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Produit de convolution des mesures opératorielles

S. K. BERBERIAN

Si, pour $i=1, 2$, A_i est un opérateur normal dans l'espace hilbertien H_i, avec la représentation spectrale $A_i=\int \lambda\, dE_i$, l'opérateur

$$A = A_1 \otimes 1 + 1 \otimes A_2 \tag{1}$$

dans l'espace produit tensoriel hilbertien $H_1 \otimes H_2$ est aussi normal, donc possède une représentation spectrale $A=\int \lambda\, dE$. Dans un article récent [6], D. W. Fox a montré qu'on peut regarder E comme un produit de convolution $E=E_1 * E_2$ dans un sens convenable (Fox ne considère que des opérateurs hermitiens, mais ses raisonnements se généralisent immédiatement). Le but de cet article est de pousser les raisonnements de Fox vers leurs limites naturelles, et indiquer quelques applications aux représentations des groupes abéliens et aux représentations intégrales des contractions.

Par *espace à mesure p. o.* (sur un espace hilbertien H) nous entendons un triple $(X, \mathscr{S}, E)$, où X est un ensemble, $\mathscr{S}$ est un σ-anneau de sous-ensembles de X, et E est une mesure positive opératorielle (c'est-à-dire une PO-mesure au sens de [1, Def. 1]) définie sur $\mathscr{S}$, dont les valeurs sont des opérateurs positifs dans H. Ainsi, pour chaque couple de vecteurs x, y dans H, il est défini sur $\mathscr{S}$ une mesure complexe bornée $\mu^E_{x,y}$ telle que

$$\mu^E_{x,y}(M) = (E(M)x|y) \quad \text{pour tout} \quad M \in \mathscr{S}. \tag{2}$$

Si, de plus, X est un espace topologique et si la mesure E est birégulière [1, p. 88] au sens que, pour chaque $M \in \mathscr{S}$, on a

$$E(M) = \sup \{E(C)\colon C \subset M, \ \ C \text{ compact}, \ \ C \in \mathscr{S}\}, \tag{3a}$$

$$E(M) = \inf \{E(U)\colon U \supset M, \ \ U \text{ ouvert}, \ \ U \in \mathscr{S}\}, \tag{3b}$$

nous appellerons $(X, \mathscr{S}, E)$ un *espace à mesure p. o. birégulière* (sur H).

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Lemme 1. *Si, pour* $i=1, 2$, $(X_i, \mathscr{S}_i, E_i)$ *est un espace à mesure p. o. birégulière sur l'espace hilbertien* H_i, *alors il existe une mesure p. o.* $E_1 \otimes E_2$ *(sur* $H_1 \otimes H_2$*) définie sur* $\mathscr{S}_1 \times \mathscr{S}_2$, *et une seule, telle que*

$$(E_1 \otimes E_2)(M_1 \times M_2) = E_1(M_1) \otimes E_2(M_2) \tag{4}$$

pour chaque rectangle mesurable $M_1 \times M_2$.

Démonstration. Définissons les mesures p. o. $E_1 \otimes 1$ et $1 \otimes E_2$ sur $\mathscr{S}_1$ et $\mathscr{S}_2$, suivant les cas, par les formules

$$(E_1 \otimes 1)(M_1) = E_1(M_1) \otimes 1_{H_2}, \quad (1 \otimes E_2)(M_2) = 1_{H_1} \otimes E_2(M_2).$$

On voit immédiatement que $(X_1, \mathscr{S}_1, E_1 \otimes 1)$ et $(X_2, \mathscr{S}_2, 1 \otimes E_2)$ sont tous les deux des espaces à mesure p. o. birégulière sur $H_1 \otimes H_2$, et que $E_1 \otimes 1 \leftrightarrow 1 \otimes E_2$, c'est-à-dire que chaque valeur de $E_1 \otimes 1$ est permutable avec chaque valeur de $1 \otimes E_2$; l'existence et l'unicité de $E_1 \otimes E_2$ dérivent donc immédiatement de [1, Th. 33]. ∎

Il y a une relation naturelle entre le produit tensoriel des mesures opératorielles et le produit des mesures numériques:

Lemme 2. *Avec les notations du lemme* 1 *on a*

$$\mu^{E_1 \otimes E_2}_{x_1 \otimes x_2, y_1 \otimes y_2} = \mu^{E_1}_{x_1, y_1} \times \mu^{E_2}_{x_2, y_2} \tag{5}$$

pour tous $x_1, y_1 \in H_1$ *et* $x_2, y_2 \in H_2$.

Démonstration. Les deux membres de l'équation sont des mesures bornées sur $\mathscr{S}_1 \otimes \mathscr{S}_2$, et il est immédiat des définitions qu'elles sont égaux pour chaque rectangle mesurable. ∎

Avec les notations du lemme 1, supposons donné une application $\varphi : X_1 \times X_2 \to Y$ (Y un ensemble quelconque). Soit $\mathscr{T} = \{N \subset Y : \varphi^{-1}(N) \in \mathscr{S}_1 \times \mathscr{S}_2\}$; alors $\mathscr{T}$ est un σ-anneau de sous-ensembles de Y, et la correspondance $N \to (E_1 \otimes E_2)(\varphi^{-1}(N))$ définit une mesure p. o. sur $\mathscr{T}$, qu'on appelle l'image de $E_1 \otimes E_2$ par φ et que l'on note $\varphi(E_1 \otimes E_2)$; $\varphi(E_1 \otimes E_2)$ s'appelle aussi le *produit de convolution* de E_1 et E_2 pour φ et se note $E_1 *_\varphi E_2$ [cf. 4, Ch. VIII, § 1, Déf. 1]; lorsque $X_1 = X_2 = Y$ est un groupe abélien et $\varphi(s, t) = s + t$ (la loi du groupe), nous supprimons le φ et écrivons simplement $E_1 * E_2$. La convolution des mesures opératorielles et la convolution des mesures numériques sont liées par une formule naturelle:

Lemme 3. *Si* $F = E_1 *_\varphi E_2$, *avec les notations précédentes, on a*

$$\mu^{F}_{x_1 \otimes x_2, y_1 \otimes y_2} = \mu^{E_1}_{x_1, y_1} *_\varphi \mu^{E_2}_{x_2, y_2} \tag{6}$$

pour $x_1, y_1 \in H_1$ *et* $x_2, y_2 \in H_2$.

Démonstration. Les deux membres de (6) sont des mesures complexes bornées sur $\mathscr{T}$. Écrivons μ pour la mesure à gauche; pour tout $N \in \mathscr{T}$ on a (en citant le lemme 2 et les définitions)

$$\begin{aligned}\mu(N) &= \big(F(N)(x_1 \otimes x_2) | y_1 \otimes y_2\big) = \big((E_1 \otimes E_2)(\varphi^{-1}(N)) x_1 \otimes x_2 | y_1 \otimes y_2\big) = \\ &= \mu^{E_1 \otimes E_2}_{x_1 \otimes x_2, y_1 \otimes y_2}(\varphi^{-1}(N)) = (\mu^{E_1}_{x_1, y_1} \times \mu^{E_2}_{x_2, y_2})(\varphi^{-1}(N)) = \\ &= \varphi(\mu^{E_1}_{x_1, y_1} \times \mu^{E_2}_{x_2, y_2})(N) = (\mu^{E_1}_{x_1, y_1} *_\varphi \mu^{E_2}_{x_2, y_2})(N). \ \blacksquare\end{aligned}$$

