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INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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On intertwining dilations. II

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1. In this paper we shall consider only (linear bounded) operators on (either all real, or all complex) Hilbert spaces. As usual, $L(\mathfrak{H}', \mathfrak{H})$ will denote the space of all operators from $\mathfrak{H}'$ into $\mathfrak{H}$ and by $L(\mathfrak{H})$ the space $L(\mathfrak{H}, \mathfrak{H})$. Let $T_i \in L(\mathfrak{H}_i)$ be a contraction; and let $U_i \in L(\mathfrak{K}_i)$ be its minimal isometric dilation $(i=1, 2)$. Also, let us denote by $I(T_1; T_2)$ the set of all operators $A \in L(\mathfrak{H}_2, \mathfrak{H}_1)$ intertwining T_1 and T_2 (i.e. $T_1A = AT_2$). By an *exact intertwining dilation* (EID) of $A \in I(T_1; T_2)$ we mean any $B \in L(\mathfrak{K}_2, \mathfrak{K}_1)$ satisfying

$$(1.1) \qquad P_{\mathfrak{H}_1}B = AP_{\mathfrak{H}_2}, \quad B \in I(U_1; U_2) \quad \text{and} \quad \|B\| = \|A\|,$$

(where $P_{\mathfrak{H}_i}$ is the orthogonal projection of $\mathfrak{K}_i$ onto $\mathfrak{H}_i$ $(i=1, 2)$).

In order to state our sufficient and necessary conditions for the uniqueness of the EID of a contraction $\in I(T_1; T_2)$ we also need the concept of the regularity of a factorization of a contraction as a product of two contractions (see [9], Ch. VII, § 3 and [10]). Namely, for two contractions $A_1 \in L(\mathfrak{A}, \mathfrak{B})$, $A_2 \in L(\mathfrak{B}, \mathfrak{A}_*)$ the factorization of $A_2A_1 \in L(\mathfrak{A}, \mathfrak{A}_*)$ as the product of A_2 and A_1 is called *regular* if

$$(1.2) \qquad \{D_{A_2}A_1a \oplus D_{A_1}a : a \in \mathfrak{A}\}^- = (D_{A_2}\mathfrak{B})^- \oplus (D_{A_1}\mathfrak{A})^-,$$

where, as usual, for any contraction C, D_C denotes the defect operator $(1-C^*C)^{1/2}$.

Our main result which was suggested by [1], [2] and [3] is given by the following

Theorem 1.1. *Let $A \in L(\mathfrak{H}_2, \mathfrak{H}_1)$, $\|A\|=1$, intertwine the contractions T_1 and T_2. A sufficient and necessary condition for A to have a unique exact intertwining dilation is that at least one of the factorizations $A \cdot T_2$ or $T_1 \cdot A$ (of $AT_2 = T_1A$) be r egular.*

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The next three sections are devoted to the proof of this theorem. Some complements and connections with results of [1], [2], [3] and [5] will be discussed in sections 5 and 6.

The authors take this opportunity to express their thanks to Prof. B. Sz.-Nagy for his stimulating interest in this research.

2. Let us start with some simple preliminaries. For a contraction $T_i \in L(\mathfrak{H}_i)$ we denote, as above, by $U_i \in L(\mathfrak{K}_i)$ its minimal isometric dilation; and we shall denote by $\hat{U}_i \in L(\hat{\mathfrak{K}}_i)$ the minimal unitary dilation of U_i, which is also the minimal unitary dilation of T_i $(i=1, 2)$.

By the construction of $\hat{U}_i$ (see [9], Ch. I and II) it is known that $\hat{U}_i$ is the minimal unitary dilation and $U_i^{(*)} = \hat{U}_i^{-1}|\mathfrak{K}_i^{(*)}$ is the minimal isometric dilation, of T_i^*, where

$$\mathfrak{K}_i^{(*)} = \hat{\mathfrak{K}}_i \ominus \bigvee_{n=0}^{\infty} U_i^n \mathfrak{L}_i \quad \text{and} \quad \mathfrak{L}_i = ((U_i - T_i)\mathfrak{H}_i)^- \quad (i = 1, 2).$$

Also, it is well known that any EID B of A has a unique extension $\hat{B} \in L(\hat{\mathfrak{K}}_2, \hat{\mathfrak{K}}_1)$ satisfying: $\hat{B}\hat{U}_2 = \hat{U}_1 \hat{B}$, $\|\hat{B}\| = \|A\|$ and $\hat{P}_{\mathfrak{H}_1} \hat{B}|\mathfrak{H}_2 = A$, where $\hat{P}_{\mathfrak{H}_1}$ denotes the orthogonal projection of $\hat{\mathfrak{K}}_1$ onto $\mathfrak{H}_1$ ([9], Ch. II, § 2). Now, it is easy to see that if $B_* \in I(U_2^{(*)}; U_1^{(*)})$ is an EID of $A^* \in I(T_2^*; T_1^*)$ then $(\hat{B}_*)^*|\mathfrak{K}_2$ is an EID of A, and conversely, if $B \in I(U_1; U_2)$ is an EID of $A \in I(T_1; T_2)$ then $(\hat{B})^*|\mathfrak{K}_1^*$ is an EID of A^*. So we can conclude with the following

Lemma 2.1. *$A \in I(T_1; T_2)$ has a unique EID if and only if $A^* \in I(T_2^*; T_1^*)$ has a unique EID.*

Another simple fact is condensed in the following

Remark 2.1. With the above notations, let $A \in I(T_1; T_2)$ be a contraction and let $\tilde{A} = AP_{\mathfrak{H}_2}$. Plainly, $\tilde{A} \in I(T_1; U_2)$; and any EID of $\tilde{A}$ is an EID of A and vice-versa (see [9], Ch. II, § 2). Consequently, A has a unique EID if and only if $\tilde{A}$ enjoys the same property.

Finally, in the sequel we shall also use the following

Lemma 2.2. *Let $A \in L(\mathfrak{A}, \mathfrak{B})$, $T \in L(\mathfrak{A})$ be contractions and U the minimal isometric dilation of T on $\tilde{\mathfrak{A}} = \bigvee_{n=0}^{\infty} U^n \mathfrak{A}$. Let $\tilde{A} = AP \in L(\tilde{\mathfrak{A}}, \mathfrak{B})$, where P is the orthogonal projection of $\tilde{\mathfrak{A}}$ onto $\mathfrak{A}$. Then, the factorization $\tilde{A} \cdot U$ of $\tilde{A}U$ is regular if and only if so is the factorization $A \cdot T$ of AT.*

Proof. Let us first observe that

$$\begin{aligned} (2.1) \qquad \|D_{\tilde{A}}(\tilde{a} - U\tilde{a}')\|^2 &= \|\tilde{a} - U\tilde{a}'\|^2 - \|AP(\tilde{a} - U\tilde{a}')\|^2 = \\ &= \|D_A P(\tilde{a} - U\tilde{a}')\|^2 + \|(I-P)(\tilde{a} - U\tilde{a}')\|^2 = \\ &= \|D_A(P\tilde{a} - TP\tilde{a}')\|^2 + \|(I-P)(\tilde{a} - U\tilde{a}')\|^2, \end{aligned}$$

for all $\tilde{a}, \tilde{a}' \in \tilde{\mathfrak{A}}$. Now, let us assume that the factorization $\tilde{A} \cdot U$ of $\tilde{A}U$ is regular, i.e.

$$(D_{\tilde{A}} U \tilde{\mathfrak{A}})^- = (D_{\tilde{A}} \tilde{\mathfrak{A}})^-. \tag{2.2}$$

For any $a, a' \in \mathfrak{A}$, we consider

$$\tilde{a} = a + (U-T)a' \in \tilde{\mathfrak{A}}. \tag{2.3}$$

Then, from (2.2) it follows that there exists a sequence $(\tilde{a}_j)_{j=1}^{\infty} \subset \tilde{\mathfrak{A}}$ such that

$$\|D_{\tilde{A}}(\tilde{a} - U\tilde{a}_j)\| \to 0 \quad (j \to \infty). \tag{2.4}$$

Also, for $\tilde{a}$ and $\tilde{a}_j$ satisfying (2.3) and (2.4), we have, by (2.1)

$$\begin{aligned}\|D_{\tilde{A}}(\tilde{a} - U\tilde{a}_j)\|^2 &= \|D_A(a - TP\tilde{a}_j)\|^2 + \|(U-T)a' - (I-P)U\tilde{a}_j\|^2 = \\ &= \|D_A(a - TP\tilde{a}_j)\|^2 + \|(U-T)(a' - P\tilde{a}_j)\|^2 + \|(I-P)U(I-P)\tilde{a}_j\|^2 = \\ &= \|D_A(a - TP\tilde{a}_j)\|^2 + \|D_T(a' - P\tilde{a}_j)\|^2 + \|(I-P)\tilde{a}_j\|^2.\end{aligned}$$

From this and from (2.4) we infer that

$$\{D_A Ta \oplus D_T a : a \in \mathfrak{A}\}^- = (D_A \mathfrak{A})^- \oplus (D_T \mathfrak{A})^-, \tag{2.5}$$

i.e., the factorization $A \cdot T$ of AT is regular. Conversely, let us assume that (2.5) holds. Hence, for any $a, a' \in \mathfrak{A}$ there exists $(a_j)_{j=1}^{\infty} \subset \mathfrak{A}$ such that

$$\|D_A(a - Ta_j)\|^2 + \|D_T(a' - a_j)\|^2 \to 0 \quad (j \to \infty). \tag{2.6}$$

Then, for any $\tilde{a} \in \tilde{\mathfrak{A}}$ of the form

$$\tilde{a} = a + (U-T)a' + \tilde{a}'', \tag{2.7}$$

where $a, a' \in \mathfrak{A}$ and $\tilde{a}'' \in U(I-P)\tilde{\mathfrak{A}}$, consider the elements

$$\tilde{a}_j = a_j + U^* \tilde{a}'' \in \tilde{\mathfrak{A}} \quad (j = 1, 2, \ldots), \tag{2.8}$$

where $(a_j)_{j=1}^{\infty} \subset \mathfrak{A}$ is the sequence occurring in (2.6). By virtue of (2.1) we have for $\tilde{a}$ and $\tilde{a}_j$ given in (2.7) and (2.8)

$$\begin{aligned}\|D_{\tilde{A}}(\tilde{a} - U\tilde{a}_j)\|^2 &= \|D_A(a - Ta_j)\|^2 + \|(U-T)a' + \tilde{a}'' - (I-P)U\tilde{a}_j\|^2 = \\ &= \|D_A(a - Ta_j)\|^2 + \|(U-T)(a' - a_j)\|^2 + \|\tilde{a}'' - (I-P)UU^*\tilde{a}''\|^2 = \\ &= \|D_A(a - Ta_j)\|^2 + \|D_T(a' - a_j)\|^2.\end{aligned}$$

Thus, from (2.6), it follows that $D_{\tilde{A}}\tilde{a} \in (D_{\tilde{A}} U \tilde{\mathfrak{A}})^-$, for any $\tilde{a}$ of the form (2.7). Since the set of these $\tilde{a}$ is dense in $\tilde{\mathfrak{A}}$, (2.2) follows at once.

Remark 2.2. In the sequel we shall also use the following characterization of regular factorization. Namely, (1.2) is equivalent to any one of the relations

$$D_{A_2} \mathfrak{B} \cap D_{A_1^*} \mathfrak{B} = \{0\}, \tag{2.9}$$

$$D_{A_2} \mathfrak{B} \cap \ker A_1^* = \{0\} \quad \text{and} \quad D_{A_1} \mathfrak{A} \cap A_1^* D_{A_2} \mathfrak{B} = \{0\}. \tag{2.10}$$

For the equivalence of (1.2) and (2.9) we refer to [6] and [10]. On the other hand, if (2.9) holds then the first relation of (2.10) follows from the inclusion $\ker A_1^* \subset D_{A_1^*}\mathfrak{B}$ while if $D_{A_1}a = A_1^* b$ for some $b \in D_{A_2}\mathfrak{B}$ then by virtue of the relation $A_1 D_{A_1} = D_{A_1^*} A_1$ we have

$$b = D_{A_1^*}^2 b + A_1 A_1^* b = D_{A_1^*}(D_{A_1^*} b + A_1 a),$$

hence $b=0$. Thus (2.9) implies (2.10). Conversely if (2.10) holds and if $D_{A_2} b = D_{A_1^*} b'$ for some $b, b' \in \mathfrak{B}$, then $A_1^* D_{A_2} b = D_{A_1} A_1^* b'$, therefore $D_{A_2} b = 0$, i.e. (2.9) holds too.

Remark 2.3. Let $A \in L(\mathfrak{A}, \mathfrak{B})$, $\tilde{A} \in L(\tilde{\mathfrak{A}}, \mathfrak{B})$ be as in Lemma 2.2 and let $T' \in L(\mathfrak{B})$ be a contraction. Then, since $D_{\tilde{A}^*} = D_{A^*}$, it is obvious (by virtue of the preceding remark) that the factorization $T' \cdot \tilde{A}$ of $T'\tilde{A}$ is regular if and only if so is the factorization $T' \cdot A$ of $T'A$.

3. In order to prove the sufficiency of the condition in Theorem 1.1, we shall firstly consider the case when T_2 is an isometry. For the simplification of the notations, we shall introduce the following notations: $\mathfrak{H}_1 = \mathfrak{H}$, $T_1 = T$, $U \in L(\mathfrak{K})$ — the minimal isometric dilation of T, and $\mathfrak{H}_2 = \mathfrak{G}$, $T_2 = Z$.

Let us also denote by $P_{(n)}$ the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}_{(n)} = \mathfrak{H} \oplus \mathfrak{L} \oplus \ldots \oplus U^{n-1}\mathfrak{L}$, where $\mathfrak{L} = ((U-T)\mathfrak{H})^-$, $P_{(0)} = P_{\mathfrak{H}}$, and $T_{(n)} = P_{(n)} U | P_{(n)}\mathfrak{K}$ $(n=1, 2, \ldots)$, $T_{(0)} = T$; also for any $A \in I(T; Z)$, $\|A\| = 1$, let us set

$$\mathscr{B}_{T_{(1)}}(A) = \{B_1 \in L(\mathfrak{G}, \mathfrak{H}_{(1)}) : T_{(1)} B_1 = B_1 Z,\ \|B_1\| = 1,\ P_{\mathfrak{H}} B_1 \doteq A\}. \tag{3.1}$$

In order to show that $\mathscr{B}_{T_{(1)}}(A)$ is not empty we recall the first step of the construction of an EID of A (see [9], Ch. II, § 2). We have to determine an operator of the form

$$B_1 = \begin{bmatrix} A \\ X \end{bmatrix} : \mathfrak{G} \to \mathfrak{H}_{(1)} = \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathfrak{L} \end{matrix} \tag{3.2}$$

satisfying the conditions

$$\|Xg\| \leqq \|D_A g\| \quad (g \in \mathfrak{G}), \tag{3.3}$$

$$T_{(1)} B_1 = B_1 Z, \tag{3.4}$$

where

$$T_{(1)} = \begin{bmatrix} T & 0 \\ U-T & 0 \end{bmatrix} : \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathfrak{L} \end{matrix} \to \begin{matrix} \mathfrak{H} \\ \oplus \\ \mathfrak{L} \end{matrix}.$$

The last condition is equivalent to

$$(U-T)A = XZ \quad (\text{and } TA = AZ). \tag{3.4'}$$

Since the space $\mathfrak{L}$ can be identified with $(D_T\mathfrak{H})^-$ and then the operator corresponding to $U-T$ is D_T, (3.4′) becomes

$$D_T A = XZ; \tag{3.4''}$$

here X is an operator from $\mathfrak{G}$ into $(D_T\mathfrak{H})^-$ (namely, the operator corresponding to the "original operator X"). Conditions (3.3) and (3.4″) are equivalent to the existence of a contraction $C\colon (D_A\mathfrak{G})^- \to (D_T\mathfrak{H})^-$ satisfying

$$X = CD_A, \tag{3.5}$$

$$D_T A = CD_A Z. \tag{3.6}$$

Since $\|D_T Ag\|^2 \leqq \|D_A Zg\|^2$ for all $g \in \mathfrak{G}$, it results that there exists a contraction defined on $(D_A Z\mathfrak{G})^-$ such that (3.6) holds. Obviously, this can be extended to a contraction $C\colon (D_A\mathfrak{G})^- \to (D_T\mathfrak{H})^-$. Then, if we define by (3.5) an operator $X\colon \mathfrak{G} \to (D_T\mathfrak{H})^-$, it is clear that $B_1 = \begin{bmatrix} A \\ X \end{bmatrix} \in \mathscr{B}_{T_{(1)}}(A)$.

By recurrence, we define, for every $n \geqq 1$,

$$\mathscr{B}_{T_{(n)}}(B_{n-1}) = \{B_n \in L(\mathfrak{G}, \mathfrak{H}_{(n)}) : T_{(n)} B_n = B_n Z,\ \|B_n\| = 1,\ P_{\mathfrak{H}_{(n-1)}} B_n = B_{n-1}\}, \tag{3.7}$$

where $B_0 = A$.

Remark 3.1. It is easy to show that if $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ $(n = 1, 2, \ldots)$ and if all B_n's are considered in $L(\mathfrak{G}, \mathfrak{K})$, then the strong limit $B = \lim_{n\to\infty} B_n$ exists; obviously, B is a dilation of A with $\|B\| = 1$. Also, since U is the strong limit of $(T_{(n)} P_{(n)})_{n=1}^\infty$, we clearly have $B \in I(U; Z)$. Thus, B defined as the strong limit of $(B_n)_{n=1}^\infty$, where $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ $(n = 1, 2, \ldots)$, is an EID of A. Conversely, for any EID B of A, the compression $B_n = P_{(n)} B$ belongs to $\mathscr{B}_{T_{(n)}}(B_{n-1})$ and B is the strong limit of $(B_n)_{n=1}^\infty$. [1])

Remark 3.2. It is plain that by the canonical identifications we have $(T_{(n)})_{(1)} = T_{(n+1)}$ and that for any $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$

$$\mathscr{B}_{T_{(n+1)}}(B_n) = \mathscr{B}_{(T_{(n)})_{(1)}}(B_n)$$

(for all $n = 1, 2, \ldots$).

Using the above remarks we shall obtain

Lemma 3.1. *A sufficient condition in order that* $A \in I(T; Z)$, $\|A\| = 1$, *have a unique EID is*

$$(D_A Z\mathfrak{G})^- = (D_A\mathfrak{G})^-. \tag{3.8}$$

Proof. We shall show by induction that, by virtue of (3.8), $B_n \in \mathscr{B}_{T_{(n)}}(B_{n-1})$ (where $\mathscr{B}_{T_{(n)}}(B_{n-1})$ is defined by (3.7)) is uniquely determined by A for every $n \geqq 1$. First, it is obvious by the construction of $B_1 = \begin{pmatrix} A \\ X \end{pmatrix} \in \mathscr{B}_{T_{(1)}}(A)$, where X is

[1]) This iterative explication of the construction of an EID, firstly given in [8], was inspired by [4].

defined by (3.5), that the contraction C of this formula is uniquely defined on $(D_A Z\mathfrak{G})^-$ by (3.6); therefore if (3.8) holds, then C is uniquely determined on the whole $(D_A\mathfrak{G})^-$. Consequently X, and thus B_1, is uniquely determined by $A=B_0$. From here, by the construction of $B_n\in\mathscr{B}_{T_{(n)}}(B_{n-1})$ $(n=1, 2, \ldots)$ and by virtue of Remark 3.2, we infer the following sufficient condition that B_n should be uniquely determined by its preceding B_{n-1}:

$$(D_{B_{n-1}}Z\mathfrak{G})^- = (D_{B_{n-1}}\mathfrak{G})^-. \tag{3.9}$$

Also we notice that

$$\begin{aligned}\|D_{B_n}(g-Zg')\|^2 &= \|g-Zg'\|^2-\|B_n(g-Zg')\|^2 \leqq \\ &\leqq \|g-Zg'\|^2-\|P_{\mathfrak{H}_{(n-1)}}B_n(g-Zg')\|^2 = \|D_{B_{n-1}}(g-Zg')\|^2 \leqq \ldots \\ &\ldots \leqq \|D_{B_1}(g-Zg')\|^2 \leqq \|D_A(g-Zg')\|^2,\end{aligned}$$

for all $g, g'\in\mathfrak{G}$ $(n=1, 2, \ldots)$. Hence, if (3.8) holds, (3.9) holds too, for all $n=1, 2, \ldots$. Now, let us assume that B_{n-1} is uniquely determined by A. Then, since by the above remark B_n is uniquely determined by B_{n-1}, it readily follows by our induction hypothesis that it is uniquely determined by A. From this and by virtue of Remark 3.1 we infer that A has a unique EID.

Now, returning to the original situation we can easily prove that the regularity condition imposed on one of the factorizations $A\cdot T_2$ or $T_1\cdot A$ implies the uniqueness of the EID of A. First, let us assume that the factorization $A\cdot T_2$ of AT_2 is regular. Then, by Lemma 2.2, the factorization $\tilde{A}\cdot U_2$ of $\tilde{A}U_2$ is regular, and then, by Lemma 3.1, $\tilde{A}$ has a unique EID. Thus, by Remark 2.1, A also has a unique EID. Now, assume that the factorization $T_1\cdot A$ of T_1A is regular. Then, it is known ([9], Ch. VII, § 2) that the factorization $A^*\cdot T_1^*$ is regular, and thus, by the same rasons as above, A^* has a unique EID. Consequently, by virtue of Lemma 2.1, so has A.

4. For the remaining part of Theorem 1.1, we have only to prove that if none of the factorizations $T_1\cdot A$ and $A\cdot T_2$ (of $T_1A=AT_2$) is regular, then the contraction A has at least two different EID's.

By virtue of Lemma 2.2 and Remark 2.3, our present assumption concerning the factorizations $T_1\cdot A$ and $A\cdot T_2$ implies that the factorizations $T_1\cdot\tilde{A}$ and $\tilde{A}\cdot U_2$, where $\tilde{A}=AP_{\mathfrak{H}_2}\in I(T_1;\, U_2)$ are not regular either. Also, by virtue of Remarks 2.1 and 3.1, it suffices to show that if the above conditions hold then $\mathscr{B}_{T_{(1)}}(\tilde{A})$ (defined by (3.1)) is not a singleton. We must show, by virtue of (3.2), (3.5), and (3.6), that the contraction C defined by

$$CD_{\tilde{A}}U_2 = D_{T_1}\tilde{A} \tag{4.1}$$

has at least one contractive extension $C'\colon(D_{\tilde{A}}\mathfrak{K}_2)^- \to (D_{T_1}\mathfrak{H}_1)^-$ such that

$$C'|(D_{\tilde{A}}\mathfrak{K}_2)^- \ominus (D_{\tilde{A}}U_2\mathfrak{K}_2)^- \neq 0. \tag{4.2}$$

Since the factorization $T_1 \cdot \tilde{A}$ does not satisfy (2.9), there exist $h_0 \in (D_{T_1}\mathfrak{H}_1)^-$ and $k_0 \in \mathfrak{H}_2$ such that

(4.3) $$D_{T_1} h_0 = D_{\tilde{A}^*} k_0 \neq 0;$$

also, since the factorization $\tilde{A} \cdot U_2$ does not satisfy (1.2), there exists $0 \neq d_0 \in (D_{\tilde{A}}\mathfrak{K}_2)^- \ominus \ominus (D_{\tilde{A}} U_2 \mathfrak{K}_2)^-$, where we can suppose that $\|h_0\| = 1$ and $\|d_0\| = 1$. Now, we define $C' : (D_{\tilde{A}}\mathfrak{K}_2)^- \to (D_{T_1}\mathfrak{H}_1)^-$ by

(4.4) $$C' = CQ + \theta d_0^* \otimes h_0$$

where Q is the orthogonal projection of $(D_{\tilde{A}}\mathfrak{K}_2)^-$ onto $(D_{\tilde{A}} U_2 \mathfrak{K}_2)^-$, $d_0^* \otimes h_0$ is the operator defined on $(D_{\tilde{A}}\mathfrak{K}_2)^-$ by $(d_0^* \otimes h_0) d = (d, d_0) h_0$, and $0 < \theta < 1$ will be chosen later. Obviously, $C' d_0 \neq 0$, thus (4.2) holds. Also, we shall show that θ can be chosen such that C' defined by (4.4) be a contraction, i.e.

$$\|CQd + \theta((I-Q)d,\, d_0) h_0\| \leq \|d\|,$$

or equivalently,

(4.5) $$\|CQd\|^2 + 2\theta \operatorname{Re} (CQd, h_0) \overline{((I-Q)d, d_0)} + \theta^2 |((I-Q)d, d_0)|^2 \leq$$
$$\leq \|Qd\|^2 + \|(I-Q)d\|^2, \quad \text{for all} \quad d \in (D_{\tilde{A}}\mathfrak{K}_2)^-.$$

Obviously, it is enough to verify (4.5) for d of the form $D_{\tilde{A}} U_2 k + \lambda d_0$ $(k \in \mathfrak{K}_2, \lambda \in \mathbf{C})$, for which (4.5) becomes

$$\|CD_{\tilde{A}} U_2 k\|^2 + 2\theta \operatorname{Re} \bar{\lambda} (CD_{\tilde{A}} U_2 k, h_0) + \theta^2 |\lambda|^2 \leq \|D_{\tilde{A}} U_2 k\|^2 + |\lambda|^2,$$

or according to (4.1),

(4.6) $$2\theta \operatorname{Re} \bar{\lambda} (D_{T_1} \tilde{A} k, h_0) \leq \|D_{\tilde{A}} U_2 k\|^2 - \|D_{T_1} \tilde{A} k\|^2 + |\lambda|^2 (1-\theta^2) =$$
$$= \|D_{\tilde{A}} k\|^2 + |\lambda|^2 (1-\theta^2) \quad (k \in \mathfrak{K}_2, \lambda \in \mathbf{C}).$$

It is elementary to deduce that (4.6) is true if

(4.7) $$|(D_{T_1} \tilde{A} k, h_0)|^2 \leq \|D_{\tilde{A}} k\|^2 (1-\theta^2) \theta^{-2} \quad (k \in \mathfrak{K}_2).$$

Since by (4.3) we have $(D_{T_1} \tilde{A} k, h_0) = (D_{\tilde{A}} k, \tilde{A}^* k_0)$ for all $k \in \mathfrak{K}_2$, it is easy to prove that (4.7) will be true if we choose $0 < \theta < (1 + \|\tilde{A}^* k_0\|^2)^{-1/2}$. This concludes the proof of Theorem 1.1.

Remark 4.1. Plainly, the whole proof in this section works for any contraction $A \in I(T_1; T_2)$. Also, if for such an A, one of the factorizations $A \cdot T_2$ and $T_1 \cdot A$ of $T_1 A = A T_2$ is regular then either $\|A\| = 1$ or T_2 is a coisometry or T_1 is an isometry. By virtue of Theorem 1.1 and Lemma 2.1 we infer that in any of these cases A has exactly one contractive intertwining dilation $\in I(U_1; U_2)$. Thus, we can reformulate Theorem 1.1 in the following, slightly more general form: *A contraction $A \in I(T_1; T_2)$ has a unique contractive intertwining dilation $\in I(U_1; U_2)$ if and only if at least one of the factorizations $T_1 \cdot A$ and $A \cdot T_2$ of $T_1 A = A T_2$ is regular.*

Remark 4.2. We give an example showing that it is not necessary that both factorizations $A\cdot T_2$ and $T_1\cdot A$ be regular in order to have the uniqueness property of the EID of A.

To this purpose we define $A\in L(l^2)$, by

$$A(c_0, c_1, \dots, c_n, \dots) = (c_0, (1-d_1^2)^{1/2}c_1, \dots (1-d_n^2)^{1/2}c_n, \dots)$$

where $x=(c_n)_{n=0}^{\infty}\in l^2$ and $0<d_n<d_{n+1}<1$ $(n=1, 2, \dots)$ are fixed. Also we denote by $T\in L(l^2)$ the weighted shift

$$T(c_0, c_1, \dots, c_n, \dots) = (0, (1-d_1^2)^{1/2}c_0, \dots, (1-d_n^2)^{1/2}(1-d_{n-1}^2)^{-1/2}c_{n-1}, \dots)$$

and by U the unilateral shift

$$U(c_0, c_1, \dots, c_n, \dots) = (0, c_0, \dots, c_{n-1}, \dots)$$

on l^2. Then, clearly, A and T are contractions on l^2 and U is an isometry. Also, it is easy to verify that $TA=AU, A^*=A$, $\|A\|=1$ and

$$T^*(c_0, c_1, \dots, c_n, \dots) = ((1-d_1^2)^{1/2}c_1, \dots, (1-d_{n+1}^2)^{1/2}(1-d_n^2)^{-1/2}c_{n+1}, \dots).$$

Then, we obtain

$$\begin{aligned} &D_A(c_0, c_1, \dots, c_n, \dots) = (0, d_1c_1, \dots, d_nc_n, \dots), \\ &D_T(c_0, c_1, \dots, c_n, \dots) = \\ &= (d_1c_0, (d_2^2-d_1^2)^{1/2}(1-d_1^2)^{-1/2}c_1, \dots, (d_{n+1}^2-d_n^2)^{1/2}(1-d_n^2)^{-1/2}c_n, \dots). \end{aligned}$$

Whence, obviously

$$D_A l^2 \cap D_{U^*} l^2 = D_A l^2 \cap \ker U^* = \{0\}, \tag{4.8}$$

$$D_T l^2 \cap D_{A^*} l^2 \ni (0, 1, 0, \dots). \tag{4.9}$$

Therefore, by virtue of Remark 2.2, we infer from (4.8), respectively from (4.9), that the factorization $A\cdot U$, respectively $T\cdot A$, (of $AU=TA$) is regular, respectively nonregular.

5. Let us notice that Theorem 1.1 has the following direct consequences:

Corollary 5.1. *Let A and T be double commuting* (i.e. $AT=TA$, $AT^*=T^*A$) *contractions on $\mathfrak{H}$, $\|A\|=1$. Then A has a unique exact intertwining dilation (with respect to $T_1=T=T_2$) if and only if there is a decomposition $\mathfrak{H}=\mathfrak{H}_A\oplus\mathfrak{H}_T$ reducing A and T, such that $A|\mathfrak{H}_A$ and $T^*|\mathfrak{H}_T$ are isometric or that $A^*|\mathfrak{H}_A$ and $T|\mathfrak{H}_T$ are isometric.*

Indeed, the splitting properties obviously imply

$$D_A D_{T^*} = D_{T^*} D_A = 0, \tag{5.1}$$

respectively

$$D_T D_{A^*} = D_{A^*} D_T = 0. \tag{5.2}$$

Conversely, if (5.1), respectively (5.2), is satisfied, then defining $\mathfrak{H}_A$ as the smallest (linear closed) subspace of $\mathfrak{H}$ reducing T and containing $D_{T^*}\mathfrak{H}$, respectively reducing A and containing $D_{A^*}\mathfrak{H}$, we obtain the splitting properties stated above.

By the double commuting property, (5.1), respectively (5.2), is equivalent to

$$D_A\mathfrak{H}\cap D_{T^*}\mathfrak{H} = \{0\}, \quad \text{respectively} \quad D_T\mathfrak{H}\cap D_{A^*}\mathfrak{H} = \{0\},$$

thus, by Remark 2.2, to the regularity of the factorization $A\cdot T$, respectively $T\cdot A$, of $AT=TA$.

Corollary 5.2. *Let $A, T\in L(\mathfrak{H})$ be commuting contractions. Then A has a unique contractive intertwining dilation (with respect to T) if and only if T has a unique contractive intertwining dilation (with respect to A).*

Indeed, by Remark 4.1 each of the two assertions above is equivalent to the regularity of at least one of the factorizations $A\cdot T$ or $T\cdot A$ of $AT=TA$.

Corollary 5.3. *Let $A\in L(\mathfrak{H}_2, \mathfrak{H}_1)$, $\|A\|=1$, intertwine the coisometry T_1 and the isometry T_2. Then A has a unique exact intertwining dilation if and only if at least one of the following two conditions holds:*

$$D_A\mathfrak{H}_2\cap\ker T_2^* = \{0\}, \quad D_{A^*}\mathfrak{H}_1\cap\ker T_1 = \{0\}.$$

Indeed, under the present assumptions, these conditions are equivalent to the regularity of the factorizations $A\cdot T_2$, respectively $T_1\cdot A$ of $AT_2=T_1A$ (see Remark 2.2).

Remark 5.1. The preceding corollary is a slight extension of the uniqueness theorem of Adamjan, Arov and Krein, [2] Theorem 3.1, which concerns the case when T_2 and T_1^* are unilateral shifts. However, in case $T_2\in C_{\cdot 0}$, $T_1\in C_{0\cdot}$ (i.e. if $T_2^{*n}\to 0$, $T_1^n\to 0$ strongly, for $n\to\infty$) our Theorem 1.1 is an easy consequence of [2], Theorem 3.1 and [9], Ch. II, Theorem 1.2.

Let us also indicate how one of the main results of [3] follows from our Theorem 1.1. To this purpose we recall that according to [3], a contraction $A\in L(\mathfrak{H}_2, \mathfrak{H}_1)$ is said to *Harnack-dominate* a contraction $B\in L(\mathfrak{H}_2, \mathfrak{H}_1)$ if there exists a positive constant γ such that

$$\|D_Bh\| \leqq \gamma\|D_Ah\| \quad \text{and} \quad \|(B-A)h\| \leqq \gamma\|D_Ah\| \quad (h\in\mathfrak{H}_2). \tag{5.3}$$

Plainly, relations (5.3) imply that

$$D_B\mathfrak{H}_2\subset D_A\mathfrak{H}_2 \quad \text{and} \quad (B-A)^*\mathfrak{H}_1\subset D_A\mathfrak{H}_2. \tag{5.4}$$

Corollary 5.4. ([3], Theorem 3.2) *Let $A, B\in L(\mathfrak{H}_2, \mathfrak{H}_1)$ intertwine the contractions T_1 and T_2, $\|A\|=1$, and such that A Harnack-dominates B. Then if A has a unique EID so has B.*

Proof. By Theorem 1.1, one of the factorizations $A\cdot T_2$ and $T_1\cdot A$ is regular. If the first one is regular, then from (2.9) (with $A_2=A$, $A_1=T$ and $A_2=B$, $A_1=T$) and from the first relation (5.4) we readily infer that the factorization $B\cdot T_2$ is regular, thus by Theorem 1.1, B has a unique EID. In case $T_1\cdot A$ is regular, from (2.10) (with $A_2=T_1$, $A_1=A$) we obtain

$$D_{T_1}\mathfrak{H}_1\cap\ker A^* = \{0\},\quad D_A\mathfrak{H}_2\cap A^*D_{T_1}\mathfrak{H}_1 = \{0\}. \tag{5.5}$$

If

$$B^*D_{T_1}h_1 = 0 \quad\text{and}\quad D_B h_2 = B^*D_{T_1}h_1'$$

for some $h_1, h_1'\in\mathfrak{H}_1$, $h_2\in\mathfrak{H}_2$, then from (5.4) we infer at once that

$$A^*D_{T_1}h_1\in D_A\mathfrak{H}_2 \quad\text{and}\quad A^*D_{T_1}h_1'\in D_A\mathfrak{H}_2;$$

by (5.5), it follows $D_{T_1}h_1=0=D_{T_1}h_1'$. We conclude that $A_2=T_1$, $A_1=B$ satisfy (2.10), thus that the factorization $T_1\cdot B$ is regular. Since (5.3) also implies $\|B\|=1$, the proof is achieved by referring to Theorem 1.1.

6. A less direct consequence of our preceding results is the following

Proposition 6.1. *Let $A\in L(\mathfrak{H}_2,\mathfrak{H}_1)$, $\|A\|=1$, intertwine the contractions $T_1\in$ $\in L(\mathfrak{H}_1)$ and $T_2\in L(\mathfrak{H}_2)$ and let $\mathfrak{M}$ be a subspace of $\mathfrak{H}_2$, cyclic for the minimal unitary dilation U_2 of T_2. If, moreover, $\mathfrak{M}$ enjoys also the property*

$$D_A\mathfrak{M}\oplus\{0\}\subset\{D_AT_2h\oplus D_{T_2}h : h\in\mathfrak{M}\}^-, \tag{6.1}$$

then A has a unique exact intertwining dilation.

Proof. We shall use the notations of the preceding sections. In particular we set $\tilde{A}=AP_{\mathfrak{H}_2}$. Also we set

$$\mathfrak{K}_2' = \bigvee_{n=0}^{\infty} U_2^n\mathfrak{M} \tag{6.2}$$

and

$$U_2' = U_2|\mathfrak{K}_2',\quad \tilde{A}' = \tilde{A}|\mathfrak{K}_2'.$$

For elements $h\in\mathfrak{M}$ and $k\in\mathfrak{K}_2'$ of the form

$$k = \sum_{n=0}^{\infty} U_2^n k_n, \tag{6.3}$$

where $k_n\in\mathfrak{M}$ $(n=0, 1, 2, \ldots)$ and only a finite number of k_n's are $\neq 0$, we have

$$\begin{aligned}
&\|D_{\tilde{A}'}[k-U_2'(k_1+h+\sum_{n=2}^{\infty}U_2^{n-1}k_n)]\|^2 = \\
&= \|D_{\tilde{A}}(k-\sum_{n=1}^{\infty}U_2^n k_n - U_2h)\|^2 = \|D_{\tilde{A}}(k_0-U_2h)\|^2 = \\
&= \|k_0-T_2h\|^2+\|(U_2-T_2)h\|^2-\|A(k_0-T_2h)\|^2 = \\
&= \|D_A(k_0-T_2h)\|^2+\|D_{T_2}h\|^2 = \|D_Ak_0\oplus 0-D_AT_2h\oplus D_{T_2}h\|^2.
\end{aligned} \tag{6.4}$$

The last quantity can be made, by virtue of (6.1), as small as we want if $h\in\mathfrak{M}$ is suitably chosen. Thus, we can deduce from (6.4) that the factorization $\tilde{A}'\cdot U_2'$ is regular. Consequently, from Theorem 1.1 it follows that $\tilde{A}'$ has a unique EID; let B' be this EID. It enjoys the property

$$(6.5)\qquad P_{\mathfrak{H}_1}B' = \tilde{A}' \quad\text{and}\quad U_1B' = B'U_2'.$$

Let now B_j $(j=1, 2)$ be two EID of A. As we already pointed out in Section 2, there exists a unique contractive extension $\hat{B}_j\in L(\hat{\mathfrak{K}}_2, \hat{\mathfrak{K}}_1)$ such that

$$(6.6)\qquad \|\hat{B}_j\| = \|B_j\|,\quad \hat{B}_j\hat{U}_2 = \hat{U}_1\hat{B}_j \quad (j = 1, 2).$$

Since $\hat{B}_j|\mathfrak{K}_2'$ is a contraction from $\mathfrak{K}_2'$ into $\mathfrak{K}_1$ enjoying property (6.5), by the uniqueness of B' we infer

$$(6.7)\qquad \hat{B}_1|\mathfrak{K}_2' = B_1|\mathfrak{K}_2' = B' = B_2|\mathfrak{K}_2' = \hat{B}_2|\mathfrak{K}_2';$$

whence, by (6.6),

$$(6.8)\qquad \hat{B}_1 g = \hat{B}_2 g$$

for any element $g\in\hat{\mathfrak{K}}_2$ of the form

$$(6.9)\qquad g = \hat{U}_2^n k' \quad (\text{with } n = 0, \pm 1, \pm 2, \dots;\ k'\in\mathfrak{K}_2').$$

Since $\mathfrak{K}_2'$ contains $\mathfrak{M}$ which is cyclic for U_2, the elements g of the form (6.8) span $\hat{\mathfrak{K}}_2$, thus from (6.6) and (6.8) we deduce that $\hat{B}_1=\hat{B}_2$, and hence $B_1=B_2$. This shows that A has a unique EID and thus the proof is achieved.

Remark 6.1. In case $\mathfrak{M}$ is an invariant subspace for T_2, then (6.1) is equivalent to the regularity of the factorization $(A|\mathfrak{M})\cdot(T_2|\mathfrak{M})$ of $AT_2|\mathfrak{M}$.

Corollary 6.1. *Let A be a contraction intertwining the contractions T_1 and T_2. Then, if* $\ker D_A$ *is cyclic for the unitary dilation $\hat{U}_2$ of T_2, A has a unique exact intertwining dilation.*

Indeed, in this case, for $\mathfrak{M}=\ker D_A$, the left hand side of (6.1) is $\{0\}\oplus\{0\}$ and consequently (6.1) is trivially satisfied.

Remark 6.2. Corollary 6.1 (which however can be easily proved in a direct way by an argument similar to the last part of the proof of Proposition 6.1) contains as particular cases some uniqueness theorems of [1] and [5].

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Star-algebras induced by non-degenerate inner products

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§ 1. Introduction

Let $\mathscr{B}(\mathfrak{E})$ denote the algebra of all bounded linear operators on the Banach space $\mathfrak{E}$. According to a classical theorem of Kawada, Kakutani and Mackey (see e.g. [1; Corollary 4.10.8]), if * is an involution on $\mathscr{B}(\mathfrak{E})$ satisfying the condition $T^*T\neq 0$ for every non-zero $T\in\mathscr{B}(\mathfrak{E})$, then there is a positive definite inner product $(\cdot,\cdot)$ on $\mathfrak{E}$ such that

(i) $(Tx, y)=(x, T^*y)$ for every $x, y\in\mathfrak{E}$ and $T\in\mathscr{B}(\mathfrak{E})$;

(ii) the norm induced by $(\cdot,\cdot)$ is equivalent to the original norm on $\mathfrak{E}$.

In our paper [2] we generalized this theorem to a class of indefinite inner products and began similar investigations for wider classes.

J. Saranen [3] has obtained numerous further improvements and generalizations involving non-symmetric bilinear forms as well as operator algebras different from $\mathscr{B}(\mathfrak{E})$ on normed or non-normed vector spaces.

Below, combining the stand-point and methods of [2] with achievements of [3], we try to give a unified, elementary and possibly complete treatment of those aspects of the subject which are relevant to the general theory of indefinite inner product spaces [4]. For this purpose, we single out certain results explicit or implicit in [3], regroup, reformulate, extend or restrict them, modify their proofs, and add some new observations (cf. especially Theorems 3.5, 3.7, 4.7, 4.10 and some corollaries).

It should be noted that representations for involutions of general (i.e., not operator) algebras by means of indefinite inner products have been known prior to [2] (see [1; Theorem 4.3.7]). However, they seem to be of a different nature, since their representation space is not fixed in advance.

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§ 2. Preliminaries

1. Admissible *-algebras. Let $\mathfrak{E}$ be a vector space over the complex field $\mathbf{C}$. The algebra of all linear operators (i.e., all homomorphisms) $T:\mathfrak{E}\to\mathfrak{E}$ will be denoted by $\mathscr{L}(\mathfrak{E})$.

Let $\mathscr{A}$ be a subalgebra of $\mathscr{L}(\mathfrak{E})$. The mapping *: $\mathscr{A}\to\mathscr{A}$ is said to be an *involution* if for all T_1, T_2, $T\in\mathscr{A}$ and $\alpha\in\mathbf{C}$ the following conditions are satisfied: (i) $(T_1+T_2)^*=T_1^*+T_2^*$, (ii) $(\alpha T)^*=\bar{\alpha}T^*$, (iii) $(T_1T_2)^*=T_2^*T_1^*$, (iv) $T^{**}=T$.

An algebra equipped with an involution * is called a **-algebra.*

The algebra (or *-algebra) $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ is said to be *dense* if, for any positive integer n, linearly independent vectors $x_1, \ldots, x_n\in\mathfrak{E}$ and vectors $y_1, \ldots, y_n\in\mathfrak{E}$, there is an operator $T\in\mathscr{A}$ such that $Tx_j=y_j$ $(j=1, \ldots, n)$.

$\mathscr{L}(\mathfrak{E})$ itself is a dense algebra. What is more, the finite-rank elements of $\mathscr{L}(\mathfrak{E})$ also form a dense algebra (see [3; Lemma 2.2]). If $\mathfrak{E}$ is a Banach space, the algebra $\mathscr{B}(\mathfrak{E})$ of all *bounded* linear operators T: $\mathfrak{E}\to\mathfrak{E}$ is dense. Even the finite-rank elements of $\mathscr{B}(\mathfrak{E})$ form a dense algebra ([3; Lemma 2.2]).

We say the algebra (*-algebra) $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ is *admissible* if $\mathscr{A}$ is dense and contains an operator of rank 1.

2. Non-degenerate inner products. Let $\mathfrak{E}$ be a vector space over $\mathbf{C}$. We say a mapping of $\mathfrak{E}\times\mathfrak{E}$ into $\mathbf{C}$ is an *inner product* if, denoting the image of the ordered pair $x, y\in\mathfrak{E}$ by (x, y), for any $x_1, x_2, x, y\in\mathfrak{E}$ and $\alpha\in\mathbf{C}$ the following conditions are fulfilled: (i) $(x_1+x_2, y)=(x_1, y)+(x_2, y)$, (ii) $(\alpha x, y)=\alpha(x, y)$, (iii) $(y, x)=\overline{(x, y)}$.

The inner product $(\cdot,\cdot)$ is said to be *non-degenerate* if for $x\neq 0$ there exists $y\in\mathfrak{E}$ such that $(x, y)\neq 0$.

A norm $|\cdot|$ is said to be *compatible* with the non-degenerate inner product $(\cdot,\cdot)$ on $\mathfrak{E}$ if (i) for any fixed $y\in\mathfrak{E}$ the linear form $\varphi_y(x)=(x, y)$ $(x\in\mathfrak{E})$ is continuous in the norm $|\cdot|$, (ii) for any $|\cdot|$-continuous linear form φ there exists $y\in\mathfrak{E}$ satisfying the relation $\varphi=\varphi_y$.

Here we note that if (x, y) is $|\cdot|$-continuous in the variable x then it is $|\cdot|$-continuous in y as well (since $(y, x)=\overline{(x, y)}$) and, in case $\mathfrak{E}$ is complete for $|\cdot|$, it is jointly $|\cdot|$-continuous in x and y (a consequence of the principle of uniform boundedness; see [4; Theorem IV. 2.3]).

3. Induced *-algebras. Let $(\cdot,\cdot)$ be a non-degenerate inner product on the vector space $\mathfrak{E}$. Given a linear operator $T\in\mathscr{L}(\mathfrak{E})$ it may happen that for each $y\in\mathfrak{E}$ there is a $y_T\in\mathfrak{E}$ with the property

$$(Tx, y) = (x, y_T) \quad (x\in\mathfrak{E}). \tag{2.1}$$

By the non-degeneracy of the inner product, the vector y_T is unique. The relation

$T^{()}y=y_T$ $(y\in\mathfrak{E})$ defines a linear operator $T^{()}\in\mathscr{L}(\mathfrak{E})$. Thus the existence, for each $y\in\mathfrak{E}$, of a vector $y_T\in\mathfrak{E}$ with property (2.1) is equivalent to the existence of a linear operator $T^{()}\in\mathscr{L}(\mathfrak{E})$ satisfying the condition

(2.2) $$(Tx, y) = (x, T^{()}y) \quad (x, y\in\mathfrak{E}).$$

Obviously, $T^{()}$ is unique; it is called the *adjoint* of T relative to the inner product $(\cdot, \cdot)$.

We write $\mathrm{Ind}_{()}$ for the set of all operators $T\in\mathscr{L}(\mathfrak{E})$ which do have an adjoint relative to $(\cdot, \cdot)$:

(2.3) $$\mathrm{Ind}_{()} = \{T\in\mathscr{L}(\mathfrak{E}) : T^{()} \text{ exists}\}.$$

It is easy to see that $\mathrm{Ind}_{()}$ is an algebra and that the mapping $T\mapsto T^{()}$ $(T\in\mathrm{Ind}_{()})$ is an involution on $\mathrm{Ind}_{()}$. We say $\mathrm{Ind}_{()}$ is the *-algebra *induced* by the non-degenerate inner product $(\cdot, \cdot)$.

4. Inner products representing a *-algebra. Let $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ be a *-algebra. If there exists a non-degenerate inner product $(\cdot, \cdot)$ on $\mathfrak{E}$ satisfying

(2.4) $$(Tx, y) = (x, T^*y) \quad (T\in\mathscr{A};\ x, y\in\mathfrak{E}),$$

we say $(.,.)$ *represents* the *-algebra $\mathscr{A}$ (or the involution *).

In other words, $(\cdot, \cdot)$ represents $\mathscr{A}$ if $\mathscr{A}$ is a *-subalgebra of $\mathrm{Ind}_{()}$. In this case we write $\mathscr{A}\subset\mathrm{Ind}_{()}$. (More generally, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are *-algebras, then $\mathscr{A}_1\subset\mathscr{A}_2$ will signify that $\mathscr{A}_1$ is a *-subalgebra of $\mathscr{A}_2$.)

5. Decomposable inner products. Let $(\cdot, \cdot)$ be a non-degenerate inner product on the vector space $\mathfrak{E}$.

Two vectors $x, y\in\mathfrak{E}$ are said to be *orthogonal* if $(x, y)=0$. Two subsets $\mathfrak{A}$, $\mathfrak{B}\subset\mathfrak{E}$ are said to be orthogonal if $(x, y)=0$ for all $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$.

We say that the (linear) subspace $\mathfrak{M}\subset\mathfrak{E}$ is *positive definite* (or that $(\cdot, \cdot)$ is positive definite on $\mathfrak{M}$) if $(x, x)>0$ for all $x\in\mathfrak{M}$, $x\neq 0$. The definition of *negative definite* subspace is similar. A subspace is said to be *definite* if it is either positive definite or negative definite.

The subspace $\mathfrak{M}\subset\mathfrak{E}$ is said to be *neutral* if $(x, x)=0$ for all $x\in\mathfrak{M}$.

In case $\mathfrak{E}$ is the orthogonal direct sum of a positive definite subspace $\mathfrak{E}^+$ and a negative definite subspace $\mathfrak{E}^-$,

(2.5) $$\mathfrak{E} = \mathfrak{E}^+(\dot{+})\mathfrak{E}^-,$$

we say the space $\mathfrak{E}$ (or: the inner product $(\cdot, \cdot)$) is *decomposable* and (2.5) is a *fundamental decomposition.*

Let P^+ denote the projection to $\mathfrak{E}^+$ along $\mathfrak{E}^-$, and set $P^-=I-P^+$. Then the operator $J=P^+-P^-$ has the properties $J^2=I$, $(Jx, y)=(x, Jy)$ for all $x, y\in\mathfrak{E}$.

Moreover, $(Jx, x)>0$ if $x\neq 0$. The positive definite inner product

$$(x, y)_J = (Jx, y) \quad (x, y\in\mathfrak{E}) \tag{2.6}$$

and the norm

$$x)^{1/2}\|x\|_J = (Jx, \quad (x\in\mathfrak{E}) \tag{2.7}$$

will be called the *fundamental inner product* and the *fundamental norm* corresponding to (2.5).

If, for some fundamental norm, $\mathfrak{E}$ is complete, we say $\mathfrak{E}$ is a *Krein space.*

All fundamental norms on a Krein space are topologically equivalent (cf. [4; Corollary IV. 6.3]).

If $\mathfrak{E}$ is a decomposable space and $\mathfrak{E}^+$ or $\mathfrak{E}^-$ has finite dimension, the space (or: the inner product) is said to be *quasi-definite;* the non-negative integer

$$\varkappa(\mathfrak{E}) = \min\{\dim\mathfrak{E}^+, \dim\mathfrak{E}^-\} \tag{2.8}$$

is called the *rank of indefiniteness.*

Quasi-definiteness and the value (2.8) do not depend on the choice of the fundamental decomposition (2.5) (cf. [4; Corollary II.10.4]).

A quasi-definite Krein space is called a *Pontrjagin space.*

§ 3. Star-algebras on vector spaces

In this section, $\mathfrak{E}$ is a vector space over **C**.

1. Admissible *-algebras in general. We first examine the problem of representing admissible *-algebras on $\mathfrak{E}$ without any additional assumption.

Theorem 3.1 (cf. [3; Folgerung 3.2 and relations (2.3a)—(2.3b)]). *Let* $(\cdot, \cdot)$ *be a non-degenerate inner product on* $\mathfrak{E}$. *Then* $\mathrm{Ind}_{()}$ *is an admissible *-algebra on* $\mathfrak{E}$.

Proof. We mentioned in Section 2 that $\mathrm{Ind}_{()}$ is an algebra and $T\mapsto T^{()}$ $(T\in\mathrm{Ind}_{()})$ is an involution on $\mathrm{Ind}_{()}$.

Set

$$Rx = \sum_{j=1}^{r}(x, y_j)z_j \quad (x\in\mathfrak{E}), \tag{3.1}$$

where $y_j, z_j\in\mathfrak{E}$ $(j=1, \dots, r)$. Then $R\in\mathrm{Ind}_{()}$, since

$$(Rx, u) = \left(x, \sum_{j=1}^{r}(u, z_j)y_j\right) \quad (x, u\in\mathfrak{E}).$$

Moreover, in the case $r=1$ the operator R has rank 1. Finally, let us be given $2n$ vectors $x_k, w_k\in\mathfrak{E}$ $(k=1, \dots, n)$, the system $\{x_1, \dots, x_n\}$ being linearly independent.

Choose $y_1, \ldots, y_n$ such that $(x_k, y_j) = \delta_{kj}$ $(j, k = 1, \ldots, n$; see e.g. [4; Lemmas I.10.4 and I.10.6]). Then (3.1) with $r = n$ and $z_j = w_j$ $(j = 1, \ldots, n)$ yields $Rx_k = w_k$ $(k = 1, \ldots, n)$.

The following result is, in a certain sense, converse to Theorem 3.1.

Theorem 3.2 (cf. [3; Satz 5.1 and Satz 5.3]). *Let $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ be an admissible *-algebra. Then there is one and, up to a constant real factor, only one non-degenerate inner product on $\mathfrak{E}$ which represents $\mathscr{A}$.*

Proof (cf. [2; pp. 56—60]). We first show that there is an operator T_0 with the properties

(3.2) $$T_0 \in \mathscr{A}, \quad \dim T_0 \mathfrak{E} = 1, \quad T_0^* T_0 \neq 0.$$

By assumption, $\mathscr{A}$ contains an operator T_1 of rank 1. The operator T_1^* is non-zero, since $T_1^{**} = T_1$ is non-zero whereas $0^* = 0$. Choose vectors $e \notin \mathfrak{N}(T_1)$, $e_* \notin \mathfrak{N}(T_1^*)$, where $\mathfrak{N}(T)$ denotes the kernel of T. The algebra $\mathscr{A}$ being dense, there exists $Q \in \mathscr{A}$ such that $QT_1 e = e_*$. Set $T_2 = QT_1$.

As $\mathfrak{N}(T_2) \supset \mathfrak{N}(T_1)$, the operators T_2, $T_1 + T_2$, $T_1 + iT_2$ have rank not greater than 1. Therefore if (3.2) cannot be fulfilled then

$$T_1^* T_1 = T_2^* T_2 = (T_1 + T_2)^* (T_1 + T_2) = (T_1 + iT_2)^* (T_1 + iT_2) = 0.$$

Hence $T_1^* T_2 = 0$. On the other hand, the vector $T_1^* T_2 e = T_1^* Q T_1 e = T_1^* e_*$ is non-zero. Contradiction.

So let T_0 satisfy (3.2). Take

(3.3) $$f \notin \mathfrak{N}(T_0^* T_0), \quad g = T_0^* T_0 f.$$

By assumption, for every $x \in \mathfrak{E}$ there exists $Q_x \in \mathscr{A}$ such that $Q_x g = x$. Set

(3.4) $$P_x = Q_x T_0^* T_0.$$

Then

(3.5) $$P_x \in \mathscr{A}, \quad P_x f = x, \quad P_x \mathfrak{N}(T_0) = 0 \quad (x \in \mathfrak{E}).$$

Relation (3.4) implies $P_x^* = T_0^* T_0 Q_x^*$. In particular, $P_x^* \mathfrak{E} \subset T_0^* T_0 \mathfrak{E} = \langle g \rangle$, the span of g. Thus

(3.6) $$P_x^* y = \varphi_x(y) g \quad (x, y \in \mathfrak{E}),$$

where $\varphi_x : \mathfrak{E} \to \mathbf{C}$ is a linear form depending on x.

From (3.6), (3.5), (3.3) and (3.2) we obtain

(3.7) $$P_x^* P_y = \varphi_x(y) T_0^* T_0 \quad (x, y \in \mathfrak{E}).$$

Really, the two sides of (3.7) coincide on f and $\mathfrak{N}(T_0)$ while the span of f and $\mathfrak{N}(T_0)$ equals $\mathfrak{E}$.

Suppose $(\cdot, \cdot)$ is a non-degenerate inner product representing $\mathscr{A}$. Then, in particular, $(P_x f, y)=(f, P_x^* y)$ for all x, y. Hence, on account of (3.5) and (3.6),

$$(x, y) = \overline{\varphi_x(y)}(f, g) \quad (x, y \in \mathfrak{E}), \tag{3.8}$$

where $(f, g)=(f, T_0^* T_0 f)=(T_0 f, T_0 f)$, a real number. This proves the uniqueness assertion.

To prove existence, choose a non-zero real number λ and set

$$(x, y) = \lambda \overline{\varphi_x(y)} \quad (x, y \in \mathfrak{E}). \tag{3.9}$$

From (3.5)—(3.6) it follows that $\varphi_{x_1+x_2}=\varphi_{x_1}+\varphi_{x_2}$ and $\varphi_{\alpha x}=\alpha\varphi_x$. From (3.7), applying the involution $*$ to both sides, interchanging the vectors x, y and comparing the result with (3.7) we find $\overline{\varphi_y(x)}=\varphi_x(y)$. Therefore (3.9) really defines an inner product on $\mathfrak{E}$.

Let the vector x satisfy $(x, y)=0$ for all $y \in \mathfrak{E}$. Then relations (3.9) and (3.6) yield $P_x^*=0$ i.e. $P_x=0$. Thus, in view of (3.5), $x=0$. Therefore the inner product (3.9) is non-degenerate.

Consider an operator $T \in \mathscr{A}$. From (3.5) it follows that $P_{Tx}=TP_x$ for all $x \in \mathfrak{E}$. Consequently, $P_{Tx}^*=P_x^* T^*$. Hence, making use of (3.6), we obtain $\varphi_{Tx}(y)=\varphi_x(T^* y)$ for all $y \in \mathfrak{E}$. Therefore the inner product (3.9) satisfies (2.4); in other words, it represents $\mathscr{A}$.

2. Maximal admissible $*$-algebras. Theorem 3.2 says that an admissible $*$-algebra is represented by one and, in essence, only one non-degenerate inner product. On the other hand, a non-degenerate inner product $(\cdot, \cdot)$ can represent several admissible $*$-algebras. It will turn out, however, that $(\cdot, \cdot)$ represents only one *maximal* admissible $*$-algebra.

We say the admissible $*$-algebras $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{L}(\mathfrak{E})$ are *equivalent*, $\mathscr{A}_1 \sim \mathscr{A}_2$, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are represented by the same non-degenerate inner products. This relation defines a partition of the class of all admissible $*$-algebras on $\mathfrak{E}$.

Lemma 3.3. *Let Ω be an equivalence class of admissible $*$-algebras on $\mathfrak{E}$. Let $(\cdot, \cdot)$ denote a non-degenerate inner product representing the elements of Ω. Then each element of Ω is a $*$-subalgebra of* $\mathrm{Ind}_{(\,)}$.

Proof. By definition, $(\cdot, \cdot)$ represents $\mathscr{A}$ if and only if $\mathscr{A} \subset \mathrm{Ind}_{(\,)}$ (the inclusion being meant in the sense of $*$-algebras).

Lemma 3.4. *If $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{L}(\mathfrak{E})$ are admissible $*$-algebras such that $\mathscr{A}_1 \subset \mathscr{A}_2$, then $\mathscr{A}_1 \sim \mathscr{A}_2$.*

Proof. If $\mathscr{A}_1 \subset \mathscr{A}_2$, then the non-degenerate inner products representing $\mathscr{A}_2$ represent $\mathscr{A}_1$ too.

Theorem 3.5. *Any equivalence class Ω of admissible *-algebras on $\mathfrak{E}$ contains exactly one maximal admissible *-algebra, namely* $\mathrm{Ind}_{(\,)}$, *where* $(\cdot,\,\cdot)$ *is a non-degenerate inner product representing the elements of* Ω.

Proof. Obviously, $\mathrm{Ind}_{(\,)}\in\Omega$. Let $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ be an admissible *-algebra with $\mathscr{A}\supset\mathrm{Ind}_{(\,)}$. Lemma 3.4 implies that $\mathscr{A}\in\Omega$. Hence, by Lemma 3.3, $\mathscr{A}\subset\mathrm{Ind}_{(\,)}$. Thus $\mathrm{Ind}_{(\,)}$ is maximal.

Conversely, for any $\mathscr{A}\in\Omega$ Lemma 3.3 yields $\mathscr{A}\subset\mathrm{Ind}_{(\,)}$. Therefore $\mathscr{A}$ cannot be maximal unless $\mathscr{A}=\mathrm{Ind}_{(\,)}$.

From Theorems 3.2 and 3.5 we obtain:

Corollary 3.6. *The mapping* $(\cdot,\,\cdot)\mapsto\mathrm{Ind}_{(\,)}$ *is a one-to-one correspondence between all non-degenerate inner products and all maximal admissible *-algebras on the same space $\mathfrak{E}$ provided we do not distinguish between inner products which are constant multiples of each other.*

Theorem 3.7. *Any admissible *-algebra $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ can uniquely be extended to a maximal one. Namely, if $\mathscr{A}$ is represented by* $(\cdot,\,\cdot)$, *then the maximal extension is* $\mathrm{Ind}_{(\,)}$.

Proof. According to Lemma 3.3, $\mathscr{A}\subset\mathrm{Ind}_{(\,)}$. Theorem 3.5 assures that $\mathrm{Ind}_{(\,)}$ is a maximal admissible *-algebra. Let $\mathscr{A}\subset\mathscr{A}_1$, where $\mathscr{A}_1$ is a maximal admissible *-algebra on $\mathfrak{E}$. Then, in view of Lemma 3.4, $\mathscr{A}_1$ is also represented by $(\cdot,\,\cdot)$. So, again by Theorem 3.5, $\mathscr{A}_1=\mathrm{Ind}_{(\,)}$.

Theorems 3.2 and 3.7 yield:

Corollary 3.8. *Two admissible *-algebras $\mathscr{A}_1, \mathscr{A}_2\subset\mathscr{L}(\mathfrak{E})$ are represented by the same inner products if and only if $\mathscr{A}_1$ and $\mathscr{A}_2$ have the same maximal extension.*

3. Admissible *-algebras represented by quasi-definite inner products. Next we impose certain conditions on the inner product and ask the resulting features of the *-algebras they represent.

Theorem 3.9. *The non-degenerate inner products representing the admissible *-algebra $\mathscr{A}\subset\mathscr{L}(\mathfrak{E})$ are definite if and only if*

$$T^*T\neq 0\quad (T\in\mathscr{A};\ T\neq 0). \tag{3.10}$$

Proof. Suppose the non-degenerate inner product $(\cdot,\,\cdot)$ represents $\mathscr{A}$, i.e.

$$(Tx, y)=(x, T^*y)\quad (T\in\mathscr{A};\ x, y\in\mathfrak{E}).$$

Let $(\cdot,\,\cdot)$ be definite. If for some $T_0\in\mathscr{A}$ we have $T_0^*T_0=0$, then $(T_0x, T_0x)=$ $=(T_0^*T_0x, x)=0$ for all $x\in\mathfrak{E}$. Hence $T_0=0$.

Let $(\cdot,\cdot)$ be non-definite. Then there is a vector $z \in \mathfrak{E}$, $z \neq 0$, with $(z, z)=0$ (cf. [4; Lemma I.2.1]). As $\mathscr{A}$ is admissible, there exist an operator $R \in \mathscr{A}$ of rank 1 and an operator $Q \in \mathscr{A}$ satisfying $QR\mathfrak{E}=\langle z\rangle$. Setting $T_1=QR$ we have $(T_1^* T_1 x, y)=(T_1 x, T_1 y)=0$ for all $x, y \in \mathfrak{E}$. Hence $T_1^* T_1=0$ though $T_1 \neq 0$.

Theorem 3.10. *Let k be a non-negative integer. The non-degenerate inner products representing the admissible *-algebra $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ are quasi-definite with rank of indefiniteness $\leqq k$ if and only if*

$$T^*T \neq 0 \quad (T \in \mathscr{A};\ \dim T\mathfrak{E} > k). \tag{3.11}$$

Proof. Let $(\cdot,\cdot)$ represent $\mathscr{A}$.

Suppose $(\cdot,\cdot)$ is quasi-definite with rank of indefiniteness $\varkappa(\mathfrak{E}) \leqq k$. If for some $T_0 \in \mathscr{A}$ we have $T_0^* T_0=0$, then $(T_0 x, T_0 x)=(T_0^* T_0 x, x)=0$ for all $x \in \mathfrak{E}$. Hence $T_0\mathfrak{E}$ is a neutral subspace, so that [2; Lemma 2] yields $\dim T_0\mathfrak{E} \leqq k$.

Suppose, conversely, that the non-degenerate inner product $(\cdot,\cdot)$ belongs to the complementary set of quasi-definite inner products with rank of indefiniteness $\leqq k$. By [2; Lemma 2] $\mathfrak{E}$ contains a neutral subspace $\mathfrak{M}$ of dimension $k+1$.

Let $x_1, \ldots, x_{k+1}$ be a linearly independent system in $\mathfrak{E}$, and let $y_1, \ldots, y_{k+1}$ be a basis of $\mathfrak{M}$. As $\mathscr{A}$ is admissible, it contains an operator R of rank 1. For such an R there exist vectors $x_0, y_0 \neq 0$ with

$$R\mathfrak{E} = \langle y_0\rangle, \quad Rx_0 = y_0.$$

Moreover, as $\mathscr{A}$ is dense, we can find operators $Q_j, S_j \in \mathscr{A}$ $(j=1, \ldots, k+1)$ such that

$$Q_j x_l = \delta_{jl} x_0, \quad S_j y_0 = y_j \quad (j, l = 1, \ldots, k+1).$$

The operator

$$T_1 = \sum_{j=1}^{k+1} S_j R Q_j$$

belongs to $\mathscr{A}$ and satisfies the relations

$$T_1 x_l = S_l R x_0 = y_l \quad (l = 1, \ldots, k+1),$$

$$\dim T_1\mathfrak{E} \leqq \sum_{j=1}^{k+1} \dim S_j R Q_j \mathfrak{E} = k+1.$$

Consequently,

$$T_1\mathfrak{E} = \mathfrak{M}. \tag{3.12}$$

As $\mathfrak{M}$ is neutral, (3.12) yields $(T_1^* T_1 x, x)=(T_1 x, T_1 x)=0$ for all $x \in \mathfrak{E}$, and by the polarization formula (see e.g. [4; relation (I.2.3)]) also $(T_1^* T_1 x, y)=0$ for all $x, y \in \mathfrak{E}$. Therefore $T_1^* T_1=0$. At the same time, $\dim T_1\mathfrak{E} > k$.

Remark 3.11. For decomposable inner products in general, the only relevant result consists of a reduction to the definite case by means of an operator $J \in \mathscr{A}$ satisfying $J^*=J^{-1}=J$ (see [3; Satz 6.1]).

§ 4. Star-algebras on Banach spaces

In this section $\mathfrak{E}$ is a Banach space over **C**. The norm of $x \in \mathfrak{E}$ will be denoted by $|x|$. Further, we denote by $\mathscr{L}(\mathfrak{E})$ the algebra of all (bounded or unbounded) linear operators on $\mathfrak{E}$, and by $\mathscr{B}(\mathfrak{E})$ the algebra of bounded linear operators on $\mathfrak{E}$.

1. Admissible *-algebras represented by continuous inner products. We are going to study how the mutual behaviour of the norm $|\cdot|$ and of a non-degenerate inner product $(\cdot, \cdot)$ is reflected on the relationship between $\mathscr{B}(\mathfrak{E})$ and $\mathrm{Ind}_{(\,)}$.

Theorem 4.1 (cf. [1; p. 196], [2; Theorem 2] and [3; Folgerung 5.6]). *The non-degenerate inner products representing the admissible *-algebra $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ are continuous if and only if $\mathscr{A} \subset \mathscr{B}(\mathfrak{E})$.*

Proof. Suppose $(\cdot, \cdot)$ represents $\mathscr{A}$.

Let $(.,.)$ be continuous. Consider an operator $T \in \mathscr{A}$. If the sequence $(x_n) \subset \mathfrak{E}$ satisfies $x_n \to 0$ and, for some z, $Tx_n \to z$, then for all $y \in \mathfrak{E}$ we have $(Tx_n, y) \to$ $\to (z, y)$ and $(Tx_n, y) = (x_n, T^* y) \to 0$; hence $z = 0$. Thus T is closed and, by the closed graph principle, bounded.

Let, conversely, $\mathscr{A} \subset \mathscr{B}(\mathfrak{E})$. According to the proof of Theorem 3.2 (see especially (3.8)) we have $(x, y) = \overline{\varphi_x(y)}\,(f, g)$, where $\varphi_x(y)$ is defined by the relation $P_x^* y = \varphi_x(y) g$ with some $g \neq 0$ and $P_x \in \mathscr{A}$. In particular, $P_x^* \in \mathscr{A}$ and, consequently, $P_x^* \in \mathscr{B}(\mathfrak{E})$. Thus $\varphi_x(y)$ and (x, y) are continuous functions of y. It follows (see subsection 2.2) that (x, y) is jointly continuous in x and y.

Setting $\mathscr{A} = \mathrm{Ind}_{(\,)}$ we find:

Corollary 4.2. *The non-degenerate inner product $(\cdot, \cdot)$ is continuous on $\mathfrak{E}$ if and only if* $\mathrm{Ind}_{(\,)} \subset \mathscr{B}(\mathfrak{E})$.

From Theorem 4.1 and Lemma 3.4 we obtain:

Corollary 4.3. *If an admissible *-algebra is contained in $\mathscr{B}(\mathfrak{E})$, then its extensions are also contained in $\mathscr{B}(\mathfrak{E})$.*

In particular:

Corollary 4.4. *If $\mathscr{B}(\mathfrak{E})$ is a *-algebra, then it is maximal.*

Theorem 4.5 (cf. [3; Satz 3.7]). *The non-degenerate inner product $(\cdot, \cdot)$ is compatible with $|\cdot|$ on $\mathfrak{E}$ if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$.

Proof. Let $(\cdot, \cdot)$ be compatible with $|\cdot|$. Consider an operator $T \in \mathscr{B}(\mathfrak{E})$. By compatibility, (Tx, y) is a continuous function of Tx and therefore, by the boundedness of T, it is a continuous function of x. Hence, again by compatibility, there exists $y_T \in \mathfrak{E}$ such that $(Tx, y) = (x, y_T)$ for all $x \in \mathfrak{E}$. Thus the adjoint $T^{(\,)}$

exists, i.e. $T \in \mathrm{Ind}_{(\,)}$, $\mathscr{B}(\mathfrak{E}) \subset \mathrm{Ind}_{(\,)}$. On the other hand, Corollary 4.2 yields $\mathrm{Ind}_{(\,)} \subset \subset \mathscr{B}(\mathfrak{E})$.

Let, conversely, $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$. Then, on account of Corollary 4.2, $(\cdot\,,\,\cdot)$ is continuous. On the other hand, let φ be a continuous linear form on $\mathfrak{E}$. Set $Tx = =\varphi(x)z$ $(x \in \mathfrak{E})$, where $z \neq 0$ is fixed. Obviously, $T \in \mathscr{B}(\mathfrak{E})$. Therefore, by assumption, the adjoint $T^{(\,)}$ exists:

$$\big(\varphi(x)z,\, y\big) = (x,\, T^{(\,)} y)$$

for all $y \in \mathfrak{E}$. In particular, if $(z, y) = 1$, then

$$\varphi(x) = (x,\, T^{(\,)} y) \quad (x \in \mathfrak{E}).$$

Theorems 4.5 and 3.7 yield:

Corollary 4.6. *The non-degenerate inner products representing the admissible* **-algebra* $\mathscr{A} \subset \mathscr{L}(\mathfrak{E})$ *are compatible with* $|\cdot|$ *if and only if the maximal extension of* $\mathscr{A}$ *equals* $\mathscr{B}(\mathfrak{E})$.

2. Admissible *-algebras represented by decomposable, continuous inner products. In Theorems 3.9—3.10 we dealt with admissible *-algebras $\mathscr{A}$ represented by certain kinds of decomposable inner products. Below we obtain additional information in the special case $\mathscr{A} = \mathscr{B}(\mathfrak{E})$.

Our starting point is the following application of Theorem 4.5:

Theorem 4.7. *The fundamental norms corresponding to the decomposable, non-degenerate inner product* $(\cdot\,,\,\cdot)$ *are topologically equivalent to the given norm* $|\cdot|$ *on* $\mathfrak{E}$ *if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$. *In this case,* $\mathfrak{E}$ *equipped with* $(\cdot\,,\,\cdot)$ *is a Krein space.*

Proof. Consider a fundamental inner product $(\cdot\,,\,\cdot)_J$ associated with $(\cdot\,,\,\cdot)$, and the corresponding fundamental norm $\|\cdot\|_J$ (see (2.6)—(2.7)). By Theorem 4.5 we must prove that $\|\cdot\|_J$ is equivalent to $|\cdot|$ if and only if $(\cdot\,,\,\cdot)$ is compatible with $|\cdot|$.

