d'où $\left(I-\frac{1}{\varrho}TT^*\right)h=0$. Vu (7) cela donne $h=0$, c.q.f.d.

L'auteur tient à remercier M. le professeur B. Sz.-Nagy pour l'aide accordée dans la rédaction définitive de cette Note.*)

*) *Remarque par la rédaction.* La majeure partie des résultats de cette Note dérive aussi des résultats de la Note précédente de E. Durszt, On the spectrum of unitary ϱ-dilations, *Acta Sci. Math.*, **28** (1967), 299—304. Les deux travaux étant indépendants et leurs méthodes différentes, on a jugé justifié d'insérer toutes les deux Notes dans ces *Acta.*

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Remarks to a paper of D. Gaier on gap theorems

By GÁBOR HALÁSZ in Budapest

In his paper [1] cited, D. GAIER proved several gap theorems, including the high indices theorem for Borel summability. However, in its original form, his method is not applicable to the theory of Abel summation. In this paper we show how to eliminate this difficulty by a slight modification. At the same time we complete the series of the theorems of GAIER with some more, obtainable by the same modification.

Theorem 1. (HARDY—LITTLEWOOD [2].) *If a series is Abel summable to* 0 *and has Hadamard gaps, then it converges to its Abel sum* 0. *I. e., if*

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}, \quad \frac{\lambda_{n+1}}{\lambda_n} \geqq q > 1, \quad \lambda_1 > 0$$

and

$$\lim_{x \to +0} f(x) = 0,$$

then

$$\sum_{n=1}^{\infty} a_n = 0.$$

Or in an almost equivalent form:

Theorem 1′. *If $f(x)$ of Theorem* 1 *is bounded for $x > 0$, then*

$$\sup_{N \geqq 1} \left| \sum_{n=1}^{N} a_n \right| \leqq c_1 \cdot \sup_{x>0} |f(x)|,$$

where the constant c_1 depends only on the sequence $\{\lambda_n\}$.

(Such positive constants, independent of the quantities a_n, x, N etc. will be denoted in the sequel by $c_2, c_3, \ldots$.)

Without any special difficulty, we get the following more precise information about the rapidity of convergence in Theorem 1:

Theorem 2. *Let $r(x)$ $(x > 0)$ be an increasing positive function such that with some $\alpha \geqq 0$ $r(x)x^{-\alpha}$ is decreasing. $f(x)$ should satisfy, in addition to the hypotheses of Theorem* 1,

$$|f(x)| \leqq r(x) \qquad (x > 0).$$

In this case

$$\left| \sum_{\lambda_n < X} a_n \right| \leqq c_2 r(X^{-1}).$$

The condition imposed on $r(x)$ implies that it is larger than constant times x^α near 0. As to smaller remainder terms, let us restrict ourselves to extremely small ones in the following connection.

Theorem 2 suggests that $\sum_{\lambda_n < X} a_n$ and $f(X^{-1})$ have approximately the same order of magnitude and one would expect that an estimation $|f(x)| \leqq \exp\left(-\frac{\text{const.}}{x}\right)$ implies $|\sum_{\lambda_n < X} a_n| \leqq \exp(-\text{const.}\ X)$ so that $f(z)$ is regular in a larger half plane and by its vanishing at $z=0$ of infinite order, $f(z) \equiv 0$. We can deduce this from a weaker estimation, e.g. from $|f(x)| \leqq \exp\left(-\frac{1}{x \log^2 x}\right)$ as is shown by

Theorem 3. *If for the error term we have* $r(x) = e^{-s(x)}$ *where* $s(x)$ *is convex from below and*

$$\int_0^1 \sqrt{-s'(x)}\, dx = +\infty$$

then $f(z) \equiv 0$, $a_n \equiv 0$ *even if the weaker gap condition*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} < +\infty \qquad (\lambda_n - \lambda_{n-1} < d)$$

is assumed.

Concerning absolute summability, we prove

Theorem 4. (Zygmund [3].) *If* $f(x)$ *has Hadamard gaps and is of bounded variation on* $(0, +\infty)$, *i. e.*

$$\int_0^\infty |f'(x)|\, dx < +\infty,$$

then

$$\sum_{n=1}^{\infty} |a_n| < +\infty.$$

Theorem 5. *Theorem* 4 *is valid if* $f(x)$, *instead of being of bounded variation satisfies the condition*

$$\int_0^\infty \frac{|f(x)|}{x}\, dx < +\infty.$$

Proof of Theorem 1′. Suppose first

$$\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} < +\infty$$

and consider the Laplace transform

$$F(s) = \int_0^\infty f(x) e^{xs}\, dx \qquad (\operatorname{Re} s < 0).$$

Substituting the series representation of $f(x)$, we get $F(s)$ in the form

$$F(s) = \int_0^\infty \sum_{n=1}^\infty a_n e^{-\lambda_n x + sx}\, dx = \sum_{n=1}^\infty a_n \int_0^\infty e^{(s-\lambda_n)x}\, dx = -\sum_{n=1}^\infty \frac{a_n}{s-\lambda_n},$$

where the change of order of summation and integration is justified by $\sum \int |\ | < +\infty$, a consequence of $\sum_{n=1}^\infty \frac{|a_n|}{\lambda_n} < +\infty$. The sum on the right represents a continuation of $F(s)$ into the right half plane with poles at $s = \lambda_n$ and corresponding residues $-a_n$. An application of the residue theorem gives therefore

$$\sum_{n=1}^N a_n = -\frac{1}{2\pi i} \int_{|s|=R} F(s)\, ds,$$

provided $\lambda_N < R < \lambda_{N+1}$, and our task is to estimate $F(s)$.

On the negative real axis a bound is provided by the original integral representation:

$$|F(-\sigma)| \leqq \int_0^\infty |f(x)| e^{-\sigma x}\, dx \leqq \sup_{x>0} |f(x)| \int_0^\infty e^{-\sigma x}\, dx = \sup_{x>0} |f(x)| \frac{1}{\sigma} \quad (\sigma > 0).$$

To be able to extend this, we form the Blaschke product for the region D obtained from the plane by omitting the negative real axis:

$$G(s) = \prod_{n=1}^\infty \frac{1 - \frac{\sqrt{s}}{\sqrt{\lambda_n}}}{1 + \frac{\sqrt{s}}{\sqrt{\lambda_n}}} \qquad (\sqrt{s} \text{ is determined by } \sqrt{1} = 1).$$

By its vanishing at the poles of $F(s)$, the function

$$H(s) = F(s) G(s) s$$

is regular in the whole of D, satisfying on both sides of the boundary line the inequality

$$|H(-\sigma)| = |F(-\sigma)|\sigma \leqq \sup_{x>0} |f(x)|$$

because $|G(-\sigma)| = 1$. Well-known theorems of Lindelöf type (see e. g. [4]) state that the same bound is valid for $H(s)$ in the whole region D if we know in advance some mild estimation on a sequence of circles around 0, tending to infinity. Circles at a distance of at least 1 from all the λ_n's can serve for such a sequence, for we have on them

$$|H(s)| \leqq |F(s)|\,|s| \leqq |s| \sum_{n=1}^{\infty} \frac{|a_n|}{|s-\lambda_n|} \leqq$$

$$\leqq |s| \left\{ \sum_{\lambda_n < \frac{|s|}{2}} \frac{|a_n|}{\lambda_n} + \sum_{\frac{|s|}{2} \leqq \lambda_n < 2|s|} |a_n| + 2 \sum_{2|s| \leqq \lambda_n} \frac{|a_n|}{\lambda_n} \right\} \leqq$$

$$\leqq |s| \left\{ 2 \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} + 2|s| \sum_{\frac{|s|}{2} \leqq \lambda_n < 2|s|} \frac{|a_n|}{\lambda_n} \right\} \leqq 2|s|(|s|+1) \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n}$$

and $O(|s|^2)$ is well below the limit allowed in those theorems of Lindelöf type. Therefore, we may write

$$|H(s)| \leqq \sup_{x>0} |f(x)| \qquad (s \in D).$$

Let $R = \frac{\lambda_{N+1}}{\sqrt{q}}$ in the residue theorem written down above. We shall prove in a Lemma *), using heavily the gap condition (with its notations $\nu_n = \sqrt{\lambda_n}$, $p = \sqrt{q}$, $z = \sqrt{s}$, $m = \sqrt[4]{q}$), that on $|s| = R$ we have $|G(s)| > c_3$, thus by the definition of $H(s)$

$$|F(s)| = \frac{|H(s)|}{|G(s)|\,|s|} \leqq \frac{1}{c_3 R} \sup_{x>0} |f(x)|.$$

The residue theorem then gives

$$\left| \sum_{n=1}^{N} a_n \right| \leqq \frac{1}{2\pi} \int_{|s|=R} |F(s)|\,|ds| \leqq \frac{1}{c_3} \sup_{x>0} |f(x)|.$$

To get rid of the supposition $\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} < +\infty$ we first consider the function $f(x+\delta)$ $(\delta > 0)$ with the coefficients $a_n e^{-\lambda_n \delta}$. For this the above argument holds and hence

$$\left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n \delta} \right| \leqq \frac{1}{c_3} \sup_{x>0} |f(x+\delta)| \leqq \frac{1}{c_3} \sup_{x>0} |f(x)|,$$

and we may let $\delta \to 0$, the bound being uniform in δ, thus completing the proof of Theorem 1′.