Adaptons les résultats précédents au contexte des espaces localements compacts et des mesures (de Radon) au sens de [4]. Soient X un espace localement compact, $\mathscr{B}(X)$ la σ-algèbre engendrée par les ensembles fermés de X, c'est-à-dire la tribu des ensembles boréliens de X (appelés les ensembles « faiblement boréliens » dans [1]). Si E est une mesure p. o. (sur un espace hilbertien H) définie sur $\mathscr{B}(X)$, la formule $T_f = \int f\,dE$ définit une application linéaire $f \to T_f$ de l'espace $\mathscr{K}(X)$ des fonctions complexes continues f sur X à support compact, dans l'espace $\mathscr{L}(H)$ des opérateurs linéaires continus dans H; cette application est *positive* au sens que $f \geqq 0$ entraîne $T_f \geqq 0$, et *bornée* au sens que $\|T_f\| \leqq M \|f\|_\infty$ pour tout $f \in \mathscr{K}(X)$, où $M = \|E(X)\|$. Inversement:

Lemme 4. (Théorème de Riesz) *Si X est un espace localement compact, H un espace hilbertien, et $f \to T_f$ une application linéaire positive bornée de $\mathscr{K}(X)$ dans $\mathscr{L}(H)$, il existe une mesure p. o. birégulière E sur $\mathscr{B}(X)$, et une seule, telle que*

(7) $$T_f = \int f\,dE \quad \textit{pour tout} \quad f \in \mathscr{K}(X).$$

Démonstration. D'abord on définit E sur le σ-anneau engendré par les ensembles G_δ compacts de X [1, Th. 19], puis on l'étend à une mesure opératorielle positive « régulière à l'intérieur » sur $\mathscr{B}(X)$ [1, Ths. 21, 22], puis on observe que E est en fait birégulière (parce qu'une mesure positive, bornée, régulière à l'intérieur sur la tribu $\mathscr{B}(X)$ est automatiquement régulière à l'extérieur [3, Th. 3]). $\blacksquare$

Des lemmes 1 et 4 on déduit aisément (cf. 5, p. 105, Prop. 3.1]:

Lemme 5. *Si, pour $i=1, 2$, X_i est un espace localement compact, H_i un espace hilbertien, et $f \to T^i_f$ une application linéaire positive bornée de $\mathscr{K}(X_i)$ dans $\mathscr{L}(H_i)$, alors il existe une application linéaire positive bornée $f \to T_f$ de $\mathscr{K}(X_1 \times X_2)$ dans $\mathscr{L}(H_1 \otimes H_2)$, et une seule, telle que*

(8) $$T_{f_1 \otimes f_2} = T^1_{f_1} \otimes T^2_{f_2}$$

pour tout $f_1 \in \mathscr{K}(X_1)$, $f_2 \in \mathscr{K}(X_2)$, où $(f_1 \otimes f_2)(s_1, s_2) = f_1(s_1) f_2(s_2)$ pour $s_1 \in X_1$, $s_2 \in X_2$. En effet, si E_i est la mesure p. o. birégulière sur $\mathscr{B}(X_i)$ qui représente $f \to T^i_f$, alors l'application $f \to T_f$ est représentée par la mesure p. o. birégulière unique E sur $\mathscr{B}(X_1 \times X_2)$ qui prolonge la mesure p. o. $E_1 \otimes E_2$ sur $\mathscr{B}(X_1) \times \mathscr{B}(X_2)$.

Avec les notations du lemme 5, écrivons simplement $E = E_1 \otimes E_2$ (abus de notation); ainsi, par $E_1 \otimes E_2$ on entend la mesure p. o. birégulière unique sur $\mathscr{B}(X_1 \times X_2)$ qui associe à chaque rectangle borélien $M_1 \times M_2$ l'opérateur $E_1(M_1) \otimes \otimes E_2(M_2)$. Continuant avec ces notations, considérons un espace localement compact Y et une application $\varphi : X_1 \times X_2 \to Y$. Pour simplicité supposons que φ est *continue* (ce qui suffira pour nos buts), donc borélienne. Si $f: Y \to \mathbf{C}$ est une fonction borélienne bornée, alors $f \circ \varphi$ est une fonction borélienne bornée sur $X_1 \times X_2$, et on peut former l'opérateur $\int (f \circ \varphi) dE$; en particulier, $f \to \int (f \circ \varphi) dE$ définit une application linéaire, positive, bornée de $\mathscr{K}(Y)$ dans $\mathscr{L}(H_1 \otimes H_2)$, donc (lemme 4) une mesure p. o. birégulière $\varphi(E)$ sur $\mathscr{B}(Y)$, telle que

$$\int f d(\varphi(E)) = \int (f \circ \varphi) dE \tag{9}$$

pour tout $f \in \mathscr{K}(Y)$. Conformément aux notations antérieures, la mesure p. o. $\varphi(E)$ aussi s'écrit $E_1 *_\varphi E_2$. On a encore (cf. lemme 2) la formule

$$\mu^{E_1 \otimes E_2}_{x_1 \otimes x_2, y_1 \otimes y_2} = \mu^{E_1}_{x_1, y_1} \otimes \mu^{E_2}_{x_2, y_2} \tag{10}$$

(le produit à droite est pris au sens de [4, Ch. III, § 4, n° 2]) et, en écrivant $F = E_1 *_\varphi E_2$, on a

$$\mu^{F}_{x_1 \otimes x_2, y_1 \otimes y_2} = \mu^{E_1}_{x, y_1} *_\varphi \mu^{E_2}_{x_2, y_2} \tag{11}$$

(cf. lemme 3 et [4, Ch. VIII, § 1, Déf. 1]). La formule (9) devient alors

$$\int f d(E_1 *_\varphi E_2) = \int (f \circ \varphi) d(E_1 \otimes E_2); \tag{12}$$

cette formule reste valable pour toute fonction borélienne bornée f sur Y, et, en l'appliquant à un couple de vecteurs élémentaires $x_1 \otimes x_2$, $y_1 \otimes y_2$, l'intégration numérique indiquée à droite peut être calculée par des intégrations simples successives [4, Ch. V, § 6, Th. 1 et § 8, Th. 1]. Si les supports S_1, S_2 de E_1, E_2 sont compacts, il est évident de (11) que le support de $E_1 *_\varphi E_2$ est contenu dans $\varphi(S_1 \times S_2)$, donc est aussi compact [4, Ch. VIII, § 1, Prop. 5a)]; la formule (12) est alors valable pour toute fonction borélienne f qui est bornée sur le support de $E_1 *_\varphi E_2$, en particulier pour toute fonction continue f. On voit aisément que les constructions précédentes, appliquées aux mesures p. o. normalisées (resp. dont les valeurs sont des projecteurs) produisent des mesures p. o. de la même sorte.

Considérons maintenant quelques applications.