Let $\alpha_1|x| \leqq \|x\|_J \leqq \alpha_2|x|$ $(x \in \mathfrak{E})$, where $\alpha_1, \alpha_2 > 0$. Then $\mathfrak{E}$ is a Hilbert space relative to $(\cdot\,,\,\cdot)_J$. Since J is the difference of two orthogonal complementary projectors P^+, P^- in this Hilbert space, we have

$$|(x, y)| = |(J^2x, y)| = |(Jx, y)_J| \leqq \|Jx\|_J \|y\|_J = \|x\|_J \|y\|_J \leqq \alpha_2^2 |x|\,|y| \quad (x, y \in \mathfrak{E}).$$

On the other hand, if the linear form φ is continuous for $|\cdot|$, then it is continuous for $\|\cdot\|_J$, so that by the Riesz representation theorem there is a $y \in \mathfrak{E}$ satisfying the relations $\varphi(x) = (x, y)_J = (x, Jy)$ $(x \in \mathfrak{E})$.

Let, conversely, $(\cdot\,,\cdot)$ be compatible with $|\cdot|$. Then $\|\cdot\|_J$ is continuous relative to $|\cdot|$ (see [4; Lemma IV.5.4]). Hence, if φ is a linear form continuous for $\|\cdot\|_J$, then φ is continuous for $|\cdot|$ and, consequently, there exists $y \in \mathfrak{E}$ such that $\varphi(x) = =(x, y)$ $(x \in \mathfrak{E})$. Thus $\varphi(x) = (x, J^2y) = (x, Jy)_J$ $(x \in \mathfrak{E})$. On the other hand, by the

Schwarz inequality, $|(x, y)_J| \leqq \|x\|_J \|y\|_J$ $(x, y \in \mathfrak{E})$. As a result, $(\cdot, \cdot)_J$ is compatible with $\|\cdot\|_J$. In other words, the relation $\varphi(x) = (x, z)_J$ $(x \in \mathfrak{E})$ defines an isomorphism between $\mathfrak{E}$ and $\mathfrak{E}_J^*$, the Banach space of all linear forms which are continuous for $\|\cdot\|_J$. Moreover, by the elements of Hilbert space theory, $\|\varphi\|_J = \|z\|_J$, i.e. the isomorphism is isometrical. Therefore $\mathfrak{E}$ is complete with respect to $\|\cdot\|_J$. Once more recalling that $\|\cdot\|_J$ is continuous relative to $|\cdot|$, the closed graph principle guarantees the equivalence of $\|\cdot\|_J$ and $|\cdot|$.

From Theorem 4.7 by the aid of Theorems 3.9 and 3.10, respectively, we obtain the following results.

Corollary 4.8 (cf. [1; Corollary 4.10.8]). *The non-degenerate inner product* $(\cdot, \cdot)$ *turns* $\mathfrak{E}$ *into a Hilbert space with norm equivalent to* $|\cdot|$ *if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ *and*

$$T^{(\,)} T \neq 0 \quad (T \in \mathscr{B}(\mathfrak{E});\ T \neq 0).$$

Corollary 4.9 (cf. [2; Theorem 3]). *Let k be a non-negative integer. The non-degenerate inner product* $(\cdot, \cdot)$ *turns* $\mathfrak{E}$ *into a Pontrjagin space with rank of indefiniteness* $\leqq k$ *and fundamental norms equivalent to* $|\cdot|$ *if and only if* $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ *and*

$$T^{(\,)} T \neq 0 \quad (T \in \mathscr{B}(\mathfrak{E});\ \dim T\mathfrak{E} > k).$$

As we have no good criterion for decomposability of inner products representing a given *-algebra (see Remark 3.11), for Krein spaces we can give only the following characterization:

Theorem 4.10. *The non-degenerate inner product* $(\cdot, \cdot)$ *turns* $\mathfrak{E}$ *into a Krein space with fundamental norms equivalent to* $|\cdot|$ *if and only if* (i) $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ *and* (ii) $\mathfrak{E}$ *is topologically isomorphic to a Hilbert space.*

Proof. Suppose that $(\cdot, \cdot)$ turns $\mathfrak{E}$ into a Krein space with fundamental norms equivalent to $|\cdot|$. Then, in particular, $\mathfrak{E}$ is decomposable, and Theorem 4.7 yields $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$. Moreover, $\mathfrak{E}$ is a Hilbert space with respect to any fundamental inner product.

Suppose, conversely, that $\mathrm{Ind}_{(\,)} = \mathscr{B}(\mathfrak{E})$ and $\mathfrak{E}$ is topologically isomorphic to a Hilbert space. The norm $|\cdot|$ being involved in the theorem up to topological equivalence only, we may regard $\mathfrak{E}$ as a Hilbert space with inner product $[\cdot, \cdot]$ and norm $|x| = [x, x]^{1/2}$. On the other hand, by Theorem 4.1, $(\cdot, \cdot)$ is continuous on $\mathfrak{E}$. Consequently, there exists a bounded self-adjoint operator G on $\mathfrak{E}$ satisfying $(x, y) = [Gx, y]$ $(x, y \in \mathfrak{E})$. It is easy to see that the positive and negative spectral subspaces of G are the components of a fundamental decomposition of $\mathfrak{E}$ (cf. [4; Theorem IV.5.2]). Hence $(\cdot, \cdot)$ is decomposable and Theorem 4.7 applies.

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О тождестве Л. Б. Редеи для полиномов Лагерра

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Пусть $D=d/dx$ — оператор дифференцирования и

(1) $$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} D^n (e^{-x} x^{n+\alpha}), \quad n = 0, 1, 2, \ldots,$$

— обобщенные полиномы Лагерра (см. [1], 10.12). В настоящей заметке устанавливается справедливость следующего представления для полиномов $L_n^\alpha(x)$:

(2) $$L_n^\alpha(x) = \frac{(-1)^n}{n!} e^x (xD^2+\alpha D+D)^n (e^{-x}).$$

Представление (2) в частном случае $\alpha=0$ было доказано Л. Б. Редеи в недавней работе [2]. Приводимое ниже доказательство (2) существенно отличается от предложенного в [2] подхода, сохраняя в то же время его элементарность.

В заключение работы приводится ещё одно представление для обобщенных полиномов Лагерра. Именно, устанавливается, что

(3) $$L_n^\alpha(x) = \frac{1}{n!} \{(1+\alpha-x+xD)(1-D)\}^n (1).$$

Доказательство (2) опирается на соотношение (1) и следующую лемму, представляющую независимый интерес.

Лемма. *Пусть* A *и* B *— линейные операторы такие, что*

(4) $$AB = (B+1)A.$$

Тогда

(5) $$(BA)^n = (B)_n A^n, \quad n = 0, 1, 2, \ldots,$$

где обозначено $(B)_0=1$ *и* $(B)_n=B(B+1)\ldots(B+n-1)$ *при* $n=1, 2, \ldots$.

Доказательство леммы проводится по индукции в два этапа. Во-первых, легко показывается, что

(6) $$A(B)_n = (B+1)_n A, \quad n = 0, 1, 2, \ldots.$$

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В самом деле, равенство (6) при $n=0$ тривиально. Если же оно имеет место для $n=0, 1, \ldots, m$, то, принимая во внимание (4), получаем

$$A(B)_{m+1} = A(B)_m(B+m) = (B+1)_m A(B+m) = (B+1)_{m+1} A.$$

Таким образом, (6) доказано. В свою очередь, равенство (5) тривиально верно при $n=0$. Если же оно справедливо при $n=0, 1, \ldots, m$, то, в силу (6),

$$(BA)^{m+1} = BA(B)_m A^m = B(B+1)_m A^{m+1} = (B)_{m+1} A^{m+1}.$$

Лемма доказана.

Положим теперь $A=D$ и $B=1+\alpha+xD$. Легко проверяется, что условия леммы удовлетворяются, и потому, в силу (5),

$$(7)\qquad (x^{-\alpha}Dx^{1+\alpha}D)^n = (xD^2+\alpha D+D)^n = (1+\alpha+xD)_n D^n = x^{-\alpha}D^n x^{n+\alpha}D^n.$$

Первое и последнее равенства в (7) являются следствием простого тождества

$$(\beta+xD)\big(f(x)\big) = x^{1-\beta}D\big(x^\beta f(x)\big).$$

Записав представление (1) в виде

$$L_n^\alpha(x) = \frac{(-1)^n}{n!} e^x x^{-\alpha} D^n x^{n+\alpha} D^n(e^{-x})$$

и воспользовавшись (7), получаем (2).

Доказательство (3) основывается на элементарном тождестве

$$(8)\qquad -(xD^2+\alpha D+D)\big(e^{-x}f(x)\big) = e^{-x}\{1+\alpha-x+(2x-\alpha-1)D-xD^2\}\big(f(x)\big)$$

и повторном использовании представления (2). Действительно, согласно (2), имеем

$$(n+1)L_{n+1}^\alpha(x) = \frac{(-1)^{n+1}}{n!} e^x(xD^2+\alpha D+D)^{n+1}(e^{-x}) = -e^x(xD^2+\alpha D+D)\big(e^{-x}L_n^\alpha(x)\big),$$

откуда, в силу (8), приходим к рекуррентной формуле

$$(9)\qquad (n+1)L_{n+1}^\alpha(x) = (1+\alpha-x+xD)(1-D)\big(L_n^\alpha(x)\big).$$

Из (9) с учетом того, что $L_0^\alpha(x)=1$, следует (3).

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On minimal bi-ideals and minimal quasi-ideals in compact semigroups

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In what follows, S will denote a compact topological semigroup. A non-empty subset $B\subset S$ is a *bi-ideal* of S if $BSB\cup B^2\subset B$; a non-empty subset $Q\subset S$ is a *quasi-ideal* of S if $QS\cap SQ\subset Q$. A bi-ideal (quasi-ideal) is said to be *minimal* if it does not contain properly a bi-ideal (quasi-ideal) of S. DELANGHE [1] has established the existence of minimal quasi-ideals and minimal bi-ideals in S, and, moreover, shown that the family of minimal quasi-ideals and that of minimal bi-ideals coincide. In this note, we are concerned with the relations between primitive idempotents in S and minimal quasi-ideals and minimal bi-ideals; consequently we obtain a theorem which implies all the results in [1].

An idempotent $e\in S$ is called *primitive* if e and the zero of S (which may not exist) are the only idempotents in the set eSe.

Theorem 1. *Suppose S has no zero and e is an idempotent of S. Then the following are equivalent:*

(a) *e is primitive.*

(b) *eSe is a minimal bi-ideal of S.*

(c) *eSe is a minimal quasi-ideal of S.*

Proof. (a) implies (b). Observe that eSe is a bi-ideal of S. By [3, p. 43], eSe is a group; so eSe is a minimal bi-ideal.

(b) implies (c). Let Q be a quasi-ideal of S contained in eSe. Since Q is also a bi-ideal of S, Q coincides with eSe, giving (c).

(c) implies (a). Let K be the minimal ideal of S (see [3, p. 32]); then $K\cap eSe$ is clearly a (non-empty) quasi-ideal of S and so $K\cap eSe=eSe$, whence $e\in K$. This together with [3, p. 43] completes the proof.

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Corollary [1]. *If B is a bi-ideal (quasi-ideal) of S, then B contains a minimal bi-ideal (minimal quasi-ideal) of S.*

Proof. We prove the corollary for bi-ideals, and the case for quasi-ideals can be derived in a similar way. First, if S has zero 0, it is obvious that $\{0\}$ is the minimal bi-ideal contained in B. Next, if S has no zero, then $B\cap K$ is a bi-ideal of S, where K denotes the minimal ideal of S. Let $x\in B\cap K$, implying that $x\in eSe$ for some idempotent $e\in K$ [3, p. 30]. Since eSe is a group containing the bi-ideal $B\cap eSe$, we see that $eSe=B\cap eSe$. Thus $eSe\subset B$, and the result follows from Theorem 1 and [3, p. 43].

Now suppose S contains a zero 0. Then an element $x\in S$ is called *nilpotent* if $x^n\to 0$ as $n\to\infty$, and a subset $A\subset S$ is said to be *nil* if every element in A is nilpotent.

Theorem 2. *If S has zero 0 and $e\in S$ is a non-zero idempotent of S, then the following conditions are equivalent:*

(a) *e is primitive.*

(b) *eSe is a minimal non-nil bi-ideal of S.*

(c) *eSe is a minimal non-nil quasi-ideal of S.*

Proof. The equivalence of (a) and (b) has been shown by Koch in [2], and we want to show the equivalence of (b) and (c). Suppose (b) holds, and let Q be a non-nil quasi-ideal of S contained in eSe. Since Q is also a non-nil bi-ideal, we have $Q=eSe$ so that (c) follows. Conversely, if (c) is true, then take a non-nil bi-ideal B in eSe. It is easy to see that B contains a non-zero idempotent f, in view of Lemma 2 of [2]. Hence $fSf\subset B\subset eSe$. Since fSf is a non-nil quasi-ideal of S, we have $fSf=eSe$. This yields $B=eSe$, giving the result.

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Completeness of eigenfunctions of seminormal operators

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Let $\mathfrak{H}$ be a separable complex Hilbert space and $\mathscr{B}(\mathfrak{H})$ the algebra of bounded linear operators on $\mathfrak{H}$. An operator T in $\mathscr{B}(\mathfrak{H})$ is called a seminormal operator in case its self-commutator $D=T^*T-TT^*$ is semidefinite. In the case $D\geqq 0$ (respectively, $D\leqq 0$) the operator T is said to be hyponormal (respectively, cohyponormal). The operator T in $\mathscr{B}(\mathfrak{H})$ will be said to be completely non-normal in case the only subspace reducing the operator T on which T is a normal operator is the zero subspace. The notations $\mathrm{sp}(T)$, $\mathrm{sp}_e(T)$ and $\pi_0(T)$ will be used for the spectrum, essential spectrum and the set of eigenvalues of the operator T, respectively.

Let T be a hyponormal operator on $\mathfrak{H}$. It is easy to verify that $\ker T$ (the kernel of T) is a reducing subspace for T. Consequently, $\pi_0(T)$ must be empty whenever T is completely non-normal. On the other hand, $\pi_0(T^*)$ is sometimes non-empty. The following result will be proved in Section 1.

Theorem 1. *Let T be a completely non-normal cohyponormal operator. Assume that the planar Lebesgue measure of $\mathrm{sp}_e(T)$ is zero, then*

$$\underset{\lambda\notin \mathrm{sp}_e(T)}{\mathrm{c.l.m.}}\{\ker(T-\lambda)\}=\mathfrak{H},$$

where c.l.m.$\{\dots\}$ *denotes the closed linear manifold generated by* $\{\dots\}$.

If T is an operator with a rank one self-commutator, then T is either hyponormal or cohyponormal. It is still an open question as to whether such an operator T has a non-trivial invariant subspace. In certain cases T is known to possess an invariant subspace. (See [2] and [3].) On the other hand there are not many operators with a rank one self-commutator that are known to possess cyclic vectors. Theorem 1 can be used to provide examples in this direction.

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In Sections 2 and 3 we will study the singular integral operator S_b defined on $L^2(c, d)$ by

$$S_b f(s) = sf(s) + \frac{b(s)}{\pi} \int_c^d \frac{\bar{b}(t) f(t)}{t-s} dt,$$

where b is a non-vanishing smooth function on the interval $[c, d]$. The operator S_b is an irreducible cohyponormal operator that satisfies $S_b^* S_b - S_b S_b^* = -\frac{2}{\pi}(\ , b)b$; here, $(\ ,\ )$ denotes the inner product in $L^2(c, d)$.

In Section 3 it will be shown that b is a cyclic vector for the operator S_b. The method will entail constructing a pair of analytic continuations of the local resolvent $(S_b - \lambda)^{-1} b$ onto portions of $\pi_0(S_b)$. This leads to a discussion of solutions of singular integral equations in Section 2.

The interest in the operator S_b stems from the fact that every completely non-normal seminormal operator has a singular integral representation (see, e.g., [8], [9] and [10]).

§ 1. Completeness of eigenfunctions. PUTNAM [11] established the following remarkable inequality. Let T be a seminormal operator on $\mathfrak{H}$. Then

$$\pi \|T^* T - T T^*\| \leqq \operatorname{meas}_2(\operatorname{sp}(T)), \tag{1}$$

where meas_2 denotes planar Lebesgue measure. Below we will show how Theorem 1 follows from the inequality (1).

Proof of Theorem 1. Let $\mathfrak{M} = \underset{\lambda \notin \operatorname{sp}_e(T)}{\text{c.l.m.}} \{\ker(\lambda - T)\}$. Relative to the decomposition $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, the operator T has the matrix form

$$T = \begin{pmatrix} T_{\mathfrak{M}} & X \\ 0 & T_{\mathfrak{M}^\perp} \end{pmatrix},$$

here $T_{\mathfrak{M}}$ is the restriction of T to $\mathfrak{M}$ and $T_{\mathfrak{M}^\perp}$ denotes the compression of T to $\mathfrak{M}^\perp$. The operator $T_{\mathfrak{M}^\perp}$ is cohyponormal.

Let $\lambda \notin \operatorname{sp}_e(T)$. It follows from the continuity of the orthogonal projection onto $\ker(\mu - T)$, on the complement of $\operatorname{sp}_e(T)$, that $(\lambda - T)_{\mathfrak{M}}$ has dense range. It is easy to see that $(\lambda - T)_{\mathfrak{M}}$ has closed range and therefore $(\lambda - T)_{\mathfrak{M}}$ is onto. The surjectivity of $(\lambda - T)_{\mathfrak{M}}$ and the fact that $\ker(\lambda - T) \subset \mathfrak{M}$ imply $\lambda \notin \operatorname{sp}(T_{\mathfrak{M}^\perp})$.

The last paragraph shows that $\operatorname{meas}_2(\operatorname{sp}(T_{\mathfrak{M}^\perp})) = 0$. Thus Putnam's inequality (1) applied to the operator $T_{\mathfrak{M}^\perp}$ shows that $T_{\mathfrak{M}^\perp}$ is a normal operator. Since T is completely non-normal it must be that $\mathfrak{M}^\perp$ is the zero subspace. This completes the proof.

Let A be an operator on $\mathfrak{H}$ and let Ω be an open subset of the complex plane such that for every $\lambda \in \Omega$ the operator $A - \lambda$ is surjective. G. R. ALLAN [1] has shown that it is possible to construct an analytic right resolvent for A on Ω. This means there is a $\mathscr{B}(\mathfrak{H})$-valued analytic function $R(\lambda)$ defined on Ω such that $(A-\lambda)R(\lambda)=I$. The operator $P(\lambda)=I-R(\lambda)(A-\lambda)$ then defines an analytic projection valued function on Ω. It is clear that the range of $P(\lambda)$ is the kernel of $A-\lambda$.

Suppose now that T is an irreducible cohyponormal operator satisfying $\mathrm{meas}_2(\mathrm{sp}_e(T))=0$. Let $\Omega(T)=\mathrm{sp}(T)\setminus\mathrm{sp}_e(T)$ and assume $\Omega(T)$ is connected. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an infinite sequence that accumulates in $\Omega(T)$. Then

$$\text{(2)}\qquad \underset{n}{\mathrm{c.l.m.}}\,\{\ker(T-\lambda_n)\} = \mathfrak{H}.$$

This last identity follows from Theorem 1 and the discussion in the preceding paragraph which demonstrates the existence of an analytic projection valued map onto $\ker(\lambda-T)$ for $\lambda\in\Omega(T)$.

It is interesting to note that if T is a seminormal operator and $\mathrm{meas}_2(\mathrm{sp}_e(T))=0$, then the self-commutator of T is compact. This follows because the projection $\hat{T}$ of T into the Calkin algebra $\mathscr{C}$ is a seminormal element in the C^*-algebra $\mathscr{C}$ with $\mathrm{meas}_2(\mathrm{sp}(\hat{T}))=0$. Since Putnam's inequality (1) holds for seminormal elements in any C^* algebra, then $\hat{T}$ must be a normal element in $\mathscr{C}$. This shows that T^*T-TT^* is compact. This last observation was pointed out to the author by D. D. Rogers.

In the case where T is an irreducible cohyponormal operator with rank one self-commutator it is easy to show that the dimension of $\ker(T)$ is at most one. It follows that if X is an element commuting with T, then X leaves $\ker T$ invariant. The following is an immediate corollary of this last remark and Theorem 1.

Corollary 1. *Let T be an irreducible operator with a rank one self-commutator such that* $\mathrm{meas}_2(\mathrm{sp}_e(T))=0$. *Then the commutant of T is abelian.*

We remark that there are very few operators T satisfying the hypothesis of Corollary 1 for which an exact description of the commutant is known.

§ 2. Seminormal singular integral operators. Let E be a bounded measurable subset of the real line having positive measure. Let a and b be bounded measurable functions on E such that $a(t)$ is real and $b(t)\neq 0$ almost everywhere. For f in $L^2(E)$ define the singular integral operator

$$\text{(3)}\qquad Sf(s) = sf(s)+i\left[a(s)f(s)+\frac{b(s)}{\pi i}\int_E \frac{\bar{b}(t)f(t)}{t-s}\,dt\right].$$

The singular integral is interpreted as a Cauchy principal value. The operator S satisfies $S^*S-SS^*=-\frac{2}{\pi}(\,,b)b$; where $(\,,)$ denotes the inner product in $L^2(E)$.

The fact that $b(t)\neq 0$ ensures that S is irreducible. For a description of $\operatorname{sp}(S)$ and $\operatorname{sp}_e(S)$ the reader is referred to [7] and [6].

It should be remarked that if T is an irreducible cohyponormal operator with a rank one self-commutator such that the real part of T has simple spectrum, then T is unitarily equivalent to an operator of the form S. In particular, if $E=[-1, 1]$, $a\equiv 0$ and $b(t)=(1-t^2)^{1/4}$, then theoperator S defined by (3) is unitarily equivalent to the unilateral shift.

We will be concerned with the case where $E=I=[c, d]$ is an interval, $a\equiv 0$ and the function b is a non-vanishing real valued element in $C'(I)$. In this case we will denote the operator S defined by (3) as S_b. The spectrum and essential spectrum of the operator S_b can be described as follows:

$$\operatorname{sp}(S_b) = \{\lambda = \mu+i\nu : \ \mu\in I, \ |\nu| \leqq b^2(\mu)\}$$

and $\operatorname{sp}_e(S_b)$ is the boundary of $\operatorname{sp}(S_b)$. Moreover, $\pi_0(S_b)=\operatorname{sp}(S_b)\setminus\operatorname{sp}_e(S_b)$ and in view of the fact that $S_b^*S_b-S_bS_b^*$ is one dimensional, then each eigenvalue of the operator S_b has multiplicity one.

Below we will establish the existence of two analytic continuations of the local resolvent $b(\lambda)=(S-\lambda)^{-1}b\,(\lambda\notin\operatorname{sp}(S_b))$ onto portions of $\operatorname{sp}(S_b)$. In fact, we will construct two weakly analytic $L^2(I)$-valued functions b_+ and b_-, where b_+ is analytic in $J_+=(c,\infty)$ and b_- is analytic on $J_-=(-\infty, d)$, such that $(S_b-\lambda)b_\pm(\lambda)=b, \lambda\in J_\pm$. Further, $e(\lambda)=b_-(\lambda)-b_+(\lambda)$ will be a non-zero eigenfunction of the operator corresponding to λ in $J_+\cap J_-=(c, d)$.

The construction of the local resolvent necessitates solving the singular integral equation $(S_b-\lambda)x=b$. The basic method employed is discussed in the book of Tricomi [12] (see, also [4] and [5]).

Let H denote the Hilbert transform on the real line $\mathbf{R}$. Thus for $f\in L^1(\mathbf{R})$, $Hf(x)$ is defined at almost every real x by the Cauchy principal value integral

$$Hf(x) = \frac{1}{\pi}\int\frac{f(t)}{t-x}\,dt.$$

It is well known that the operator H defines a bounded linear operator on $L^p(\mathbf{R})$, $p>1$.

Let E be a bounded measurable subset of the real line and let θ be a real valued bounded measurable function supported on E. It is known that if $\exp[H\theta]$ belongs to $L^p(J)$, for some $p>1$, where J is a bounded interval containing the (essential) closure of E in its interior, then

$$\cos\theta\exp H\theta = H[\sin\theta\exp H\theta]+1. \tag{4}$$

Now for $\lambda \in J_\pm$, we define the function

$$\theta_\lambda^\pm(s) = \arg\left[\pm\frac{(\lambda-s)+ib^2(s)}{[(\lambda-s)^2+b^4(s)]^{1/2}}\right], \quad s\in I.$$

The branch of the argument is chosen such that $-\pi<\arg z\leqq\pi\,(z\neq 0)$. We remark that for λ fixed in $J_\pm$, the function $\theta_\lambda^\pm$ belongs to $C'(I)$. We will tacitly assume, whenever necessary, that the function $\theta_\lambda^\pm$ is extended to be zero off I.

Fix λ in $J_\pm$. The function $\exp H\theta_\lambda^\pm$ is easily seen to be bounded in a neighborhood of every point on $\mathbf{R}$ except possibly the points c and d. Similarly one can check that when $\lambda\in J_+$ the function $\exp H\theta_\lambda^+$ is bounded in a neighborhood of the point d and that when $\lambda\in J_-$ the function $\exp H\theta_\lambda^-$ is bounded in a neighborhood of the point c. In order to conclude that $\exp H\theta_\lambda^+$ is square integrable in a neighborhood of the point c and $\exp H\theta_\lambda^-$ is square integrable in a neighborhood of the point d, one needs only to apply Lemma 1 of [5].

Making the substitution $\theta_\lambda^\pm$ for θ in equation (4), one obtains

$$(5)\qquad (s-\lambda)f_\lambda^\pm(s)+\frac{1}{\pi}\int_I\frac{b^2(t)f_\lambda^\pm(t)}{t-s}\,dt = 1, \quad s\in I;$$

here

$$f_\lambda^\pm(t) = \frac{\mp\exp[H\theta_\lambda^\pm]}{[(\lambda-t)^2+b^4(t)]^{1/2}}.$$

It follows that $b_\pm(\lambda)=bf_\lambda^\pm$ satisfies $(S_b-\lambda)b_\pm(\lambda)=b$ and further, $b_\pm(\lambda)\in L^2(I)$, for all $\lambda\in J_\pm$.

Note that for $\lambda\in J_+\cap J_-$, the function $e(\lambda)=b_-(\lambda)-b_+(\lambda)$ is a non-zero $L^2(I)$ eigenfunction of the operator S_b corresponding to the eigenvalue λ.

It is possible to extend the functions $\lambda\to\theta_\lambda^\pm$ to domains of the form

$$\hat{J}_\pm = \{\lambda=\mu+iv:\ \mu\in J_\pm,\ |v|<\varepsilon_0\},$$

where $\varepsilon_0>0$ is chosen sufficiently small. This is accomplished by defining for $\lambda\in\hat{J}_\pm$

$$(5)\qquad \theta_\lambda^\pm(s) = \frac{1}{i}\log\left[\frac{\pm(\lambda-s)+ib^2(s)}{[(\lambda-s)^2+b^4(s)]^{1/2}}\right], \quad s\in I.$$

Here, if $z=re^{i\theta}, r>0, -\pi<\theta\leqq\pi$, then $\log z=\log r+i\theta$ and $\sqrt{z}=r^{1/2}e^{i\theta/2}$. The exact choice of ε_0 will depend only on the function b. The constant ε_0 is chosen such that for every s fixed on I the functions $\lambda\to\theta_\lambda^\pm(s)$ are analytic on $\hat{J}_\pm$.

It is not difficult to verify that

$$b_\pm(\lambda)(s) = \frac{\mp b(s)\exp H\theta_\lambda^\pm(s)}{[(\lambda-s)^2+b^4(s)]^{1/2}}$$

belong to $L^2(I)$ whenever $\lambda \in \hat{J}_\pm$. Moreover the map $\lambda \to b_\pm(\lambda)$ is a weakly analytic $L^2(I)$-valued mapping on $\hat{J}_\pm$. It follows that $(S_b-\lambda)b_\pm(\lambda)=b$, $\lambda \in \hat{J}_\pm$. We have therefore constructed two distinct analytic continuations b_+ and b_- of the local resolvent $(S_b-\lambda)^{-1}b$ onto $\hat{J}_+$ and $\hat{J}_-$, respectively.

§ 3. **Cyclic vectors.** We are now in a position to establish the following:

Proposition 1. *Let $I=[c,d]$ and assume b is a nonvanishing function in $C'(I)$. Let S_b be the cohyponormal operator defined on $L^2(I)$ by* (3). *The vector b is a cyclic vector for the operator S_b.*

Proof. The operator S_b is a completely non-normal cohyponormal operator with a rank one self-commutator. Moreover, $\operatorname{meas}_2(\operatorname{sp}_e(S_b))=0$. Let $\lambda \in (c,d)$ and let $e(\lambda)$ be the eigenfunction corresponding to the value λ described in the preceding section. It follows from the identity (2) that $\underset{c<\lambda<d}{\text{c.l.m.}}\{e(\lambda)\}=L^2(I)$.

Suppose that f is in $L^2(I)$ and f is orthogonal to $\underset{n}{\text{c.l.m.}}\{S_b^n b\}$, then $((S_b-\lambda)^{-1}b,f)=0$, for $|\lambda|$ large. It follows that $(b_\pm(\lambda),f)=0$, for $\lambda \in \hat{J}_\pm$. Consequently, $(e(\lambda),f)=0$ for every $\lambda \in (c,d)$, and we conclude $f=0$. This completes the proof.

It would be interesting to find the exact conditions on an element b in $C'(I)$ which ensure that b is a cyclic vector for the operator S_b. Similarly one can ask for necessary and sufficient conditions for the function b in $L^2(E)$ to be cyclic for the operator S defined by (3).

§ 4. **Conclusion.** It is not difficult to construct irreducible cohyponormal operators T such that $\operatorname{sp}(T)\setminus \operatorname{sp}_e(T)$ is non-empty and possesses the property that $\underset{\lambda \notin \operatorname{sp}_e(T)}{\text{c.l.m.}}[\ker(T-\lambda)] \neq \mathfrak{H}$. The following is such an example.

Example. Let K be a perfect nowhere dense set of positive measure in $[0,1]$, and let J be a closed interval disjoint from K. Set $E=J\cup K$ and let S_0 be the singular integral operator defined on $L^2(E)$ by (3) with the choice $a\equiv 0$ and $b\equiv 1$. The $\operatorname{sp}(S_0)=E\times[-1,1]$ and $\operatorname{sp}(S_0)\setminus \operatorname{sp}_e(S_0)$ is the interior of $J\times[-1,1]$. Using the usual functional calculus it is possible to obtain a non-trivial invariant subspace M for the operator S_0 such that the spectrum of S_0 restricted to M is $J\times[-1,1]$. Any vector in $\ker(S_0-\lambda)$ for $\lambda \in J\times I$ must be in M. It follows that c.l.m. $\ker(S_0-\lambda)\neq L^2(E)$ for $\lambda \notin \operatorname{sp}_e(S_0)$.

Corollary 1 leads to an interest in describing the commutant of an irreducible operator T with a rank one self-commutator. In particular, one can ask if the commutant of such an operator is abelian.

More specific questions can be asked about the commutants of the operators S_b, where b is a non-vanishing element in $C'(I)$. In particular, one can ask if the commutant of S_b equals the weakly closed algebra generated by S_b and the identity.

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Periodic automorphisms of the hyperfinite factor of type II_1

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Introduction

There are many constructions of factors which give rise to the hyperfinite factor of type II_1, that we shall throughout denote by R. For instance 1) any infinite tensor product of a countable number of matrix algebras with respect to their traces, 2) the group measure space construction from an ergodic measure preserving transformation, 3) the left regular representation of a locally finite discrete group with infinite conjugacy classes.

To each of those ways of obtaining R correspond automorphisms of R. Two automorphisms α and β of R are conjugate when for some automorphism σ of R one has $\sigma\alpha\sigma^{-1}=\beta$.

The simplest nontrivial problem of noncommutative ergodic theory is certainly the problem of classifying, up to conjugacy, the periodic automorphisms of R.

It turns out that a complete classification is possible, by means of very simple invariants that we shall now describe.

We note first that the problem of conjugacy splits into two problems:

a) The problem of outer conjugacy: decide when given $\alpha, \beta \in \mathrm{Aut}\, R$ there exists an inner automorphism $\mathrm{Ad}\, W$ such that β is conjugate to $\mathrm{Ad}\, W \cdot \alpha$.

b) The problem of inner conjugacy: for $\alpha \in \mathrm{Aut}\, R$ decide which W, unitaries in R, are such that $\mathrm{Ad}\, W \cdot \alpha$ is conjugate to α.

For solving problem a) we first define two invariants of outer conjugacy:

1. $p_0(\alpha)$ is the outer period of α defined as the integer such that, for $n \in \mathbf{Z}$, $\alpha^n \in \mathrm{Int}\, R \Leftrightarrow n \in p_0(\alpha)\mathbf{Z}$.

2. $\gamma(\alpha)$ is a complex number of modulus 1 defined by the implication: U unitary in R, $\alpha^{p_0(\alpha)} = \mathrm{Ad}\, U \Rightarrow \alpha(U) = \gamma U$. One checks by direct computation (prop. 1.4) that p_0 and γ are invariants of outer conjugacy classes and that $\gamma(\alpha)^{p_0(\alpha)} = 1$.

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We exhibit for each couple $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, with $\gamma^p = 1$, an automorphism of R, s_p^γ, of period equal to $p \cdot \text{Order}\,\gamma$ and such that

$$p_0(s_p^\gamma) = p, \quad \gamma(s_p^\gamma) = \gamma \quad \text{(prop. 1.6)}.$$

We prove that the invariants p_0, γ completely classify the periodic automorphisms of R, up to outer conjugacy, so that any periodic automorphism of R is outer conjugate to one (and only one) of the s_p^γ (thm. 6.2).

The proof relies on the introduction of a group structure on the set Br_p of outer conjugacy classes of automorphisms with outer period p. One checks that if α and β are such classes then $\alpha \otimes \beta$ is also a class belonging to Br_p, as well as the class of the opposite α^0 of α, once $R \otimes R$ and R^0 (the opposite factor of R) are identified with R by some isomorphism (the classes $\alpha \otimes \beta$ and α^0 being of course independent of this isomorphism).

Once this is done one proves that Br_p is a group with inverse operation $\alpha \to \alpha^0$ and that γ is an isomorphism of Br_p onto the group of pth roots of 1 in $\mathbf{C}$.

The proof of the injectivity of γ, i.e., of the uniqueness of the outer conjugacy class with outer invariants $(p, 1)$, is obtained thanks to the technique of central sequences, as used by D. McDuff in [7] (see thm. 5.1).

The reader who is familiar with the construction of the Brauer group $B(k)$ of an arbitrary commutative field k will recognise the analogy with the construction of Br_p above — the objects that we study are periodic automorphisms of R, to the concept of similarity of simple central algebras over k corresponds the concept of outer conjugacy of two periodic automorphisms. The role of division algebras is played by the *minimal* periodic automorphism: α is called minimal periodic when its period is the smallest period of its outer conjugacy class. Exactly as any central simple algebra over k is the tensor product of a unique division algebra by a matrix algebra $M_n(k)$, we have that any periodic automorphism of R is the tensor product of a minimal periodic automorphism (uniquely determined up to conjugacy) by an inner automorphism — (thm. 1.11). Moreover the minimal automorphisms are also characterised by their fixed point algebra being a factor (thm. 2.5).

For each $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$, the automorphism s_p^γ is the unique minimal automorphism of the outer conjugacy class with outer invariants p, γ.

Also, the tensor product of two division algebras over k can fail to be a division algebra and in the same way the tensor product of two minimal automorphisms can fail to be minimal. The answer to problem b) is obtained by defining the inner invariant $\varepsilon(\alpha)$ of an arbitrary periodic automorphism α of R as the spectral measure (defined only up to rotation) corresponding to the trace vector and an arbitrary $U \in R$ such that $\alpha^{p_m(\alpha)} = \operatorname{Ad} U$, where $p_m(\alpha)$ is the minimal period of α. It turns out that p_0, γ and ε form a complete system of invariants for periodic automorphisms of R, the only relations being that $\gamma^{p_0}=1$ and that the support of ε lies in the nth roots of 1 for some n (thm. 1.11).

This also allows to solve the problem of weak equivalence:

c) For α and $\beta \in \operatorname{Aut} R$, when is there a $\sigma \in \operatorname{Aut} R$ such that $\sigma[\alpha]\sigma^{-1}=[\beta]$? (where the full group $[\alpha]$ of α is defined in classical terms (see for instance [4] def. 1.5.4)). The invariants of weak equivalence are $p_0(\alpha)$, Order $\gamma(\alpha)=c(\alpha)$ and the symbols of Legendre $(j/p)=\pm 1$, where p is a prime dividing $c(\alpha)$, and $\gamma(\alpha)=\exp i2\pi j/c$, with two additional symbols $\varepsilon(j), \omega(j)$ (resp. one: $\varepsilon(j)$) when $c(\alpha)$ is divisible by 4 (resp. 2 but not 4) which are classical in elementary arithmetic.

It turns out that they are complete invariants of weak equivalence the only relation being that $c(\alpha)$ divides $p(\alpha)$ (thm. 6.5).

We then apply these results to simple questions of noncommutative ergodic theory and we get the following answers: A periodic α is conjugate to the opposite of its inverse if and only if $\gamma(\alpha)^2=1$ (thm. 7.2).

A periodic α is an infinite tensor product of inner automorphisms if and only if $\gamma(\alpha)=1$ (thm. 7.9).

Also we determine conditions, of an arithmetical nature, under which α is an infinite tensor product of automorphisms of finite dimensional factors (thm. 7.4 (c)).

Then we prove that any periodic automorphism α of R admits very good approximation by finite dimensional automorphisms, in the sense that $\alpha(P_n)=P_n$, $\forall n \in \mathbf{N}$ for some increasing sequence of finite dimensional subalgebras of R with $\bigcup_{n=1}^{\infty} P_n$ dense in R.

This, of course, implies that the cross products or fixed point von Neumann algebras of arbitrary periodic automorphisms of R are hyperfinite (see remark 7.10 on this point).

Finally we give an example of a (periodic) automorphism of R which has no square root, and give the conditions (thm. 7.7) (namely $c(=\text{Order } \gamma)$ odd) under which s_p^γ has a square root.

I. Construction of the automorphisms $s_p^\gamma, p \in \mathbf{N}, \gamma^p=1$

Let N be a factor, $\alpha \in \operatorname{Aut} N$, then we define two numbers, $p_0(\alpha)$ and $\gamma(\alpha)$ as follows: [1])

$$(1.1) \qquad \{n \in \mathbf{Z}, \alpha^n \in \operatorname{Int} N\} = p_0(\alpha)\mathbf{Z} \quad \text{and} \quad p_0(\alpha) \in \mathbf{N},$$

$$(\alpha^{p_0(\alpha)} = \operatorname{Ad} U, \; U \text{ unitary in } N) \Rightarrow \alpha(U) = \gamma(\alpha) U.$$

We see that for each α, $p_0(\alpha)$ is an integer, that we call the outer period of α; it is 0 if all the nonzero powers of α are outer.

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[1]) See remark 6.8 for a cohomological interpretation of $\gamma(\alpha)$ as an obstruction.

Also we see that $\gamma(\alpha)$ is a complex number of modulus 1, independent of the choice of U such that $\alpha^{p_0(\alpha)}=\mathrm{Ad}\, U$, and satisfying

(1.2) $$\gamma(\alpha)^{p_0(\alpha)} = 1$$

because $\alpha^{p_0(\alpha)}(U)=\gamma(\alpha)^{p_0(\alpha)}U$ and $\alpha^{p_0(\alpha)}(U)=UUU^*=U$.

Definition 1.3. α and $\beta\in\mathrm{Aut}\, N$ are called *outer conjugate* iff there exists a $\sigma\in\in\mathrm{Aut}\, N$ such that β and $\sigma\alpha\sigma^{-1}$ have the same image in $\mathrm{Out}\, N=\mathrm{Aut}\, N/\mathrm{Int}\, N$.

For W unitary in N, put for $\alpha\in\mathrm{Aut}\, N$, ${}_W\alpha=\mathrm{Ad}\, W\cdot\alpha$. When W varies, the ${}_W\alpha$ form the class of α in $\mathrm{Out}\, N$ hence the $\beta\in\mathrm{Aut}\, N$ which are outer conjugate to α are all automorphisms conjugate to some ${}_W\alpha$, W unitary in N.

Proposition 1.4. *If α and β are outer conjugate then $p_0(\alpha)=p_0(\beta)$, $\gamma(\alpha)=\gamma(\beta)$; $(p_0(\alpha), \gamma(\alpha))$ is called the outer invariant of α.*

Proof. The first equality is clear. To prove the second we can assume that $\beta={}_W\alpha$ for some W unitary in N. Then let $p=p_0(\alpha)$, $\gamma=\gamma(\alpha)$; $\alpha^p=\mathrm{Ad}\, U$, $\alpha(U)=\gamma U$, then we have

$$({}_W\alpha)^p = \mathrm{Ad}\left(W\alpha(W)\ldots\alpha^{p-1}(W)U\right),$$

$${}_W\alpha\left(W\alpha(W)\ldots\alpha^{p-1}(W)U\right) = W\alpha(W)\ldots\alpha^{p-1}(W)UWU^*\alpha(U)W^*,$$

hence ${}_W\alpha\left(W\alpha(W)\ldots\alpha^{p-1}(W)U\right)=W\alpha(W)\ldots\alpha^{p-1}(W)U\gamma$ Q.E.D

We shall now fix our notations, as far as the simple classification of *inner* periodic automorphisms is concerned, for the case of factor N of type II_1 with canonical trace τ $(\tau(1)=1)$.

Let $\alpha=\mathrm{Ad}\, U$ be periodic, then the unitary U which is uniquely determined by α up to multiplication by a $\lambda\in\mathbf{C}$, $|\lambda|=1$, has the property that U^p is a scalar λ_0 for $p=$period α. It follows that U is a finite linear combination of its spectral projections corresponding to the pth roots a_j of λ_0, say $U=\sum_{j=1}^{p} a_j e_j$, where e_j is the spectral projection of U corresponding to $\{a_j\}$.

We define now the inner invariant $\varepsilon(\alpha)$ to be the probability measure $\Sigma\tau(e_j)\varepsilon_{a_j}$, determined up to a rotation on $\mathbf{T}=\{z\in\mathbf{C}, |z|=1\}$. It is easy to see that two inner automorphisms α and β are conjugate iff $\varepsilon(\alpha)=\varepsilon(\beta)$ and that all probability measures on $\mathbf{T}$ which have support contained in the pth roots of some $\lambda_0\in\mathbf{T}$ arise as $\varepsilon(\alpha)$. For $\alpha\in\mathrm{Aut}\, N$, α periodic with outer invariants p_0, γ we put $p_m=p_0\cdot$ Order γ and we put $\varepsilon(\alpha)=\varepsilon(\alpha^{p_m})$. (Check that $\alpha^{p_m}\in\mathrm{Int}\, N$.)

Theorem 1.5. *Let N be the II_1 hyperfinite factor R. Two periodic automorphisms α, $\beta\in\mathrm{Aut}\, R$ are conjugate if and only if they have the same outer and inner invariants* (i.e. $p_0(\alpha)$, $\gamma(\alpha)$ *and* $\varepsilon(\alpha)$).

This paper is entirely devoted to prove theorem 1.5. We spend the rest of this paragraph in giving a simple description of automorphisms, the $s_p^\gamma \otimes \mathrm{Ad}\, V$, having prescribed outer and inner invariants. Our first task is to describe the automorphisms $s_p^\gamma, p \in \mathbf{N}, \gamma^p = 1$ which have outer invariant (p, γ) and trivial inner invariant. For $p=1$ we let s_1^1 be the identity automorphism of R. Let $p \neq 1$. Then we write R as the infinite tensor product, indexed by $\mathbf{N}$, of the couples (F_p, Canonical trace on F_p) where F_p is the $p \times p$ matrix algebra over $\mathbf{C}$ with matrix units $(e_{i,j})_{i,j=1\dots p}$.

For $q \in \mathbf{N}$, let π_q be the canonical isomorphism of F_p onto a subfactor F_p^q of R, such that $\pi_q(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \cdots$.

Let $e_{ij}^q = \pi_q(e_{ij})$ and θ be the shift: $\theta\pi_q(x) = \pi_{q+1}(x), x \in F_p$. The shift θ is an isomorphism of R onto the commutant of F_p^1 in R.

Let $\gamma \in \mathbf{C}, \gamma^p = 1$, and $U_\gamma \in F_p^1$ be the unitary:

$$U_\gamma = \sum_{j=1}^{p} \gamma^j e_{jj}^1.$$

We define a unitary $v_\gamma \in (F_p^1 \cup F_p^2)''$ [2]) by the formula:

$$v_\gamma = e_{p1}^1 \theta(U_\gamma^*) + \sum_{j=1}^{p-1} e_{j,j+1}^1.$$

The following proposition is at the same time the definition of the automorphism s_p^γ of R.

Proposition 1.6. *Let p and γ be as above.*

(a) *The sequence of inner automorphisms of R defined by*

$$\alpha_n = \mathrm{Ad}\left(v_\gamma \theta(v_\gamma) \theta^2(v_\gamma) \theta^3(v_\gamma) \cdots \theta^n(v_\gamma)\right)$$

converges pointwise strongly to an automorphism s_p^γ of R.

(b) *The pth power $(s_p^\gamma)^p$ of this automorphism is equal to* $\mathrm{Ad}\, U_\gamma$ *and* $s_p^\gamma(U_\gamma) = \gamma U_\gamma$.

(c) *The outer invariant of s_p^γ is equal to (p, γ), its inner invariant is* $\{\varepsilon_1\}$.

Proof. (a) Let m be given and $x \in F_p^{(1,m)} = (\bigcup_1^m F_p^q)''$. Then for $n \geqq m$ we have $[\theta^n(v), x] = 0$ for any $v \in R$. It follows that $\alpha_n(x) = \mathrm{Ad}\left(v_\gamma \theta(v_\gamma) \cdots \theta^{m-1}(v_\gamma)\right)(x) = \alpha_{m-1}(x)$ so that the sequence $(\alpha_n(x))_{n \in \mathbf{N}}$ is constant for $n \geqq m$. For each n, α_n is an isometry in the L^2 norm of R; it follows hence from the strong density in R of the subalgebra $\bigcup_{m=1}^{\infty} F_p^{(1,m)}$ that there exists an homomorphism s_p^γ of R into R such that

$$s_p^\gamma(x) = \lim_{n \to \infty} \alpha_n(x), \quad x \in R.$$

[2]) From now we let $F_p^{(i,j)} = (\sum_{i \leqq q \leqq j} F_p^q)''$, hence for instance $v_\gamma \in F_p^{(1,2)}$.

We shall now prove that $(s_p^\gamma)^p = \mathrm{Ad}\, U_\gamma$. It will hence follow that s_p^γ is surjective and is an automorphism of R.

(b) We first have, using the equality $s_p^\gamma(U_\gamma) = \alpha_0(U_\gamma)$, that:

$$s_p^\gamma(U_\gamma) = \mathrm{Ad}\, v_\gamma(U_\gamma) = \mathrm{Ad}\left(\sum_{j=1}^{p} e_{j,j+1}\right)(U_\gamma) = \left(\sum_1^p e_{j,j+1}\right)\left(\sum_1^p \gamma^k e_{kk}\right) \cdot$$

$$\left(\sum_1^p e_{l+1,l}\right) = \sum_1^p \gamma^{j+1} e_{jj} = \gamma U_\gamma.\ ^{3)}$$

We end the proof of (b) by showing by induction on m that the following statement is true:

(1.7) $\qquad \forall \gamma \in \mathbf{C},\ \gamma^p = 1, \quad x \in F_p^{(1,m)}$ one has $(s_p^\gamma)^p(x) = U_\gamma x U_\gamma^*$.

We assume that the statement is true for m, we prove it for $m+1$, its truth for $m=1$ also follows from this computation.

Put, for $x \in R$, $\beta(x) = \lim_{n\to\infty} \mathrm{Ad}\,(\theta(v_\gamma)\theta^2(v_\gamma)\cdots\theta^n(v_\gamma))\,(x)$ then, as above, β is an homomorphism of R into R, which leaves F_p^1 pointwise invariant and satisfies the equality:

(1.8) $$\beta(\theta(x)) = \theta(s_p^\gamma(x)), \quad x \in R.$$

Take $x \in F_p^{(1,m+1)}$, $x = \sum_{i,j} e_{ij}^1 \theta(x_{ij})$ with $x_{ij} \in F_p^{(1,m)}$. From (1.8) and the induction hypothesis we conclude that:

$$\beta^p \theta(x_{ij}) = \theta((s_p^\gamma)^p(x_{ij})) = \theta(U_\gamma)\theta(x_{ij})\theta(U_\gamma)^* \quad \text{for} \quad i,j = 1, \ldots, p;$$

and hence, using the equalities $\beta^p(e_{ij}^1) = e_{ij}^1$, for $i,j = 1, \ldots, p$:

(1.9) $$\beta^p(x) = \theta(U_\gamma) x \theta(U_\gamma)^*.$$

But we have $s_p^\gamma = \mathrm{Ad}\, v_\gamma \cdot \beta$, hence (1.7) will follow from

(1.10) $$v_\gamma \beta(v_\gamma) \ldots \beta^{p-1}(v_\gamma) = U_\gamma \theta(U_\gamma^*).$$

To prove (1.10) we just have to use the equality $\beta\theta(U_\gamma) = \theta(s_p^\gamma(U_\gamma)) = \theta(\gamma U_\gamma)$, so that we have: $\beta^k \theta(U_\gamma^*) = \gamma^{-k}\theta(U_\gamma^*)$,

$$\beta^k(v_\gamma) = \gamma^{-k} e_{p1}^1 \theta(U_\gamma^*) + \sum_{j=1}^{p-1} e_{j,j+1}^1,$$

$$v_\gamma(\beta^p v_\gamma) \ldots \beta^{-1}(v_\gamma) = \sum \gamma^j e_{jj}^1 \theta(U_\gamma^*) = U_\gamma \theta(U_*^\gamma).$$

(c) We just have to prove that $(s_p^\gamma)^q$ is outer for $q \in \{1, \ldots, p-1\}$. To do this,

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3) In all sums like $\sum_1^p e_{j,j+1}$ one takes $e_{p,p+1} = e_{p,1}$, say more generally that $e_{i+p_1,j+p_2} = e_{i,j}$ whenever p_1 and p_2 are multiples of p.

note that $v_{\gamma''}$ commutes with $\theta^j(U_{\gamma'})$ for $j\geqq 1$, $\{\gamma'\}^p=1$, $(\gamma'')^p=1$. Also $v_{\gamma''}U_{\gamma'}v_{\gamma''}^*=\gamma' U_{\gamma'}$ as seen above, so that:

$$s_p^\gamma(\theta^n(U_{\gamma'})) = \gamma'\theta^n(U_{\gamma'}), \quad \forall n\in\mathbf{N}, \quad \forall\gamma',\ \gamma'^p = 1.$$

This shows that for $q\in\{1,\ldots,p-1\}$ we have $\|(s_p^\gamma)^q\theta^n(U_{\gamma'})-\theta^n(U_{\gamma'})\|_2=|\gamma'^q-1|$, hence that $(s_p^\gamma)^q$ cannot be an inner automorphism because the sequence $(\theta^n(U_{\gamma'}))_{n\in\mathbf{N}}$ is a central sequence in R.

We can now state an important consequence of theorem 1.5 and proposition 1.6:

Theorem 1.11. *Let R be the hyperfinite factor of type II_1. Let $p\in\mathbf{N}$, $\gamma\in\mathbf{C}$ with $\gamma^p=1$ and ε be a probability measure on $\mathbf{T}$ such that* Support $\varepsilon\subset\{n$ th *roots of $\lambda_0\}$ for some $\lambda_0\in\mathbf{T}$ and $n\in\mathbf{N}$. Then there exists some periodic automorphism $\alpha\in\mathrm{Aut}\,R$, satisfying the conditions $p_0(\alpha)=p$, $\gamma(\alpha)=\gamma$, $\varepsilon(\alpha)=\varepsilon$. Moreover let β be an inner automorphism of R such that $\varepsilon(\beta^{p\,\mathrm{Order}\,\gamma})=\varepsilon$ then any $\alpha\in\mathrm{Aut}\,R$ periodic, with invariants $p_0(\alpha)=p$, $\gamma(\alpha)=\gamma$, $\varepsilon(\alpha)=\varepsilon$ is conjugate to*

$$s_p^\gamma\otimes\beta\in\mathrm{Aut}\,R\otimes R.$$

Proof. We just have to check that the outer invariant of $s_p^\gamma\otimes\beta$ is (p,γ) which is easy and to check that its inner invariant is ε. But $(s_p^\gamma\otimes\beta)^{p\,\mathrm{Order}\,\gamma}=$ $=1\otimes\beta^{p\,\mathrm{Order}\,\gamma}$.

Q.E.D.

II. Minimal periodic automorphisms

Throughout N is a factor, countably decomposable for simplicity. For $\alpha\in\mathrm{Aut}\,N$ let $\mathrm{Sp}\,\alpha$ be the spectrum of α in the Banach algebra $B(N)$ of weakly continuous linear mappings from N to N. Then $\mathrm{Sp}\,\alpha$ is a closed subset of $\mathbf{T}=\{z\in\mathbf{C},\ |z|=1\}$ and is equal to the spectrum in the sense of [1] [4] of the representation $n\to\alpha^n$ of $\mathbf{Z}$ on N [4]) (cf. [4] 2.3.8).

For any nonzero projection $e\in N^\alpha$ we put as in [4] p.170, $\alpha^e=\alpha$ restricted to N_e, and we have by [4] 2.2.1 and 2.3.17 that

$$\Gamma(\alpha) = \bigcap_{e\in N^\alpha}\mathrm{Sp}\,\alpha^e = \bigcap_{W\text{ unitary in }N}\mathrm{Sp}\,{}_W\alpha \tag{2.1}$$

where ${}_W\alpha=\mathrm{Ad}\,W\cdot\alpha$ by definition.

By [4] thm. 2.2.4, $\Gamma(\alpha)$ is a closed subgroup of $\mathbf{T}$ and by [4] thm. 2.3.1 we have:

$$\Gamma(\alpha)^\perp = \{n\in\mathbf{Z},\ \alpha^n \text{ is inner: } \alpha^n(x) = uxu^*,\ \forall x\in N \text{ with } u\in N^\alpha\}. \tag{2.2}$$

[4]) Identifying $\mathbf{T}$ with the dual group of $\mathbf{Z}$, by $(n,\lambda)=\lambda^n$, $\lambda\in\mathbf{T}$, $n\in\mathbf{Z}$.

Proposition 2.3. *Let* α *be a periodic automorphism of* N *and let* $p_m(\alpha) = p_0(\alpha) \cdot \text{Order}\,\gamma(\alpha)$, *where* (p_0, γ) *are the outer invariants of* α. *Then* $p_m(\alpha)$ *is the smallest period of automorphisms outer conjugate to* α. *(We call* $p_m(\alpha)$ *the minimal period of* α.*)*

Proof. As $\alpha^{p_m(\alpha)} = \text{Ad}\, U^{\text{Order}\,\gamma(\alpha)}$, where $\alpha^{p_0(\alpha)} = \text{Ad}\, U$, we see that $p_m(\alpha) \in \Gamma(\alpha)^\perp$. Conversely, if $q \in \Gamma(\alpha)^\perp$ then q is a multiple $np_0(\alpha)$ of $p_0(\alpha)$ and necessarily $\gamma(\alpha)^n = 1$ so that q is a multiple of $p_m(\alpha)$. Using [4] corollary 2.3.11 we get proposition 2.3 because

$$\Gamma(\alpha)^\perp = \{p_m(\alpha)\mathbf{Z}\}, \quad \Gamma(\alpha) = \{\lambda \in \mathbf{C}\colon \lambda^{p_m(\alpha)} = 1\}. \tag{2.4}$$

The following equivalent conditions define the minimal periodic automorphisms:

Theorem 2.5. *Let* α *be a periodic automorphism of* N, *then the following conditions are equivalent:*

(a) Period of α = Minimal period of α,

(b) $\text{Sp}\,\alpha = \Gamma(\alpha)$,

(c) N^α *is a factor*,

(d) *For* $n \in \{1, \dots, (\text{period}\,\alpha) - 1\}$ *and* $\alpha^n = \text{Ad}\, U$, $U \in N$ *one has* $\alpha(U) \neq U$.

Proof. That (a)$\Leftrightarrow$(b) follows from 2.4; that (b)$\Leftrightarrow$(c) is a corollary of [4] thm. 2.4.1, also (a)$\Leftrightarrow$(d) follows from 2.2.

Corollary 2.6. *Let* α *be a minimal periodic automorphism of* N *with minimal period* p, *then*

(a) *A unitary* $U \in N$ *is of the form* $v^*\alpha(v)$, v *unitary in* N, *if and only if* $U\alpha(U) \cdots \alpha^{p-1}(U) = 1$.

(b) *Any minimal periodic* $\beta \in \text{Aut}\, N$ *which is outer conjugate to* α *is conjugate to* α.

Proof. The condition (a) is clearly necessary since for any v one has

$$v^*\alpha(v)\alpha(v^*\alpha(v)) \cdots \alpha^{p-1}(v^*\alpha(v)) = v^*v = 1.$$

To prove that it is sufficient, let ([4] lemma 2.2.6) β be the automorphism of $N \otimes F_2$ [5]) such that:

$$\beta(x \otimes e_{11}) = \alpha(x) \otimes e_{11}, \quad \beta(x \otimes e_{22}) = U\alpha(x)U^*, \quad \beta(1 \otimes e_{21}) = U \otimes e_{21}.$$

Then the condition (a) and the computation in [4] p. 176 show that $\beta^p = 1$. Hence, as $\Gamma(\beta) = \Gamma(\alpha) = \{p\mathbf{Z}\}$ we see that β is minimal periodic.

[5]) F_2 is a type I_2 factor with a system of matrix units $(e_{ij})_{i,j=1,2}$.

If N is finite, then $1\otimes e_{11}$ and $1\otimes e_{22}$ have the same trace in the finite factor $(N\otimes F_2)^\beta$ and hence are equivalent in $(N\otimes F_2)^\beta$.

If N is properly infinite, then so are N^α and $N^{(U^\alpha)}$ which means that $1\otimes e_{11}$ and $1\otimes e_{22}$ are properly infinite, and hence equivalent, in $(N\otimes F_2)^\beta$.

In all cases there hence exists a partial isometry $v^*\otimes e_{21}\in(N\otimes F_2)^\beta$ with $1\otimes e_{11}$ as initial support and $1\otimes e_{22}$ as final support. But then $\beta(v^*\otimes e_{21})=v^*\otimes e_{21}$ means $U\alpha(v^*)=v^*$ so that v is the required unitary. (b) We can assume that $\beta={}_W\alpha$ for some W. As $\Gamma(\beta)=\Gamma(\alpha)$ we see that the period of β is equal to p=period α. It follows that $W\alpha(W)...\alpha^{p-1}(W)$ is a scalar. Replacing W by λW, $\lambda\in\mathbf{C}$, for a suitable λ we can assume that $W\alpha(W)...\alpha^{p-1}(W)=1$. Then statement (a) of the corollary shows that β is conjugate to α.

Corollary 2.7. *Let α be a minimal periodic automorphism of N.*

(a) *Let e_1, e_2 be two projections in N^α which are equivalent relative to N. Then for any $\lambda\in\operatorname{Sp}\alpha$ there exists a partial isometry $U\in N$ such that:*

$$\alpha(U)=\lambda U,\ U^*U=e_1,\ UU^*=e_2.$$

(b) *If N is continuous, then for each integer m dividing* period $\alpha=p$ *and each λ, $\lambda^m=1$ there exists a system of $m\times m$ matrix units $e_{ij}\in N$ such that*

$$\alpha(e_{ij})=\lambda^{i-j}e_{ij}.^{6)}$$

Proof. (a) As we have seen above, either N is finite and $e_1\sim e_2(N^\alpha)$ or N is properly infinite and still $e_1\sim e_2(N^\alpha)$ so there exists a partial isometry $v\in N^\alpha$, $v^*v=e_1$, $vv^*=e_2$. Now α^{e_1} has period at most the period of α while $\Gamma(\alpha^{e_1})=\Gamma(\alpha)$ by [4] 2.3.3. Hence α^{e_1} is minimal periodic with the same period as α. But then by corollary 2.6 (a), there exists a unitary operator $W\in N^{e_1}$ such that $\alpha(W)=\lambda W$ (apply 2.6 (a) to $U=\lambda$). We then have $W^*W=e_1$, $WW^*=e_1$ and $\alpha(W)=\lambda W$ so that $U=vW$ satisfies the condition (a) of 2.7. (b) The factor N^α is continuous because a minimal projection in N^α is automatically minimal in N.

So we let $(e_j)_{j=1,...,m}$ be a family of m projections of N^α equivalent in N. Then by 2.7 (a), let U_j satisfy $U_j\in N$, $U_j^*U_j=e_j$, $U_jU_j^*=e_{j+1}$ and $\alpha(U_j)=\lambda U_j$, for $j=1,2,...,m-1$.

It follows that $e_{j+1,j}=U_j$ generates a system of matrix units satisfying the required conditions. Q.E.D.

Corollary 2.8. *Let α and β be periodic automorphisms of a factor N of type II_1 with canonical trace τ. Then α and β are conjugate if and only if they are outer conjugate and the inner automorphisms $\alpha^{p_m(\alpha)}$ and $\beta^{p_m(\alpha)}$ are conjugate.*

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[6]) Throughout $i\pm j$ for $i,j\in\{1,...,m\}$ means $i\pm j$ modulo m.

In other words two elements of an outer conjugacy class are conjugate if and only if they have the same inner invariant.

Proof. The condition is clearly necessary. To prove that it is sufficient first note that if α and β are outer conjugate we have $p_m(\alpha)=p_m(\beta)=p$ for some $p\in\mathbf{N}$. Write now $\alpha^p=\operatorname{Ad}U$, $U=\sum_{i=1}^k \lambda_i e_i$ where the e_i are projections belonging to the center of N^α (we use 2.4 and 2.2), with say $\tau(e_i)=\mu_i$, and the λ_i are complex numbers of modulus 1. For each λ_i choose a pth root and call it $\lambda_i^{1/p}$, then $U^{1/p}=\sum_{i=1}^k \lambda_i^{1/p} e_i$ belongs to N^α so that $\tilde{\alpha}=\operatorname{Ad}U^{-1/p}\alpha$ satisfies $\tilde{\alpha}^p=\operatorname{Ad}U^*\alpha^p=1$.

Write then $\beta^p=\operatorname{Ad}v$, with $v=\sum_{i=1}^k \lambda_i f_i$ where the λ_i are the same as the λ_i used above for U and where for each i, f_i is a projection (belonging to N^β) which is equivalent to e_i relative to the factor N. We have used the fact that α^p and β^p are conjugate inner automorphisms of N.

Choose $v^{1/p}=\sum_{i=1}^k \lambda_i^{1/p} f_i$ and put as above:

$$\tilde{\beta}=\operatorname{Ad}v^{-1/p}\beta.$$

We have $\tilde{\beta}^p=1$. Hence $\tilde{\alpha}$ and $\tilde{\beta}$ are outer conjugate and minimal periodic, so that by 2.6 they are conjugate, say $\tilde{\beta}=\sigma\tilde{\alpha}\sigma^{-1}$, $\sigma\in\operatorname{Aut}N$. Now $\alpha=\tilde{\alpha}\cdot\operatorname{Ad}U^{1/p}=\operatorname{Ad}U^{1/p}\tilde{\alpha}$, and:

$$\sigma\alpha\sigma^{-1}=\tilde{\beta}\operatorname{Ad}\sigma(U^{1/p})=\operatorname{Ad}\sigma(U^{1/p})\tilde{\beta}.$$

As we have $\beta=\operatorname{Ad}v^{1/p}\tilde{\beta}$ we just have to find an automorphism θ of N commuting with $\tilde{\beta}$ and such that $\theta\sigma(U^{1/p})=v^{1/p}$. Both $\sigma(U^{1/p})$ and $v^{1/p}$ belong to $N^{\tilde{\beta}}$ and we look for θ as an inner automorphism defined by a unitary $X\in N^{\tilde{\beta}}$.

We have $\sigma(U^{1/p})=\sum_{i=1}^k \lambda_i^{1/p}\sigma(e_i)$, $v^{1/p}=\sum_{i=1}^k \lambda_i^{1/p} f_i$ so it is enough to check that for each i, $\sigma(e_i)$ is equivalent to f_i relative to $N^{\tilde{\beta}}$. But this is true because $\tau(e_i)=\tau(f_i)$ hence $\tau(\sigma(e_i))=\tau(f_i)$ for $i=1,\dots,k$.

III. Action of automorphisms of R on central sequences

Definition 3.1. [7] *Let N be a II_1 factor with canonical trace τ, and ω be a free ultrafilter on* $\mathbf{N}$.

(a) *We let $N_{\tau,\omega}$ be the quotient of $l^\infty(\mathbf{N},N)$ by the two sided ideal $J_\omega=\{(x_n)_{n\in\mathbf{N}}$, $x_n\to 0$* strongly when $n\to\omega\}$.

(b) *We let N_ω be the commutant in $N_{\tau,\omega}$ of the image $\tilde{N}$ of N in $N_{\tau,\omega}$, where for $x\in N$, $\tilde{x}\in N_{\tau,\omega}$ is represented by the sequence $(x_n)_{n\in\mathbf{N}}$, $x_n=x$, $\forall n\in\mathbf{N}$.*

This definition is exactly the one given by McDuff in [7], except for a change of notations which matches with [5] part II. By [7] we know that $N_{\tau,\omega}$ is a II_1 factor with canonical trace τ_ω: $\tau_\omega((x_n)_{n\in\mathbf{N}})=\lim_{n\to\omega}\tau(x_n)$.

Let $\theta\in\operatorname{Aut}N$ then the automorphism $(x_n)_{n\in\mathbf{N}}\to(\theta(x_n))_{n\in\mathbf{N}}$ of $l^\infty(\mathbf{N},N)$ leaves J_ω globally invariant and thus defines an automorphism $\theta_{\tau,\omega}$ of $N_{\tau,\omega}$. Moreover $\theta_{\tau,\omega}(\tilde{N})=\tilde{N}$ so that $\theta_{\tau,\omega}$ leaves N_ω globally invariant and thus defines an automorphism θ_ω of N_ω.

Now take $N=R$, the hyperfinite II_1 factor. We know that all hypercentral sequences on R are trivial [7], hence by [7] thm. 4, R_ω is a factor of type II_1.

Theorem 3.2. *Let α be an automorphism of the* II_1 *hyperfinite factor R, and ω be a free ultrafilter, then:*

(1) *α_ω is inner on R_ω if and only if α is inner on R, in which case $\alpha_\omega=1$.*

(2) *There exists a unitary $U\in R_{\tau,\omega}$ such that $(\alpha(x))^\sim=U\tilde{x}U^*$, $\forall x\in R$.*

Before we proceed and prove this theorem, let us note one consequence for periodic automorphisms:

Corollary 3.3. *Let $\alpha\in\operatorname{Aut}R$ be periodic, with* Outer period $\alpha=$period $\alpha=n$, *then there exists a unitary $v\in R_{\tau,\omega}$ such that:*

$$\alpha_{\tau,\omega}(v)=v;\quad v^n=1;\quad (\alpha(x))^\sim=v\tilde{x}v^*,\quad \forall x\in R.$$

Proof. Let U be a unitary, $U\in R_{\tau,\omega}$ such that $(\alpha(x))^\sim=U\tilde{x}U^*$, $x\in R$. We hence have that $\alpha_{\tau,\omega}(\tilde{x})=U\tilde{x}U^*$, $\forall x\in R$ and replacing x by $\alpha^{-1}(x)$: $\alpha_{\tau,\omega}(\tilde{x})=\alpha_{\tau,\omega}(U)\tilde{x}\alpha_{\tau,\omega}(U^*)$, $\forall x\in R$. Then $U^*\alpha_{\tau,\omega}(U)\in(\tilde{R})'\cap R_{\tau,\omega}=R_\omega$.

Put $w=U^*\alpha_{\tau,\omega}(U)$. We have $w\alpha_{\tau,\omega}(w)\dots\alpha_{\tau,\omega}^{n-2}(w)\alpha_{\tau,\omega}^{n-1}(w)=1$. Now as Outer period $\alpha=n$ we have Outer period $\alpha_\omega=n$ using part (1) of theorem 3.2. So it follows from corollary 2.6 that this $w\in R_\omega$ such that $w\alpha_\omega(w)\dots\alpha_\omega^{n-1}(w)=1$ can be written $w=X^*\alpha_\omega(X)$ for some unitary $X\in R_\omega$.

Put $Y=UX^*$. Then $\alpha_{\tau,\omega}(Y)=\alpha_{\tau,\omega}(U)\alpha_{\tau,\omega}(X^*)=Uww^*X^*=Y$ and $(\alpha(x))^\sim=Y\tilde{x}Y^*$, $\forall x\in R$, because $X^*\in(\tilde{R})'$.

Now $\alpha_{\tau,\omega}^n(\tilde{x})=Y^n\tilde{x}(Y^*)^n$, $\forall x\in R$, so that $Y^n\in R_\omega$. As $R_\omega^{\alpha_\omega}$ is a von Neumann algebra, there exists a $Z\in R_\omega^{\alpha_\omega}$ such that $Z^n=Y^n$ and Z is in the von Neumann algebra generated by Y^n in $R_\omega^{\alpha_\omega}$. In particular Z and Y commute as elements of $R_{\tau,\omega}$. Put $U'=YZ^*$ then we have:

$$\alpha_{\tau,\omega}(U')=\alpha_{\tau,\omega}(Y)\alpha_\omega(Z^*)=YZ^*=U',$$
$$U'^n=Y^n(Z)^{-n}=1,\quad U'\tilde{x}U'^*=Y\tilde{x}Y^*=(\alpha(x))^\sim,\quad \forall x\in R.\qquad \text{Q.E.D.}$$

The proof of the theorem, part (1), relies on the following simple adaptation of the proof given in [10] p. 156—157 of the derivation theorem which can also be found in [3] with $\|\ \ \|$ instead of $\|\ \ \|_2$.

Lemma 3.4. *Let P be a factor of type* II_1, *and K be a finite dimensional subfactor of P. Let $\alpha \in \operatorname{Aut} P$, then if* $\sup_{U \text{ unitary in } K' \cap P} \|\alpha(U) - U\|_2 < 1$ *the automorphism α is inner.*

Proof. Let U_0 be the unitary group of $K' \cap P$. Then, exactly as in [10] p. 156—157 we define an action of U_0 on the vector space P by the formula

$$\varphi_u(x) = ux\alpha(u^*), \quad x \in P, \quad u \in U_0.$$

As $\|\varphi_u(x)\|_2 = \|x\|_2$, $\forall x \in P$ we can extend this action to an action of U_0 on $L^2(P, \tau)$ where τ is the canonical trace of P. If the hypothesis of the lemma is satisfied the orthogonal projection y of 0 on $\overline{\operatorname{Conv}}\{\varphi_u(1), u \in U_0\}$ [7]) is different from 0 and is a fixed point for φ_u, for all $u \in U_0$. In other words we have $uy = y\alpha(u)$, $\forall u \in U_0$, hence $xy = y\alpha(x)$ $\forall x \in K' \cap P$. Now there exists a unitary $v \in P$ such that ${}_v\alpha = \operatorname{Ad} v \cdot \alpha$ is of the form $1_K \otimes \beta$ where $\beta \in \operatorname{Aut}(K' \cap P)$, and we have:

$$xyv^* = yv_v^* \alpha(x), \quad \forall x \in K' \cap P.$$

Let $yv^* = \sum e_{ij} \otimes y_{ij}$, with $e_{ij} \in K$ and $y_{ij} \in K' \cap P$, then if the e_{ij} are matrix units in K we get $xy_{ij} = y_{ij}\beta(x)$, $\forall x \in K' \cap P$, $\forall i, j$. It follows then that there exists a nonzero $z \in K' \cap P$ such that

$$xz = z\beta(x), \quad \forall x \in K' \cap P.$$

Hence by [9] $\beta = ({}_v\alpha$ restricted to $K' \cap P)$ is an inner automorphism, so that ${}_v\alpha$ is inner on P, being identity on K, and finally α is inner on P.

Proof of part (1) of theorem 3.2. If α is an inner automorphism then easily $\alpha_\omega = 1$. Let α be an outer automorphism, and u_n be a sequence of unitaries of R. We construct a central sequence $(v_n)_{n \in \mathbf{N}}$ of unitaries such that $\|u_n v_n u_n^* - v_n\|_2 \to 0$ as $n \to \infty$ and $\|\alpha(v_n) - v_n\|_2 \geqq \frac{1}{2}$, $\forall n$. It follows that for any unitary $u \in R_{\tau, \omega}$ there exists a unitary $v \in R_\omega$ such that $uvu^* = v$ while $\alpha(v) \neq v$. This will hence show that α_ω is not inner on R_ω.

To construct the sequence $(v_n)_{n \in \mathbf{N}}$, let K_n be an increasing sequence of finite dimensional subfactors of R such that: $u_n \overset{1/n}{\in} K_n$, [8]) $\forall n \in \mathbf{N}$, $(\bigcup_{n \in \mathbf{N}} K_n)^- = R$. Let then, for each n, v_n be a unitary in K_n' such that $\|\alpha(v_n) - v_n\|_2 \geqq \frac{1}{2}$ (we apply lemma 3.4).