P r o o f o f T h e o r e m 1. From Theorem 1′ we know already that the partial sums, hence also the coefficients, are bounded so that $\sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n} < +\infty$ and we may repeat the argument of the previous proof. The only difference is that $f(x)$ is not

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*) See after the proof of Theorem 3.

only bounded but tends to 0 with x, which implies for $\sigma \to +\infty$ the better estimations

$$|F(-\sigma)| \leqq \int_0^\infty |f(x)| e^{-\sigma x}\, dx = \int_0^\infty o(1) e^{-\sigma x}\, dx = o\left(\frac{1}{\sigma}\right),$$

$$|H(-\sigma)| = |F(-\sigma)|\sigma = o(1).$$

Another variant of the Lindelöf type theorem used states that in this case

$$H(s) = o(1) \quad \text{as} \quad |s| \to +\infty,$$

uniformly in s in the whole region D. Returning to $F(s)$, this means

$$F(s) = o\left(\frac{1}{R}\right) \quad \text{for} \quad |s| = R$$

and by the residue theorem

$$\sum_{n=1}^{N} a_n = o(1). \qquad \text{Q. e. d.}$$

Proof of Theorem 2. First we remark that the constant 1 in $r\left(\frac{1}{X}\right)$ has no special significance since if $0 < c < 1$

$$r(cx) \leqq r(x) = x^\alpha r(x) x^{-\alpha} \leqq x^\alpha r(cx)(cx)^{-\alpha} = c^{-\alpha} r(cx),$$

$r(x)$ being increasing, $r(x)x^{-\alpha}$ decreasing.

Now, the proof will consist of a repetition of that of Theorem 1, $o(1)$ replaced everywhere by explicit estimations.

First,

$$|F(-\sigma)| = \int_0^\infty r(x) e^{-\sigma x}\, dx = \int_0^{1/\sigma} + \int_{1/\sigma}^\infty \leqq r\left(\frac{1}{\sigma}\right)\frac{1}{\sigma} + r\left(\frac{1}{\sigma}\right)\sigma^\alpha \int_{1/\sigma}^\infty x^\alpha e^{-\sigma x}\, dx =$$

$$= r\left(\frac{1}{\sigma}\right)\frac{1}{\sigma}\left(1 + \int_1^\infty x^\alpha e^{-x}\, dx\right) = c_4 r\left(\frac{1}{\sigma}\right)\frac{1}{\sigma},$$

i. e.

$$|H(-\sigma)| \leqq c_4 r\left(\frac{1}{\sigma}\right).$$

Owing to the properties of $r(x)$ we can write further with a temporarily fixed R

$$|H(-\sigma)| = c_4 r\left(\frac{1}{R}\right) \quad \text{for} \quad \sigma \geqq R,$$

$$|H(-\sigma)| \leqq c_4 r\left(\frac{1}{\sigma}\right)\sigma^\alpha \frac{1}{\sigma^\alpha} \leqq c_4 r\left(\frac{1}{R}\right) R^\alpha \frac{1}{\sigma^\alpha} = c_4 r\left(\frac{1}{R}\right)\left(\frac{R}{\sigma}\right)^\alpha \quad \text{for} \quad \sigma \leqq R.$$

The test function

$$W(s) = \left(1 + \frac{\sqrt{R}}{\sqrt{s}}\right)^{2\alpha}$$

is regular in D and since $\operatorname{Re}\frac{1}{\sqrt{s}} \geqq 0$ there, we have on both sides of the boundary

$$|W(-\sigma)| \geqq 1 \quad \text{for} \quad \sigma \geqq R,$$

$$|W(-\sigma)| \geqq \left(\frac{\sqrt{R}}{\sqrt{\sigma}}\right)^{2\alpha} = \left(\frac{R}{\sigma}\right)^{\alpha} \quad \text{for} \quad \sigma \leqq R.$$

On account of the estimations of $H(-\sigma)$ from above and those of $W(-\sigma)$ from below, the quotient

$$\frac{H(s)}{W(s)}$$

(which is bounded since $H(s)$ is bounded and $|W(s)| \geqq 1$ in D) has the uniform bound on the negative axis

$$c_4 r\left(\frac{1}{R}\right).$$

By the much used Lindelöf theorem this provides a bound in the whole region and applying it to $|s|=R$

$$|H(s)| \leqq c_4 r\left(\frac{1}{R}\right)|W(s)| = c_4 r\left(\frac{1}{R}\right)\left|1+\frac{\sqrt{R}}{\sqrt{s}}\right|^{2\alpha} \leqq c_4 r\left(\frac{1}{R}\right)2^{2\alpha} = c_5 r\left(\frac{1}{R}\right).$$

For $F(s)$, $|s|=R$ this implies

$$|F(s)| \leqq c_6 r\left(\frac{1}{R}\right)\frac{1}{R},$$

and by the residue theorem

$$\left|\sum_{n=1}^{N} a_n\right| \leqq c_6 r\left(\frac{1}{R}\right) = c_6 r\left(\frac{\sqrt{q}}{\lambda_{N+1}}\right) \leqq c_7 r\left(\frac{1}{\lambda_{N+1}}\right)$$

which is only an alternative formulation of the theorem.

Proof of Theorem 3. The gap condition enables us (and it is its only use here) to form again the product

$$G(s) = \prod_{n=1}^{\infty}\left(1-\frac{\sqrt{s}}{\sqrt{\lambda_n}}\right)\Big/\left(1+\frac{\sqrt{s}}{\sqrt{\lambda_n}}\right).$$

Let further

$$F_\delta(s) = \int_0^\infty f(x+\delta)e^{sx}\,dx, \quad H_\delta(s) = F_\delta(s)G(s)s.$$

As proved in Theorem 1′, the last function is regular and bounded (uniformly with respect to δ) in D with boundary values

$$H_\delta(-\sigma) = -\sigma G(-\sigma)\int_0^\infty f(x+\delta)e^{-\sigma x}\,dx.$$

Since $f(x)\to 0$ as $x\to +0$ and $O(e^{-\text{const}\,x})$ as $x\to +\infty$, these boundary values tend uniformly to

$$H(-\sigma)\overset{\text{def}}{=}-\sigma G(-\sigma)\int_0^\infty f(x)e^{-\sigma x}\,dx\overset{\text{def}}{=}-\sigma G(-\sigma)F(-\sigma)$$

as $\delta\to 0$. But then the convergence is uniform inside D and we get that $H_\delta(s)$ tends to a regular and bounded function $H(s)$ with the above boundary values $H(-\sigma)$. To prove $f(x)\equiv 0$ it is enough to show that $H(s)\equiv 0$. According to a well-known theorem*) this will follow if we can show that

$$\left(\int^\infty+\int^\infty\right)\frac{\log|H(-\sigma)|}{\sigma^{3/2}}\,d\sigma=-\infty,$$

where the integration is on both sides of the boundary line. Recalling $|G(-\sigma)|=1$ and the definition of $H(-\sigma)$, we have to prove this with $F(-\sigma)$ in place of $H(-\sigma)$. Now,

$$|F(-\sigma)|\leqq\int_0^1|f(x)|e^{-\sigma x}\,dx+\int_1^\infty|f(x)|e^{-\sigma x}\,dx\leqq\int_0^1 e^{-s(x)-\sigma x}+O(e^{-\sigma}).$$

For σ large enough the integrand attains its maximum when

$$-s'(x)-\sigma=0.$$

$-s'(x)$ being decreasing, this value $x=x(\sigma)$ is a well-defined and decreasing function of σ and with it the integral is less than

$$1\cdot e^{-s(x)-\sigma x}\leqq e^{-\sigma x}.$$

Since $x(\sigma)$ tends to 0 with $1/\sigma$, the second term in the estimation of $F(-\sigma)$ cannot exceed this bound and hence

$$|F(-\sigma)|=O(e^{-\sigma x})\qquad\log|F(-\sigma)|\leqq-\sigma x+O(1),$$

and it suffices to show

$$\int^\infty\frac{\sigma x}{\sigma^{3/2}}\,d\sigma=\int^\infty\frac{x}{\sqrt{\sigma}}\,d\sigma=+\infty.$$

Introducing x as a new variable, we have, since $-s'(x)\equiv\sigma$,

$$\int^\infty\frac{x}{\sqrt{\sigma}}\,d\sigma=-\int_0\frac{x}{\sqrt{\sigma}}\,\sigma'(x)\,dx=\int_0\frac{xs''(x)}{\sqrt{-s'(x)}}\,dx$$

and partial integration shows that this is

$$\geqq O(1)+\varliminf_{x=0}x2\sqrt{-s'(x)}+2\int_0\sqrt{-s'(x)}\,dx\geqq O(1)+2\int_0\sqrt{-s'(x)}\,dx=+\infty.$$

Qu. e. d.

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*) This is JENSEN's inequality $\int_{|z|=1}\log|f(z)||d_z|\geqq\log|f(0)|$, after a conformal mapping of the unit disk onto our region D.

Lemma. *If $\{v_n\}$ is an Hadamard sequence $\left(\frac{v_{n+1}}{v_n} \geqq p > 1\right)$ of positive numbers then with the definition*

$$B(z) = \prod \left(1-\frac{z}{v_n}\right) \Big/ \left(1+\frac{z}{v_n}\right)$$

we have

$$c_8 < |B(z)| < c_9$$

for $v_N m < |z| < \frac{v_{N+1}}{m}$ $(m > 1)$, uniformly in z and N.