Théorème 1. *Soit $X = \mathbf{R}$ ou $\mathbf{C}$, et soient E_1, E_2 des mesures p. o. (nécessairement birégulières) définies sur $\mathscr{B}(X)$, à support compact. On a alors*

$$\int \lambda d(E_1 * E_2) = \left(\int \lambda dE_1\right) \otimes E_2(X) + E_1(X) \otimes \left(\int \lambda dE_2\right). \tag{13}$$

Démonstration. Ici $E_1 * E_2 = \varphi(E_1 \times E_2)$, où $\varphi(s_1, s_2) = s_1 + s_2$. Si $f: X \to \mathbf{C}$ est l'injection identique, on a $(f \circ \varphi)(s_1, s_2) = s_1 + s_2$. En écrivant $A_1 = \int \lambda dE_1$, $A_2 = \int \lambda dE_2$, $A = \int \lambda d(E_1 * E_2)$, il résulte de (12) et (10) que

$$\begin{aligned}(A(x_1 \otimes x_2) | y_1 \otimes y_2) &= \iint (s_1 + s_2)\, d\mu^{E_1}_{x_1, y_1}(s_1)\, d\mu^{E_2}_{x_2, y_2}(s_2) = \\ &= \int [(A_1 x_1 | y_1) + (E_1(X) x_1 | y_1) s_2]\, d\mu^{E_2}_{x_2, y_2}(s_2) = \\ &= (A_1 x_1 | y_1)(E_1(X) x_2 | y_2) + (E_1(X) x_1 | y_1)(A_2 x_2 | y_2) = \\ &= ([A_1 \otimes E_2(X) + E_1(X) \otimes A_2](x_1 \otimes x_2) | y_1 \otimes y_2),\end{aligned}$$

d'où $A = A_1 \otimes E_2(X) + E_1(X) \otimes A_2$. ∎

Si de plus E_1, E_2 sont des mesures spectrales normalisées sur $\mathscr{B}(\mathbf{R})$ [resp. $\mathscr{B}(\mathbf{C})$], on obtient la formule de Fox pour un couple A_1, A_2 d'opérateurs hermitiens (resp. normaux). Alternativement, si E_1, E_2 sont des mesures spectrales sur $\mathscr{B}(\mathbf{R})$ et si l'on pose $A_1 = \int \lambda\, dE_1$, $A_2 = \int \lambda\, dE_2$, $A = \int \lambda\, d(E_1 * E_2)$, en appliquant (12) à la fonction bornée $f(s) = e^{its}$ (s, t réels, t fixé) on obtient la formule $e^{itA} = e^{itA_1} \otimes e^{itA_2}$; en dérivant cette formule par rapport à t, divisant par i, et substituant $t=0$, on obtient la formule (1). Cette méthode ouvre la porte pour les opérateurs hermitiens non bornés (ce qui étaient en effet considérés par Fox).

Dans le théorème suivant, les lois du groupe sont notées multiplicativement:

Théorème 2. *Soit G un groupe localement compact abélien et, pour $i = 1, 2$, soit $s \to U^i_s$ une représentation unitaire fortement continue de G dans l'espace hilbertien H_i. Soit X le groupe des caractères de G et, selon le théorème de Stone, soit E_i la mesure spectrale normalisée sur $\mathscr{B}(X)$ telle que*

$$U^i_s = \int \hat{s}\, dE_i \quad \textit{pour tout} \quad s \in G, \tag{14}$$

où $\hat{s}(\alpha) = \alpha(s)$ pour $s \in G$ et $\alpha \in X$ [cf. 2, p. 182]. *Alors la mesure spectrale sur $\mathscr{B}(X)$ pour la représentation unitaire fortement continue $s \to U^1_s \otimes U^2_s$ est $E_1 *_\varphi E_2$, où $\varphi: X \times X \to X$ est l'application $\varphi(\alpha_1, \alpha_2) = \alpha_1 \alpha_2$; c'est-à-dire,*

$$U^1_s \otimes U^2_s = \int \hat{s}\, d(E_1 * E_2) \quad \textit{pour tout} \quad s \in G. \tag{15}$$

Démonstration. On a $(\hat{s} \circ \varphi)(\alpha_1, \alpha_2) = \hat{s}(\alpha_1)\hat{s}(\alpha_2)$, et on voit que $\int (\hat{s} \circ \varphi)\, d(E_1 \otimes E_2) = U^1_s \otimes U^1_s$ tout comme dans la démonstration du théorème 1. ∎

Pour $G = \mathbf{Z}$ (le groupe additif des entiers) on a $X = \mathbf{T} = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ (le groupe du cercle), l'élément $\lambda \in \mathbf{T}$ s'identifiant au caractère $n \to \lambda^n$ de $\mathbf{Z}$; pour $n \in \mathbf{Z}$ et $\lambda \in \mathbf{T}$, on a donc $\hat{n}(\lambda) = \lambda^n$. Rappelons qu'une mesure p. o. E sur $\mathscr{B}(\mathbf{T})$ s'appelle une *mesure opératorielle de Sz.-Nagy* [2, p. 181] si $E(\mathbf{T}) = 1$ et si

$$\int \lambda^n\, dE = \left(\int \lambda dE\right)^n \quad \text{pour} \quad n = 2, 3, 4, \ldots. \tag{16}$$

Si T est une contraction dans un espace hilbertien, il existe une mesure opératorielle de Sz.-Nagy E, et une seule, telle que $T = \int \lambda dE$ (théorème de Sz.-Nagy [2, p. 181]); E s'appelle la mesure opératorielle de Sz.-Nagy *associée à T*.

Théorème 3. *Si T_1, T_2 sont des contractions dans les espaces hilbertiens H_1, H_2, et si E_1, E_2 sont les mesures opératorielles de Sz.-Nagy associées, alors la mesure opératorielle de Sz.-Nagy associée à la contraction $T_1 \otimes T_2$ dans $H_1 \otimes H_2$ est $E_1 *_\varphi E_2$, où $\varphi: \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ est l'application $\varphi(\lambda_1, \lambda_2) = \lambda_1 \lambda_2$; donc*

$$T_1 \otimes T_2 = \int \lambda \, d(E_1 * E_2) \tag{17}$$

est la représentation de Sz.-Nagy pour $T_1 \otimes T_2$.

Démonstration. Fixons un entier positif n et posons $f(\lambda) = \lambda^n$ pour $\lambda \in \mathbf{T}$. Alors $(f \circ \varphi)(\lambda_1, \lambda_2) = \lambda_1^n \lambda_2^n$, donc $f \circ \varphi = f \otimes f$, donc

$$\int \lambda^n d(E_1 * E_2) = \iint \lambda_1^n \lambda_2^n \, d(E_1 \otimes E_2) = \left(\int \lambda^n dE_1\right) \otimes \left(\int \lambda^n dE_2\right) =$$

$$= \left(\int \lambda dE_1\right)^n \otimes \left(\int \lambda dE_2\right)^n = \left[\left(\int \lambda dE_1\right) \otimes \left(\int \lambda dE_2\right)\right]^n$$

ce qui montre que $E_1 * E_2$ est une mesure opératorielle de Sz.-Nagy pour la contraction $\int \lambda \, d(E_1 * E_2) = \left(\int \lambda \, dE_1\right) \otimes \left(\int \lambda \, dE_2\right) = T_1 \otimes T_2$. ∎

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On intertwining dilations. V

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Dedicated to the 65th anniversary of Professor Béla Sz.-Nagy

Introduction. The interest of a functional labelling of the intertwining dilations[1]) of a given contraction A intertwining two contractions T' and T (i.e. $T'A=AT$) was stressed in [18], where such a labelling, involving analytic and non-analytic operator-valued functions, was used in the study of some pure operator theory questions. More recently, in [11], a functional labelling, by means of contractive analytic operator-valued functions, was shown to play a central role in an electrical engineering problem, in the case when $T'=T$ are contractions of class $C_0(N)$ (in the sense of [16], Ch. IX, Sec. 3). However, in the cases $T'=S^*$, $T=S$, where S is a Jordan operator (on a finite dimensional space) or a unilateral shift, this kind of labelling was already obtained by SCHUR (implicitly, for the numerical case, in his classical research on extrapolation [14]) and by ADAMJAN—AROV—KREIN (explicitly, for the operatorial case, in their basic research on Hankel operators [1], [2], [3], [4]).