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[7]) The canonical image of the C^* unit ball of P in $L^2(P, \tau)$ is weakly closed and contains the $\varphi_u(1)$, $u \in U_0$. So y belongs to this unit ball.

[8]) $x \overset{\varepsilon}{\in} K$ means, as in [7], the existence of some $y \in K$ such that $\|x - y\|_2 < \varepsilon$.

Clearly $(v_n)_{n\in\mathbf{N}}$ is a central sequence in R and $\|[u_n, v_n]\|_2 \leqq \frac{2}{n}$, $\forall n\in\mathbf{N}$, so that $\|u_n v_n u_n^* - v_n\|_2 \xrightarrow[n\to\infty]{} 0$.

Q.E.D.

Proof of part (2) of theorem 3.2. Let $(K_n)_{n\in\mathbf{N}}$ be an increasing sequence of finite dimensional subfactors of R. For each $n\in\mathbf{N}$ there exists a unitary u_n' such that $\alpha(K_n)=u_n' K_n u_n'^*$ hence a unitary u_n such that $\alpha(x)=u_n x u_n^*$, $\forall x\in K_n$.

It follows that $\alpha(x)=\lim_{n\to\infty} u_n x u_n^*$, $\forall x\in \bigcup_{n=1}^{\infty} K_n$ and hence for all $x\in R$, provided $\bigcup_{n=1}^{\infty} K_n$ is strongly dense in R.

Q.E.D.

IV. Some technical lemmas

Lemma 4.1. *Let* $\varepsilon\in]0,1[$, *and* N *be a factor of type* II_1. *Let* $e_1, \dots, e_n$ *be projections of* N *such that* $\left\|\sum_{j=1}^{n} e_j - 1\right\|_2 \leqq \varepsilon$. *Then*

(a) $f_j = \bigvee_1^j e_l - \bigvee_1^{j-1} e_k$ *is a family of pairwise orthogonal projections such that* $\|f_j - e_j\|_2 \leqq 10 n \varepsilon^{1/4}$.

(b) *There exists a family of pairwise orthogonal projections* $E_j \sim e_j$ *with* $\|E_j - e_j\|_2 \leqq 14 n \varepsilon^{1/8}$, *provided* $\Sigma\tau(e_j)\leqq 1$.

Proof. (a) Put $T_j = \sum_1^j e_l$, and $F_j = \bigvee_1^j e_l$. We have $F_j = \text{Support } T_j$. Let f^j be the spectral projection of T_j corresponding to the interval $[1+\sqrt{\varepsilon}, \infty[$. As we have $T_j \leqq T_n$, we get $\tau(f^j)\leqq\tau(f^n)$ by the minimax theorem, hence $\tau(f^j)\leqq\varepsilon$. Also f^j commutes with T_j and F_j and as $\|(1+\sqrt{\varepsilon})F_j - T_j\|\leqq n$, we have $\tau((1-f^j)((1+\sqrt{\varepsilon})F_j - T_j)) \leqq \tau((1+\sqrt{\varepsilon})F_j - T_j) + n\tau(f^j) \leqq \sqrt{\varepsilon} + n\varepsilon$. (Because $\tau(F_j)\leqq \sum_1^j \tau(e_{j'}) = \tau(T_j)$.) As $0\leqq (1-f^j)((1+\sqrt{\varepsilon})F_j - T_j) \leqq 1+\sqrt{\varepsilon}\leqq 2$ we have $\|(1-f^j)((1+\sqrt{\varepsilon})F_j - T_j)\|_2 \leqq \sqrt{2}(n+1)^{1/2}\varepsilon^{1/4}$.

But $\|f^j((1+\sqrt{\varepsilon})F_j - T_j\|_2 \leqq n(\tau(f^j))^{1/2} \leqq n\varepsilon^{1/4}$, so that $\|F_j - T_j\|_2 \leqq \|(1+\sqrt{\varepsilon})F_j - T_j\|_2 + \sqrt{\varepsilon} \leqq (n\sqrt{2} + (n+1)^{1/2}\sqrt{2}+1)\varepsilon^{1/4}$. As $T_j - T_{j-1} = e_j$ and $F_j - F_{j-1} = f_j$ we get (a).

(b) We have $\Sigma\tau(e_j)\leqq 1$. Take f_j as in (a). Let $I=\{j\in\{1,\dots,n\}, \tau(f_j)\leqq\tau(e_j)\}$ and $J=\{j\in\{1,\dots,n\}, \tau(f_j)>\tau(e_j)\}$. Let for each $j\in J$, f_j' be a subprojection of f_j such that $\tau(f_j')=\tau(f_j)-\tau(e_j)$. Then the f_j' are pairwise orthogonal with $\sum_{j\in J}\tau(f_j') + \tau(1-\bigvee_{j=1}^n f_j) = \sum_{j\in J}\tau(f_j') + 1 - \sum_{j=1}^n \tau(f_j) = 1 - \sum_{j\in I}\tau(f_j) - \sum_{j\in J}\tau(e_j) \geqq \sum_{j\in I}(\tau(e_j)-\tau(f_j))$. Let $(f_j'')_{j\in I}$ be a family of pairwise orthogonal projections, with $f_j'' \leqq \sum_{j\in J} f_j' + 1 - \sum_{j=1}^n f_j$, $\tau(f_j'')=\tau(e_j - f_j)$, $\forall j\in I$. Put $E_j = f_j + f_j''$ for $j\in I$, $R_j = f_j - f_j'$ for

$j \in J$. Then each E_j is a projection equivalent to e_j, the E_j's are pairwise orthogonal, and

$$\|E_j - f_j\|_2 \leqq |\tau(e_j) - \tau(f_j)|^{1/2} \leqq 4n^{1/2}\varepsilon^{1/8}, \quad \|E_j - e_j\|_2 \leqq (10n + 4n^{1/2})\varepsilon^{1/8} \leqq 14n\varepsilon^{1/8}.$$

Lemma 4.2. *Let $n, m \in \mathbf{N}$ with n dividing m. Let α be a minimal periodic automorphism of period m, of a II_1 factor N. Let $\eta \in]0, 1/m[$ and $\lambda \in \mathbf{C}$, $\lambda^n = 1$. Let $(u_j)_{j=1,\dots,n-1}$ be a family of $n-1$ elements of N of norm less than 1 such that*

(a) $\|\alpha(u_j) - \lambda u_j\|_2 \leqq \eta$, $j \in \{1, \dots, n-1\}$,

(b) $\left\|\left(\sum_1^{n-1} u_j^* u_j\right) + u_{n-1}u_{n-1}^* - 1\right\|_2 \leqq \eta$,

(c) $\|u_j^* u_j - (u_j^* u_j)^2\|_2 \leqq \eta$ *for* $j \in \{1, \dots, n-1\}$, *and*

$\|u_{n-1}u_{n-1}^* - (u_{n-1}u_{n-1}^*)^2\|_2 \leqq \eta$,

(d) $\|u_j u_j^* - u_{j+1}^* u_{j+1}\|_2 \leqq \eta$ *for* $j = 1, \dots, n-2$.

Then there exists a system of matrix units $(e_{ij})_{i,j \in \{1,\dots,n\}}$ of N such that $\alpha(e_{ij}) = \lambda^{i-j} e_{ij}$ and

$$\|u_j - e_{j+1,j}\|_2 \leqq 142n(m\eta)^{1/256} \quad \textit{for all} \quad j \in \{1, \dots, n-1\}.$$

Proof. For $x \in N$, we put $x^\lambda = \frac{1}{m} \sum_{j=1}^m \bar{\lambda}^j \alpha^j(x)$. Then if $\|x\| \leqq 1$, $\|\alpha(x) - \lambda x\|_2 \leqq \eta$ one has $\|x^\lambda\| \leqq 1$, $\alpha(x^\lambda) = \lambda x^\lambda$ and $\|x - x^\lambda\|_2 \leqq \frac{1}{m} \sum_1^m k\eta \leqq \frac{m-1}{2}\eta$. Take $v_j = u_j^\lambda$, then we have

(e) $\|u_j - v_j\|_2 \leqq \frac{m-1}{2}\eta$, $\|u_j^* u_j - v_j^* v_j\| \leqq (m-1)\eta$,

$\|u_{n-1}u_{n-1}^* - v_{n-1}v_{n-1}^*\| \leqq (m-1)\eta$ and the v_j satisfy a condition like (b) with $nm\eta$ instead of η, like (c) with $3m\eta$ and (d) with $2m\eta$.

Put $T_j = v_j^* v_j$ for $j = 1, \dots, n-1$ and $T_n = v_{n-1}v_{n-1}^*$. Then we have $T_l \in N^\alpha$ $(l = 1, \dots, n)$ and $\|T_l^2 - T_l\|_2 \leqq 3m\eta$ $(l = 1, \dots, m)$.

Then by [6] p. 273—274 there exists for $j \in \{1, \dots, n\}$ a spectral projection F_j of T_j, $F_j \in N^\alpha$ such that:

(f) $\|T_j - F_j\|_2 \leqq 8(m\eta)^{1/2}$, $\|T_j^{1/2} - F_j\|_2 \leqq 6(m\eta)^{1/4}$.

If for $j \in \{1, \dots, n-1\}$ we let $v_j = V_j T_j^{1/2}$ be the polar decomposition of v_j we get:

(f') $\|v_j - V_j F_j\|_2 \leqq 6(m\eta)^{1/4}$.

Put $a = \mathrm{Inf}(1/n, \tau(F_1), \dots, \tau(F_n))$. We have $|\tau(F_j) - \tau(T_j)| \leqq 8(m\eta)^{1/2}$, $j \in \{1, \dots, n\}$;

hence using condition (b) for the v's:

$$\left|\sum_{j=1}^{n} \tau(F_j)-1\right| \leqq \left|\sum_{j=1}^{n} \tau(T_j)-1\right|+8n(m\eta)^{1/2} \leqq nm\eta+8n(m\eta)^{1/2} \leqq 9n(m\eta)^{1/2}.$$

(Because $m\eta \leqq 1$.)

For $j<n-1$

$$|\tau(F_j)-\tau(F_{j+1})| \leqq |\tau(v_j v_j^*)-\tau(v_{j+1}^* v_{j+1})|+16(m\eta)^{1/2} \leqq 2m\eta+16(m\eta)^{1/2} \leqq 18(m\eta)^{1/2}.$$

Moreover $|\tau(F_{n-1})-\tau(F_n)| \leqq 18(m\eta)^{1/2}$. So that

$$\left|\tau(F_j)-\frac{1}{n}\right| \leqq \left|\tau(F_j)-\frac{1}{n}\sum_{i=1}^{n}\tau(F_i)\right|+9(m\eta)^{1/2} \leqq \frac{n-1}{2}\,18(m\eta)^{1/2}+9(m\eta)^{1/2} \leqq 9n(m\eta)^{1/2}.$$

Hence $\left|a-\frac{1}{n}\right| \leqq 9n(m\eta)^{1/2}$ and $|\tau(F_j)-a| \leqq 18n(m\eta)^{1/2}$. For each j, let E_j be a projection in N^α with $E_j \leqq F_j$, $\tau(E_j)=a$. Put $W_j=V_jE_j$, $j<n$. Then

$$\|W_j-v_j\|_2 \leqq \|F_j-E_j\|_2+\|v_j-V_jF_j\|_2 \leqq 5n^{1/2}(m\eta)^{1/4}+6(m\eta)^{1/4} \leqq (16n)(m\eta)^{1/4},$$
$$\|W_{n-1}W_{n-1}^*-T_n\|_2 \leqq 32n(m\eta)^{1/4};$$

for $j<n-1$,

$$\|W_jW_j^*-W_{j+1}^*W_{j+1}\|_2 \leqq 4\cdot 16n(m\eta)^{1/4}+2m\eta \leqq 66n(m\eta)^{1/4}.$$

Now we have $\tau(E_j)=a \leqq \frac{1}{n}$,

$$\left\|\sum_{j=1}^{n} E_j-1\right\|_2 \leqq 5n^{3/2}(m\eta)^{1/4}+8n(m\eta)^{1/2}+\left\|\sum_{j=1}^{n} T_j-1\right\|_2 \leqq 14n^{3/2}(m\eta)^{1/4}.$$

Hence there exists a system G_j of pairwise orthogonal projections of N^α, with $\tau(G_j)=\tau(E_j)=a$, and $\|G_j-E_j\|_2 \leqq 22n^2(m\eta)^{1/32}$ (Lemma 4.1). For each $j<n$, we let, using [6] lemma 7 p. 275, X_j be a unitary in N^α such that $X_jG_jX_j^*=E_j$, Y_j be unitary in N^α with $Y_jW_jW_j^*Y_j^*=G_{j+1}$ and such that:

$$\|X_j-1\|_2 \leqq 50n^{1/4}(m\eta)^{1/256}, \quad \|Y_j-1\|_2 \leqq 72n^{1/4}(m\eta)^{1/256}.$$

Choose n pairwise orthogonal projections G'_j such that $G'_j \in N^\alpha$, $\tau(G'_j)=\frac{1}{n}-a$, $G'_j \leqq 1-\sum_{k=1}^{n} G_k$, and $n-1$ partial isometries $(U'_j)_{j=1,n-1}$ where $U'^*_jU'_j=G'_j$, $U'_jU'^*_j= =G'_{j+1}$, $\alpha(U'_j)=\lambda U'_j$ (apply 2.7a). Put $U_j=Y_jW_jX_j+U'_j$ for $j=1,\dots,n-1$. Then U_j is a partial isometry with initial support $G_j+G'_j$ and final support $G_{j+1}+G'_{j+1}$.

Also $\alpha(U_j)=\lambda U_j$ and we have:

$$\|U_j-W_j\|_2 \leqq 122(n^{1/4}(m\eta)^{1/256})+\left(\frac{1}{n}-a\right)^{1/2}, \quad \text{and using (e),}$$

$$\|U_j-u_j\|_2 \leqq \frac{m-1}{2}\eta+16n(m\eta)^{1/4}+122n^{1/4}(m\eta)^{1/256}+3n^{1/2}(m\eta)^{1/4} \leqq$$

$$\leqq (1+16n+122n^{1/4}+3n^{1/2})(m\eta)^{1/256} \leqq 142n(m\eta)^{1/256}. \qquad \text{Q.E.D.}$$

Lemma 4.3. *Let $n, m\in\mathrm{N}$ with n dividing m. Let α be a minimal periodic automorphism of period m of a II_1 factor N. Let $\delta>0$, and $(e_j)_{j=1,\dots,n}$ be a partition of unity in N such that $\alpha(e_j)=e_{j+1}$, $j=1,\dots,n$ $(e_{n+1}=e_1)$, and $U\in N$, $\|U\|\leqq 1$ with*

(1) $\qquad \|U^{n-l}-(U^*)^l\|_2 \leqq \delta, \quad l=0,1,2,\dots,n-1 \quad (\text{read } (U^*)^0=1),$

(2) $\qquad \|\alpha(U)-U\|_2 \leqq \delta,$

(3) $\qquad \|Ue_iU^*-e_{i+1}\|_2 \leqq \delta \quad (i=1,\dots,n).$

Then there exists a partition of unity $(E_j)_{j=1,\dots,n}$ of N, such that $\alpha(E_j)=E_{j+1}$ for $j=1,\dots,n$ $(E_{n+1}=E_1)$, and a unitary V, $V^n=1$, $V\in N^\alpha$ such that $VE_iV^*=E_{i+1}$ for $i=1,\dots,n$ and that

$$\|V-U\|_2 \leqq \varepsilon, \quad \|E_i-e_i\| \leqq \varepsilon, \quad i=1,\dots,n,$$

where $\varepsilon=143n^4(2mn^2\delta)^{1/256}$ (provided $2n^2\delta\leqq 1/m$).

Proof. Let $\lambda=\exp(i2\pi/n)$. Put $W=\sum\bar{\lambda}^j e_j$. Then W is a unitary of N such that $\alpha(W)=\sum\bar{\lambda}^j e_{j+1}=\lambda W$.

For $k\in\{0,\dots,n-1\}$, put $f_k=\frac{1}{n}\sum_{j=1}^{n-1}\bar{\lambda}^{jk}U^j$, where U^0 is taken to be 1. Then $U^l=\sum_{k=0}^{n-1}\lambda^{kl}f_k$ for $l=0,1,\dots,n-1$.

Moreover $\|f_k\|\leqq 1$ $(k=0,1,\dots,n-1)$ and

$$\|f_k^*-f_k\|_2 \leqq \frac{1}{n}\left\|\sum_{j=1}^{n-1}(\lambda^{-k})^{n-j}(U^{n-j}-(U^*)^j)\right\|_2+\frac{1}{n}\|U^n-1\|_2 \leqq 2\delta,$$

$$n^2f_k^*f_k=\sum_{j,l=0,\dots,n-1}(\lambda^k)^j(\lambda^k)^{-l}(U^*)^jU^l.$$

For $l\geqq j$ we have $\|(U^*)^jU^l-U^{l-j}\|_2\leqq\|(U^*)^jU^l-U^{n-j}U^l\|_2+\|U^n-1\|_2\leqq 2\delta$. And for $l<j$, we have $\|(U^*)^jU^l-U^{n-(j-l)}\|_2\leqq\delta$. It follows that, in the above sum for $n^2f_k^*f_k$ one can replace each $(U^*)^jU^l$ by $U^{\underline{l-j}}$ where $\underline{l-j}=l-j$, mod n, and $0\leqq\underline{l-j}<n$ to get

$$\|n^2f_k^*f_k-n^2f_k\|_2 \leqq 2n^2\delta, \quad \|f_k^*f_k-f_k\|_2 \leqq 2\delta.$$

Put $U_j = Wf_j$ with f_j as above. Then as W is unitary we get: $U_j^* U_j = f_j^* f_j$, $\|U_j^* U_j - f_j\|_2 \leqq 2\delta$. Also $U_{n-2} U_{n-2}^* = Wf_{n-2} f_{n-2}^* W^*$ so $\|U_{n-2} U_{n-2}^* - (U_{n-2} U_{n-2}^*)^2\|_2 \leqq 4\delta$. We get $\|U_j^* U_j - (U_j^* U_j)^2\|_2 \leqq 4\delta$ (because $\|U_j^* U_j - f_j^*\|_2 \leqq 2\delta$, $\|U_j^* U_j - f_j\| \leqq 2\delta$).

Also $UWU^* = \Sigma \bar{\lambda}^j U e_j U^*$ so that $\|UWU^* - \lambda W\|_2 \leqq n\delta$ and $\|UW^*U^* - \bar{\lambda} W^*\|_2 \leqq n\delta$, $\|WUW^* - \bar{\lambda} U\|_2 \leqq n\delta + \|U^* U - 1\|_2 \leqq (n+2)\delta$ because $\|U^* - U^{n-1}\|_2 \leqq \delta$ and $\|U^n - 1\|_2 \leqq \delta$.

So for $j \in \{0, 1, \ldots, n-1\}$ we get $\|WU^j W^* - \bar{\lambda}^j U^j\|_2 \leqq j(n+2)\delta$. Hence $\|Wf_k W^* - f_{k+1}\|_2 \leqq \frac{1}{n} \Sigma \|WU^j W^* - \bar{\lambda}^j U^j\|_2 \leqq \frac{(n-1)}{n}(n+2)\delta \leqq n^2\delta$. It follows that $\|U_k U_k^* - U_{k+1}^* U_{k+1}\|_2 \leqq \|Wf_k W^* - f_{k+1}\|_2 + 4\delta \leqq (n^2+4)\delta$ for all $k = 0, 1, \ldots, n-1$. Hence also $\|U_{n-2} U_{n-2}^* - f_{n-1}\|_2 \leqq (n^2+2)\delta$ and $\left\|\left(\sum_0^{n-2} U_j^* U_j\right) + U_{n-2} U_{n-2}^* - 1\right\|_2 \leqq$ $\leqq 2(n-1)\delta + (n^2+2)\delta$ because $\sum_{j=0}^{n-1} f_j = 1$. As $\alpha(W) = \lambda W$ and $\|\alpha(f_j) - f_j\|_2 \leqq$ $\leqq \frac{1}{n} \sum_{l=0}^{n-1} \|U^l - \alpha(U^l)\|_2 \leqq \frac{n-1}{2}\delta$ we have $\|\alpha(U_j) - \lambda U_j\|_2 \leqq \frac{n-1}{2}\delta$.

We have shown that the family $(U_j)_{j=0,\ldots,n-2}$ satisfies the conditions (a), (b), (c), (d) of Lemma 4.2 with $\eta = 2n^2\delta$. By hypothesis $2n^2\delta \leqq \frac{1}{m}$ so that we can find a family $(e_{ij})_{i,j \in \{0,\ldots,n-1\}}$ of partial isometries of N, such that $\alpha(e_{ij}) = \lambda^{i-j} e_{ij}$ and

$$\|U_j - e_{j+1,j}\|_2 \leqq 142n(2mn^2\delta)^{1/256} = \delta' \quad (j = 0, 1, \ldots, n-2).$$

Moreover we have $U_{n-1} = Wf_{n-1} = (W^*)^{n-1} f_{n-1}$ and as $\|f_k^* W^* - W^* f_{k+1}^*\|_2$ is smaller than $n^2\delta$ for all k we get:

$$\|U_{n-1} - U_0^* U_1^* \ldots U_{n-2}^*\|_2 \leqq n(n^2\delta) + \sum_{j=1}^{n-1} \|(f_j^*)^2 - f_j^*\|_2 \leqq nn^2\delta + 4n\delta,$$

$$\|U_{n-1} - e_{0,n-1}\|_2 \leqq 2n^3\delta + n\delta' \leqq \delta'' = 143n^2(2mn^2\delta)^{1/256}.$$

Put $W_1 = \sum_{j=0}^{n-1} e_{j+1,j}$. Then W_1 is a unitary such that $W_1^n = 1$ and $\alpha(W_1) = \lambda W_1$. Put $E_j = \frac{1}{n} \sum_{l=0}^{n-1} \lambda^{jl} W_1^l$, for $j = 0, \ldots, n-1$, so that $W_1 = \Sigma \bar{\lambda}^j E_j$ and the E_j are the spectral projections of W_1 corresponding to $\bar{\lambda}^j$, $j = 0, \ldots, n-1$. We have $\alpha(E_j) = E_{j+1}$ $(j = 0, \ldots, n-1)$.

Put $V = \Sigma \lambda^k e_{kk}$, then V is unitary, $V^n = 1$, $\alpha(V) = V$ and $VE_j V^* = E_{j+1}$ $(j = 0, 1, \ldots, n-1)$. (Because $VW_1V^* = \lambda W_1$.)

We have

$$\|W_1 - W\|_2 \leqq \left\|\sum_0^{n-1} (U_j - e_{j+1,j})\right\|_2 \leqq n\delta'',$$

and hence

$$\|E_j - e_j\|_2 \leqq \frac{1}{n} \sum_0^{n-1} \|W_1^l - W^l\|_2 \leqq \left(\frac{n-1}{2}\right) n\delta''.$$

Also $\|V-U\|_2 \leqq \sum_{k=0}^{n-1} \|e_{kk} - f_k\|_2 \leqq n(2\delta') + (n-1)2\delta + n^2\delta \leqq 2\delta''$. Hence we get the conclusion of the lemma, taking $\varepsilon = n^2\delta'' = 143 n^4 (2mn^2\delta)^{1/256}$. Q.E.D.

V. Actions of finite cyclic groups, by outer automophisms, on the II_1 hyperfinite factor

The fact that for each n there exists only one action by outer automorphisms of $\mathbf{Z}/n$ on R follows from statement (b) of

Theorem 5.1. *Let R be the hyperfinite II_1 factor and $p, q \in \mathbf{N}$.*

(a) *Let $\alpha \in \operatorname{Aut} R$ be minimal periodic with* (outer period α) $= pq$ *then $\alpha \otimes s_q^1$ is conjugate to α, also $\alpha \otimes 1_R$ is conjugate to α.*

(b) *Any periodic $\alpha \in \operatorname{Aut} R$ such that* period α = outer period $\alpha = p$ *is conjugate to s_p^1.*

Proof. Let $(x_j)_{j \in \mathbf{N}}$ be a strongly dense sequence in the unit ball R_1 of R. We shall, in the proof of (a) and (b), construct a sequence (K_j) of type I_n subfactors of R pairwise commuting, with:

$$\|E_{K'_m}(x_l) - x_l\|_2 \leqq \frac{1}{2^m}, \quad l < m. \text{ [9]} \tag{5.2}$$

We recall that using [7], we then have for each l and

$$l' \geqq l, \quad x_l \overset{1/2^{l'-1}}{\in} \bigcap_{m \geqq l'} K'_m = \left((K_1 \cup \ldots \cup K_{l'})'' \cup \left(\bigcup_{j=1}^{\infty} K_j\right)'\right)''$$

because $(\prod_{m \geqq l'} E_{K'_m})(x)$ belongs to $\bigcap_{m \geqq l'} K'_m$.

Hence we know ([7]) that letting $K = \left(\bigcup_{k=1}^{\infty} K_j\right)''$ the factor R splits as the tensor product of K by its commutant K' in R.

(a) Let λ be an nth root of 1 where n is an integer dividing the outer period of α. We construct by induction on m a sequence K_m of pairwise commuting type I_n subfactors of R satisfying condition 5.2 and:

(5.3) $\alpha(K_m) = K_m$ and there exists a system of matrix units e_{ij}^m of K_m such that $\alpha(e_{ij}^m) = \lambda^{i-j} e_{ij}^m$.

The existence of K_1 follows from corollary 2.7.

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[9]) From now on, if K is a von Neumann subalgebra of R, we let E_K be the trace preserving conditional expectation of R onto K and let K' be the relative commutant of K in R.

Assume we have constructed the K_j's up to K_m included. We are looking for K_{m+1} such that:

(1) $K_{m+1} \subset (K_1 \cup \ldots \cup K_m)'$,

(2) K_{m+1} is generated by a system of $n \times n$ matrix units (e_{ij}) such that $\alpha(e_{ij}) = \lambda^{i-j} e_{ij}$ $(i, j = 1, \ldots, n)$.

(3) $\|[e_{i+1,i}, x_l]\|_2 < \varepsilon, \quad i = 1, \ldots, n-1; \quad l = 1, \ldots, m.$

Clearly, if ε is chosen small enough, the conditions (1), (2), (3) will imply conditions (5.2) and (5.3).

To get K_{m+1}, restrict to $P = (K_1 \cup \ldots \cup K_m)'$ which is a II_1 hyperfinite factor on which the restriction β of α has outer period equal to $p_0(\alpha)$. Then by corollary 2.7 and the fact that β_ω is minimal periodic of period $p_0(\alpha)$ (theorem 3.2.1) there exists a system $(v_j)_{j=1,\ldots,n}$ of partial isometries in P_ω such that, with $v_{n+1} = v_1$:

$$\beta_\omega(v_j) = \lambda v_j, \quad \sum_{j=1}^{n} v_j^* v_j = 1, \quad v_j v_j^* = v_{j+1}^* v_{j+1} \quad (j = 1, \ldots, n).$$

Then apply lemma 4.2 to construct the $(e_{j,k})_{j,k=1,\ldots,n}$ satisfying conditions (1), (2), (3) above.

Now let $K = \left(\bigcup_{j=1}^{\infty} K_j\right)''$. Take first $n = q$ and $\lambda = e^{i2\pi/q}$ then the restriction of α to K is obviously conjugate to s_q^1. Now $s_q^1 \otimes s_q^1$ is conjugate to s_q^1 so that $s_q^1 \otimes \alpha$ is conjugate to α, because $\alpha = \alpha/K \otimes \alpha/K'$.

Take then $n =$ outer period α and $\lambda = 1$, then the restriction of α to K is identity so that $\alpha \otimes 1_R$ is conjugate to α because $1_R \otimes 1_R = 1_R$.

(b) For each $n \in \mathbf{N}$ we choose a positive ε_n having the following property:

(5.4) Let U be a unitary in R with $U^p = 1$, $(e_j)_{j=1,\ldots,p}$ be a partition of unity in R with $U e_j U^* = e_{j+1}$, $j = 1, \ldots, p-1$, and K be the type I_p subfactor of R generated by U and the e_j's.

Then $x \in R$, $\|x\|_\infty \leqq 1$, $\|[x, U]\|_2 \leqq 2\varepsilon_n$, $\|[x, e_j]\|_2 \leqq 2\varepsilon_n$, $j = 1, \ldots, p$ implies

$$\|E_{K'}(x) - x\|_2 \leqq \frac{1}{2^{n+1}}.$$

We can moreover assume that $\varepsilon_{n+1} \leqq \varepsilon_n$ for each $n \in \mathbf{N}$ and $\varepsilon_n \xrightarrow[n \to \infty]{} 0$. Then we construct by induction a sequence $(K_n)_{n \in \mathbf{N}}$ of pairwise commuting type I_p subfactors of R satisfying condition 5.2 and

(5.5) For each $n \in \mathbf{N}$ one has $\alpha(K_n) = K_n$, α restricted to K_n equals Ad U_n, U_n unitary in K_n and:

$$\left\| \alpha(x_j) - \left(\prod_1^n \operatorname{Ad} U_k \right)(x_j) \right\|_2 \leqq \varepsilon_n, \quad j = 1, \ldots, n.$$

We directly prove the existence of K_{n+1} assuming $K_1, \ldots, K_n$ have already been constructed. This will also show how to build K_1. Let $P=(K_1 \cup \ldots \cup K_n)' \cap R$ and P_1 its unit ball. P is globally invariant under α, let $\beta=\alpha/P$, then $p_0(\beta)=p$ so that by 5.1(a) there exists a partition of unity $(e_j)_{j=1,\ldots,n}$ in P, with $\beta(e_j)=e_{j+1}$ $(j=1, \ldots, p)$ and:

$$\|[e_j, x_l]\|_2 \leqq \varepsilon_n \quad (j=1, \ldots, p;\ l=1, \ldots, n).$$

We then choose a system of matrix units $(f_r)_{r=1,\ldots,n^{2n}}$ in $(K_1 \cup \ldots \cup K_n)''$ and write $x_j = \sum_r \lambda_{r,j} f_r y_{r,j}$ where the λ's are scalars, the y's belong to P_1, for $j=1, \ldots$ $\ldots, n+1$. Clearly we thus have a finite number k of elements of P_1, say $y_1, \ldots, y_k$, and an $\eta>0$ such that

(5.6)
$$\begin{aligned} &\big(v \text{ unitary in } P,\ \|\beta(y_j)-v y_j v^*\|_2 \leqq \eta \ \ (j=1, \ldots, k)\big) \Rightarrow \\ &\Rightarrow \Big(\|\alpha(x_j)-\Big(\prod_1^n \operatorname{Ad} U_l\Big) \operatorname{Ad} v\,(x_j)\|_2 \leqq \varepsilon_{n+1} \ \ (j=1, \ldots, n+1)\Big). \end{aligned}$$

We moreover assume that $\eta \leqq 2\varepsilon_n$.

We choose $\delta>0$ from the above $\eta>0$ and the lemma 4.3 with $\varepsilon=\frac{1}{4}\eta$. Now by corollary 3.3 we can find an element U of P, $\|U\| \leqq 1$, satisfying the following conditions:

(5.7)
$$\|U^{p-l}-(U^*)^l\|_2 \leqq \delta, \quad \|\beta(U)-U\|_2 \leqq \delta,$$

$$\|U^p-1\|_2 \leqq \delta, \quad \|U e_i U^* - \beta(e_i)\|_2 \leqq \delta \quad (i=1, \ldots, p)$$

and

$$\|U y_j U^* - \beta(y_j)\|_2 \leqq \frac{1}{2}\eta \quad (j=1, \ldots, k).$$

It then follows from lemma 4.3 applied to $\beta \in \operatorname{Aut} P$ that there exists a partition of unity $(E_j)_{j=1,\ldots,p}$ in P, a unitary $V \in P$ such that $V E_j V^* = E_{j+1}$ for all j, $V^p=1$, that the type I_p subfactor K of P generated by $(E_j)_{j=1,\ldots,p}$ and V is globally invariant under β, with $\beta/K = \operatorname{Ad} V/K$ and that

$$\|E_j - e_j\|_2 \leqq \frac{1}{4}\eta, \quad \|V-U\|_2 \leqq \frac{1}{4}\eta.$$

We shall take $K_{n+1}=K$, $U_{n+1}=V$. We have for $j \in \{1, \ldots, k\}$:

$$\|V y_j V^* - \beta(y_j)\| \leqq 2\|V-U\|_2 + \|U y_j U^* - \beta(y_j)\|_2 \leqq \eta$$

so that by 5.6 we get:

$$\Big\|\alpha(x_j) - \Big(\prod_1^{n+1} \operatorname{Ad} U_l\Big)(x_j)\Big\|_2 \leqq \varepsilon_{n+1} \quad (j=1, \ldots, n+1).$$

But by the induction hypothesis we had:

$$\left\|\alpha(x_j)-\left(\prod_1^n \mathrm{Ad}\, U_l\right)(x_j)\right\|_2 \leqq \varepsilon_n \quad (j=1, \ldots, n).$$

This, using the fact that $\prod\limits_1^n \mathrm{Ad}\, U_l$ preserves the $\|\ \|_2$ norm shows that $\|Vx_jV^*-$ $-x_j\|_2 \leqq \varepsilon_n+\varepsilon_{n+1} \leqq 2\varepsilon_n$ $(j=1, \ldots, n)$.

Also we have $\|[E_i, x_j]\|_2 \leqq \|[e_i, x_j]\|_2+2\|E_i-e_i\|_2 \leqq 2\varepsilon_n$ $(j=1, \ldots, n$ and $i=1, \ldots, p)$.

So it follows from 5.4 that:

$$\|E_{K'_{n+1}}(x_j)-x_j\|_2 \leqq \frac{1}{2^{n+1}} \quad (j=1, \ldots, n).$$

We have shown how to construct the sequence $(K_n)_{n\in\mathbf{N}}$ satisfying 5.2 and 5.5. Let $K=(\bigcup\limits_{n\in\mathbf{N}} K_n)''$, then by (5.2) we have a splitting of R as a tensor product of K by K'_R. Let us note also by 5.5 that:

$$\alpha(x)=\left(\prod_1^\infty \mathrm{Ad}\, U_l\right)(x), \quad \forall x\in R.$$

This shows that α is conjugate to $s_p^1\otimes$(identity of K'_R); but α is conjugate to $\alpha\otimes 1_R$ (thm. 5.1(a)) and (Identity on K'_R)$\otimes$(Identity on R) is clearly 1_R because $K'_R\otimes R$ is isomorphic to R. So α is conjugate to $s_p^1\otimes 1_R$ which again by 5.1 (a) is conjugate to s_p^1. Q.E.D.

VI. The cyclic group of outer conjugacy classes with given outer period p

In this section we shall prove that for given $p\in\mathbf{N}$ and γ, $\gamma^p=1$, there is only one outer conjugacy class with outer invariants (p, γ). The proof relies on the study of the tensor product as a law of composition between outer conjugacy classes with outer period p.

Definition 6.1. Let R be the hyperfinite II_1 factor, a and b be outer conjugacy classes in $\mathrm{Aut}\, R$, then we let $a\times b$ be the outer conjugacy class of any automorphism $\alpha\otimes\beta\in\mathrm{Aut} R\otimes R$ with $\alpha\in a$, $\beta\in b$, brought back to $\mathrm{Aut} R$ by any isomorphism Π of R on $R\otimes R$.

In other words, for $\alpha\in a$, $\beta\in b$ and $\pi\colon R\to R\otimes R$ the automorphism $\pi^{-1}(\alpha\otimes\beta)\pi$ belongs to $a\times b$. Clearly changing α to $\alpha'\in a$, β to $\beta'\in b$, and π to $\pi'\colon R\to R\otimes R$ does not change the outer conjugacy class of $\pi^{-1}(\alpha\otimes\beta)\pi$, so that 6.1 makes sense.

Theorem 6.2. *For each* $p \in \mathbf{N}$ *the set* Br_p *of outer conjugacy classes in* Aut R, *with outer period equal to p, endowed with the law of composition* $(a, b) \to a \times b$ *is an abelian group and* γ *is an isomorphism of this group on the group of* pth *roots of* 1 *in* $\mathbf{C}$.

Corollary 6.3. *For each* $p \in \mathbf{N}$, Br_p *is a cyclic group of order* p *with unit the outer conjugacy class of* s_p^1 *and generator the outer conjugacy class of* s_p^γ *if* γ *is a primitive* pth *root of* 1.

Proof. Immediate from 6.2 and proposition 1.6.

Corollary 6.4. *Let* R *be the hyperfinite* II_1 *factor,* $\alpha, \beta \in \mathrm{Aut}\, R$ *be periodic, then*

$$(\alpha \text{ conjugate to } \beta) \Leftrightarrow (p_0(\alpha) = p_0(\beta),\ \gamma(\alpha) = \gamma(\beta),\ \varepsilon(\alpha) = \varepsilon(\beta)).$$

Proof. By 6.2, α and β are outer conjugate iff they have the same outer invariants. By 2.8 if α and β are outer conjugate and $\varepsilon(\alpha) = \varepsilon(\beta)$ then α and β are conjugate. Q.E.D.

Proof of theorem 6.2. Let us first check that $a \in Br_p$, $b \in Br_p \Rightarrow a \times b \in Br_p$. By [9] Cor. 6 we know that the tensor product $\alpha \otimes \beta$ of two automorphisms α and β is inner if and only if both are inner. It follows in general that $p_0(\alpha \otimes \beta)$ equals l.c.m.$(p_0(\alpha), p_0(\beta))$ and in particular that Br_p is stable under $(a, b) \to a \times b$. Next we show that the class e of s_p^1 is a unit in Br_p. Let $a \in Br_p$ and let α be a minimal periodic automorphism. Then by theorem 5.1 (a) we know that $\alpha \otimes s_p^1$ is conjugate to α and hence that $a \times e$ is equal to a.

Let us now check that γ is an homomorphism; let $a, b \in Br_p$, $\alpha \in a$, $\beta \in b$, and $\alpha^p = \mathrm{Ad}\, u$, $\beta^p = \mathrm{Ad}\, v$, $u, v \in R$ with $\alpha(u) = \gamma(\alpha) u$, $\beta(v) = \gamma(\beta) v$. We then have $(\alpha \otimes \beta)^p = \mathrm{Ad}(u \otimes v)$ and $\alpha \otimes \beta(u \otimes v) = \alpha(u) \otimes \beta(v) = \gamma(\alpha)\gamma(\beta) u \otimes v$.

Let us prove that e is characterized in Br_p by the condition $\gamma(e) = 1$. Take $a \in Br_p$ with $\gamma(a) = 1$, and let α be minimal periodic. Then the period of α is equal to its outer period, equal to p. Hence by theorem 5.1 (b) α is conjugate to s_p^1, so that $a = e$.

We know therefore that Br_p is a group, we can in fact give a description of the inverse of an element a of Br_p: Let R^0 be the opposite von Neumann algebra of R, i.e., R^0 coincides with R as a complex vector space but the product is $(x, y) \to yx$ instead of $(x, y) \to xy$. Then let $\alpha \in \mathrm{Aut}\, R$, obviously α as a linear transformation of R defines an automorphism α^0 of R^0, which, because R^0 is hyperfinite and hence isomorphic to R, defines a conjugacy class in Aut R, called the opposite of α. Clearly $p(\alpha) = p(\alpha^0)$. Let $\alpha^p = \mathrm{Ad}\, U$, $\alpha(U) = \gamma U$, then the equality $\alpha^p(x) = UxU^*$, $x \in R$, means that $(\alpha^0)^p(x) = U^* x U$, $x \in R^0$, so that, as $\alpha^0(U^*) = \alpha(U^*) = \bar{\gamma} U^*$, we get $\gamma(\alpha^0) = \gamma(\alpha)^{-1}$.

Of course a^0 is meaningful for $a \in Br_p$ and $a \times a^0$ is equal to e because $\gamma(a \times a^0) = 1$.

The end of the proof of 6.2 is now easy. We know that Br_p is a group, that γ is an homomorphism with trivial kernel and that γ is surjective by 1.6 (c). Q.E.D.

We now apply theorem 6.2 to determine the conditions under which two periodic automorphisms $\alpha, \beta \in \mathrm{Aut}\, R$ are weakly equivalent, i.e. there exists a $\sigma \in \mathrm{Aut}\, R$ such that $\sigma[\alpha]\sigma^{-1} = [\beta]$, where $[\alpha]$ is the full group, [4] p. 163, of the group $\{\alpha^n, n \in \mathbf{Z}\} \subset \subset \mathrm{Aut}\, R$.

Let $n = 2^l m$, m odd, be an integer. Let S_n be the set consisting of all prime divisors of m with in addition an element ε if $l = 2$ and two elements ε, ω if $l > 2$. Let for each integer k prime relative to n, $\left(\frac{k}{n}\right) \in \{-1, 1\}^{S_n}$ be such that $\left(\frac{k}{n}\right)_\varepsilon = (-1)^{\varepsilon(k)}$, $\varepsilon(k) = \frac{k-1}{2}$, $\left(\frac{k}{n}\right)_\omega = (-1)^{\omega(k)}$ where $\omega(k) = \frac{k^2-1}{8}$, and $\left(\frac{k}{n}\right)_p = \left(\frac{k}{p}\right)$ as in [11] p. 14 otherwise.

Theorem 6.5. *For a periodic $\alpha \in \mathrm{Aut}\, R$ define $c(\alpha) = \mathrm{Order}\, \gamma(\alpha)$ and $q(\alpha) = \left(\frac{k}{c(\alpha)}\right)$ where $\gamma(\alpha) = \exp(2\pi i k / c(\alpha))$.*

(a) *Two periodic automorphisms α and β are weakly equivalent if and only if $p_0(\alpha) = p_0(\beta)$, $c(\alpha) = c(\beta)$ and $q(\alpha) = q(\beta)$.*

(b) *Let c and d be integers $\geqq 1$, S_c be defined as above, and $q \in \{-1, 1\}^{S_c}$. Then there exists a periodic $\alpha \in \mathrm{Aut}\, R$ such that $p_0(\alpha) = cd$, $c(\alpha) = c$, $q(\alpha) = q$.*

Proof. (a) If α is weakly equivalent to β then $p_0(\alpha) = p_0(\beta) = p$ because $p_0(\alpha)$ is the order of the image of $[\alpha]$ in $\mathrm{Out}\, R$. Also there exists an integer s, necessarily prime relative to p, and such that α is outer equivalent to β^s.

We have, with $p = p(\alpha) = p(\beta)$, that $\beta^p = \mathrm{Ad}\, U$, $\beta(U) = \gamma(\beta) U$ for some unitary $U \in R$. Hence $(\beta^s)^p = \mathrm{Ad}\, U^s$, $\beta^s(U^s) = \gamma(\beta)^{s^2} U$ so that $\gamma(\beta^s) = \gamma(\beta)^{s^2}$.

As s is prime relative to p it follows that the order of $\gamma(\beta)^{s^2}$ is the same as the order of $\gamma(\beta)$ so that $c(\alpha) = c(\beta) = c$. Put $\gamma(\alpha) = \exp\left(\frac{2\pi i k}{c}\right)$, $\gamma(\beta) = \exp\left(\frac{2\pi i k'}{c}\right)$. Then we have $\gamma(\alpha) = \gamma(\beta)^{s^2}$ so that in $\mathbf{Z}/c$ we get $k = s^2 k'$, where k, k', s^2 are units in $\mathbf{Z}/c$. It hence follows that $q(\alpha) = q(\beta)$.

Conversely, assume that $p_0(\alpha) = p_0(\beta)$, $c(\alpha) = c(\beta)$, $q(\alpha) = q(\beta)$, and put $\gamma(\alpha) = \exp\left(\frac{2\pi i k}{c}\right)$, $\gamma(\beta) = \exp\left(\frac{2\pi i k'}{c}\right)$. Then by [11] thm. 3, p. 39 and 1, p. 46, take s, prime relative to $p_0(\alpha)$, such that $ks^2 = k'\,(c(\alpha))$. It follows that α^s has the same outer invariants as β and that α and β are weakly equivalent.

(b) Let $k \in \mathbf{Z}/c$ be a unit in $\mathbf{Z}/c$ such that $\left(\frac{k}{c}\right) = q$ ([11] lemma 1, p. 46) then take $\gamma = \exp\left(\frac{2\pi i k}{c}\right)$ and $\alpha = s_{cd}^{\gamma}$.

It follows that $p_0(\alpha)=cd$, $c(\alpha)=\text{Order}\,\gamma=c$, and $q(\alpha)=q$. Q.E.D.

Remark 6.7. By 6.5 there are automorphisms α of R, the simplest being s_3^j where $j^3=1, j\neq 1$, which are not weakly equivalent to their opposite α^0. One can deduce from this that the pair $R^\alpha\subset R$ is not isomorphic to the opposite pair.

Remark 6.8. Let $n\in\mathbf{N}$ and M be an arbitrary factor. Let $\alpha\in\text{Aut}M$, $p_0(\alpha)=n$. Then consider the abstract kernel

$$q\in\mathbf{Z}/n\to\varepsilon(\alpha^q)\in\text{Out}\ M$$

where ε is the canonical map from $\text{Aut}\,M$ to $\text{Out}\,M$. To this abstract kernel there corresponds an obstruction $k\in H^3(\mathbf{Z}/n,\mathbf{T})$ ([12] p. 216) with $\mathbf{T}$=center of the unitary group of M (i.e. $\mathbf{T}=\{z\in\mathbf{C}, |z|=1\}$) with trivial action of $\mathbf{Z}/n$. To get k one takes for $q\in\mathbf{Z}/n$ an arbitrary $\beta_q\in\text{Aut}\,M$ with $\varepsilon(\beta_q)=\varepsilon(\alpha^q)$, then one takes for $q_1,q_2\in\mathbf{Z}/n$ a unitary u_{q_1,q_2} of M with $\text{Ad}\,u_{q_1,q_2}=\beta_{q_1}\beta_{q_2}\beta_{q_1+q_2}^{-1}$ and: $k(q_1,q_2,q_3)=\beta_{q_1}(u_{q_2,q_3})u_{q_1,q_2+q_3}u_{q_1+q_2,q_3}^{-1}u_{q_1,q_2}^{-1}\in\mathbf{T}$. With the choice $\beta_q=\alpha^q$, one gets:

$$k(q_1,q_2,q_3)=\gamma(\alpha)^{q_1\eta(q_2,q_3)}\quad\text{where}\quad \eta(q_2,q_3)=\begin{cases}0 & \text{if}\quad q_2+q_3<n,\\ 1 & \text{if}\quad q_2+q_3\geqq n.\end{cases}$$

Comparing the bar resolution of the trivial $\mathbf{Z}/n$ module $\mathbf{Z}$ with its periodic resolution of period 2 one brings back k to the element $\gamma(\alpha)$ of $\{a\in\mathbf{T}, a^n=1\}$.

VII. Applications to various questions of noncommutative ergodic theory

Throughout we let R be the hyperfinite factor of type II_1. This section is devoted to apply theorem 1.5 to answer the following questions.

Problem 7.1. Is any periodic $\alpha\in\text{Aut}\,R$ conjugate to the opposite of its inverse? (It is easy to show that there are inner automorphisms which are neither conjugate to their opposite nor their inverse. However they are always conjugate to the opposite of their inverse, when they are periodic).

Theorem 7.2. *Let $\alpha\in\text{Aut}\,R$ be periodic. Then α is conjugate to $(\alpha^0)^{-1}$ if and only if $\gamma(\alpha)^2=1$.*

Proof. We have $\gamma(\alpha^{-1})=\gamma(\alpha)$ $\big(\alpha^{-p}=\text{Ad}\,U^*$ where $p=p_0(\alpha)$ and $\alpha(U)==\gamma(\alpha)\,U$, so that $\alpha^{-1}(U^*)=\gamma(\alpha)\,U^*\big)$.

Hence, $\gamma\big((\alpha^{-1})^0\big)=\overline{\gamma(\alpha)}$ so that if $\gamma(\alpha)^2\neq 1$, α is not even *outer* conjugate to $(\alpha^0)^{-1}$.

We have $\varepsilon(\alpha^0)=\varepsilon(\alpha^{-1})$ for any $\alpha\in\text{Int}\,R$, α periodic. Hence $\varepsilon(\alpha^0)=\varepsilon(\alpha^{-1})$ holds for any periodic $\alpha\in\text{Aut}\,R$ $\big($because $(\alpha^0)^{p_m}=(\alpha^{p_m})^0$, $(\alpha^{-1})^{p_m}=(\alpha^{p_m})^{-1}\big)$.

So 7.2 follows from 1.5 and the equalities:

$$p_0((\alpha^0)^{-1}) = p_0(\alpha), \quad \gamma((\alpha^0)^{-1}) = \overline{\gamma(\alpha)}, \quad \varepsilon((\alpha^0)^{-1}) = \varepsilon(\alpha).$$

In particular if $\gamma(\alpha)^2 \neq 1$, α cannot be outer conjugate to an infinite tensor product of automorphisms of finite dimensional factors, because such automorphisms are conjugate to the opposite of their inverse. This drives to:

Problem 7.3. Which automorphisms $\alpha \in \mathrm{Aut}\, R$, α periodic, are conjugate (resp. outer conjugate) to an infinite tensor product of automorphisms of finite dimensional factors? To infinite tensor product of inner automorphisms of arbitrary factors?

Theorem 7.4. *Let $\alpha \in \mathrm{Aut}\, R$, α periodic, then:*

(a) *If α is an infinite tensor product of inner automorphisms* $\mathrm{Ad}\, U_j$ *of finite factors R_j, then $\gamma(\alpha) = 1$.*[10]

(b) *If $\gamma(\alpha) = 1$, α is the tensor product of an inner automorphism of R by an infinite tensor product of automorphisms of finite dimensional factors.*

(c) *Let α be periodic of period p, with $\gamma(\alpha) = 1$. Put $p = qp_0(\alpha)$, assume q prime, let $\varepsilon = \sum_{j=0}^{q-1} \lambda_j\, \varepsilon(e^{i2\pi j/n})$ be the inner invariant of α. Then α is an infinite tensor product of automorphisms of finite dimensional factors if and only if either all the λ'_j are rational numbers or they are all $\neq 0$ as well as the $\hat{\lambda}_k$:*

$$\hat{\lambda}_k = \Sigma \lambda_j \exp \frac{i2\pi jk}{n} \quad k = 0, \ldots, q-1.$$

Proof. Can be left to the reader.

One has the following positive general result concerning the approximation of periodic automorphisms of R by automorphisms of finite dimensional von Neumann algebras:

Theorem 7.5. *Let α be a periodic automorphism of R, then there exists an increasing sequence of finite dimensional subalgebras P_n of R such that $\alpha(P_n) = P_n$ for all n and that $\bigcup_{n=1}^{\infty} P_n$ is strongly dense in R.*

Proof. First take $p \in \mathbf{N}$, $\gamma \in \mathbf{C}$, $\gamma^p = 1$ and s_p^γ as constructed in part 1. Consider $P_n = (F_p^{(1,n)} \cup \{\theta^n(U_\gamma)\})''$ with the notations of proposition 1.6. Then, as $\alpha\theta^n(U_\gamma) = \gamma\theta^n(U_\gamma)$ with $\alpha = s_p^\gamma$ and as $\alpha(F_p^{(1,n)})$ is contained in the algebra generated by $F_p^{(1,n)}$ and $\theta^{n-1}(v_\gamma)$, i.e., by $F_p^{(1,n)}$ and $\theta^n(U_\gamma^*)$, we see that P_n is globally invariant under α for each n.

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[10]) Hence if $\gamma(\alpha) \neq 1$, α is not outer conjugate to such a tensor product.

Having proven 7.5 for the s_p^γ's we just have to prove it for periodic inner automorphisms and conclude using 1.11.

Let $\operatorname{Ad} U$ be a periodic inner automorphism of R, with $U=\sum_{j=1}^m a_j e_j$ where $a_j \in \mathbf{C}$, $a_j^m=1$ and the e_j's are projections in R.

Choose an increasing sequence of projections $f_n \in R$ commuting with U, such that for each n, j: $\tau(f_n e_j)$ is a dyadic rational, and with $f_n \to 1$ when $n \to \infty$.

Let $(x_k)_{k\in\mathbf{N}}$ be a dense sequence in the unit ball of R (dense for the strong topology). Then by induction on n one builds a sequence $(K_n)_{n\in\mathbf{N}}$ where K_n is a subfactor of type $\mathrm{I}_{2^{p_n}}$ of R_{f_n}, containing Uf_n and such that $K_{n-1}+\mathbf{C}(f_n-f_{n-1})\subset \subset K_n$ and that it approximates the $f_n x_j f_n$ $(j=1, \dots, n)$ up to $\frac{1}{n}$ in the trace norm.

It follows that the sequence $P_n=K_n+\mathbf{C}(1-f_n)$ is increasing, that it generates R and that $UP_nU^*=P_n$ for each $n\in\mathbf{N}$. Q.E.D.

Problem 7.6. Which periodic automorphisms of R have a square root?

Clearly, any inner automorphism of R has a square root, and hence by 1.11 we see that a periodic α with outer invariants (p, γ) has a square root in $\operatorname{Aut} R$ if s_p^γ has one. To compute $p_0(\alpha^2)$, $\gamma(\alpha^2)$ we distinguish two cases:

(1) $p_0(\alpha)$ is odd. Then α^{2q} is outer for $q=1, \dots, p_0(\alpha)-1$ because $2q$ is not a multiple of $p_0(\alpha)$. So:

$$p_0(\alpha^2) = p_0(\alpha), \quad \gamma(\alpha^2) = \gamma(\alpha)^4.$$

(2) $p_0(\alpha)$ is even. Then $(\alpha^2)^{\frac{p_0(\alpha)}{2}}$ is inner, and

$$p_0(\alpha^2) = \frac{1}{2}\, p_0(\alpha), \quad \gamma(\alpha^2) = \gamma(\alpha)^2.$$

Theorem 7.7.[11] *Let $p\in\mathbf{N}$, $\gamma\in\mathbf{C}$, $\gamma^p=1$. If the order of γ is odd, then any periodic automorphism with outer invariant (p, γ) has a square root.*

If the order of γ is even then s_p^γ has no square root.

Proof. Assume $\operatorname{Order}\gamma=2q+1$. Put $\gamma'=\gamma^{-q}$, then γ' is a root of 1 of an order dividing the order of γ and $\gamma'^2=\gamma^{-2q}=\gamma$. So $s_{2p}^{\gamma'}$ has a period dividing $2\cdot$ period of s_p^γ. As $2p$ is even we have $p_0((s_{2p}^{\gamma'})^2)=p$, $\gamma((s_{2p}^{\gamma'})^2)=\gamma'^2=\gamma$, and

$$(s_{2p}^{\gamma'})^{2p\cdot\operatorname{Order}\gamma} = 1.$$

So $(s_{2p}^{\gamma'})^2$ has outer invariants (p, γ) and trivial inner invariant, so that it is con-

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[11]) Clearly any $\alpha\in\operatorname{Aut} R$ with odd period, say $2m+1$, has a square root, namely α^{m+1}.

jugate to s_p^γ by theorem 1.5. Hence $s_p^\gamma \otimes \mathrm{Ad}\, U$ has a square root for all inner automorphisms and theorem 1.11 applies.

If the square roots γ', γ'' of γ satisfy $\mathrm{Order}\, \gamma' = \mathrm{Order}\, \gamma'' = 2\, \mathrm{order}\, \gamma$, take α such that $\alpha^2 = s_p^\gamma$. Then we must have $p_0(\alpha) = 2p_0(\alpha^2)$ because $p_0(\alpha)$ must be even. Then also $\gamma(\alpha)^2 = \gamma$, so that, say, $\gamma(\alpha) = \gamma'$. We have:

$$(\text{period } \alpha) \text{ is a multiple of } (\text{period } s_{2p}^{\gamma'}).$$

Hence, as $(\text{period } s_{2p}^{\gamma'}) = 2p \cdot \mathrm{Order}\, \gamma' = 4p \cdot \mathrm{Order}\, \gamma$ we see that we cannot have $\alpha^{2p\, \mathrm{Order}\, \gamma} = 1$ as required by $\alpha^2 = s_p^\gamma$. Q.E.D.

Remark 7.8. In $\mathrm{Out}\, R$ any periodic element has a qth root for any $q \in \mathbf{N}$, $q \neq 0$, because $(s_p^{\gamma q})^q$ is outer conjugate to s_{pq}^γ for all p, q and $\gamma, (\gamma^q)^p = 1$.

Remark 7.9. In [2] H. Borchers studies automorphisms α of von Neumann algebras M and their relations with inner automorphisms. For each $n \in \mathbf{N}$ he introduces a class K_n of automorphisms, and theorem 4.1 of [2] states that, when M is a factor for simplicity, $(\alpha^i$ is inner iff $i = 0\, (n)) \Leftrightarrow \alpha \in K_n$.

However the automorphisms s_p^γ, $\gamma \neq 1$, give a counterexample to this theorem because by [2] prop. 4.7, if $\alpha \in K_n$ then α^n is of the form $\mathrm{Ad}\, U$ with $U \in M^\alpha$. (In the notations of [2] $U \in Z_0$ where (Def. 2.1) Z_0 denotes the center of the fixed point algebra.) However if in [2] one replaces everywhere the word "inner" by "inner implemented in Z_0" then all the argument goes through.

Remark 7.10. In [8] thm. 1, V. Ya. Golodets claims that the cross product of the hyperfinite II_1 factor R by any cyclic group G of outer automorphisms is again hyperfinite. This theorem is true from our above results. (Apply 7.5.) However the proof given in [8] does not work. To see this we take the notations of [8]. The automorphism h of $\mathcal{M} = G \times M$ corresponds to the dual action of Takesaki, of the generator of G associated to ε (ε is a primitive nth root of 1). Hence in $G_h \times \mathcal{M}$ the commutant of the type I_n factor generated by V_g and V_h is, by the duality, the von Neumann algebra $\tilde{\Pi}(M)$, where $\tilde{\Pi}$ is an isomorphism of M into $G_h \times \mathcal{M}$ defined by

$$\tilde{\Pi}(x) = \Sigma \hat{x}_q V_h^q, \quad x = \Sigma \hat{x}_q, \quad g(\hat{x}_q) = \varepsilon^q \hat{x}_q.$$

Now $\mathcal{M}$, as a subfactor of $G_h \times \mathcal{M}$, has $\mathbf{C}$ as relative commutant so that the normalizer of $\mathcal{M}$ in $G_h \times \mathcal{M}$ consists only of unitaries of the form $v V_h^m$, v unitary in $\mathcal{M}$, $0 \leqq m < n$.

We can hence find a unitary X in $G_h \times \mathcal{M}$ which commutes with V_g and V_h, but for which $X \mathcal{M} X^* \neq \mathcal{M}$.

The claim in [8] is that for any automorphism φ of $G_h \times \mathcal{M}$ for which $\varphi(W_1) = V_h$, $\varphi(W_2) = V_g$, the family of operators $\varphi(W_2) = V_g$, $V_k = \varphi(W_k)$, $k = 3, 4, \ldots$ generates $\mathcal{M}$.

But if this is true for some φ, replace φ by $\varphi' = (\mathrm{Ad}\, X)\varphi$ with X as above, then certainly $\varphi'(W_1) = V_h$, $\varphi'(W_2) = V_g$, but the $\varphi'(W_p)$ $(p=2, 3, 4, \ldots)$ generate $X\mathcal{M}X^*$ which is different from $\mathcal{M}$, so that the condition would fail for φ'.

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PARIS VI U.E.R. 48
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A note on quasisimilarity of operators

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1. Introduction. Let $\mathfrak{H}$ be a separable, infinite dimensional complex Hilbert space, and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all bounded, linear operators on $\mathfrak{H}$. An operator X in $\mathcal{L}(\mathfrak{H})$ is *quasi-invertible* [1]) if X is injective and has dense range (i.e., $\ker(X)=\ker(X^*)=\{0\}$). Operators A and B in $\mathcal{L}(\mathfrak{H})$ are *quasisimilar* if there exist quasi-invertible operators X and Y in $\mathcal{L}(\mathfrak{H})$ such that $AX=XB$ and $YA=BY$. Two operators that are similar are clearly quasisimilar, and similar operators have equal spectra; one purpose of this note is to study the relationships between the spectra of quasisimilar operators.

There are several cases in which the quasisimilarity of two operators A and B implies the equality of their spectra: this is true if A and B are decomposable [7] or if A and B are hyponormal [6]. In section 4 we give necessary and sufficient conditions for two injective weighted shifts to be quasisimilar. We prove that if shifts W_α and W_β are quasisimilar, then they have equal spectra; if, in addition, W_α or W_β is invertible, then W_α is similar to W_β.

Contrasting with these results is an example, due to HOOVER [15], of two quasisimilar non-injective weighted shifts A and B such that $\sigma(A)=\{0\}$ and $\sigma(B)==D=\{z\in\mathbf{C}:|z|\leqq 1\}$, In [18] SZ.-NAGY and FOIAŞ gave necessary and sufficient conditions for a contraction to be quasisimilar to a unitary operator, and they gave an example of such an operator whose spectrum equals the disk D. The general result governing all of these cases is the following well-known corollary of Rosenblum's Theorem: The intersection of the spectra of quasisimilar operators is non-empty [15]. In Theorem 2.5 we prove the following refinement of this result: If $AX=XB$, where X is injective, and S is a part of B, then each non-empty closed-and-open subset of $\sigma(S)$ has non-empty intersection with $\sigma(A)$. In Theorem 2.6, Lemma 2.8, and Lemma 2.11 we give partial analogues of this result for the essential spectra of A and B.

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[1]) „Quasiaffinity" in [18].

In [13] Foiaş and Pearcy established a model for quasinilpotent operators up to similarity, and in [19] Pearcy inquired whether an analogous model could be given for quasinilpotent operators up to quasisimilarity. Since quasisimilarity is a transitive relation, such a model would apply to each operator in $\mathcal{Q}_{qs}=\{T\in\mathcal{L}(\mathfrak{H}): T$ is quasisimilar to some quasinilpotent operator in $\mathcal{L}(\mathfrak{H})\}$; in particular, the hyperinvariant subspace problem for operators in $\mathcal{Q}_{qs}$ is equivalent to the hyperinvariant subspace problem for operators in $\mathcal{Q}$ (see [15]). In section 3 we study properties of operators in $\mathcal{Q}_{qs}$. While quasisimilarity does not, in general, preserve quasitriangularity [24], we prove that each operator in $\mathcal{Q}_{qs}$ is quasitriangular; in addition, $\mathcal{Q}_{qs}$ is a proper subset of the norm closure of the set of all nilpotent operators (i.e., $\mathcal{Q}_{qs}\subsetneqq\mathcal{N}^-$). We prove that $\mathcal{Q}_{qs}$ contains no non-quasinilpotent decomposable or hyponormal operators. On the other hand, $\mathcal{Q}_{qs}$ is closed under countable direct sums (Proposition 3.10), and this result is used to prove that a subset $X\subset\mathbf{C}$ is the spectrum of an operator in $\mathcal{Q}_{qs}$ if and only if X is compact, connected, and contains 0 (Theorem 3.11).

We conclude this section with some terminology and notation. Let $\mathcal{K}$ denote the ideal of all compact operators in $\mathcal{L}(\mathfrak{H})$; if T is in $\mathcal{L}(\mathfrak{H})$, let $\tilde{T}$ denote the image of T in the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}$. The essential spectrum of T, $\sigma_e(T)$, is the spectrum of $\tilde{T}$ with respect to the Calkin algebra [11]. We will use results from [9] about semi-Fredholm operators and quasitriangular operators. We denote by $\mathcal{N}$ and $\mathcal{Q}$ the sets of all nilpotent and, respectively, quasinilpotent operators in $\mathcal{L}(\mathfrak{H})$. If T is in $\mathcal{L}(\mathfrak{H})$, then a *part* of T is an operator S of the form $S=T|\mathfrak{M}$, where $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}$ such that $T\mathfrak{M}\subset\mathfrak{M}$ and $\mathfrak{M}\neq\{0\}$ ($\mathfrak{M}=\mathfrak{H}$ is permitted). We denote the spectrum of T by $\sigma(T)$ and the spectral radius of T by $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}=\lim\|T^n\|^{1/n}$; thus $\mathcal{Q}=\{T$ in $\mathcal{L}(\mathfrak{H}): r(T)=0\}$.

2. On the spectra of quasisimilar operators. Let $\mathcal{A}$ denote a complex Banach algebra with identity and let $\mathcal{M}(\mathcal{A})$ denote the Banach algebra consisting of all 2×2 matrices with entries from $\mathcal{A}$ (where the norm of a matrix is its norm as an operator on the Banach space $\mathcal{A}\oplus\mathcal{A}$). Let a, b, and x denote elements of $\mathcal{A}$. Let $\sigma(y)$ denote the spectrum of an element y of $\mathcal{A}$.

Lemma 2.1. *If f is a function that is analytic in a neighborhood of $\sigma(a)\cup\sigma(b)$, and $ax=xb$, then $f(a)x=xf(b)$.*

Proof. Let M and N denote, respectively, the elements of $\mathcal{M}(\mathcal{A})$ whose matrices are

$$\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 0 & x\\ 0 & 0\end{bmatrix}.$$

Since f is analytic in a neighborhood of $\sigma(M)=\sigma(a)\cup\sigma(b)$, then $f(a)$, $f(b)$, and

$f(M)$ are defined by the Riesz functional calculus, and it is easy to prove that

$$f(M)=\begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix} \text{ (see, e.g., the proof of [10, Lemma 2.1]).}$$

Since $ax=xb$, N commutes with M, and Theorem 7.4 of [5, page 33] implies that N commutes with $f(M)$. A matrix calculation now shows that $f(a)x=xf(b)$ and the proof is complete.

The following well-known result is usually proved as a corollary of Rosenblum's Theorem [20, Theorem 0.12, page 8]; we give an elementary proof based on Lemma 2.1.

Lemma 2.2. *If* $ax=xb$ *and* $\sigma(a)\cap\sigma(b)=\emptyset$, *then* $x=0$.

Proof. Without loss of generality we may replace a and b, respectively, by $a-\lambda$ and $b-\lambda$, where λ is any complex number, and we may thus assume that a is invertible. Let $f(z)$ be an analytic function such that $f(z)=z$ in a neighborhood of $\sigma(a)$ and $f(z)=0$ in a neighborhood of $\sigma(b)$. Since $f(a)=a$ and $f(b)=0$, Lemma 2.1 implies that $ax=0$, and the invertibility of a implies that $x=0$.

Using Lemma 2.2 and basic properties of the spectral measure of a normal operator, we can prove the following refinement of Lemma 2.2. The proof, which is not needed in the sequel, will be omitted.

Proposition 2.3. *Suppose that* T, X, *and* N *are in* $\mathscr{L}(\mathfrak{H})$, *where* N *is normal and* $TX=XN$ *or* $XT=NX$. *Let* $E(\cdot)$ *denote the spectral measure of* N. *If* $E(\sigma(T))=0$, *then* $X=0$.

We note that the preceding result is also valid if N is a spectral operator. An element e in $\mathscr{A}$ is said to be *idempotent* if $e^2=e$.

Lemma 2.4. *If* $ax=xb$ *and if there exists no non-zero idempotent* e *such that* $xe=0$, *then each non-empty closed-and-open subset of* $\sigma(b)$ *has non-empty intersection with* $\sigma(a)$.

Proof. Suppose that τ is a non-empty closed-and-open subset of $\sigma(b)$ that is disjoint from $\sigma(a)$. Since $\mathscr{A}$ has an identity, $x\neq 0$, and Lemma 2.2 implies that $\tau\neq\sigma(b)$. Thus there exists an analytic function f such that $f(z)=0$ in a neighborhood of $\sigma(a)\cup(\sigma(b)-\tau)$ and $f(z)=1$ in a neighborhood of τ. Then $f(a)=0$ and [5, Prop. 7.9, page 36] implies that $f(b)$ is a non-zero idempotent in $\mathscr{A}$. Lemma 2.1 implies that $0=f(a)x=xf(b)$, and the hypothesis on x implies that $f(b)=0$, which is a contradiction.

Theorem 2.5. *Let* A, B, *and* X *be in* $\mathscr{L}(\mathfrak{H})$. *Suppose that* $AX=XB$, X *is injective, and* P *is a non-zero projection such that* $P\mathfrak{H}$ *is invariant for* B $(P=1$ *is* *permitted). Then each non-empty closed-and-open subset of $\sigma(B|P\mathfrak{H})$ has non-empty intersection with $\sigma(A)$.*

Proof. We may assume from Lemma 2.4 that $P\neq 1$. Suppose that τ is a non-empty closed-and-open subset of $\sigma(B|P\mathfrak{H})$ such that $\tau\cap\sigma(A)=\emptyset$. Let λ be chosen so that $(B-\lambda)|P\mathfrak{H}$ is invertible; since $PBP=BP$, we have $(*)$ $(A-\lambda)(XP)=(XP)(B-\lambda)P$. Let f be an analytic function such that $f(z)=1$ in a neighborhood of $\tau-\lambda$ and $f(z)=0$ in a neighborhood of $\sigma(A-\lambda)\cup(\sigma((B-\lambda)|P\mathfrak{H})-(\tau-\lambda))\cup\{0\}$. (This definition of f is valid since $\tau-\lambda$ is a non-empty closed-and-open subset of $\sigma((B-\lambda)|P\mathfrak{H})$ such that $(\tau-\lambda)\cap\sigma(A-\lambda)=\emptyset$ and $0\notin\sigma((B-\lambda)|P\mathfrak{H})$.) Since $\sigma((B-\lambda)P)=\sigma((B-\lambda)|P\mathfrak{H})\cup\{0\}$, f is defined in a neighborhood of $\sigma(A-\lambda)\cup\cup\sigma((B-\lambda)P)$, and Lemma 2.1 and $(*)$ imply that $f(A-\lambda)(XP)=(XP)f((B-\lambda)P)$. Now $f(A-\lambda)=0$ and $E=f((B-\lambda)P)$ is a non-zero idempotent; thus we have $0=XPE$. Further, [20, Theorem 2.10, page 31] implies that $(B-\lambda)P$ commutes with E, that the range of E is invariant for $(B-\lambda)P$, and that $\sigma((B-\lambda)P|E\mathfrak{H})==\tau-\lambda$. With respect to the decomposition $\mathfrak{H}=P\mathfrak{H}\oplus(1-P)\mathfrak{H}$, the operator matrices of B and P are, respectively,

$$\begin{bmatrix} B_1 & * \\ 0 & * \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the operator matrix of $(B-\lambda)P$ is

$$\begin{bmatrix} B_1-\lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where $B_1-\lambda$ is invertible in $\mathscr{L}(P\mathfrak{H})$. Let

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$$

denote the operator matrix of E. Since E commutes with $(B-\lambda)P$, a calculation shows that $E_2=0$ and $E_3=0$. We claim that $E_4=0$ in $\mathscr{L}((1-P)\mathfrak{H})$. Indeed, if x is a nonzero vector in $(1-P)\mathfrak{H}$ such that $E_4x\neq 0$, then

$$\begin{bmatrix} B_1-\lambda & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & E_4 \end{bmatrix}\begin{bmatrix} 0 \\ x \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and so $0\in\sigma((B-\lambda)P|E\mathfrak{H})=\tau-\lambda$, which is a contradiction since $\lambda\notin\sigma(B|P\mathfrak{H})$. Now $E_4=0$, so we have $PE=E$ and $0=XPE=XE$. Since X is injective, $E=0$, and we have a contradiction which completes the proof.

Remark. If X is non-injective, then the conclusion of Theorem 2.5 is no longer valid; if X is a projection in $\mathscr{L}(\mathfrak{H})$, $X\neq 0, 1$, then $1X=X^2$.

In contrast to Theorem 2.5, it can be shown that quasisimilarity does not preserve the connectedness of spectra. Indeed, HOOVER [15] gives an example of quasisimilar operators A and B such that $\sigma(A)=\{0\}$ while $\sigma(B)$ equals the closed unit disk. Then $A\oplus(A-1/2)$ is quasisimilar to $B\oplus(B-1/2)$; the spectrum of the first operator is disconnected and the spectrum of the second operator is connected.

The analogue of Theorem 2.5 for essential spectra is false. Let U denote a unilateral (unweighted) shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$ and let W_α denote the unilateral weighted shift defined by $\alpha_n=1/n$ for $n\geqq 1$ (see section 4 for notation). Let X denote the injective diagonalizable operator defined by $Xe_n=\beta_n e_n$, where $\beta_1=\beta_2=1$ and $\beta_n=1/(n-1)!$ for $n\geqq 3$. Now $W_\alpha X=XU$; however, $\sigma_e(W_\alpha)$ and $\sigma_e(U)$ are disjoint, since $\sigma_e(W_\alpha)=\{0\}$ and $\sigma_e(U)$ is the unit circle.

Despite the preceding example we have the following perhaps surprising result.

Theorem 2.6. *If A and B are quasisimilar operators in $\mathscr{L}(\mathfrak{H})$, then $\sigma_e(A)$ and $\sigma_e(B)$ have non-empty intersection.*

Before proceding with the proof of Theorem 2.6, the following observation seems pertinent. If X is in $\mathscr{L}(\mathfrak{H})$, and if $\tilde{X}$ is "injective" in the Calkin algebra (i.e., if there exists no non-zero idempotent $\tilde{E}$ in the Calkin algebra such that $\tilde{X}\tilde{E}=0$), then $\tilde{X}$ is left invertible in the Calkin algebra (see [11, Theorem 1.1]); thus if X is also quasi invertible, then X is invertible. This fact implies that if two operators are quasi-similar but not similar, then the intertwining quasi invertible operators are both non-injective in the Calkin algebra. Thus it appears to be difficult to directly adapt the proof of Lemma 2.4 to the setting of the Calkin algebra in order to prove Theorem 2.6.