Proof. For $n \geqq N+1$

$$\left|\frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| \leqq \frac{1+\frac{|z|}{v_n}}{1-\frac{|z|}{v_n}} < e^{c_{10}\frac{|z|}{v_n}},$$

$$\left|\prod_{n \geqq N+1} \frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| < \exp\left[c_{10}|z| \sum_{n \geqq N+1} \frac{1}{v_n}\right] \leqq \exp\left[c_{10}\frac{|z|}{v_{N+1}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)\right] \leqq$$

$$\leqq \exp\left[c_{11}\frac{|z|}{v_{N+1}}\right] < e^{c_{11}}.$$

For $n \leqq N$

$$\left|\frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| \leqq \frac{\frac{|z|}{v_n}+1}{\frac{|z|}{v_n}-1} = 1+\frac{2}{\frac{|z|}{v_n}-1} \leqq 1+\frac{2}{\left(1-\frac{1}{m}\right)\frac{|z|}{v_n}} \leqq e^{c_{12}\frac{v_n}{|z|}},$$

$$\left|\prod_{n \leqq N} \frac{1-\frac{z}{v_n}}{1+\frac{z}{v_n}}\right| \leqq \exp\left(c_{12}\frac{1}{|z|}\sum_{n \leqq N} v_n\right) \leqq \exp\left[c_{12}\frac{v_N}{|z|}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)\right] \leqq$$

$$\leqq \exp\left(c_{13}\frac{v_N}{|z|}\right) < e^{c_{13}},$$

thus $B(z)$ is bounded above. Since $B(z) = \frac{1}{B(-z)}$, boundedness from below follows.

Proof of Theorem 4. The hypotheses of Theorem 1′ are satisfied here and we can use the results of its proof.

Separating real and imaginary parts, we may suppose a_n real. If and only if

a_n and a_{n+1} are of different sign we pick out a number between λ_n and λ_{n+1}, e. g. their geometric mean and denote the sequence constructed this way by $\{\mu_k\}$. On the positive real axis the function

$$P(s) = \prod_{k=1}^{\infty}\left(1-\frac{s}{\mu_k}\right)\Big/\left(1+\frac{s}{\mu_k}\right)$$

changes sign at its simple zeros $s=\mu_k$ so that according to the construction, $P(\lambda_n)$ has the same changes of sign as a_n, and $a_nP(\lambda_n)$ is either always positive or always negative. Now the μ_k's form an Hadamard sequence and since λ_n is "far" from all of them in the sense required in the lemma, applying it to $P(s)$,

$$|P(\lambda_n)| > c_{14}.$$

Consequently

$$\sum_{n=1}^{N}|a_n| \leqq \frac{1}{c_{14}}\sum_{n=1}^{N}|a_nP(\lambda_n)| = \frac{1}{c_{14}}\left|\sum_{n=1}^{N}a_nP(\lambda_n)\right|,$$

and it is enough to prove that the sum on the right is bounded.

This is a partial sum of the residues of the function $F(s)P(s)$ originating from the poles of $F(s)$ at $s=\lambda_n$. Taking into account the remaining poles $s=-\mu_k$ provided by $P(s)$:

$$\sum_{\lambda_n<R} a_nP(\lambda_n) = -\frac{1}{2\pi i}\int_{|s|=R} F(s)P(s)\,ds + \sum_{\mu_k<R} F(-\mu_k)\operatorname*{Res}_{s=-\mu_k} P(s).$$

Let here $R=\dfrac{\lambda_N}{\sqrt[4]{q}}$. This choice guarantees that R is far from both the λ_n's and the μ_k's and we have

$$F(s) = O\left(\frac{1}{R}\right) \qquad (|s|=R)$$

as we proved in Theorem 1′ while

$$|P(s)| < c_{15} \ (|s|=R)$$

by the Lemma so that the integral on the right hand side is bounded.

To estimate the sum on the right, we return to the original integral representation of $F(s)$:

$$F(-\mu_k) = \int_0^{\infty} f(x)e^{-\mu_k x}\,dx,$$

$$\sum_{\mu_k<R} F(-\mu_k)p_k = \int_0^{\infty} f(x)\sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx.$$

Here we have introduced the notation $p_k = \operatorname*{Res}_{s=-\mu_k} P(s)$. Integrating by parts where we may assume that $\lim_{x=0} f(x)=0$ making the integrated part vanish,

$$\left|\int_0^\infty f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\, dx\right| = \left|\int_0^\infty f'(x) \sum_{\mu_k<R} \frac{p_k}{\mu_k} e^{-\mu_k x}\, dx\right| \leqq$$

$$\leqq \max_{x\geqq 0} \left|\sum_{\mu_k<R} \frac{p_k}{\mu_k} e^{-\mu_k x}\right| \cdot \int_0^\infty |f'(x)|\, dx \leqq \sup_K \left|\sum_{k=1}^K \frac{p_k}{\mu_k}\right| \int_0^\infty |f'(x)|\, dx,$$

the last inequality by a simple Abelian theorem. Appealing once again to the residue theorem, we have

$$\sum_{\mu_k<R} \frac{p_k}{\mu_k} = -\frac{1}{2\pi i} \int_{|s|=R} \frac{P(s)}{s}\, ds + 1$$

where the estimation $|P(s)| < c_{15}$ assures that the integral is bounded and so are the partial sums of $\sum_{k=1}^\infty \frac{p_k}{\mu_k}$. Thus we proved

$$\sum_{k=1}^K F(-\mu_k) \operatorname*{Res}_{s=-\mu_k} P(s) = O(1),$$

hence

$$\sum_{n=1}^N a_n P(\lambda_n) = O(1),$$

from which according to an earlier remark the conclusion already follows.

Proof of Theorem 5. We try to repeat the previous proof up to the stage where the boundedness of variation was exploited.

The results of Theorem 1′ were used there and we do not know in advance if its hypotheses are fulfilled here. But a simple consequence of our assumptions is

$$\int_0^\infty |f(x)|\, dx < +\infty$$

and thus the primitive function

$$\int f(x)\, dx = -\sum_{n=1}^\infty \frac{a_u}{\lambda_n} e^{-\lambda_n x}$$

is of bounded variation on $(0, +\infty)$, implying by Theorem 4

$$\sum_{n=1}^\infty \frac{|a_n|}{\lambda_n} < +\infty.$$

This was a prerequisite for the considerations in the proof of Theorem 1′. The boundedness of $f(x)$ was used to deduce $F(-\sigma)=O\left(\frac{1}{\sigma}\right)$. Here we have, however, the same bound by

$$|F(-\sigma)| \leqq \int_0^\infty |f(x)|e^{-\sigma x}\,dx = \int_0^\infty \frac{|f(x)|}{x} xe^{-\sigma x}\,dx \leqq$$

$$\leqq \max_{x\geqq 0} xe^{-\sigma x} \int_0^\infty \frac{|f(x)|}{x}\,dx = \frac{e^{-1}}{\sigma}\int_0^\infty \frac{|f(x)|}{x}\,dx.$$

Therefore the results of the proof of Theorem 1′ are valid,

$$F(s) = O\left(\frac{1}{R}\right) \qquad (|s| = R)$$

and the proof holds unaltered until the partial integration. This was performed to prove the boundedness of

$$\int_0^\infty f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx.$$

Here we proceed more roughly:

$$\left|\int_0^\infty f(x) \sum_{\mu_k<R} p_k e^{-\mu_k x}\,dx\right| \leqq \int_0^\infty |f(x)| \sum_{k=1}^\infty |p_k| e^{-\mu_k x}\,dx.$$

As we showed, the partial sums of $\sum_{k=1}^\infty \frac{p_k}{\mu_k}$ are bounded, hence $|p_k|<c_{16}\mu_k$ and we get further, $\{\mu_k\}$ being an Hadamard sequence,

$$\sum_{k=1}^\infty |p_k| e^{-\mu_k x} \leqq c_{16} \sum_{k=1}^\infty \mu_k e^{-\mu_k x} \leqq c_{17} \sum_{k=1}^\infty (\mu_k-\mu_{k-1}) e^{-\mu_k x} \leqq$$

$$\leqq \frac{c_{17}}{x} \sum_{k=1}^\infty \int_{\mu_{k-1}x}^{\mu_k x} e^{-u}\,du \leqq \frac{c_{17}}{x}.$$

The integral to be estimated is, therefore, less than

$$c_{17}\int_0^\infty \frac{|f(x)|}{x}\,dx$$

and with this the proof is completed.

*

I wish to express my thanks to Prof. Dr D. Gaier for his reading the manuscript and making some valuble suggestion among them calling my attention to a mistake in my original proof of Theorem 3 wich is corrected here.

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On the absolute summability factors of Fourier series

By LÁSZLÓ LEINDLER in Szeged*)

Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\lambda=\{\lambda_n\}$ be a monotone non-decreasing sequence of natural numbers with $\lambda_{n+1}-\lambda_n \leqq 1$ and $\lambda_1=1$.

The sequence-to-sequence transformation

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n+1}^{n} s_\nu$$

defines the sequence $\{V_n(\lambda)\}$ of generalized de la Vallée Poussin means of the sequence $\{s_n\}$ generated by the sequence $\{\lambda_n\}$. The series $\sum a_n$ is said to be summable $|V, \lambda|$, if the series

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda) - V_n(\lambda)|$$

is convergent. Let $\lambda(x)$ $(x \geqq 1)$ be a continuous function linear between n and $n+1$, furthermore $\lambda(n)=\lambda_n$.