The general case (considered for instance in [17], [16], [10], [8], [5], etc.), namely for arbitrary contractions A, T', T and arbitrary contractive (but not necessarily strictly contractive) intertwining dilations, seems to have not been considered. The first aim of this paper is to fill this gap by showing that in this most general case there exists also a labelling by contractive analytic operator valued functions. This labelling was suggested, by the previous papers [6], [9], [7].

In establishing this labelling (in Sec. 4 below) we shall establish a new one. Namely we shall show that the contractive intertwining dilations can be labelled by sequences $\{\Gamma_n\}_1^\infty$ of contractions such that Γ_1 acts between two suitable spaces while, for $n\geqq 2$, Γ_n acts between the closed ranges of $I-\Gamma_{n-1}^*\Gamma_{n-1}$ and $I-\Gamma_{n-1}\Gamma_{n-1}^*$ (see Sec. 3 below). This labelling was imposed to us by a problem in geophysics

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[1]) For the terminology and partly for notations, which are essentially those of [16], [8], [6], see the next Section 1.

(where the Γ_n's have a concrete physical meaning) and by its numerical treatment. These connections will be discussed elsewhere. However, in Sec. 5 we give an application of our results to the classification of ANDO's isometric dilations of a pair of commuting contractions [5].

Finally, let us remark that at this stage of our research the explicit connection of this paper with [18] is still an open (and seemingly, basic) question.

Also, we take this opportunity to thank our colleague Gr. Arsene for the useful discussions on the subject of this Note.

1. We start by giving the main notations and recalling some basic facts concerning contractive intertwining dilations.

Let $\mathfrak{H}$ and $\mathfrak{H}'$ be some Hilbert spaces[2]) and let $L(\mathfrak{H}, \mathfrak{H}')$ denote the algebra of all operators from $\mathfrak{H}$ to $\mathfrak{H}'$; in case $\mathfrak{H}=\mathfrak{H}'$, $L(\mathfrak{H}, \mathfrak{H})$ will be denoted simply by $L(\mathfrak{H})$. For two contractions, $T\in L(\mathfrak{H})$, $T'\in L(\mathfrak{H}')$ we denote by $\mathscr{I}(T', T)$ the set of the $A\in L(\mathfrak{H}, \mathfrak{H}')$ intertwining T' and T, i.e. such that $T'A=AT$. Let $U\in L(\mathfrak{K})$, $U'\in L(\mathfrak{K}')$ be the minimal isometric dilations of T and T', respectively; for $n\geqq 0$ let P_n, P'_n denote the orthogonal projections of $\mathfrak{K}$ and $\mathfrak{K}'$ onto

$$\mathfrak{H}_n=\begin{cases}\mathfrak{H}\\ \mathfrak{H}+\mathfrak{L}+U\mathfrak{L}+\ldots+U^{n-1}\mathfrak{L}\end{cases} \text{ and } \mathfrak{H}'_n=\begin{cases}\mathfrak{H}' & (n=0)\\ \mathfrak{H}'+\mathfrak{L}'+U'\mathfrak{L}'+\ldots+U'^{n-1}\mathfrak{L}' & (n\geqq 1),\end{cases}$$

respectively, where $\mathfrak{L}=((U-T)\mathfrak{H})^-$, $\mathfrak{L}'=((U'-T'\mathfrak{H}')^-$. We also set $P=P_0$, $P'=P'_n$ and

$$T_n=P_nU|\mathfrak{H}_n,\quad T'_n=P'_nU'|\mathfrak{H}_n\quad (n=0,1,2,\ldots,);$$

obviously $T_0=T$, $T'_0=T'$ and U, U' are also minimal isometric dilations of T_N, T'_N, respectively ($N=0, 1, 2, \ldots,$). In the sequel A will be a contraction $\in\mathscr{I}(T', T)$.

By a *contractive intertwining dilation* (CID), or an N^{th} *partial contractive intertwining dilation* (N-PCID) of A we respectively mean operators $A_\infty\in L(\mathfrak{K}, \mathfrak{K}')$ and $A_N\in L(\mathfrak{H}_N, \mathfrak{H}'_N)$ such that

$$\|A_\infty\|\leqq 1,\quad A_\infty\in\mathscr{I}(U', U),\quad P'A_\infty=AP, \tag{1.1}$$

$$\|A_N\|\leqq 1,\quad A_N\in\mathscr{I}(T'_N, T_N),\quad P'A_N=A(P|\mathfrak{H}_N). \tag{1.1$_N$}$$

Thus, the operator

$$A_n=P'_nA_\nu|\mathfrak{H}_n,\quad \text{where}\quad n=0,1,2,\ldots \text{ if } \nu=\infty, \text{ and } n=0,1,2,\ldots,N \text{ if } \nu=N, \tag{1.2$_n$}$$

[2]) Hilbert spaces will be considered complex and their subspaces, if not specified, will be assumed to be linear and closed. Operators will always be assumed to be linear and bounded; also when confusion might occur the identity operator I and the null operator O on a Hilbert space $\mathfrak{G}$ will be denoted by $I_{\mathfrak{G}}$ and $O_{\mathfrak{G}}$, respectively.

is an n-PCID of A, and

$$P_n' A_{n+1} = A_n(P_n|\mathfrak{H}_{n+1}) \quad \text{for} \quad 0 \leqq n < \nu; \tag{1.3$_n$}$$

moreover, in the first case we have

$$A_\infty = \text{strong}\lim A_n P_n \quad (n \to \infty). \tag{1.4}$$

It is also easy to verify that, conversely, if a sequence of n-PCID's A_n satisfies conditions $(1.3)_n$ $(n=0, 1, 2, \dots,)$, then the strong limit in (1.4) exists and defines a CID of A. Therefore we can make the following

Remark 1.1. There exists a one-to-one correspondence (given by $(1.2)_n$ and (1.4)) between the CID's of A and the sequences $\{A_n\}_0^\infty$ of n-PCID's of A, A_n satisfying $(1.3)_n$ $(n=0, 1, 2, \dots,)$.

In order to facilitate the exposition, we shall give several useful facts, which actually resume the original construction of a CID (see [17], [10], [16], [7]). To this aim, let T, T' and A be as above. We set[3])

$$\mathfrak{F}_A = \{D_A Th + (U-T)h : h \in \mathfrak{H}\}^-, \quad \mathfrak{R}_A = (\mathfrak{D}_A + \mathfrak{L}) \ominus \mathfrak{F}_A \tag{1.5}$$

and

$$\mathfrak{F}_A' = \{D_A h \oplus (U'-T')Ah : h \in \mathfrak{H}\}^-, \quad \mathfrak{R}_A' = (\mathfrak{D}_A \oplus \mathfrak{L}') \ominus \mathfrak{F}_A'. \tag{1.5$'$}$$

Lemma 1.1. *Let T, T' and A be as above. Then*

$$C(D_A Th + (U-T)h) = (U'-T')Ah \quad (h \in \mathfrak{H}) \tag{1.6}$$

defines a contraction $C = C_A \in L(\mathfrak{F}_A, \mathfrak{L}')$. Moreover, the formula

$$W_A D_{C_A^*} l' = R_A'(0_{\mathfrak{D}_A} \oplus l') \quad (l' \in \mathfrak{L}'), \tag{1.7}$$

where R_A' denotes the orthogonal projection of $\mathfrak{D}_A \oplus \mathfrak{L}'$ onto $\mathfrak{R}_A'$, defines a unitary operator from $\mathfrak{D}_{C_A^*}$ onto $\mathfrak{R}_A'$.