Our proof of Theorem 2.6 is instead inspired by the techniques and terminology of [19]. We next summarize some of the results and terminology from [19]. Let T be in $\mathscr{L}(\mathfrak{H})$. A subset $H\subset \mathbf{C}$ is said to be a *hole* in $\sigma_e(T)$ if H is a bounded connected component of $\mathbf{C}-\sigma_e(T)$; thus $\mathrm{bdry}(H)\subset\sigma_e(T)$.

Lemma 2.7. *If H is a hole in $\sigma_e(T)$ and $H\cap\sigma(T)$ is uncountable, then $H\subset\sigma(T)$. In this case, if S is quasisimilar to T, then $H\subset\sigma(S)$ and* $\mathrm{bdry}(H)\subset\sigma_e(T)\cap\sigma(S)$. *If H is a component of $\mathbf{C}-\sigma_e(T)$ and $H\cap\sigma(T)$ is finite or countably infinite, then each point of $H\cap\sigma(T)$ is an isolated point of $\sigma(T)$ and an eigenvalue of finite multiplicity; moreover, if K is the unbounded component of $\mathbf{C}-\sigma_e(T)$, then $K\cap\sigma(T)$ is either empty, finite, or countably infinite.*

Proof. The proof follows immediately from the results of [19].

Lemma 2.8. *If A and B are quasisimilar, then each non-empty closed-and open subset of $\sigma_e(B)$ has non-empty intersection with $\sigma(A)$.*

Proof. Let τ be a non-empty closed-and-open subset of $\sigma_e(B)$. If τ is open in $\sigma(B)$, then Theorem 2.5 implies that $\tau\cap\sigma(A)\neq\emptyset$. Otherwise, there exists t in τ, and a sequence $\{t_n\}\subset\sigma(B)-\tau$, such that $t_n\to t$. Since τ is open in $\sigma_e(B)$, we may assume that each t_n is in $\sigma(B)-\sigma_e(B)$. Thus t_n is an eigenvalue of B (and thus of A) for infinitely many n, or $\bar{t}_n$ is an eigenvalue of B^* (and thus of A^*) for infinitely many n. In either case, t is in $\sigma_e(B)\cap\sigma(A)$, and the proof is complete.

Remark. Let X denote a non-empty, bounded, open, connected subset of the complex plane; let $\varphi(X)$ denote the unbounded component of the complement of the closure of X, and let $\beta(X)=\text{bdry}(\varphi(X))$; note that $\beta(X)\subset\text{bdry}(X)$. It is a result of the topology of the plane that $\beta(X)$ is connected [23, Theorem 14.2, page 123]. In particular, if T is in $\mathcal{L}(\mathfrak{H})$ and $\beta(X)\subset\sigma(T)-\sigma_e(T)$, then the connectedness of $\beta(X)$ implies that $\beta(X)$ is contained in some component H of $\mathbf{C}-\sigma_e(T)$; further, since $\beta(X)$ is uncountable, Lemma 2.7 implies that H is a hole in $\sigma_e(T)$.

Lemma 2.9. *If A and B are quasisimilar operators in $\mathcal{L}(\mathfrak{H})$, and if there exists a hole H_0 in $\sigma_e(A)$ such that $H_0\subset\sigma(A)$, then $\sigma_e(A)\cap\sigma_e(B)\neq\emptyset$.*

Proof. Suppose to the contrary that $\sigma_e(A)$ and $\sigma_e(B)$ are disjoint. Since $H_0\subset\subset\sigma(A)$, then $H_0\subset\sigma(B)$, and thus $\beta(H_0)\subset\sigma_e(A)\cap\sigma(B)\subset\sigma(B)-\sigma_e(B)$. The above Remark implies that there exists a hole K_1 in $\sigma_e(B)$ such that $\beta(H_0)\subset K_1$, and it follows by a connectedness argument that $\varphi(K_1)^-\subset\varphi(H_0)$. Now $\beta(K_1)$ is an uncountable connected subset of $\sigma_e(B)$; thus, as above, there exists a hole H_1 in $\sigma_e(A)$ such that $\beta(K_1)\subset H_1$, and we also have $\varphi(H_1)^-\subset\varphi(K_1)$. Moreover, H_1 and H_0 are disjoint; indeed, otherwise H_1 and H_0 (components of $\mathbf{C}-\sigma_e(A)$) are equal, and since $\beta(K_1)\subset H_1$, it follows that there is a point in $\varphi(K_1)\cap H_1=$ $=\varphi(K_1)\cap H_0$. Since $\varphi(K_1)\subset\varphi(H_0)\subset\mathbf{C}-H_0$, we have a contradiction, and thus $H_1\cap H_0=\emptyset$.

The above procedure may now be used to inductively define two sequences $\{H_i\}$ $(i\geqq 0)$ and $\{K_i\}$ $(i\geqq 1)$ such that:

i) H_i is a hole in $\sigma_e(A)$; $\beta(H_i)\subset\sigma_e(A)$ $(i\geqq 0)$;

ii) K_i is a hole in $\sigma_e(B)$; $\beta(K_i)\subset\sigma_e(B)$ $(i>0)$;

iii) $\beta(H_i)\subset K_{i+1}$, $\beta(K_{i+1})\subset H_{i+1}$ $(i\geqq 0)$;

iv) $\varphi(H_i)^-\subset\varphi(K_i)$, $\varphi(K_i)^-\subset\varphi(H_{i-1})$ $(i\geqq 1)$;

v) $K_i\cap K_j=\emptyset$, $H_i\cap H_j=\emptyset$ for all $i\neq j$.

Now iii) and iv) imply that $\beta(H_i)\cap\beta(H_j)=\emptyset$ for all $i\neq j$. Let $\{h_i\}$ $(i\geqq 0)$ denote a sequence such that h_i is in $\beta(H_i)$ for $i\geqq 0$. Since these points are distinct, there exists a convergent subsequence $h_{i_k}\to h$, and i) implies that h is in $\sigma_e(A)$. Since $i_k>i_{k-1}$, iv) implies that $\varphi(H_{i_k})\subset\varphi(K_{i_k})\subset\varphi(H_{i_k-1})\subset\ldots\subset\varphi(H_{i_{k-1}})$; now if L denotes the line segment from h_{i_k} to $h_{i_{k-1}}$, then L contains a point g_{i_k} from $\beta(K_{i_k})$. Since $|g_{i_k}-h|\leqq|g_{i_k}-h_{i_k}|+|h_{i_k}-h_{i_{k-1}}|+|h_{i_{k-1}}-h|\leqq 2|h_{i_k}-h_{i_{k-1}}|+|h_{i_{k-1}}-h|$, it follows that $g_{i_k} \to h$. Now ii) implies that h is in $\sigma_e(B)$. Since h is also in $\sigma_e(A)$, we have a contradiction, which completes the proof.

Lemma 2.10. *If A and B are quasisimilar, and if there exists an infinite sequence $\{z_n\}$ of distinct isolated points of $\sigma(A)$ such that* $\dim(\ker(A-z_n))>0$ *or* $\dim(\ker((A-z_n)^*))>0$ *for each n, then $\sigma_e(A)\cap\sigma_e(B)\neq\emptyset$.*

Proof. Since A and B are quasisimilar, $\{z_n\}\subset\sigma(B)$; by passing, if necessary, to a subsequence, we may assume that $z_n\to z$, where z is in $\sigma(B)$. Since z is an accumulation point of bdry $(\sigma(A))$, [19, Corollary 1.26] implies that z is in $\sigma_e(A)$, and we claim that z is also in $\sigma_e(B)$. For otherwise, since z is in $\sigma(B)-\sigma_e(B)$ and z is not an isolated point of $\sigma(B)$, Lemma 2.7 implies that there exists an open disk D centered at z, such that $B-w$ or $(B-w)^*$ is non-injective for each w in D. Since $D\subset\sigma(A)$, and since there exists some z_n in D, it follows that z_n is not an isolated point of $\sigma(A)$, which is a contradiction. Thus z is in $\sigma_e(A)\cap\sigma_e(B)$, and with the proof is complete.

Lemma 2.11. *Let A, B, and X be in $\mathscr{L}(\mathfrak{H})$, with X injective and $AX=XB$. If H is a component of $\mathbf{C}-\sigma_e(A)$ such that $K=H\cap\sigma(B)$ is a non-empty closed-and-open subset of $\sigma(B)$, and $K\cap\sigma_e(B)\neq\emptyset$, then $H\subset\sigma(A)$.*

Proof. The hypothesis implies that K is a closed subset of the open set H; thus there exists an open set U such that $K\subset U\subset U^-\subset H$. If we assume that $H\not\subset\sigma(A)$, then Lemma 2.7 implies that H contains no limit points of $\sigma(A)$; in particular, $L=U\cap\sigma(A)$ is a finite set. Since U contains no limit points of $\sigma(A)$, L is an open subset of $\sigma(A)$. Since K is a non-empty closed-and-open subset of $\sigma(B)$, and $L\supset K\cap\sigma(A)$, Lemma 2.4 implies that L is non-empty: moreover, since $L\cap\sigma_e(A)=\emptyset$, then $L\neq\sigma(A)$.

Thus K and L are, respectively, non-empty closed-and-open subsets of $\sigma(B)$ and $\sigma(A)$. Now there exists an analytic function f such that $f(z)=1$ in a neighborhood of $K\cup L$, and $f(z)=0$ in a neighborhood of $(\sigma(A)-L)\cup(\sigma(B)-K)$. As in the proof of Theorem 2.5, $f(A)$ is an idempotent commuting with A, $\sigma(A|f(A)\mathfrak{H})= =L$, and $\sigma(A|(1-f(A))\mathfrak{H})=\sigma(A)-L$. Since each idempotent operator in $\mathscr{L}(\mathfrak{H})$ is similar to an orthogonal projection, there exists an invertible operator J such that $P=J^{-1}f(A)J$ is an orthogonal projection; then $R=J^{-1}AJ$ commutes with P, and $R|P\mathfrak{H}$ is similar to $A|f(A)\mathfrak{H}$. We assert that $P\mathfrak{H}$ is finite dimensional; otherwise, $\sigma_e(R|P\mathfrak{H})$ is a nonempty subset of $\sigma(R|P\mathfrak{H})=\sigma(A|f(A)\mathfrak{H})=L$. Since $R|P\mathfrak{H}$ is a direct summand of R, it follows that some point of L is in $\sigma_e(R)=\sigma_e(A)$, which is a contradiction.

Since $AX=XB$, Lemma 2.1 implies that $f(A)X=Xf(B)$. Since P has finite rank, so does $f(A)$, and since X is injective it follows that $f(B)$ also has finite rank. In particular, $f(B)\neq 1$ and so $K\neq\sigma(B)$. Now $f(B)$ is a nontrivial idempotent that commutes with B. Proceeding as above, there exists an invertible operator M such that $Q=M^{-1}f(B)M$ is an orthogonal projection, Q commutes with $S=M^{-1}BM$, $\sigma(S|Q\mathfrak{H})=K$, and $\sigma(S|(1-Q)\mathfrak{H})=\sigma(B)-K$ (since $S|Q\mathfrak{H}$ is similar to $B|f(B)\mathfrak{H}$ and $S|(1-Q)\mathfrak{H}$ is similar to $B|(1-f(B))\mathfrak{H}$). If z is in $K\cap\sigma_e(B)$, then with respect to the orthogonal decomposition $\mathfrak{H}=Q\mathfrak{H}\oplus(1-Q)\mathfrak{H}$, we have $S-z=$ $=((1|Q\mathfrak{H})\oplus((S-z)|(1-Q)\mathfrak{H}))+(((S-z-1)|Q\mathfrak{H})\oplus(0|(1-Q)\mathfrak{H}))$. Since the first term on the right is invertible, while the second term in the sum is a finite rank operator, it follows that $S-z$ is a Fredholm operator, which contradicts the assumption that z is in $\sigma_e(B)=\sigma_e(S)$. Thus $H\subset\sigma(A)$, and the proof is complete.

Proof of Theorem 2.6. By Lemma 2.9 we may assume that if there exists a hole H in $\sigma_e(A)$, then $H\not\subset\sigma(A)$, for otherwise the proof is complete. Moreover, we may assume from Lemmas 2.7 and 2.10 that $H\cap\sigma(A)$ is at most finite, and that if K denotes the unbounded component of $\mathbf{C}-\sigma_e(A)$, then $K\cap\sigma(A)$ is at most finite. Let $X=\sigma_e(B)\cap\sigma(A)$; Lemma 2.8 implies that X is non-empty. If we assume that $X\cap\sigma_e(A)=\emptyset$, then there exists a component H of $\mathbf{C}-\sigma_e(A)$ such that $X\cap H\neq\emptyset$; from the preceding remarks we may assume that $H\cap\sigma(A)$ is a finite set. Since $(\sigma_e(B)\cap H)^-\cap\operatorname{bdry}(H)\subset\sigma_e(B)\cap\sigma_e(A)$, we may assume that there is an open set U such that $\sigma_e(B)\cap H\subset U\subset U^-\subset H$; in particular, $Y=\sigma_e(B)\cap H$ is a closed subset of $\sigma(B)$.

We assert that Y is also an open subset of $\sigma(B)$; indeed, if Y is not open in $\sigma(B)$, then there exists an infinite sequence of distinct points $\{z_n\}\subset\sigma(B)-Y$ such that $z_n\to z$, where z is some point in Y. We may assume (excluding at most a finite number of points) that each z_n is in U; thus each z_n is in $\sigma(B)-\sigma_e(B)\subset$ $\subset\sigma(A)$. Now each z_n is in $H\cap\sigma(A)$, which contradicts the fact that $H\cap\sigma(A)$ is finite. Thus Y is a non-empty closed-and-open subset of $\sigma(B)$, and Lemma 2.11 implies that $H\subset\sigma(A)$, which also contradicts the fact that $H\cap\sigma(A)$ is finite. The proof is now complete.

Remark. In a preliminary version of this paper, the author was unable to prove Theorem 2.6, and instead posed it as a question. L. R. Williams, meanwhile, independently found a somewhat different proof of Theorem 2.6, which will appear in his note [22].

Corollary 2.12. *Let A, B, and X be in $\mathscr{L}(\mathfrak{H})$ with X injective and $AX=XB$. If S is a part of B and S is decomposable, then $\sigma(S)\subset\sigma(A)$.*

Proof. Let $S=B|\mathscr{L}$, where $\mathscr{L}\neq\{0\}$ and $B\mathscr{L}\subset\mathscr{L}$. If $\sigma(S)\not\subset\sigma(A)$, then there exists an open subset $U\subset\mathbf{C}$ such that $U\cap\sigma(S)\neq\emptyset$ and $U\cap\sigma(A)=\emptyset$. Since S is decomposable, [7, Lemma 1.2, page 30] implies that there exists an S-invariant closed subspace $\mathfrak{M}\subset\mathscr{L}$ such that $\mathfrak{M}\neq\{0\}$ and $\sigma(S|\mathfrak{M})\subset U$. Since $\mathfrak{M}\subset\mathfrak{H}$ is also invariant for B and $\sigma(A)\cap\sigma(B|\mathfrak{M})\subset\sigma(A)\cap U=\emptyset$, we have a contradiction to Theorem 2.5, and the proof is complete.

3. On quasisimilarity and quasinilpotent operators. In this section we give some properties of operators in $\mathcal{Q}_{qs}$. An operator T in $\mathcal{L}(\mathfrak{H})$ is called a *quasiaffine transform* of the operator S if there exists a quasi-invertible operator X in $\mathcal{L}(\mathfrak{H})$ such that $XT=SX$. Let $\mathcal{Q}_{af}=\{T\in\mathcal{L}(\mathfrak{H})$: T is a quasiaffine transform of some quasinilpotent operator$\}$ and let $\mathcal{Q}_{af}^*=\{T\in\mathcal{L}(\mathfrak{H}):T^*\in\mathcal{Q}_{af}\}$; thus $\mathcal{Q}_{qs}\subset\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$.

Theorem 3.1. *If T is in $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$, then T satisfies the following properties:*

i) *If P is a non-zero projection such that $(1-P)TP=0$, then $\sigma(T|P\mathfrak{H})$ is connected and contains* 0; *if additionally $P\neq 1$, then $\sigma((1-P)T|(1-P)\mathfrak{H})$ is connected and contains* 0.

ii) $\sigma(T)-\{0\}\subset\{\lambda\in\mathbf{C}$: $T-\lambda$ and $(T-\lambda)^*$ are injective$\}$.

iii) *If $\lambda\neq 0$ and $T-\lambda$ is semi-Fredholm, then $T-\lambda$ is invertible.*

iv) $\sigma(T)=\sigma_e(T)$.

v) *T is bi-quasitriangular.*

Proof. Let Q and R be quasinilpotent operators and let X and Y be quasi-invertible operators such that $QX=XT$ and $RY=YT^*$.

i) If $P\neq 0$ and $(1-P)TP=0$, then since X is injective, Theorem 2.5. implies that $\sigma(T|P\mathfrak{H})$ is connected and contains 0. If $P\neq 1$, then since $(1-P)\mathfrak{H}$ is invariant for T^* and Y is injective, $\sigma(T^*|(1-P)\mathfrak{H})$ is connected and contains 0. Since $\sigma((1-P)T|(1-P)\mathfrak{H})=\{\lambda\in\mathbf{C}$: $\bar{\lambda}\in\sigma(T^*|(1-P)\mathfrak{H})\}$, the proof is complete.

ii) Since $(Q-\lambda)X=X(T-\lambda)$, $(R-\bar{\lambda})Y=Y(T-\lambda)^*$, and $\sigma(Q)=\sigma(R)=\{0\}$, it is clear that if $\lambda\neq 0$, then $T-\lambda$ and $(T-\lambda)^*$ are injective.

iii) If $T-\lambda$ is semi-Fredholm but not invertible, then either $T-\lambda$ or $(T-\lambda)^*$ is non-injective, so the result follows from ii).

iv) Since $\sigma_e(T)$ is a non-empty subset of $\sigma(T)$, we may assume that T is not quasinilpotent. It is clear from iii) that each non-zero member of $\sigma(T)$ is in $\sigma_e(T)$; now i) implies that 0 is a limit point of $\sigma_e(T)$ and so 0 is in $\sigma_e(T)$.

v) For each vector h in $\mathfrak{H}$, we have $\|T^nY^*h\|^{1/n}=\|Y^*R^{*n}h\|^{1/n}\leqq\|Y^*\|^{1/n}\cdot\|R^{*n}\|^{1/n}\|h\|^{1/n}\to 0$. Since Y^* has dense range, Theorem 3.1 of [1] implies that T is quasitriangular. A similar argument, using the equation $T^{*n}X^*=X^*Q^{*n}$, implies that T^* is quasitriangular.

Corollary 3.2. *If T is in $\mathcal{Q}_{qs}$, then T satisfies properties* i) — v) *of Theorem* 3.1.

Corollary 3.3. *If T is in $\mathcal{Q}_{af}$ and S is a part of T that is decomposable, then S is quasinilpotent.*

Proof. The result follows from Corollary 2.12 or Theorem 3.1—i).

Corollary 3.4. *If T is a decomposable operator in $\mathcal{Q}_{af}$, then is quasinilpotent.*

Corollary 3.5. *If T is in $\mathcal{Q}_{af}$ and S is a part of T that is normal, then $S=0$.*

Theorem 3.6. *If T is a hyponormal operator in $\mathcal{Q}_{af}^{*}$, then $T=0$.*

Proof. Theorem 1 of [6] implies that if $XA=TX$ and X has dense range, then $\sigma(T)\subset\sigma(A)$; thus, if A is quasinilpotent, then so is T. Now [20, Proposition 1.8, page 24] implies that $\|T\|=r(T)=0$.

Question 3.7. Which injective weighted shifts are in $\mathcal{Q}_{qs}$? This question, which we are unable to answer, motivated the results of section 4. Theorem 4.8 implies that if an injective weighted shift W is quasisimilar to a quasinilpotent injective weighted shift, then W is quasinilpotent.

Corollary 3.8. *$\mathcal{Q}_{qs}$ is a proper subset of $\mathcal{N}^{-}$.*

Proof. Theorem 3.1 implies that if T is in $\mathcal{Q}_{qs}$, then T is bi-quasitriangular and that $\sigma(T)$ and $\sigma_e(T)$ are connected and contain 0. Now [4] implies that T is in $\mathcal{N}^{-}$. Theorem 7 of [14] implies that $\mathcal{N}^{-}$ contains non-zero normal operators, while Corollary 3.5 implies that there are no non-zero normal operators in $\mathcal{Q}_{qs}$; therefore, $\mathcal{Q}_{qs}$ is a proper subset of $\mathcal{N}^{-}$.

Question 3.9. Is the converse of Corollary 3.2 true?

We note that if T is a noninvertible operator in $\mathcal{L}(\mathfrak{H})$, and if T fails to satisfy properties i) — v) of Theorem 3.1, then T has a nontrivial invariant subspace; moreover, if T fails to satisfy properties ii) — v), then T has a nontrivial hyperinvariant subspace. (These observations are easy to prove except with regard to property v); the fact that a non-bi-quasitriangular operator has a nontrivial hyperinvariant subspace is a result of [3].) Thus, if the converse of Corollary 3.2 is true, and if each quasinilpotent operator does have a nontrivial hyperinvariant subspace, then each operator has a nontrivial invariant subspace. It is therefore of interest to determine whether the converse of Corollary 3.2 is true; we will show in Theorem 3.11 that as regards the topology of the spectra of operators in $\mathcal{Q}_{qs}$, Corollary 3.2 is indeed "best possible".

Proposition 3.10. *$\mathcal{Q}_{qs}$ is closed under countable direct sums.*

Proof. Let $\mathfrak{H}_i$ denote a separable Hilbert space $(i=1,2,\dots)$, and let T_i be in $\mathcal{Q}_{qs}$ with respect to $\mathcal{L}(\mathfrak{H}_i)$. We seek to prove that if $\{\|T_i\|\}$ is bounded, then $T=\Sigma\oplus T_i$ is in $\mathcal{Q}_{qs}$ with respect to $\mathcal{L}(\mathfrak{H})$, where $\mathfrak{H}=\Sigma\oplus\mathfrak{H}_i$.

For each $i>0$, T_i is quasisimilar to a quasinilpotent operator Q_i in $\mathcal{L}(\mathfrak{H}_i)$; Rota's Theorem [20, Proposition 3.12, page 58] implies that there exists an operator

P_i in $\mathscr{L}(\mathfrak{H}_i)$ such that P_i is similar to Q_i and $\|P_i\|<1/i$. Now [15, Theorem 2.5] implies that T is quasisimilar to $S=\Sigma\oplus P_i$, so it suffices to prove that S is quasinilpotent. Let λ be a non-zero complex number and let n be a positive integer such that $1/n<|\lambda|$. For $i>n$, $\|P_i\|<1/i<1/n<|\lambda|$, and therefore

$$\|(P_i-\lambda)^{-1}\| \leqq (|\lambda|-\|P_i\|)^{-1} < (|\lambda|-1/i)^{-1} < (|\lambda|-1/n)^{-1}.$$

Now $\sup_{i\in N}\|(P_i-\lambda)^{-1}\|\leqq\max\left(\sup_{1\leqq i\leqq n}\|(P_i-\lambda)^{-1}\|, (|\lambda|-1/n)^{-1}\right)<\infty$, and hence $\lambda\notin\sigma(S)$.

Remark. In [13, Theorem 1.1] it is proved that if T is a quasinilpotent operator on $\mathfrak{H}$, then there exists a compact, quasinilpotent backward weighted shift K in $\mathscr{L}(\mathfrak{H})$ and a closed subspace $\mathfrak{M}\subset\mathscr{L}=\mathfrak{H}\oplus\cdots\oplus\mathfrak{H}\oplus\cdots$, such that

i) $\mathfrak{M}$ is invariant for $L=K\oplus\cdots\oplus K\oplus\cdots$;
ii) T is similar to $L|\mathfrak{M}$;
iii) $\|L|\mathfrak{M}\|\leqq\|T\|$ (see [13, Theorem 1.1, inequality 11]).

Using this result and the method of the proof of Proposition 3.10, it is not difficult to prove the following analogue for direct sums operators in $\mathcal{Q}_{qs}$: let $T=\Sigma\oplus T_i$, where T_i is in $\mathcal{Q}_{qs}$ with respect to $\mathfrak{H}_i$, and let $\mathfrak{H}=\Sigma\oplus\mathfrak{H}_i$. Then there exists a compact, quasinilpotent operator K on $\mathfrak{H}$, of arbitrarily small norm, and a closed subspace $\mathfrak{M}\subset\mathscr{L}=\mathfrak{H}\oplus\cdots\oplus\mathfrak{H}\oplus\cdots$, such that
i) $\mathfrak{M}$ is invariant for $L=K\oplus K\oplus\cdots\oplus K\oplus\cdots$;
ii) T is quasisimilar to $L|\mathfrak{M}$.

Theorem 3.11. *A subset $X\subset\mathbf{C}$ is the spectrum of an operator in $\mathcal{Q}_{qs}$ if and only if X is compact, connected, and contains* 0.

Proof. Let X denote a compact, connected subset of the plane that contains 0. Theorem 3.2 of [10] implies that there exists an operator T in $\mathscr{L}(\mathfrak{H})$ such that T is a direct sum of nilpotent operators and $\sigma(T)=X$; Proposition 3.10 implies that T is in $\mathcal{Q}_{qs}$.

The converse is contained in Theorem 3.1—i).

Remark. The proof of Theorem 3.11 did not require the full force of Proposition 3.10, but only the fact that each countable direct sum of nilpotent operators is in $\mathcal{Q}_{qs}$. Using [2, Theoreml] (or [21, Theoreml]), it is not difficult to prove that each countable direct sum of nilpotent operators is quasisimilar to some compact, quasinilpotent operator. On the other hand, not every quasinilpotent operator is quasisimilar to a compact operator (see [13, Prop. 1.5]).

We conclude this section with an additional necessary condition for membership in $\mathcal{Q}_{qs}$. For T in $\mathscr{L}(\mathfrak{H})$, let $\mathfrak{M}(T)=\{x\in\mathfrak{H}: \|T^n x\|^{1/n}\to 0\}$. It is easy to prove that $\mathfrak{M}(T)$ is a linear subspace of $\mathfrak{H}$ and that $\mathfrak{M}(T)^-$ is a (possibly trivial) hyperinvariant subspace for T. For example, if U denotes a unilateral shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$, then $\mathfrak{M}(U)=\{0\}$, and since $\mathfrak{M}(U^*)$ contains an orthonormal basis for $\mathfrak{H}$, then $\mathfrak{M}(U^*)^-=\mathfrak{H}$.

Lemma 3.12. *If T is in $\mathfrak{Q}^*_{af}$, then there exists an orthonormal basis $\{e_k\}$ $(1\leqq k<\infty)$ for $\mathfrak{H}$ such that for each k, $\lim_{n\to\infty}\|T^n e_k\|^{1/n}=0$.*

Proof. Suppose that $XQ=TX$, where X is quasi-invertible and Q is in $\mathfrak{Q}$. For each t in $\mathfrak{H}$, we have $\|T^n Xt\|^{1/n}=\|XQ^n t\|^{1/n}\leqq\|X\|^{1/n}\|Q^n\|^{1/n}\|t\|^{1/n}\to 0$. Theorem 1.1 of [12] and the remarks of [12, page 280] imply that for S in $\mathscr{L}(\mathfrak{H})$, $S\mathfrak{H}$ contains an orthonormal basis for $(S\mathfrak{H})^-$. Since X has dense range, $X\mathfrak{H}$ contains an orthonormal basis for $\mathfrak{H}$, and since $X\mathfrak{H}\subset\mathfrak{M}(T)$, the proof is complete.

Proposition 3.13. *If T is in $\mathfrak{Q}_{qs}$, then $\mathfrak{M}(T)$ and $\mathfrak{M}(T^*)$ contain orthonormal bases for $\mathfrak{H}$; in particular, $\mathfrak{M}(T)^-=\mathfrak{M}(T^*)^-=\mathfrak{H}$.*

Question 3.14. Is the converse of Proposition 3.13 true? It is known that if I is in $\mathscr{L}(\mathfrak{H})$ and $\mathfrak{M}(T)=\mathfrak{H}$, then T is quasinilpotent (see [7, Lemma, page 28]).

Proposition 3.13 is related to Theorem 3.1 by the following result.

Proposition 3.15. *If T is in $\mathscr{L}(\mathfrak{H})$ and $\mathfrak{M}(T)^-=\mathfrak{M}(T^*)^-=\mathfrak{H}$, then T satisfies properties* i)—v) *of Theorem* 3.1.

Proof. Since $\mathfrak{M}(T)^-=\mathfrak{M}(T^*)^-=\mathfrak{H}$, Theorem 3.1 of [1] implies that T is bi-quasitriangular.

Let P be a non-zero projection such that $(1-P)TP=0$ and denote the operator matrix of T with respect to the decomposition $\mathfrak{H}=P\mathfrak{H}\oplus(1-P)\mathfrak{H}$ by

$$\begin{pmatrix} S & A \\ 0 & B \end{pmatrix}.$$

We will first show that $\sigma(S)$ contains 0. If S is invertible, then so is S^*, and there exists $\varepsilon>0$ such that $\|S^*x\|\geqq\varepsilon\|x\|$ for each x in $P\mathfrak{H}$. If z is in $\mathfrak{H}$, then $z=x+y$, where x is in $P\mathfrak{H}$ and y is in $(1-P)\mathfrak{H}$. Now we have $\|T^{*n}z\|^{1/n}\geqq\|S^{*n}x\|^{1/n}\geqq$ $\geqq\varepsilon\|x\|^{1/n}$, which implies that $\mathfrak{M}(T^*)\subset(1-P)\mathfrak{H}$. Since $\mathfrak{M}(T^*)$ is dense, this contradiction implies that $0\in\sigma(S)$; a similar argument, using the relation $\mathfrak{M}(T)^-=$ $=\mathfrak{H}$, implies that if S is a part of T^*, then S is noninvertible. In particular, T and T^* have no non-zero eigenvalues, and thus T satisfies ii)—iv).

To complete the proof we must show that if S is a part of T, then $\sigma(S)$ is connected. Since $0\in\sigma(S)$, if $\sigma(S)$ is not connected, then there exists a non-empty, closed-and-open subset $\tau\subset\sigma(S)$ such that $0\notin\tau$. If E denotes the spectral idempotent for S associated with τ, then $\sigma(T|E\mathfrak{H})=\sigma(S|E\mathfrak{H})=\tau$, which contradicts the fact that $T|E\mathfrak{H}$ is noninvertible.

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Question 3.16. Is $\mathcal{Q}_{qs}=\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$?

If the answer to Question 3.9 is affirmative, then it is clear from Proposition 3.15 that the answers to Questions 3.14 and 3.16 would also be affirmative.

4. Quasisimilarity of weighted shifts. In this section we give necessary and sufficient conditions for two injective weighted shifts to be quasisimilar, and we prove that quasisimilar injective weighted shifts have equal spectra. Several authors have considered cases in which quasisimilarity of two operators implies their similarity or the equality of their spectra. Let S, T, and X be in $\mathcal{L}(\mathfrak{H})$ with X quasi-invertible and $SX=XT$. In [6, Theorem 4.4, page 55], Colojoară and Foiaş proved that if S and T are decomposable, then $\sigma(S)=\sigma(T)$. Each normal operator is decomposable [6, Example 1.6—ii, p. 33], and in [8] Douglas proved that if S and T are normal, then S is unitarily equivalent to T. Concerning operators that are not necessarily decomposable, Hoover [15, Theorem 3.1.] proved that if S and T are quasisimilar isometries, then S is unitarily equivalent to T; Clary [6, Theorem 2] proved that if S and T are quasisimilar hyponormal operators, then $\sigma(S)=\sigma(T)$.

Let $I=\mathbf{Z}$ or $\mathbf{Z}^+$ and let $\alpha=\{\alpha_n\}$ $(n\in I)$ denote a bounded sequence of non-zero complex numbers. An operator T in $\mathcal{L}(\mathfrak{H})$ is said to be an *(injective) weighted shift with weight sequence* α if there exists an orthonormal basis $\{e_n\}$ $(n\in I)$ for $\mathfrak{H}$ such that $Te_n=\alpha_n e_{n+1}$ $(n\in I)$. If $I=\mathbf{Z}^+$, T is a *unilateral* shift, while if $I=\mathbf{Z}$, T is a *bilateral* shift.

In [17, Appendix] Lambert proved that if S and T are quasisimilar injective unilateral weighted shifts, then S and T are similar. In the sequel we therefore consider only bilateral weighted shifts; thus we set $I=\mathbf{Z}$ and let $\{e_n\}$ $(n\in\mathbf{Z})$ denote a fixed orthonormal basis for $\mathfrak{H}$. Let W_α denote the bilateral shift with weight sequence α corresponding to this basis. It T is a bilateral shift in $\mathcal{L}(\mathfrak{H})$ with weight sequence α, then T is unitarily equivalent to W_α; moreover, W_α is unitarily equivalent to W_β, where $\beta_n=|\alpha_n|$ $(n\in\mathbf{Z})$. Thus, for questions concerning quasisimilarity of injective bilateral weighted shifts, it suffices to consider shifts of the form W_α, where $\alpha_n>0$ $(n\in\mathbf{Z})$, and in the sequel we implicitly assume that the shifts are of this form.

Lemma 4.1. *The following are equivalent for shifts* W_α *and* W_β:
i) *There exists an integer* k *such that*

$$\sup_{i \geqq \max(1-k,1)} (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) < \infty$$

and

$$\sup_{i \geqq \max(1-k,1)} (\beta_{-1} \dots \beta_{-(i+k)})/(\alpha_{-1} \dots \alpha_{-i}) < \infty.$$

ii) *There exists a quasi-invertible operator* X *such that* $W_\alpha X = X W_\beta$.

Proof. Suppose that there is an integer k such that i) is satisfied. We consider five cases for the values of k and define X in each case by giving the values of X on the basis vectors.

Case 1. If $k \geqq 2$ we set

a) $Xe_i = (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) e_{i+k}$ for $i \geqq 1$;

b) $Xe_0 = \alpha_0 \dots \alpha_{k-1} e_k$;

c) $Xe_i = (\beta_i \dots \beta_{-1} \alpha_0 \dots \alpha_{k+i-1}) e_{k+i}$ for $-k+1 \leqq i \leqq -1$;

d) $Xe_{-k} = \beta_{-k} \dots \beta_{-1} e_0$;

e) $Xe_{-(k+i)} = (\beta_{-(k+i)} \dots \beta_{-1})/(\alpha_{-i} \dots \alpha_{-1}) e_{-i}$ for $i \geqq 1$.

Case 2. If $k=1$ equation c) may be deleted.

Case 3. If $k=0$, equations b)—d) may be replaced by the equation $Xe_0 = e_0$.

Case 4. If $k \leqq -2$ we set

a) $Xe_i = (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) e_{i+k}$ for $i \geqq 1-k$;

b) $Xe_{-k} = 1/(\beta_0 \dots \beta_{-k-1}) e_0$;

c) $Xe_{-k-i} = 1/(\alpha_{-i} \dots \alpha_{-1} \beta_0 \dots \beta_{-k-i-1}) e_{-i}$ for $1 \leqq i \leqq -(k+1)$;

d) $Xe_0 = 1/(\alpha_k \dots \alpha_{-1}) e_k$;

e) $Xe_{-i-k} = (\beta_{-(i+k)} \dots \beta_{-1})/(\alpha_{-i} \dots \alpha_{-1}) e_{-i}$ for $i \geqq 1-k$.

Case 5. If $k=-1$ equation c) of case 4) may be deleted. Condition i) implies that X may be extended to a quasi-invertible operator X in $\mathcal{L}(\mathfrak{H})$, and a calculation shows that $W_\alpha X = X W_\beta$.

For the converse, let X denote a quasi-invertible operator such that $W_\alpha X = X W_\beta$, and denote the matrix of X with respect to the basis (e_n) $(n \in Z)$ by (x_{ij}) $(-\infty < i, j < \infty)$. X has dense range, so there exists an integer m such that $x_{0,m} \neq 0$. An easy matrix calculation shows that for each pair of integers i and j we have (*) $\alpha_{i-1} x_{i-1,j-1} = x_{ij} \beta_{j-1}$. Successive application of (*) gives the identity (**)

$x_{-i,m-i}=x_{0,m}(\beta_{m-1}\cdots\beta_{m-i})/(\alpha_{-1}\cdots\alpha_{-i})$ for $i\geqq 1$. We consider the case $m\leqq 0$; if we set $k=-m$, then for $i\geqq 1$ we have

$$(\beta_{-1}\dots\beta_{-(i+k)})/(\alpha_{-1}\dots\alpha_{-i})=(\beta_{-1}\dots\beta_{-k})(\beta_{m-1}\dots\beta_{m-i})/(\alpha_{-1}\dots\alpha_{-i})=$$
$$=(\beta_{-1}\dots\beta_{-k})x_{-i,m-i}/x_{0,m}\leqq\|W_\beta\|^k\|X\|/x_{0,m}.$$

Now (*) also implies that (***) $x_{i,m+i}=(\alpha_{i-1}\cdots\alpha_0)/(\beta_{m+i-1}\cdots\beta_m)x_{0,m}$ for $i\geqq 1$, and therefore

$$(\alpha_0\dots\alpha_{k+i-1})/(\beta_0\dots\beta_{i-1})=(x_{k+i,i}/x_{0,m})(\beta_m\dots\beta_{-1})\leqq\|X\|\,\|W_\beta\|^k/x_{0,m},$$

which completes the proof when $m\leqq 0$. The proof for the case $m>0$ may be given similarly, by dividing (**) and (***) by $(\beta_0\cdots\beta_{m-1})$.

Theorem 4.2. *The following are equivalent for shifts W_α and W_β:*

i) *W_α is quasisimilar to W_β;*

ii) *There exists an integer k such that*

$$\sup_{i\geqq\max(1,1-k)}(\alpha_0\dots\alpha_{i-1+k})/(\beta_0\dots\beta_{i-1})<\infty$$

and

$$\sup_{i\geqq\max(1,1-k)}(\beta_{-1}\dots\beta_{-(i+k)})/(\alpha_{-1}\dots\alpha_{-i})<\infty,$$

and there exists an integer m such that

$$\sup_{i\geqq\max(1,1-m)}(\beta_0\dots\beta_{i-1+m})/(\alpha_0\dots\alpha_{i-1})<\infty,$$

$$\sup_{i\geqq\max(1,1-m)}(\alpha_{-1}\dots\alpha_{-(i+m)})/(\beta_{-1}\dots\beta_{-i})<\infty.$$

We state for ease of reference the following result concerning similarity of bilateral shifts.

Theorem 4.3. (Kelley [16]) *The shifts W_α and W_β are similar if and only if there exist an integer k and constants M and N such that*

$$0<M\leqq\prod_{j=0}^{n-1}(\alpha_{j+k}/\beta_j)\leqq N<\infty\quad for\quad n>0$$

and

$$0<M\leqq\prod_{j=1}^{-n}(\beta_{-j}/\alpha_{-j+k})\leqq N<\infty\quad for\quad n<0.$$

The next example shows that there exist shifts W_α and W_β that are quasisimilar but not similar.

Example 4.4. Let α be defined by $\alpha_n=1/2^{2n}$ for $n\geqq 0$ and $\alpha_n=1$ for $n<0$; let β be defined by $\beta_n=1/2^{2n-1}$ for $n\geqq 0$ and $\beta_n=1$ for $n<0$. With the values $k=0$ and $m=1$, α and β satisfy the inequalities of Theorem 4.2 ii), and thus W_α is quasisimilar to W_β.

If W_α is similar to W_β, let k, M, and N be as in Theorem 4.3. If $k \geqq 0$ and $n > 0$, then

$$0 < M \leqq (\alpha_k \dots \alpha_{n-1+k})/(\beta_0 \dots \beta_{n-1}) \leqq (\alpha_0 \dots \alpha_{n-1})/(\beta_0 \dots \beta_{n-1})$$

$=1/2^n$; if $k<0$ and $n>-k$, then

$$\infty > N \geqq (\alpha_k \dots \alpha_{n-1+k})/(\beta_0 \dots \beta_{n-1+k} \dots \beta_{n-1}) =$$
$$= 1/(2^{n+k}\beta_{n+k} \dots \beta_{n-1}) > 1/(2^{n+k}\beta_{n+k}) = 2^{n+k-1}.$$

In either case, since n is arbitrary, we have a contradiction, and Theorem 4.3 implies that W_α is not similar to W_β.

In Theorem 4.8 (below) we prove that quasisimilar shifts have equal spectra. We now show that this equality of spectra is not a consequence of the results of [6] or [7] by proving that both W_β and W_β^* are non-hyponormal and non-decomposable. Since $\beta_{-1}=1, \beta_0=2$, and $\beta_1=1/2$, the weight sequence β is neither increasing nor decreasing and thus neither W_β nor W_β^* is hyponormal.

Let U denote a unilateral (unweighted) shift of multiplicity one in $\mathscr{L}(\mathfrak{H})$. Since $\lim_{n\to\infty} \beta_n=0$, it is clear that W_β^* is unitarily equivalent to a compact perturbation of $T=U\oplus 0_{\mathfrak{H}}$. The results of [9] imply that T is non-quasitriangular, and thus W_β^* is non-quasitriangular. Theorem 3.1 of [1] states that each decomposable operator is quasitriangular, and it follows that W_β^* is non-decomposable.

To prove that W_β is non-decomposable, we recall from [7, Corollary 1.4, p. 31] that each decomposable operator has the single-valued extension property (in the sense of [7]). Let $D=\{\lambda\in\mathbb{C} \mid 0<|\lambda|<1\}$ and for $\lambda\in D$ let

$$f(\lambda) = e_1 + \sum_{n=1}^{\infty} (1/\beta_0)\lambda^n e_{-n+1} + \sum_{n=1}^{\infty} (\beta_1 \dots \beta_n)\lambda^{-n} e_{n+1}.$$

A straightforward series calculation shows that $f(\lambda)$ converges in $\mathfrak{H}$ and that $f: D \to \mathfrak{H}$ is analytic. Since $(W_\beta-\lambda)f(\lambda)=0$ for each λ in D, W_β does not satisfy the single-valued extension property, and is thus non-decomposable. (Note, however, that W_β is quasitriangular.)

Lemma 4.5. *If W_α is quasisimilar to W_β and W_α is invertible, then W_β is invertible.*

Proof. Since W_α is invertible, $\epsilon \equiv \inf_{i\in Z} \alpha_i > 0$, and it clearly suffices to prove that $\inf_{j\in Z} \beta_j > 0$. Theorem 4.2 implies that there are integers k and m, and a constant $M>0$, such that

i) $(\alpha_0 \dots \alpha_{i+k}) < M(\beta_0 \dots \beta_i), \quad i \geqq \max(0, -k);$

ii) $(\beta_{-1} \dots \beta_{-i-k}) < M(\alpha_{-1} \dots \alpha_{-i}), \quad i > \max(0, -k);$

iii) $(\beta_0 \dots \beta_{j+m}) < M(\alpha_0 \dots \alpha_j), \quad j \geqq \max(0, -m);$

iv) $(\alpha_{-1} \dots \alpha_{-j-m}) < M(\beta_{-1} \dots \beta_{-j}), \quad j > \max(0, -m).$

We consider first the case when $k+m \geqq 0$. For $j > \max(-m-1, 0)$, let $i = j+m+1$; now i) and iii) imply that $(\alpha_{j+1} \dots \alpha_{j+1+m+k})/\beta_{j+m+1} = (\alpha_0 \dots \alpha_{j+1+m+k} \times \beta_0 \dots \beta_{j+m})/(\beta_0 \dots \beta_{j+m+1}\alpha_0 \dots \alpha_j) < M^2$, and thus $\beta_{j+m+1} > (1/M^2)\epsilon^{m+k+1}$. For $j \geqq \max(1-m, k+2, 2)$, let $i = j-k-1$; now ii) and iv) imply that $(\alpha_{-j+k} \dots \alpha_{-j-m})/\beta_{-j} = (\alpha_{-1} \dots \alpha_{-j-m}\beta_{-1} \dots \beta_{-j+1})/(\beta_{-1} \dots \beta_{-j}\alpha_{-1} \dots \alpha_{-j+k+1}) < M^2$, and thus $\beta_{-j} > (1/M^2)\epsilon^{m+k+1}$. It now follows that $\inf_{j \in Z} \beta_j > 0$ in case $k+m \geqq 0$.

For $\delta > 0$, the shifts δW_α and δW_β are quasisimilar, and are invertible if and only if W_α and, respectively, W_β are invertible. We may thus assume that $\|W_\alpha\| \leqq 1$ and $\|W_\beta\| \leqq 1$; since $\alpha_n \leqq 1$ and $\beta_n \leqq 1$ $(n \in Z)$, we may also assume in i)—iv) that $k \geqq 0$ and $m \geqq 0$. Since the result is true when $k+m \geqq 0$, the proof is now complete.

Theorem 4.6. *If W_α is quasisimilar to W_β, and W_α is invertible, then W_α is similar to W_β.*

Proof. From Lemma 4.5, we may assume that W_β is also invertible. It is now straightforward to show that the inequalities of Theorem 4.2—ii) imply that the inequalities of Theorem 4.3 are satisfied for suitable values of k, M, and N, and thus W_α is similar to W_β. (The value for k in Theorem 4.3 may be taken to be that of either k or m from Theorem 4.2—ii); we omit the details.)

Lemma 4.7. *Let A and B be in $\mathscr{L}(\mathfrak{H})$. Suppose that there exist positive integers p and N, integers $a_1, \dots, a_p$, and positive numbers $c_1, \dots, c_p$, such that for each $n \geqq N, n+a_i > 0$ $(1 \leqq i \leqq p)$ and $\|A^n\| \leqq \max_{1 \leqq i \leqq p} c_i \|B^{n+a_i}\|$. Then $r(A) \leqq r(B)$.*

Proof. If T is in $\mathscr{L}(\mathfrak{H})$, then $r(T) = \lim \|T^n\|^{1/n}$, and it suffices to verify that for each integer a, $r(T) = \lim \|T^{n+a}\|^{1/n}$ $(n > -a)$. If $r(T) > 0$, then $\lim (\|T^{n+a}\|^{1/(n+a)})^{1/n} = 1$, so $\lim \|T^{n+a}\|^{1/n} = \lim \|T^{n+a}\|^{1/(n+a)} (\|T^{n+a}\|^{1/(n+a)})^{a/n} = r(T)$. If $r(T) = 0$, then $0 \leqq \overline{\lim} \|T^{n+a}\|^{1/n} \leqq \overline{\lim} \|T^{n+a}\|^{1/(n+a)} \|T\|^{a/n} = 0 = r(T)$, and the proof is complete.

Theorem 4.8. *If W_α is quasisimilar to W_β, then $\sigma(W_\alpha) = \sigma(W_\beta)$.*

Proof. From Theorem 4.6, we may assume that both W_α and W_β are non-invertible. In this case the spectra of W_α and W_β consist of closed disks centered at 0 [16], and therefore, by symmetry, it suffices to prove that $r(W_\alpha) \leqq r(W_\beta)$. For each $\epsilon > 0$, the shifts ϵW_α and ϵW_β are quasisimilar; moreover $r(\epsilon W_\alpha) =$

$=\epsilon r(W_\alpha)$ and $r(\epsilon W_\beta)=\epsilon r(W_\beta)$. We may therefore assume that $\|W_\alpha\|\leqq 1$ and $\|W_\beta\|\leqq 1$. Theorem 4.2 implies that there exists $M>0$ and integers k and m such that $\alpha_0\ldots\alpha_{i-1+k}\leqq M\beta_0\ldots\beta_{i-1}$ and $\beta_{-1}\ldots\beta_{-(i+k)}\leqq M\alpha_{-1}\ldots\alpha_{-i}$ for $i\geqq\max(1,1-k)$, and such that $\beta_0\ldots\beta_{i-1+m}\leqq M\alpha_0\ldots\alpha_{i-1}$ and $\alpha_{-1}\ldots\alpha_{-(i+m)}\leqq M\beta_{-1}\ldots\beta_{-i}$ for $i\geqq\max(1,1-m)$. Since $\alpha_j\leqq 1$ and $\beta_j\leqq 1$ for each j, we may assume that $k\geqq 0$ and $m\geqq 0$. To prove that $r(W_\alpha)\leqq r(W_\beta)$ we will show that the hypothesis of Lemma. 4.7 is satisfied with $A=W_\alpha$ and $B=W_\beta$. Since $\|W_\alpha^n\|=\sup\limits_{j\in Z}\alpha_{j+1}\ldots\alpha_{j+n}$, we may replace $\|A^n\|$ in Lemma 4.7 by an arbitrary product $\alpha_{j+1}\ldots\alpha_{j+n}$, and we now estimate these products.

Let $N=k+m+1$ and $n>N$. We consider several special cases for the product $\alpha_{j+1}\ldots\alpha_{j+n}$.

i) Suppose that $j\geqq 0$. Since $j\geqq 0\geqq k-n, j+m\geqq 0$, and $n-k\geqq m+1$, then

$$\alpha_{j+1}\ldots\alpha_{j+n}\leqq M\big((\beta_0\ldots\beta_{j+m})/(\alpha_0\ldots\alpha_j)\big)(\beta_{j+m+1}\ldots\beta_{j+n-k})\leqq M^2\|W_\beta^{n-k-m}\|.$$

ii) Suppose that $j\geqq 1$. Since $j=0\geqq -n+1+m$, we have $-j-n+m<-1$, and since $-j-k\leqq -1$ and $n\geqq m+k+1$, then

$$\alpha_{-j-1}\ldots\alpha_{-j-n}\leqq M\big((\beta_{-1}\ldots\beta_{-j-k})/(\alpha_{-1}\ldots\alpha_{-j})\big)(\beta_{-j-k-1}\ldots\beta_{-j-n+m})\leqq M^2\|W_\beta^{n-k-m}\|.$$

iii) We also have $\alpha_0\ldots\alpha_{n-1}\leqq M\beta_0\ldots\beta_{n-1-k}\leqq M\|W_\beta^{n-k}\|$, and $\alpha_{-1}\ldots\alpha_{-n}\leqq$ $\leqq M\beta_{-1}\ldots\beta_{-n+m}\leqq M\|W_\beta^{n-m}\|$.

The remaining products are of the form $\alpha_{j+1}\ldots\alpha_{-1}\alpha_0\ldots\alpha_{j+n}$ for $-n\leqq j\leqq -2$. Since $j+n\geqq 0$ and $j+1\leqq -1$, there are $p=-j-1\geqq 1$ factors with a negative subscript and $q=j+n+1\geqq 1$ factors with a nonnegative subscript. We consider the possible values of p and q.

iv) If $p>m$ and $q>k$, then $-1-j>m$ and $j+n-k>-1$, and therefore

$$\alpha_{j+1}\ldots\alpha_{-1}\alpha_0\ldots\alpha_{j+n}\leqq M^2\beta_{j+1+m}\ldots\beta_{-1}\beta_0\ldots\beta_{j+n-k}\leqq M^2\|W_\beta^{n-k-m}\|.$$

v) If $p=-j-1\leqq m$, then $q=j+n+1>k$ since $p+q=n>m+k$. Now

$$\alpha_{j+1}\ldots\alpha_{j+n}\leqq\alpha_0\ldots\alpha_{j+n}\leqq M\beta_0\ldots\beta_{j+n-k}\leqq M\|W_\beta^{j+n-k+1}\|,\ \text{where}\ -1-m\leqq j\leqq -2.$$

Thus $\alpha_{j+1}\ldots\alpha_{j+n}\leqq(M)\max\{\|W_\beta^{n+a}\|:-m-k\leqq a\leqq -1-k\}$.

vi) If $q=n+j+1\leqq k$, then $p=-j-1>m$, and $\alpha_{j+1}\ldots\alpha_{-1}\alpha_0\ldots\alpha_{j+n}\leqq$ $=M\beta_{j+1+m}\ldots\beta_{-1}\alpha_0\ldots\alpha_{j+n}\leqq M\|W_\beta^{-j-m-1}\|$. Since $n+1-k\leqq -j\leqq n$, then $n-k-m\leqq$ $\leqq -j-m-1\leqq n-m-1$, and so $\alpha_{j+1}\ldots\alpha_{j+n}\leqq(M)\max\{\|W_\beta^{n+a}\|:-k-m\leqq a\leqq$ $\leqq -m-1\}$. The proof is now complete.

Remark. The example just before Corollary 2.6 shows that the conclusion of Theorem 4.8 is false if we only have a single equation $SX=XT$ (where S and T are injective weighted shifts and X is quasi-invertible).

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On reductive operator algebras

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In this paper, all Hilbert spaces are assumed to be separable.

The following conjectures are well-known in operator theory (see, e.g. [4]):

(I) *The invariant subspace conjecture:* Every operator on a Hilbert space of dimension larger than one has a (non-trivial, proper, closed) invariant subspace.

(R) *The reductive operator conjecture:* Every reductive operator is normal.

(TA) *The transitive algebra conjecture:* The only weakly closed transitive algebra on a Hilbert space is the whole algebra $B(H)$.

(RA) *The reductive algebra conjecture:* Every weakly closed reductive algebra is self-adjoint.

(H) *The hyperinvariant subspace conjecture:* Every operator other than a scalar has a hyperinvariant subspace.

(Recall that an operator T is *reductive* if every invariant subspace of T reduces T. A subspace is *hyperinvariant* for T if it is invariant for every operator commuting with T. An algebra $\mathscr{A}$ of operators is said to be *reductive* if every subspace which is invariant under all the operators in $\mathscr{A}$ reduces all the operators in $\mathscr{A}$.)

It is obvious that

$$\begin{array}{ccccc} (R) & \Rightarrow & (I) & & \\ \Uparrow & & \Uparrow & \nwarrow & \\ (RA) & \Rightarrow & (TA) & \Rightarrow & (H) \end{array}$$

DYER and PORCELLI [2] proved that $(R)\Leftrightarrow(I)$. In what follows, we consider other inter-relationships of these conjectures. First we introduce some notation.

For an operator T, we write $T^{(n)}$ for the direct sum of n copies of T, and for an algebra $\mathscr{A}$, we write $\mathscr{A}^{(n)}$ for $\{T^{(n)}\colon T\in\mathscr{A}\}$ and $M_n(\mathscr{A})$ for the $n\times n$ matrices with entries in $\mathscr{A}$. For an algebra $\mathscr{A}$, we write Lat $\mathscr{A}$ for the set of all

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subspaces which are invariant under all the operators in $\mathscr{A}$ and we write $\mathscr{A}'$ for the commutant of $\mathscr{A}$.

Note that (RA) is the strongest statement among the above conjectures. We divide it into the following two weaker statements:

(RA)′ If an algebra $\mathscr{A}$ is reductive, then $\mathscr{A}'$ is self-adjoint.

(RA)″ If a weakly closed algebra $\mathscr{A}$ is reductive and $I\in\mathscr{A}$, then $\mathscr{A}=\mathscr{A}''$. Obviously (RA)$\Leftrightarrow$((RA)′ & (RA″)).

Theorem 1. *Statement* (RA)′ *is equivalent to each of the following:*

(1) *If an algebra* $\mathscr{A}$ *is reductive, then so is* $\mathscr{A}'$.

(2) *If an algebra* $\mathscr{A}$ *is reductive and* $\mathscr{A}=\mathscr{A}''$, *then* $\mathscr{A}$ *is self-adjoint.*

To prove this, we need two lemmas. The first is quite well-known (e.g., see [4] Theorem 7.1).

Lemma a. *An operator* T *is in the weak closure of an albegra* $\mathscr{A}$ *if and only if*

$$\operatorname{Lat}\mathscr{A}^{(n)} \subseteqq \operatorname{Lat} T^{(n)}$$

for infinitely many positive integers n.

It follows from the above lemma that a weakly closed algebra is self-adjoint if and only if $\mathscr{A}^{(n)}$ is reductive for all n.

Lemma b. *If* $\mathscr{A}$ *is reductive, then so is* $\mathscr{A}''$.

Proof. Let $M\in\operatorname{Lat}\mathscr{A}''$. Then $M\in\operatorname{Lat}\mathscr{A}$ since $\mathscr{A}\subseteqq\mathscr{A}''$. As $\mathscr{A}$ is reductive, $P_M\in\mathscr{A}'$ where P_M is the projection associated with M. Hence $P_M\in(\mathscr{A}'')'$ $(=\mathscr{A}')$, i.e., M reduces $\mathscr{A}''$.

Proof of Theorem 1. Obviously (RA)′$\Rightarrow$(1) and (RA)′$\Rightarrow$(2). Now assume (2). Let $\mathscr{A}$ be a reductive algebra and $\mathscr{B}=\mathscr{A}''$. Then, by Lemma b, $\mathscr{B}$ is reductive and $\mathscr{B}=\mathscr{B}''$. By our assumption, $\mathscr{B}$ is self-adjoint. Hence $\mathscr{A}'=\mathscr{B}'$ is also self-adjoint. Thus (2)$\Rightarrow$(RA)′.

Assume (1) and let $\mathscr{A}$ be a reductive algebra. Then, for any positive integer n, $M_n(\mathscr{A})$ is also a reductive algebra. Hence $\mathscr{A}'^{(n)}=M_n(\mathscr{A})'$ is reductive for every n. Therefore, by the remark following Lemma a, $\mathscr{A}'$ is self-adjoint.

The conjecture (TA) can also be separated into weaker statements:

(TA)′ If an algebra is transitive, then $\mathscr{A}'$ consists of scalars.

(TA)″ If a weakly closed algebra is transitive, then $\mathscr{A}=\mathscr{A}''$.

Obviously (RA)′$\Rightarrow$(TA)′. (RA)″$\Rightarrow$(TA)″ and (TA)$\Leftrightarrow$((TA)′ & (TA)″). Note that (TA)′ is equivalent to the hyperinvariant subspace conjecture (H).

The following theorem is the main result of the present paper. It is a generalization of the following result in [1]: The hyperinvariant subspace conjecture is equivalent to the statement: if $\{T\}'$ is reductive, then $\{T\}'$ is self-adjoint. The proof is inspired by [5].

Theorem 2. *The hyperinvariant subspace conjecture* (H) *is equivalent to* (RA)$'$.

Proof. We have seen that (RA)$'\Rightarrow$(H). It remains to show that (TA)$'\Rightarrow$(RA)$'$. To prove this, we need some results from [1]. Let $\mathscr{A}$ be a reductive algebra. Take a maximal direct integral decomposition of $\mathscr{A}$:

$$\mathscr{A} \sim \int_Z^{\oplus} \mathscr{A}(z)\,dm(z).$$

By Theorem 4.1 in [1], $\mathscr{A}(z)$ is transitive a.e. (m). Let $T\in\mathscr{A}'$. We are going to show that $T^*\in\mathscr{A}'$.

For convenience, we call a finite collection $\mathscr{P}=\{P_1, P_2, \ldots, P_n\}$ of hermitian projections a *partition* if: (1) Each P_j is a diagonal operator with respect to the above decomposition, (2) $P_jP_k=0$ for $j\neq k$, and (3) $P_1+P_2+\ldots+P_n=I$. A partition $\mathscr{P}$ is a refinement of a partition $\mathscr{Q}$ and we write $\mathscr{P}\geqq\mathscr{Q}$ if for each P in $\mathscr{P}$ there is some Q in $\mathscr{Q}$ such that $PQ=P$. It is easy to see that there is a sequence of partitions $\{\mathscr{P}_n\}$ such that:

$$\mathscr{P}_1 \leqq \mathscr{P}_2 \leqq \mathscr{P}_3 \leqq \ldots$$

and the abelian von Neumann algebra generated by $\bigcup_{n=1}^{\infty}\mathscr{P}_n$ is the diagonal algebra $\mathscr{Z}$. Suppose $\mathscr{P}_n=\{P_{n,1}, P_{n,2}, \ldots, P_{n,m(n)}\}$. Put

$$T_n = \sum_{k=1}^{m(n)} P_{n,k}TP_{n,k}.$$

Note that, for $j\neq k$, $P_{n,j}TP_{n,k}$ is a nilpotent operator in $\mathscr{A}'$. Hence by [4] Lemma 9.2, $P_{n,j}T^*P_{n,k}\in\mathscr{A}'$. Therefore, $T^*-T_n^*\in\mathscr{A}'$ for each n.

Obviously $\|T_n\|\leqq\|T\|$ for each n. Hence $\{T_n\}_n$ has a subsequence, say $\{T_{n_k}\}_k$, which converges in the weak operator topology to S, say. It is easy to see that $SP=PS$ for each $P\in\bigcup_{n=1}^{\infty}\mathscr{P}_n$. Hence $S\in\mathscr{Z}'$. Therefore, S is decomposable, say $S=\int_Z^{\oplus} S(z)\,dm(z)$. Since $T_{n_k}\in\mathscr{A}'$ for each k, we also have $S\in\mathscr{A}'$. Therefore $S(z)\in\mathscr{A}(z)'$ a.e. (m). Since $\mathscr{A}(z)$ is transitive a.e. (m), by our assumption (TA)$'$, $S(z)$ is a scalar. Therefore S is a normal operator in $\mathscr{A}'$. By Fuglede's theorem, $S^*\in\mathscr{A}'$.

Since $T^* - T^*_{n_k} \in \mathscr{A}'$ for each k, we have $T^* - S^* \in \mathscr{A}'$. Hence $T^* = (T^* - S^*) + S^* \in \mathscr{A}'$.

By using the same argument, we can show:

Theorem 3. *If $\mathscr{A}$ is a reductive algebra and $\mathscr{A} \sim \int_Z^{\oplus} \mathscr{A}(z)\,dm(z)$ is a direct integral decomposition of $\mathscr{A}$ such that $\mathscr{A}(z)'$ is self-adjoint a.e. (m), then $\mathscr{A}'$ is self-adjoint.*

Corollary. *If $\mathscr{Z}$ is an abelian von Neumann algebra, n a positive integer and $\mathscr{A}$ a reductive algebra contained in $M_n(\mathscr{Z})$, then $\mathscr{A}'$ is self-adjoint.*

Proof. There is a finite measure space (Z, m) such that $\mathscr{Z}$ corresponds to multiplication operators acting on $L^2(Z, m)$. (See, for example, Radjavi and Rosenthal [4] p. 124.) Let K be an n-dimensional Hilbert space and $z \to H(z)$ be the constant field of Hilbert spaces with $H(z) \equiv K$. Then $\mathscr{A}$ becomes a reductive algebra consisting of decomposable operators. We write

$$\mathscr{A} \sim \int_Z^{\oplus} \mathscr{A}(z)\,dm(z).$$

By Theorem 4.1 in [1], $\mathscr{A}(z)$ is reductive a.e. (m). Since $\mathscr{A}(z) \subseteq B(K)$ and dim $K = n < \infty$, $\mathscr{A}(z)$ is self-adjoint a.e. (m). Now the corollary follows from Theorem 3.

Let $\mathscr{A}$ be a reductive algebra. The von Neumann algebra $\mathscr{I}(\mathscr{A})$ generated by $\{P_M : M \in \mathrm{Lat}\,\mathscr{A}\}$ is called the *invariant algebra* of $\mathscr{A}$ and was introduced by Hoover [3]. Let $\mathscr{Z}(\mathscr{A})$ be the centre of $\mathscr{I}(\mathscr{A})$. Then the reductive algebra conjecture (RA) can be rendered into the following two weaker statements:

(RA1) If $\mathscr{A}$ is a reductive algebra, and $\mathscr{Z}(\mathscr{A}) \subseteq \mathscr{A}$, then $\mathscr{A}$ is self-adjoint.

(RA2) If $\mathscr{A}$ is a reductive algebra, then $\mathscr{Z}(\mathscr{A}) \subseteq \mathscr{A}$.

Obviously (RA)$\Leftrightarrow$((RA1) & (RA2)).

Since, for a reductive algebra $\mathscr{A}$, $\mathscr{Z}(\mathscr{A}) \subseteq \mathscr{A}''$ (see e.g. [3] Corollary 1), we have (RA)$''\Rightarrow$(RA2). By the same reasoning it follows from Theorem 1 that (RA1)$\Rightarrow$(RA)$'$.

In [1], it was proved that the following two statements are equivalent:

(CTA) An abelian algebra on a Hilbert space is intransitive.

(CRA) If $\mathscr{A}$ is an abelian reductive algebra, then $\mathscr{A}$ is self-adjoint.

Remark. If $\{\mathscr{A}_j\}$ is a collection of reductive algebras, then the weakly closed algebra generated by $\bigcup_j \mathscr{A}_j$ is also reductive. Thus, by Zorn's lemma, every abelian reductive algebra is contained in a maximal abelian reductive algebra.

Theorem 4. *If $\mathscr{A}$ is a maximal abelian reductive algebra, then $\mathscr{A}=\mathscr{A}'$.*

Proof. Suppose the contrary. Then there is an operator T in $\mathscr{A}'$ which is not in $\mathscr{A}$. Let $\mathscr{B}$ be the algebra generated by $\mathscr{A}$ and T. Then $\mathscr{B}$ is an abelian algebra properly containing $\mathscr{A}$. By the first sentence of the above remark, $\mathscr{A}$ contains all projections in $\mathscr{A}'$. Let P be a projection onto an invariant subspace of $\mathscr{B}$. Then $P\in\mathscr{A}'$ since $\mathscr{A}\subseteq\mathscr{B}$ and $\mathscr{A}$ is reductive. Hence $P\in\mathscr{A}$. As $T\in\mathscr{A}'$, we have $TP=PT$. Therefore $P\in\mathscr{B}'$. We see that $\mathscr{B}$ is also an abelian reductive algebra. Contradiction.

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Concerning the uniqueness lemma for absolutely continuous functions

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1. We recall the classical lemma from the elements of real analysis bearing on the uniqueness of absolutely continuous functions [1], [2] which we restate in terms of vector-valued functions:

Given f: $[a, b] \to X$ where $-\infty < a < b < +\infty$ and X is a Banach space over R. If f is absolutely continuous and $f'(t) = 0$ (the zero element of X) for almost all $t \in [a, b]$, then f is constant. (For the vector-valued situation we cannot assert in general the almost everywhere existence of $f'(t)$.)

The object of this note is to show that the lemma as stated may be established in a very simple way without the introduction of ancillary considerations such as the Vitali covering theorem or the "rising sun lemma" of F. Riesz (taken with the Hahn-Banach theorem). To be sure, these powerful approaches would appear to be indispensable to develop fully the theory of absolutely continuous functions of a single real variable and its relation to the theory of the Lebesgue integral.

2. We start with two arbitrary positive numbers ε and η in a manner reminiscent of the classical approach which uses the notion of a Vitali covering and let δ denote a positive number such that whenever $[x_k, y_k]$, $x_k < y_k$, $k = 1, \ldots, n$, are nonoverlapping segments in $[a, b]$ which satisfy $\sum (y_k - x_k) \leqq \delta$, we have $\sum \| f(y_k) - f(x_k) \| \leqq \eta$. Here $\| \;\|$ denotes the norm of X. Let Ω denote an open subset of R containing $[a, b] - \{f'(t) = 0\}$ whose Lebesgue measure is at most δ. We introduce the class $\mathfrak{S}$, of finite sequences s that satisfy: (1) the domain of s is an initial segment $\langle 1, n(s) \rangle$ of the positive integers, (2) s maps its domain in a monotone strictly increasing fashion into $[a, b]$ with $s(1) = a$, and finally, (3) for each integer k satisfying $1 \leqq k < n(s)$ either $[s(k), s(k+1)] \subset \Omega$ or

$$\| f[s(k+1)] - f[s(k)] \| \leqq \varepsilon [s(k+1) - s(k)].$$

We note that $\mathfrak{S}$ is not empty and that

$$\| f\{s[n(s)]\} - f(a) \| \leqq \eta + \varepsilon (b-a).$$

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Let $c=\sup s[n(s)]$. We are assured that

$$\|f(c)-f(a)\| \leqq \eta+\varepsilon(b-a).$$

Clearly $a<c\leqq b$ as we see on noting that either $a\in\Omega$ or $f'(a)=0$. The assumption $c<b$ leads to a contradiction. For if $c\in\Omega$ and $s[n(s)]$ is sufficiently near c, we may extend s to a member $\sigma\in\mathfrak{S}$ with domain $\langle 1, n(s)+1\rangle$ such that $[\sigma[n(s)], \sigma[n(s)+1]]\subset\Omega$ and $\sigma[n(s)+1]>c$, while if $c\notin\Omega$ and $s[n(s)]$ is sufficiently near c, we may this time extend s to a σ satisfying $\sigma[n(s)+1]>c$ and

$$\|f\{\sigma[n(s)+1]\}-f\{\sigma[n(s)]\}\| \leqq \varepsilon\big(\sigma[n(s+1)]-\sigma[n(s)]\big).$$

Hence $c=b$. It follows that $f(b)=f(a)$, given the arbitrariness of ε and η. The same argument applies when b is replaced by a point of (a, b).

The lemma is thereby established.

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An integrability theorem for power series

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1. One of the first results concerning integrability theorems for power series is due P. HEYWOOD [6] who proved that

$$\int_0^1 (1-x)^{-\gamma} f(x)\,dx < \infty \quad \text{for} \quad f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_k \geqq 0, \quad \gamma < 1$$

if and only if

$$\sum_{n=1}^{\infty} n^{\gamma-2} \sum_{k=0}^{n} a_k < \infty.$$

A theorem, which states only an implication, was proved earlier by HARDY and LITTLEWOOD [5], as follows:

If $a_k \geqq 0$, $r \geqq p > 1$, $q > 0$ *and*

$$A(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then

$$\int_0^1 (1-x)^{\frac{r}{q}-1} A^r(x)\,dx \leqq K \left(\sum_{k=1}^{\infty} k^{-\frac{p+q-pq}{q}} a_k^p \right)^{r/p},$$

where $K = K(p, q, r)$ *depends on* p, q *and* r *only.*

Henceforth — to our knowledge — P. B. KENNEDY [9], R. P. BOAS and J. M. GONZÁLEZ-FERNÁNDEZ [3], P. HEYWOOD [7], Y. M. CHEN [4], R. ASKEY [1], R. S. KHAN [10], L. LEINDLER [11], R. ASKEY and S. KARLIN [2] and P. JAIN [8] have proved similar theorems.

Very recently one of the authors ([12]) generalized most of the results known up to that time as follows:

Theorem A. *Let* $\lambda(t) > 0$ *be a nonincreasing function on the interval* $0 < t \leqq 1$ *such that*

$$\sum_{n=k}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \leqq M \lambda\left(\frac{1}{k}\right) k^{-1},$$

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and let

$$F(x)=\sum_{n=0}^{\infty} c_n x^n;\quad 0\leqq x<1.$$

Suppose there is a positive monotonic sequence $\{\varrho_n\}$ *with* $\sum_{n=1}^{\infty}\frac{1}{n\varrho_n}<\infty$ *such that*

$$c_n>\frac{-K}{\left(\varrho_n\lambda\left(\frac{1}{n}\right)\right)^{1/p} n^{1-\frac{1}{p}}}\qquad (0<p<\infty,\ K>0)$$

for all sufficiently large values of n. *Then* $\lambda(1-x)(|F(x)|)^p\in L(0,1)$ *if and only if*

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\left(\sum_{k=0}^{n}|c_k|\right)^p<\infty.$$

In the particular case $\lambda(t)=t^{-\gamma}$ $(\gamma<1)$ and $\varrho_n=n^{\varepsilon}$ Theorem A reduces to a theorem of JAIN [8], which, for $p=1$, was previously proved by HEYWOOD [7].

In the present paper we give a generalization of Theorem A.

2. We use the following notations:

$\Phi=\Phi(p)$ $(p\geqq 1)$ denotes the set of all nonnegative functions $\varphi(u)$ having the properties: $\varphi(u)/u$ is nondecreasing and $\varphi(u)/u^p$ is nonincreasing on $(0,\infty)$.

$\Psi=\Psi(p)$ denotes the set of all functions $\psi(u)$ whose inverse functions belong to Φ.

$P=P(R)$ denotes the set of all nonnegative nondecreasing functions $\varrho(u)$ with $\varrho(u^2)\leqq R\cdot\varrho(u)$ $(u\in(0,\infty))$.

We use the notation $\bar{f}(x)$ to denote the inverse of $f(x)$.

3. We prove the following

Theorem. *Let* $\lambda(t)$ *be a positive nonincreasing function on the interval* $0<t\leqq 1$ *such that*

$$\sum_{n=k}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\leqq M\lambda\left(\frac{1}{k}\right)k^{-1}\tag{1}$$

and let $\{\alpha_n\}$ *be a positive increasing sequence with*

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot\alpha_n}<\infty.\tag{2}$$

Suppose that $\varrho(u)\in P$, *that* $\eta(u)$ *denotes either a function of* Φ *or a function of* Ψ, *and that*

$$F(x)=\sum_{n=0}^{\infty}c_n x^n,\quad 0\leqq x<1.\tag{3}$$

Then, under the condition

$$c_n > -Kn^{-1}\cdot\bar{\eta}\left(\frac{n}{\alpha_n\lambda(1/n)\varrho_n}\right), \quad (K>0), \tag{4}$$

$\lambda(1-x)\eta(|F(x)|)\varrho(|F(x)|)\in L(0,1)$ *if and only if*

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\eta\left(\sum_{k=0}^{n}|c_k|\right)<\infty. \tag{5}$$

It is clear that this theorem includes Theorem A, namely, if $\alpha_n=\varrho_n$, $\varrho(x)\equiv 1$ and $\eta(x)=x^p$ or $\eta(x)=x^{1/p}$ $(p\geqq 1)$ then it reduces to Theorem A.