Let $f(x)$ be a function integrable in the sense of Lebesgue and periodic with period 2π and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum A_n(x). \tag{1}$$

For a fixed of x, we write

$$\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$$

and

$$\Phi(t) = \int_0^t |\varphi(u)|\, du.$$

*) This research was made while the author stood at the University of Toronto, by a grant of the National Research Council of Canada.

Let $\{\mu_\nu\}$ be a convex and bounded sequence. CHOW [1] proved that *the series*

$$\sum A_n(z)\mu_n \tag{2}$$

is summable $|C, 1|$ [1]) *almost everywhere, if the series* $\sum n^{-1}\mu_n$ *converges.* CHOW [1] proved also that, *if* $f(x)$ *belongs to the class* H, *i. e. if* $f(x)$ *and its conjugate function are both L-integrable, and if* $\{\mu_n\}$ *is a sequence such that the series*

$$\sum_{n=1}^{\infty} n(\Delta\mu_n)^2, \quad \sum_{n=1}^{\infty}\frac{\mu_n^2}{n} \tag{3}$$

converge, then the series (2) *is summable* $|C, 1|$ *for almost all values of* z. *If* μ_n *is convex, it is sufficient to assume the convergence of the second series* (3).

Later PATI [4] proved: *if* $\sum n^{-1}\mu_n<\infty$ *and at a fixed point of* x $\Phi(t)=O(t)$ $(t\to 0)$, *then the series* (2) *is summable* $|C, \alpha|$ *for every* $\alpha>1$ *at this point* $z=x$.

Recently HSIANG [2] demonstrated: *if at a fixed point of* x $\Phi(t)=\big(t(\log 1/t)^{-1}\big)$ $(t\to 0)$, *then the series* $\sum A_n(x)/(\log n)^{1+\varepsilon}$ *is summable* $|C, 1|$ *for every* $\varepsilon>0$.

In this field many other interesting results have been obtained mostly by Indian and Chinese mathematicians.

In the present paper, we are going to give some theorems of $|V, \lambda|$-summability, similar to the cited ones. Since in some of our structural conditions both the magnitude of the factor sequence $\{\mu_n\}$ and the strength of the summability appear we obtain new results even in the classical case of $|C, 1|$-summability.

Theorem 1. *If* $\{\mu_n\}$ *is a monotone convex sequence and the series* $\sum\mu_n\lambda_n^{-1}$ *converges, then the series* (2) *is summable* $|V, \lambda|$ *almost everywhere.*

Theorem 2. *If* $f(x)$ *belongs to the class* H *and if* $\{\mu_n\}$ *is monotone convex and satisfies the condition* $\sum n\mu_n^2\lambda_n^{-2}<\infty$, *then the series* (2) *is summable* $|V, \lambda|$ *almost everywhere.*

The following theorems concern the summability at a given point.

Theorem 3. *Let* $\mu(x)$ $(x\geqq 0)$ *be a function monotone decreasing and satisfying the condition*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\lambda_n}<\infty. \tag{4}$$

If

$$\Phi(t)=O\left(\lambda^{-1}\left(\frac{1}{t}\right)\mu\left(\frac{1}{t}\right)\right) \tag{5}$$

as $t\to+0$, *then the series*

$$\sum_{n=1}^{\infty}\mu(n)A_n(x) \tag{6}$$

is summable $|V, \lambda|$ *at the point* x.

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[1]) A series $\sum c_n$ is said to be summable $|C, \alpha|$ $(\alpha\geqq 0)$ if $\sum|\sigma_{n+1}^\alpha-\sigma_n^\alpha|$ where σ_n^α is its n-th Cesàro mean of order α converges.

Theorem 4. *If instead of* (4) *the condition*

$$\sum_{n=4}^{\infty} \frac{\mu(n) \log\log n}{\lambda_n} < \infty \tag{7}$$

is fulfilled, then the condition

$$\Phi(t) = O\left(t\left(\log \frac{1}{t}\right)^{-1}\right) \tag{8}$$

also suffices for the $|V, \lambda|$*-summability of the series* (6).

In this theorem the structural condition (8) is independent of the factor sequence and the summability.

In the special case $\lambda_n = n$, i. e. in the case of $|C, 1|$-summability, Theorems 3 and 4 give the following result:

If either $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} < \infty$ and $\Phi(t) = O\left(t\mu\left(\frac{1}{t}\right)\right)$, or $\sum_{n=4}^{\infty} \frac{\mu(n)\log\log n}{n} < \infty$ and (8) are fulfilled, then the series (6) is summable $|C, 1|$.

From this, by Lemma 4, we have as

Corollary. *Let* $\{p_n\}$ *be a non-increasing sequence of positive numbers and let* $P_n = p_0 + p_1 + \dots + p_n$. *If either* $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} < \infty$ *and* $\Phi(t) = O\left(t\mu\left(\frac{1}{t}\right)\right)$, *or* $\sum_{n=4}^{\infty} \frac{\mu(n)\log\log n}{n} < \infty$ *and* (8) *are fulfilled, then the series*

$$\sum_{n=1}^{\infty} \frac{\mu(n) P_n A_n(x)}{n}$$

is summable $|N, p_n|$. [2])

Finally we prove the following

Theorem 5. *Let* $0 < \alpha \leqq 1$ *and let* $\mu(x)$ $(x \geqq 0)$ *be a function monotone decreasing and satisfying either*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} < \infty \quad \textit{and} \quad \Phi(t) = O\left(t^\alpha \mu\left(\frac{1}{t}\right)\right), \tag{9}$$

or

$$\sum_{n=4}^{\infty} \frac{\mu(n)\log\log n}{n^\alpha} < \infty \quad \textit{and} \quad \Phi(t) = O\left(t\left(\log\frac{1}{t}\right)^{-1}\right), \tag{10}$$

then the series (6) *is summable* $|C, \alpha|$.

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[2]) A series $\sum a_n$ is said to be summable $|N, p_n|$ if $\sum |t_{n+1} - t_n|$, where $t_n = \sum_{k=0}^{n} \frac{p_k s_{n-k}}{P_n}$ is the n-th Nörlund mean, converges.

From the condition (9), it is easy to see the close connection existing between the power of the summability and the magnitude of the factor sequence.

It is clear, too, that in the special case $\mu(x)=(\log x)^{-1-\varepsilon}$ and $\alpha=1$ the second half of Theorem 5 includes the theorem of HSIANG.

It seems worth while to observe that we can derive analogous structural theorems for the conjugate series

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n(x).$$

Write

$$\Psi(t)=\int_0^t|f(x+u)-f(x-u)|\,du.$$

Then, for example we have the following:

Theorem 6. *Let $\mu(x)$ $(x\geqq 0)$ be a function monotone decreasing and satisfying either*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\lambda_n}<\infty \quad \text{and} \quad \Psi(t)=O\left(t\mu\left(\frac{1}{t}\right)\right) \qquad (t\to 0),$$

or

$$\sum_{n=4}^{\infty}\frac{\mu(n)\log\log n}{\lambda_n}<\infty \quad \text{and} \quad \Psi(t)=O\left(t\left(\log\frac{1}{t}\right)^{-1}\right) \qquad (t\to 0),$$

then the series

$$\sum_{n=1}^{\infty}\mu(n)B_n(x)$$

is $|V,\lambda|$-summable.

The second half of this theorem in case $\lambda_n=n$ and $\mu(x)=(\log x)^{-1-\varepsilon}$ includes the theorem of HSIANG [2] given for the conjugate series.

§ 1. Lemmas

We require the following lemmas.

Lemma 1. (cf. [1], Lemma 2.) *Let*

$$t_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}kA_k(x).$$

Then

$$\sum_{k=1}^{n}|t_k(x)|=o(n)$$

for almost all values of x.

Lemma 2. (cf [1], Lemma 7.) *If $f(x)$ belongs to the class H, the series $\sum n^{-1}|t_n(x)|^2$ converges for almost all values of x.*

Lemma 3. *If* $\{\mu_n\}$ *is convex and* $\sum n^{-1}\mu_n^2<\infty$, *then* $\sum n(\Delta\mu_n)^2$ *converges.*

This lemma holds by the proof of Theorem 2 of Chow [1].

Lemma 4. (cf. [3].) *If a series* $\sum a_n$ *is summable* $|C, 1|$ *and if* $\{p_n\}$ *is a non-increasing sequence of real and non-negative numbers, then the series* $\sum a_n P_n n^{-1}$ *is summable* $|N, p_n|$.

Lemma 5. (cf. [2], Lemma 2.) *Denote*

$$C_n(t) = \sum_{k=1}^{n} k\cos kt,$$

then

$$C_n(t) = O(nt^{-1})$$

for $nt \geqq 1$.