Proof. Let $i_{\mathfrak{L}'}$ and ω be the operators defined by

$$i_{\mathfrak{L}'}(l') = 0_{\mathfrak{D}_A} \oplus l' \in \mathfrak{D}_A \oplus \mathfrak{L}' \quad (l' \in \mathfrak{L}'),$$
$$\omega(D_A Th + (U-T)h) = D_A h \oplus (U'-T')Ah \quad (h \in \mathfrak{H}).$$

Obviously $i_{\mathfrak{L}'}$ is unitary from $\mathfrak{L}'$ to $\{0\} \oplus \mathfrak{L}' \subset \mathfrak{D}_A \overline{\oplus} \mathfrak{L}'$; also, ω is unitary from $\mathfrak{F}_A$ to $\mathfrak{F}_A'$ since, by virtue of [16], Sec. II. 1, we have

$$\begin{aligned}
\|D_A Th + (U-T)h\|^2 &= \|D_A Th\|^2 + \|(U-T)h\|^2 = \|Th\|^2 - \|ATh\|^2 + \|D_T h\|^2 = \\
&= \|h\|^2 - \|ATh\|^2 = \|D_A h\|^2 + \|Ah\|^2 - \|T'Ah\|^2 = \|D_A h\|^2 + \|D_{T'} Ah\|^2 = \\
&= \|D_A h\|^2 + \|(U'-T')Ah\|^2 = \|D_A h \oplus (U'-T')Ah\|^2
\end{aligned}$$

[3]) Recall that for any operator C from a Hilbert space $\mathfrak{G}$ to another one $\mathfrak{G}'$, D_C denotes the defect operator $((I-C^*C)^2)^{1/4}$ and $\mathfrak{D}_C = (D_C \mathfrak{G})^-$; if $\|C\| \leqq 1$, then obviously $D_C = (I - C^*C)^{1/2}$.

for all $h \in \mathfrak{H}$. We shall consider $i_{\mathfrak{L}'}$ as operator from $\mathfrak{L}'$ to $\mathfrak{D}_A \oplus \mathfrak{L}'$ and extend ω on the whole of $\mathfrak{D}_A + \mathfrak{L}$ by setting $\omega r = 0_{\mathfrak{D}_A \oplus \mathfrak{L}'}$ for $r \in \mathfrak{R}_A$. Then $C_A = i_{\mathfrak{L}'}^* \omega | \mathfrak{F}_A$; hence C_A is a contraction and

$$C_A^* = \omega^* i_{\mathfrak{L}'}, \quad C_A C_A^* = i_{\mathfrak{L}'}^* \omega\omega^* i_{\mathfrak{L}'}, \quad D_{C_A^*}^2 = i_{\mathfrak{L}'}^* R_A' i_{\mathfrak{L}'}.$$

It follows that

$$\|D_{C_A^*} l'\|^2 = (D_{C_A^*}^2 l', l') = (i_{\mathfrak{L}'}^* R_A' i_{\mathfrak{L}'} l', l') = \|R_A' i_{\mathfrak{L}'} l'\|^2 = \|R_A'(0_{\mathfrak{D}_A} \oplus l')\|^2 \quad (l' \in \mathfrak{L}')$$

and consequently that W_A is an isometric operator from $\mathfrak{D}_{C_A^*}$ to $\mathfrak{R}_A'$. If $d_0 \oplus l_0' \in \mathfrak{R}_A'$ is orthogonal to the range of W_A then

$$(l_0', l') = (d_0 \oplus l_0', 0 \oplus l') = (d_0 \oplus l_0', W_A D_{C_A^*} l') = 0 \quad (l' \in \mathfrak{L}')$$

whence $l_0' = 0$. But, by (1.5)$'$, $(d_0, D_A h) = (d_0 \oplus l_0', D_A h \oplus (U' - T') A h) = 0$ $(h \in \mathfrak{H})$, whence $d_0 = 0$, since $d_0 \in \mathfrak{D}_A$. Thus $d_0 \oplus l_0' = 0$ and consequently we conclude that W_A is unitary.

Lemma 1.2. *Let T, T' and A be as in Lemma 1.1. Then the formula*

$$C(A_1)(D_A P + I - P)|\mathfrak{H}_1 = (I - P')A_1 \tag{1.8}$$

establishes a one-to-one correspondence between the 1-PCID A_1 of A and all contractions

$$C: \mathfrak{D}_A + \mathfrak{L} \to \mathfrak{L}', \quad C|\mathfrak{F}_A = C_A. \tag{1.9}$$

Moreover, the formula

$$X(A_1) D_{C(A_1)}(D_A P + I - P)|\mathfrak{H}_1 = D_{A_1} \tag{1.10}$$

defines a unitary operator from $\mathfrak{D}_{C(A_1)}$ to $\mathfrak{D}_{A_1}$.

Proof. Let A_1 be a 1-PCID of A.

Then, since by $(1.1)_1$,

$$\begin{aligned} \|(I-P')A_1 h_1\|^2 &= \|A_1 h_1\|^2 - \|P' A_1 h_1\|^2 = \\ = \|A_1 h_1\|^2 - \|A P h_1\|^2 &\leq \|h_1\|^2 - \|A P h_1\|^2 = \|h_1\|^2 - \|P h_1\|^2 + \|D_A P h_1\|^2 = \\ &= \|(I-P) h_1\|^2 + \|D_A P h_1\|^2 = \|(D_A P + I - P) h_1\|^2 \quad (h_1 \in \mathfrak{H}_1), \end{aligned} \tag{1.11}$$

we infer that formula (1.8) defines a contraction $C = C(A_1)$ from $\mathfrak{D}_A + \mathfrak{L}$ to $\mathfrak{L}' = (I - P)\mathfrak{H}_1$. Moreover, since $(D_A P + I - P) T_1 h = D_A T h + (U - T) h$ $(h \in \mathfrak{H})$, we also have

$$\begin{aligned} C(D_A T h + (U-T)h) &= (I-P')A_1 T_1 h = (I-P')T_1' A_1 h = \\ &= (I-P')T_1' P' A_1 h = (I-P')T_1' A h = (U'-T')Ah = \\ &= C_A(D_A T h + (U-T)h) \quad (h \in \mathfrak{H}), \end{aligned}$$

i.e. $C|\mathfrak{F}_A=C_A$. Also, from $A_1=P'A_1+(I-P')A_1=AP|\mathfrak{H}_1+(I-P')A_1$ we obtain

$$A_1 = \big(AP+C(D_A P+I-P)\big)|\mathfrak{H}_1. \tag{1.12}$$

This formula shows that A_1 is uniquely determined by $C=C(A_1)$. Let now C be any contraction with the properties (1.9) and let $A_1\in L(\mathfrak{H}_1)$ be defined by (1.12). Then, the relation $P'A_1=AP|\mathfrak{H}_1$ is obvious and therefore

$$\begin{aligned} T_1'A_1h_1 &= T_1'P'A_1h_1 = T_1'APh_1 = T'APh_1+(U'-T')APh_1 = \\ &= T'APh_1+C_A(D_AT+U-T)Ph_1 = ATPh_1+C_A(D_AT+U-T)Ph_1 = \\ &= ATPh_1+C(D_AT+U-T)Ph_1 = APT_1h_1+C(D_APT_1+(I-P)T_1)h_1 = \\ &= \big(AP+C(D_AP+I-P)\big)T_1h_1 = A_1T_1h_1 \quad (h_1\in\mathfrak{H}_1), \end{aligned}$$

i.e. $A_1\in\mathscr{I}(T_1', T_1)$. Moreover, from (1.12) we infer

$$\begin{aligned} \|h_1\|^2-\|A_1h_1\|^2 &= \|h_1\|^2-\|APh_1\|^2-\|C(D_AP+I-P)h_1\|^2 = \\ &= \|D_C(D_AP+I-P)h_1\|^2 \quad (h_1\in\mathfrak{H}_1) \end{aligned} \tag{1.13}$$

This shows that A_1 is a contraction. We have thus verified that A_1 has the properties $(1.1)_1$. The last statement of the lemma follows now readily from (1.13).