4. We require the following

Lemma. *Let $\lambda(t)$, $\varrho(u)$ and $\eta(u)$ be defined as in our Theorem, and be*

$$f(x)=\sum_{k=0}^{\infty}a_kx^k \quad with \quad a_k\geqq 0, \quad 0\leqq x<1. \tag{6}$$

Then

$$\lambda(1-x)\eta(f(x))\varrho(f(x))\in L(0,1) \tag{7}$$

if and only if

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(n)<\infty; \tag{8}$$

or equivalently

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(A_n)<\infty \tag{9}$$

where

$$A_n=\sum_{k=1}^{n}a_k.$$

Proof. First of all we show that (8) and (9) are equivalent.

It is easy to verify that (8) implies (9). Namely, (8) implies the existence of a natural number k such that for all $n(\geqq 2)$

$$A_n\leqq n^k$$

so the implication (8)⇒(9) is obvious. In order to show that (9) also implies (8) we use the following property of the function $\varrho(u)$ for any integer r there exists a constant C_r such that for any numbers $\alpha>0$, $\beta>0$

$$\alpha\cdot\varrho(\beta)\leqq C_r\alpha\cdot\varrho(\alpha)+\sqrt[r]{\beta}\varrho(\beta) \tag{10}$$

(this property may be proved as the statement (2.38) of Lemma 13 in [15]). Using (10) and considering that $\eta(u)/u^p\downarrow$ and $\varrho(u)\in P$ we obtain for any integer r that

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(n)\leqq C_r\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(\eta(A_n))+\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}n^{1/r}\varrho(n)\leqq$$

$$\leqq K(p,r,R)\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(A_n)+\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}n^{1/r}\varrho(n). \tag{11}$$

An easy computation gives by (1) and $\varrho(u)\in P$ that

$$\sum_{n=1}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}n^{1/r}\varrho(n)<\infty \tag{12}$$

for all sufficiently large values of r. So, by (9), (11) and (12), we have (8). Now we prove the equivalence of (7) and (9). Set $y=1-x$. Since $\left(1-\frac{1}{n}\right)^n$ is an increasing sequence, we have for $\frac{1}{n+1}\leqq y\leqq\frac{1}{n}$ $(n\geqq2)$:

$$f(1-y)\geqq\sum_{k=0}^{n}a_k(1-y)^k\geqq\sum_{k=0}^{n}a_k\left(1-\frac{1}{n}\right)^k\geqq\left(1-\frac{1}{n}\right)^n\sum_{k=0}^{n}a_k\geqq\frac{1}{4}A_n.$$

Using this we obtain for $m\geqq2$:

$$\sum_{n=1}^{m}\lambda\left(\frac{1}{n}\right)n^{-2}\eta(A_n)\varrho(A_n)\leqq2\sum_{n=1}^{m}\int_{\frac{1}{n+1}}^{1/n}\lambda(y)\,dy\,\eta(A_n)\varrho(A_n)\leqq$$

$$\leqq2\int_{1/2}^{1}\lambda(y)\,dy\,\eta(A_1)\varrho(A_1)+\sum_{n=2}^{m}\int_{\frac{1}{n+1}}^{1/n}\lambda(y)\,dy\,\eta(A_n)\varrho(A_n)\leqq$$

$$\leqq O(1)+K\sum_{n=2}^{m}\int_{\frac{1}{n+1}}^{1/n}\lambda(y)\eta(f(1-y))\varrho(f(1-y))\,dy\leqq$$

$$\leqq O(1)+K\int_0^1\lambda(1-x)\eta(f(x))\varrho(f(x))\,dx.$$

This proves that (9) follows from (7). To prove the inverse statement of the equivalence we consider the following estimations for $m\geqq 1$

$$\int_0^{1-\frac{1}{m+1}} \lambda(1-x)\eta(f(x))\varrho(f(x))\,dx = \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta(f(1-y))\varrho(f(1-y))\,dy =$$

$$= \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\left(\sum_{k=0}^{\infty} a_k(1-y)^k\right)\varrho\left(\sum_{k=0}^{\infty} a_k(1-y)^k\right)dy \leqq$$

(13)
$$\leqq \sum_{n=1}^{m} \int_{\frac{1}{n+1}}^{1/n} \lambda(y)\eta\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right)\varrho\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right)dy \leqq$$

$$\leqq O(1)\sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2}\eta\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right)\varrho\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k\right).$$

Since $\frac{1}{2}\geqq\left(1-\frac{1}{n+1}\right)^n\geqq\left(1-\frac{1}{n+2}\right)^{n+1}$ for $n=1, 2, \ldots$ we have

$$\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{n+1}\right)^k \leqq \sum_{j=0}^{\infty}\sum_{k=nj}^{n(j+1)} a_k\left(1-\frac{1}{n+1}\right)^k \leqq$$

(14)
$$\leqq \sum_{j=0}^{\infty}\left(1-\frac{1}{n+1}\right)^{nj}\sum_{k=nj}^{n(j+1)} a_k \leqq 2\sum_{i=1}^{\infty} 2^{-i}A_{ni}.$$

Henceforth we split the proof into two parts. If $\eta(u)=\varphi(u)$ then we use the inequality

(15)
$$\varphi\left(\frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i}\right)\varrho\left(\frac{\sum_{i=1}^{\infty} a_i b_i}{\sum_{i=1}^{\infty} a_i}\right) \leqq K\cdot\frac{\sum_{i=1}^{\infty} a_i\varphi(b_i)\varrho(b_i)}{\sum_{i=1}^{\infty} a_i}.$$

This property of the function $\varphi(u)\varrho(u)$ immediately follows from results of H. P. Mulholland [13] (see Theorem 1 and Remark (2.34)) and from the properties of the functions $\varphi(u)$ and $\varrho(u)$. By (15) we get:

(16)
$$\varphi\left(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\right)\varrho\left(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\right) \leqq K\sum_{i=1}^{\infty} 2^{-i}\varphi(A_{ni})\varrho(A_{ni}).$$

Hence and from (13), (14) and (16) we deduce, for $m \geqq 1$, that

$$\int_0^{1-\frac{1}{m+1}} \lambda(1-x)\varphi(f(x))\varrho(f(x))\,dx \leqq O(1) \sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2} \sum_{i=1}^{\infty} 2^{-i}\varphi(A_{ni})\varrho(A_{ni}) \leqq$$

$$\leqq O(1) \sum_{i=1}^{\infty} 2^{-i} \sum_{n=1}^{m} \lambda\left(\frac{1}{n}\right) n^{-2}\varphi(A_{ni})\varrho(A_{ni}) \leqq$$

$$\leqq O(1) \sum_{i=1}^{\infty} 2^{-i} i^2 \sum_{n=1}^{m} \lambda\left(\frac{1}{ni}\right) (ni)^{-2}\varphi(A_{ni})\varrho(A_{ni}) \leqq O(1) \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\varphi(A_n)\varrho(A_n).$$

If $\eta(u)=\psi(u)$ the proof runs similarly but we use the following inequality

$$\psi\left(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\right)\varrho\left(\sum_{i=1}^{\infty} 2^{-i}A_{ni}\right) \leqq O(1) \sum_{i=1}^{\infty} 2^{-\frac{i}{p}}\psi(A_{ni})\varrho\left(\sum_{i=1}^{\infty} \frac{(ni)^t}{2^i}\right) \leqq \tag{17}$$

$$\leqq O(1) \sum_{i=1}^{\infty} 2^{-\frac{i}{p}}\psi(A_{ni})\varrho(n)$$

instead of (16). Inequality (17) is just an easy consequence of the following elementary facts:

$$\psi(a+b) \leqq \psi(a)+\psi(b), \quad \psi(kx) \leqq k^{1/p}\psi(x) \quad \text{for} \quad k<1,$$

and that, by (8), there exists an integer t such that $A_n \leqq n^t$ for any $n (\geqq 2)$. Thus the proof of Lemma is completed.

5. Proof of the theorem.

Let $A(x)=\sum_{k=0}^{\infty} a_k x^k$ for $0 \leqq x < 1$ with $a_0=0$ and

$$a_k = K \cdot k^{-1} \cdot \bar{\eta}\left(\frac{k}{\alpha_k \lambda(1/k)\varrho(k)}\right).$$

First we consider the case $\eta(u)=\varphi(u)$.

We show that these coefficients a_k satisfy condition (8). Using the inequality

$$\sum_{n=1}^{\infty} \lambda_n \varphi(A_n) \leqq K_1 \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{a_n}{\lambda_n} \sum_{k=n}^{\infty} \lambda_k\right) \tag{18}$$

which holds for any $\lambda_n > 0$ and $a_n \geqq 0$ (see the inequality (8) of [14]) with $\lambda_n = \lambda(1/n)n^{-2}\varrho(n)$, and the following consequence of (1)

$$\sum_{n=k}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\varrho(n) \leqq M\lambda\left(\frac{1}{k}\right) k^{-1}\varrho(k)$$

we have

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \varphi(A_n) \varrho(n) \leqq O(1) \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \varrho(n) \varphi(n \cdot a_n) \leqq$$

$$\leqq O(1) \sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \varrho(n) n \frac{1}{\alpha_n \lambda(1/n) \varrho(n)} \leqq O(1) \sum_{n=1}^{\infty} \frac{1}{n \cdot \alpha_n} < \infty.$$

Hereby we proved that the coefficients of the function $A(x)$ satisfy condition (8), so by Lemma

(19) $$\lambda(1-x)\varphi\big(A(x)\big)\varrho\big(A(x)\big) \in L(0, 1).$$

By (4) the coefficients $a_n + c_n$ are positive for all sufficiently large values of n, thus the functions

$$A(x) + F(x) = \sum_{n=0}^{\infty} (a_n + c_n) x^n$$

has the property

(20) $$\lambda(1-x)\varphi\big(A(x)+F(x)\big)\varrho\big(A(x)+F(x)\big) \in L(0, 1)$$

if and only if

(21) $$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \varphi\left(\sum_{k=0}^{n} (a_k + c_n)\right) \varrho(n) < \infty.$$

If $\lambda(1-x)\varphi(|F(x)|)\varrho(|F(x)|) \in L(0, 1)$, then (19) implies (20) which implies (21). But by (4) we have

$$|c_n| \leqq 2a_n + c_n,$$

whence, by (8) and (21), (5) follows.

If (5) holds, then this implies (21) because from (15) immediately follows that

(22) $$\varphi(a+b)\varrho(a+b) \leqq K\big(\varphi(a)\varrho(a) + \varphi(b)\varrho(b)\big), \quad a > 0, \ b > 0.$$

But from (21) follows (20). By (19) and (20)

$$\lambda(1-x)\varphi(|F(x)|)\varrho(|F(x)|) \in L(0, 1)$$

follows obviously.

Thus the theorem is proved for $\eta(u) = \varphi(u)$. The proof for $\eta(u) = \psi(u)$ runs similarly. To prove (8) we use the inequality

$$\psi(a+b) \leqq \psi(a) + \psi(b) \quad \text{for all} \quad a > 0, \ b > 0;$$

thus

$$\sum_{n=2}^{\infty}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\psi\left(\sum_{k=1}^{n}a_k\right)\leqq\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda\left(\frac{1}{n}\right)n^{-2}\varrho(n)\psi\left(\sum_{k=1}^{2^{m+1}}a_k\right)\leqq$$

$$\leqq O(1)\sum_{m=0}^{\infty}\lambda\left(\frac{1}{2^{m+1}}\right)2^{-m}\varrho(2^{m+1})\psi\left(\sum_{k=1}^{m+1}\bar{\psi}\left(\frac{2^k}{\lambda(1/2^k)\alpha_{2^k}\varrho(2^k)}\right)\right)\leqq$$

$$\leqq O(1)\sum_{k=1}^{\infty}\frac{2^k}{\lambda(1/2^k)\alpha_{2^k}\varrho(2^k)}\sum_{m=k}^{\infty}\lambda\left(\frac{1}{2^{m+1}}\right)2^{-m}\varrho(2^m)\leqq O(1)\sum_{k=1}^{\infty}\frac{1}{\alpha_{2^k}}<\infty.$$

From this point the proof runs on the same line as before. The proof is thus completed.

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Über die Struktur der Hauptidealhalbgruppen. II

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Dem Andenken von Prof. A. Kertész gewidmet

In diesem Teil möchten wir einen kleinen Umweg machen und die kommutativen Hauptidealhalbgruppen beschreiben. Diese Beschreibung wird — bis auf Abelsche Gruppen und ihre Homomorphismen — vollständig sein. Die Bezeichnungen werden mit denjenigen von [1] übereinstimmen.

In [1] haben wir gezeigt, dass in einer Hauptidealhalbgruppe H die Greensche Relation $\mathcal{J}$ eine Kongruenz ist, und nach Satz 4 derselben Arbeit ist $H/\mathcal{J}$ einer Halbgruppe $H_{\mathfrak{F}}$ isomorph, d.h. einer Kette von zyklischen Halbgruppen Z_σ ($\sigma<\vartheta$, ϑ eine Ordnungszahl $\geqq 1$) von endlicher oder unendlicher Ordnung n_σ, wobei $\mathfrak{F}$ die Folge $(n_0, \ldots, n_\sigma, \ldots)_{\sigma<\vartheta}$ bedeutet, und im Falle $n_\sigma<\infty$ die maximale Untergruppe von Z_σ trivial ist. Ist z_σ das Erzeugende von Z_σ, so bezeichne $I_{\sigma l}$ $(1\leqq l\leqq n_\sigma)$ das Urbild von z_σ^l bei dem Isomorphismus $\nu: H/\mathcal{J}\to H_{\mathfrak{F}}$. $I_{\sigma l}$ ist eine $\mathcal{J}$-Klasse. Bezeichnet $\prec$ die lexikographische Ordnung der Paare (σ, l), so gilt $\mathbf{I}_{\varrho k}\supset \mathbf{I}_{\sigma l}\Leftrightarrow(\varrho, k)\prec(\sigma, l)$.

Wir wollen jetzt ein Repräsentantensystem $c_{\sigma l}$ der Klassen $I_{\sigma l}$ auswählen, um dann die Elemente von H in der Form $c_{\sigma l}g$ anzugeben, wo g ein Element der Schützenbergergruppe $\Gamma_{\sigma l}$ von $I_{\sigma l}$ ist; in dieser Form wird die Multiplikation ziemlich einfach. Zu diesem Zweck wähle man $c_{\sigma 1}=c_\sigma$ ganz beliebig, falls $n_\sigma>1$; für $n_\sigma=1$ soll $c_{\sigma 1}$ etwa das einzige Idempotent in $I_{\sigma 1}$ sein. Aus $I_{\sigma l}$ $(1<l\leqq n_\sigma)$ wähle man $c_{\sigma l}=c_\sigma^l$.

Ist $(\varrho, k)\prec(\sigma, l)$, so sei $\varphi_{\varrho k}^{\sigma l}:\Gamma_{\varrho k}\to\Gamma_{\sigma l}$ der kanonische Homomorphismus zwischen den Schützenbergergruppen, der also dadurch definiert ist, dass die (partiellen) Translationen $g\in\Gamma_{\varrho k}$ und $g\varphi_{\varrho k}^{\sigma l}\in\Gamma_{\sigma l}$ durch dasselbe Element von H induziert werden können. Dann bilden die Gruppen $\Gamma_{\sigma l}$ und die Homomorphismen $\varphi_{\varrho k}^{\sigma l}$ ein direktes System $\Delta=\{\Gamma_{\varrho k};\ \varphi_{\varrho k}^{\sigma l}\}$ (wie übrigens immer für kommutative Halbgruppen), d.h. $\varphi_{\varrho k}^{\sigma l}\varphi_{\sigma l}^{\tau m}=\varphi_{\varrho k}^{\tau m}$ und $\{(\varrho, k);\ \prec\}$ ist nach oben gerichtet (in unserem Fall sogar linear geordnet).

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Ist $g\in\Gamma_{\sigma l}, g'\in\Gamma_{\sigma' l'}$, $(\sigma, l)\leqq(\sigma', l')$, so gibt es ein Element $c(g)\in H$ derart, dass für $a\in I_{\sigma l}$ stets $ag=a\cdot c(g)$ gilt, also hat man für $a'\in I_{\sigma' l'}$

$$ag\cdot a'g' = a\cdot c(g)\cdot(a'g') = a\cdot[(a'g')(g\varphi_{\sigma l}^{\sigma' l'})] = a\cdot[a'(g\varphi_{\sigma l}^{\sigma' l'}\cdot g')].$$

Ist ferner $c'\in H$ so beschaffen, dass $x'(g\varphi_{\sigma l}^{\sigma' l'}\cdot g')=x'\cdot c'$ für jedes $x'\in I_{\sigma' l'}$ besteht, so gilt

$$ag\cdot a'g' = aa'c' = (aa')[(g\varphi_{\sigma l}^{\sigma' l'}\cdot g')\varphi_{\sigma' l'}^{\sigma' l''}] = (aa')(g\varphi_{\sigma l}^{\sigma' l''}\cdot g'\varphi_{\sigma' l'}^{\sigma' l''}),$$

wo l'' durch die Relation $I_{\sigma l}I_{\sigma' l'}\subseteqq I_{\sigma' l''}$ definiert ist, d.h.

$$l'' = \begin{cases} l', & \text{falls} \quad \sigma<\sigma', \\ \min(l+l', n_\sigma), & \text{falls} \quad \sigma=\sigma'. \end{cases}$$

Deshalb genügt es die Produkte $c_\varrho^k\cdot c_\sigma^l$, $n_\varrho>1$ für $\varrho\leqq\sigma$ anzugeben, um die vollständige Multiplikationstafel zu erhalten. Anders gesagt, muss man einerseits für jedes Tripel ϱ, σ, l mit $\varrho<\sigma<\vartheta$, $l\leqq n_\sigma$ die durch c_ϱ induzierte partielle Translation $g_{\varrho\sigma l}\in\Gamma_{\sigma l}$, andererseits für jede ϱ mit $n_\varrho<\infty$ die ebenfalls durch c_ϱ induzierte $g_\varrho\in\Gamma_{\varrho n_\varrho}$ angeben. Man kann sich dabei auf den Fall $n_\varrho>1$ beschränken, da für $n_\varrho=1$ das Element c_ϱ idempotent und somit $g_\varrho, g_{\varrho\sigma l}$ identisch sind.

Man beachte, dass durch die Angabe dieser Translationen auch die Multiplikation in $\bigcup\limits_{k=1}^{n_\varrho} I_{\varrho k}$ bekannt wird. In der Tat, es gilt:

Satz 1. *Die Elemente der kommutativen Hauptidealhalbgruppe H sind eindeutig in der Form $c_\varrho^k g$ $(0\leqq\varrho<\vartheta, 1\leqq k\leqq n_\varrho, k<\infty, g\in\Gamma_{\varrho k})$ darstellbar. Die Multiplikation ist durch*

$$c_\varrho^k g\cdot c_\varrho^{k'}g' = \begin{cases} c_\varrho^{k+k'}(g\varphi_{\varrho k}^{\varrho, k+k}\cdot g'\varphi_{\varrho k'}^{\varrho, k+k'}) & \text{für} \quad k+k'\leqq n_\varrho, \\ c_\varrho^{n_\varrho}(g\varphi_{\varrho k}^{\varrho n_\varrho}\cdot g'\varphi_{\varrho k'}^{\varrho n_\varrho}\cdot g_\varrho^{k+k'-n_\varrho}) = e_\varrho(g\varphi_{\varrho k}^{\varrho n_\varrho}\cdot g'\varphi_{\varrho k'}^{\varrho n_\varrho}\cdot g_\varrho^{k+k'}) & \text{für} \quad k+k'>n_\varrho \end{cases}$$

$$c_\varrho^k g\cdot c_\sigma^l g'' = c_\sigma^l(g\varphi_{\varrho k}^{\sigma l}\cdot g_\varrho^k\varphi_{\varrho' m_\varrho}^{\sigma l}\cdot g'') \quad \text{für} \quad \varrho<\sigma$$

gegeben, wobei $g'\in\Gamma_{\varrho k'}, g''\in\Gamma_{\sigma l}, g_\varrho\in\Gamma_{\varrho' m_\varrho}$ ist $(\varrho'=\varrho, m_\varrho=n_\varrho$ falls $n_\varrho<\infty$ und $\varrho'=\varrho+1$, $m_\varrho=1$ sonst), e_ϱ das Idempotent von $I_{\varrho n_\varrho}$ bezeichnet und g_ϱ durch $ag_\varrho=a\cdot c_\varrho$ für $a\in I_{\varrho' m_\varrho}$ definiert ist.

Nach den obigen Erläuterungen ist der Beweis eine einfache Rechenaufgabe. Die Rolle von $g_{\varrho\sigma l}$ wird natürlich von $g_\varrho\varphi_{\varrho' m_\varrho}^{\sigma l}$ gespielt.

Bemerkung. Statt c_ϱ konnte man natürlich ein anderes Element c_ϱ' aus $I_{\varrho 1}$ als Repräsentant wählen, und dazu würde eine andere Translation g_ϱ' gehören. Ist $c_\varrho'=c_\varrho h, h\in\Gamma_{\varrho 1}$, so hat man für $a\in I_{\varrho' m_\varrho}$

$$ag_\varrho' = a\cdot c_\varrho' = (a\cdot c_\varrho)(h\varphi_{\varrho 1}^{\varrho' m_\varrho}) = a(g_\varrho\cdot h\varphi_{\varrho 1}^{\varrho' m_\varrho}),$$

also $g_\varrho' \in (\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\varrho' m_\varrho})g_\varrho$. Dies bedeutet, dass g_ϱ mod der Untergruppe $\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\varrho' m_\varrho}$ von $\Gamma_{\varrho' m_\varrho}$ eindeutig bestimmt ist, unabhängig von Wahl des c_ϱ.

Satz 1 zeigt, dass die Folge $\mathfrak{F}=(n_1, \ldots, n_\sigma, \ldots)_{\sigma<\vartheta}$, das direkte System $\Delta = \{\Gamma_{\varrho k}; \varphi_{\varrho k}^{\sigma l}\}_{\varrho,\sigma<\vartheta}$ und die Restklassen $\bar{g}_\varrho=(\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\varrho' m_\varrho})\cdot g_\varrho \in \Gamma_{\varrho' m_\varrho}/\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\varrho' m_\varrho}$ ein vollständiges System von Invarianten für die kommutativen Hauptidealhalbgruppen bilden. Dieses System ist aber noch nicht unabhängig. Die Homomorphismen $\varphi_{\varrho k}^{\sigma l}$ sind nämlich nicht ganz beliebig, und zwar gilt

Satz 2. a) *Γ_{01} ist die Einsgruppe, wenn $n_0>1$.* b) *Für $1<l<n_\sigma$ ist $\varphi_{\sigma,l-1}^{\sigma l}$ sürjektiv. Ist ferner $n_\sigma>1$ und* c) *$\sigma=\varrho+1, n_\varrho<\infty$, so ist $\varphi_{\varrho n_\varrho}^{\sigma 1}$ sürjektiv;* d) *$\sigma=\varrho+1$, $n_\varrho=\infty$, so ist $\Gamma_{\sigma 1}/\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\sigma 1}$ eine endliche zyklische Gruppe für jede natürliche Zahl k;* e) *σ eine Limeszahl, so ist $\Gamma_{\sigma 1}$ ein homomorphes Bild des direkten Limits $\Lambda_\sigma = \varinjlim \{\Gamma_{\varrho k}; \varphi_{\varrho k}^{\varrho' k'}\}_{\varrho,\varrho'<\sigma}$, wobei $\varphi_\sigma: \Lambda_\sigma \to \Gamma_{\sigma 1}$ von den Homomorphismen $\varphi_{\varrho k}^{\sigma 1}$ induziert wird, d.h., für den kanonischen Homomorphismen $\psi_{\varrho k}$: $\Gamma_{\varrho k}\to\Lambda_\sigma$ gilt*

$$\psi_{\varrho k}\varphi_\sigma = \varphi_{\varrho k}^{\sigma 1}. \tag{1}$$

Beweis. a) ist trivial.

Es sei $I_{\sigma l}$ eine nichtidempotente $\mathcal{J}$-Klasse. Dann ist $I_{\sigma l}c_\sigma \subseteqq I_{\sigma l}I_{\sigma 1} \subseteqq I_{\sigma,l+1}$, also $I_{\sigma l}\mathbf{I}_{\sigma 1} \subseteqq \mathbf{I}_{\sigma,l+1}$, und jede Translation $g\in\Gamma_{\sigma l}$ muss durch ein Element $a(g)\in \in H\setminus \mathbf{I}_{\sigma 1}$, d.h. $a(g)\in I_{\sigma' h}$ mit $\sigma'<\sigma$ induziert werden. Dann aber induziert $a(g)$ auch eine Translation $g'\in\Gamma_{\sigma 1}$ und somit gilt $g=g'\varphi_{\sigma 1}^{\sigma l}$, also ist $\varphi_{\sigma 1}^{\sigma l}$ eine Sürjektion. Ist hier $l>1$, so hat man $\varphi_{\sigma 1}^{\sigma l}=\varphi_{\sigma 1}^{\sigma, l-1}\cdot\varphi_{\sigma,l-1}^{\sigma l}$, und $\varphi_{\sigma,l-1}^{\sigma l}$ ist auch sürjektiv, womit b) bewiesen ist.

Für $l=1, \sigma=\varrho+1, n_\varrho<\infty$ haben wir $I_{\varrho n_\varrho}\cdot a(g)\subseteqq I_{\varrho n_\varrho}$, d.h. $x\mapsto x\cdot a(g)$ $(x\in I_{\varrho n_\varrho})$ ist eine Translation $g'\in\Gamma_{\varrho n_\varrho}$, für welche $g'\varphi_{\varrho n_\varrho}^{\sigma 1}=g$ gilt, also ist $\varphi_{\varrho n_\varrho}^{\sigma 1}$ sürjektiv. Damit ist auch c) bewiesen.

Jetzt sei $\sigma=\varrho+1, l=1, n_\varrho=\infty$ und $g\in\Gamma_{\varrho 1}$. Dann ist wieder $a(g)\in I_{\sigma' h}$ mit $\sigma'<\sigma$. Ist $\sigma'<\varrho$, so gilt $I_{\varrho 1}\cdot a(g)\subseteqq I_{\varrho 1}$, und man zeigt wie oben, dass $g=g'\varphi_{\sigma_1}^{\sigma 1}$ für eine $g'\in\Gamma_{\varrho 1}$. Ist andererseits $a(g)\in I_{\varrho l}$, so haben wir

$$a(g) = c_\varrho^l g'' \quad (g''\in\Gamma_{\varrho l}).$$

Bezeichne $g_\varrho\in\Gamma_{\sigma 1}$ die Transformation $x\mapsto xc_\varrho$ $(x\in I_{\sigma 1})$. Da es in $\Gamma_{\varrho 1}$ ein g' mit $g''=g'\varphi_{\varrho 1}^{\sigma l}$ gibt, und $g=g'\varphi_{\varrho 1}^{\sigma 1}\cdot g_\varrho^l$ gilt, erhielt man $g\in\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\sigma 1}\cdot g_\varrho^l$. Nun gilt aber dies Enthaltensein, eventuell mit $l=0$, auch für $g=g_\varrho^{-1}$, also muss $\Gamma_{\sigma 1}/\Gamma_{\varrho 1}\varphi_{\varrho 1}^{\sigma 1}$ endlich sein.

Endlich sei σ eine Limeszahl, $g\in\Gamma_{\sigma 1}$ und, wie vorher, $a(g)\in I_{\sigma' h}$. Wegen $\sigma'<\sigma$ gibt es eine π mit $\sigma'<\pi<\sigma$ und $a(g)$ induziert eine Translation $g'\in\Gamma_{\pi k}$ $(k\leqq n_\pi$ beliebig), wobei also $g'\varphi_{\pi k}^{\sigma 1}=g$ gilt. Da aber Δ ein direktes System ist, gibt es einen Homomorphismus $\varphi_\sigma: \Lambda_\sigma\to\Gamma_{\sigma 1}$, so dass (1) gilt, und φ_σ sürjektiv ist (weil ja $\varphi_{\pi k}^{\sigma 1}$ es ist). Dies ergibt e), und vollendet den Beweis des Satzes.

Das so erhaltene Invariantensystem ist schon vollständig und unabhängig. Es gilt nämlich:

Satz 3. *Es seien eine Halbgruppe* $H_{\mathfrak{F}}$ *mit* $\mathfrak{F}=(n_0, \dots, n_\sigma, \dots)_{\sigma<\vartheta}$, *ein direktes System von Abelschen Gruppen* $D=\{G_{\varrho k}; f_{\varrho k}^{\sigma l} \mid \varrho, \sigma<\vartheta, 1 \leqq k \leqq n_\varrho, 1 \leqq l \leqq n_\sigma, (\varrho, k) \prec (\sigma, l)\}$, *und je ein Element* $a_\varrho \in G_{\varrho' m_\varrho}$ $((\varrho', m_\varrho)=(\varrho, n_\varrho)$ *falls* $n_\varrho<\infty$ *und* $(\varrho', m_\varrho)=(\varrho+1, 1)$ *sonst*$)$ *gegeben, wobei folgende Bedingungen erfüllt sind:*

a) G_{01} *ist die Einsgruppe falls* $n_0>1$;

b) $f_{\varrho k}^{\varrho, k+1}$ *für* $k+1<n_\varrho$ *und* $f_{\varrho n_\varrho}^{\varrho+1,1}$ *für* $n_\varrho<\infty, n_{\varrho+1} \neq 1$ *sind sürjektiv;*

c) *ist* $n_\varrho=\infty, n_{\varrho+1} \neq 1$, *so ist* $G_{\varrho+1,1}/G_{\varrho 1} f_{\varrho 1}^{\varrho+1,1}$ *eine endliche zyklische Gruppe, und zwar von* $(G_{\varrho 1} f_{\varrho 1}^{\varrho+1,1}) a_\varrho$ *erzeugt;*

d) *ist* σ *eine Limeszahl,* $n_\sigma \neq 1$, *so ist der von den Abbildungen* $f_{\varrho k}^{\sigma 1}$ $(\varrho<\sigma)$ *induzierte Homomorphismus* f_σ *von* $L_\sigma=\varinjlim \{G_{\varrho k}; f_{\varrho k}^{\pi l}\}_{\varrho, \pi<\sigma}$ *in* $G_{\sigma 1}$ *sürjektiv.*

Man definiere in $U_D=\bigcup\limits_{\substack{\varrho<\vartheta \\ k \leqq n_\varrho}} G_{\varrho k}$ *eine Verknüpfung* $\circ$ *durch*

$$
a \circ b = \begin{cases} (a f_{\varrho k}^{\varrho, k+k'})(b f_{\varrho k'}^{\varrho, k+k'}) & \text{für } a \in G_{\varrho k},\ b \in G_{\varrho k'},\ k+k'<n_\varrho \quad (2_1) \\ (a f_{\varrho k}^{\varrho n_\varrho})(b f_{\varrho k'}^{\varrho n_\varrho}) & \text{für } a \in G_{\varrho k},\ b \in G_{\varrho k'},\ k+k' \geqq n_\varrho \quad (2_2) \\ (a f_{\varrho k}^{\sigma l})\, b\, (a_\varrho^k f_{\varrho' m_\varrho}^{\sigma l}) & \text{für } a \in G_{\varrho k},\ b \in G_{\sigma l},\ \varrho<\sigma. \quad (2_3) \end{cases}
$$

Dann ist $H=H(\mathfrak{F}, D, a_\varrho)=\{U_D, \circ\}$ *eine kommutative Hauptidealhalbgruppe und es gelten:* $H/\mathscr{J} \cong H_{\mathfrak{F}}$, $I_{\varrho k}$ *gleich der Trägermenge von* $G_{\varrho k}$ *und* $\Delta \cong D$ *im Sinne dass es Isomorphismen* $\gamma_{\varrho k}: \to \Gamma_{\varrho k}$ *gibt, für welche* $\gamma_{\varrho k} \varphi_{\varrho k}^{\sigma l}=f_{\varrho k}^{\sigma l} \gamma_{\sigma l}$ *gilt. Ferner besteht* $H \cong H'=H(\mathfrak{F}', D', a_\varrho')$ *dann und nur dann, wenn* $\mathfrak{F}=\mathfrak{F}'$, $D \cong D'$ *im Sinne, dass es Isomorphismen* $\iota_{\varrho k}: G_{\varrho k} \to G'_{\varrho k}$ *derart geben, dass* $f_{\varrho k}^{\sigma l} \iota_{\sigma l}=\iota_{\varrho k} f_{\varrho k}^{\prime \sigma l}$, *endlich* $a_\varrho \iota \in (G'_{\varrho 1} f_{\varrho 1}^{\varrho' m_\varrho}) a'_\varrho$ *für den Isomorphismus* $\iota: H \to H'$ *gilt, falls* $\{G_{\varrho' m_\varrho}, \circ\}$ *eine Gruppe ist.*

Beweis. Die Kommutativität der Verknüpfung $\circ$ folgt aus (2) unmittelbar, die Assoziativität kann man prüfen. Ebenfalls ersieht man aus (2), dass aus $a \in G_{\varrho k}$, $a \circ b \in G_{\sigma l}$ immer $(\varrho, k) \leqq (\sigma, l)$ folgt. Wir zeigen, dass auch umgekehrt, aus $a \in G_{\varrho k}$, $c \in G_{\sigma l}$, $(\varrho, k) \leqq (\sigma, l)$ stets $c \in (a)$ folgt.

Ist nämlich $\sigma>\varrho$, so braucht man nur die Gleichung $c=(a f_{\varrho k}^{\sigma l})(a_\varrho^k f_{\varrho' m_\varrho}^{\sigma l}) x$ in $G_{\sigma l}$ zu lösen. Ist $\sigma=\varrho, l=n_\varrho$ so hat man ein $x \in G_{\varrho n_\varrho}$ mit $(a_{\varrho k} f_{\varrho k}^{\varrho n_\varrho}) a_\varrho^k \cdot x=c$ zu finden, was auch immer möglich ist, Für $\sigma=\varrho, n_\varrho>l>k$ gibt es wegen b) ein $x \in G_{\varrho, l-k}$ mit $a \circ x=(a f_{\varrho k}^{\varrho l})(x f_{\varrho, l-k}^{\varrho l})=c$. Es bleibt also der Fall $\sigma=\varrho, l=k<n_\varrho$. Wir unterscheiden mehrere Möglichkeiten.

1. $\varrho=0$. Nach a), b) gilt $G_{\varrho k}=G_{\sigma l} \cong E$ und somit besteht die Behauptung trivialerweise.

2. ϱ ist von erster Art und $n_{\varrho-1}<\infty$. Dann ist $f_{\varrho-1, n_{\varrho-1}}^{\varrho k}$ sürjektiv nach b), also lasst sich ein $x \in G_{\varrho-1, n_{\varrho-1}}$ mit

$$
a \circ x=(x f_{\varrho-1, n_{\varrho-1}}^{\varrho k})\, a\, (a_{\varrho-1}^{n_{\varrho-1}} f_{\varrho-1, n_{\varrho-1}}^{\varrho k})=c \tag{3}
$$

finden.

3. ϱ ist von erster Art und $n_{\varrho-1}=\infty$, also $a_{\varrho-1}\in G_{\varrho 1}$. In $G_{\varrho k}$ gibt es ein y_1 mit $ay_1=c$ und y_1 hat kraft b) ein Urbild y in $G_{\varrho 1}$: $y_1=yf_{\varrho 1}^{\varrho k}$. Laut c) ist $y=x_1 a_{\varrho-1}^s$ für ein $x_1\in G_{\varrho-1,1}f_{\varrho-1,1}^{\varrho 1}$ und eine natürliche Zahl s. Es sei $x_1=zf_{\varrho-1,1}^{\varrho 1}$ $(z\in G_{\varrho-1,1})$. Für $x=zf_{\varrho-1,1}^{\varrho-1,s}$ gilt dann

$$(3')\ a\circ x=(xf_{\varrho-1,s}^{\varrho k})a(a_{\varrho-1}^s f_{\varrho 1}^{\varrho k})=a\cdot(zf_{\varrho-1,1}^{\varrho 1}\cdot a_{\varrho-1}^s)f_{\varrho 1}^{\varrho k}=a\cdot(x_1 a_{\varrho-1}^s)f_{\varrho 1}^{\varrho k}=ay_1=c.$$

4. ϱ ist von zweiter Art. y_1 und y seien wie im Falle 3. Kraft d) gilt $y=zf_\varrho$ für ein $z\in L_\varrho$, also $x_1 f_{\pi t}^{\varrho 1}=x_1 p_{\pi t} f_\varrho=zf_\varrho=y$ mit einem x_1 aus einem geeigneten $G_{\pi t}$, wobei $p_{\pi t}:G_{\pi t}\to L_\varrho$ die kanonische Abbildung ist. Da x_1 durch ein beliebiges $x_1 f_{\pi t}^{\pi s}$ $(s>t)$ ersetzt werden kann, darf man annehmen, dass $t=n_\pi$ falls $n_\pi<\infty$ und $[G_{\pi+1,1}:G_{\pi 1}.f_{\pi 1}^{\pi+1,1}]|t$ falls $n_\pi=\infty$. Im ersten Falle setzen wir $x_1=xa_\pi^{n_\pi}$ (in $G_{\pi n_\pi}$) und erhalten

$$(3'')\qquad a\circ x=(xf_{\pi n_\pi}^{\varrho k})a(a_\pi^{n_\pi}f_{\pi n_\pi}^{\varrho k})=a(x_1 f_{\pi n_\pi}^{\varrho k})=ay_1=c.$$

Im zweiten Fall sei $[G_{\pi+1,1}:G_{\pi 1}f_{\pi 1}^{\pi+1,1}]=r$, $t=rn$. In $G_{\pi+1,1}$ gibt es ein w für welches $x_1 f_{\pi t}^{\pi+1,1}=wa_\pi^t$. Da hier $a_\pi^t=(a_\pi^r)^n\in G_{\pi 1}f_{\pi 1}^{\pi+1,1}$ laut c), liegt auch w in $G_{\pi 1}f_{\pi 1}^{\pi+1,1}=G_{\pi t}f_{\pi t}^{\pi+1,1}$ und hat deshalb ein Urbild x in $G_{\pi t}$ (siehe Fall 3). Dann haben wir

$$(3''')\qquad a\circ x=(xf_{\pi t}^{\varrho k})a(a_\pi^t f_{\pi+1,1}^{\varrho k})=a[(wa_\pi^t)f_{\pi+1,1}^{\varrho k}]=a(x_1 f_{\pi+1,1}^{\varrho k})=ay_1=c.$$

Somit ist $c\in(a)\Leftrightarrow(\varrho,k)\leqq(\sigma,l)$ in allen Fällen bewiesen. Hieraus folgt, dass $H=$ $=H(\mathfrak{F},D,a_\varrho)$ eine Hauptidealhalbgruppe ist (weil ja ihre Hauptideale bezüglich der Inklusion dual wohlgeordnet sind), dass die $\mathcal{J}$-Klassen von H mit den Trägermengen der Gruppen $G_{\varrho k}$ zusammenfallen, und dass $H/\mathcal{J}\cong H_{\mathfrak{F}}$ auch erfüllt ist.

Um $G_{\varrho k}\cong\Gamma_{\varrho k}$ zu beweisen, man bemerke zuerst, dass für $k=n_\varrho<\infty$ die durch $a\gamma_{\varrho n_\varrho}=\tau_x$ $(x=aa_\varrho^{-n_\varrho})$ definierte Abbildung $\gamma_{\varrho n_\varrho}:G_{\varrho n_\varrho}\to\Gamma_{\varrho n_\varrho}(\cong\{G_{\varrho n_\varrho},\circ\})$ ein Isomorphismus ist, weil aus (2_2) sich

$$aa_\varrho^{-n_\varrho}\circ ba_\varrho^{-n_\varrho}=aba_\varrho^{-n_\varrho}\quad\text{für}\quad a,b\in G_{\varrho n_\varrho}$$

ergibt. Ist nun $k<n_\varrho$, so nehmen wir wieder die obigen Fälle 1)–4) vor. Für $\varrho=0$ ist die Behauptung trivial. In den übrigen Fällen haben wir immer ein $x\in H$ gefunden, so dass $a\circ x=c$, wobei x nicht von a und c selber, sondern nur von $y_1=ca^{-1}$ abhing. Deshalb haben wir

$$a'\circ x=a'y_1\quad\text{für jedes}\quad a'\in G_{\varrho k}.$$

Bezeichnet also $\tau_x\in\Gamma_{\varrho k}$ die durch x auf $I_{\varrho k}$ hervorgerufene partielle Translation, so ist $\gamma_{\varrho k}:y_1\mapsto\tau_x$ ein Isomorphismus zwischen $G_{\varrho k}$ und $\Gamma_{\varrho k}$.

Ist ausserdem $(\varrho,k)\prec(\sigma,l)$, $z\in G_{\sigma l}$, so gilt im Falle $k=n_\varrho$

$$z\circ aa_\varrho^{-n_\varrho}=(aa_\varrho^{-n_\varrho}f_{\varrho n_\varrho}^{\sigma l})z(a_\varrho^{n_\varrho}f_{\varrho n_\varrho}^{\sigma l})=z(af_{\varrho n_\varrho}^{\sigma l}),$$

d.h. $a\gamma_{\varrho n_\varrho}\varphi^{\sigma l}_{\varrho n_\varrho}=af^{\sigma l}_{\varrho n_\varrho}\gamma_{\sigma l}$. Für $k<n_\varrho$ ist $y_1\gamma_{\varrho k}=\tau_x$ mit $x\in G_{\pi t}$, $\pi<\varrho$, d.h.

$$ay_1 = a\circ x = (xf^{\varrho k}_{\pi t})a(a^t_\pi f^{\varrho k}_{\pi n_\pi}) \quad \text{für} \quad a\in G_{\varrho k},$$

also $y_1=(xf^{\varrho k}_{\pi t})\,(a^t_\pi f^{\varrho k}_{\pi n_\pi})$ und dann

$$z\circ x = (xf^{\sigma l}_{\pi t})z(a^t_\pi f^{\sigma l}_{\pi n_\pi}) = z(y_1 f^{\sigma l}_{\varrho k}),$$

womit $\gamma_{\varrho k}\varphi^{\sigma l}_{\varrho k}=f^{\sigma l}_{\varrho k}\gamma_{\sigma l}$ gilt. Hiermit ist die erste Hälfte des Satzes bewiesen.

Jetzt sei $H'=H(\mathfrak{F}', D', a'_\varrho)\cong H$. Aus den oben bewiesenen geht hervor, dass $H_{\mathfrak{F}}, D'$ mit Strukturinvarianten isomorph sind, also muss $\mathfrak{F}'=\mathfrak{F}, D'\cong D$ erfüllt sein. Ist dabei $\iota:H\to H'$ ein Isomorphismus, so induziert er natürlicherweise Isomorphismen $\nu_{\varrho k}:\Gamma_{\varrho k}\to\Gamma'_{\varrho k}$ $(\nu_{\varrho k}:\tau_x\mapsto\tau_{x\iota})$ zwischen den Schützenbergergruppen, und $\iota_{\varrho k}=\gamma_{\varrho k}\nu_{\varrho k}\gamma'^{-1}_{\varrho k}$ sind die erwünschten Isomorphismen, wo $\gamma'_{\varrho k}$ die zu den $\gamma_{\varrho k}$ analogen Isomorphismen zwischen $G'_{\varrho k}$ und $\Gamma'_{\varrho k}$ sind. Was die letzte Behauptung betrifft, haben wir $a^{-1}_\varrho\iota=a'^{-1}_\varrho$ im Falle, wo $\{G_{\varrho' m_\varrho}, \circ\}$ eine Gruppe ist, weil dann a^{-1}_ϱ und a'^{-1}_ϱ die Einselemente der Gruppen $\{G_{\varrho' m_\varrho}, \circ\}$ bzw. $\{G'_{\varrho' m_\varrho}, \circ\}$ sind. Das Einselement von $G_{\varrho k}$ bezeichnen wir durch $e_{\varrho k}$. Dann folgt wegen $e_{\varrho 1}\iota\in G'_{\varrho 1}$

$$e_{\varrho' m_\varrho}\iota = (e_{\varrho 1}\circ a^{-1}_\varrho)\iota = e_{\varrho 1}\iota\circ a^{-1}_\varrho\iota = (e_{\varrho 1}\iota f'^{\varrho' m_\varrho}_{\varrho 1})\cdot a^{-1}_\varrho\iota\cdot a'_\varrho = e_{\varrho 1}\iota f'^{\varrho' m_\varrho}_{\varrho 1}$$

und

$$a_\varrho\iota = (e_{\varrho 1}\circ e_{\varrho' m_\varrho})\iota = e_{\varrho 1}\iota\circ e_{\varrho' m_\varrho}\iota = (e_{\varrho 1}\iota f'^{\varrho' m_\varrho}_{\varrho 1})^2 a'_\varrho\in(G'_{\varrho 1}f'^{\varrho' m_\varrho}_{\varrho 1})\cdot a'_\varrho.$$

Die Umkehrung ist leicht zu kontrollieren.

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The lattice of translations on a lattice

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1. Introduction and preliminaries. The purpose of this paper is to consider the lattice of all translations on a lattice and to illuminate the decomposition of lattices generated by translations on lattices. Also some properties of translations on meet-semilattices are given.

Let S be a meet-semilattice and φ a single-valued mapping of S into itself. φ is called a meet-translation, briefly a translation, on S, if $\varphi(x\wedge y)=\varphi(x)\wedge y$ for each pair x, y of elements in S. A translation φ on a lattice L is defined analogously. Each translation φ on S (and on L) has the following properties [7]: $\varphi(x)\leqq x$, $\varphi(x)=\varphi(\varphi(x))$, and $x\leqq y\Rightarrow\varphi(x)\leqq\varphi(y)$. In a lattice L the fixelements of φ, i.e. the elements $t=\varphi(t)$, constitute an ideal K_φ of L, which determines φ uniquely.

A non-empty subset J of a meet-semilattice S is called a semi-ideal of S, if (i) $a\leqq b$ and $b\in J$ imply $a\in J$, and (ii) $a, b\in J$ imply $a\vee b\in J$ whenever $a\vee b$ exists in S. As one can easily conclude from [7, Thm. 1], the fixelements of a translation φ on a meet-semilattice S form a semi-ideal K_φ of S, and K_φ determines φ uniquely [7, Thm. 3].

We denote by $\mathscr{I}(L)$ the lattice of all ideals of a lattice L, $(a]=\{x|x\leqq a, x, a\in S\}$ is the principal ideal generated by a. The semi-ideals of a meet-semilattice S constitute a lattice $\mathscr{J}(S)$ with respect to the set-theoretical inclusion; $I\vee J$ means the least semi-ideal containing I and J of $\mathscr{J}(S)$.

A translation $s_a(x)=a\wedge x$ is called a specified translation.

The following lemma was proved in [6]:

Lemma 1. *An ideal I of a lattice L generates a translation φ on L, i.e. $K_\varphi=I$, if and only if for each $y\in L$ there is an element $k_y\in I$ such that $I\wedge(y]=(k_y]$.*

A direct analogy holds for translations φ on a meet-semilattice S and semi-ideals J of S.

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2. Translations on a lattice. We denote by $\Phi(L)$ the set of all translations on L. As shown by SZÁSZ and SZENDREI [8, Thm. 3], $\Phi(L)$ is a meet-semilattice.

Theorem 1. *Let φ and λ be two translations on a lattice L. The mapping β on L, defined by $\beta(x)=\varphi(x)\vee\lambda(x)$, is a translation on L if and only if $(K_\varphi\vee K_\lambda)\wedge\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])$ for each $x\in L$.*

Proof. Let $(K_\varphi\vee K_\lambda)$ have the property of the theorem. Then $(K_\varphi\vee K_\lambda)\wedge\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])=(\varphi(x)]\vee(\lambda(x)]=(\varphi(x)\vee\lambda(x)]$, and so $K_\varphi\vee K_\lambda$ generates a translation on L with values $\varphi(x)\vee\lambda(x)$, i.e. $K_\varphi\vee K_\lambda$ generates a translation β on L. Conversely, let β be a translation on L. The fixelements of β are the elements $\varphi(x)\vee\lambda(x)$ $(x\in L)$, and so $K_\beta=K_\varphi\vee K_\lambda$. According to Lemma 1, $(\beta(x)]==(K_\varphi\vee K_\lambda)\wedge(x]=(\varphi(x)]\vee(\lambda(x)]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])$, and the latter part of the theorem follows.

Corollary 1. *Let φ be a translation on L. The mapping $\varphi\vee\lambda$ is a translation on L for each $\lambda\in\Phi(L)$ if and only if K_φ is a standard element of $\mathscr{I}(L)$.*

Proof. If K_φ is standard, then $(K_\varphi\vee K_\lambda)\wedge(x]=(K_\varphi\wedge(x])\vee(K_\varphi\wedge(x])$ for each $\lambda\in\Phi(L)$. Hence $\beta(x)=\varphi(x)\vee\lambda(x)$ is a translation on L. Conversely, if $\varphi\vee\lambda$ is a translation for each $\lambda\in\Phi(L)$, then, in particular the relation $((a]\vee K_\varphi)\wedge\wedge(x]=((a]\wedge(x])\vee(K_\varphi\wedge(x])$ holds for each specified translation s_a, $a\in L$, and for each $x\in L$. But already this equation implies the standardness of K_φ according to [1, Thm. 2(α'')].

Corollary 2. *The meet-semilattice $\Phi(L)$ is a lattice if and only if L is a distributive lattice.*

Proof. If L is a distributive lattice, each $I\in\mathscr{I}(L)$ is a standard element in $\mathscr{I}(L)$, and the first part of the assertion follows. Conversely, if $\Phi(L)$ is a lattice, then each ideal $(a]$ generating a specified translation s_a on L is a standard element of $\mathscr{I}(L)$, from which the distributivity of L follows.

Lemma 2. *$\Phi(L)$ contains always a greatest element ω, and there is a least elemetn τ in $\Phi(L)$ if and only if $0\in L$.*

Proof. The identical mapping $\omega(x)=x$ is a translation on L and $K_\omega=L$; evidently it is the greatest translation on L. The mapping $\tau(x)=0$ is obviously the least translation on L whenever a least element 0 exists in L, and $k_\tau=(0]$. If there is no least element in L, then there exists for each $a_1\in L$ an infinite chain $a_1>a_2>\dots$ and the corresponding specified translations form an infinitely descending chain, whence $\tau\notin\Phi(L)$.

In the following we shall consider a decomposition of a lattice by means of translations on this lattice. In [2] JANOWITZ considered the decomposition of a lattice into a direct sum; this decomposition is generalized for join-semilattices in [5]. Let L be a lattice with 0. $a \triangledown b$ denotes the fact that $a \wedge b = 0$ and $(a \vee x) \wedge b = x \wedge b$ for all $x \in L$. For a subset H of L we denote by $H^{\triangledown}$ the set of elements $a \in L$ such that $a \triangledown b$ for all $b \in H$. In a lattice L with 0, let $H_1, \ldots, H_n$ be subsets of L, each of which contains 0. We say that L is the direct sum of $H_1, \ldots, H_n$ and write $L = H_1 \oplus \ldots \oplus H_n$ when

(1) every element $a \in L$ can be expressed in the form $a = a_1 \vee \ldots \vee a_n$, $a_i \in H_i$, $i = 1, \ldots, n$, and

(2) $H_i \subset H_j^{\triangledown}$ for $i \neq j$.

The subsets $H_1, \ldots, H_n$ are called direct summands of L. If $L = H_1 \oplus \ldots \oplus H_n$, then the expression in (1) unique and the sets $H_1, \ldots, H_n$ are ideals of L [4, Lemma 4.8]. Moreover, in a lattice L with 0, an ideal J of L is a central element of $\mathscr{I}(L)$ if and only if it is a direct summand of L [2, Thm. 1]. Now we are able to prove a theorem on direct sums of a lattice.

Theorem 2. *A lattice L with 0 has a decomposition into non-trivial direct summands if and only if there are at least two non-trivial translations φ and λ on L such that $\varphi \vee \lambda = \omega$ and $\varphi \wedge \lambda = \tau$, and φ and λ have join with each translation on L.*

Proof. Let $L = J \oplus K$. According to [2, Thm. 1], J and K are standard elements of $\mathscr{I}(L)$, and $J \wedge K = (0]$ and $J \vee K = L$ in $\mathscr{I}(L)$. Consider the meet $J \wedge (x]$, $x \in L$. As $L = J \oplus K$, $x = a_1 \vee a_2$, $a_1 \in J$ and $a_2 \in K$, and the expression $x = a_1 \vee a_2$ is unique. So $J \wedge (x] = (a_1]$, $a_1 \in J$, and hence J generates a translation φ on L. As J is standard in $\mathscr{I}(L)$, the join $\varphi \vee \mu$ exists for each translation $\mu \in \Phi(L)$. Similar facts hold also for the translation λ on L generated by K. $\varphi \wedge \lambda$ corresponds to the translation generated by the ideal $J \wedge K = (0]$, i.e. τ, and $\varphi \vee \lambda$ that of $J \vee K = L$, i.e. ω. As $J, K \neq L, (0]$, φ and λ are non-trivial translations on L, and the first part of the theorem follows.

Conversely, let φ and λ be two translations with the properties given in the theorem. As $\varphi \vee \mu$ exists for each translation $\mu \in \Phi(L)$, the ideal J generating φ is a standard element of the lattice $\mathscr{I}(L)$ (by Corollary 1 to Theorem 1), and this holds also for the ideal K generating λ. As $\varphi \wedge \lambda = \tau$ and $\varphi \vee \lambda = \omega$, $J \wedge K = (0]$ and $J \vee K = L$, respectively. As J and K are standard and complements, they belong to the center of $\mathscr{I}(L)$ [3, Thm. 7.2] and, accordingly, $L = J \oplus K$ [2, Thm. 1]. As φ and λ are non-trivial, $J, K \neq L, (0]$, and the decomposition is also non-trivial.

3. Translations on partial lattices. We call a meet-semilattice S a partial lattice if $a \vee b$ exists for any two $a, b \in S$ having a common upper bound in S. At first we consider the structure of meet-semilattices S for which $\Phi(L)$ is a lattice.

Let $\varphi(x)$ and $\lambda(x)$ be translations on a partial lattice S. As in the case of lattices, one can show that $\beta(x)=\varphi(x)\vee\lambda(x)$ is a translation on S if and only if $(K_\varphi\vee K_\lambda)\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])$ for each $x\in S$, $K_\varphi, K_\lambda, (x]\in\mathscr{I}(S)$.

Theorem 3. *Let S be a partial lattice. Then the following three assumptions are equivalent:*

(i) *The meet-semilattice of all translations on S is a lattice.*

(ii) *Each translation on S is a join-endomorphism on S.*

(iii) $(x]$ *is a distributive sublattice of S for each $x\in S$.*

Proof. We shall show that (i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iii).

(iii)$\Rightarrow$(i). We shall show that $\mathscr{I}(S)$ is a distributive lattice, from which the validity of the assertion follows.

Let $I, J\in\mathscr{I}(S)$. $I\wedge J=I\cap J$, and $I\vee J=\{x|x\leqq i\vee j, i\in I, j\in J$ and $i\vee j\in S\}$. We must only show that $F\wedge(I\vee J)\subseteqq(F\wedge I)\vee(F\wedge J)$ when $F, I, J\in\mathscr{I}(S)$. Clearly, $x\in F\wedge(I\vee J)\Leftrightarrow x\in F$ and $x\leqq i\vee j$, where $i\in I$ and $j\in J$. By assumption, $(i\vee j]$ is a distributive sublattice of S and $i, j, x\in(i\vee j]$. So $x=x\wedge(i\vee j)=(x\wedge i)\vee(x\wedge j)$, where $(x\vee i)\in F\wedge I$ and $x\vee j\in F\wedge J$. Therefore, $x\in(\wedge I)\vee(F\wedge J)$.

(i)$\Rightarrow$(iii). Let $\Phi(S)$ be a lattice and $w, y, z\in(x]$ in S. Then the mapping $s_y\vee s_z$ is a translation on S, whence $(y\vee z]\wedge(u]=(y\wedge u]\wedge(z\vee u]$ for each $u\in S$ by the analogy of Theorem 1. The distributivity of $(x]$ follows now by putting $u=w$.

(iii)$\Rightarrow$(ii). Let J be a semi-ideal of S generating a translation φ on S, and assume that $x\vee y$ exists in S. As $x\vee y$ exists and $x\leqq\varphi(x), y\leqq\varphi(y)$, then $\varphi(x)\vee\vee\varphi(y)$ exists in S. As shown in the proof (iii)$\Rightarrow$(i), $\mathscr{I}(S)$ is a distributive lattice. Let us consider now $\varphi(x\vee y)$, i.e. the meet $J\wedge(x\vee y]=(J\wedge(x])\vee(J\wedge(y])$, which implies that $\varphi(x\vee y)=\varphi(x)\vee\varphi(y)$. Thus φ is also a join-endomorphism on S.

(ii)$\Rightarrow$(iii). Let $u, w, z\in(x]$. As the mapping s_u is also a join-endomorphism, $s_u(w\vee z)=(u]\wedge(w\vee z]=s_u(w)\vee s_u(z)=((u]\wedge(w])\vee((u]\wedge(z])$, from which the distributivity of $(x]$ follows.

As above, one can easily prove that in a partial lattice S each $(x]$ is a modular lattice of S if and only if $\mathscr{I}(S)$ is a modular lattice. The proof of the following theorem is analogous to that of Theorem 3, and hence we omit it.

Theorem 4. *Let S be a partial lattice. Each translation on S has the property that $\varphi(\varphi(z)\vee y)=\varphi(z)\vee\varphi(y)$ when $\varphi(z)\vee y$ exists in S, if and only if $(x]$ is a modular sublattice of S for each $x\in S$.*

The equivalenc (ii)$\Leftrightarrow$(iii) in Theorem 3 and Theorem 4 are generalizations of Theorems 4 and 5 in SzÁsz's paper [7].

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Algebras intertwining compact operators

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The main result of this note is that if $\mathscr{A}$ is a norm-closed algebra of (bounded) operators on a (complex) Banach space $\mathfrak{X}$, if K is a nonzero compact operator on $\mathfrak{X}$, and if $\mathscr{A}K=K\mathscr{A}$, then $\mathscr{A}$ has a non-trivial (closed) invariant subspace. This is an extension of Lomonosov's theorem that every compact operator has a non-trivial hyperinvariant subspace. For injective compact operators we shall prove stronger results.

LOMONOSOV's result [4] quoted above implies that if $AK=KA$ for every A in the algebra $\mathscr{A}$, then $\mathscr{A}$ has invariant subspaces. It is very easy to construct uniformly closed algebras $\mathscr{A}$ with $\mathscr{A}K=K\mathscr{A}$ for some compact operator K, where $\mathscr{A}$ has members not commuting with K. (See Remark (i) below.) In Section 2 we shall mention other contrasts with the case of Lomonosov's result.

In what follows $\mathfrak{X}$ and $\mathfrak{Y}$ will always denote Banach spaces, $\mathscr{B}(\mathfrak{X})$ the algebra of all operators on $\mathfrak{X}$, and $\mathscr{A}$ a subalgebra of $\mathscr{B}(\mathfrak{X})$; subalgebras are not assumed to have identities.

1. Main Results. We start with the following lemma which may be of some independent interest. (See also Remark (iv) in Section 2.)

Lemma 1. *Let K be a compact operator on $\mathfrak{X}$ and let S be any operator on $\mathfrak{Y}$. Let T be a bounded linear transformation of $\mathfrak{Y}$ into $\mathfrak{X}$ such that $KTS=T$. Then T has finite rank.*

Proof. Let C be the circle of radius $r>0$ centered at the origin in the complex plane; assume that (i) r is sufficiently small so that $1-\lambda S$ is invertible for λ inside C; and (ii) C does not intersect the spectrum of K. Let P be the Riesz projection

$$P=\frac{1}{2\pi i}\int_C (K-\lambda)^{-1}\,d\lambda.$$

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(See, e.g., [7, p. 31].) We show that $T\mathfrak{Y}$ is contained in the finite-dimensional space $(1-P)\mathfrak{X}$. Let y be any vector in $\mathfrak{Y}$ and let $x=Ty$. Then, for λ inside C,

$$Px = (K-\lambda)PTS(1-\lambda S)^{-1}y = (K_P-\lambda)PTS(1-\lambda S)^{-1}y,$$

where $K_P=PK|P\mathfrak{X}$.

Since $\sigma(K_P)$ lies inside C, it follows that $(K_P-\lambda)^{-1}Px$ has an analytic extension to the entire plane and thus $Px=0$. Hence $x\in(1-P)\mathfrak{X}$.

Lemma 2. *Let A and B be bounded linear transformations from $\mathfrak{X}$ into $\mathfrak{Y}$. If $A\mathfrak{X}\subseteqq B\mathfrak{X}$ and if B is injective, then there exists an operator S on $\mathfrak{X}$ such that $A=BS$.*

Proof. The proof given in [2] for a stronger theorem on Hilbert spaces works for this lemma also: just observe that $B^{-1}A$ is a closed operator.

Theorem 3. *Let $\mathscr{A}$ be a norm-closed subalgebra of $\mathscr{B}(\mathfrak{X})$ and let K be an injective, non-quasinilpotent compact operator on $\mathfrak{X}$ such that $\mathscr{A}K\subseteqq K\mathscr{A}$. Then $\mathscr{A}$ has a nonzero finite-dimensional invariant subspace.*

Proof. Assume, with no loss of generality, that there exists a nonzero x_0 in $\mathfrak{X}$ with $Kx_0=x_0$. Then $\mathscr{A}x_0\subseteqq K(\mathscr{A}x_0)$. The linear manifold $\mathscr{A}x_0$ of $\mathfrak{X}$ is the range of the linear transformation T of $\mathscr{A}$ (considered as a Banach space) into $\mathfrak{X}$ defined by

$$T(A) = Ax_0.$$

Clearly T is bounded, and $T(\mathscr{A})\subseteqq KT(\mathscr{A})$. Let $\mathscr{A}_0$ be the null space of T and let $\mathfrak{Y}$ be the quotient space $\mathscr{A}/\mathscr{A}_0$. Then $\hat{T}\mathfrak{Y}\subseteqq K\hat{T}\mathfrak{Y}$, where $\hat{T}$ is the induced injective linear transformation from $\mathfrak{Y}$ into $\mathfrak{X}$. Since K is injective, so is $K\hat{T}$. Thus there exists, by Lemma 2, an operator S on $\mathfrak{Y}$ with $\hat{T}=K\hat{T}S$. It follows from Lemma 1 that $\hat{T}$ has finite rank. But

$$\hat{T}(\mathfrak{Y}) = T(\mathscr{A}) = \mathscr{A}x_0,$$

and thus $\mathscr{A}x_0$ is a finite-dimensional invariant subspace for $\mathscr{A}$. (If $\mathscr{A}x_0=0$, then x_0 generates a 1-dimensional invariant subspace.)

The following lemma is a special case of a result of Foiaş [3].

Lemma 4. *Let $\mathscr{A}K\subseteqq K\mathscr{A}$, where $\mathscr{A}$ is norm-closed and K is injective (and not necessarily compact). Then the map φ on $\mathscr{A}$ defined by*

$$AK = K\varphi(A).$$

is a continuous algebra homomorphism.

Proof. The map φ is clearly a homomorphism. To prove that φ is continuous, it suffices to show that it is a closed map: if $A=\lim A_n$ and $B=\lim \varphi(A_n)$, then

$$AK=\lim A_n K=\lim K\varphi(A_n)=KB,$$

and thus $B=\varphi(A)$.

Theorem 5. *Let $\mathcal{A}K\subseteqq K\mathcal{A}$, where $\mathcal{A}$ is norm-closed and K is injective, compact, and quasinilpotent. Then $\mathcal{A}$ has a non-trivial invariant subspace.*

Proof. The main idea in the proof is that used in the simple, elegant proof given by H. M. Hilden for Lomonosov's result quoted above (cf. [7], p. 165).

We start, as in Lomonosov's proof, by assuming with no loss of generality that $\|K\|=1$, and that $\mathcal{A}$ is transitive if possible. Fix x_0 in $\mathfrak{X}$ such that $\|Kx_0\|>1$ (and thus also $\|x_0\|>1$), and let $\mathfrak{S}$ be the open ball of radius 1 centered at x_0. It follows from the transitivity of $\mathcal{A}$ that the open sets $A^{-1}(\mathfrak{S}), A\in\mathcal{A}$, cover $\mathfrak{X}\setminus\{0\}$, and thus they also cover the compact set $\overline{K\mathfrak{S}}$. Hence there is a finite subset $\mathcal{F}$ of $\mathcal{A}$ such that

$$\overline{K\mathfrak{S}}\subseteqq\bigcup_{A\in\mathcal{F}}A^{-1}(\mathfrak{S}).$$

Let $r=\max\{\|A\|:A\in\mathcal{F}\}$. Given any positive integer n, one can inductively obtain $A_1,\dots,A_n$ in $\mathcal{F}$ such that

$$A_nKA_{n-1}K\dots A_2KA_1Kx_0\in\mathfrak{S}.$$

But, again by induction,

$$A_nKA_{n-1}K\dots A_2KA_1K=K^n\varphi\big(\dots(\varphi(\varphi(\varphi(A_n)A_{n-1})A_{n-2})\dots A_1\big).$$

Therefore

$$\|A_nKA_{n-1}K\dots A_1K\|\leqq\|K^n\|\cdot\|\varphi\|^n\cdot r^n=\big\|(r\|\varphi\|\cdot K)^n\big\|.$$

Since $r\|\varphi\|K$ is quasinilpotent, the vector $A_nK\dots A_1Kx_0$ in $\mathfrak{S}$ would be arbitrarily small for sufficiently large n. This contradicts the fact that $\|x_0\|>1$. Thus $\mathcal{A}$ cannot be transitive.

Theorem 6. *Let $\mathcal{A}$ be a norm-closed subalgebra of $\mathcal{B}(\mathfrak{X})$ which intertwines a nonzero compact operator. Then $\mathcal{A}$ has a non-trivial invariant subspace.*

Proof. Let $\mathcal{A}K=K\mathcal{A}$, where K is compact and nonzero. If K is injective, the assertion follows from Theorems 3 and 5; otherwise the null space of K is a nontrivial subspace invariant under $\mathcal{A}$.

2. Remarks.

(i) Let $\mathfrak{H}_1$ be a finite-dimensional subspace of the infinite-dimensional Hilbert space $\mathfrak{H}$. Let K_1 be an invertible, non-scalar operator on $\mathfrak{H}_1$, and let K_2 be an injective compact operator in $\mathfrak{H}\ominus\mathfrak{H}_1$. Let $K=K_1\oplus K_2$, and let $\mathscr{A}=\mathscr{B}(\mathfrak{H}_1)\oplus\mathscr{A}_2$, where $\mathscr{A}_2$ is the commutant of K_2. Then $\mathscr{A}K=K\mathscr{A}$, but not every member of $\mathscr{A}$ commutes with K. (In this example $\mathscr{A}$ is also weakly closed.)

A less trivial example can be constructed as follows. Let K be any nonzero compact operator on $\mathfrak{H}$ and let $\mathscr{A}$ be the weakly closed algebra of all operators of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ on $\mathfrak{H}\oplus\mathfrak{H}$, where A and C commute with K but B is arbitrary. Then $\mathscr{A}$ intertwines the operator $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ but is not contained in the commutant of *any* nonscalar operator.

(ii) There are examples of $\mathscr{A}$ and K as in Theorem 3 with $\mathscr{A}K$ *properly* contained in $K\mathscr{A}$. Let $\mathscr{A}=\mathscr{B}^*$, where $\mathscr{B}$ is the algebra of all analytic Toeplitz operators on a Hilbert space $\mathfrak{H}$, and represent $\mathscr{A}$ as an algebra of uppertriangular matrices. Let K be the compact operator represented by a diagonal matrix $\mathrm{Diag}\{\lambda^n\}_{n=0}^{\infty}$ with $|\lambda|<1$. Then it can be verified that $\mathscr{A}K\subsetneqq K\mathscr{A}$.

(iii) In contrast with the case of Lomonosov's Theorem, it is essential in our results that $\mathscr{A}$ be closed. Let, for instance, $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathfrak{H}$, and let $\mathscr{A}$ be the algebra of all operators on $\mathfrak{H}$ whose matrices elative to $\{e_i\}_{i=1}^{\infty}$ have only finitely many nonzero entries. Then $\mathscr{A}$ is clearly (topologically) transitive, but $\mathscr{A}K=K\mathscr{A}$ for any injective operator (compact or not) whose matrix relative to $\{e_i\}_{i=1}^{\infty}$ is diagonal.

(iv) Using properties of decomposable operators [1, pp. 30—31] one can prove another version of Lemma 1 as follows:

Lemma 1′. *Let K be a non-invertible, decomposable operator on $\mathfrak{X}$ and let S be any operator on $\mathfrak{Y}$. Let T be a bounded linear transformation of $\mathfrak{Y}$ into $\mathfrak{X}$ such that $KTS=T$. Then the range of T is not dense in $\mathscr{X}$.*

A corresponding version of Theorem 3 follows.

Theorem 3′. *Let $\mathscr{A}$ be a norm-closed subalgebra of $\mathscr{B}(\mathfrak{X})$ and let K be an injective, non-invertible, decomposable operator with a nonzero eigenvalue, such that $\mathscr{A}K\subseteq K\mathscr{A}$. Then $\mathscr{A}$ has a non-trivial invariant subspace.*

Lemma 1′ and Theorem 3′ remain true if "decomposable" is replaced by "hyponormal" or by "subspectral". (Use [5, Lemma 1] and [6, Proposition 1].)

(v) We conjecture that if a norm-closed algebra $\mathscr{A}$ leaves invariant the range of a compact operator K, then it has a non-trivial invariant subspace. The hypothesis is equivalent to the inclusion $\mathscr{A}K \subseteqq K\mathscr{B}(\mathfrak{X})$.

A weaker version of the conjecture is obtained by assuming $\mathscr{A}$ to be closed in the strong operator topology. Validity of this version would follow from that of the transitive-algebra conjecture [7, p. 138]: every strongly closed transitive algebra of operators on $\mathfrak{X}$ is $\mathscr{B}(\mathfrak{X})$.

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Differentiability for Rademacher series on groups

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In the paper [1] P. L. BUTZER and H. J. WAGNER defined the derivative of real-valued functions f defined on the dyadic group, both in the pointwise sense and in the strong sense, that is, with respect to the norm of the space to which f belongs. They proved that this derivative has many properties similar to properties of the ordinary derivative of functions on the circle group. In the present paper we shall extend this definition to functions defined on groups G that are the direct product of countably many groups of prime order. Furthermore, we shall give some applications to functions that are defined as the sum of a Rademacher series on G.

1. Introduction

Let $\{p_n\}$ be a sequence of prime numbers and let G be the direct product of groups of order p_n, that is, $G=\prod_{n=1}^{\infty} Z(p_n)$. Thus the elements of G are of the form $x=(x_1, x_2, \ldots)$, with $0\leqq x_n<p_n$ for each $n\geqq 1$ and for x, y in G the n-th coordinate of their sum $x+y$ is obtained by adding the n-th coordinates of x and y modulo p_n. Furthermore, if we define the subgroups G_n of G by $G_0=G$ and for $n\geqq 1$

$$G_n = \{x\in G;\ x_1 = \ldots = x_n = 0\},$$

then the G_n's form a basis for the neighborhoods of $0=(0, 0, \ldots)$ in G. Finally, for $n\geqq 1$ we define the elements e_n of G by $(e_n)_i=0$ if $i\neq n$ and $(e_n)_n=1$.

Next, let $\hat{G}$ denote the character group of G. We enumerate the elements of $\hat{G}$ as follows. For each $k\geqq 0$ and each x in G let $\varphi_k(x)$ be defined by

$$\varphi_k(x) = \exp(2\pi i x_{k+1}/p_{k+1}).$$

Thus, $\varphi_k(e_j)=1$ if $j\neq k+1$ and $\varphi_k(e_{k+1})=\exp(2\pi i/p_{k+1})=\omega_k$. We observe here

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that for each $j \geqq 0$

$$\sum_{l=0}^{p_{j+1}-1} (\omega_j)^{lk} = \begin{cases} 0 & \text{if } 0 < k < p_{j+1}, \\ p_{j+1} & \text{if } k = 0. \end{cases} \tag{1}$$

Let the sequence $\{m_n\}$ be defined by $m_0=1$ and $m_n=p_n \cdot m_{n-1}$ for $n \geqq 1$. Next, if $n \geqq 0$ is represented as $n=a_0m_0+\ldots+a_km_k$, with $0 \leqq a_i < p_{i+1}$ for each $i \geqq 0$, then we define χ_n by

$$\chi_n(x) = \varphi_0^{a_0}(x) \cdot \ldots \cdot \varphi_k^{a_k}(x) = \prod_{\nu=0}^{k} \exp(2\pi i a_\nu x_{\nu+1}/p_{\nu+1}). \tag{2}$$

The χ_n's are precisely the elements of $\hat{G}$. The functions φ_n are called the Rademacher functions on G and the χ_n are called the generalized Walsh functions on G.

Remark 1. If $p_n=2$ for all n, then G is the so-called dyadic group. The elements of the character group $\hat{G}$, when ordered as indicated here, are the Walsh (-Paley) functions, see [3].