§ 2. Proofs of the theorems

Proof of Theorem 1. An easy computation gives that

$$V_{n+1}(\lambda)-V_n(\lambda) = \frac{1}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n+1}[(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n]a_k.$$

Let $V_n(\lambda; z)$ denote the n-th de la Vallèe Poussin mean of the series (2). Then we have that

$$\sum_{n=1}^{\infty}|V_{n+1}(\lambda; z)-V_n(\lambda; z)| =$$

$$= \sum_{n=1}^{\infty}\left|\frac{1}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n+1}[(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n]\mu_k A_k(z)\right|.$$

Let $\sum_n'$ be the summation over all n satisfying $\lambda_{n+1}=\lambda_n$; and $\sum_n''$ the summation over all n where $\lambda_{n+1}>\lambda_n$. Then, Abel's transformation gives that

$$\sum_n{}' = \sum_n{}'\frac{1}{\lambda_{n+1}}\left|\sum_{k=n-\lambda_n+2}^{n+1}\frac{\mu_k}{k}kA_k(z)\right| =$$

$$= \sum_n{}'\frac{1}{\lambda_{n+1}}\left\{\sum_{k=n-\lambda_n+2}^{n}\left|\sum_{\nu=1}^{k}\nu A_\nu(z)\right|\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)-\right.$$

$$\left.-\frac{\mu_{n-\lambda_n+2}}{n-\lambda_n+2}\left|\sum_{\nu=1}^{n-\lambda_n+1}\nu A_\nu(z)\right|+\frac{\mu_{n+1}}{n+1}\left|\sum_{\nu=1}^{n+1}\nu A_\nu(z)\right|\right\} \equiv \sum_n{}'_1+\sum_n{}'_2+\sum_n{}'_3.$$

Since the inside lower indices $n-\lambda_n+2$ in $\sum_n{}'_1$ are strictly increasing, we have

$$\sum_n{}'_1 = O(1)\sum_{k=1}^{\infty}k|t_k(z)|\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)\sum_{n=k}^{k+\lambda_k-1}\frac{1}{\lambda_n} = O(1)\sum_{k=1}^{\infty}k|t_k(z)|\,\Delta\left(\frac{\mu_k}{k}\right) \equiv A(z).$$

It is easy to see that

$$\sum_n{}_2' + \sum_n{}_3' = O(1)\sum_{n=1}^{\infty}\frac{\mu_n}{\lambda_n}|t_n(z)| \equiv B(z).$$

Using Abel's transformation again, by Lemma 1, we get

$$A(z) = O(1)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}k|t_k(z)|\right)\Delta^2\left(\frac{\mu_n}{n}\right) = O(1)\sum_{n=1}^{\infty}n^2\Delta^2\left(\frac{\mu_n}{n}\right)$$

for almost all z. Since

$$\Delta^2\left(\frac{\mu_n}{n}\right) \leqq \frac{1}{n}\Delta^2\mu_n + \frac{2}{n^2}\Delta\mu_n + \frac{\mu_n}{n^3},$$

we have

$$A(z) = O(1)\sum_{n=1}^{\infty}n\Delta^2\mu_n + O(1) = O(1)\sum_{n=1}^{\infty}\Delta\mu_n + O(1) = O(1) \tag{2.1}$$

for almost all z. Similarly, by Lemma 1, it follows that

$$B(z) = O(1)\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}|t_k(z)|\right)\Delta\left(\frac{\mu_n}{\lambda_n}\right) = O(1)\sum_{n=1}^{\infty}n\,\Delta\left(\frac{\mu_n}{\lambda_n}\right) = O(1) \tag{2.2}$$

holds for almost all z. This means that the sum $\sum_n{}'$ converges almost everywhere.

The estimation of $\sum_n{}''$ is somewhat more tricky. We obtain, with the aid of the Abel transformation, that

$$\sum_n{}'' = \sum_n{}''\frac{1}{\lambda_n\lambda_{n+1}}\left|\sum_{k=n-\lambda_n+2}^{n+1}(\lambda_n-n-1+k)\frac{\mu_k}{k}kA_k(z)\right| =$$

$$= O(1)\sum_n{}''\frac{1}{\lambda_n^2}\left\{\sum_{k=n-\lambda_n+2}^{n}k|t_k(z)|\left|(\lambda_n-n-1+k)\frac{\mu_k}{k}-(\lambda_n-n+k)\frac{\mu_{k+1}}{k+1}\right|+\right.$$

$$\left.+(n-\lambda_n)|t_{n-\lambda_n+1}(z)|\frac{\mu_{n-\lambda_n+2}}{n-\lambda_n+2}+(n+1)|t_{n+1}(z)|\frac{\lambda_n\mu_{n+1}}{n+1}\right\} \equiv \sum_n{}_1''+\sum_n{}_2''+\sum_n{}_3''.$$

Since $\left|(\lambda_n-n-1+k)\frac{\mu_k}{k}-(\lambda_n-n+k)\frac{\mu_{k+1}}{k+1}\right| \leqq \lambda_k\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)+\frac{\mu_k}{k}$ we have that

$$\sum_n{}_1'' \leqq \sum_{k=2}^{\infty}|t_k(z)|\left(k\lambda_k\left(\frac{\mu_k}{k}-\frac{\mu_{k+1}}{k+1}\right)+\mu_k\right)\sum_{n\geqq k}{}''\frac{1}{\lambda_n^2}.$$

Because Σ'' has only the indices n having the property $\lambda_{n+1}>\lambda_n$, it follows that

$$\sum_{n\geqq k}{}''\frac{1}{\lambda_n^2} \leqq \sum_{\nu=\lambda_k}^{\infty}\frac{1}{\nu^2} = O(1)\frac{1}{\lambda_k},$$

hence we obtain, by (2. 1) and (2. 2) that

$$\Sigma_n'' 1 = O(1) \sum_{k=2}^{\infty} k|t_k(z)| \left(\frac{\mu_k}{k} - \frac{\mu_{k+1}}{k+1}\right) + \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda_k} |t_k(z)| = O(1)$$

for almost all z. Since, by (2. 2) $B(z) = O(1)$ almost everywhere,

$$\Sigma_n'' 2 + \Sigma_n'' 3 = O(1) \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} |t_n(z)| \equiv O(1) B(z) = O(1)$$

for almost all z, i. e.

$$\Sigma_n'' = O(1)$$

for almost all z, too.

This completes the proof of Theorem 1.

Proof of Theorem 2. As in the proof of Theorem 1, we have

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda; z) - V_n(\lambda; z)| =$$

$$= O(1) \left\{ \sum_{k=1}^{\infty} k|t_k(z)| \Delta\left(\frac{\mu_k}{k}\right) + \sum_{k=1}^{\infty} \frac{\mu_k}{\lambda_k} |t_k(z)| \right\} \equiv O(1)(A(z) + B(z)).$$

By CAUCHY's inequality and Lemma 2, we get

$$B(z) \leqq \left\{ \sum_{k=1}^{\infty} \frac{k\mu_k^2}{\lambda_k^2} \right\}^{1/2} \left\{ \sum_{k=1}^{\infty} \frac{|t_k(z)|^2}{k} \right\}^{1/2} = O(1)$$

for almost all z and

$$A(z) \leqq \left\{ \sum_{k=1}^{\infty} k^3 \left(\Delta\left(\frac{\mu_k}{k}\right)\right)^2 \right\}^{1/2} \left\{ \sum_{k=1}^{\infty} \frac{|t_k(z)|^2}{k} \right\}^{1/2}.$$

In order to prove the theorem, it is sufficient to demonstrate that

$$\sum_{k=1}^{\infty} k^3 \left(\Delta\left(\frac{\mu_k}{k}\right)\right)^2 < \infty. \tag{2. 3}$$

Since

$$\left(\Delta\left(\frac{\mu_n}{n}\right)\right)^2 \leqq 4\left(\frac{1}{n^2}(\Delta\mu_n)^2 + \frac{\mu_n^2}{n^4}\right),$$

so we have that

$$\sum_{k=1}^{\infty} k^3 \left(\Delta\left(\frac{\mu_k}{k}\right)\right)^2 \leqq 4 \sum_{k=1}^{\infty} k(\Delta\mu_k)^2 + O(1).$$

From this, by Lemma 3, (2. 3) follows, that is, the theorem is proved.

Proof of Theorem 3. Let $V_n(\lambda; x)$ denote the n-th de la Vallée Poussin mean of series (6). Using that

$$A_n(x) = \frac{1}{\pi} \int_0^{\pi} \varphi(t) \cos nt \, dt. \tag{2. 4}$$

we obtain the equality

$$d_n(x) \equiv \pi |V_{n+1}(\lambda; x) - V_n(\lambda; x)| =$$

$$= \left| \int_0^\pi \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k) \cos kt \, dt \right|.$$

We write

$$d_n(x) \leqq d_n^1(x) + d_n^2(x) \equiv \left| \int_0^{1/n} \right| + \left| \int_{1/n}^{\pi} \right|.$$

By (5), we get that

$$d_n^1(x) = O(1) \left(\frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k) \int_0^{1/n} |\varphi(t)| \, dt \right) =$$

$$= O(1) \left(\frac{1}{\lambda_n^2 \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k) \right).$$

Let $\alpha_k^{(n)} = [(\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n] \mu(k)/k$. For $d_n^2(x)$ with the aid of the Abel transformation, we obtain

$$d_n^2(x) \leqq \left| \int_{1/n}^{\pi} \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \left\{ \sum_{k=n-\lambda_n+2}^{n} C_k(t) \Delta \alpha_k^{(n)} \right\} dt \right| +$$

$$+ \left| \int_{1/n}^{\pi} \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \frac{(\lambda_{n+1} - \lambda_n)(1 - \lambda_n) + \lambda_n}{n - \lambda_n + 2} \mu(n - \lambda_n + 2) C_{n-\lambda_n+1}(t) \, dt \right| +$$

$$+ \left| \int_{1/n}^{\pi} \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \frac{\lambda_n \mu(n+1)}{n+1} C_{n+1}(t) \, dt \right| \equiv I_1 + I_2 + I_3.$$