Remark 1.2. The basic existence theorem [17], [16] for a CID A_∞ of a contraction $A\in\mathscr{I}(T', T)$, where T', T are as above, follows from the preceding lemmas by the following simple recurrent construction.

Set $A_0=A$ and set $C_1=C_AQ_1$ where Q_1 denotes the orthogonal projection of $\mathfrak{D}_{A_0}+\mathfrak{L}$ onto $\mathfrak{F}_{A_0}$. Define A_1 as the 1-PCID such that $C(A_1)=C_1$. Repeat the same precedure with A_1, $U\mathfrak{L}$ and $U'\mathfrak{L}'$ in the roles of A_0, $\mathfrak{L}$ and $\mathfrak{L}'$ and obtain A_2, and so on. Finally one obtains a sequence $\{A_n\}_{n=0}^\infty$ of n-PCID's A_n of A satisfying the conditions $(1.3)_n$ $(n=0, 1, 2, \ldots)$ and consequently a CID A_∞ of A, by virtue of Remark 1.1.

2. Let T, T' and A, $\|A\|\leq 1$, be as in Sec. 1. Let, moreover, A_N be an N-PCID of A and A_n be the operator defined by $(1.2)_n$ $(n=1, 2, \ldots, N-1)$. From Lemma 1.2 it follows readily that A_N is uniquely determined by, and also uniquely determines, a string $\{C_n\}_{n=1}^N$ of $U'^{n-1}\mathfrak{L}'$-valued contractions C_n $(n=1, 2, \ldots, N-1)$, namely the string $\{C(A_n)\}_1^N$. However, the definition of the string $\{C(A_n)\}_1^N$ explicitly involves, besides the operators U, U' (i.e. the minimal isometric dilations of T, T' respectively) and A, also the operators $A_1, \ldots, A_{N-1}$. In order to get rid of the explicit reference to $A_1, \ldots, A_{N-1}$ in the characterization of A_N by a string of $U'^{n-1}\mathfrak{L}'$-valued contractions C_n $(n=1, 2, \ldots, N)$, we firstly introduce the following:

Definition 2.1. A string or a sequence $\{C_n\}_{1\leqq n<\nu}$ (where $\nu=1, 2, \ldots, \infty$; in the case $\nu=\infty$ we set $\nu-1=\infty$) of operators

$$C_n: \mathfrak{D}_{n-1}+U^{n-1}\mathfrak{L} \to U'^{n-1}\mathfrak{L}' \quad (1 \leqq n < \nu) \tag{2.1}$$

is called an *A-cascade* if each $C_n\cdot (1\leqq n\leqq \nu)$ is a contraction,

$$\mathfrak{D}_0 = \mathfrak{D}_A, \quad \mathfrak{D}_n = \mathfrak{D}_{C_n} \quad (1 \leqq n < \nu-1), \tag{2.2}$$

$$C_1|\mathfrak{F}_A = C_A, \tag{2.3$_1$}$$

$$C_2\big(D_{C_1}(D_A Th+(U-T)h)+Ul_1\big) = U'C_1(D_A h+l_1) \quad (h\in\mathfrak{H},\ l_1\in\mathfrak{L}) \tag{2.3$_2$}$$

(in case $\nu>2$), and

$$\begin{aligned}(2.3)_n\ & C_n\big(D_{C_{n-1}}\big(D_{C_{n-2}}\big(\ldots\big(D_{C_1}(D_A Th+(U-T)h)+Ul_1\big)\ldots\big)+U^{n-2}l_{n-2}\big)+U^{n-1}l_{n-1}\big) = \\ &= U'C_{n-1}\big(D_{C_{n-2}}\big(\ldots\big(D_{C_1}(D_A h+l_1)+Ul_2\big)\ldots\big)+U^{n-2}l_{n-1}\big) \quad (h\in\mathfrak{H},\ l_1, \ldots, l_{n-1}\in\mathfrak{L})\end{aligned}$$

for all $3\leqq n<\nu$ (in case $\nu>3$).

In the next two lemmas, $\{C_n\}_{1\leqq n<\nu}$ will be any fixed A-cascade string or sequence; also the spaces $\mathfrak{D}_n (0\leqq n<\nu)$ will have the same meaning as in the preceding definition.

Lemma 2.1. *There exists a unique string (or sequence, respectively)* $\{Y_n\}_{1\leqq n<\nu}$ *of isometric operators*

$$Y_n: \mathfrak{D}_{n-1} \to \mathfrak{D}_n \quad (1 \leqq n < \nu) \tag{2.4}$$

such that

$$Y_1 D_A h = D_{C_1}\big(D_A Th+(U-T)h\big) \quad (h\in\mathfrak{H}), \tag{2.5$_1$}$$

and (if $\nu>2$ *then) for* $2\leqq n<\nu$:

$$Y_n D_{C_{n-1}}(d+U^{n-2}l) = D_{C_n}(Y_{n-1}d+U^{n-1}l) \quad (d\in\mathfrak{D}_{n-2},\ l\in\mathfrak{L}). \tag{2.5$_n$}$$

Proof. We have, by $(2.3)_1$,

$$\begin{aligned}\|D_A h\|^2 &= \|Th\|^2-\|ATh\|^2+\|D_T h\|^2-\|D_{T'}Ah\|^2 = \\ &= \|D_A Th+(U-T)h\|^2-\|(U'-T')Ah\|^2 = \\ = \|D_A Th+(U-T)h\|^2-&\|C_A(D_A Th+(U-T)h)\|^2 = \|D_{C_1}(D_A Th+(U-T)h)\|^2 \\ &\qquad (h\in\mathfrak{H}),\end{aligned}$$

thus, indeed, $(2.5)_1$ defines an isometric operator from $\mathfrak{D}_0=\mathfrak{D}_A$ to $\mathfrak{D}_1=\mathfrak{D}_{C_1}$. We assume now that $(2.5)_n$ (for $n=m-1$, $2\leqq m<\nu$) defines an isometric operator Y_{m-1}, obviously in a unique manner. Then definition $(2.3)_m$ can be written under the form

$$C_m(Y_{m-1}d+U^{m-1}l) = U'C_{m-1}(d+U^{m-2}l) \quad (d\in\mathfrak{D}_{m-2},\ l\in\mathfrak{L});$$

consequently we have

$$\|D_{C_{m-1}}(d+U^{m-2}l)\|^2 = \|d+U^{m-2}l\|^2-\|C_{m-1}(d+U^{m-2}l\|^2 =$$
$$= \|d\|^2+\|U^{m-2}l\|^2-\|U'C_{m-1}(d+U^{m-2}l)\|^2 =$$
$$= \|Y_{m-1}d\|^2+\|U^{m-1}l\|^2-\|C_m(Y_{m-1}d+U^{m-1}l)\|^2 =$$
$$= \|Y_{m-1}d+U^{m-1}l\|^2-\|C_m(Y_{m-1}d+U^{m-1}l)^2 = \|D_{C_m}(Y_{m-1}d+U^{m-1}l)\|^2$$
$$(d\in\mathfrak{D}_{m-1},\ l\in\mathfrak{L}).$$

These relations show that, indeed, $(2.5)_m$ defines the searched isometric operator Y_m. Thus the lemma is proved by recurrence.