Let dx or m denote normalized Haar measure on G. For f in $L_1(G)$ we define its generalized Walsh-Fourier series by

$$\sum_{k=0}^{\infty} \hat{f}(k)\chi_k(x), \quad \text{where} \quad \hat{f}(k) = \int_G f(t)\overline{\chi_k(t)}\,dt.$$

In a number of previous papers, [4] and [5], we have studied several properties of such generalized Walsh-Fourier series. Among other things we defined the concept of r-generalized bounded fluctuation. We recall the definition here. For each subgroup G_n of G we denote the m_n cosets of G_n in G by $z_{q,n}+G_n$, $q=0, 1, \ldots, m_n-1$, with $z_{0,n}+G_n=G_n$. If f is a function on G and if $H \subset G$ then

$$\operatorname{osc}(f; H) = \sup\{|f(x)-f(y)|;\, x, y \in H\}.$$

Definition 1. Let f be a function on G, r a real number with $r \geqq 1$, and

$$V_r(f) = \sup\left\{\left\{\sum_{q=0}^{m_n-1} (\operatorname{osc}(f;\, z_{q,n}+G_n))^r\right\}^{1/r};\ n = 0, 1, \ldots\right\}.$$

The function f is of *r-generalized bounded fluctuation* ($f \in r$-GBF) if $V_r(f) < \infty$.

In [6] and [5] it was shown that functions in r-GBF have many properties similar to properties of functions of r-bounded variation (r-BV) on the circle group T. However, we shall show that the differentiability properties of functions in GBF, that is, in 1-GBF, are unlike those of functions in BV.

2. Differentiation of functions on G

Definition 2. If for a complex-valued function f on G and for x in G

$$\lim_{m\to\infty}\sum_{j=0}^{m} m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} f(x+le_{j+1})$$

exists, then we call this limit the *pointwise derivative* of f at x, denoted by $f^{[1]}(x)$.

Definition 3. Let $X(G)$ denote either $C(G)$ or $L_p(G)$, $1\leqq p<\infty$, with the usual norm. If for f in $X(G)$ there exists a g in $X(G)$ such that

$$\lim_{m\to\infty}\left\|\sum_{j=0}^{m} m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} f(.+le_{j+1})-g(.)\right\|_{X(G)}=0,$$

then g is the *strong derivative* of f, denoted by $D^{[1]}f$.

Higher order derivatives are defined recursively. If $p_n=2$ for all n then these definitions agree with the definitions of Butzer and Wagner [1]. These authors showed that the Walsh functions $\chi_n(x)$ have the property that $D^{[1]}\chi_n=n\chi_n$ in each space $X(G)$ and $\chi_n^{[1]}(x)=n\chi_n(x)$ for all x in G. Further results in [1] are largely based on these identities. Therefore we shall prove that the derivatives as presently defined for functions on G satisfy the same identities, after which it is easy to extend most of the results in [1] to functions on G.

Remark. We would like to thank the referee for bringing the paper by Gibbs and Ireland [4] to our attention. In it the authors define the derivative for functions on groups G which are the direct product of finitely many cyclic groups. Their definition closely resembles our Definition 2, see [4, Section VI].

Theorem 1. *For each $n\geqq 0$ and each x in G we have $\chi_n^{[1]}(x)=n\chi_n(x)$.*

Proof. Since $\chi_0(x)\equiv 1$, the theorem is clearly true for $\chi_0(x)$.

Assume $n=a_0m_0+\ldots+a_rm_r$, with $0\leqq a_i<p_{i+1}$ for each $i\geqq 0$ and $a_r\neq 0$. Take a fixed j with $0\leqq j\leqq r$. Then

$$m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} \chi_n(x+le_{j+1}) =$$

$$= m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} \chi_n(x)(\chi_n(e_{j+1}))^l =$$

$$= m_j\chi_n(x) \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{l(a_j-k)},$$

according to (2). Using (1) we see that this expression can be simplified further into

$$m_j\chi_n(x)a_jp_{j+1}^{-1}p_{j+1} = a_jm_j\chi_n(x).$$

Next, for each $j>r$ we have

$$m_j\sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1}\sum_{l=0}^{p_{j+1}-1}(\omega_j)^{-lk}\chi_n(x+le_{j+1}) = m_j\chi_n(x)\sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1}\sum_{l=0}^{p_{j+1}-1}(\omega_j)^{-lk} = 0.$$

Therefore,

$$\chi_n^{[1]}(x) = \sum_{j=0}^{r} a_jm_j\chi_n(x) = n\chi_n(x).$$

It is clear that a similar result holds for the strong derivative of χ_n in each of the spaces $X(G)$.

3. Rademacher series on G

In this section we shall consider Rademacher series on G, that is, functions defined by a series $R(x)=\sum_{i=0}^{\infty} c_i\varphi_i(x)$. We shall assume that c_k is real for each $k\geqq 0$ and that $R(x)$ exists for all x in G. The last assumption is equivalent to the condition that $\sum_{i=0}^{\infty}|c_i|<\infty$, as can be seen as follows. Define the element $x=(x_1, x_2, \ldots)$ in G by $x_{i+1}=0$ if $c_i\geqq 0$ and $x_{i+1}=1$ if $c_i<0$ and $p_{i+1}=2$, whereas $x_{i+1}=(p_{i+1}-1)/2$ if $c_i<0$ and $p_{i+1}\neq 2$. Then $\varphi_i(x)=1$ if $c_i\geqq 0$ and $\operatorname{Re}[\varphi_i(x)]\leqq -1/2$ if $c_i<0$. Consequently, for all $i\geqq 0$ we have $c_i\operatorname{Re}[\varphi_i(x)]\geqq |c_i|/2$ and this shows that $\sum_{i=0}^{\infty}|c_i|<\infty$. We also observe that for Rademacher series on G the following proposition holds. Its proof is similar to the proof for the case of Rademacher series on the dyadic group [7, page 212] and will not be given here.

Proposition 1. *If* $R(x)=\sum_{i=0}^{\infty}c_i\varphi_i(x)$ *is a Rademacher series on* G *then* (i) *if* $\sum_{i=0}^{\infty}|c_i|^2<\infty$, *then* $R(x)$ *converges a.e.*, (ii) *if* $\sum_{i=0}^{\infty}|c_i|^2=\infty$, *then* $R(x)$ *diverges a.e.*

Now we turn to the differentiability of such Rademacher series.

Theorem 2. *R is differentiable at a point x in G if and only if* $\sum_{k=0}^{\infty} m_kc_k\varphi_k(x)$ *converges.*

Proof. For each $j\geqq 0$ and each l with $0\leqq l<p_{j+1}$ we have

$$R(x+le_{j+1}) = \sum_{i=0}^{\infty} c_i\varphi_i(x+le_{j+1}) = \sum_{i=0}^{j-1} c_i\varphi_i(x)+c_j\varphi_j(x)(\omega_j)^l+\sum_{i=j+1}^{\infty} c_i\varphi_i(x).$$

Hence, for each $j \geqq 0$ we find, using (1), that

$$m_j \sum_{k=0}^{p_{j+1}-1} k p_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} R(x+le_{j+1}) =$$

$$= m_j \sum_{k=0}^{p_{j+1}-1} k p_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} c_j \varphi_j(x)(\omega_j)^l = m_j c_j \varphi_j(x) p_{j+1}^{-1} p_{j+1} = m_j c_j \varphi_j(x).$$

Consequently, R is differentiable at x if and only if $\sum_{j=0}^{\infty} m_j c_j \varphi_j(x)$ converges.

We now mention a number of corollaries of Theorem 2. For a Rademacher series R let

$$\Delta_R = \{x \in G;\ R \text{ differentiable at } x\}.$$

Corollary 1. *If for some x in G we have $x \in \Delta_R$ and if y is a rational element of G, that is, y has the property that there exists a constant K so that $y_k = 0$ for $k > K$, then $x+y \in \Delta_R$.*

Proof. Since for $j \geqq K$ we have $\varphi_j(y) = 1$, we see that

$$\sum_{j=0}^{\infty} m_j c_j \varphi_j(x+y) = \sum_{j=0}^{K-1} m_j c_j \varphi_j(x+y) + \sum_{j=K}^{\infty} m_j c_j \varphi_j(x).$$

Hence, if $x \in \Delta_R$, then Theorem 1 implies that $x+y \in \Delta_R$.

Because the rational elements of G are dense in G we have

Corollary 2. *If Δ_R is not empty, then Δ_R is dense in G.*

In view of Proposition 1 we have

Corollary 3. *For each Rademacher series R we have $m(\Delta_R) = 0$ or $m(\Delta_R) = 1$.*

An argument similar to the one in the beginning of this section shows the following.

Corollary 4. *R is differentiable for all x in G if and only if $\sum_{j=0}^{\infty} m_j |c_j| < \infty$.*

Finally we give the analogue on G of the well-known example of Weierstrass of a continuous nowhere differentiable function on T, namely $f(x) = \sum_{n=0}^{\infty} 2^{-n} \cos 2^n x$.

Corollary 5. *There exists a continuous nowhere differentiable function on G.*

Proof. Let $R(x) = \sum_{k=0}^{\infty} (m_k)^{-1} \varphi_k(x)$. Clearly, R is continuous on G and, according to Theorem 1, R is differentiable at x if and only if $\sum_{k=0}^{\infty} \varphi_k(x)$ converges. Hence, Δ_R is the empty set.

As mentioned earlier, the functions in GBF on G have many properties in common with the functions in BV on T. However, we shall now show that this is not the case with the differentiability property.

Theorem 3. (a) *If R is differentiable at a point x in G then $R \in r$-GBF for all $r \geqq 1$.* (b) *There exists a function R in GBF for which $m(\Delta_R)=0$.*

Proof. (a) Consider a fixed coset $z_{q,n}+G_n$ of G. Since R is continuous on G there are points x, y in $z_{q,n}+G_n$ such that

$$\operatorname{osc}(R;\, z_{q,n}+G_n) = R(x)-R(y) = \sum_{i=0}^{\infty} c_i(\varphi_i(x)-\varphi_i(y)).$$

Since $x_i=y_i$ for $1 \leqq i \leqq n$, we have $\varphi_i(x)=\varphi_i(y)$ for $1 \leqq i \leqq n$; also, $\varphi_0(x)= =\varphi_0(y)=1$. Therefore,

$$\operatorname{osc}(R;\, z_{q,n}+G_n) = \left|\sum_{i=n+1}^{\infty} c_i(\varphi_i(x)-\varphi_i(y))\right| \leqq 2 \sum_{i=n+1}^{\infty} |c_i|.$$

Hence,

$$\left\{\sum_{q=0}^{m_n-1} (\operatorname{osc}(R;\, z_{q,n}+G_n))^r\right\}^{1/r} \leqq \left\{m_n\left(2\sum_{i=n+1}^{\infty}|c_i|\right)^r\right\}^{1/r} \leqq 2(m_n)^{1/r} \sum_{i=n+1}^{\infty} |c_i|.$$

Next, if $R^{[1]}(x)$ exists for at least one x in G, then Theorem 2 implies that there exists a natural number K such that for all $i>K$ we have $|c_i|<(m_i)^{-1}$. Hence, if $n \geqq K$, then

$$2(m_n)^{1/r} \sum_{i=n+1}^{\infty} |c_i| \leqq 2(m_n)^{1/r} \sum_{i=n+1}^{\infty} (m_i)^{-1} \leqq 2(m_n)^{1/r}(m_n)^{-1} \sum_{k=1}^{\infty} 2^{-k} = 2(m_n)^{(1-r)/r}.$$

Thus, $R \in r$-GBF if $r \geqq 1$.

(b) Let R be defined by

$$R(x) = \sum_{k=1}^{\infty} (-1)^k (k^{1/2} m_k)^{-1} \varphi_k(x).$$

According to Theorem 2, $R^{[1]}(x)$ exists if and only if $\sum_{k=1}^{\infty} (-1)^k k^{-1/2} \varphi_k(x)$ converges. Since $\varphi_k(0)=1$ for all $k \geqq 0$ we see that $R^{[1]}(0)$ exists and, hence, Theorem 3(a) implies that $R \in$ GBF. However, it follows from Proposition 1(b) that $m(\Delta_R)=0$.

In case G is the dyadic group we have obtained some slightly stronger results than those of Theorem 3. Since this case is especially interesting we mention these results briefly.

Proposition 2. *If R is a Rademacher function on the dyadic group and if $r \geqq 1$ then*

$$V_r(R) = \sup\left\{2^{(n+r)/r} \sum_{i=n+1}^{\infty} |c_i|;\ n = 0, 1, \ldots\right\}.$$

Proof. Like in the proof of Theorem 3(a) we see that for each coset $z_{q,n}+G_n$ in G we have x, y in $z_{q,n}+G_n$ such that

$$\operatorname{osc}(R;\ z_{q,n}+G_n) = \left|\sum_{i=n+1}^{\infty} c_i(\varphi_i(x)-\varphi_i(y))\right|.$$

Now we observe that if G is the dyadic group we can find elements x and y in this coset so that for $i>n$ we have $x_i=0$ if $c_i \geqq 0$ and $x_i=1$ if $c_i<0$, whereas $y_i=1$ if $c_i \geqq 0$ and $y_i=0$ if $c_i<0$. For this choice of x and y we see that

$$\sum_{i=n+1}^{\infty} c_i(\varphi_i(x)-\varphi_i(y)) = 2\sum_{i=n+1}^{\infty} |c_i|.$$

The rest of the proof is obvious.

In [2, p. 323] J. E. Coury raised the question whether or not there exists a function on [0, 1) which can be expressed as a Rademacher series on [0, 1) and which is differentiable in the classical sense on an uncountable set of measure zero. Though we are unable to solve this problem we have obtained an affirmative answer in the present context of functions and their derivatives on the dyadic group.

Proposition 3. *There exists a Rademacher series on the dyadic group which is differentiable on an uncountable set of measure zero.*

Proof. Let $R(x)=\sum_{k=1}^{\infty} k^{-1/2}2^{-k}\varphi_k(x)$. Clearly, R is well-defined and it follows from Theorem 2 that $x\in\Delta_R$ if and only if $\sum_{k=1}^{\infty} k^{-1/2}\varphi_k(x)$ converges. So, Proposition 1(b) implies that $m(\Delta_R)=0$. Next, in order to show that Δ_R is uncountable we observe that for every real number α we can find a sequence $\{\alpha_n\}$ with $\alpha_n\in\{+1,-1\}$ for all $n\geqq 1$ and so that $\sum_{n=1}^{\infty}\alpha_n n^{-1/2}=\alpha$. Moreover, these sequences can be chosen so that if $\alpha\neq\beta$ then $\{\alpha_n\}\neq\{\beta_n\}$. Also, for every such sequence $\{\alpha_n\}$ there exists a uniquely determined x in the dyadic group such that $\varphi_n(x)=\alpha_n$ for all n. Hence, for each real number α we can find a corresponding x in the dyadic group for which $\sum_{k=1}^{\infty} k^{-1/2}\varphi_k(x)$ converges. This shows that Δ_R is an uncountable set.

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On a representation of a C*-algebra in a Lorentz algebra

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§ 1. **Introduction.** Let $\mathscr{K}$ be a Hilbert space with the usual inner product (,) and let J be an hermitian unitary aperator on $\mathscr{K}$. A *Lorentz algebra* on $\{\mathscr{K}, J\}$ is defined as a Banach subalgebra of the full operator algebra $\mathfrak{B}(\mathscr{K})$ on $\mathscr{K}$, invariant under the involution $a \to Ja^*J$ [3]. A non-zero closed subspace $\mathscr{M}$ of $\mathscr{K}$ is said to be J-uniformly positive if there is a constant $\lambda \in (0, 1]$ with $\lambda \|x\|^2 \leqq (Jx, x)$ for all x in $\mathscr{M}$.

In this paper we shall study, for a given C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ and ist derivation δ, a certain representation π_δ (defined in § 3) of $\mathfrak{A}$ on $\{\mathscr{H} \oplus \mathscr{H}, J_0\}$ with $\pi_\delta(a^*) = J_0 \pi_\delta(a)^* J_0$ for all a in $\mathfrak{A}$. In Section 2 we shall show that there is a bijective correspondence between the set $\mathscr{M}(J_0)$ of all maximal $\mathscr{J}$-uniformly positive subspaces of $\mathscr{H} \oplus \mathscr{H}$ and a certain class of operators on $\mathscr{H}$. In Section 3 we shall investigate the relationship between globally $\pi_\delta(\mathfrak{A})$-invariant elements of $\mathscr{M}(J_0)$ and derivations of $\mathfrak{A}$.

The author would like to express his sincere gratitude to Dr. Y. Nakagami for valuable discussions. He is deeply indebted to Professor M. Tomita for his encouragement.

§ 2. **The set $\mathscr{M}(J_0)$.** Let $\mathscr{H}$ be a Hilbert space and J_0 the operator on $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$ defined by $J_0(\xi \oplus \eta) = \eta \oplus \xi$ for all $\xi, n \in \mathscr{H}$. Then $J_0 = J_0^* = J_0^{-1}$. For an operator S on $\mathscr{H}$ we denote by $G(S)$ the graph of S in $\tilde{\mathscr{H}}$, i.e., the set of all $\xi \oplus S\xi \in \tilde{\mathscr{H}}$ with $\xi \in D(S)$, where $D(S)$ denotes the domain of S.

Let J_1 be the hermitian unitary operator on $\tilde{\mathscr{H}}$ defined by $J_1(\xi \oplus \eta) = \xi \oplus (-\eta)$ for all $\xi, \eta \in \mathscr{H}$ and let $\mathscr{M}(J_1)$ be the set of all maximal J_1-uniformly positive subspaces of $\tilde{\mathscr{H}}$. The following lemma is shown in [2].

Lemma 2.1. *G is a bijection from the set of all $S \in \mathfrak{B}(\mathscr{H})$ with $\|S\| < 1$ onto $\mathscr{M}(J_1)$.*

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Theorem 2.2. *G is a bijection from the set of all* $T\in\beta(\mathscr{H})$ *with*

(1) $$(T+1)^{-1}\in\mathfrak{B}(\mathscr{H}) \quad \text{and} \quad \|(T-1)(T+1)^{-1}\|<1$$

onto $\mathscr{M}(J_0)$. *In this case* $T^{-1}\in\mathfrak{B}(\mathscr{H})$.

Proof. We shall show that the following correspondences

$$\begin{array}{ccccccc} \{T:(1)\} & \xrightarrow{(i)} & \{S:\|S\|<1\} & \xrightarrow{(ii)} & \mathscr{M}(J_1) & \xrightarrow{(iii)} & \mathscr{M}(J_0) \\ T & \mapsto & S & \mapsto & \mathscr{M}_1 & \mapsto & \mathscr{M} \end{array}$$

are bijections given by (i) $S=(T-1)(T+1)^{-1}$, (ii) $G(S)=\mathscr{M}_1$ (by Lemma 2.1), (iii) $\mathscr{M}=u\mathscr{M}_1$; and that $G(T)=\mathscr{M}$, where $u=2^{-1/2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$.

It is clear that (i) and (ii) are bijections. We shall show that (iii) is also a bijection. If a closed subspace $\mathscr{M}_1$ of $\tilde{\mathscr{H}}$ is J_1-uniformly positive, then there exists a constant $\lambda\in(0,1]$ such that

$$\lambda\|ux\|^2=\lambda\|x\|^2\leqq(J_1x,x)=(J_0ux,ux)$$

for all $x\in\mathscr{M}_1$. Hence $u\mathscr{M}_1$ is J_0-uniformly positive. Conversely, if $\mathscr{M}$ is a J_0-uniformly positive closed subspace of $\tilde{\mathscr{H}}$, $u^*\mathscr{M}$ is J_1-uniformly positive. As the correspondence preserves the order of set inclusion, if $\mathscr{M}_1$ is maximal, so is $u\mathscr{M}_1$ and vice versa.

We shall show that $G(T)=\mathscr{M}$. Since $(1-S)\mathscr{H}=\mathscr{H}$, it follows that

$$G(T)=G((1+S)(1-S)^{-1})=\{(1-S)\xi\oplus(1+S)\xi:\xi\in\mathscr{H}\}=$$

$$=u\{\xi\oplus S\xi:\xi\in\mathscr{H}\}=uG(S)=u\mathscr{M}_1=\mathscr{M}.$$

Finally, since $T=(1+S)(1-S)^{-1}$, it is clear that $T^{-1}\in\mathfrak{B}(\mathscr{H})$. The proof is complete.

§ 3. Representations of C*-algebras in a Lorentz algebra on $\{\tilde{\mathscr{H}},J_0\}$. Let $\mathfrak{A}$ be a C^*-algebra acting on $\mathscr{H}$ and δ be a $*$-derivation on $\mathfrak{A}$ $(\delta(a^*)=\delta(a)^*$ for all $a\in\mathfrak{A})$. We shall define a mapping π_0 of $\mathfrak{A}$ into $\mathfrak{B}(\tilde{\mathscr{H}})$ by

$$\pi_\delta(a)=\begin{bmatrix}a & \delta(a)\\ 0 & a\end{bmatrix}$$

for all a in $\mathfrak{A}$. Then it is easily seen that π_δ is a faithful representation of $\mathfrak{A}$ on $\tilde{\mathscr{H}}$ with $\pi_\delta(a^*)=J_0\pi_\delta(a)^*J_0$ for $a\in\mathfrak{A}$, and $\pi_\delta(\mathfrak{A})$ is a Lorentz algebra on $\{\tilde{\mathscr{H}},J_0\}$. Therefore $\pi_\delta(\mathfrak{A})$ is C^*-*equivalent* (that is, isomorphic to some C^*-algebra as an involutive Banach algebra).

Lemma 3.1. *Let $\mathcal{K}$ be a Hilbert space with dim $\mathcal{K}=2$ and π the natural representation of $\mathfrak{A}$ onto $\mathfrak{A}\otimes 1_{\mathcal{K}}$.*

(i) *π_δ is similar to π.*

(ii) *π_δ is similar to $\pi_{\delta'}$ for any $*$-derivation δ'.*

Proof. (i) Since δ is implemented by some $k\in\mathfrak{B}(\mathcal{H})$ [6], we have

$$\pi_\delta(a)=\begin{bmatrix} a & ka-ak \\ 0 & a\end{bmatrix}=\begin{bmatrix} 1 & k \\ 0 & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a\end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1\end{bmatrix}^{-1}$$

for all $a\in\mathfrak{A}$. Hence π_δ is similar to π.

(ii) Clear from (i).

Theorem 3.2. *Let T be an operator with* (1) *of Theorem 2.2 and $\mathcal{M}=G(T)$. Then $\mathcal{M}$ is invariant under $\pi_\delta(\mathfrak{A})$ if and only if $\delta(a)=[T^{-1},a]$ for all a in $\mathfrak{A}$.*

Proof. Since we have

$$\pi_\delta(a)(\xi\oplus T\xi)=\big(a\xi+\delta(a)T\xi\big)\oplus aT\xi=\big(aT^{-1}+\delta(a)\big)T\xi\oplus aT\xi \tag{2}$$

for all $a\in\mathfrak{A}$ and $\xi\in\mathcal{H}$, it follows that the invariance of $\mathcal{M}=G(T)$ under $\pi_\delta(\mathfrak{A})$ is equivalent to the fact that $\delta(a)=[T^{-1},a]$ for all $a\in\mathfrak{A}$, which completes the proof.

Any operator T with $\mathcal{M}=G(T)$ in Theorem 3.2 (or Theorem 2.2) can not be skew-adjoint. Otherwise $S=(T-1)(T+1)^{-1}$ is unitary, which is impossible by Theorem 2.2.

In our previous paper [3], we have shown that a Lorentz algebra with identity with respect to J is C^*-equivalent if it has a maximal J-uniformly positive invariant subspace. Even in the case of a Lorentz algebra without identity, we remark, this statement holds by [4; Corollary 12] and the same way as a proof of [3; Theorem 3.5]. The converse holds whenever the C^*-equivalent Lorentz algebra with identity is commutative by [5; Theorem 6.1]. As for Lorentz algebras $\pi_\delta(\mathfrak{A})$, we have

Corollary 3.3. *$\pi_\delta(\mathfrak{A})$ always has a maximal J_0-uniformly positive invariant subspace.*

Proof. Since δ is a $*$-derivation, there exists an invertible skew-adjoint operator k in the double commutant $\mathfrak{A}''$ of $\mathfrak{A}$ implementing δ, [6]. If we set $T=(k+\varepsilon 1)^{-1}$ for any constant $\varepsilon>0$, then T satisfies (1) of Theorem 2.2 and $\delta(a)=[T^{-1},a]$ for all $a\in\mathfrak{A}$. Therefore $\mathcal{M}=G(T)\in\mathcal{M}(J_0)$ by Theorem 2.2 and it is invariant under $\pi_\delta(\mathfrak{A})$ by Theorem 3.2. This completes the proof.

Example. For any fixed non-zero skew-adjoint operator k and a given C^*-

algebra $\mathfrak{A}$ acting on $\mathscr{H}$, we define an algebra on $\tilde{\mathscr{H}}$ as follows;

$$\tilde{\mathfrak{A}}_k = \left\{ \begin{bmatrix} a & kb-ak \\ 0 & b \end{bmatrix} : a, b \in \mathfrak{A} \right\}.$$

Then it is a Lorentz algebra on $\{\tilde{\mathscr{H}}, J_0\}$, but it has no maximal J_0-uniformly positive invariant subspace. In fact, if $\tilde{\mathfrak{A}}_k$ has a maximal J_0-uniformly positive invariant subspace $\mathscr{M}=G(T)$, by the same computation as the proof of Theorem 3.2, $kb-ak=T^{-1}b-aT^{-1}$ for every $a, b\in\mathfrak{A}$, which implies $k=T^{-1}$ since a C^*-algebra has an approximately identity. This is a contradiction to the non-skew-adjointness of T.

Let $\operatorname{Rep}\mathfrak{A}$ be the set of all *-representations of $\mathfrak{A}$ on Hilbert spaces (in a usual sense) and $\sim$ the unitary equivalence in $\operatorname{Rep}\mathfrak{A}$. Let $\operatorname{Rep}_\delta\mathfrak{A}$ be the subset of all $\pi\in\operatorname{Rep}\mathfrak{A}$ similar to π_0. If $\pi\in\operatorname{Rep}_\delta\mathfrak{A}$ then there exists an intertwining operator A such that $\pi(a)=A\pi_\delta(a)A^{-1}$ for all $a\in\mathfrak{A}$. Then we have

Theorem 3.4. *There is a bijection of $\pi\in\operatorname{Rep}_\delta\mathfrak{A}/\sim$ onto the set of all positive operators B on $\tilde{\mathscr{H}}$ with $B^{-1}\in B(\tilde{\mathscr{H}})$ and $J_0B\in\pi_\delta(\mathfrak{A})'$ (the commutant of $\pi_\delta(\mathfrak{A})$) by the condition $B=A^*A$, where A denotes the intertwining operator mentioned above. Furthermore if we put $\langle x, y\rangle_B=(Bx, y)$ for $x, y\in\tilde{\mathscr{H}}$ then π_δ is a *-representation on a Hilbert space $\{\tilde{\mathscr{H}}, \langle\ ,\ \rangle_B\}$.*

Proof. If $\pi\in\operatorname{Rep}_\delta\mathfrak{A}$, there exists an invertible operator A such that $\pi(a)= =A\pi_\delta(a)A^{-1}$ for all $a\in\mathfrak{A}$. Since we have

$$\begin{aligned} J_0A^*A\pi_\delta(a)A^{-1}A^{*-1}J_0 &= J_0A^*\pi(a)A^{*-1}J_0 = \left(J_0A^{-1}\pi(a^*)AJ_0\right)^* = \\ &= \left(J_0\pi_\delta(a^*)J_0\right)^* = \pi_\delta(a). \end{aligned}$$

for all $a\in\mathfrak{A}$, it follows that $JA^*A\in\pi_\delta(\mathfrak{A})'$.

Suppose that $\pi'(a)=A'\pi_\delta(a)A'^{-1}$ and $A'^*A'=A^*A$. Then we have $\pi'(a)= =U'U^{-1}\pi(a)(U'U^{-1})^{-1}$ for all $a\in\mathfrak{A}$, where $A=U|A|$ and $A'=U'|A'|$ are the polar decompositions of A and A' respectively. Thus $\pi'\sim\pi$ and hence bijectivity follows.

On the other hand, since $J_0B\in\pi_\delta(\mathfrak{A})'$ we have

$$\langle\pi_\delta(a^*)x, y\rangle = \left(BJ_0\pi_\delta(a^*)J_0x, y\right) = \left(\pi_\delta(a)^*Bx, y\right) = \langle x, \pi_\delta(a)y\rangle$$

for all $a\in\mathfrak{A}$ and $x, y\in\tilde{\mathscr{H}}$. Therefore π_δ is a *-representation of $\mathfrak{A}$ on $\{\tilde{\mathscr{H}}, \langle\ ,\ \rangle_B\}$. This completes the proof.

Remark. The above proof shows that the result is valid for any representation ψ of $\mathfrak{A}$ on $\{\mathscr{K}, J\}$ with $\psi(a^*)=J\psi(a)^*J$. Therefore a C^*-equivalent Lorentz algebra $\psi(\mathfrak{A})$ on $\{\mathscr{K}, J\}$ has a maximal J-uniformly positive invariant subspace if and only if ψ is similar to some $*$-representation of $\mathfrak{A}$ on a Hilbert space by [5; the proof of Theorem 6.1 and Remark 1].

On the other hand, as easily seen from Theorem 3.4, a representation π of a C^*-algebra $\mathfrak{A}$ on $\{\mathcal{K}, J\}$ with $\pi(a^*)=J\pi(a)^*J$ $(J\neq 1, -1)$ is not similar to any irreducible $*$-representation of $\mathfrak{A}$.

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Mean Ergodic Theorem in reflexive spaces

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The mean ergodic theorem proved by LORCH [4] states that if T is a linear operator on a reflexive Banach space X with $\|T\|\leqq 1$ then

$$\frac{1}{n}(I+T+T^2+\ldots+T^{n-1})x \to Px, \tag{1}$$

for each $x\in X$, P being a projection onto the subspace $\{x\in X\colon Tx=x\}$. BLUM and others in a series of papers [1, 2, 3] studied the question of the convergence

$$\frac{1}{n}(T^{k_1}+T^{k_2}+\ldots+T^{k_n})x \to Px, \tag{2}$$

where (k_n) is a given subsequence of the positive integers and X is a Hilbert space. The definitive result due to these authors is that if X is a Hilbert space and $\|T\|\leqq 1$ then (2) holds for each $x\in X$ if for each z on the unit circle it is true that

$$\frac{1}{n}(z^{k_1}+z^{k_2}+\ldots+z^{k_n})(1-z) \to 0. \tag{3}$$

This result is the best possible in the sense that if (2) holds for each contraction T then (3) must follow. The methods used to prove these results depend heavily on the Hilbert space structure and do not apply in the case where X is not a Hilbert space. We prove below a theorem which enables us to obtain a condition on the subsequence (k_n) which is sufficient for the truth of (2) where T now acts on any reflexive Banach space. Since it involves no additional effort we have stated our theorem for a sequence of polynomials more general than the one appearing in (3).

Theorem. *Let X be a reflexive Banach space, T a linear contraction on X, $(p_k)_1^\infty$ a sequence of complex polynomials and* $q(z)=(z-\lambda_1)\ldots(z-\lambda_n)$, $\lambda_1=1$, $|\lambda_i|=1$,

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$1 \leqq i \leqq n$; $\lambda_i \neq \lambda_j$ if $i \neq j$. Suppose that

$$\text{(i)} \qquad p_k(1) \to 1,\ p_k(\lambda_i) \to 0 \ (2 \leqq i \leqq n), \quad \text{as} \quad k \to \infty,$$

$$\text{(ii)} \qquad \sup_k \|p_k(T)\| < \infty,$$

$$\text{(iii)} \qquad q(T)p_k(T)x \to 0 \quad \text{as} \quad k \to \infty,\ x \in X.$$

Then for each $x \in X$, $p_k(T)x \to Px$ where P is the bounded projection onto the subspace $\{x \in X: Tx = x\}$ such that the range of $I-P$ is the closure of the range of $I-T$.

Proof. In the following, for an operator S on a reflexive space B we will denote by $R(S)$ and $N(S)$ the closure of the range of S and the null space of S, respectively. We note the well-known result that if $\|S\| \leqq 1$, then

$$B = R(I-S) \oplus N(I-S). \tag{4}$$

We now claim that the following relations hold:

$$X = N(q(T)) \oplus R(q(T)), \tag{5}$$

and

$$(Rq(T)),\ N((T-\lambda_2 I)\dots(T-\lambda_n I)) \subseteqq R(I-T). \tag{6}$$

Assuming the truth of (5) and (6), we will prove the theorem.

First, the relation (5) implies that $p_k(T)x$ converges for each $x \in X$. This is so since for $x \in R(q(T))$ and $\varepsilon > 0$, $x = q(T)y + y'$ with $\|y'\| < \varepsilon$. By (iii) and (ii) we will then have that $p_k(T)x \to 0$. If $x \in N(q(T))$ then $x = x_1 + \dots + x_n$ with $Tx_i = \lambda_i x_i$, $(1 \leqq i \leqq n)$. Thus $p_k(T)x = p_k(\lambda_1)x_1 + \dots + p_k(\lambda_n)x_n$, and by the relations in (i), the sequence $p_k(T)x$ converges to x_1.

Next, if we also have the relation (6), then noting that $N(q(T)) = N(I-T) \oplus \oplus N((T-\lambda_2 I)\dots(T-\lambda_n I))$ we have in view of the decomposition (4) that $p_k(T)x \to Px$ where P is as in the statement of the theorem.

We will now prove by induction on n that

$$X = N(I-T) \oplus \dots \oplus N(I-\bar{\lambda}_n T) \oplus Y, \tag{7}$$

where $\overline{(I-T)Y} = \dots = \overline{(I-\bar{\lambda}_n T)Y} = Y$. This surely implies (5) and (6).

Let us suppose that for $n-1$ there exists such a $Y = Y_{n-1}$. This Y_{n-1} is necessarily invariant under T, and by (4), we have

$$Y_{n-1} = R(I-\bar{\lambda}_n T|Y_{n-1}) \oplus N(I-\bar{\lambda}_n T|Y_{n-1}).$$

Now it is immediate that $N(I-\bar{\lambda}_n T) \subseteqq Y_{n-1}$, thus $N-(I-\bar{\lambda}_n T|Y_{n-1}) = N(I-\bar{\lambda}_n T)$ and we only have to show that for $Y_n = R(I-\bar{\lambda}_n T|Y_{n-1})$ we have $\overline{(I-T)Y_n} = \dots = \overline{(I-\bar{\lambda}_n T)Y_n} = Y_n$. The last equality is immediate, the others follow from the corresponding equalities for Y_{n-1}, from the fact that $N(I-\bar{\lambda}_n T)$ is invariant under

T and from the boundedness of the projections defined by the decomposition of Y_{n-1}. Thus the proof of (7) and therefore that of the theorem are complete.

The following corollaries now follow directly from the theorem. These corollaries are stated in such a way that the conditions on the operator T and the sequence (p_k) are independent of each other.

For $p(z)=\sum_0^N a_n z^n$, set $\|p\|_A=\sum_0^N |a_n|$ and $\|p\|_\infty=\sup\{|p(z)|:\ |z|\leqq 1\}$.

Corollary 1. *Let X be a reflexive Banach space and T a linear contraction on X. Let (p_k), q be as in the theorem and suppose that the relations* (i) *of the theorem hold. Suppose further that*

(ii)′ $$\sup_k \|p_k\|_A < \infty,$$

(iii)′ $$\|qp_k\|_A \to 0 \quad as \quad k\to\infty.$$

Then $p_k(T)x\to Px$ $(x\in X)$ where P is as in the theorem.

Corollary 2. *Let X be a reflexive Banach space and T a linear operator on X such that for every polynomial p, $\|p(T)\|\leqq\|p\|_\infty$. Let (p_k), q be as in the theorem and suppose that the relations* (i) *of the theorem hold. Suppose further that*

(ii)″ $$\sup \|p_k\|_\infty < \infty,$$

(iii)″ $$\|qp_k\|_\infty \to 0 \quad as \quad k\to\infty.$$

Then $p_k(T)x\to Px$ $(x\in X)$ where P is as in the theorem.

We now return to the problem discussed in the introduction. Let (k_n) be a subsequence of the positive integers satisfying (3) and take $p_n(z)=\frac{1}{n}(z^{k_1}+\ldots+z^{k_n})$, $q(z)=z^\nu-1$, ν a positive integer. Then all the conditions except (iii)′ of Corollary 1 are satisfied. The condition (iii)′ will also be fulfilled if

(8) $$\lim_{n\to\infty}\frac{1}{N}\operatorname{card}\big(E_N\cap(E_N+\nu)\big)=1,$$

where $E_N=\{k_1,\ldots,k_N\}$ and $E_N+\nu$ is the translate of E_N by ν. We can therefore conclude that for a linear contraction T on a reflexive space X if a sequence (k_n) satisfies (3) then the condition (8) is sufficient for the convergence of (2). The example in [2], p. 428 is of a sequence (k_n) satisfying (3) and (8) with $\nu=2$.

We note that any linear contraction T on a Hilbert space satisfies the hypothesis (on T) of Corollary 2. However, as shown in [3], the conclusion of Corollary 2 holds under weaker hypothesis on (p_n). Thus the Corollary 2 has significance only when the reflexive space X is not a Hilbert space.

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Every subring R of N with $A(\bar{D}) \subset R$ is not adequate

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Let N (Nevanlinna class) be the ring of all functions of bounded characteristic and $A(\bar{D})$ (disc algebra) the subring of all holomorphic functions in the open unit disc D, which are continuously extendible to $\bar{D}$. (For details see [1], p. 16 ff and [3].)

In answering a question raised by Szűcs ([5], p. 201) E. A. Nordgren [2] showed that the ring N^+ is not adequate in showing (by an example) that there exists a finitely generated ideal, which is not principal. In his review [7] N. Yanagihara remarks that the same is true in the ring $F^+ \cap N$ by the same construction. We remark that for other rings R (e.g. $R=N$) the construction does not carry over.

The purpose of this note is to give an example which works in any subring R of N with $R \supset A(\bar{D})$ and is for some reason simpler than the example in [2]. Also the construction works after appropriate modifications in other rings of holomorphic functions.

Theorem. *In every subring R of N with $A(\bar{D}) \subset R$ there exists a finitely generated ideal which is not principal.*

Proof. Take $f_i(z) = (1-z)B_i(z)$ $(i=1, 2)$ with the Blaschke products

$$B_1(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z} \quad \text{and} \quad B_2(z) = \prod_{n=1}^{\infty} \frac{b_n - z}{1 - b_n z}$$

where $a_n = 1 - n^{-2}$, $b_n = a_n + \varepsilon_n$ and $\varepsilon_n > 0$ is tending very rapidly to zero, e.g. $\varepsilon_n = n^{-2} \exp[-(1-a_n)^{-2}]$.

Clearly $f_1, f_2 \in R$, since $f_1, f_2 \in A(\bar{D}) \subset R$. We claim that the ideal (f_1, f_2) is not principal. Assume the contrary, i.e. that there exist $d, g_1, g_2 \in R$ such that $d = f_1 g_1 + f_2 g_2$ and $(d) = (f_1, f_2)$. Then there exist $h_1, h_2 \in R$ with $f_1 = h_1 d, f_2 = h_2 d$. Since B_1 and B_2 have no common zero and in view of the factorization theorem in N ([1], p. 25) there exist $d_1, d_2 \in N$ such that $h_1 = d_1 B_1$, $h_2 = d_2 B_2$. This yields $d = d_1 B_1 d g_1 + d_2 B_2 d g_2$ or $1 = d_1 B_1 g_1 + d_2 B_2 g_2$. For $z = a_m$ it follows $1 = (d_2 B_2 g_2)(a_m)$

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or $|(g_2 d_2)(a_m)| = |B_2(a_m)|^{-1}$.

$$|B_2(a_m)| = \frac{b_m - a_m}{1 - b_m a_m} \left| \prod_{n \neq m} \frac{b_n - a_m}{1 - b_n a_m} \right| < \frac{\varepsilon_m}{1 - a_m} = m^2 \varepsilon_m = \exp[-(1-a_m)^{-2}].$$

It follows $|(g_2 d_2)(a_m)| > \exp[(1-a_m)^{-2}]$. But this is a contradiction to the fact that every function $f \in N$ (here $f = g_2 d_2$) fulfills $|f(z)| \leqq \exp[C(1-|z|)^{-1}]$ for some constant $C > 0$ (see [3], p. 57).

Remark. The idea behind the proof is not new (it seems that WHITTAKER [6], p. 256 was the originator) and has the advantage to carry over to other rings of holomorphic functions restricted by a growth condition and with some type of canonical factorization, for example the Hadamard-Weierstrass factorization in the ring of all entire functions of exponential type (see [4], p. 10 for an analogous construction).

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Approximation by unitary and essentially unitary operators

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Introduction. In [9] P. R. HALMOS formulated the problem of normal spectral approximation in the algebra of bounded linear operators on a Hilbert space. One special case of this problem is the problem of unitary approximation; this case has been studied in [3], [7, Problem 119], and [13]. The main purpose of this paper is to continue this study of unitary approximation and some related problems.

In Section 1 we determine the distance (in the operator norm) from an arbitrary operator on a separable infinite-dimensional Hilbert space to the set of unitary operators in terms of familiar operator parameters. We also study the problem of the existence of unitary approximants. Several conditions are given that are sufficient for the existence of a unitary approximant, and it is shown that some operators fail to have a unitary approximant. This existence problem is solved completely for weighted shifts and compact operators.

Section 2 studies the problem of approximation by two sets of essentially unitary operators. It is shown that both the set of compact perturbations of unitary operators and the set of essentially unitary operators are proximinal; this latter fact is shown to be equivalent to the proximinality of the unitary elements in the Calkin algebra.

Notation. Throughout this paper H will denote a fixed separable infinite-dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For an arbitrary operator T, we write $\|T\|=\sup\{\|Tf\|:f \text{ in } H \text{ and } \|f\|=1\}$ and $m(T)=\inf\{\|Tf\|:f \text{ in } H \text{ and } \|f\|=1\}$. The spectral radius of T is $r(T)$. We write $|T|=(T^*T)^{1/2}$, and $E(\cdot)$ is the spectral measure for $|T|$.

The index of an operator T is defined by $\operatorname{ind}(T)=\dim\ker(T)-\dim\ker(T^*)$ if at least one of these numbers is finite, and we use the convention that $\operatorname{ind}(T)=0$ if both number are $\aleph_0$.

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The ideal of compact operators is denoted by $K(H)$, and π is the canonical homomorphism from $B(H)$ onto the Calkin algebra $C(H)=B(H)/K(H)$. The operator T is Fredholm if $\pi(T)$ is invertible in $C(H)$. The spectrum of $\pi(T)$ is $\sigma_e(T)$ with spectral radius $r_e(T)$; the complement of $\sigma_e(T)$ is denoted by $\varrho_e(T)$. We write $\|T\|_e=\|\pi(T)\|$ and $m_e(T)=$ the infimum of $\sigma_e(|T|)$. The unilateral weighted shift of multiplicity one with weight sequence $(\alpha_1, \alpha_2, \ldots)$ is denoted shift $(\alpha_1, \alpha_2, \ldots)$. If $\mathcal{M}$ is a set of operators, then an operator X_0 in $\mathcal{M}$ is an *$\mathcal{M}$-approximant* of the operator T if $\|T-X_0\|=\inf\{\|T-X\|:X \text{ in } \mathcal{M}\}$. The set $\mathcal{M}$ is *proximinal in* $B(H)$ (or simply *proximinal*) if every operator T has an $\mathcal{M}$-approximant.

1. Unitary operators. We shall frequently use the following theorem. It appears in [5, Theorem 2.2] for the case that T_1 is a Fredholm operator; a slightly different proof is given below for completeness.

1.1. Theorem. *If T_1 and T_2 are in $B(H)$ and if $\|T_1-T_2\|_e<m_e(T_1)$, then* $\operatorname{ind}(T_1)=\operatorname{ind}(T_2)$.

Proof. Consider first the case $T_1=1$. If $\|1-T_2\|_e<1$, then there exists K in $K(H)$ such that $\|1-T_2-K\|<1$. Hence T_2+K is invertible, so clearly $\operatorname{ind}(T_2+K)=0$ Thus T_2 is Fredholm and $\operatorname{ind}(T_2)=\operatorname{ind}(T_2+K)=0$ [2, Lemma 5.20]. Thus $\operatorname{ind}(T_2)=\operatorname{ind}(1)=0$.

For the general case, we can assume that $m_e(T_1)>0$. Then there exists L in $B(H)$ such that $\|L\|_e=1/m_e(T_1)$ and LT_1 is a compact perturbation of the identity (this can be seen by looking at the polar decomposition of T_1). Then $\|1-LT_2\|_e=\|LT_1-LT_2\|_e\leqq\|L\|_e\cdot\|T_1-T_2\|_e<1$. Hence LT_2 is Fredholm of index 0 by the above result.

Consequently T_1 is Fredholm if and only if T_2 is Fredholm, and in this case $\operatorname{ind}(T_1)=-\operatorname{ind}(L)=\operatorname{ind}(T_2)$ by the additivity of the index for Fredholm operators [2, Theorem 5.36].

If both T_1 and T_2 fail to be Fredholm, then $\dim\ker T_1^*=\aleph_0=\dim\ker T_2^*$. This follows because both LT_1 and LT_2 are Fredholm, which implies that both T_1 and T_2 have closed range and finite-dimensional kernel [2, proof of Theorem 5.17]. Hence both T_1 and T_2 are Fredholm unless $\dim\ker T_1^*=\aleph_0=\dim\ker T_2^*$. Thus in this one remaining case it follows that $\operatorname{ind}(T_1)=\operatorname{ind}(T_2)=-\aleph_0$.

1.2. Corollary. *If* $\operatorname{ind}(T)<0$ *and U is a unitary operator, then* $\|T-U\|\geqq 1+m_e(T)$.

Proof. Clearly $\|T-U\|=\|U^*T-1\|\geqq r(U^*T-1)$. Assertion: Each number in the open ball $\{\zeta:|\zeta|<m_e(T)\}$ is an eigenvalue of T^*U. To see this, let $|\zeta|<m_e(T)$ and apply Theorem 1.1 to the operators $T_1=U^*T$ and $T_2=U^*T-\zeta$; notice

that $m_e(T)=m_e(U^*T)$ and ind $(U^*T)=\mathrm{ind}(T)$. Hence ind $(U^*T-\zeta)=\mathrm{ind}(U^*T)<0$. This proves the assertion, and the assertion implies 1.2.

We can now determine the distance from an arbitrary operator T to the set of unitary operators. Write $u(T)=\inf\{\|T-U\|:U$ a unitary operator$\}$.

1.3. Theorem.

(i) *If* $\mathrm{ind}(T)=0$, *then* $u(T)=\max\{\|T\|-1, 1-m(T)\}$.

(ii) *If* $\mathrm{ind}(T)<0$, *then* $u(T)=\max\{\|T\|-1, 1+m_e(T)\}$.

(The case $\mathrm{ind}(T)>0$ follows from (ii) by considering the adjoint of T).

Proof. Assertion (i) is true also in finite dimensions [3] and is proved here in a similar manner. The main point is that it is possible to find a unitary operator U such that $T=U|T|$ by enlarging the partial isometry in the polar decomposition of T (if necessary). Then $\|T-U\|=\||T|-1\|$, and it is easy to see that $\||T|-1\|=\max\{\|T\|-1, 1-m(T)\}$. That this maximum is a lower bound for $u(T)$ is also easy to see by using the triangle inequality. This proves assertion (i).

To prove assertion (ii), let $E(\cdot)$ be the spectral measure for $|T|$, and for $\varepsilon>0$ let E_ε denote the projection $E([0, m_e(T)+\varepsilon])$. Then $\dim E_\varepsilon(H)=\aleph_0$ since $m_e(T)$ has the equivalent definition $m_e(T)=\inf\{x\geqq 0:\dim E([0,x])H=\aleph_0\}$ (see [4, p. 185]).

Because ind $(T)<0$, there exists a (non-unitary) isometry S such that $T=S|T|$. Because $E_\varepsilon(H)$ and $\ker S^*\oplus SE_\varepsilon(H)$ have equal dimension and co-dimension, there exists an isometry V_ε in $B(H)$ that maps $E_\varepsilon(H)$ onto $\ker S^*\oplus SE_\varepsilon(H)$. Define the operator $U_\varepsilon=V_\varepsilon E_\varepsilon+S(1-E_\varepsilon)$.

Assertion: U_ε *is a unitary operator.*

Proof. It is easy to see that U_ε is an isometry; that U_ε is onto follows since

$$U_\varepsilon(E_\varepsilon(H)) = \ker S^*\oplus SE_\varepsilon(H)$$

and

$$U_\varepsilon(H\ominus E_\varepsilon(H)) = S(H\ominus E_\varepsilon(H)).$$

Assertion: $\|T-U_\varepsilon\|\leqq\max\{\|T\|-1, 1+m_e(T)+\epsilon\}$.

Proof. Clearly $\|T-U_\varepsilon\|=\|U_\varepsilon^*T-1\|$; we examine the operator U_ε^*T. It is not difficult to see from the definition of U_ε that $E_\varepsilon(H)$ reduces $U_\varepsilon^*T=U_\varepsilon^*S|T|$. With respect to the decomposition $H=E_\varepsilon(H)\oplus(1-E_\varepsilon)(H)$, if follows that $U_\varepsilon^*T=X_\varepsilon\oplus Y_\varepsilon$ with $\|X_\varepsilon\|\leqq m_e(T)+\epsilon$ and Y_ε=restriction of $|T|$ to the (reducing) subspace $(1-E_\varepsilon)(H)$.

Thus $\|U_\varepsilon^*T-1\|=\max\{\|X_\varepsilon-1\|, \|Y_\varepsilon-1\|\}$. Clearly $\|X_\varepsilon-1\|\leqq 1+m_e(T)+\epsilon$ and $\|Y_\varepsilon-1\|\leqq\||T|-1\|$. The fact that $\max\{1+m_e(T)+\epsilon, \||T|-1\|\}=\max\{1+m_e(T)+\epsilon, \|T\|-1\}$ follows easily. This proves $u(T)\leqq\max\{1+m_e(T), \|T\|-1\}$.

The reverse inequality follows from Corollary 1.2 and the triangle inequality. This proves Theorem 1.3.

In [8] it was shown that every operator has a positive approximant that is in the C*-algebra generated by the identity and the operator. For approximation by unitary operators, however, the situation is considerably different.

1.4. Theorem.

(i) *If* $\operatorname{ind}(T)=0$, *then* $T=U|T|$ *for some unitary approximant* U.

If the index of T *is non-zero, then* $u(T)\geqq 1$; *we consider the following two cases.*

(ii) *If* $\operatorname{ind}(T)\neq 0$ *and* $u(T)=1$, *then* T *fails to have a unitary approximant.*

(iii) *If* $\operatorname{ind}(T)<0$ *and* $u(T)>1$, *then each one of the following conditions is sufficient for* T *to have a unitary approximant:*

(a) $$\|T\|-1>1+m_e(T),$$

(b) $$\dim E([0,m_e(T)])(H)=\aleph_0,$$

(c) $m_e(T)$ *is a cluster point of eigenvalues of* $|T|$.

(The case $\operatorname{ind}(T)>0$ and $u(T)>1$ follows from (iii) by considering the adjoint of T).

Proof. Assertion (i) follows easily from the proof of Theorem 1.3 (i).

Assertion (ii) is a consequence of [14, p. 408]. For if $\operatorname{ind}(T)\neq 0$ and U is a unitary operator such that $\|U-T\|=u(T)=1$, then $\|1-U^*T\|=1$ and hence [14] implies $\operatorname{ind}((1-U^*T)-1)=0=\operatorname{ind}(-U^*T)$. It is easy to see, however, that $\operatorname{ind}(-U^*T)=\operatorname{ind}(T)$. Hence no such unitary operator U exists.

For the proof of (iii) (a), choose $\epsilon>0$ such that $m_e(T)+1+\epsilon\leqq\|T\|-1=u(T)$. Then the unitary operator U_ε constructed in the proof of Theorem 1.3 (ii) is shown by that proof to be a unitary approximant of T.

If (iii) (b) holds, then the construction of U_ε can be carried out in exactly the same way as above with $\epsilon=0$; again, this can be seen from the proof of Theorem 1.3 (ii).

If (iii) (c) holds, the construction is as follows. If $m_e(T)=0$, then (iii) (a) gives a unitary approximant since $\|T\|-1=u(T)>1$ by hypothesis (iii). If $m_e(T)>0$, then we use the following lemma to construct a unitary approximant of T; after this lemma is proved, the proof of (iii) (c) is straightforward.

1.5. Lemma. *If* $\alpha>0$, *then there exists a sequence* $\{\alpha_k\}$ *of real numbers such that* $\alpha_k>\alpha$ *for all* k *and such that* $\|1+\operatorname{shift}(\alpha_1,\alpha_2,\ldots)\|=1+\alpha$.

Proof. Notation: Let $\{e_1,e_2,\ldots\}$ be an orthonormal basis that is shifted. For any sequence $\{\alpha_k\}$, let A_n be the compression of the operator $|1+\operatorname{shift}(\alpha_1,\alpha_2,\ldots)|^2$ to the span of $\{e_1,\ldots,e_n\}$.

We prove below that there is some choice of $\{\alpha_k\}$ such that $\alpha_k > \alpha$ for all k and $A_n < (1+\alpha)^2$ for all n (where $<$ is the usual partial order for Hermitian operators). Since the norm is weakly lower semicontinuous, this proves $\|1+\text{sihft}(\alpha_1, \alpha_2, \ldots)\| \leqq 1+\alpha$; the reverse inequality follows from Theorem 1.3 (ii) since $m_e(\text{shift}(\alpha_1, \alpha_2, \ldots)) \geqq \alpha$.

The n-by-n matrix A_n is a tridiagonal matrix, about which the following two facts are known [11, p. 180]:

(1) For the characteristic polynomials $p_0(x)=1$ and $p_n(x)=\det(A_n-x)$, $n=1, 2, \ldots$, there are recursion relations $p_{n+1}(x)=\{1+\alpha_{n+1}^2-x\}p_n(x)-\alpha_n^2 p_{n-1}(x)$, $n=1, 2, \ldots$.

(2) For any real number x, the number of eigenvalues of A_n that are less than x is equal to the number of sign changes between consecutive terms of the sequence $\{p_0(x), p_1(x), \ldots, p_n(x)\}$.

By (2) we shall prove 1.5 if we show there exists some choice of $\{\alpha_k\}$ such that $\alpha_k > \alpha$ and sign $p_n((1+\alpha)^2) \neq$ sign $p_{n+1}((1+\alpha)^2)$ for all n. This is because there will be n sign changes (with $x=(1+\alpha)^2$), and hence all n (positive) eigenvalues of A_n will be less than $(1+\alpha)^2$. Write $q_n=p_n((1+\alpha)^2)$. It thus suffices to define $\{\alpha_k\}$ such that $\alpha_k > \alpha$ and such that for all integers n we have $q_n/q_{n+1} < 0$.

We define such a sequence $\{\alpha_k\}$ by induction.

To begin, choose $\alpha_1 > \alpha$ such that $\alpha_1^2 < \alpha^2((\alpha+2)/(\alpha+1))$.

We shall use the fact that this upper bound on α_1^2 implies the pair of inequalities $-1 < (\alpha_1^2 q_0)/\alpha q_1) < 0$.

To see this, note that $\alpha_1^2(\alpha+1)-\alpha(\alpha^2+2\alpha) < 0$ so that $\alpha(\alpha_1^2-\alpha^2-2\alpha) < -\alpha_1^2$ or $\alpha(\alpha_1^2-\alpha^2-2\alpha)/\alpha_1^2 < -1$ and thus $(\alpha q_1)/(\alpha_1^2 q_0) < -1$ since $q_0=1$ and $q_1=\alpha_1^2-\alpha^2-2\alpha$. The desired pair of inequalities now follow by inverting the above inequality.

Next, assume that $\alpha_1, \ldots, \alpha_k$ have been chosen $> \alpha$ such that for $j=1, \ldots, k$, there are the pair of inequalities $-1 < (\alpha_j^2 q_{j-1})/(\alpha q_j) < 0$.

Choose $\alpha_{k+1} > \alpha$ such that $\alpha_{k+1}^2 < \alpha^2\{\alpha+2+((\alpha_k^2 q_{k-1})/(\alpha q_k))\}/(\alpha+1)$.

Assertion. *This upper bound on* α_{k+1}^2 *implies the pair of inequalities*

$$-1 < (\alpha_{k+1}^2 q_k)/(\alpha q_{k+1}) < 0.$$

Proof. The upper bound clearly implies that $\alpha_{k+1}^2(\alpha+1) < \alpha^2(\alpha+2)+ +((\alpha q_{k-1}\alpha_k^2)/q_k)$ and thus $\alpha(\alpha_{k+1}^2-\alpha^2-2\alpha)-((\alpha q_{k-1}\alpha_k^2)/q_k) < -\alpha_{k+1}^2$ so that $\alpha\{(\alpha_{k+1}^2-\alpha^2-2\alpha)q_k-\alpha_k^2 q_{k-1}\}/q_k < -\alpha_{k+1}^2$. Thus by (1) $(\alpha q_{k+1})/q_k < -\alpha_{k+1}^2$ or $(\alpha q_{k+1})/(q_k\alpha_{k+1}^2) < -1$.

By inverting the above inequality, the assertion follows.

Thus we can define by induction a sequence $\{\alpha_j\}$ such that for all j both $\alpha_j > \alpha$ and

$$(\alpha_j^2 q_{j-1})/(\alpha q_j) < 0.$$

This clearly implies $\operatorname{sign}(q_{j-1}) \neq \operatorname{sign}(q_j)$ for all j, and completes the proof of Lemma 1.5.

1.6 Remark. If $0 \leqq \beta_k \leqq \alpha_k$, with α and α_k as in 1.5, then $\|1+\operatorname{shift}(\beta_1, \beta_2, \ldots)\| \leqq \leqq 1+\alpha$. Proof: Write $\beta_k = (1/2)(\alpha_k' + \alpha_k'')$ with $|\alpha_k'| = |\alpha_k''| = \alpha_k$. Then $\operatorname{shift}(\beta_1, \beta_2, \ldots)$ is the average of two shifts each unitarily equivalent to $\operatorname{shift}(\alpha_1, \alpha_2, \ldots)$ (see [7], Problem 75); this is sufficient to prove above inequality.

By using Lemma 1.5, it is straightforward to complete the proof of Theorem 1.4 (iii) (c). Write $\alpha = m_e(T) > 0$ and choose $\{\alpha_k\}$ as in 1.5. Choose a strictly decreasing sequence $\{a_k\}$ of eigenvalues of $|T|$ such that $\alpha < a_k \leqq \alpha_k$ for $k = 1, 2, \ldots$ (if it is possible to choose eigenvalues a_k with $0 \leqq a_k \leqq \alpha$ for all k, then Theorem 1.4 (iii) (b) gives a unitary approximant). Let $\{f_k\}$ be a sequence of (orthogonal) unit vectors such that $|T| f_k = a_k f_k$ and put $M = \operatorname{span}\{f_1, f_2, \ldots\}$. Because $\operatorname{ind}(T) < 0$, there exists an isometry S such that $T = S|T|$ and $-\dim\ker S^* = \operatorname{ind}(T)$.

If the index of T is finite, proceed as follows. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\ker S^*$, where $n = \dim\ker S^*$. Define an operator U by $Uf_k = -e_k$ for $k = 1, \ldots, n$ and $Uf_k = -Sf_{k-n}$ for $k = n+1, n+2, \ldots$ and $Ug = Sg$ for g in $H \ominus M$. It is not difficult to see that U is a unitary operator.

Assertion. *U is a unitary approximant of T.*

Proof. Define $M_k = \operatorname{span}\{f_j : j \equiv k \pmod n\}$, $k = 1, \ldots, n$; clearly $M = M_1 \oplus \ldots \oplus M_n$. It is straightforward to verify that each M_k reduces U^*T and the part of U^*T on M_k is $-\operatorname{shift}(a_k, a_{n+k}, a_{2n+k}, \ldots)$. It is also straightforward to verify that the part of U^*T on the (reducing) subspace $H \ominus M$ is the restriction of $|T|$ to this (reducing) subspace. Since $\{a_j\}$ is a strictly decreasing sequence, the norm of the identity plus $\operatorname{shift}(a_k, a_{n+k}, a_{2n+k}, \ldots)$ is $\leqq 1+\alpha$ (cf. Remark 1.6); it follows that $\|T-U\| = \|U^*T - 1\| \leqq \max\{1+\alpha, \||T|-1\|\}$. It is not difficult to see that this maximum equals $\max\{1+\alpha, \|T\|-1\}$, which is $u(T)$.

If the index of T is $-\aleph_0$, proceed as follows. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\ker S^*$. Define the operator U by $Uf_{2^{k-1}} = -e_k$ and $Uf_{2^{k-1}(2n+1)} = -Sf_{2^{k-1}(2n-1)}$ for $k, n = 1, 2, \ldots$ and $Ug = Sg$ for g in $H \ominus M$. It is not difficult to see that U is a unitary operator.

Assertion. *U is a unitary approximant of T.*

Proof. For $k = 1, 2, \ldots$ define the subspace $M_k = \operatorname{span}\{f_j : j = 2^{k-1}(2n+1), n = 0, 1, 2, \ldots\}$; then $M = M_1 \oplus M_2 \oplus \ldots$. Each M_k reduces U^*T, and the restriction of U^*T to M_k is $-\operatorname{shift}(a_{2^{k-1}}, a_{3\cdot 2^{k-1}}, a_{5\cdot 2^{k-1}}, \ldots)$. Since $\{a_j\}$ is a strictly decreasing sequence, the norm of the identity plus each of these shifts is $\leqq 1+\alpha$. The part of U^*T on $H \ominus M$ is the part of $|T|$ on this reducing subspace, and we can conclude as before that

$$\|T-U\| = \|U^*T-1\| = \max\{1+\alpha, \|T\|-1\} = u(T).$$

This completes the proof of Theorem 1.4.

Theorem 1.4 implies the following result, which applies in particular to weighted shifts and compact operators.

1.7. Theorem. *If T is an operator such that $m_e(T)$ is a cluster point of eigenvalues of $|T|$, then T has a unitary approximant if and only if* $\operatorname{ind}(T)=0$ *or* $u(T)>1$.

Proof. If $u(T)>1$ and $\operatorname{ind}(T)\neq 0$, then Theorem 1.4 (iii) (c) applied to T (or else T^*) gives a unitary approximant; if $\operatorname{ind}(T)=0$, then 1.4 (i) gives an approximant. The one remaining case is covered by 1.4 (ii).

1.8. Example. The compact operator shift $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$ has index -1 and is at distance 1 from the unitary operators by Theorem 1.3 (ii). Hence, by Theorem 1.4 (ii), it fails to have a unitary approximant.

1.9. Example. If S is the (unweighted) unilateral shift and 0 is the zero operator on H, then the operator $S\oplus 0$ on $H\oplus H$ does not have a unitary approximant that is in the von Neumann algebra it generates. Proof: By Theorem 1.3 (i), $S\oplus 0$ is at distance 1 from the unitary operators, and, by Theorem 1.4 (i), it has an approximant. The von Neumann algebra generated by $S\oplus 0$ and the identity on $H\oplus H$ is $\{T\oplus\zeta : T$ in $B(H)$ and ζ a complex number$\}$, and the unitary operators in this algebra are $\{U\oplus\zeta_1 : U$ a unitary operator in $B(H)$ and $|\zeta_1|=1\}$. It follows from [7, Problem 119] that $\|(S\oplus 0)-(U\oplus\zeta_1)\|=2$; hence the algebra fails to contain a unitary approximant of $S\oplus 0$.

1.10. Remark. Theorem 1.4 does not describe all operators that have unitary approximants. For example, if S is the (unweighted) unilateral shift and $0<x<1$, then the operator $S+x$ has index -1 [2, Theorem 7.26], fails to satisfy (a), (b), or (c) of 1.4 (iii) and has the identity as a unitary approximant. A similar anyalysis works for the operator S^2+x and fails for the operator $S(S+x)$; the existence of a unitary approximant for $S(S+x)$ is apparently not known.

2. Essentially unitary operators. We shall use the following theorem to prove two results on approximation by essentially unitary operators (*i.e.* operators whose image in $C(H)$ is a unitary element).

2.1. Theorem. *If T is any operator and W is a maximal partial isometry such that* $\operatorname{ind}(W)\neq\operatorname{ind}(T)$, *then* $\|W-T\|_e\geqq 1+m_e(T)$.

Proof. It is sufficient to prove this result only for $\operatorname{ind}(T)\leqq 0$, since in this case $m_e(T)\geqq m_e(T^*)$. Hence we assume $\operatorname{ind}(T)\leqq 0$.

If $m_e(T)=0$, then $m_e(T^*)=0$; since $\pi(W)$ or $\pi(W^*)$ is an isometry in $C(H)$, this implies $\|\pi(W)-\pi(T)\|\geqq 1$. Thus we can and do assume $m_e(T)>0$.

With these two assumptions, the proof is divided into four cases depending on whether T or W is Fredholm. Write $\mathcal{O}=\{\zeta\colon |\zeta|<m_e(T)\}$.

Case (i). If both T and W are Fredholm, then $\pi(W)$ is a unitary element in $C(H)$ and $\|W-T\|_e=\|\pi(W)-\pi(T)\|=\|1-\pi(TW^*)\|$.

Assertion. *The set $\mathcal{O}$ is included in a bounded component of $\varrho_e(TW^*)$.*

Proof. Because $\pi(T)$ is invertible and $\pi(W)$ is a unitary element, it follows that $m_e(T)=m_e(T^*)=m_e(TW^*)=m_e(WT^*)$. Hence if $|\zeta|<m_e(T)$, then both $\pi(TW^*-\zeta)$ and $\pi(WT^*-\bar{\zeta})$ are bounded below by $m_e(T)-|\zeta|>0$; this implies $\pi(TW^*-\zeta)$ is invertible, *i.e.* $\mathcal{O}$ is included in $\varrho_e(TW^*)$. Note that $\operatorname{ind}(TW^*)\neq 0$ by the additivity of the index for Fredholm operators. Since the index is constant on components of $\varrho_e(TW^*)$ and is zero on the unbounded component, it follows that $\mathcal{O}$ is included in a bounded component. This assertion implies that $r_e(1-TW^*)\geqq \geqq 1+m_e(T)$; hence $\|W-T\|_e\geqq 1+m_e(T)$.

In each of the three remaining cases we prove that $\mathcal{O}$ is included in $\sigma_e(TW^*)$ because the index is $-\aleph_0$ in $\mathcal{O}$.

Case (ii). If T is Fredholm and W is not Fredholm, then either $\dim\ker W^*= =\aleph_0$ or $\dim\ker W=\aleph_0$.

Assume $\dim\ker W=\aleph_0$. Then W^* is an isometry and hence $\|\pi(W)-\pi(T)\|\geqq \geqq\|1-\pi(TW^*)\|$. Note that $m_e(TW^*)\geqq m_e(T)$ since WT^*TW^* is unitarily equivalent to the compression of T^*T to the range of W^*. Thus Theorem 1.1 implies that if ζ is in $\mathcal{O}$, then $\operatorname{ind}(TW^*-\zeta)=\operatorname{ind}(TW^*)$.

Assertion. $\operatorname{ind}(TW^*)=-\aleph_0$.

Proof. $\dim\ker TW^*<\aleph_0$ since W^* is an isometry and $\dim\ker T<\aleph_0$. The fact that $\dim\ker WT^*=\aleph_0$ follows since the kernel of W has dimension $\aleph_0$ and the range of T^* is a closed subspace of finite co-dimension (since T^* is Fredholm); the intersection of any two such closed subspaces has dimension $\aleph_0$. This proves the assertion.

Since $\operatorname{ind}(TW^*-\zeta)=-\aleph_0$ for each ζ in $\mathcal{O}$, it follows that $\pi(TW^*-\zeta)$ is not invertible; hence $\mathcal{O}$ is included in $\sigma_e(TW^*)$. This implies $r_e(1-TW^*)\geqq \geqq 1+m_e(T)$, and hence $\|W-T_e\|\geqq 1+m_e(T)$.

If $\dim\ker W^*=\aleph_0$, then $\|W-T\|_e\geqq 1+m_e(T)$ follows by symmetry since the proof above used only that T^* is Fredholm and that $m_e(T)=m_e(T^*)$, but not the hypothesis $\operatorname{ind}(T)\leqq 0$.

Case (iii). If T fails to be Fredholm and W is Fredholm, then $\dim\ker T^*=\aleph_0$ (because if $\ker T^*$ is finite-dimensional, then the assumptions $\operatorname{ind}(T)\leqq 0$ and $m_e(T)>0$ imply T is Fredholm). Hence $\operatorname{ind}(TW^*)=-\aleph_0$ since $\dim\ker TW^*\leqq$

$\leqq \dim \ker T + \dim \ker W^* < \aleph_0$ and $\dim \ker WT^* = \aleph_0$. Note that $m_e(TW^*) = m_e(T)$ since $\pi(W^*)$ is a unitary element of $C(H)$. Again, Theorem 1.1 implies that $\operatorname{ind}(TW^* - \zeta) = -\aleph_0$ for ζ in $\mathcal{O}$, and consequently $\mathcal{O} \subset \sigma_e(TW^*)$. Thus $\|W - T\|_e = \|1 - TW^*\|_e \geqq r_e(1 - TW^*) \geqq 1 + m_e(T)$.

Case (iv). If both T and W fail to be Fredholm, then $\dim \ker T^* = \aleph_0$ (for the same reasons as in Case (iii)) and $\dim \ker W = \aleph_0$ (since $\operatorname{ind}(W) \neq \operatorname{ind}(T)$). Thus $\operatorname{ind}(TW^*) = -\aleph_0$ since $\dim \ker TW^* \leqq \dim \ker T < \aleph_0$ (since W^* is an isometry and $\operatorname{ind}(T) \leqq 0$) and $\dim \ker WT^* = \aleph_0$. Furthermore, $m_e(TW^*) \geqq m_e(T)$ since WT^*TW^* is unitarily equivalent to the compression of T^*T to the range of W^*. Again, Theorem 1.1 implies that $\mathcal{O} \subset \sigma_e(TW^*)$. Hence $\|W - T\|_e \geqq \|1 - TW^*\|_e \geqq r_e(1 - TW^*) \geqq 1 + m_e(T)$. This completes the proof of Theorem 2.1.

2.2 Corollary [12]. *If T_1 and T_2 are isometries such that $\|T_1 - T_2\| < 2$, then* $\dim \ker (T_1^*) = \dim \ker (T_2^*)$.

Proof. For any isometry T, $\dim \ker (T^*) = -\operatorname{ind}(T)$ since $\ker(T) = \{0\}$, and $m_e(T) = 1$ since $T^*T = 1$. Thus if $\dim \ker (T_1^*) \neq \dim \ker (T_2^*)$, then Theorem 2.1 asserts $\|T_1 - T_2\| \geqq \|T_1 - T_2\|_e \geqq 2$; this proves the corollary.

The next theorem follows from Theorem 2.1 and the results of Section 1.

2.3 Theorem. *The set* $\{U + K : U$ *a unitary operator and* K *a compact operator*$\}$ *is a proximinal subset of* $B(H)$. *For* T *in* $B(H)$, *write* $v(T)$ *for the distance from* T *to this set; there are two cases:*

(i) *If* $\operatorname{ind}(T) = 0$, *then* $v(T) = \max\{\|T\|_e - 1, 1 - m_e(T)\}$

(ii) *If* $\operatorname{ind}(T) < 0$, *then* $v(T) = \max\{\|T\|_e - 1, 1 + m_e(T)\}$.

(The case $\operatorname{ind}(T) > 0$ follows from (ii) by considering the adjoint of T).

Proof. We prove the distance assertion and show that each distance is attained, which proves the proximinality assertion.

To prove (i), Let U be a unitary operator such that $T = U|T|$; let K_1 be the compact operator $E[0, m_e(T)) \cdot (|T| - m_e(T)) + E(\|T\|_e, \|T\|] \cdot (|T| - \|T\|_e)$, where $E(\cdot)$ is the spectral measure of $|T|$. Then $\|T - U - UK_1\| = \||T| - 1 - K_1\| = \||T| - 1\|_e$, and it is easy to see that this number is equal to $\max\{\|T\|_e - 1, 1 - m_e(T)\}$. That this maximum is a lower bound for $v(T)$ is easy to see by using the triangle inequality. This proves (i).

To prove (ii), let S be an isometry such that $T = S(T)$. We shall obtain a lower bound for $v(T)$ and prove it is attained. Since $|T|$ is the sum [2, Exercise 5.17] of a diagonal operator and a compact operator, there exists a compact operator K_2 such that $|T| - K_2 \geqq 0$, $\sigma(|T| - K_2) \subset [m_e(T), \|T\|_e]$ and $m_e(T)$ is an eigenvalue of $|T| - K_2$ of multiplicity $\aleph_0$. Then $\|S(|T| - K_2)\| = \|T\|_e$ and $m(S(|T| - K_2)) = m_e(S(|T| - K_2)) = m_e(T)$.

If $m_e(T) > 0$, then T is semi-Fredholm and $\operatorname{ind}(T - SK_2) = \operatorname{ind}(T) < 0$.

Theorem 1.3 (ii) and Theorem 1.4 (iii) (b) then imply that there is a unitary operator U_0 such that $\|T-SK_2-U_0\|=u(T-SK_2)=\max\{\|T\|_e-1, 1+m_e(T)\}$.