In the following steps we shall use that

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = O(1); \tag{2.5}$$

in fact, considering (4) and (5), we have

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t} dt = \left(\frac{\Phi}{t} \right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2} dt = O(1) + \int_{1/n}^{\pi} \frac{\mu\left(\frac{1}{t}\right)}{t^2 \lambda\left(\frac{1}{t}\right)} dt =$$

$$= O(1) + \int_{1/\pi}^{n} \frac{\mu(x)}{\lambda(x)} dx = O(1).$$

By the Lemma 5 and (2. 5), we have

$$I_1 = O\left(\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)}\right),$$

$$I_2 = O\left(\frac{1}{\lambda_n^2}[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\mu(n-\lambda_n+2)\right)$$

and

$$I_3 = O\left(\frac{\mu(n)}{\lambda_n}\right).$$

From the above analysis we obtain that

$$d_n(x) = O\left(\frac{1}{\lambda_n^3}\sum_{k=n-\lambda_n+2}^{n+1} k\alpha_k^{(n)} + \frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} + \right.$$

$$\left. + \frac{1}{\lambda_n^2}[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\mu(n-\lambda_n+2) + \frac{\mu(n)}{\lambda_n}\right).$$

From this it follows

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda; x) - V_n(\lambda; x)| =$$

$$= O\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\sum_{k=n-\lambda_n+2}^{n+1} k\alpha_k^{(n)} + \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} + \right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\mu(n-\lambda_n+1) + \sum_{n=1}^{\infty}\frac{\mu(n)}{\lambda_n}\right) = O(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4).$$

The fourth sum is finite by (4). Let $\underset{n}{\Sigma'_k}$ and $\underset{n}{\Sigma''_k}$ $(k=1, 2, 3)$ be defined as in Theorem 1. In the estimations of $\underset{n}{\Sigma'_k}$, we shall use that the inside lower indices $(n-\lambda_n+2)$ are strictly increasing. So we have by (4)

$$\underset{n}{\Sigma'_1} = \sum_n{}' \frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n+1}\mu(k) \leqq \sum_{k=1}^{\infty}\frac{\mu(k)}{\lambda_k} = O(1),$$

$$\underset{n}{\Sigma'_2} = \sum_n{}' \frac{1}{\lambda_n}\sum_{k=n-\lambda_n+2}^{n} k\left(\frac{\mu(k)}{k}-\frac{\mu(k+1)}{k+1}\right) \leqq \sum_{k=1}^{\infty} k\left(\frac{\mu(k)}{k}-\frac{\mu(k+1)}{k+1}\right) \leqq$$

$$\leqq \sum_{k=1}^{\infty}\left(\Delta\mu(k)+\frac{\mu(k)}{k}\right) = O(1)$$

and

$$\underset{n}{\Sigma'_3} = \sum_n{}' \frac{1}{\lambda_n}\mu(n-\lambda_n+1) \leqq \sum_n{}' \frac{\mu(n-\lambda+1)}{\lambda_{n-\lambda_n+1}} \leqq \sum_{k=1}^{\infty}\frac{\mu(k)}{\lambda_k} = O(1).$$

In the case $\lambda_{n+1}>\lambda_n$, $\alpha_k^{(n)}=(\lambda_n+k-n-1)\dfrac{\mu(k)}{k}$, so we get that

$$\Sigma_{n}''_1=\sum_n''\frac{1}{\lambda_n^3}\sum_{k=n-\lambda_n+2}^{n+1}(\lambda_n+k-n-1)\mu(k)\leqq$$

$$\leqq\sum_n''\frac{1}{\lambda_n^3}\sum_{k=n-\lambda_n+2}^{n+1}\lambda_k\mu(k)\leqq\sum_{k=2}^{\infty}\lambda_k\mu(k)\sum_{n>k}''\frac{1}{\lambda_n^3}.$$

Because in Σ'' there are only the indices n having the property $\lambda_{n+1}>\lambda_n$, it holds

$$\sum_{n>k}''\frac{1}{\lambda_n^3}\leqq\sum_{\nu=\lambda_k}^{\infty}\frac{1}{\nu^3}=O\left(\frac{1}{\lambda_k^2}\right),$$

so it is easy to see that

$$\Sigma_{n}''_1=O\left(\sum_{k=2}^{\infty}\frac{\mu(k)}{\lambda_k}\right)=O(1).$$

ABEL's transformation gives that

$$\sum_{k=n-\lambda_n+2}^{n}k\Delta\alpha_k^{(n)}\leqq\sum_{k=n-\lambda_n+2}^{n}\alpha_k^{(n)},\tag{2.6}$$

so we have by $n-\lambda_n\geqq k-\lambda_k$ $(n>k)$

$$\Sigma_{n}''_2\leqq\sum_n''\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n}(\lambda_n+k-n-1)\frac{\mu(k)}{k}\leqq$$

$$\leqq\sum_n''\frac{1}{\lambda_n^2}\sum_{k=n-\lambda_n+2}^{n}\frac{\lambda_k\mu(k)}{k}\leqq\sum_{k=2}^{\infty}\frac{\lambda_k\mu(k)}{k}\sum_{n\geqq k}''\frac{1}{\lambda_n^2}\leqq$$

$$\leqq\sum_{k=2}^{\infty}\frac{\lambda_k\mu(k)}{k}\sum_{\nu=\lambda_k}^{\infty}\frac{1}{\nu^2}=O\left(\sum_{k=2}^{\infty}\frac{\mu(k)}{k}\right)=O(1).$$

Finally

$$\Sigma_{n}''_3=\sum_n''\frac{1}{\lambda_n^2}\mu(n-\lambda_n+1)\leqq\mu(1)\sum_n''\frac{1}{\lambda_n^2}\leqq\mu(1)\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}=O(1).$$

From the above analysis we obtain the statement of Theorem 3.

P r o o f o f T h e o r e m 4. The proof is similar to the proof of Theorem 3. The first difference steps at (2. 5); that is, under the conditions (7) and (8), we can only say that

$$\int_{1/n}^{\pi}\frac{|\varphi|}{t}\,dt=O(\log\log n).$$

In fact

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t}\,dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2}\,dt = O(1) + \int_{1/n}^{\pi} \frac{1}{t\log\frac{1}{t}}\,dt =$$

$$\leqq O(1) + O(\log\log n) = O(\log\log n).$$

So we have only the following estimations:

$$I_1 = O\left(\frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)}\right),$$

$$I_2 = O\left(\frac{\log\log n}{\lambda_n^2}\,[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\,\mu(n-\lambda_n+2)]\right)$$

and

$$I_3 = O\left(\frac{(\log\log n)\,\mu(n)}{\lambda_n}\right).$$

With the aid of these and the estimations obtained in the proof of Theorem 3, by (7), we obtain that

$$\sum_{n=4}^{\infty} |V_{n+1}(\lambda; x) - V_n(\lambda; x)| = O(1) +$$

$$O\Bigg(\sum_{n=4}^{\infty} \frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)} +$$

$$+ \sum_{n=4}^{\infty} \frac{\log\log n}{\lambda_n^2}\,[(\lambda_{n+1}-\lambda_n)(1-\lambda_n)+\lambda_n]\,\mu(n-\lambda_n+1)\Bigg).$$

We can demonstrate the finitness of these sums as in Theorem 3. Let us see e. g. the case of the first sum. Let Σ_1' and Σ_1'' denote the suitable sums as in Theorem 3. (7), $\Delta\left(\frac{\mu(k)}{k}\right) \leqq \frac{1}{k}\Delta\mu(k) + \frac{\mu(k)}{k^2}$ and $n-\lambda_n < n+1-\lambda_{n+1}$ give

$$\Sigma_1' = \sum_n{}' \frac{\log\log n}{\lambda_n} \sum_{k=n-\lambda_n+2}^{n} k\left(\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right) \leqq$$

$$\leqq 2\sum_{k=2}^{\infty} k\left(\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right)\log\log k \leqq 2\sum_{k=2}^{\infty} (\log\log k)\,\Delta\mu(k) + O(1) =$$

$$= O(1)\sum_{k=2}^{\infty}\left(\sum_{n=2}^{k} \frac{1}{n\log n}\right)\Delta\mu(k) + O(1) = O(1)\sum_{k=2}^{\infty} \frac{\mu(k)}{k\log k} < \infty.$$

Using (2. 6), $\alpha_k^{(n)} = (\lambda_n + k - n - 1)\frac{\mu(k)}{k}$ (for $\lambda_{n+1} > \lambda_n$) and $n-\lambda_n \geqq k-\lambda_k$ $(n>k)$, we have

$$\Sigma_1'' = \sum_n'' \frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} (\lambda_n+k-n-1)\frac{\mu(k)}{k} \leqq$$

$$\leqq \sum_n'' \frac{\log\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} \frac{\lambda_k\mu(k)}{k} \leqq \sum_{k=2}^{\infty} \frac{\lambda_k\mu(k)}{k} \sum_{n\geqq k}^{n-\lambda_n+2\leqq k} \frac{\log\log n}{\lambda_n^2} \leqq$$

$$\leqq \sum_{k=2}^{\infty} \frac{\lambda_k\mu(k)}{k} \sum_{\nu=\lambda_k}^{\infty} \frac{\log\log(\nu+k)}{\nu^2} = O\left(\sum_{k=2}^{\infty} \frac{\mu(k)\log\log k}{k}\right) = O(1).$$

We can prove similarly that the second sum is finite, so Theorem 4 follows.

Proof of Theorem 5. If the sequence $\left\{\mu(n)n\Big/\binom{n+\alpha}{n}\right\}$ is monotone, then the statement of Theorem 5 follows from the Corollary with this sequence, namely it is well known that if

$$p_n = \binom{n+\alpha-1}{\alpha-1} \qquad (\alpha>0)$$

then the Nörlund mean reduces to the Cesàro mean of order α.