Remark 2.1. By virtue of Lemma 2.1, it is easy to infer that the definitions $(2.3)_n$ (for $2\leqq n<\nu$) can be written under the compact form

$(2.6)_n$ $$C_n(Y_{n-1}d+U^{n-1}l) = U'C_{n-1}(d+U^{n-2}l) \quad (d\in\mathfrak{D}_{n-2},\ l\in\mathfrak{L}).$$

Also let us notice that

$(2.7)_n$ $$\mathfrak{D}_n\subset\mathfrak{D}_{n-1}+U^{n-1}\mathfrak{L},\quad \mathfrak{D}_n\subset\mathfrak{H}_n$$

$(1\leqq n<\nu-1)$ and $\mathfrak{D}_0\subset\mathfrak{H}$.

Now let us consider any isometric operator $X:\mathfrak{D}_1\to\mathfrak{H}_1$. By $(2.7)_n$ $(2\leqq n<\nu)$, we can attach to X the string (or sequence) $\{X_n\}_{1\leqq n<\nu}$ of unitary operators

(2.8) $$X_n:\mathfrak{D}_n+U^n\mathfrak{L}\to\mathfrak{R}_n = \text{Range}\,(X_n)\quad (1\leqq n<\nu)$$

by the following recurrent manner:

(2.9) $$\mathfrak{R}_1 = \text{Range}\,(X)+U\mathfrak{L},$$

$(2.10)_n$ $$X_n|U^n\mathfrak{L} = I_{U^n\mathfrak{L}}\quad (1\leqq n<\nu),$$

$(2.11)_1$ $$X_1|\mathfrak{D}_1 = X\quad (2\leqq n<\nu)$$

and, in case $\nu>2$,

$(2.11)_n$ $$X_n|\mathfrak{D}_n = X_{n-1}|\mathfrak{D}_n$$

This sequence will be improperly called the $\{C_n\}_{1\leqq n<\nu}$-*extension* of X.

Let, moreover, A_1 be the 1-PCID of A such that $C(A_1)=C_1$ (see Lemma 1.2) and let $\{X_n\}_{1\leqq n<\nu}$ be the $\{C_n\}_{1\leqq n<\nu}$-extension of $X(A_1)$; then by virtue of (2.1) and (2.8) $C_n'=C_{n+1}X_n^*\in L(\mathfrak{R}_n,\ U'^n\mathfrak{L}')$ $(1\leqq n<\nu-1)$. In case $\nu>2$, the string (or sequence) $\{C_n'\}_{1\leqq n<\nu-1}$ will be called, for convenience, the *reduced* string (or sequence) of $\{C_n\}_{1\leqq n<\nu}$.

Lemma 2.2. *The reduced string (or sequence) is an A_1-cascade.*

Proof. It is plain that C_n' $(1\leqq n<\nu)$ are contractions from $\mathfrak{D}_{n-1}'+U^{n-1}(U\mathfrak{L})$ to $U'^{n-1}(U'\mathfrak{L}')$, where $\mathfrak{D}_0'=\mathfrak{D}_{A_1}$ and $\mathfrak{D}_{n-1}'=X_n\mathfrak{D}_n$ $(1\leqq n<\nu)$. Moreover, since

$C'_n X_n = C_{n+1}$ $(1 \leqq n < \nu - 1)$, we have $X_n^* C_n'^* C_n' X_n = C_{n+1}^* C_{n+1}$, $C_n'^* C_n' X_n = X_n C_{n+1}^* C_{n+1}$ whence

$$D_{C'_n} X_n = X_n D_{C_{n+1}} \quad (1 \leqq n < \nu - 1). \tag{2.12$_n$}$$

Since, by $(2.11)_2$, (2.1), (2.2), $(2.10)_1$, $(2.11)_1$ and (1.10), we have

$$\mathfrak{D}'_1 = X_2 \mathfrak{D}_2 = X_1 \mathfrak{D}_2 \subset X_1(\mathfrak{D}_1 + U\mathfrak{L}) = X(A_1)\mathfrak{D}_{C_1} + U\mathfrak{L} = \mathfrak{D}_{A_1} + U\mathfrak{L} = \mathfrak{D}' + U\mathfrak{L}$$

and, since (if $\nu > 3$ and $2 \leqq n < \nu - 1$) we have, by $(2.11)_{n+1}$, (2.2), $(2.12)_n$ and (2.8)

$$\mathfrak{D}'_n = X_{n+1}\mathfrak{D}_{n+1} = X_n \mathfrak{D}_{n+1} = X_n \mathfrak{D}_{C_{n+1}} = X_n(D_{C_{n+1}}(\mathfrak{D}_n + U^n \mathfrak{L}))^- =$$
$$= (D_{C'_n}\mathfrak{R}_n)^- = \mathfrak{D}_{C'_n},$$

relations (2.1) and (2.2) are satisfied by $\{C'_n\}_{1 \leqq n < \nu}$ (of course with ν, $\mathfrak{D}_n$, $\mathfrak{L}$ and $\mathfrak{L}'$ replaced by $\nu - 1$, $\mathfrak{D}'_n$, $U\mathfrak{L}$ and $U'\mathfrak{L}'$). Also, by virtue of $(2.10)_1$ and (1.10), we have

$$C'_1(D_{A_1}(h + l_1) + Ul_2) = C_2 X_1^{-1}(D_{A_1}(h + l_1) + Ul_2) = \tag{2.13$_1$}$$
$$= C_2(X_1^{-1} D_{A_1}(h + l_1) + Ul_2) = C_2(D_{C_1}(D_A h + l_1) + Ul_2) \quad (h \in \mathfrak{H},\ l_1,\ l_2 \in \mathfrak{L}),$$

$$C'_n(D_{C'_{n-1}}(\ldots D_{C'_1}(D_{A_1}(h + l_1) + Ul_2) + U^2 l_3)\ldots) + U^n l_{n+1}) = \tag{2.13$_n$}$$
$$= C_{n+1} X_n^{-1}(D_{C'_{n-1}}(\ldots) + U^n l_{n+1}) = C_{n+1}(X_n^{-1} D_{C'_{n-1}}(\ldots) + U^n l_{n+1}) =$$
$$= C_{n+1}(X_n^{-1} X_{n-1} D_{C_n} X_{n-1}^{-1}(\ldots) + U^n l_{n+1}) = C_{n+1}(D_{C_n} X_{n-1}^{-1}(\ldots) + U^n l_{n+1}) = \ldots =$$
$$= C_{n+1}(D_{C_n}(\ldots X_2^{-1}(D_{C'_1}(D_{A_1}(h + l_1) + Ul_2) + U^2 l_3)\ldots) + U^n l_{n+1}) =$$
$$= C_{n+1}(D_{C_n}(\ldots(X_2^{-1} X_1 D_{C_2} X_1^{-1}(D_{A_1}(h + l_1) + Ul_2) + U^2 l_3)\ldots) + U^n l_{n+1}) =$$
$$= C_{n+1}(D_{C_n}(\ldots(D_{C_2} X_1^{-1}(D_{A_1}(h + l_1) + Ul_2) + U^2 l_3)\ldots) + U^n l_{n+1}) =$$
$$= C_{n+1}(D_{C_n}(\ldots(D_{C_2}(X_1^{-1} D_{A_1}(h + l_1) + Ul_2) + U^2 l_3)\ldots) + U^n l_{n+1}) =$$
$$= C_{n+1}(D_{C_n}(\ldots(D_{C_2}(D_{C_1}(D_A h + l_1) + Ul_2) + U^2 l_3)\ldots) + U^n l_{n+1}) \quad (h \in \mathfrak{H},\ l_1, \ldots, l_{n+1} \in \mathfrak{L}),$$