That this maximum is a lower bound for $v(T)$ is easy to see: $v(T)\geqq\|T\|_e-1$ by the triangle inequality, and $v(T)\geqq 1+m_e(T)$ from Theorem 2.1. Thus U_0+SK_2 is an approximant of T and $v(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$ if $m_e(T)>0$.

If $m_e(T)=0$, then $\dim\ker(T-SK_2)=\dim\ker(T-SK_2)^*=\aleph_0$ and hence $\operatorname{ind}(T-SK_2)=0$. Theorem 1.3 (i) and Theorem 1.4 (i) imply that there is a unitary operator U_0 such that $\|T-SK_2-U_0\|=u(T-SK_2)=\max\{\|T\|_e-1, 1\}$ (since $m_e(T-SK_2)=0$).

That this maximum is a lower bound for $v(T)$ is again easy to see. Hence U_0+SK_2 is an approximant of T, and $v(T)=\max\{\|T\|_e-1, 1\}=\max\{\|T\|_e-1, 1+m_e(T)\}$. This completes the proof of Theorem 2.3.

The set of compact perturbations of unitary operators is precisely the set of essentially unitary operators of index zero [1], and the previous theorem shows that this is a proximinal subset of $B(H)$. The next theorem shows that the same is true of the set of all essentially unitary operators.

2.4. Theorem. *The set* $\{W \text{ in } B(H):\pi(W)$ a unitary element of $C(H)\}$ *is a proximinal subset of* $B(H)$. *For* T *in* $B(H)$, *write* $u_e(T)$ *for the distance from* T *to this set; there are two cases:*

(i) *If* $\operatorname{ind}(T)$ *is finite, then* $u_e(T)=\max\{\|T\|_e-1, 1-m_e(T)\}$

(ii) *If* $\operatorname{ind}(T)=-\aleph_0$, *then* $u_e(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$.

(The case $\operatorname{ind}(T)=+\aleph_0$ follows from (ii) by considering the adjoint of T).

Proof. (i) If the index of T is finite, then T can be written $T=W|T|$ with W a maximal partial isometry such that $\operatorname{ind}(W)=\operatorname{ind}(T)$; then $\pi(W)$ is a unitary element in $C(H)$. Let K_1 be the compact operator in the proof of Theorem 2.3 (i). Then $W+WK_1$ is an essentially unitary operator, and by the definition of K_1, $\|T-W-WK_1\|\leqq\||T|-1\|_e=\max\{\|T\|_e-1, 1-m_e(T)\}$; that this maximum is a lower bound for $u_e(T)$ is easy to see. This proves part (i).

To prove (ii), note that Theorem 2.1 implies $\|T-W\|\geqq 1+m_e(T)$ for every essentially unitary operator W, and clearly $\|T-W\|\geqq\|T\|_e-1$. By Theorem 2.3 (ii) there exists a unitary operator U and a compact operator K such that $\|T-U-K\|= =v(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$. Hence $U+K$ is also an essentially unitary approximant of T, and $u_e(T)=\max\{\|T\|_e-1, 1+m_e(T)\}$. This proves Theorem 2.4.

Theorem 2.4 together with the following observation shows that the set of unitary elements in $C(H)$ is a proximinal subset in $C(H)$.

2.5. Proposition. *If* $\mathscr{S}$ *is a non-empty subset of* $C(H)$ *and* T *is in* $B(H)$, *then* $\operatorname{dist}(T, \pi^{-1}(\mathscr{S}))=\operatorname{dist}(\pi(T), \mathscr{S})$, *and* $\pi(T)$ *has an* $\mathscr{S}$*-approximant if and only if* T *has a* $\pi^{-1}(\mathscr{S})$*-approximant.*

Proof. The equality of distances is basically a consequence of the definition of the norm in $C(H)$:$\inf\{\|T-S'\|:S'$ in $\pi^{-1}(\mathcal{S})\}=\inf\{\|T-S-K\|:\pi(S)$ in $\mathcal{S}$ and K in $K(H)\}=\inf\{\|\pi(T)-s\|:s$ in $\mathcal{S}\}$.

If $\pi(T)$ has an $\mathcal{S}$-approximant s, then $s=\pi(S)$ for some S in $B(H)$. Since the set of compact operators (the case $\mathcal{S}=\{0\}$) is proximinal in $B(H)$ ([6], [10]), there exists a compact operator K such that $\|T-S-K\|=\|\pi(T-S)\|$. Then $S+K$ is in $\pi^{-1}(\mathcal{S})$ and

$$\|T-(S+K)\|=\|\pi(T)-\pi(S)\| = \operatorname{dist}(\pi(T), \mathcal{S}) = \operatorname{dist}(T, \pi^{-1}(\mathcal{S})).$$

Conversely, let S be a $\pi^{-1}(\mathcal{S})$-approximant of T. Then $\pi(S)$ is an $\mathcal{S}$-approximant of $\pi(T)$ since $\operatorname{dist}(\pi(T), \mathcal{S})\leqq\|\pi(T)-\pi(S)\|\leqq\|T-S\|=\operatorname{dist}(T, \pi^{-1}(\mathcal{S}))$. This proves the proposition.

If in the above proposition $\mathcal{S}$ is the set of unitary elements in $C(H)$, then Theorem 2.4 shows that the set of unitary elements in $C(H)$ is proximinal.

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Обратимые мультиоперации и подстановки

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Мультиоперация $\mathscr{A}$ на непустом множестве Q определяется как отображение декартовых степеней этого множества, $\mathscr{A}: Q^n \to Q^m$ (см. [3]). Здесь $n=\delta\mathscr{A}$ называется степенью, $m=\varrho\mathscr{A}$ рангом, $|\mathscr{A}|=\delta\mathscr{A}-\varrho\mathscr{A}+1$ — арностью мультиоперации $\mathscr{A}$. Ясно, что мультиоперация $\mathscr{A}$ однозначно определяется упорядоченной последовательностью $\varrho\mathscr{A}$ операций одинаковой арности $\delta\mathscr{A}$, т.е., $\mathscr{A}=[A_1^{\varrho\mathscr{A}}]$; именно, если $\mathscr{A}(x_1^{\delta\mathscr{A}})=y_1^{\varrho\mathscr{A}}$, то $A_i(x_1^{\delta\mathscr{A}})=y_i$, $i=1, 2, \dots, \varrho\mathscr{A}$. Операция A_i называется i-той компонентой мультиоперации $\mathscr{A}$.

Условимся в следующих обозначениях: множество, наделенное мультиоперацией, обозначим буквой Q, элементы этого множества — малыми, мультиоперации — большими прописными, операции — большими заглавными буквами латинского алфавита. Совокупность всех мультиопераций одинаковой степени n и одинакового ранга m, определенные на множестве Q, обозначим через $Q^{(n,m)}$. Совокупность всех мультиопераций, определенных на множестве Q, обозначим через $\mathbf{T}_Q=\bigcup_{n=0}^{\infty}\bigcup_{m=1}^{\infty} Q^{(n,m)}$, см. [3]. Там, где не могут появляться недоразумения, степень $\delta\mathscr{A}$ обозначим через n, а ранг $\varrho\mathscr{A}$ — через m. Буквой F_i обозначим i-тый селектор: $F_i(x_1^n)=x_i$, а последовательность $[F_k^i]$ — через $\mathscr{F}_{(i,k)}$. Напомним, что если $i\leqq j$, то x_i^j означает краткую запись последовательности $x_i, x_{i+1}, \dots, x_j$; если $i>j$, тогда x_i^j означает пустую последовательность.

1. По аналогии с определением операции i (см. [3]), на множестве $\mathbf{T}_Q$ определим композицию $\odot_{tk}$ следующим образом:

$$\mathscr{A}\odot_{tk}\mathscr{B}(x_1^l)=\begin{cases}\left(\mathscr{A}(x_1^{t-1}, y_1^{\varrho\mathscr{B}}, x_{k+1}^{|\mathscr{A},\mathscr{B}|+k-t}),\ x^{\delta\mathscr{B}}_{|\mathscr{A},\mathscr{B}|+k-t+1}\right), & \text{если } t<|\mathscr{A},\mathscr{B}|,\\ \left(\mathscr{A}(x_1^{t-1}, y_1^{\delta\mathscr{A}-t+1}),\ y^{\varrho\mathscr{B}}_{\delta\mathscr{A}-t+2},\ x_{k+1}^{\delta\mathscr{B}}\right), & \text{если } |\mathscr{A},\mathscr{B}|\leqq t\leqq\delta\mathscr{A},\\ \left(\mathscr{A}(x_1^{\delta\mathscr{A}}),\ x^{t-1}_{\delta\mathscr{A}+1},\ y_1^{\delta\mathscr{B}},\ x_{k+1}^{\delta\mathscr{B}}\right), & \text{если } t>\delta\mathscr{A},\end{cases}$$

где $|\mathscr{A},\mathscr{B}|=\delta\mathscr{A}-\varrho\mathscr{B}+1$; $y_1^{\varrho\mathscr{B}}=\mathscr{B}(x_1^{\delta\mathscr{B}})$; $l=\delta(\mathscr{A}\odot_{tk}\mathscr{B})$.

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Замечание 1. Закон композиции $\odot_{tk}$ определен для любых $\mathscr{A}, \mathscr{B} \in \mathrm{T}_Q$ и для любых натуральных чисел t, k $(t \leqq k)$.

Если $t=k=i$, тогда композицию $\odot_{tk}$ обозначим через $\odot_i$.

Таким образом, множество T_Q относительно закона композиции $\odot_{tk}$ образует алгебру, которую обозначим через $(\mathrm{T}_Q;\ \odot_{tk})$. В [3] дана абстрактная характеристика алгебры $(\mathrm{T}_Q;\ \odot_i)$. Здесь мы не зададим себе целью описания алгебры $(\mathrm{T}_Q;\ \odot_{tk})$. Определение закона композиции $\odot_{tk}$ приведено здесь только потому что ниже оно нам понадобится.

Замечание 2. Соотношение между степенями и рангами мультиоперации $\mathscr{A} \odot_{tk} \mathscr{B}$ и ее факторов $\mathscr{A}$ и $\mathscr{B}$ устанавливается без труда.

Замечание 3. Если $t=1$, $k=\delta\mathscr{A}$ и $\varrho\mathscr{B} \geqq \delta\mathscr{A}$, тогда

$$\mathscr{A} \odot_{tk} \mathscr{B}(x_1^{\delta\mathscr{B}}) = \left(\mathscr{A}(y_1^{\delta\mathscr{A}}),\ y_{\delta\mathscr{A}+1}^{\varrho\mathscr{B}}\right).$$

Мультиоперации $\mathscr{A}$ и $\mathscr{B}$ можно рассмотреть как отображения декартовых степеней множества Q. В этом смысле $\mathscr{A} \odot_{tk} \mathscr{B}$ означает последовательное применение этих отображений, т.е., их произведение. Поэтому, когда $t=1$, $k=\delta\mathscr{A}$ и $\varrho\mathscr{B} \geqq \delta\mathscr{A}$, мультиоперацию $\mathscr{A} \odot_{tk} \mathscr{B}$ имеет смысл обозначить естественным образом — $\mathscr{A} \cdot \mathscr{B}$.

Определение. Мультиоперация $\mathscr{A} \in Q^{(n,m)}$ называется (i,j)-*обратимой мультиоперацией*, если в равенстве $\mathscr{A}(x_1^{i-1}, x_i^j, x_{j+1}^n) = y_1^m$ любая тройка последовательностей $x_1^{i-1}, y_1^m, x_{j+1}^n$ элементов Q однозначно определяет последовательность x_i^j.

Из определения следует, что если мультиоперация $\mathscr{A}$ является (i,j)-обратимой, тогда существует закон согласно которому любой последовательности $(x_1^{i-1}, y_1^m, x_{j+1}^n) \in Q^{n+m-(j-i+1)}$ однозначно ставится в соответствие последовательность элементов $x_i^j \in Q$. Этот закон не что иное как мультиоперация, отображающая $Q^{n+m-(j-i+1)}$ в Q^{j-i+1}. Обозначим эту мультиоперацию через $\mathscr{A}^{-(i,j)}$ и назовём её (i,j)-*обратной мультиоперацией* к мультиоперации $\mathscr{A}$.

Замечание 4. Если мультиоперация $\mathscr{A}$ — (i,j)-обратима и $j-i+1=n$ (т.е., $i=1$, $j=n$), тогда $\mathscr{A}$ является взаимнооднозначным отображением Q^n на Q^m. Очевидно, что при $n=m$ множество Q может быть конечным или бесконечным, а при $n \neq m$ оно может быть только бесконечным множеством.

Замечание 5. Пусть $j-i+1<n$. Тогда (i,j)-обратимая мультиоперация $\mathscr{A}$ не может быть взаимно однозначным отображением множества Q^n на множество Q^m.

Действительно, пусть $\mathscr{A}$ является $(2,n)$-обратимой и взаимно однозначной. Предположим, что $\mathscr{A}(c_1^n)=b_1^m$. Уравнение $\mathscr{A}(a_1, x_2^n)=b_1^m$ однозначно разрешимо для любых фиксированных элементов $a_1, b_1^m \in Q$. Возмём $a_1 \neq c_1$. Пусть для этих фиксированных элементов решение уравнения есть $x_2^n=d_2^n$:

$$\mathscr{A}(a_1, d_2^n) = b_1^m.$$

Так как $a_1 \neq c_1$, то последовательности (a_1, d_2^n) и c_1^n различны. Однако эти последовательности мультиоперацией $\mathscr{A}$ отображаются в одну и ту же последовательность b_1^m — противоречие с предположенным.

Рассмотрим ещё вопрос: в каких случаях (i, j)-обратимая мультиоперация может быть определена на конечном множестве?

Возможны два случая: 1) $j-i+1=n$, на который даёт ответ замечание 4; 2) $j-i+1<n$.

Второй случай распадается на два подслучая: 2a) $n=m$, и 2b) $n \neq m$. Пусть мы имеем условия подслучая 2a). Тогда уравнение

$$\mathscr{A}(a_1^{i-1}, x_i^j, a_{j+1}^n) = b_1^n$$

однозначно разрешимо. При фиксированных a_1^{i-1}, a_{j+1}^n мультиоперация $\mathscr{A}$ индуцирует мультиоперацию $\bar{\mathscr{A}}(x_i^j)=\mathscr{A}(a_1^{i-1}, x_i^j, a_{j+1}^n)$, которая $(1, j-i+1)$-обратима и $\bar{\mathscr{A}} \in Q^{(n,\, j-i+1)}$. Однако в таком случае, согласно замечания 1, множество Q может быть только бесконечным множеством.

Следующие примеры показывают, что в условиях подслучая 2b) (i, j)-обратимые мультиоперации могут быть определены и на конечных множествах.

Пример 1. Пусть $Q=\{1, 2\}$; $n=3$; $m=2$; $i=2$; $j=3$. Определим $\mathscr{A}: Q^3 \to Q^2$ следующим образом:

$$\mathscr{A} = \begin{pmatrix} (111), & (112), & (121), & (122), & (211), & (212), & (221), & (222) \\ (22), & (11), & (21), & (12), & (22), & (11), & (21), & (12) \end{pmatrix}.$$

Легко убедиться в том, что $\mathscr{A}$ — (2, 3)-обратима.

Пример 2. Пусть Q — конечное множество, i, j, m — некоторые натуральные числа такие, что $1<i \leqq j$, $j-i+1=m$. Возмём произвольную взаимно однозначную подстановку $\bar{\mathscr{A}}: Q^{j-i+1} \to Q^m$. Для любого $n \geqq j$ определим мультиоперацию $\mathscr{A} \in Q^{(n,\, m)}$ следующим образом: $\mathscr{A}(x_1^{i-1}, x_i^j, x_{j+1}^n)=y_1^m$ если только $\bar{\mathscr{A}}(x_i^j)=y_1^m$. Мультиоперация $\mathscr{A}$ будет (i, j)-обратимой.

Из вышеизложенного можно делать следующее заключение:

Предложение 1. *(i, j)-обратимая мультиоперация $\mathscr{A} \in Q^{(n,\, m)}$ может быть определена на конечном множестве, если только $j-i+1=m$.*

Предложение 2. *Если* $\mathscr{A}=[A_1^m]$ — (i,j)-*обратимая мультиоперация, тогда её компоненты попарно различны.*

Действительно, если $\mathscr{A}$ — (i,j)-обратимая, тогда уравнение $\mathscr{A}(a_1^{i-1}, x_i^j, a_{j+1}^n)=b_1^m$, то есть система

$$A_p(a_1^{i-1}, x_i^j, a_{j+1}^n) = b_p; \quad p = 1, 2, \ldots, m$$

однозначно разрешима для произвольных фиксированных элементов a_1^{i-1}, a_{j+1}^n, $b_1^m \in Q$. Пусть зафиксированы такие элементы $b_p \neq b_k$, относительно которых a_i^j является решением системы. В таком случае $A_p(a_1^n)=b_p \neq b_k = A_k(a_1^n)$, т.е. $A_p \neq A_k$.

Предложение 3. (i,j)-*обратимая мультиоперация* $\mathscr{A} \in Q^{(n,m)}$ *связана со своей обратной мультиоперацией* $\mathscr{A}^{-(i,j)}$ *соотношением*

$$\mathscr{A} \odot_{ij} \mathscr{A}^{-(i,j)} = \mathscr{F}_{(i,m+i-1)}.$$

На самом деле, если

$$\mathscr{A}(x_1^{i-1}, x_i^j, x_{j+1}^n) = y_1^m, \tag{1}$$

тогда по определению $\mathscr{A}^{-(i,j)}$

$$x_i^j = \mathscr{A}^{-(i,j)}(x_1^{i-1}, y_1^m, x_{j+1}^n).$$

Подставляя эти значения последовательности x_i^j в равенстве (1), получаем

$$\mathscr{A}\big(x_1^{i-1}, \mathscr{A}^{-(i,j)}(x_1^{i-1}, y_1^m, x_{j+1}^n), x_{j+1}^n\big) = y_1^m.$$

На основании определения композиции $\odot_{tk}$, последнее равенство принимает вид:

$$\mathscr{A} \odot_{tk} \mathscr{A}^{-(i,j)}(x_1^{i-1}, y_1^m, x_{j+1}^n) = y_1^m.$$

Этим и доказано наше утверждение.

При $i=j$, $m=1$ (i,j)-обратимая мультиоперация становится (i)-квазигруппой*. Извествно, что множество всех (i)-квазигрупп (при фиксированном i) одинаковой арности, определённые на одном и том же множестве относительно композиции $\odot_i$ образует группу. Аналогичное утверждение справедливо и для некоторых (i,j)-обратимых мультиопераций.

Предложение 4. (i,j)-*обратная мультиоперация к мультиоперации* $\mathscr{A}$ *является* (i,j')-*обратимой, где* $j'=m+1-i$.

Пусть выполняются условия предложения и

$$\mathscr{A}^{-(i,j)}(a_1^{i-1}, x_i^{j'}, a_{j+1}^n) = b_1^{m'},$$

---

* Операция A называется (i)-квазигруппой, если в равенстве $A(x_1^{i-1}, x_i, x_{i+1}^n)=x_{n+1}$ любые заданные элементы $x_1^{i-1}, x_{i+1}^{n+1} \in Q$ однозначно определяют элемент x_i.

где $m'=j-i+1$. Тогда

$$\mathscr{A}(a_1^{i-1}, \mathscr{A}^{-(i,j)}(a_1^{i-1}, x_i^{j'}, a_{j+1}^n), a_{j+1}^n) = \mathscr{A}(a_1^{i-1}, b_1^{m'}, a_{j+1}^n),$$
$$\mathscr{A} \odot_{ij} \mathscr{A}^{-(i,j)}(a_1^{i-1}, x_i^{j'}, a_{j+1}^n) = \mathscr{A}(a_1^{i-1}, b_1^{m'}, a_{j+1}^n).$$

Так как $\mathscr{A} \odot_{ij} \mathscr{A}^{-(i,j)} = \mathscr{F}_{(i,j')}$, то $x_i^{j'} = \mathscr{A}(a_1^{i-1}, b_1^{m'}, a_{j+1}^n)$. Это означает, что $\mathscr{A}^{-(i,j)}$ является (i,j')-обратимой.

Предложение 5. *Пусть i, j — натуральные числа ($i \leqq j$), а $\Lambda_{(i,j)}$ — множество всех (i,j)-обратимых мультиопераций одинаковой степени n и одинакового ранга m, определённые на одном и том же множестве Q.*

(I). *Если $j-i+1 \neq m$, тогда множество $\Lambda_{(i,j)}$ относительно операции $\odot_{ij}$ не замкнуто.*

(II). *Если $j-i+1=m$, тогда множество $\Lambda_{(i,j)}$ относительно операции $\odot_{ij}$ образует группу.*

Пусть сперва выполняется условие пункта (I). Если $j-i+1 \neq m$, тогда либо $j-i+1>m$, либо $j-i+1<m$. Предположим, что $j-i+1>m$; тогда $|\mathscr{A}, \mathscr{B}| = \delta\mathscr{A} - \varrho\mathscr{B} + 1 = n-m+1 > n-j+i > i$, т.е., $i < |\mathscr{A}, \mathscr{B}|$. Отсюда получим $\delta(\mathscr{A} \odot_{ij} \mathscr{B}) = \max[n, n+(j-i+1)-m] > n$. Следовательно $\mathscr{A} \odot_{ij} \mathscr{B} \notin Q^{(n,m)}$. При $j-i+1<m$ утверждение (I) доказывается аналогично.

Докажем втрорую часть предложения. Если $j-i+1=m$, тогда $\delta(\mathscr{A} \odot_{ij} \mathscr{B}) = \max[n, n+(j-i+1)-m] = n$, если $i < |\mathscr{A}, \mathscr{B}|$, и $\delta(\mathscr{A} \odot_{ij} \mathscr{B}) = \max(n, i-1) = n$, если $|\mathscr{A}, \mathscr{B}| \leqq i \leqq \delta\mathscr{A}$ (заметим, что в данном случае число i не может быть больше $\delta\mathscr{A}$). Из доказанных выше условий и на основании замечания 1 имеем:

В случае $i < |\mathscr{A}, \mathscr{B}|$ имеем $\delta\mathscr{B} = n = n+(j-i+1)-m = (n-m+1)-j-i = |\mathscr{A}, \mathscr{B}| + j - i$ и поэтому $\varrho(\mathscr{A} \odot_{ij} \mathscr{B}) = \varrho\mathscr{A}$.

В остальном случае получаем $\delta\mathscr{B} = n \geqq i = k$ и поэтому

$$\varrho(\mathscr{A} \odot_{ij} \mathscr{B}) = n+m-n+m-1+i-j = m+m-(j-i+1) = m.$$

Доказано, что $\delta(\mathscr{A} \odot_{ij} \mathscr{B}) = n$ и $\varrho(\mathscr{A} \odot_{ij} \mathscr{B}) = m$, т.е. $\mathscr{A} \odot_{ij} \mathscr{B} \in Q^{(n,m)}$. Покажем, что $\mathscr{A} \odot_{ij} \mathscr{B}$ — (i,j)-обратима.

Пусть

(2) $$\mathscr{A} \odot_{ij} \mathscr{B}(a_1^{i-1}, x_i^j, a_{j+1}^n) = b_1^m,$$

т.е.

$$\mathscr{A}[a_1^{i-1}, \mathscr{B}(a_1^{i-1}, x_i^j, a_{j+1}^n), a_{j+1}^n] = b_1^m.$$

Обозначим $\mathscr{B}(a_1^{i-1}, x_i^j, a_{j+1}^n) = y_i^j$. Уравнение

$$\mathscr{A}(a_1^{i-1}, y_i^j, a_{j+1}^n) = b_1^m$$

однозначно разрешимо. Пусть $y_i^j = c_i^j$, т.е.

$$\mathscr{B}(a_1^{i-1}, x_i^j, a_{j+1}^n) = c_i^j.$$

Это уравнение также однозначно разрешимо. Её решение будет и решением уравнения (2). Однозначность этого решения очевидно. Таким образом $\mathscr{A} \odot_{ij} \mathscr{B} \in \in \Lambda_{(i,j)}$, т.е., $\Lambda_{(i,j)}$ замкнуто относительно операции $\odot_{ij}$, если $j-i+1=m$. Из самого определения операции $\odot_{ij}$ следует её ассоциативность. Существование единичного элемента $\mathscr{F}_{(i,m+i-1)}$ и обратного элемента $\mathscr{A}^{-(i,j)}$ доказано в предложении 3.

Предложение 5 доказано.

Следствие 1. *Если* $\Lambda_{(i,j)}$ *— группа, то*

$$\mathscr{A}^{-(i,j)}[x_1^{i-1}, \mathscr{A}(x_1^n), x_{j+1}^n] = x_i^j.$$

Следствие 2. *Множество всех* (i)*-квазигрупп (при фиксированном* i*) одинаковой арности образует группу относительно операции* $\odot_i$.

2. Рассмотрим теперь мультиоперации, являющиеся подстановками декартовных степеней множества Q и связь между их компонентами.

Пусть $\delta\mathscr{A}=\varrho\mathscr{A}$, тогда мультиоперация $\mathscr{A}$ является отображением множества $Q^{\delta\mathscr{A}}$ в себе. Если оно взаимно однозначное, назовём его подстановкой. Ясно, что если $\mathscr{A}$ подстановка, то она $(1, \delta\mathscr{A})$-обратима и наоборот.

Имеет место следующее

Предложение 6. *Операция* A *является* (i)*-квазигруппой тогда и только тогда, когда* $[F_1^{i-1}, A, F_{i+1}^n]$ *— подстановка множества* Q^n.

На самом деле, если $[F_1^{i-1}, A, F_{i+1}^n]$ подстановка, то уравнение

$$[F_1^{i+1}, A, F_{i+1}^n](x_1^n) = a_1^n$$

однозначно разрешимо. Другими словами, система

$$\begin{cases} F_j(x_1^n) = a_j; \quad j = 1, 2, \ldots, i-1, i+1, \ldots, n \\ A(x_1^n) = a_i \end{cases}$$

однозначно разрешима. Следовательно, однозначно разрешимым будет и уравнение $A(a_1^{i-1}, x_i, a_{i+1}^n)=a_i$. Так же просто доказывается и обратное утверждение.

Предложение 7. а) *Если две из мультиопераций*

$$[F_1^{i-1}, A_i^{i+1}, F_{i+2}^n], [F_1^{i-1}, A_i, F_{i+1}^n], [F_1^i, A_{i+1} \odot_i A_i^{-(i)}, F_{i+2}^n]$$

являются подстановками, тогда и третья мультиоперация — подстановка

b) *Если две из следующих трёх мультиопераций являются подстановками:*

$$[F_1^{i-1}, A_1^{i+1}, F_{i+2}^n], [F_1^i, A_{i+1}, F_{i+2}^n], [F_1^{i-1}, A_i \odot_{i+1} A_{i+1}^{-(i+1)}, F_{i+2}^n],$$

тогда третья мультиоперация также является подстановкой.

На самом деле, пусть выполняются условия нашего утверждения. Тогда мультиоперация $[F_1^{i-1}, A_i^{-(i)}, F_{i+1}^n]$, где $A_i^{-(i)}$ обратная операция к операции A_i в группе (i)-квазигрупп относительно операции $\odot_i$, также является подстановкой. Подстановкой будет и произведение подстановок $[F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]$ и $[F_1^{i-1}, A_i^{-(i)}, F_{i+1}^n]$, которое, однако, равняется $[F_1^i, A_{i+1}\odot_i A_i^{-(i)}, F_{i+2}^n]$. На самом деле,

$$\begin{aligned}
&[F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]\cdot[F_1^{i-1}, A_i^{-(i)}, F_{i+1}^n](x_1^n) = \\
&= [F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]\big(x_1^{i-1}, A_i^{-(i)}(x_1^n), x_{i+1}^n\big) = \\
&= \big(x_1^{i-1}, A_i(x_1^{i-1}, A_i^{-(i)}(x_1^n), x_{i+1}^n), A_{i+1}(x_1^{i-1}, A_i^{-(i)}, (x_1^n), x_{i+1}^n), x_{i+2}^n\big) = \\
&= \big(x_1^{i-1}, A_i\odot_i A_i^{-(i)}(x_1^n), A_{i+1}\odot_i A_i^{-(i)}(x_1^n), x_{i+2}^n\big) = \\
&= \big(x_1^{i-1}; x_i, A_{i+1}\odot_i A_i^{-(i)}(x_1^n), x_{i+2}^n\big) = \\
&= [F_1^i, A_{i+1}\odot_i A_i^{-(i)}, F_{i+2}^n](x_1^n),
\end{aligned}$$

т.е.

$$(3)\qquad [F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]\cdot[F_1^{i-1}, A_i^{-(i)}, F_{i+1}^n] = [F_1^i, A_{i+1}\odot_i A_i^{-(i)}, F_{i+2}^n].$$

Здесь мы использовали равенство $A_i\odot_i A_i^{-(i)}(x_i)=x_i$, которое всегда верно, так как A_i является (i)-квазигруппой, и $A_i^{-(i)}$ — её обратный элемент в группе всех (i)-квазигрупп (см. следствие 2) с единицей $F_i=A_i\odot_i A_i^{-(i)}$.

Аналогично доказывается и равенство

$$(4)\qquad [F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]\cdot[F_1^i, A_{i+1}^{-(i+1)}, F_{i+2}^n] = [F_1^{i-1}, A_i\odot_{i+1}A_{i+1}^{-(i+1)}, F_{i+1}^n].$$

Из доказанных равенств (3) и (4) следуют и остальные утверждения предложения 7.

Следствие 3. a) *Если A_i является (i)-квазигруппой, то $[F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]$ является подстановкой если и только если $A_{i+1}\odot_i A_i^{-(i)}$ является $(i+1)$-квазигруппой.*

b) Если A_{i+1} $(i+1)$-квазигруппа, то $[F_1^{i-1}, A_i^{i+1}, F_{i+2}^n]$ является подстановкой если и только если $A_i\odot_{i+1}A_{i+1}^{-(i+1)}$ является (i)-квазигруппой.

Замечание 6. Бинарные операции A и B ортогональны тогда и только тогда, когда $\mathscr{A}=[A, B]$ является подстановкой множества Q^2.

При $i=1$ и $n=2$ (т.е., для бинарных операций) получаем следующее обобщение леммы 2 из [1]:

Следствие 4. *Если A_1 является (1)-квазигруппой (левой квазигруппой), то $[A_1, A_2]$ является подстановкой множества Q^2 (другими словами, A_1 и A_2 ортогональны) тогда и только тогда, когда $A_2\odot_1 A_1^{-(1)}$ будет (2)-квазигруппой (правой квазигруппой).*

Аналогичное утверждение получаем и в случае b) следствия 1.

Заметим, что на основании определения операции $\cdot=\odot_{1,n}$ (см. следствие 3) для мультиопераций $\mathscr{A}=[A_1^m]$ и $\mathscr{B}=[B_1^n]$, таких что $\delta\mathscr{A}=\varrho\mathscr{B}$, следует справедливость равенства:

(5) $$[A_1^m]\cdot[B_1^n] = \big[A_1\cdot[B_1^n], A_2\cdot[B_1^n], \ldots, A_m\cdot[B_1^n]\big].$$

Также очевидно, что если B — (i)-квазигруппа, тогда для любой операции K, однаковой арности что и B, справедливы равенства:

(6) $$K\odot_i B^{-(i)}\cdot[F_1^{i-1}, B, F_{i+1}^n] = K,$$

(7) $$F_i\cdot[A_1^m] = A_i.$$

На основании равенств (5) и (6) без особых трудностей доказывается равенство (8)

$$[B_1^n] = [B_1\odot_i B_i^{-(i)}, \ldots, B_{i-1}\odot_i B_i^{-i}, F_i, B_{i+1}\odot B_i^{-(i)}, \ldots, B_n\odot_i B_i^{-(i)}]\cdot[F_1^{i-1}, B_i, F_{i+1}^n],$$

если B_i является (i)-квазигруппой.

Говорим, что мультиоперация $\mathscr{A}=[F_1^{k-1}, A_k^n]$ *удовлетворяет условию* α_k, если $\mathscr{A}\in Q^{(n,n)}$ и её компоненты таковы, что A_i является (i)-, $(i+1)$-квазигруппой для $i=k+1, k+2, \ldots, n-1$; A_k — $(k+1)$-квазигруппа и A_n — (n)-квазигруппа.

Имеет место

Предложение 8. *Если мультиоперация* $\mathscr{A}=[A_1^n]$ *удовлетворяет условию* α_1 *и* $B_j=A_j\odot_{j+1}A_{j+1}^{-(j+1)}$, $j=1, 2, \ldots, n-1$, $B_n=A_n$ *является* (j)*-квазигруппой, тогда* $\mathscr{A}$ *— подстановка, и*

$$\mathscr{A} = [A_1^n] = \prod_{j=1}^{n}[F_1^{j-1}, B_j, F_{j+1}^n].$$

Действительно, пусть решением уравнения $X\odot_{i+1}A_{i+1}=A_i$, $i=1, 2, \ldots, n-1$ является операция B_i (B_i всегда существует и является $(i+1)$-квазигруппой, так как нам дано, что A_i, A_{i+1} — $(i+1)$-квазигруппы, которые принадлежат группе всех $(i+1)$-квазигрупп $\Lambda(\odot_{i+1})$). Следовательно $A_i=B_i\odot_{i+1}A_{i+1}$. Обозначим $A_n=B_n$. Тогда

(9) $$A_i = B_i\odot_{i+1}B_{i+1}\odot_{i+2}\ldots\odot_{n-1}B_{n-1}\odot_n B_n.$$

Поэтому

$$\mathscr{A} = [A_1^n] = [B_1\odot_2 B_2\odot_3\ldots\odot_n B_n, B_2\odot_3 B_3\odot_4\ldots\odot_n B_n, \ldots, B_n].$$

По предположению B_i является (i)-квазигруппой, и тогда $B_i\odot_i B_i^{-(i)}=F_i$. На основании равенства (8) имеем при $i=n$

$$\mathscr{A} = [B_1\odot_2\ldots\odot_{n-1}B_{n-1}\odot_n F_n, B_2\odot_3\ldots\odot_{n-1}B_{n-1}\odot_n F_n, \ldots$$
$$\ldots, B_{n-1}\odot_n F_n, F_n]\cdot[F_1^{n-1}, B_n],$$

а при $i=n-1$, используя равенство (6), получаем

$$\mathscr{A} = [B_1\odot_2\ldots\odot_{n-2}B_{n-2}\odot_{n-1}F_{n-1}\odot_n F_n, B_2\odot_3\ldots\odot_{n-2}B_{n-2}\odot_{n-1}F_{n-1}\odot_n F_n, \ldots$$
$$\ldots, B_{n-2}\odot_{n-1}F_{n-1}\odot_n F_n, F_{n-1}\odot_n F_n, F_n]\cdot[F_1^{n-2}, B_{n-1}, F_n]\cdot[F_1^{n-1}, B_n],$$

Продолжая этот процесс, на n-ом шаге получаем:

$$\mathscr{A} = [F_1 \odot_2 \ldots \odot_n F_n, F_2 \odot_3 \ldots \odot_n F_n, \ldots, F_n] \cdot \prod_{j=1}^{n} [F_1^{j-1}, B_j, F_{j+1}^n].$$

Однако $F_j \odot_{j+1} F_{j+1} \odot_{j+2}, \ldots, \odot_n F_n = F_j$, так как F_k единица в группе $A(\odot_k)$, $k=j+1, j+2, \ldots, n$. Кроме того, $[F_1^n]$ тождественная подстановка. Поэтому

$$[A] = \prod_{j=1}^{n} [F_1^{j-1}, B_j, F_{j+1}^n]. \tag{10}$$

Замечание 7. Очевидно, что если B_i $(i=1, 2, \ldots, n)$ является (i)-квазигруппой и $A_i = B_i \odot_{i+1} B_{i+1} \odot_{i+2} \ldots \odot_n B_n$, $B_n = A_n$, тогда $[A_1^n]$ является подстановкой множества Q^n.

Другими словами, при наличии n (i)-квазигрупп $(i=1, 2, \ldots, n)$ арности n, мы всегда можем построить подстановку множества Q^n.

При $n=2$ получаем следующее

Следствие 5. *Если A_1, A_2 — бинарные операции, то $[A_1, A_2]$ — подстановка множества Q^2 тогда и только тогда, когда $B_1 = A_1 \odot_1 A_2^{-(2)}$ является (1)-квазигруппой, A_1 и $B_2 = A_2$ являются (2)-квазигруппами.*

Необходимость этого утверждения доказано утверждением предложения 8. Достаточность сразу следует из того, что при $n=2$ равенство (9) принимает вид:

$$[A_1, A_2] = [B_1, F_2] \cdot [F_1, B_2].$$

Если B_j — (j)-квазигруппа, тогда $[B_1, F_2]$ и $[F_1, B_2]$ — подстановки множества Q^2; подстановкой будет и их произведение $[B_1, F_2] \cdot [F_1, B_2] = [A_1, A_2]$.

Возникает вопрос: будут ли справедливы утверждения обратные к утверждениям предложения 8 для любого n?

Следствие 6. *Если компоненты мультиопераций*

$$\mathscr{A}' = [F_1^k, A_{k+1}^n], \quad \mathscr{A}'' = [F_1^{k+1}, A_k^{k+1}, F_{k+2}^n]$$

удовлетворяют условиям α_k и α_{k+1} соответственно, тогда $\mathscr{A}''' = [F_1^{k-1}, A_k^n]$ является подстановкой.

Действительно, если компоненты мультиоперации $\mathscr{A}'$ удовлетворяют условию α_k, то $[F_1^k, A_{k+1}^n]$ — подстановка и равенство (10) принимает вид:

$$\mathscr{A}' = \prod_{j=k+1}^{n} [F_1^{j-1}, B_j, F_{j+1}^n].$$

На основании пункта b) предложения 7, $[F_1^{k-1}, B_k, F_{k+1}^n]$, где $B_k = A_k \odot_{k+1} A_{k+1}^{-(k+1)}$ также является подстановкой. Произведение подстановок

$$[F_1^{k-1}, B_k, F_{k+1}^n] \cdot \prod_{j=k+1}^{n} [F_1^{j-1}, B_j, F_{j+1}^n] = \prod_{j=k}^{n} [F_1^{j-1}, B_j, F_{j+1}^n] = \mathscr{A}'''$$

— подстановка. Становится очевидным и обратное утверждение, что даёт нам возможность в итоге утверждать справедливость следующего предложения.

Предложение 9. *Если* $[F_1^{k-1}, A_k^{k+1}, F_{k+2}^n]$ — *подстановка, то* $[F_1^{k-1}, A_k^n]$ *будет подстановкой тогда и только тогда, когда её компоненты удовлетворяют условию* α_k.

Предложение 10. *Если компоненты мультиоперации* $\mathscr{A}=[F_1^{k-1}, A_k^n]$ *удовлетворяют условию* α_k, *то*

$$\overline{\mathscr{A}} = [A_k^n] = [B_k, F_{k+1}^n]\cdot \prod_{j=k+1}^{n} [F_1^{j-1}, B_j, F_{j+1}^n], \quad B_j = A_j \odot_{j+1} A_{j+1}^{-(j+1)}$$

и, следовательно, $\mathscr{A}$ *является взаимно однозначным отображением* $Q^n \to \to Q^{n-k+1}$.

Действительно, из условиях предложения и на основании равенства (9) следует, что

$$\overline{\mathscr{A}} = [A_n^k] = [B_k \odot_{k+1} B_{k+1} \odot_{k+2} \dots \odot_n B_n, \dots, B_{n-1} \odot_n B_n, B_n].$$

В силу (7), из последнего равенства имеем

$$\overline{\mathscr{A}} = [A_k^n] = [B_k, F_{k+1}^n]\cdot \prod_{j=k+1}^{n} [F_1^{j-1}, B_j, F_{j+1}^n]. \tag{11}$$

Замечание 8. Если $[F_1^k, A_{k+1}^n]=\mathscr{A}$ взаимно однозначное отображение $Q^n \to Q^n$, то $[A_{k+1}^n]=\overline{\mathscr{A}}$ является взаимно однозначным отображением $Q^n \to Q^{n-k}$ и обратно.

На основании этого замечания и доказанного равенства (11) заключаем, что от взаимно однозначного отображения $\overline{\mathscr{A}}$: $Q^n \to Q^{n-k+1}$, компоненты которой удовлетворяют условию α_k, можно переходить к взаимно однозначному отображению $\mathscr{A}$: $Q^n \to Q^{n-k+1}$ и обратно.

Замечание 9. Если B_i $(i=k, \dots, n)$ является (i)-квазигруппой и $A_i= =B_i \odot_{i+1} B_{i+1} \odot_{i+2} \dots \odot_n B_n$, $B_n=A_n$, тогда $[A_k^n]=\overline{\mathscr{A}}$ — взаимно однозначное соответствие между Q^n и Q^{n-k+1}.

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КАФ. ВЫСШЕЙ МАТЕМАТИКИ
ПОЛИТЕХНИЧЕСКИЙ ИНСТИТУТ ИМ. С. ЛАЗО
КИШИНЕВ, СССР

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Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wenn seine Weylsche Projektivkrümmung verschwindet

Z. I. SZABÓ

L. BERWALD hat bewiesen, dass die Finslerschen Räume skalarer Krümmung die Weylsche Projektivkrümmung Null haben. Sie lassen sich daher für dim>2 durch bahntreue Abbildung in allgemeine affine Räume mit der Krümmung Null überführen [1]. BERWALD, und ihn folgend mehrere Verfasser haben aber gemeint, dass nicht jeder Finslersche Raum mit verschwindendem Weylschem Krümmungstensor von skalarer Krümmung ist [1], [2]. Z. B. hat L. BERWALD den folgenden Satz behauptet [1]:

„Die Räume skalarer Krümmung sind für $n=\dim>2$ unter den Finslerschen Räumen mit der Projektivkrümmung Null durch

$$R_0{}^p{}_{0h\|p}-\frac{1}{n-1}R_0{}^p{}_{0p\|h}+R_0{}^p{}_{0p}l_h=0,$$

oder durch die äquivalenten Bedingungen:

$$(n+1)(R_0{}^p{}_{hp}-R_0{}^p{}_{0p}l_h)+(n-2)(R_0{}^p{}_{0h}A_p+A_{h|0|0})=0$$

gekennzeichnet.“

Das Hauptergebnis der vorliegenden Arbeit ist der folgende

Satz. *Ein Finslerscher Raum F_n ist für* $\dim F_n>2$ *genau dann von skalarer Krümmung, wenn sein Weylscher Tensor W^i_{jk} verschwindet.*

Aus dieser Aussage folgt offensichtlich, dass die obigen Gleichungen für Finslersche Räume mit $W^i_{jk}=0$ automatisch erfült sind.

§ 1. Einleitung

Bezeichne $L(x^1, \dots x^n, y^1, \dots, y^n)$, oder $L(x, y)$ die metrische Grundfunktion eines n-dimensionalen Finslerschen Raumes F_n. $L(x, y)$ wird als positiv, und in y als positiv homogen von erster Ordnung vorausgesetzt. Es ist auch vorausgesetzt

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dass der Finslersche Tensor $g_{ij}=(1/2)\dot{\partial}_i\dot{\partial}_j L^2$ vom $(0, 2)$-Typ positiv definit ist, wobei $\dot{\partial}$ die partielle Ableitung $\partial/\partial y_i$ bezeichnet.

Der Einheitsvektor im Punkte (x) in der Richtung des Linienelementes (x, y) hat die kontravarianten, bzw. kovarianten Komponenten

$$l^i = y^i/L, \quad l_i = \dot{\partial}_i L = g_{ij} l^i.$$

Es gilt auch $l^i = g^{ij} l_j$, wobei der reziproke Tensor g^{il} durch $g^{ij} g_{lj} = \delta^i_l$ definiert ist.

Es bezeichne ∂_i die partielle Ableitung: $\partial/\partial x^i$. Die Zusammenhangsobjekte $G^i(x, y)$ des Raumes definieren wir durch:

$$G^i(x, y) = (1/4)\, g^{ih}(y^m \dot{\partial}_h \partial_m L^2 - \partial_h L^2),$$

ferner sei

$$G^i_j \overset{\text{def}}{=} \dot{\partial}_j G^i, \quad G^i_{jk} \overset{\text{def}}{=} \dot{\partial}_j G^i_k, \dots \text{ u.s.w.}$$

Mit Hilfe dieser Objekte definieren wir den Grundtensor der affinen Krümmung, bzw. den Weylschen Krümmungstensor durch

(1.1) $$H^i_{jk} = \partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj},$$

(1.2) $$W^i_{jk} = H^i_{jk} - \frac{1}{n+1} H_r{}^r{}_{jk} y^i + \frac{1}{n^2-1} \delta^i_j (nH_k + H_{km} y^m) - \frac{1}{n^2-1} \delta^i_k (nH_j + H_{jm} y^m),$$

wobei

$$H_h{}^i{}_{jk} = \dot{\partial}_h H^i_{jk}, \quad H_j = H^r_{jr}, \quad H_{hj} = H_h{}^r{}_{jr}.$$

Der Skalar $H = \dfrac{1}{n-1} H^r_r$ ist der affine Krümmungsskalar des Raumes. Wir benötigen später die Formeln [1]:

(1.3) $$H = \frac{1}{n-1} y^i H^k_{ik}, \quad H_{hj} = \dot{\partial}_h H_j, \quad H_{jk} - H_{kj} = -H_r{}^r{}_{jk}.$$

L. Berwald hat das Krümmungsmass $R(x, y, \eta)$ des Raumes im Linienelement (x, y) nach der 2-Richtung (y, η) durch

(1.4) $$R(x, y, \eta) = H_{ikhm} y^i y^h \eta^k \eta^m / (g_{ih} g_{km} - g_{im} g_{hk}) y^i y^h \eta^k \eta^m,$$

und den Krümmungsskalar $R(x, y)$ im Linienelement (x, y) durch

(1.5) $$R(x, y) = H(x, y)/L^2$$

definiert. Der Raum heisst von skalarer Krümmung, wenn $R(x, y, \eta)$ von der Wahl des Vektors η unabhängig ist. In diesem Fall gilt die Gleichung: $R(x, y, \eta) = R(x, y)$. Ein Satz von Schur [1] lautet: Ist der Raum von skalarer Krümmung, und ist $R(x, y)$ nur eine Funktion des Ortes, so ist $R(x, y)$ konstant. Diese Räume sind die Räume von konstanter Krümmung.

Bezeichnet $R_j{}^i{}_{kh}$ bzw. $g_{si}R_j{}^s{}_{kh}=R_{jikh}$ den Cartanschen Krümmungstensor des Raumes, so gelten die Relationen [1]:

$$H^i_{jk} = y^l R_l{}^i{}_{jk}, \tag{1.6}$$

$$H^i_j = y^l y^k R_l{}^i{}_{kj}. \tag{1.7}$$

Aus der wohlbekannten schiefsymmetrischen Eigenschaft $R_{jikh}=-R_{jikh}$ folgt auch: $y^i y^j R_{jikh}=0$, und somit $l_i H^i_{kj}=0$.

Berwald hat auch folgendes bewiesen [1]:

Ein Finslerscher Raum F_n, dim $F_n>2$ ist genau dann von skalarer Krümmung, wenn der Tensor $R_0{}^i{}_{0j} \overset{\text{def}}{=} l^k l^m R_k{}^i{}_{mj}=(1/L^2)H^i_j$ die folgende Form hat:

$$R_0{}^i{}_{0j} = R(x, y)(\delta^i_j - l^i l_j). \tag{1.8}$$

Die Antiderivationen d_h und d_v auf den schiefsymmetrischen Finslerschen Tensoren vom $(0, s)$-Typ führt man folgendermassen ein:

$$(d_h\omega)_{i_0 i_1 \dots i_s} = \sum_{k=0}^{s} (-1)^k \omega_{i_0 i_1 \dots i_{k-1} i_{k+1} \dots i_s | i_k}, \tag{1.9}$$

$$(d_v\omega)_{i_0 i_1 \dots i_s} = \sum_{k=0}^{s} (-1)^k \dot{\partial}_{i_k} \omega_{i_0 \dots i_{k-1} i_{k+1} \dots i_s}, \tag{1.10}$$

wobei $\omega_{i_0 i_1 \dots i_s}$ ein schiefsymmetrischer Finslerscher Tensor vom $(0, s)$-Typ ist, ferner das Symbol „I" die Berwaldschen kovarianten Ableitung bezüglich den Objekten G^i_{jk} bezeichnet. Aus den obigen Definitionen erhalten wir mühelos:

Satz 1. 1) *d_h und d_v sind beide Antiderivationen bezüglich des äusseren Tensorproduktes $\wedge$, ferner gilt $d_h d_v + d_v d_h = 0$.*

2) *Die Gleichung $d_h^2=0$ besteht genau dann, wenn $H^i_{jk}=0$.*

3) $d_v^2=0$.

Der Tensor ω ist in horizontaler bzw. in vertikaler Weise geschlossen, wenn es $d_h\omega=0$ bzw. $d_v\omega=0$ gilt. Es gilt auch der interessante Satz von Poincaré:

Ist ω ein Finslerscher, in vertikaler Weise geschlossener Tensor vom $(0, s)$-Typ, so gibt es einen globalen Tensor ω^* vom $(0, s-1)$-Typ, für den $d_v\omega^*=\omega$ gilt.

Diesen Satz benötigen wir in folgendem nicht.

§ 2. Die Umkehrung des Satzes von Berwald. Der kanonische Krümmungstensor

Das Hauptergebnis dieses Paragraphen ist:

Satz 2. *Ein Finslerscher Raum F_n, mit dim $F_n>2$, ist genau dann von skalarer Krümmung, wenn sein Weylscher Tensor W^i_{jk} verschwindet.*

Zum Beweis dieses Satzes benötigen wir das

Lemma. *Der Grundtensor* H^i_{jk} *eines beliebigen Finslerschen Raumes lässt sich eindeutig in der Form*

$$H^i_{jk} = W^i_{jk} + \omega_j \delta^i_k - \omega_k \delta^i_j - (d_v \omega)_{jk} y^i \tag{2.1}$$

darstellen, wobei ω *ein positiv homogener (von 1-ter Ordnung) Finslerscher kovarianter Vektor ist, ferner gelten:*

$$y^i \omega_i = H, \tag{2.2}$$

$$y^i (d_v \omega)_{ij} = 2\omega_j - \dot{\partial}_j H. \tag{2.3}$$

Beweis. Aus (1.2) folgt

$$H^i_{jk} = W^i_{jk} + \omega_j \delta^i_k - \omega_k \delta^i_j - \Omega_{jk} y^i$$

mit

$$\omega_k = \frac{1}{n^2-1} (nH_k + H_{km} y^m), \tag{2.4}$$

$$\Omega_{jk} = -\frac{1}{n+1} H_r{}^r{}_{jk}. \tag{2.5}$$

In diesem Fall:

$$(d_v \omega)_{jk} = \dot{\partial}_j \omega_k - \dot{\partial}_k \omega_j = \frac{1}{n^2-1} \big((n-1)(H_{jk} - H_{kj}) + y^m (\dot{\partial}_j H_{km} - \dot{\partial}_k H_{jm})\big).$$

Wegen $H_{km} = \dot{\partial}_k H_m$ verschwindet das letzte Glied im Klammer, so folgt aus (1.3)

$$\Omega_{jk} = -\frac{1}{n+1} H_r{}^r{}_{jk} = \big((n-1)/(n^2-1)\big)(H_{jk} - H_{kj}) = (d_v \omega)_{jk}.$$

Somit ist H^i_{jk} von der Form (2.1). Wir beweisen jetzt (2.2) und (2.3). Aus (1.3) und (2.1) folgt:

$$H = \big(1/(n-1)\big) y^i H^k_{ik} = \big(1/(n-1)\big)\big(y^i W^k_{ik} + (n-1) y^i \omega_i - y^j y^i (\dot{\partial}_j \omega_i - \dot{\partial}_i \omega_j)\big) = y^i \omega_i,$$

da

$$W^k_{ik} = 0, \quad y^i y^j (d_v \omega)_{ij} = 0.$$

Ähnlich folgt:

$$y^i (d_v \omega)_{ij} = y^i (\dot{\partial}_i \omega_j - \dot{\partial}_j \omega_i) = \omega_j - y^i \dot{\partial}_j \omega_i = 2\omega_j - \dot{\partial}_j H.$$

Wir beweisen noch, dass der Tensor ω in (2.1) eindeutig bestimmt ist. In der Tat, aus (2.1), (2.2) und (2.3) erhalten wir:

$$y^i H^k_{ij} = y^i W^k_{ij} + H\delta^k_j - \omega_j y^k - 2(\omega_j - \dot{\partial}_j H) y^i.$$

Komponiert man diese Gleichung mit l_k, so bekommt man

$$\omega_j = (1/3)\big(l_k l^i W^k_{ij} + (H/L) l_j + \dot{\partial}_j H\big).$$

Beweis des Satzes 2. Wenn der Raum von skalarer Krümmung ist, so hat er einen verschwindenen Weylschen Tensor. Diesen Tat hat Berwald bewiesen [1], so werden wir uns mit diesem Teil des Satzes nicht beschäftigen.

Umgekehrt, wenn für F_n die Gleichung $W^i_{jk}=0$ gilt, dann folgt aus (2.1):

$$H^i_{jk} = \omega_j \delta^i_k - \omega_k \delta^i_j - \omega_k \delta^i_j - (d_v \omega)_{jk} y^i.$$

Komponiert man diese Gleichung mit l_i, so erhält man:

$$L(d_v \omega)_{jk} = \omega_j l_k - \omega_k l_j,$$

und daraus folgt:

$$y^j (d_v \omega)_{jk} = \omega_j l_k - \omega_k l_j.$$

Mit Rücksicht auf (2.3) erhalten wir:

$$\omega_j = (1/3)\big((H/L) l_k + \dot{\partial}_j H\big) = R L_{\|i} + (1/3) L R_{\|i}, \tag{2.6}$$

und daraus:

$$(d_v \omega)_{ij} = (1/3)(l_j R_{\|i} - l_i R_{\|j}), \tag{2.7}$$

wobei $R=H/L^2$, und die Operation $\|i$ durch $L\dot{\partial}_i$ definiert ist. Substituiert man (2.6) und (2.7) in (2.1) so rechnet man mühelos aus:

$$R_0{}^j{}_{0m} = (1/L^2) y^i H^j_{im} = (1/L^2)\big(H\delta^j_m - (RL_{\|m} + (1/3) L R_{\|m}) y^j + (1/3) L R_{\|m} y^j\big) =$$
$$= R(\delta^j_m - l^j l_m),$$

damit ist der Raum von skalarer Krümmung. Q.E.D.

Den Tensor ω nennen wir den kanonischen Krümmungstensor des Finslerschen Raumes. Im allgemeinen gilt die Formel:

$$\omega_i = \frac{1}{n^2-1}(nH_i + H_{im} y^m) = \frac{1}{n+1}(H_i + \dot{\partial}_i H), \tag{2.8}$$

und wenn der Raum von skalarer Krümmung ist, so gilt:

$$\omega_i = R L_{\|i} + (1/3) L R_{\|i}. \tag{2.9}$$

Aus (2.9) folgt unmittelbar:

Satz 3. *Der Raum* F_n, dim $F_n > 2$, *ist genau dann von konstanter Krümmung wenn* $W^i_{jk}=0$, $(d_v\omega)_{jk}=0$. *In diesem Fall gilt:*

$$\omega_i = (1/2)\dot{\partial}_i H = RL l_i.$$

Satz 4. *In einem allgemeinen Finslerschen Raum gelten die Folgenden:*

1) $d_v \omega = 0$ *genau dann, wenn* $H_k{}^k{}_{ij} = 0$.

2) $d_h \omega = 0$ *genau dann, wenn* $W_i{}^s{}_{j|s} = 0$.

3) *In einem Raum von skalarer Krümmung gelten die Gleichungen:*

$$(d_h \omega)_{ij} = 0; \quad (d_v L\omega)_{ij} = 0,$$

und in einem Raum von konstanter Krümmung:

$$(d_h\omega)_{ij} = 0; \quad (d_v\omega)_{ij} = 0; \quad l_i \wedge \omega_j = 0.$$

Beweis. 1) folgt aus (2.5), und 3) folgt aus 1) und 2) unmittelbar. Der Beweis des Punktes 2) ist, wie folgt.

Substitutiert man in die Bianchi-Identität

$$H^i_{jk|l} + H^i_{kl|j} + H^i_{lj|k} = 0$$

den Ausdruck (2.1), so erhält man:

$$-\sigma W^i_{jk|l} = \sigma(d_h\omega)_{jk}\delta^i_l - (d_h d_v \omega)_{jkl}\, y^i,$$

wobei σ dei zyklische Summe bezüglich den Indizes k, j, l bezeichnet. Wendet man die Gleichung $-d_h d_v \omega = d_v d_h \omega$ an, dann bekommt man die Behauptung 2) mit der Kontraktion $i \to j$. Q.E.D.

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Vecteurs cycliques et commutativité des commutants. II

BÉLA SZ.-NAGY et CIPRIAN FOIAŞ

1. Dans la Note I (*Acta Sci. Math.*, **32** (1971), 177—183) on a démontré que pour toute contraction complètement non-unitaire T dans l'espace de Hilbert $\mathfrak{H}$, de classe $C_{\cdot 1}$, la condition

(i$_*$) T^* admet un vecteur cyclique

entraîne que

(i) T admet un vecteur cyclique,

(ii) le commutant $\{T\}'$ est commutatif.

Dans la Note présente on va compléter ce résultat comme il suit:

Théorème. *Pour une contraction T de classe C_{01}, telle que $I-TT^*$ est de trace finie, la condition* (i$_*$) *entraîne même que $\{T\}'$ est constitué des fonctions de T, notamment*

(iii) $\{T\}'=\{u(T): u\in H^\infty\}$.

Tout comme dans la Note I, la démonstration sera basée sur des éléments de la théorie des dilatations.

2. Pour une contraction quelconque T de l'espace $\mathfrak{H}$, désignons par U la dilatation unitaire minimum de T, opérant dans un espace $\mathfrak{K}(\supset\mathfrak{H})$, et par U_+ la dilatation isométrique minimum de T, opérant dans l'espace

$$\mathfrak{K}_+ = \bigvee_{n\geq 0} U^n\mathfrak{H}. \tag{1}$$

Soit R la partie unitaire de U_+, opérant dans l'espace

$$\mathfrak{R} = \bigcap_{n\geq 0} U_+^n\mathfrak{K}_+ (\subset\mathfrak{K}_+).$$

L'opérateur $X=P_{\mathfrak{R}}|\mathfrak{H}$ $(\mathfrak{H}\to\mathfrak{R})$ et son adjoint $X^*=P_{\mathfrak{H}}|\mathfrak{R}$ $(\mathfrak{R}\to\mathfrak{H})$ vérifient alors les relations (cf. Note I)

$$XT^{*n} = R^{*n}X, \quad T^nX^* = X^*R^n \qquad (n=0,1,\ldots). \tag{2}$$

Il s'ensuit que $R^*\overline{X\mathfrak{H}}\subset\overline{X\mathfrak{H}}$, d'où $R(\mathfrak{R}\ominus\overline{X\mathfrak{H}})\subset\mathfrak{R}\ominus\overline{X\mathfrak{H}}$.

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Soit $\mathfrak{R}'$ l'espace de la partie unitaire de l'isométrie $R_0=R|(\mathfrak{R}\ominus\overline{X\mathfrak{H}})$. On a $R_0\mathfrak{R}'=\mathfrak{R}'$, $R\mathfrak{R}'=\mathfrak{R}'$, $U_+\mathfrak{R}'=\mathfrak{R}'$, d'où il s'ensuit que $\mathfrak{R}'$ réduit U_+ aussi. Or on a

$$\mathfrak{R}' \perp X\mathfrak{H} = P_{\mathfrak{R}}\mathfrak{H}, \quad \mathfrak{R}' \perp \mathfrak{H}.$$

Comme la dilatation U_+ est minimum cela entraîne $\mathfrak{R}'=\{0\}$. Ainsi dans la condition

$$\text{(a)} \qquad \mathfrak{R}_0 = \mathfrak{R}\ominus\overline{X\mathfrak{H}} \neq \{0\}$$

l'opérateur R_0 est une translation unilatérale non banale (c'est-à-dire de multiplicité $\geqq 1$). Toujours dans la condition (a), posons

$$\mathfrak{R}_1 = \bigvee_{n\geqq 0} R^{-n}\mathfrak{R}_0, \quad R_1 = R|\mathfrak{R}_1;$$

R_1 est évidemment une translation bilatérale: prolongement unitaire minimum de la translation unilatérale R_0.

Faisons aussi l'hypothèse:

(b) T^* admet un vecteur cyclique, soit h.

On déduit alors de (2) et (1) que

$$\overline{P_{\mathfrak{R}}\mathfrak{H}} = \overline{P_{\mathfrak{R}} \bigvee_{n\geqq 0} T^{*n}h} = \bigvee_{n\geqq 0} R^{*n}r \quad \text{où} \quad r = P_{\mathfrak{R}}h, \tag{3}$$

$$\mathfrak{R} = P_{\mathfrak{R}}\mathfrak{K}_+ = \bigvee_{m\geqq 0} P_{\mathfrak{R}}U_+^m\mathfrak{H} = \bigvee_{m\geqq 0} U_+^m P_{\mathfrak{R}}\mathfrak{H} = \bigvee_{m\geqq 0} R^m P_{\mathfrak{R}}\mathfrak{H} = \bigvee_{j=-\infty}^{\infty} R^j r. \tag{4}$$

Supposons de plus que

(c) T est complètement non-unitaire.

Dans ce cas U et par conséquent R ont leurs mesures spectrales E^U et $E^R=E^U|\mathfrak{R}$ absolûment continues. Comme, d'autre part, dans nos hypothèses R contient une translation bilatérale non banale, nous concluons en particulier que la fonction $(E_t^R r, r)$ est absolûment continue et que

$$\alpha(t) = \frac{d}{dt}(E_t^R r, r) > 0 \quad \text{p.p.} \tag{5}$$

Vu que pour j, k entiers quelconques on a

$$(R^j r, R^k r) = \int_0^{2\pi} e^{i(j-k)t}\alpha(t)\,dt,$$

la correspondance

$$\sum_j c_j R^j r \mapsto \sum_j c_j e^{ijt}\cdot\sqrt{\alpha(t)}$$

(pour des sommes finies) est isométrique; en vertu de (4) et (5) elle s'étend par continuité à un opérateur unitaire

$$\tau : \mathfrak{R} \to L^2(0,2\pi).$$

R se transforme par τ en l'opérateur de multiplication par e^{it} dans $L^2(0,2\pi)$. On conclut que R est une translation bilatérale simple dans $\mathfrak{R}$.

Comme R_1 est aussi une translation bilatérale, restriction de R à $\mathfrak{R}_1$, on a nécessairement $\mathfrak{R}_1=\mathfrak{R}$, $R_1=R$; *cf.* [H], Proposition I.2.1. Ainsi, R_0 est une translation unilatérale simple dans $\mathfrak{R}_0$ et R est une extension unitaire minimum de R_0.

Cela étant, envisageons, toujours dans les hypothèses (a)—(c), un $A\in\{T\}'$. On y peut attacher un $B\in\{U_+\}'$ tel que

$$AP_{\mathfrak{H}} = P_{\mathfrak{H}}B, \quad \|B\| = \|A\|, \tag{6}$$

et on a $B\mathfrak{R}\subset\mathfrak{R}$, $C=B|\mathfrak{R}\in\{R\}'$; *cf.* Note I, (17). Par (6) on a

$$AX^* = X^*C, \quad XA^* = C^*X, \quad \text{d'où} \quad C^*\overline{X\mathfrak{H}}\subset \overline{X\mathfrak{H}}, \quad C\mathfrak{R}_0\subset\mathfrak{R}_0. \tag{7}$$

En posant $C_0=C|\mathfrak{R}_0$ on aura $C_0\in\{R_0\}'$. Comme R_0 est une translation unilatérale simple, cela entraîne qu'il existe $u\in H^\infty$ tel que

$$C_0 = u(R_0), \quad \text{d'où} \quad C|\mathfrak{R}_0 = u(R)|\mathfrak{R}_0$$

Puisque R permute à C et à $u(R)$, et que

$$\mathfrak{R} = \bigvee_{n\geqq 0} R^{-n}\mathfrak{R}_0,$$

il vient:

$$C = u(R), \tag{8}$$

Par (7) et (8), et par la relation $TP_{\mathfrak{H}}=P_{\mathfrak{H}}U_+$ entre T et U_+ il s'ensuit:

$$AX^* = X^*u(R) = P_{\mathfrak{H}}u(U_+)|\mathfrak{R} = u(T)P_{\mathfrak{H}}|\mathfrak{R}, \quad \big(A-u(T)\big)P_{\mathfrak{H}}|\mathfrak{R} = 0.$$

Lorsque $T\in C_{\cdot 1}$, on a $\ker P_{\mathfrak{R}}|\mathfrak{H}=\{0\}$ (*cf.* [H], Prop. II.3.1) et par conséquent $(P_{\mathfrak{R}}|\mathfrak{H})^*=P_{\mathfrak{H}}|\mathfrak{R}$ a ses valeurs denses dans $\mathfrak{H}$, donc dans ce cas

$$A-u(T) = 0, \quad A = u(T). \tag{9}$$

On a donc démontré le suivant

Lemme 1. *Pour toute contraction T dans $\mathfrak{H}$, de classe $C_{\cdot 1}$, vérifiant les conditions* (a)—(c), *et pour tout $A\in\{T\}'$ on a la représentation* (9), *avec un $u\in H^\infty$.*

Remarque. On aboutit au même résultat si, au lieu de la condition (c), on suppose seulement que la partie unitaire de T ait sa mesure spectrale absolument continue.

3. Afin d'élucider la condition (a) rappelons que pour une contraction T quelconque dans $\mathfrak{H}$ on a les décompositions

$$\mathfrak{K}_+ = \mathfrak{H}\oplus M_+(\mathfrak{L}) \quad \text{et} \quad \mathfrak{K}_+ = M_+(\mathfrak{L}_*)\oplus\mathfrak{R}$$

où $\mathfrak{L}=\overline{(U-T)\mathfrak{H}}$, $\mathfrak{L}_*=\overline{(I-UT^*)\mathfrak{H}}$; *cf.* [H], Chap. I.

Il s'ensuit l'équivalence:

$$\{\overline{P_{\mathfrak{R}}\mathfrak{H}} = \mathfrak{R}\} \Leftrightarrow \{Q_+ : M_+(\mathfrak{L}) \to M_+(\mathfrak{L}_*) \text{ est injectif}\},$$

où Q_+ désigne la projection orthogonale de $M_+(\mathfrak{L})$ à $M_+(\mathfrak{L}_*)$. Dans la représentation de Fourier de Q_+ (*cf.* [H], Chap. VI) la dernière condition veut dire que l'opérateur

$$\Theta : H^2(\mathfrak{L}) \to H^2(\mathfrak{L}_*)$$

de multiplication par la fonction caractéristique $\Theta(\lambda)$ de T est injectif.

Ainsi, la condition (a) est équivalente à la suivante:

$$(\mathrm{a}^*) \qquad \text{il existe } h \in H^2(\mathfrak{L}), \ h \neq 0, \ \text{tel que } \Theta h = 0.$$

Lemme 2. *La condition* (a*) *est vérifiée en particulier dans le cas où* $T \in C_{01}$ *et* $I - TT^*$ *est de trace finie.*

Démonstration. Soit

$$(I - TT^*)h = \sum_{n=1}^{\infty} \mu_n (h, \varphi_n)\varphi_n \quad (h \in \mathfrak{H}) \tag{10}$$

la représentation spectrale de $I - TT^*$ suivant un système orthonormal $\{\varphi_n\}$ de vecteurs propres, où $\mu_1 \geqq \mu_2 \geqq \ldots > 0$.[1] Puisque $T \in C_{01}$ $(\subset C_{\cdot 1})$, on a $T^*\varphi_n \neq 0$ et par conséquent $\mu_n < 1$. Les vecteurs

$$\psi_n = (1 - \mu_n)^{-1/2} T^* \varphi_n \tag{11}$$

forment eux aussi un système orthonormal et on a

$$\varphi_n = (1 - \mu_n)^{-1/2} T \psi_n. \tag{12}$$

De plus, on déduit de (11) et (12)

$$\begin{aligned}(I - T^*T)\psi_n &= (1-\mu_n)^{-1/2}(I - T^*T)T^*\varphi_n = (1-\mu_n)^{-1/2}T^*(I - TT^*)\varphi_n = \\ &= (1-\mu_n)^{-1/2}\mu_n T^*\varphi_n = \mu_n \psi_n.\end{aligned} \tag{13}$$

Considérons les sous-espaces $\mathfrak{M}_n$ de $\mathfrak{D}_T (= \overline{(I - T^*T)\mathfrak{H}})$ et $\mathfrak{M}_{*n}$ de $\mathfrak{D}_{T^*} (= \overline{(I - TT^*)\mathfrak{H}})$ engendrés par les vecteurs $\psi_1, \ldots, \psi_n$ et $\varphi_1, \ldots, \varphi_n$, selon les cas. Notons que par (10) on a $\mathfrak{D}_{T^*} = \bigvee_1^\infty \varphi_n$, tandis que (13) assure seulement que $\mathfrak{M} = \bigvee_1^\infty \psi_n$ est un sous-espace de $\mathfrak{D}_T$. Soient P_n et P les projections orthogonale sde $\mathfrak{D}_T$ sur $\mathfrak{M}_n$ et $\mathfrak{M}$, selon les cas, et soit P_{*n} la projection orthogonale de $\mathfrak{D}_{T^*}$ sur $\mathfrak{M}_{*n}$.

On a donc

$$P_{*n} \to I_{\mathfrak{D}_{T^*}} \quad \text{et} \quad P_n \to P \quad (n \to \infty). \tag{14}$$

Cela étant, considérons la fonction caractéristique de T dans sa forme canonique $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$, *cf.* [H], Sec. VI.1.1. Soit $d_n(\lambda)$ le déterminant de la matrice

$$M_n(\lambda) = [m_{ij}(\lambda)]_{i,j=1,\ldots,n} \quad \text{où} \quad m_{ij}(\lambda) = (\Theta_T(\lambda)\psi_j, \varphi_i).$$

[1] Si $I - TT^*$ est de rang fini, les sommes dans (10), et dans ce qui suit, s'étendent à un nombre fini de termes.

Puisque $\Theta_T(0) = -T|\mathfrak{D}_T$, on a

$$|d_n(0)| = |\det[(T\psi_j, \varphi_i)]_{i,j=1,\dots,n}| =$$
$$= |\det[((1-\mu_j)^{1/2}\varphi_j, \varphi_i)]_{i,j=1,\dots,n}| = \prod_{j=1}^{n}(1-\mu_j)^{1/2} \geqq a,$$

où

$$a = \prod_{j=1}^{\infty}(1-\mu_j)^{1/2} > 0 \quad \text{parce que} \quad \sum_j \mu_j = \mathrm{tr}(I-TT^*) < \infty.$$

Définissons les fonctions $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_n(\lambda)\}$ par

$$\Theta_n(\lambda)f = P_{*n}\Theta_T(\lambda)P_n f + \sum_{k=n+1}^{\infty}(f, \psi_k)\varphi_k \quad (f \in \mathfrak{D}_T); \tag{15}$$

ces fonctions sont évidemment analytiques, contractives, et on a

$$\Theta_n(\lambda)^* g = P_n\Theta_T(\lambda)^* P_{*n} g + \sum_{k=n+1}^{\infty}(g, \varphi_k)\psi_k \quad (g \in \mathfrak{D}_{T^*}). \tag{16}$$

Faisant usage de ce que P_n et P_{*n} sont des projections orthogonales et convergent suivant (14), on déduit de (15) et (16) que

$$\Theta_n(\lambda)f \to \Theta_T(\lambda)Pf \quad (f \in \mathfrak{D}_T) \tag{17}$$

$$\Theta_n(\lambda)^* g \to P\Theta_T(\lambda)^* g \quad (g \in \mathfrak{D}_{T^*}) \tag{18}$$

lorsque $n \to \infty$.

Soit $\omega_n(\lambda)$ l'opérateur de $\mathfrak{M}_{*n}$ dans $\mathfrak{M}_n$ dont la matrice $[(\omega_n(\lambda)\varphi_j, \psi_i)]_{i,j=1,\dots,n}$ est l'adjoint algébrique de la matrice $M_n(\lambda)$, donc telle que

$$M_n(\lambda)\omega_n(\lambda) = \omega_n(\lambda)M_n(\lambda) = d_n(\lambda)I_n$$

où I_n désigne la matrice unité d'ordre n. En fonctions de λ ($|\lambda| < 1$) toutes ces matrices sont analytiques et contractives; *cf.* [H], Sec. V.6.1.