In the general case we give a short direct proof, but only under conditions (9) we shall detail it because the other case is similar. Using (2. 4), an easy computation gives that for the Cesàro means of series (6)

$$\pi n(\sigma_n^\alpha(x)-\sigma_{n-1}^\alpha(x)) = \int_0^\pi \varphi(t)\left(\frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1}\mu(k)k\cos kt\right)dt \equiv \tau_n^\alpha(x).$$

We write

$$\tau_n^\alpha(x) = \tau_n^\alpha(1;x)+\tau_n^\alpha(2;x) = \int_0^{1/n} + \int_{1/n}^{\pi}.$$

From (9) we get that

$$\tau_n^\alpha(1;x) = O\left(\frac{1}{n^\alpha}\sum_{\nu=1}^{n}\mu(\nu)(n+1-\nu)^{\alpha-1}\nu\int_0^{1/n}|\varphi(t)|\,dt\right) =$$

$$= O\left(\frac{1}{n^{2\alpha}}\sum_{\nu=1}^{n}\mu(\nu)(n+1-\nu)^{\alpha-1}\nu\right).$$

For $\tau_n^\alpha(2;x)$ with the aid of the Abel transformation, we obtain

$$\pi\tau_n^\alpha(2;x) = \int_{1/n}^{\pi}\varphi(t)\frac{1}{A_n^\alpha}\left\{\sum_{\nu=1}^{n-1}C_\nu(t)(\mu(\nu)A_{n-\nu}^{\alpha-1}-\mu(\nu+1)A_{n-\nu-1}^{\alpha-1})\right\}dt +$$

$$+\int_{1/n}^{\pi}\varphi(t)\frac{1}{A_n^\alpha}\mu(n)C_n(t)\,dt \equiv I_1+I_2.$$

Let $d_\nu=|\mu(\nu)A_{n-\nu}^{\alpha-1}-\mu(\nu+1)A_{n-\nu-1}^{\alpha-1}|$. As (2. 5) is valid now, so by the Lemma 5, we have

$$I_1=O\left(\frac{1}{n^\alpha}\sum_{\nu=1}^{n-1}\nu\, d_\nu \int_{1/n}^{\pi}\frac{|\varphi|}{t}\,dt\right)=O\left(\frac{1}{n^\alpha}\sum_{\nu=1}^{n-1}\nu\, d_\nu\right)$$

and

$$I_2=O\left(\frac{\mu(n)n}{n^\alpha}\int_{1/n}^{\pi}\frac{|\varphi|}{t}\,dt\right)=O\left(\frac{\mu(n)n}{n^\alpha}\right).$$

From the above analysis we obtain that

$$\sum_{n=2}^{\infty}|\sigma_n^\alpha(x)-\sigma_{n-1}^\alpha(x)|=O\left(\sum_{n=2}^{\infty}\frac{1}{n^{1+2\alpha}}\sum_{\nu=1}^{n}\mu(\nu)(n+1-\nu)^{\alpha-1}\nu+\right.$$

$$\left.+\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=1}^{n-1}\nu\, d_\nu+\sum_{n=2}^{\infty}\frac{\mu(n)}{n^\alpha}\right).$$

The third sum is finite by (9). ABEL's transformation gives that

$$\sum_{n=2}^{\infty}\frac{1}{n^{1+2\alpha}}\sum_{\nu=1}^{n}\mu(\nu)(n+1-\nu)^{\alpha-1}\nu\leqq\sum_{\nu=1}^{\infty}\mu(\nu)\nu\sum_{n=\nu}^{\infty}(n+1-\nu)^{\alpha-1}\frac{1}{n^{1+2\alpha}}=$$

$$=O(1)\sum_{\nu=1}^{\infty}\mu(\nu)\nu\frac{1}{\nu^{1+\alpha}}=O(1),$$

i. e. the first sum is finite, too. Putting $\bar n=\left[\frac{n}{2}\right]$, we can write the second sum into two sums:

$$\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=1}^{n}\nu\, d_\nu=\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=1}^{\bar n}\nu\, d_\nu+\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=\bar n+1}^{n-1}\nu\, d_\nu. \tag{2. 7}$$

The first sum under (2. 7) is less than

$$\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=1}^{\bar n}\nu(\mu(\nu)A_{n-\nu}^{\alpha-1}-\mu(\nu+1)A_{n-\nu}^{\alpha-1})+O(1)=$$

$$=O(1)\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=1}^{\bar n}\nu n^{\alpha-1}\Delta\mu(\nu)=O(1)\sum_{\nu=1}^{\infty}\nu\,\Delta\mu(\nu)\sum_{n=2\nu}^{\infty}\frac{1}{n^2}=$$

$$=O(1)\sum_{\nu=1}^{\infty}\Delta\mu(\nu)=O(1).$$

Similarly, the second sum under (2. 7) is not greater than

$$O(1)\sum_{n=2}^{\infty}\frac{1}{n^{1+\alpha}}\sum_{\nu=\bar n+1}^{n-1}\nu(n-\nu)^{\alpha-1}\Delta\mu(\nu)+O(1)=O(1)\sum_{\nu=2}^{\infty}\Delta\mu(\nu)=O(1).$$

We have also that

$$\sum_{n=2}^{\infty} |\sigma_n^\alpha(x) - \sigma_{n-1}^\alpha(x)|$$

converges, so our statement is proved.

Proof of Theorem 6. It runs similarly to the proof of Theorems 3 and 4.

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(Received December 28, 1966)

Corrigendum to the paper:

On the strong summability of orthogonal series*)

By L. LEINDLER in Szeged

Professor G. SUNOUCHI kindly drew my attention to a gap in Lemma 3. The given proof of this lemma is correct only in that case if under the condition (3. 1) $\frac{pk}{p-1}=2$ is fulfilled instead of $\frac{pk}{p-1}\geqq 2$. [1]) Consequently, the conditions of the theorems are satisfied only for $k<2$, so the theorems and corollaries are also valid for such powers k.

However, instead of Lemma 3, we can prove the following

Lemma. *Let $k>0$ and $\sum c_n^2<\infty$. If there exists a $p>1$ such that the conditions* (3. 1) *are satisfied, then for $\gamma>1-\frac{p-1}{pk}$ we have*

$$\int_a^b \left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^{\gamma}(x)|^k\right)^{1/k}\right\}^2 dx \leqq K\sum_{n=0}^{\infty}c_n^2.$$

Using this lemma we obtain that Theorem 1 and 2, for arbitrary $k>0$ but with $\gamma>1-\frac{p-1}{pk}$ instead of $\gamma>\frac{1}{2}$, remain valid.

The modified theorems also include the theorems of SUNOUCHI [2]).

Proof of Lemma. We have as in the proof of Lemma 3 that

$$\int_a^b \left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^{\gamma}(x)|^k\right)^{1/k}\right\}^2 dx \leqq$$

$$\leqq K_1\int_a^b \left(\sum_{\nu=1}^{\infty}\nu^{-1}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^{\gamma}(x)|^{qk}\right)^{2/qk} dx.$$

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*) *Acta Sci. Math.*, **27** (1966), 217—228.

[1]) We use the same notation as in the paper.

[2]) G. SUNOUCHI, On the strong summability of orthogonal series, *Acta. Sci. Math.*, **27** (1966), 71—79.

Using a theorem of FLETT [3]) we obtain that

$$\int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1} |\sigma_\nu^{\gamma-1}(x) - \sigma_\nu^{\gamma}(x)|^{qk} \right)^{2/qk} dx \leqq K_2 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1} |\sigma_\nu^{\alpha-1}(x) - \sigma_\nu^{\alpha}(x)|^2 \right) dx,$$

where $\frac{1}{2} < \alpha < \gamma - \frac{1}{2} + \frac{p-1}{pk}$.

From this, by Lemma 2, we get the statement of the Lemma.

(Received April 1, 1967)

[3]) On an extension of absolute summability and some theorems of LITTLEWOOD and PALEY, *Proc. London Math. Soc.*, 7 (1957), 113—141, cf. in particular p. 115.

Bibliographie

Eduard Čech, Topological spaces. Revised edition by ZDENĚK FROLIK and MIROSLAV KATĚTOV, 893 pages (errata insert), Publishing House of the Czechoslovak Academy of Sciences, Prague; Interscience Publishers (John Wiley and Sons), London—New York—Sydney, 1966.

E. ČECH directed in Brno from 1936 until 1939 a seminar on Topology. The substance of his book on Topological Spaces arose from this seminar, and the manuscript was completed during the war. However, this book did not appear in Czech language before 1959. Though a number of minor changes was made on the first manuscript, it was impossible to give account in it of the recent and extremely rapid development of General Topology. Therefore, after the death of the author, Z. FROLÍK and M. KATĚTOV decided to prepare a revised edition in English, re-writing the whole book in such an extent that its dimensions grew nearly to the double of the first version. In the same time, the new book got the character of a monograph, while the original was rather a text-book.

The fundamental feature of this book is — quite similarly to the first edition — the effort to investigate all concepts under circumstances as general as possible. Thus, while in most works on General Topology the basic concepts are topology and topological space, here closure operations and closure spaces are principally considered. Under a closure operation on a set E it is understood an operation u for the subsets of E such that $u\emptyset=\emptyset$, $uA\supset A$ and $u(A\cup B)=uA\cup uB$ for A, $B\subset E$; a topology is a closure operation such that $uuA=uA$ for all $A\subset E$. In a similar manner, a proximity on E denotes in the usual sense a relation p for the subsets of E such that $\emptyset$ non pE, ApB implies BpA, $A\cap B\neq\emptyset$ implies ApB, $(A\cup B)$pC holds if and only if ApC or BpC, and $(*)$ A non pB implies the existence of U and V such that A non p$E-U$, B non p$E-V$. Now, in the terminology of this book, a proximity is a relation p satisfying the above conditions with the exception of $(*)$. Finally, instead of uniformities, the present book studies semi-uniformities, where a semi-uniformity $\mathcal{U}$ on E is a filter on $E\times E$ such that $U\in\mathcal{U}$ implies $\Delta\subset U$ (where Δ denotes the diagonal of $E\times E$) and $U\in\mathcal{U}$ implies $U^{-1}\in\mathcal{U}$. As well-known, $\mathcal{U}$ is a uniformity if, moreover, $U\in\mathcal{U}$ implies the existence of $U_1\in\mathcal{U}$ with $U_1\circ U_1\subset U$.

It is worth while to note that all these generalizations of the usually investigated concepts of General Topology can be easily described by means of the theory elaborated by the reviewer (*Fondements de la topologie générale*, Budapest et Paris, 1960). As a matter of fact, topologies, proximities (in the usual sense) and uniformities are special cases of the general concept of syntopogenous structures, among which perfect and simple structures correspond to topologies, symmetrical and simple structures to proximities, and perfect and symmetrical structures to uniformities. Now, if in the definition of a syntopogenous structure, axiom (S_2) is omitted, one precisely obtains closure operations, proximites (in the sense of the present book) and semi-uniformities instead of topologies, proximities (in the usual sense) and uniformities. Consequently, it is rather natural that a very extensive part of the theory of the latter "classical" concepts admits a generalization for the former mentioned more general concepts studied in the present book.

The material is divided into seven chapters and an appendix. The first two chapters, written by M. KATĚTOV, have the titles "Classes and relations" and "Algebraic structures and order", and serve as introduction to the remaining part of the book, due to Z. FROLÍK and treating General Topology. These introductory chapters contain an axiomatic but nonformal exposition of set theory, and present much novelty in basic ideas and in methods.

The following chapters (Topological spaces, Uniform and proximity spaces, Separation, Generation of topological spaces, Generation of uniform and proximity spaces) and the appendix (Compactness and completeness) give a detailed exposition of the theory of the above mentioned general structures and their interrelations. Not only the proofs are presented in an easily readable manner but extensive introductions serve to elucidate the concepts and the results of each chapter and each section, and numerous examples and remarks illustrate the meaning of every definition and proposition. A great number of exercises furnish still more illustration and essential additional material.

A short bibliography is added, whereas the text itself does not contain any references to the literature. The terminology differs somewhat from the usually adopted one, but is consequent, and a detailed index facilitates the identification of terms and notations.

Ákos Császár (Budapest)

Hanna Neumann, Varieties of Groups (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37), XII+192 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1967.

This book introduces the reader having some familiarity with the basic concepts of the group theory, to the study of the varieties of groups and presents him the results achieved in this new, branch of research. Let us consider the class of all groups satisfying each one of a given set of identities. Such a class is called a *variety*, or a primitive class. For example, the class of all abelian groups as well as the class of all nilpotent groups of class n for any positive integer n are varieties.

The concept of variety may be applied without modification to universal algebras; as a matter of fact, this concept, just for this general case, was introduced by G. BIRKHOFF some three decades ago. The study of varieties of groups obtained an essential impetus only after 1950. Since then, it advanced, however, rapidly and in this progress the author herself, as well as her husband B. H. NEUMANN, her son PETER NEUMANN, and others belonging to this research group have played an important role.

This work, written in a concise and elegant style and having a clear setting up, will surely be a very enjoyable reading for the mathematicians interested in group theory, and wanting to get acquainted with the subject. In spite of its relatively short extent, the book contains a large material including a great number of unpublished results. At the same time, the author illuminates the connections of the subject treated with other chapters of group theory drawing hereby the reader's attention to the fact that varieties prove to be an effective tool in group-theoretical investigations.

The book consists of five chapters. The first chapter has an introductory character, the second one deals with the products of varieties. By the product of the varieties $\mathfrak{A}$ and $\mathfrak{B}$, it is meant the class of all groups which are Schreier extensions of a group in $\mathfrak{A}$ by a group in $\mathfrak{B}$. Concerning products let us mention a deep and surprising result: all non-trivial varieties of groups form a free semigroup with respect this product as multiplication. Chapter 3 is devoted to a study of the varieties of nilpotent groups. Chapter 4 deals with some properties of (relatively) free groups in varieties. Among others, all the varieties are described for which the free groups have all their subgroups free also. These are exactly the following: the class of all groups, the class of all abelian groups and the classes of abelian groups with a fixed prime exponent. Finally, Chapter 5 treats the connections between varieties and finite groups belonging to them. The book is completed by a bibliography containing about 150 items.

It may be expected that the appearing of this book gives a further impulse to investigations not only on group varieties, but also on varieties of other types of algebraical systems. To motivate this hope it suffices to refer to a recent article of A. J. MAL'CEV in which some results concerning products of group varieties are generalized for normal varieties of universal algebras.

Béla Csákány (Szeged)

N. Bourbaki, Éléments de mathématique, Fascicule XXXII, **Théories Spectrales,** Chapitres 1 et 2: Algèbres normées, Groupes localement compacts commutatifs (Actualités Scientifiques et Industrielles, 1332), 166 pages, Hermann, Paris, 1967.

Voici la première partie d'un nouveau livre de l'auteur célèbre!

Dans le Chap. 1 on étudie d'abord les algèbres à élément unité, appelées „algèbres unifères", principalement les algèbres normées et les algèbres de Banach, et le spectre d'un élément dans une telle algèbre. L'exposition du calcul fonctionnel holomorphe dans les algèbres de Banach unifères commutatives est faite dans une grande généralité; on y trouve des résultats récents. Puis on passe à l'étude de deux types fondamentaux d'algèbres de Banach: les algèbres de Banach commutatives régulières, qui ont des applications dans l'analyse harmonique, et les C^*-algèbres, appelées „algèbres stellaires" qui sont d'une grande importance dans l'étude des groupes localement compacts: à un tel groupe G on peut associer canoniquement une algèbre stellaire $\mathrm{St}(G)$. On traite aussi les algèbres de fonctions continues sur un espace compact; on a inclus plusieurs résultats de date récente.

Dans le Chap. 2 on étudie les fondements de l'analyse harmonique et la théorie des groupes localement compacts commutatifs. Pour un tel groupe G l'algèbre stellaire $St(G)$, étant commutative, est isomorphe à l'algèbre des fonctions complexes continues, zéro à l'infini, sur un espace localement compact (d'après un résultat fondamental, exposé au Chap. 1). C'est ce théorème qui apparaît ici comme l'outil principal pour démontrer le théorème de Plancherel. Alors la route est libre pour établir la théorie de la dualité pour les groupes localement compacts commutatifs et la formule d'inversion pour la transformation de Fourier des fonctions intégrables (pour la mesure de Haar). Les propriétés fonctorielles de la dualité et la théorie de structure des groupes localement compacts commutatifs sont exposées d'une manière très élégante. La formule de Poisson est établie dans une grande généralité (on peut mentionner un cas qui n'est pas traité ici: la formule s'applique à toute fonction continue, à support compact, telle que la transformée de Fourier soit intégrable pour la mesure de Haar du groupe dual; ce cas est utile dans la Théorie de Nombres et permet aussi de démontrer la Proposition 9, p. 128, d'une manière plus satisfaisante). Enfin l'auteur donne une exposition du théorème taubérien de Wiener et de ses ramifications et généralisations (l'exposition étant d'ailleurs strictement traditionnelle); plusieurs de celles-ci sont données en forme d'exercices (l'exercice 10 c, p. 158, peut s'étendre aux suites telles que la réunion soit fermée).

C'est, en somme, un exposé très riche, qui sera fort utile aux lecteurs sérieux. On attendra avec un grand intérêt les chapitres suivants.

H. Reiter (Utrecht)

William A. Veech, A second course in complex analysis, IX+246 pages, New York, N. Y., W. A. Benjamin, Inc., 1967.

For this "second course" the following topics are chosen: 1. Analytic continuation (Germs and their composition, Covering surfaces, etc.). 2. Geometric considerations (Linear transformations, Noneuclidean geometry, The Schwarz reflection principle, etc.). 3. The mapping theorems of Riemann and Koebe (Lindelöf's lemma, Continuity at the boundary, etc.). 4. The modular function (Schottky's, Picard's, and Bloch's theorems; The Koebe — Faber distortion theorem, etc.). 5. The Hadamard product theorem (Canonical products, The gamma function, etc.). 6. The prime number theorem.

Many problems are added for solution. These are extremely different in level. E. g., one problem (p. 10) is to prove that the value of the integral of $(z-z_0)^{-1}$ on a circle including z_0 does not depend on z_0. (This shows that almost nothing from a standard "first course" is assumed.) As a contrast, a problem (p. 32) asks for a proof of the "Brouwer fixed point theorem" for homeomorphisms of the closed disk.

There is a bibliography on "books which have directly influenced this work, and the foremost among these are the books of Carathéodory". Bieberbach's "Lehrbuch der Funktionentheorie" is not listed, the "Complete Poems of Robert Frost" are.

Béla Sz.-Nagy (Szeged)

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