Définissons alors les fonctions $\{\mathfrak{D}_{T^*}, \mathfrak{D}_T, \Omega_n(\lambda)\}$ par

$$\Omega_n(\lambda)g = \omega_n(\lambda)P_{*n}g + d_n(\lambda)\sum_{k=n+1}^{\infty}(g, \varphi_k)\psi_k \quad (g \in \mathfrak{D}_{T^*}). \tag{19}$$

Elles sont aussi analytiques, contractives et on a

$$\Omega_n(\lambda)\mathfrak{D}_{T^*} \subset \mathfrak{M} \quad \text{pour tout } n \text{ et } \lambda,\ |\lambda| < 1. \tag{20}$$

On déduit de (15) et (19):

$$\Theta_n(\lambda)\Omega_n(\lambda)g = d_n(\lambda)g \quad (g \in \mathfrak{D}_{T^*}). \tag{21}$$

Faisant usage du théorème de Vitali—Montel on montre qu'il existe une suite partielle $\{n_q\}$ d'indices telle que $d_{n_q}(\lambda)$ tend dans $|\lambda| < 1$ vers une fonction analytique $d(\lambda)$ et $\Omega_{n_q}(\lambda)$ tend (faiblement) vers une fonction analytique $\{\mathfrak{D}_{T^*}, \mathfrak{D}_T, \Omega(\lambda)\}$;

on a $|d(\lambda)|\leqq 1$, $|d(0)|\geqq a(>0)$ et $\Omega(\lambda)$ est aussi contractive. De plus, (20) entraîne

$$\Omega(\lambda)\mathfrak{D}_{T^*}\subset\mathfrak{M}. \tag{22}$$

Enfin, (21) entraîne, eu égard à (13) et (17), que

$$\Theta_T(\lambda)\Omega(\lambda)g=d(\lambda)g \quad (g\in\mathfrak{D}_{T^*}), \tag{23}$$

d'où, en particulier (posant $g=\Theta_T(\lambda)f$),

$$\Theta_T(\lambda)\big(\Omega(\lambda)\Theta_T(\lambda)f-d(\lambda)f\big)=0 \quad (f\in\mathfrak{D}_T). \tag{24}$$

Si la condition (a*) n'est pas vérifiée, (24) entraîne

$$\Omega(\lambda)\Theta_T(\lambda)f=d(\lambda)f \quad \text{pour tout} \quad f\in\mathfrak{D}_T,$$

ce qui, ensemble avec (23), veut dire que $\Theta_T(\lambda)$ admet le multiple scalaire $d(\lambda)$. Or, cela est impossible parce que $T\in C_{01}$.

Cette contradiction prouve que (a*) est vérifiée et achève la démonstration du Lemme 2. Les deux lemmes ensemble entraînent le théorème énoncé au commencement de cette Note.

Remarque. **1.** La condition que $I-TT^*$ soit de trace finie est vérifiée en particulier si $I-TT^*$ est de rang $\mathfrak{d}_{T^*}<\infty$. Des exemples de contractions $T\in C_{01}$ avec T^* cyclique est $\mathfrak{d}_{T^*}$ fini (notamment avec $\mathfrak{d}_{T^*}=1$) ont été construits dans [1], Proposition 2. (Prendre les adjoints des opérateurs $S(\Theta)$ qui y sont considérés.) Ces exemples sont quasi-similaires à l'adjoint S^* de la translation unilatérale simple S. Il se peut que toute contraction T vérifiant les hypothèses de notre théorème et avec T^* cyclique soit quasi-similaire à S (problème ouvert).

2. Lemme 2 n'est pas en général valable si $I-TT^*$ est compact, mais de trace infinie, même si $\sum_n \mu_n^p<\infty$ pour un exposant $p>1$. En effet, dans [2] on construit des contractions $T\in C_{01}$ telles que $\sum_n \mu_n^p<\infty$ pour un p donné d'avance et que ni T ni T^* n'ont pas de valeurs propres. Par conséquent, $\Theta_T(\lambda)$ est alors une injection pour toute valeur de λ et (a*) est impossible.

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Einfacher Beweis eines Satzes von B. S. Kašin

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1. In dieser Note werden wir einen einfachen Beweis für den folgenden Satz von KAŠIN [2] geben:

Satz. *Ist p eine genügend grosse natürliche Zahl, dann gibt es ein orthonormiertes System von Treppenfunktionen* $\varphi_1(x), \ldots, \varphi_{2p^2}(x)$ *im Intervall* $(0, 1)$ *mit den folgenden Eigenschaften:*

$$|\varphi_n(x)| = 1 \quad (x \in (0, 1);\ n = 1, \ldots, 2p^2),$$

$$\operatorname{mes}\left\{x \in (0, 1):\ \max_{1 \leqq k \leqq 2p^2} \left|\sum_{n=1}^{k} \varphi_n(x)\right| \geqq C_1 p \log p\right\} \geqq C_2,$$

wobei C_1, C_2 *positive, von p unabhängige Konstanten sind.*

(Vorher hat MENCHOFF [3] diese Behauptung für mit einer von p unabhängigen Konstante $M(>1)$ beschränktes System gezeigt.)

2. Zum Beweis benützen wir den folgenden:

Hilfssatz. (S.z.B. [2], [3]) *Es sei* $\{g_n(x)\}_1^N$ *ein System von Funktionen* $g_n(x) \in L^2(0, 1)$, *für welche Zahlen* γ_i $(i = 1, \ldots, N-1)$ *existieren, mit*

$$\left|\int_0^1 g_k(x) g_l(x)\, dx\right| \leqq \gamma_i \ (1 \leqq k, l \leqq N,\ |k-l| = i),\ \sum_{i=1}^{N-1} \gamma_i < M.$$

Dann kann man die Funktionen $g_n(x)$ *auf das Intervall* $[1, 2M+1)$ *derart fortsetzen, daß sie dort Treppenfunktionen sind, mit* $|g_n(x)| = 1$, *und im ganzen Intervall* $(0, 2M+1)$ *ein orthogonales System bilden.*

3. Beweis des Satzes. Wir brauchen die Ideen von [4] und [2]. Wir gehen von einem Funktionensystem von KACZMARZ [1] aus. Es sei

$$f_n(x) = \frac{1}{2(k-p-n-1/2)} \quad \left(x \in \left(\frac{k-1}{p}, \frac{k}{p}\right);\ k = 1, \ldots, 4p;\ n = 1, \ldots, 2p\right),$$

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und wir setzen

$$\alpha_{k,l} = \int_0^4 f_k(x) f_l(x)\, dx.$$

Dann gilt

(1) $$\alpha_{k,k} \leqq C_3/p \quad (k = 1, \dots, 2p).$$

(Im folgenden bezeichnen $C_3, C_4, \dots$ positive, von p unabhängige Konstanten.) Ferner gilt für $k > l$:

$$\alpha_{k,l} = \frac{1}{4p} \sum_{n=1}^{4p} \frac{1}{(n-p-k-1/2)(n-p-l-1/2)} =$$

$$= \frac{1}{4p(k-l)} \sum_{n=1}^{4p} \left\{ \frac{1}{n-p-k-1/2} - \frac{1}{n-p-l-1/2} \right\} =$$

$$= \frac{1}{4p(k-l)} \left\{ \sum_{n=1-p-k}^{3p-k} \frac{1}{n-1/2} - \sum_{n=1-p-l}^{3p-l} \frac{1}{n-1/2} \right\} =$$

$$= \frac{1}{4p(k-l)} \left\{ \sum_{n=1-p-k}^{-p-l} \frac{1}{n-1/2} - \sum_{n=3p-k+1}^{3p-l} \frac{1}{n-1/2} \right\}.$$

Daraus folgt

(2) $$|\alpha_{k,l}| \leqq \frac{1}{4p(k-l)} \left\{ \frac{k-l}{p+l+1/2} + \frac{k-l}{3p-k+1/2} \right\} \leqq \frac{C_4}{p^2} \quad (k, l = 1, \dots, 2p;\ k \neq l).$$

Weiterhin, auf Grund der Definition ist

(3) $$\max_{1 \leqq k \leqq 2p} \sum_{n=1}^{k} f_n(x) \geqq C_5 \log p \quad \big(x \in (2, 3)\big).$$

Es sei

$$g_{r+(s-1)p}(x) = f_s(x) \quad \big(x \in (0, 4);\ r = 1, \dots, p;\ s = 1, \dots, 2p\big).$$

Wir setzen

$$\gamma_i = C_3/p \quad (i = 1, \dots, p-1), \quad \gamma_i = C_4/p^2 \quad (i = p, \dots, 2p^2-1).$$

Dann gilt

$$\sum_{i=1}^{2p^2-1} \gamma_i = C_3(p-1)/p + C_4(2p^2-p)/p^2 \leqq C_6.$$

Auf Grund von (1), (2), und durch Anwendung des Hilfssatzes erhalten wir, daß die Funktionen $g_n(x)$ auf das Intervall $[4, 2C_6+4)$ derart fortgesetzt werden können, daß sie im Intervall $(0, 2C_6+4)$ Treppenfunktionen sind, dort ein orthogonales System bilden, und im ganzen Intervall $(0, 2C_6+4)$ die Ungleichung $|g_n(x)| \leqq 1$ genügen. Dann bilden in $(0, 1)$ die Treppenfunktionen

$$h_n(x) = g_n\big((2C_6+4)x\big) \quad \big(x \in (0, 1);\ n = 1, \dots, 2p^2,\big)$$

ein orthogonales System und aus (3) folgen

$$\text{mes}\left\{x\in(0,1):\max_{1\leq k\leq 2p^2}\left|\sum_{n=1}^{k}h_n(x)\right|\geqq C_5 p\log p\right\}\geqq C_7, \tag{4}$$

und

$$|h_n(x)|\leqq 1\quad \left(x\in(0,1);\ n=1,\dots,2p^2\right). \tag{5}$$

Es sei $I_1, \dots, I_R$ eine disjunkte Einteilung des Intervalls $(0, 1)$, so daß in jedem Intervall I_r jede Funktion $h_n(x)$ konstant ist. Es sei r $(1\leqq r\leqq R)$ ein fester Index, und wir setzen $I_r=(a_r, b_r)$, $h_n(x)=\varrho_n^{(r)}$ $(x\in I_r;\ n=1,\dots,2p^2)$. Es seien weiterhin $\chi_n^{(r)}(x)$ $(n=1,\dots,2p^2)$ stochastisch unabhängige Treppenfunktionen im Intervall $(0, 1)$ mit

$$\int_0^1 \chi_n^{(r)}(x)\,dx=0\quad (n=1,\dots,2p^2),$$

wobei $\chi_n^{(r)}(x)$ den Wertbereich $\{1-\varrho_n^{(r)}, -1-\varrho_n^{(r)}\}$ besitzt [1]). Wir setzen

$$\chi_n^{(r)}(I_r;\,x)=\begin{cases}\chi_n^{(r)}\left(\dfrac{x-a_r}{b_r-a_r}\right), & x\in I_r\\ 0 & \text{sonst}\end{cases}\qquad (n=1,\dots,2p^2).$$

Es sei endlich

$$\varphi_n(x)=h_n(x)+\sum_{r=1}^{R}\chi_n^{(r)}(I_r;\,x)\quad (n=1,\dots,2p^2).$$

Offensichtlich sind $\varphi_n(x)$ Treppenfunktionen, es gilt $|\varphi_n(x)|=1$ $\left(x\in(0,1);\ n=\right.$ $\left.=1,\dots,2p^2\right)$, weitherhin folgt für $k\neq l$:

$$\begin{aligned}\int_0^1 \varphi_k(x)\varphi_l(x)\,dx &= \int_0^1 h_k(x)h_l(x)\,dx+\sum_{r=1}^{R}\int_{I_r}h_k(x)\chi_l^{(r)}(I_r;\,x)\,dx+\\ &+\sum_{r=1}^{R}\int_{I_r}\chi_k^{(r)}(I_r;\,x)h_l(x)\,dx+\sum_{r=1}^{R}\int_{I_r}\chi_k^{(r)}(I_r;\,x)\chi_l^{(r)}(I_r;\,x)\,dx=\\ &=\sum_{r=1}^{R}\varrho_k^{(r)}\,\text{mes}\,(I_r)\int_0^1\chi_l^{(r)}(x)\,dx+\sum_{r=1}^{R}\varrho_l^{(r)}\,\text{mes}\,(I_r)\int_0^1\chi_k^{(r)}(x)\,dx+\\ &+\sum_{r=1}^{R}\text{mes}\,(I_r)\int_0^1\chi_k^{(r)}(x)\chi_l^{(r)}(x)\,dx=0.\end{aligned}$$

Also bilden die Funktionen $\varphi_n(x)$ ein orthonormiertes System in $(0, 1)$.

Es sei I_r ein Intervall, für welches

$$\max_{1\leqq k\leqq 2p^2}\left|\sum_{n=1}^{k}h_n(x)\right|\geqq C_5 p\log p\quad (x\in I_r) \tag{6}$$

[1]) Ist $|\varrho_n^{(r)}|=1$, dann soll man $\chi_n^{(r)}(x)\equiv 0$ setzen.

gilt. Daraus und aus (5), durch Anwendung der Kolmogoroffschen Ungleichung ergibt sich:

$$\operatorname{mes}\left\{x\in I_r\colon \max_{1\leqq k\leqq 2p^2}\left|\sum_{n=1}^{k}\chi_n^{(r)}(I_r;\,x)\right|\geqq C_5 p\log p/2\right\}\leqq \tag{7}$$

$$\leqq 4\sum_{n=1}^{2p^2}\int_{I_r}(\chi_n^{(r)}(I_r;\,x))^2\,dx/C_5^2p^2\log^2 p\leqq C_8\operatorname{mes}(I_r)/\log^2 p.$$

Ist p so groß, daß

$$C_8/\log^2 p\leqq 1/2$$

gilt, so ist

$$\operatorname{mes}\left\{x\in I_r\colon \max_{1\leqq k\leqq 2p^2}\left|\sum_{n=1}^{k}\varphi_n(x)\right|\geqq C_5 p\log p/2\right\}\geqq \operatorname{mes}(I_r)/2,$$

auf Grund von (6) und (7). Daraus und aus (4) erhalten wir

$$\operatorname{mes}\left\{x\in(0,\,1)\colon \max_{1\leqq k\leqq 2p^2}\left|\sum_{n=1}^{k}\varphi_n(x)\right|\geqq C_5 p\log p/2\right\}\geqq C_7/2.$$

Durch Anwendung dieses Lemmas kann man den Satz von Kašin [2] leicht herleiten.

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Hyperinvariant subspaces of operators of class $C_0(N)$

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1. Let Θ be an $n\times n$ matrix over the Hardy space H^∞ on the circle, n being a fixed natural number. Such a matrix is called *inner* if $\Theta(e^{it})$ is unitary a.e.t. Associated with an inner matrix Θ are a Hilbert space $\mathfrak{H}(\Theta)$ and an operator $S(\Theta)$ defined by $\mathfrak{H}(\Theta)=H_n^2\ominus\Theta H_n^2$ and $S(\Theta)h=P_\Theta(\chi h)$ $(h\in\mathfrak{H}(\Theta))$, where H_n^2 is the Hardy space of n dimensional (column) vector valued functions, P_Θ is the projection from H_n^2 onto $\mathfrak{H}(\Theta)$, and $\chi(e^{it})=e^{it}$. Any contraction T of class $C_0(m)$ with $m\leqq n$ (i.e. $T^k\to 0$, $T^{*k}\to 0$ as $k\to\infty$, and $\operatorname{rank}((1-T^*T)^{1/2})=\operatorname{rank}((1-TT^*)^{1/2})=m$) is unitarily equivalent to $S(\Theta)$ with a suitable inner $n\times n$ matrix Θ (see [5]).

A subspace $\mathfrak{L}$ of a Hilbert space $\mathfrak{H}$ is said to be *hyperinvariant* for an operator T on $\mathfrak{H}$ if it is invariant for all operators on $\mathfrak{H}$ that commute with T. Operators T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$ are said to be *quasi-similar* if there are quasi-affinities (i.e. operators with zero kernel and dense range) X from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ and Y from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ such that $XT_1=T_2X$ and $T_1Y=YT_2$.

Theorem 1. *Let Θ and Φ be inner matrices over H^∞. If $S(\Theta)$ and $S(\Phi)$ are quasi-similar, then there exist quasi-affinities X from $\mathfrak{H}(\Theta)$ to $\mathfrak{H}(\Phi)$ and Y from $\mathfrak{H}(\Phi)$ to $\mathfrak{H}(\Theta)$ such that*

(i) *$XS(\Theta)=S(\Phi)X$ and $S(\Theta)Y=YS(\Phi)$,*

(ii) *the correspondences $\varphi\colon\mathfrak{L}\to\overline{X\mathfrak{L}}$ and $\psi\colon\mathfrak{M}\to\overline{Y\mathfrak{M}}$ establish an isomorphism from the lattice $\mathcal{I}_\Theta$ of hyperinvariant subspaces for $S(\Theta)$ onto the lattice $\mathcal{I}_\Phi$ for $S(\Phi)$, and its inverse, $\psi=\varphi^{-1}$.*

Proof. The hypothesis of quasi-similarity implies that for $\mathfrak{L}\in\mathcal{I}_\Theta$

$$\varphi(\mathfrak{L})=\bigvee_Z\{Z\mathfrak{L}\,|\,ZS(\Theta)=S(\Phi)Z\} \tag{1}$$

belongs to $\mathcal{I}_\Phi$ (cf. [3], p. 108). By one of the MOORE-NORDGREN theorems ([1], [2]) the quasi-similarity of $S(\Theta)$ and $S(\Phi)$ implies that there exist matrices Δ, Δ', Λ,

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and Λ' each of whose determinants is relatively prime to the determinants of Θ and Φ, and such that

(2) $$\Delta\Theta = \Phi\Lambda \quad \text{and} \quad \Theta\Lambda' = \Delta'\Phi.$$

Define the operator X from $\mathfrak{H}(\Theta)$ to $\mathfrak{H}(\Phi)$ and Y from $\mathfrak{H}(\Phi)$ to $\mathfrak{H}(\Theta)$ by

(3) $$Xh = P_\Phi \Delta h \quad (h \in \mathfrak{H}(\Theta)) \quad \text{and} \quad Yg = P_\Theta \Delta' g \quad (g \in \mathfrak{H}(\Phi)).$$

Relation (2) guarantees condition (i), and X, Y are quasi-affinities (see [2]). Take an arbitrary $\mathfrak{L}$ in the lattice $\mathcal{I}_\Theta$ and let $\mathfrak{L}' = \varphi(\mathfrak{L})$. By a well-known theorem ([5]) the (hyper-) invariance of $\mathfrak{L}$ and $\mathfrak{L}'$ implies the existence of inner matrices Θ_1, Θ_2, Φ_1 and Φ_2 over H^∞ satisfying

(4) $$\Theta = \Theta_2\Theta_1 \quad \text{and} \quad \Phi = \Phi_2\Phi_1,$$

and

(5) $$\mathfrak{L} = \Theta_2(H_n^2 \ominus \Theta_1 H_n^2) \quad \text{and} \quad \mathfrak{L}' = \Phi_2(H_n^2 \ominus \Phi_1 H_n^2).$$

By the definition (1) of $\varphi(\mathfrak{L})$ we have $X\mathfrak{L} \subseteq \varphi(\mathfrak{L}) = \mathfrak{L}'$. On the other hand, since YZ commutes with $S(\Theta)$ for every Z occurring in (1), hyperinvariance of $\mathfrak{L}$ for $S(\Theta)$ implies $YZ\mathfrak{L} \subseteq \mathfrak{L}$, and therefore $Y\mathfrak{L}' = Y\varphi(\mathfrak{L}) \subseteq \mathfrak{L}$. Now the inclusions $\overline{X\mathfrak{L}} \subseteq \mathfrak{L}'$ and $Y\mathfrak{L}' \subseteq \mathfrak{L}$, and relations (2)-(5) imply $\Delta\Theta_2 H_n^2 \subseteq \Phi_2 H_n^2$ and $\Delta'\Phi_2 H_n^2 \subseteq \subseteq \Theta_2 H_n^2$; whence we deduce the exisctence of matrices A and B over H^∞ such that

(6) $$\Delta\Theta_2 = \Phi_2 A \quad \text{and} \quad \Delta'\Phi_2 = \Theta_2 B.$$

Thus it follows that $\Phi_2 AB = \Delta\Delta'\Phi_2$, and hence,

(7) $$\det A \cdot \det B = \det \Delta \cdot \det \Delta'.$$

Since $\det \Delta \cdot \det \Delta'$ is relatively prime to $\det \Phi$, (7) implies that $\det A$ is relatively prime to $\det \Phi$, hence to $\det \Phi_1$. To prove $\mathfrak{L}' = \overline{X\mathfrak{L}}$ suppose that $f \in \mathfrak{L}' \ominus \overline{X\mathfrak{L}}$. Then, again using (2)-(5), we see that f is orthogonal to $\Delta\Theta_2 H_n^2$, and hence to $\Phi_2 A H_n^2$, by (6). Moreover, (5) implies $f = \Phi_2 g$ for some $g \in H_n^2 \ominus \Phi_1 H_n^2$. Then for every $h \notin H_n^2$

$$0 = (f, \Delta\Theta_2 h) = (\Phi_2 g, \Phi_2 A h) = (g, Ah).$$

Since $\det A$ is relatively prime to $\det \Phi_1$, AH_n^2 and $\Phi_1 H_n^2$ span the whole H_n^2. This implies $g = 0$, hence $f = 0$, proving $\mathfrak{L}' = \overline{X\mathfrak{L}}$. The relation $\mathfrak{L} = \overline{Y\mathfrak{L}'} = \overline{YX\mathfrak{L}}$ is proved in a similar way. This completes the proof.

Corollary 2. *Let Θ, Φ, X and Y be as in Theorem 1, and $\mathfrak{L}$ a hyperinvariant subspace for $S(\Theta)$. If*

$$S(\Theta) = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix} \quad \textit{and} \quad S(\Phi) = \begin{bmatrix} S_1' & * \\ 0 & S_2' \end{bmatrix}$$

are the triangulations corresponding to the decompositions

$$\mathfrak{H}(\Theta) = \mathfrak{L} \oplus \mathfrak{L}^\perp \quad \textit{and} \quad \mathfrak{H}(\Phi) = \overline{X\mathfrak{L}} \oplus \overline{X\mathfrak{L}}^\perp,$$

respectively, then S_1 *and* S_2 *are quasi-similar to* S_1' *and* S_2', *respectively.*

Proof. For the quasi-similarity of S_1 and S_1', use the quasi-affinities $X|\mathfrak{L}$ and $Y|X\mathfrak{L}$. Relation (6) implies that $S(\Theta_2)$ and $S(\Phi_2)$ are quasi-similar. S_2 and S_2' are unitarily equivalent to $S(\Theta_2)$ and $S(\Phi_2)$, respectively (see [5]).

2. A *normal* matrix M over H^∞ is, by definition, of the form

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),$$

where, for each i, m_i is a scalar inner function and m_{i-1} is a divisor of m_i $(m_0=1)$. The operator $S(M)$ induced by a normal matrix M is called a *Jordan operator*. By the Sz.-Nagy—Foiaş theorem [4] every operator $S(\Theta)$ with inner Θ is quasi similar to a Jordan operator. Therefore on the basis of Theorem 1 and Corollary 2 the subsequent discussions will be confined to the case of Jordan operators.

Theorem 3. *Let* M *be a normal matrix over* H^∞. *A subspace* $\mathfrak{L}$ *of* $\mathfrak{H}(M)$ *is hyperinvariant for* $S(M)$ *if and only if there are normal matrices*

$$\Theta = \operatorname{diag}(u_1, \ldots, u_n) \quad \textit{and} \quad \Phi = \operatorname{diag}(v_1, \ldots, v_n)$$

satisfying

$$M = \Theta\Phi \quad \textit{and} \quad \mathfrak{L} = \Theta(H_n^2 \ominus \Phi H_n^2). \tag{8}$$

Proof. By the lifting theorem ([5] p. 258) for every operator X on $\mathfrak{H}(M)$, commuting with $S(M)$, there is a matrix Δ over H^∞ satisfying

$$Xh = P_M \Delta h \quad (h \in \mathfrak{H}(M)) \quad \text{and} \quad \Delta M H_n^2 \subseteqq M H_n^2. \tag{9}$$

The latter condition is equivalent to the existence of a matrix Λ over H^∞ satisfying

$$\Delta M = M \Lambda. \tag{10}$$

Conversely every matrix Δ over H^∞ that is accompanied with a matrix Λ satisfying (10) induces an operator X on $\mathfrak{H}(M)$, commuting with $S(M)$, by the first part of (9).

Suppose that $\mathfrak{L}$ is of the form (8). To prove the hyperinvariance of $\mathfrak{L}$ for $S(M)$, it suffices to show the invariance of $\mathfrak{L}$ for the operator X defined by (9). The existence of Λ satisfying (10) implies that if $i>j$ then the inner function $m_j^{-1} m_i$ is a divisor of the $\Delta_{i,j}$ that is the (i,j)-th entry of Δ. Since Θ and Φ are normal matrices with $M=\Theta\Phi$, for $i>j$ the inner function $u_j^{-1} u_i$ is a divisor of $m_j^{-1} m_i$, hence a divisor of $\Delta_{i,j}$. This gaurantees the existence of a matrix Λ' over H^∞ satisfying

$$\Delta\Theta = \Theta\Lambda', \tag{11}$$

and consequently the invariance of $\mathfrak{L}$ for X.

Suppose conversely that $\mathfrak{L}$ is hyperinvariant for $S(M)$. Let P_i be the orthogonal projection from $\mathfrak{H}(M)$ onto the i-th component space. Since P_i commutes with $S(M)$, the hyperinvariance of $\mathfrak{L}$ implies that

$$\mathfrak{L} = P_1\mathfrak{L}\oplus\ldots\oplus P_n\mathfrak{L}$$

and each $P_i\mathfrak{L}$ is an invariant subspace for $S(m_i)$. By the Beurling theorem there are inner divisors u_i and v_i of m_i satisfying

$$m_i = u_i v_i \quad \text{and} \quad P_i\mathfrak{L} = u_i(H^2\ominus v_i H^2). \tag{12}$$

Set $\Theta=\operatorname{diag}(u_1, \ldots, u_n)$ and $\Phi=\operatorname{diag}(v_1, \ldots, v_n)$, then Θ and Φ satisfy (8). It remains to prove the normality of Θ and Φ. To this end, take the matrix Δ over H^∞ whose (i,j)-th entry $\Delta_{i,j}$ is defined by

$$\Delta_{ij} = 1 \quad (i \leqq j) \quad \text{and} \quad \Delta_{ij} = m_j^{-1}m_i \quad (i > j).$$

Clearly there exists a matrix Λ over H^∞ satisfying (10). The hyperinvariance of $\mathfrak{L}$ implies the existence of a matrix Λ' satisfying (11). This means if $i<j$ then u_i is a divisor of u_j and $u_i^{-1}u_j$ is a divisor of $m_i^{-1}m_j$. The former condition guarantees the normality of Θ while the latter does the normality of Φ. This completes the proof.

Corollary 4. *Let M be a normal matrix over H^∞, and $\mathfrak{L}_1$, $\mathfrak{L}_2$ subspaces of $\mathfrak{H}(M)$ hyperinvariant for $S(M)$. If $S(M)|\mathfrak{L}_1$ is quasi-similar to $S(M)|\mathfrak{L}_2$ then $\mathfrak{L}_1=\mathfrak{L}_2$.*

Proof. Take the normal matrices Θ_i and Φ_i $(i=1, 2)$ satisfying (8) with Θ_i, Φ_i and $\mathfrak{L}_i$ in place of Θ, Φ and $\mathfrak{L}$, respectively. Since $S(\Phi_i)$ is unitarily equivalent to $S(M)|\mathfrak{L}_i$, it follows that $\Phi_1=\Phi_2$. This implies that $\Theta_1=\Theta_2$ and $\mathfrak{L}_1=\mathfrak{L}_2$. This completes the proof.

Recall that the *minimal function* m_S of an operator S of class $C_0(N)$ is defined as the greatest common inner divisor of all inner functions m for which $m(S)=0$. If $S(M)$ with normal matrix $M=\operatorname{diag}(m_1, \ldots, m_n)$ is the Jordan model of S then the minimal function m_S coincides with m_n. The minimal function is preserved under quasi-similarity.

Corollary 5. *Let M be a normal matrix over H^∞. If $\mathfrak{L}$ is a subspace of $\mathfrak{H}(M)$ hyperinvariant for $S\equiv S(M)$, then*

$$m_S = m_{S_1}\cdot m_{S_2}, \tag{13}$$

where the operators S_1 and S_2 are defined by the triangulation $S=\begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ corresponding to the decomposition $\mathfrak{H}(M)=\mathfrak{L}\oplus\mathfrak{L}^\perp$

Proof. Take normal matrices Θ and Φ over H^∞ satisfying (8). Since $S(\Phi)$ and $S(\Theta)$ are unitarily equivalent to S_1 and S_2, respectively, it follows that $m_{S_1}= =v_n$ and $m_{S_2}=u_n$, which implies (13).

Remark. In the above situation $m_S=m_{S_1}\cdot m_{S_2}$ for an arbitrary invariant subspace $\mathfrak{L}$ if and only if $M=\operatorname{diag}(1, \ldots, 1, m_n)$.

3. When m is a scalar inner function, for the operator $S(m)$ the invariance of a subspace is equivalent to its hyperinvariance. The lattice $\mathcal{I}_m$ of all (hyper-) invariant subspaces is totally ordered if and only if m is of the form

$$\left(\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}\right)^n \quad (|\alpha|<1,\ n \text{ a positive integer}) \tag{14}$$

or of the form

$$e_s(\lambda)\equiv \exp\left(s\frac{\lambda+\alpha}{\lambda-\alpha}\right) \quad (|\alpha|=1;\ s>0), \tag{15}$$

according as $\dim \mathfrak{H}(m)=n$ or $\dim \mathfrak{H}(m)=\infty$ (cf. [5] p. 136). This can be generalized to the case of inner matrices.

Theorem 6. *Let M be a normal matrix over H^∞ and* $\dim \mathfrak{H}(M)=\infty$. *The lattice $\mathcal{I}_M$ of hyperinvariant subspaces for $S(M)$ is totally ordered if and only if m_n is of the form* (15) *and each m_i coincides with either* 1 *or m_n.*

Proof. By Theorem 3 the total orderedness of the lattice $\mathcal{I}_M$ is equivalent to the condition that if normal matrices Θ_i $(i=1, 2)$ are (left) divisors of M such that $\Theta_1^{-1}M$ and $\Theta_2^{-1}M$ are normal too, then one of Θ_1 and Θ_2 is a (left) divisor of the other. Suppose that $\mathcal{I}_M$ is totally ordered. Take arbitrary inner divisors u and v of m_n, and set $u_i=u\wedge m_i$ and $v_i=v\wedge m_i$ ($a\wedge b$ denotes the greatest common inner divisor of a and b). Then the normal matrices Θ_1 and Θ_2 defined by

$$\Theta_1=\operatorname{diag}(u_1, u_2, \ldots, u_{n-1}, u) \quad \text{and} \quad \Theta_2=\operatorname{diag}(v_1, v_2, \ldots, v_{n-1}, v),$$

are (left) divisors of M and $\Theta_i^{-1}M$ $(i=1, 2)$ is a normal matrix over H^∞. The divisibility of Θ_2 by Θ_1 or Θ_1 by Θ_2 implies that one of u and v is a divisor of the other. The arbitrariness of u and v implies that m_n is of the form (15) because $\dim \mathfrak{H}(M)=\infty$ implies $\dim \mathfrak{H}(m_n)=\infty$. There exists an m_i such that $m_{i-1}^{-1}m_i=e_S$ $(1\leqq i\leqq n)$. In fact if any $m_{i-1}^{-1}m_i$ is not equal to e_S, then there exists i and j such that $1\leqq i<j\leqq n$, $m_{i-1}^{-1}m_i=e_a$ $(s>a>0)$ $m_{j-1}^{-1}m_j=e_b$ $(s>b>0)$ and $a+b\leqq s$. Now set c and d so that $0<c\leqq a, 0<d\leqq b$ and $c<d$. Consider the normal matrices Ω_1 and Ω_2 defined by

$$\Omega_1=\operatorname{diag}(1, \ldots, 1, \overset{(i)}{e_c}, \ldots, e_c) \quad \text{and} \quad \Omega_2=\operatorname{diag}(1, \ldots, 1, \overset{(j)}{e_d}, \ldots, e_d).$$

Clearly Ω_i is a (left) divisor of M and $\Omega_i^{-1}M$ is a normal matrix. By Theorem 3, the subspace $\Omega_1 H_n^2 \ominus H_n^2 M$ and $\Omega_2 H_n^2 \ominus H_n^2 M$ are hyperinvariant for $S(M)$, but any one of them is not included in the other, a contradiction. Consequently

$$M = \mathrm{diag}\,(1, \ldots, 1, e_s, \ldots, e_s).$$

The "only if" part follows from the next lemma.

Lemma 7. *Let the operator V on $\mathfrak{H}_0$ be unicellular, i.e. let the lattice of all invariant subspaces for V be totally ordered. Then for any finite direct* sum $T = V \oplus \ldots \ldots \oplus V$, *acting on $\mathfrak{H} = \mathfrak{H}_0 \oplus \ldots \oplus \mathfrak{H}_0$, the lattice of all hyperinvariant subspaces for T is totally ordered.*

Proof. Let P_i be a projection from $\mathfrak{H}$ to the i-th component space. For any subspace $\mathfrak{L}$ of $\mathfrak{H}$ hyperinvariant for T, as in the proof of Theorem 3, we have $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots \oplus \mathfrak{L}_n$, where $\mathfrak{L}_i = P_i \mathfrak{L}$. The operator D_2, which causes interhchange of the first component with the second one for each vector, commutes with T, hence $D_2 \mathfrak{L} \subseteqq \mathfrak{L}$. This implies that $\mathfrak{L}_2 \subseteqq \mathfrak{L}_1, \mathfrak{L}_1 \subseteqq \mathfrak{L}_2$ and hence $\mathfrak{L}_1 = \mathfrak{L}_2$. Similarly we have $\mathfrak{L}_1 = \mathfrak{L}_i$. Thus for arbitrary hyperinvariant subspaces $\mathfrak{L}$ and $\mathfrak{L}'$ for T we have $\mathfrak{L} = \mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_1$ and $\mathfrak{L}' = \mathfrak{L}_1' \oplus \ldots \oplus \mathfrak{L}_1'$. Since $\mathfrak{L}_1$ and $\mathfrak{L}_1'$ are invariant for V, it follows that $\mathfrak{L}_1 \subseteqq \mathfrak{L}_1'$ or $\mathfrak{L}_1' \subseteqq \mathfrak{L}_1$. Thus we have $\mathfrak{L} \subseteqq \mathfrak{L}'$ or $\mathfrak{L}' \subseteqq \mathfrak{L}$. This completes the proof of the lemma.

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A bound for the nilstufe of a group

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In this paper we are concerned solely with torsion-free abelian groups of finite rank. Such a group is said to have an (associative) multiplication defined on it if there is an (associative) ring with additive structure isomorphic to G. There may be many non-isomorphic rings all having isomorphic additive structure and most significantly a group may have associative and non-associative multiplications defined on it.

T. SZELE [5] defined $\nu(G)$, the nilstufe of G, to be the positive integer n such that there is an associative multiplication on G having a non-zero product of n group elements but there being no associative multiplication on G allowing a non-zero product of more than n group elements. If no such n exists then $\nu(G)=\infty$. Following FEIGELSTOCK [2] we define the strong nilstufe of G, $N(G)$, similarly but also considering non-associative multiplications on G. It will be seen later that the two invariants $\nu(G)$ and $N(G)$ are not necessarily equal.

The case where the rank of G, $r(G)$, is one is trivial, for all torsion-free rank one groups can be considered as subgroups of the rational numbers Q. As such they are either associative subrings of the rationals or do not admit non-trivial multiplication. Hence if $r(G)=1$ then $\nu(G)=N(G)=1$ or ∞. Other results concerning rank one groups are given in [2]. In the remainder of this paper we obtain useful bounds for $\nu(G)$ and $N(G)$ using well-known results on algebras of finite dimension.

Theorem. *If G is a torsion-free abelian group of finite rank $r(G)$, then*

$$\text{(a) } \nu(G) \leqq r(G) \quad \textit{or} \quad \nu(G)=\infty, \qquad \text{(b) } N(G) \leqq 2^{r(G)-1} \quad \textit{or} \quad N(G)=\infty.$$

To prove this result we require two lemmas concerning finite dimensional algeb-

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ras and a transition from torsion-free groups of finite rank to algebras of finite dimension. The following is a standard result and so no proof is given here.

Lemma 1. *If A is an associative algebra of finite dimension d over some field K such that, for some positive integer n, $A^n \neq 0$ and $A^{n+1}=0$ then n is at most d.*

The next lemma concerns non-associative algebras and we may no longer use A^n without ambiguity. Thus we define $A^{(n)}$ to be the subalgebra of A generated by all products of n elements of A. Clearly if A is associative then $A^{(n)}=A^n$.

For any algebra we define the associative subalgebra $E(A)$ of the endomorphism algebra of the K-module A^+ as being generated by all endomorphisms L_a, R_a over all a in A, where

$$L_a(x) = ax, \quad R_a(x) = xa \quad \text{for all} \quad x \text{ in } A.$$

Then we have the following sequence of submodules;

$$A \supseteqq AE(A) \supseteqq AE(A)^2 \supseteqq \cdots \supseteqq AE(A)^r \supseteqq \cdots.$$

But if we know that all products of $n+1$ elements of A are zero then $E(A)^n=0$ and the sequence above becomes;

$$\text{(I)} \qquad A \supseteqq AE(A) \supseteqq AE(A)^2 \supseteqq \cdots \supseteqq AE(A)^{n-1} \supset 0$$

if we suppose that $E(A)^{n-1} \neq 0$. If further we suppose that for some integer k, $1 \leqq k \leqq n-1$, we have $AE(A)^k = AE(A)^{k+1}$ then;

$$AE(A)^{k+1} = AE(A)^k E(A) = AE(A)^{k+1} E(A) = AE(A)^{k+2}.$$

Thus ultimately we get that $AE(A)^k = AE(A)^n = 0$, which is a contradiction since $k<n$. So the sequence (I) strictly decreases to zero. By considering the dimension of A we obtain that the length of the sequence, n, is at most the dimension of A, d.

Lemma 2. *For any finite dimensional algebra A over the field K we have $A^{(k)} \subseteqq AE(A)^n$ for all integers $k > 2^{n-1}$.*

Proof. If $n=1$ then $2^{n-1}=1$ and trivially $A^{(k)} \subseteqq AE(A)$ for all $k>1$. Let $n>1$ and proceed by induction on n. Take x in A to be a product of $k>2^{n-1}$ elements of A. Then $x=u \cdot v$ where at least one of u or v is a product of at least $2^{n-2}+1$ elements, u say. Then by hypothesis u is in $AE(A)^{n-1}$ and $u \cdot v$ is in $AE(A)^n$, proving the lemma.

Corollary. *If $E(A)^n=0$ for some integer n, then $A^{(k)}=0$ for all $k>2^{n-1}$.*

Recall that we saw above that if an integer n exists such that $E(A)^n=0$ then n is at most the dimension of A.

We now perform the promised transition from groups to algebras. This is done by noting that any (associative) multiplication on the group G induces an (associative) algebra structure on $A=Q^+\otimes G$ over Q. It is easy to verify that

(1) $A^{(n)}=0$ if and only if $G^{(n)}=0$.

(2) The dimension of A over Q is equal to the rank of G.

Proof of Theorem.

(a) We are dealing only with associative multiplications on G hence $A=Q\otimes G$ is an associative algebra of dimension $r(G)$ over Q and so if $\nu(G)=n$ is finite, Lemma 1 applies to give that $n\leqq r(G)$.

(b) We now admit non-associative multiplications and if $N(G)$ is finite then $E(A)^n=0$. We conclude firstly that $n\leqq r(G)$ and secondly that, applying the Corollary to Lemma 2, $A^{(k)}=0$ for all integers k such that $k>2^{n-1}$ which combined with (1) above gives $N(G)\leqq 2^{r(G)-1}$.

Thus the proof of the theorem is complete. It should be noted that the special case for G of rank two was obtained by FEIGELSTOCK [3] who seems to have overlooked that Lemma 1 of [3] drawn from BEAUMONT and WISNER [1] requires the ring to be associative, which in Theorem 1 of [3] it need not be.

Finally an example of a group G is given where $\nu(G)$ and $N(G)$ are not equal. Let $R<Q$ have type $(1, 0, 1, 1, 0, 1, \ldots)$ (for the definition of type see [4] from which the notation is borrowed), $S<Q$ have type $(2, 0, 2, 2, 0, 2, \ldots)$ and $T<Q$ have type $(\infty, 1, 4, \infty, 1, 4 \ldots)$. We recall that the type of a product is at least the product of the types. So if $\mathrm{t}(a)=(n_p)$, $\mathrm{t}(b)=(m_p)$ then $\mathrm{t}(a\cdot b)\geqq(n_p+m_p)$. Hence for any multiplication on $G=\mathrm{Ra}\oplus\mathrm{Sb}\oplus\mathrm{Tc}$ where a, b, c are linearly independent the type of each summand demands that;

$$a\cdot x\in Sb\oplus Tc \quad \text{for any} \quad x \text{ in } G, \quad b\cdot b\in Tc, \quad b\cdot c=c\cdot b=c\cdot c=0.$$

Hence both $\nu(G)$ and $N(G)$ are finite. Thus $\nu(G)\leqq 3$, $N(G)\leqq 4$ from the Theorem. The following table defines a non-associative multiplication on G in which the product $(a\cdot a)\cdot(a\cdot a)\neq 0$.

	a	b	c
a	b	c	c
b	0	c	0
c	c	0	0

Thus it can be seen that the bounds given by the Theorem are attained by at least one group.

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Bibliographie

M. Aigner, Kombinatorik, II. Matroide und Transversaltheorie (Hochschultext), XIII+324 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

The first part of this book (reviewed in these *Acta*, Vol. 38, p. 429) is an excellent introduction to modern "enumerative" combinatorics. This second part is an equally excellent treatment of matroid theory.

Matroids are a common abstraction of graphs, projective, affine and hyperbolic geometries, matrices (from the combinatorial point of view) and transversal systems. Accordingly, they can be described in many different but equivalent ways and one of the reasons of the strength of the theory is that each point of view yields a new insight to its problems. The book starts with formulating various systems of axioms for matroids and proving their equivalence, which is quite involved in some cases. This is followed by various examples of matriods and a description of the basic matroid operations (reduction, contraction, sum, extensions, duality). The second chapter deals with coordinatization and invariants like the chromatic and Tutte polynomials. In connection with graphic matroids, a considerably large part of graph theory is developed, among others planarity, flows, and chromatic number. The third chapter discusses transversal theory, including Menger's and Sperner's theorems, transversal matriods and gammoids.

The book is not only a rich, up-to-date account of this fast-growing and important field, but it is also very well-written. Exercises and references at the ends of the chapters help the reader in the further study of matroids. The book is warmly recommended to everyone learning, or doing research in, combinatorics.

L. Lovász (Szeged)

Tom M. Apostol, Introduction to Analytic Number Theory (Undergraduate Texts in Mathematics), xii+338 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1976.

This textbook is a useful introduction to analytic number theory suitable for undergraduates with the knowledge of elementary calculus, but with no previous knowledge of number theory. The last four chapters require some background in complex function theory. The clarity of exposition is due to the fact that its material evolved from a course offered at the California Institute of Technology during the last 25 years. One of the goals of the author is to nurture the intrinsic interest of young mathematics students in number theory and to give some guide for them in the current periodical literature.

Chapters: 1. The Fundamental Theorem of Arithmetics, 2. Arithmetical Functions and Dirichlet Multiplication, 3. Averages of Arithmetical Functions, 4. Some Elementary Theorems on the Distribution of Prime Numbers, containing an elemantary proof sketch of the prime number theorem based on Selberg's asymptotic formula, 5. Congruences, 6. Finite Abelian Groups and Their Characters, 7. Dirichlet's Theorem on Primes in Arithmetic Progressions, 8. Periodic Arithmetical Functions and Gauss Sums, finishing with Pólya's inequality for the partial sums of primitive characters, 9. Quadratic Residues and the Quadratic Reciprocity Law, 10. Primitive Roots, 11. Dirichlet Series and Euler Products, 12. The Functions $\zeta(s)$ and $L(s, \chi)$ with a unified treatment of both functions by the Hurwitz zeta function, 13. Analytic Proof of the Prime Number Theorem, with applications to the divisor function, Euler's totient etc., 14. Partitions, as an introduction to additive number theory.

There are exercises at the end of each chapter.

A second volume is scheduled to appear in the Springer-Verlag Graduate Texts in Mathematics series under the title "Modular Functions and Dirichlet Series in Number Theory".

F. Móricz (Szeged)

Jon Barwise, Admissible Sets and Structures, An Approach to Definability Theory (Perspectives in Mathematical Logic series), XIV+394 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1975.

Admissible set theory has been developing since the early sixties. It is a basic tool for studying definability theory over arbitrary structeres and it is also a basic source of interaction between model theory, recursion theory and set theory, theories all dealing in part with problems of definability and set existence. The book under review is the first monograph on the subject. It is written for graduate students, who are interested in mathematical logic, but because of its rich material, it can be considered as a handbook for specialists of admissible set theory, and on the other hand, because of its extremely clear, elegant, informal style, it is understandable and interesting even for all those mathematicians, who do not deal with the subject, but want to become acquainted with modern parts of mathematical logic.

In order to give an image of admissible sets to the reader of this review we quote from the introduction: "[In 1964] Kripke introduced admissible ordinals by means of an equation calculus. [In 1965] Platek gave an independent equivalent definition... by means of machines as follows: Let α be an ordinal. Imagine an idealized computer capable of performing computations involving less than α steps. A function F computed by such a machine is called α-recursive. The ordinal α is said to be admissible if, for every α-recursive function F, whenever $\beta<\alpha$ and $F(\beta)$ is defined, then $F(\beta)<\alpha$, that is, the initial segment determined by α is closed under F. The first admissible ordinal is ω. An ordinal like $\omega+\omega$ can not be admissible... The second admissible ordinal is, in fact, ω_1^c [the least non-recursive ordinal, i.e., the recursive analogue of ω_1]... Takeuti's work [in 1960—61] had shown that... the Kripke-Platek theory on an admissible ordinal α has a definability version on $L(\alpha)$, the sets constructible [in Gödel's sense] before the stage α... It leads us to consider *admissible sets*, sets A which, like $L(\beta)$ for α admissible, satisfy *closure conditions* which insure a reasonable definability theory on A."

The principles are formalized in a first order set theory KP. In order to study general definability over structures this theory is further generalized to a new theory KPU ("*K*ripke-*P*latek theory with *U*relements").

The book is supplemented by a list of references (consisting of about 150 items), a subject index, and a notation index.

A. P. Huhn (Szeged)

Christian Berg—Gunnar Forst, Potential theory on locally compact Abelian groups (Ergebnisse der Mathematik und ihrer Grenzgebiete 87), VII+197 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1975.

There are only few mathematical disciplines with so deep and vigorous connection to physics as potential theory has. It plays a central role not only in thermodynamics, electrostatics and gravitation, but also in mathematical analysis itself. Perhaps even today would physicists and mathematical analysts dispute over the proper place of potential theory, if, following a brakethrough in the mid-fifties, the whole theory had not been invaded by a new "enemy", probability theory. The probabilistic approach has not only led to new, very visual interpretations of the fundamental notions, but made many, formerly misteriously seeming relations transparent, and allowed to prove several new theorems.

The typical way of thinking in probabilistic potential theory is as follows. To any semigroup of contractive operators there belongs a potential theory, while any semigroup can be regarded as arising from some Markov process (the classical potential theory is associated with the Brownian semigroup, generated by the Laplace operator). The challenging fact for the analyst is that, though potential theory is linked with the transition semigroup only, the proofs generally use the whole Markov process, which is a much more complicated object. In order to recapture potential theory for analyis the proofs should be cleaned from arguments using sample paths of processes.

The present book is devoted to a partial solution of this problem, and presents a purely analytical treatment of the important class of transient convolution semigroups. The role of probability theory is degraded to support the reader by concrete examples only. Fourier-transform methods are systematically used as basic tools. Choosing locally compact Abelian groups for the fundamental space the authors seem to yield to the temptation of highest generality allowed by Fourier techniques. Debatable whether the gain on generality could compensate the loss of simplicity offered by the Euclidean space if the interesting noncommutative case cannot be included anyway.

The first two chapters making out about half of the book present the necessarry technical basis (I. Harmonic Analysis, II. Negative definite functions and semigroups), while the main topic is elaborated in the last chapter (III. Potential theory of transient convolution semigroups). On the reader's part a basic knowledge in functional analysis, Fourier-transform and group theory is assumed.

Besides its pioneering feature the book is distinguished by precise formulations, clear language and honest references. It will meet the interest of both analysts and probabilists.

D. Vermes (Szeged)

L. D. Berkovitz, Optimal Control Theory (Applied Mathematical Sciences, Vol. 12), IX+304 pages, Springer-Verlag, New York— Heidelberg—Berlin, 1974.

Nowadays mechanization and automation of processes of production yield so complicated systems, the control of which needs methods scientifically well-founded. Control theory, one of the most successful and interesting branches of mathematics in the last twenty years, deals with such methods for systems having a mathematical model. This book is an introduction to the mathematical theory of optimal control of processes governed by ordinary differential equations.

In the first chapter there are presented some examples of control problems drawn from different areas of application: problems of production planning, chemical and electrical engineering, flight mechanics and the classical brachistochrone problem. In the second chapter the precise and quite general formulation of the mathematical problem of optimal control is given. The treatment of the relationship between problems in the calculus of variations and control problems concludes the chapter. Then basic existence theorems follow for problems in which a certain convexity condition is present. The key theorem of the development is an improvement of Cesari's theorem invented by the author. The fourth chapter contains existence theorems without convexity assumptions. Such a result is proved for the minimization problem for inertial controllers. This is a mathematical idealization of systems in which the controls are assumed to possess inertia. We learn the method of replacing the original problem by a "relaxed problem" in which the convexity assumption is satisfied. At the end of the chapter problems linear in the state variable are studied, in which the constraint set is independent of the state variable. In such systems the so-called "bang-bang principle" is valid. The fifth chapter is devoted to the maximum principle and some of its applications. The author shows how to obtain some necessary conditions of the classical calculus of variation from the maximum principle, he takes up particularly linear time optimal problem. The sixth chapter is the proof of the maximum principle.

The treatment of the subject has the proper mathematical exactness and abstraction. The material is arranged so that the readers primarily interested in applications can omit the more advanced mathematical sections without loss of continuity. The book can be read by anyone familiar with the elements of Lebesgue integration and functional analysis.

Although it is impossible for such a book to be completely up to date as new developments are so rapid in this theory, we are sure that this book will enable its readers, students or professionals in mathematics and in areas of applications, to navigate the turbulent waters of control theory.

L. Pintér—L. Hatvani (Szeged)

T. S. Blyth—M. F. Janowitz, Residuation Theory (International Series of Monographs in Pure and Applied Mathematics, Vol. 102), IX+382 pages, Oxford—New York—Toronto—Sydney—Braunschweig, Pergamon Press, 1972.

The aim of this book is to contribute to the textbook literature in the field of ordered algebraic structures. From the Preface: "The fundamental notion which permeates the entire work is that of a residuated mapping and is indeed the first unified account of this topic".

An isotone mapping f between the ordered sets A and B is *residuated* if there exists an isotone mapping $h\colon B\to A$ such that $h\circ f\geqq \mathrm{id}_A$ and $f\circ h\leqq \mathrm{id}_B$. It can be proved that if such an h exists it is unique and called the *residual* of f. The residuated mappings on a bounded ordered set E form a semigroup with a zero and identity, and many important properties of E can be naturally characterized in this semigroup. Examples:

(i) There is a bijection between the binary relations on a set E and the residuated mappings on the power set of E.

(ii) Every bounded linear operator f on a Hilbert space H induces a residuated mapping on the lattice of closed subspaces of H, namely the mapping $M\mapsto\{f(m)|m\in M\}^{\perp\perp}$.

(iii) If A is a commutative ring with an identity element then, in the ordered semigroup of ideals of A, multiplication by a fixed ideal is a residuated mapping.

The text of this book is divided into three chapters. Chapter 1 is an introduction to residuated mappings and lattice theory. This chapter contains all the elemantary material which is required later. The lattice theoretic fundamentals are treated with the help of residuated mappings. Chapter 2 deals with the concept of the Baer semigroup and uses residuated mappings to show how these semigroups may be used to study lattices. It contains some of the important works of D. J. Foulis and

S. S. Holland Jr. on orthomodular lattices. In Chapter 3 the notion of residuated mappings is used for a discussion of residuated semigroups (an ordered semigroup is called *residuated* if each translation on it is a residuated mapping.)

This well-organized book "is designed to satisfy a variety of courses.... For example, Chapter 1 may be used as an advanced undergraduate course on ordered sets and lattice theory; Chapters 1 and 2 as a one-semester postgraduate course on lattice theory; and the whole text as an M. Sc. course on lattices and residuated semigroups."

The book only assumes that the reader is familiar with the elements of abstract algebra. There are a lot of exercises throughout the book (for a few of which some knowledge of general topology is advisable).

The book is well-readable and most of the results contained in it appear for the first time in book form and some of them are only just seeing the light of day. Most of the results have been developed in the last decade. The book certainly will inspire further research.

L. Klukovits (Szeged)

A. A. Borovkov, Stochastic processes in queuing theory (Applications of Mathematics 4), XI+280 pages, Springer-Verlag; New York—Heidelberg—Berlin, 1976.

Queuing theory was originated by the engineer Erlang in the early years of this century, as he first applied probabilistic methods in the design of telephone centers. Later his methods were successively extended to the analysis of more and more general mass service systems with random service times and requests arriving stochastically from the customers.

Till now it has become an independent mathematical discipline, generally regarded as a subfield of applied probability.

But the phrase 'applied mathematical discipline' should be considered somewhat cautiously. If a pure mathematician reads e.g. the present book with the intention of learning what mathematics is good for, he would probably have a similar impression as he had studied lattice theory in fear of housebreakers. But he may not blame the book for his defect. Though queuing theory has its roots in applications, as a result of its development during the last half century it has reached the level of an axiomatic mathematical discipline, not less abstract than any other one. If someone really wants to apply it to a practical problem, the effort, generally necessary to connect theory with praxis, cannot be spared. But once this work has been invested, the present book will prove to be an extraordinarily useful aid.

The author presents queuing theory as a subfield of axiomatic probability theory, and this way he can cover an extremely broad class of problems within a reasonable space, and treat them by uniform methods. The first chapter is an introductory one dealing with the single server queue, but also pointing out all essential features of the theory. The following three chapters deal with functionals of sequences of i. i. d. (independent identically distributed) random variables. The original factorization method of the author allows to compute the distributions of several such functionals, a result with importance reaching far beyond queuing theory. The last three chapters deal with service systems with several and infinitely many servers and with refusals. Four appendices on renewal theory, ring factorizations, asymptotics of coefficients of series and estimates of distributions are included as well as a bibliography of 74 items. As prerequisite a basic knowledge of probability theory and stochastic processes, as well as some preknowledge in the motivation of the topic are assumed on the part of the reader.

Summing up, the book is a concise, uniform presentation of modern queuing theory in an exact form and clear language. It does not suggest problems in applications but helps anybody who is faced by such a problem.

D. Vermes (Szeged)

L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, XI+238 pages (Die Grundlehren der mathematischen Wissenschaften 65), 2nd edition; Springer-Verlag, Berlin—Heidelberg—New York, 1972.

Discrete geometry, the study of arrangements of various figures with certain extremality conditions, is one of the most vivid areas of geometry. This is due to a large extent to the first (1953) edition of this excellent monograph. This can be seen also from the fact that this second edition contains an Appendix of 29 pages, which surveys the most important developments in connection with problems formulated in the original edition. The riches of new results is really spectacular, and this Appendix may be very useful even for those who have read the first edition.

As pointed out in the preface, the author concentrates on arrangements in the best-known and most graphic spaces, the euclidean 2- and 3-spaces and the sphere. The first two chapters collect those results in elementary geometry and in the theory of convex bodies which play role in the sequel. It is, however, quite an informative reading even in itself, containing many interesting and not commonly known results. Chapters III and IV discuss optimal packings and coverings of planar regions by discs and other figures. Chapter V describes extremality properties of regular polyhedra. Certain quantities for polyhedra with a given number of vertices, edges and/or faces can be estimated so that equality stands for regular polyhedra only. For the case when no regular polyhedron with the given parameters exists, it is much more difficult to find the extrema. Problems of this type are discussed in Chapter VI. Chapter VII considers optimum packings and coverings in the space.

Let us finally remark that no knowledge of higher mathematics is required to read this book, and thus we may recommend it to everyone interested in geometry.

L. Lovász (Szeged)

P. J. Higgins, Introduction to Topological Groups (London Mathematical Society Lecture Note Series 15), V+106 pages, Cambridge, 1974.

Although important applications of topological groups require only a restricted part of it, textbooks generally cover the entire theory. This Introduction is designed to meet the needs of those who for the time being want to study topological groups for the sake of those applications only which utilize but a restricted part of the theory. Actually the author has given repeatedly introductory courses for first-year postgraduate students in algebra or number theory at the University of London and this Introduction is an amplified version of his lecture notes.

Chapter I contains such preliminaries as the fundamental concepts concerning groups and topological spaces. Topological groups are introduced in Chapter II where some basic facts concerning subgroups, quotient groups, connected groups and compact groups are presented. Chapter III is a concise account of integration on locally compact groups. Some examples and applications of the Haar integral are given in Chapter IV.

The presentation of this material is done with a perfection due both to a personal skill and to a deep familiarity with the literature.

J. Szenthe (Budapest)

John L. Kelley, General topology (Graduate Texts in Mathematics), XIV+298 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1976.

This book is a reprint of the famous work of the author published by Van Nostrand in 1955. The text contains a systematic exposition of the most important topics of general topology. It is intended to provide the background material for modern analysis. It begins with a preliminary chapter (Chapter 0) which covers topics requisite to the main body of the work. The more serious results of this chapter are theorems from set theory. Chapter 1 introduces the concept of topological spaces, defines basic notions of topology and proves some simple theorems. Chapter 2 studies Moore—Smith convergence and characterizes those notions of convergence which can be described as convergence relative to some topology. The purpose of Chapter 3 is to investigate two methods of constructing new topological spaces from old ones. One of these is the standard method of topologizing the Cartesian product of spaces. The second method is based on the topological identification of the points of certain subsets of the spaces. This new topology is called the quotient topology. Both of these methods are defined by making certain functions continuous. Chapter 4 contains a systematic discussion of embedding and metrization theorems. In Chapter 5 the notions of compact and locally compact spaces are introduced. This chapter contains the most important theorems for compact spaces, and two methods of compactification of spaces: Alexandroff one point and Stone-Čech compactifications. Chapter 6 defines and discusses uniform spaces. In such spaces uniform continuity of functions and Cauchy nets can be defined. Conditions for the metrizability of uniform spaces are given and a proof can be found of the fact that any uniform space can be embedded in a complete uniform space, that is in a uniform space with the property that any Cauchy net has a limit point. This chapter ends with the Baire category theorem for metric spaces. Chapter 7 is devoted to the study of function spaces. The elements of these spaces are functions on a fixed set X to a fixed topological space Y. The various topologies of function spaces are discussed. Each chapter contains a rich collection of problems.

An Appendix deals with an axiomatic study of set theory.

L. Gehér (Szeged)

John L. Kelley—Isaac Namioka, Linear topological spaces (Graduate Tests in Matematics), XV+256 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1976 (Second corrected printing).

The main purpose of this book is to give a detailed discussion of the theory of linear topological spaces, i.e. linear spaces with a topology such that scalar multiplication and addition are continuous. The text begins with an investigation of linear spaces (Chapter 1). The geometry of convex sets is the first topic which is peculiar to the theory of linear topological spaces. One section deals with the relation between orderings and convex cones. Both the algebraic and geometric forms of the Hahn—Banach theorem are proved. In Chapter 2, after establishing the geometric theorems on convexity, the elementary theory of linear topological spaces is developed. Most of the theorems of this chapter are specializations of basic theorems on topological groups or on uniform spaces, i.e. little use is made of scalar multiplication. The most serious results based on the full linear topological structure concern the criterion on normality. Chapter 3 is devoted to give a short glimpse into the fundamental category theorems. Chapter 4 deals with convex subsets of linear topological spaces and the closely related question of the existence of continuous linear functionals. The most powerful result of this chapter is the Krein—Milman theorem on the existence of extreme points of a compact convex set. Chapter 5 studies duality which is the central part of the theory of linear topological spaces. Duality can be defined only if the class of continuous linear functionals is large enough. This fact illuminates the role played by local convexity. Various topologies for a locally convex space and for its dual space are studied. The chapter concludes with a discussion of metrizable locally convex spaces. The text ends with an Appendix which deals with partially ordered locally convex spaces. The main result of the Appendix is the Kakutani characterization of Banach lattices which are of functional type or of L^1 type.

Familiarity of the reader with general topology is required.

L. Gehér (Szeged)

Peter Lax—Samuel Burstein—Anneli Lax, Calculus with Applications and Computing. Volume I (Undergraduate Texts in Mathematics), XI+513 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

"The traditional course (of calculus) too often resembles the inventory of a workshop, here we have hammers of different sizes, there saws, yonder planes; the student is instructed in the use of each instrument, but seldom are they all put together in the building of a truly worthwhile object" — say the authors in the preface. Their purpose is to emphasize the relation of calculus to science by devoting whole chapters to single, or several related, scientific topics. They intend to help the reader learn "how the notions of calculus are used to formulate the basic laws of science and how the methods of calculus are used to deduce consequences of those basic laws". Numerical methods are presented as organic parts of calculus. The treatment is intended to be rigorous without being pedantic.

Real numbers are thought of as (i) entities that can be added, multiplied, etc; (ii) points of the real line; (iii) infinite decimals. The derivative is defined as the uniform limit of difference quotients. "This makes it evident that a function whose derivative is positive on an interval is an increasing function". After the mean value theorem, Taylor's theorem, and the characterization of maxima and minima a section is devoted to one-dimensional mechanics. Integral is introduced as an additive function of interval that has the lower-upper bound property. The exponential function is defined as modeling growth. There is an introduction to both discrete and continuous probability theory. Gauss' law of error is proved and applied to the diffusion process. Sine and cosine are treated through complex numbers. A brief discussion of two-dimensional mechanics is offered in terms of complex numbers. There is a whole chapter on vibrations and another one on populations dynamics. FORTRAN programs, instructions for their use, as well as an Index are appended.

József Szűcs (Szeged)

G. I. Marchuk, Methods of Numerical Mathematics (Applications of Mathematics, Vol. 2), xii+316 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1975.

This English translation of the Russian original is an adaptation of a series of lectures on numerical mathematics given by the author at the Novosibirsk State University. An attempt has been made to focus attention on those complicated problems of mathematical physics which can be reduced to simpler and theoretically better-developed problems allowing effective computer realization. Besides, the needs of scientists and engineers are also taken into account.

Chapter 1 is a brief survey of the fundamentals of the theory of difference schemes, used extensively in the following chapters. For differential equations with sufficiently smooth coefficients it is possible to obtain high-accuracy approximate schemes, providing approximate solutions with a given accuracy, at the expense of a formal increase in the dimensionality of the subspaces involved (for instance, by decreasing the mesh size). Since the class of problems which possess fairly smooth coefficients is somewhat small, the author pursues the idea of building a general framework for constructing the difference analogues of the equations which do not possess high smoothness properties. Even problems with discontinuous coefficients come up, e. g., when studying diffusion, heat conduction, and hydrodynamics. Ch. 2 begins with a detailed exposition of boundary problems of ordinary differential equations, and then turns to more or less general approaches to solving two- and multi-dimensional problems.

Ch. 3 treats methods for solving stationary problems given in the form $A\varphi=f$ where the operator A coincides with a matrix, φ and f are vectors. Among others, over-relaxation methods, gradient iterative methods, splitting-up methods are discussed. The main object is to present the methods for solving nonstationary problems $\partial A/\partial t+\varphi A=f$, including stabilization methods, predictor-corrector methods, component-by-component methods, etc. As an application, effective algorithms are given for equations of hyperbolic type.

Ch. 5 is devoted to numerical methods for two types of inverse problems. The first type involves determining past states of a process. In the second type of problems, one has to identify the coefficients of an operator with a known structure in terms of information provided by some functionals of the solution. Inverse problems of mathematical physics are often ill-posed in the sense that small perturbation in the observed functionals may imply large changes in the correspoding solutions. For a long time ill-posed problems had been considered uninteresting, however, a need to interpret geophysical data triggered intensive research into these problems. Broad classes of ill-posed problems, the so-called conditionally well-posed problems, are studied here.

As an illustration of the fundamental methods of numerical mathematics, the author gives in Ch. 6 an elegant summary about the simplest problems of mathematical physics, i.e., the Poisson equation, the heat equation, the wave equation, and the equations of "motion".

Ch. 7 deals with the application of the splitting-up method to one of the modern branches of mathematical physics, namely to the theory of radiative transfer, of great significance in reactor and nuclear physics.

Ch. 8 is an expanded version of the lecture held by the author at the International Congress of Mathematicians in Nice (1970). This chapter briefly reviews the fundamental directions in numerical mathematics.

The presentation is concise, but always clear and well-readable. At the end of the book there is a vast and almost complete bibliography of each chapter separately.

The book is primarily intended to benefit practicing scientists encountering truly complicated problems of mathematical physics and seeking help regarding rational approaches to their solution. It may not be an exaggeration to assert that this book is of basic importance for everybody who deals with problems of applied and numerical mathematics.

F. Móricz (Szeged)

A. R. Pears, Dimension Theory of General Spaces, XII+428 pages, Cambridge University Press, Cambridge—London—New York—Melbourne, 1975.

The book is intended to serve as a reference work for mathematicians interested in general topology. The text starts (Chapters 1 and 2) with a summary of the most important notions and results of modern general topology which are indispensable for the main body of the book.

In Chapter 3 the author defines the principal concept of dimension, called covering dimension, as the least integer n such that every finite open covering has an open refinement of order not exceeding n, if such an integer exist; in the contrary case the space is said to have dimension ∞. Though this definition concerns general topological spaces, in most results the spaces are supposed to be normal. For normal spaces, covering dimension can be defined in terms of the order of finite closed refinements of finite open coverings. For Euclidean spaces the covering dimension turns out to coincide with the usual one. Two more characterizations of covering dimension for normal spaces can be obtained in terms of mappings from the space into Euclidean spheres. The concept of dimension would be different if based on arbitrary locally finite coverings instead of finite coverings. There are normal spaces of dimenson 0 which would have infinite dimension if locally finite coverings were permitted. Sum and monotonicity theorems for covering dimension are proved. Chapter 4 introduces the concepts of the small and large inductive dimensions. The main idea of definition is based on reducing the dimension of a space to the dimensions of the boundaries of open sets. The large inductive dimension satisfies sum and subset theorems for totally normal spaces. The small inductive dimension (called Menger dimension) has the greatest intuitive appeal and satisfies the subset theorem for arbitrary spaces. For separable metric spaces the three concepts of dimensions mentioned above coincide. In Chapter 5 the concept of local dimension is defined and theorems analogous to those in Chapters 3 and 4 are proved. Chapter 6 is devoted to the study of images of zero-dimensional spaces. In this chapter two further notions of dimension are introduced. Both of these definitions are in terms of families of locally finite closed coverings of a special type. Chapter 7 shows that a very satisfactory theory of dimension can be constructed for metrizable spaces, though there exists a metrizable space the small inductive dimension of which differs from its large inductive and covering dimensions (P. Roy's example). Chapter 8 mostly deals with the pathological dimension theory of compact Hausdorff spaces. An example (due to V. V. Filippov) shows that the small and large dimensions of such spaces need not coincide. Chapter 9 is devoted to the study of various connections between dimension and mappings in spheres, and relations between the dimension of the domain and range of a continuous surjection. The product theorems for covering and large inductive dimensions are proved. In Chapter 10 the concept of covering dimension is modified for non-normal spaces. Dimension-theoretical applications of the algebra of bounded continuous real functions on a topological space are given. For the dimension of a Tihonov space an algebraic characterization can be found.

There are notes at the end of every chapter (except Chapter 1), which contain references to the original sources. The notes also survey some recent developments which are not included in the book.

The book is highly recommended to anyone interested in general topology.

L. Gehér (Szeged)

R. von Randow, Introduction to the Theory of Matroids (Lecture Notes in Economics and Mathematical Systems), IX+102 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

Matroid theory, originated by the pioneering paper of H. Whitney in 1935, has strongly developed in the last two decades. It relates basic concepts of various branches of mathematics (like linear algebra, graph theory, finite geometries, integer programming) and has applications, for example, in operations research or in electric network analysis.

However, the different terminology of various papers and the seemingly confusing situation of the existing (at least 7) different systems of axioms might discourage some readers. Even the professionals need sometimes a reference (both for research and teaching) containing the relations between the axioms (how to deduce one system from another).

The present book is therefore very useful for giving a clear introduction to the basic concepts and rigorous formal proofs for the fundamental properties of matroids. Chapter I presents five axiomatic definitions, introduces the concepts of independent set, basis, circuit, and rank. Chapter II treates further properties (span, hyperplane, dual and cocircuit), while Chapter III lists some important examples (collection of vectors, binary matriods, graphic and cographic matroids, transversal matroids and gammoids). In the third chapter most of the results are stated without proofs. The greedy algorithm is briefly presented in Chapter IV, while the last chapter is devoted to the exchange properties of the bases in a matroid.

The greedy algorithm is certainly worth being included in any book about matroids — especially if the book is published in a series entitled "Lecture Notes in Economics and Mathematical Systems". However, in the opinion of the reviewer, the last chapter covers one of the less basic areas of matriod theory. Its theorems (and especially the more sophisticated counterexamples) can be presented in graduate courses very successfully, but are perhaps less essential in an introduction to the theory. The matroid partition and intersection theorems, mentioned "per tangentem" in the proof of Theorem 26, are perhaps more important than the whole fifth chapter.

Anyhow, the book gives a clear, up to date description of the fundamental concepts and results of the theory of matroids; it is recommended to everybody interested in this area of combinatorics.

A. Recski (Budapest)

Robert R. Stoll, Sets, Logic, and Axiomatic Theories, Second Edition, XI+233 pages, W. H. Freeman and Company, San Francisco, 1974.

This is the second edition of the author's highly popular textbook on the foundations of set theory and mathematical logic. The treatment is similar to that followed in the first edition but the material is more extensive. The book is intended to serve as a textbook for undergraduate students of mathematics and computer science. Its primary aim is "to bridge the gap between an undergraduate's initial conception of mathematics as a computational theory and the abstract nature of more advanced and more modern mathematics".

In Chapter I the elements of intuitive set theory are outlined. Specifically, this chapter discusses, within the framework of set theory, the following mathematical concepts: function, equivalence relation, ordering relation and natural number. A supplementary section deals with the axiom of choice. References to original works of Cantor, Frege, Russel and others make the reading of this chapter stimulating.

Chapter II presents the most basic notions and facts concerning the predicate calculus and first order logic.

After surveying the historical evolution of the axiomatic method, Chapter III deals with axiomatic theories. Among others consistency, completeness and independence of axiom system, metamathematics, recursive functions and Church's thesis are briefly discussed.

In Chapter IV the Lindenbaum algebra of a statement calculus is defined, whereafter a study of Boolean algebras completes the book.

A. P. Huhn (Szeged)

G. Szász, Théorie des treillis, IX+227 pages, Akadémiai Kiadó, Budapest, 1971.

After its Hungarian, German and English editions this is the French translation of the author's famous textbook on lattice theory. Since its first edition in 1959 this book has proved to be one of the most successful textbooks on algebra in general, and on lattice theory in praticular.

The book is intended to serve as a textbook for students wishing to study lattices and also for those mathematicians, especially algebraists, whose studies require some knowledge of lattice theory.

The chapter headings are: Partly ordered sets, Lattices in general, Complete lattices, Distributive and modular lattices, Special subclasses of the class of modular lattices, Boolean algebras, Semimodular lattices, Ideals of lattices, Congruence relations, Direct and subdirect decompositions.

A number of well-chosen examples and exercises help the reader understand the material. There is a bibliography consisting of 250 items and there are numerous references to this bibliography in the text bringing the mathematical research closer to the student.

This book can be recommended to anybody who is interested in abstract algebra.

A. P. Huhn (Szeged)

A. H. Stroud, Numerical Quadrature and Solution of Ordinary Differential Equations (A textbook for a beginning course in numerical analysis, Applied Mathematical Sciences, Vol. 10), XI+338 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1974.

This is a textbook for a one-semester course on the topics of numerical analysis mentioned in the title. It only requires from the reader knowledge of calculus; the occurring concepts are carefully defined and the results necessary to understand the subject are fully and exactly cited.

Chapter 1 (Background Information) contains statements of results from other branches of mathematics needed for numerical analysis. In Chapter 2 (Interpolation) the methods of interpolation used for the treatment of quadratures and differential equations are introduced. In Chapter 3 (Quadrature) three types of formulas for approximating definite integrals are discussed; these are the Newton—Cotes formulas, Gauss formulas and Romberg formulas. Chapter 4 (Initial Value Problems for Ordinary Differential Equations) contains classical methods of the numerical solution and some of their improved versions.

The book is well-written and well-organized. The methods are illustrated by interesting examples. For instance, one has the total numerical solution of the earth-moon-spaceship problem. Each of the paragraphs ends with problems. Marked sections serve as guides for further study.

The book contains Fortran-programs of the most important procedures, excellently running on computer in our experience.

Edith Huhn (Szeged)

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ANNOUNCEMENT

The International Congress of Mathematicians will be held in Helsinki, Finland, during August 15—23, 1978. Further details will be issued in the autumn of 1977. Correspondence concerning the Congress should be addressed to:

International Congress of Mathematicians, ICM 78
Department of Mathematics
University of Helsinki
Hallituskatu 15
SF-00100 Helsinki 10
Finland

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