where we used in order the relations $(2.10)_n$, $(2.12)_{n-1}$, $(2.11)_n$, $(2.10)_{n-1}$, $\ldots$, $(2.10)_2$, $(2.12)_1$, $(2.11)_2$, $(2.10)_1$, $(2.11)_1$ and (1.10). Now, from $(2.13)_1$, $(2.3)_2$ and (1.8) we infer

$$C'_1(D_{A_1} T_1(h + l) + (U - T_1)(h + l)) = C'_1(D_{A_1}(Th + (U - T)h) + Ul) =$$
$$= C_2(D_{C_1}(D_A Th + (U - T)h) + Ul) = U' C_1(D_A h + l) = U'(P'_1 - P')A_1(h + l) =$$
$$= (U' - T'_1)A_1(h + l) \quad (h \in \mathfrak{H},\ l \in \mathfrak{L}),$$

thus C'_1 satisfies $(2.3)_1$ (with $\mathfrak{F}_A$ and C_A replaced by $\mathfrak{F}_{A_1}$ and C_{A_1}). Also, (in case $\nu > 3$) from $(2.13)_2$, $(2.3)_3$ and $(2.13)_1$ we infer

$$C'_2(D_{C'_1}(D_{A_1} T_1(h + l) + (U - T_1)(h + l)) + U^2 l_1) =$$
$$= C'_2(D_{C'_1}(D_{A_1}(Th + (U - T)h) + Ul) + U^2 l_1) =$$
$$= C_3(D_{C_2}(D_{C_1}(D_A Th + (U - T)h) + Ul) + U^2 l_1) =$$
$$= U' C_2(D_{C_1}(D_A h + l) + Ul_1) = U' C'_1(D_{A_1}(h + l) + Ul_1) \quad (h \in \mathfrak{H},\ l,\ l_1 \in \mathfrak{L}),$$

thus C_1', C_2' satisfy $(2.3)_2$ (with A and $\mathfrak{L}$, $\mathfrak{L}'$ replaced by A_1 and $U\mathfrak{L}$, $U'\mathfrak{L}'$ respectively). Finally, in a similar way, one verifies that (in case $\nu>4$), by virtue of $(2.13)_n$, $(2.3)_{n+1}$ and $(2.13)_{n-1}$, the string $\{C_n'\}_{1\leqq n<\nu-1}$ satisfies the relations $(2.3)_n$ for all n, $3\leqq n<\nu-1$ (again with $A, \mathfrak{L}, \mathfrak{L}'$ replaced by A_1, $U\mathfrak{L}$, $U'\mathfrak{L}'$). This finishes the proof of the lemma.

Lemma 2.3. *Let A_1 be an 1-PCID of A. Any A_1-cascade string (or sequence) $\{C_n'\}_{1\leqq n<\nu-1}$ is the reduced string of a uniquely determined A-cascade string (or sequence) of contractions $\{C_n\}_{1\leqq n<\nu}$.*

Proof. If the string (or sequence) $\{C_n'\}_{1\leqq n<\nu-1}$ is the reduced string (or sequence) of $\{C_n\}_{1\leqq n<\nu}$, then the last one must be defined in the following manner. Firstly,

$$(2.14)_1 \qquad C_1 = C(A_1), \quad X_1|\mathfrak{D}_{C_1} = X(A_1), \quad X_1|U\mathfrak{L} = I_{U\mathfrak{L}}$$

(thus $\mathfrak{D}_{C_1}\subset\mathfrak{H}_1$ and X_1 is a unitary operator from $\mathfrak{D}_{C_1}+U\mathfrak{L}$ to $\mathfrak{R}_1=\mathfrak{D}_{A_1}+U\mathfrak{L}$); then

$$(2.14)_n \qquad C_n = C_{n-1}'X_{n-1}, \quad X_n(d_n+U^nl) = X_{n-1}d_n+U^nl \quad (d_n\in\mathfrak{D}_{C_n}, l\in\mathfrak{L})$$

$(2\leqq n<\nu)$, where X_n is viewed as an operator from $\mathfrak{D}_{C_n}+U^n\mathfrak{L}$ to $\mathfrak{R}_n=(X_{n-1}\mathfrak{D}_{C_n})^-+ +U^n\mathfrak{L}$. These definitions are consistent if they imply recurrently $X_{n-1}\mathfrak{D}_{C_n}\subset\mathfrak{D}_{C_{n-1}'}$ $(2\leqq n<\nu)$.

However, we shall prove by induction even more, namely that

$$(2.15)_n \qquad X_{n-1}\mathfrak{D}_{C_n} = \mathfrak{D}_{C_{n-1}'},$$

$$(2.16)_n \qquad \mathfrak{D}_{C_n}\subset\mathfrak{H}_n$$

and that X_n is unitary (for $2\leqq n<\nu$) (for $n=2$, the last two statements are, by virtue of $(2.16)_1$, obviously true). We start by noticing that if for some, n, $2\leqq n<\nu$, the first relation $(2.14)_n$ makes sense and if X_{n-1} is unitary then, by the same argument as in the proof of Lemma 2.2, we infer the validity of $(2.12)_{n-1}$, whence that of $(2.15)_n$. Thus, by virtue of $(2.14)_1$, $(2.15)_2$ is also valid, so that we have completed the first induction step. In case $\nu>3$ we can therefore assume that, n being fixed, $2\leqq n<\nu$, the statements are always valid for $n-1$. Then, be virtue of $(2.15)_{n-1}$ and the fact that $\{C_n'\}_{1\leqq n<\nu-1}$ is an A_1-cascade, the first relation $(2.14)_n$ makes sense; thus, by virtue of the above discussion on $(2.15)_n$, we infer that this relation is valid. Therefore, using once again the fact that $\{C_n'\}_{1\leqq n<\nu-1}$ is an A_1-cascade, from the second relation $(2.14)_n$ we obtain that X_n is unitary, while from the second relation $(2.14)_{n-1}$ and $(2.16)_{n-1}$ we obtain $(2.16)_n$. Thus the n^{th} inductive step is completed and consequently the string (or sequence) $\{C_n\}_{1\leqq n<\nu}$ is consistently defined. By this very definition, it is plain that $\{C_n\}_{1\leqq n<\nu}$ satisfies conditions (2.1), (2.2) and $(2.3)_1$. Now we can establish, as in the proof of Lemma 2.2, the relations $(2.13)_n$ $(1\leqq n<\nu-1)$ and subsequently infer the relations $(2.3)_n$ $(2\leqq n<\nu)$ for $\{C_n\}_{1\leqq n<\nu}$ from the fact that $\{C'_n\}_{1\leqq n<\nu}$, being an A_1-cascade, satisfies $(2.3)_n$ $(1\leqq n<\nu-1$; of course with A, $\mathfrak{L}$ and $\mathfrak{L}'$ replaced by A_1, $U'\mathfrak{L}'$, respectively). In this manner we conclude that $\{C_n\}_{1\leqq n<\nu}$ is an A-cascade. Actually, the proof of the lemma is now completed.

We can, and shall, now define a mapping from A-cascade strings to PCID's of A, for any contractions T, T' and $A\in\mathscr{I}(T', T)$. Namely, for given T, T', A, $N=1, 2, \ldots$, and A-cascade string $\{C_n\}_1^N$ (of length N) we shall define an N-PCID $A_N(A; C_1, \ldots, C_N)$ by the following recurrent formula

$$(2.17)_1 \qquad A_1(A; C_1) = A_1$$

where A_1 is the 1-PCID (yielded by Lemma 1.2) such that $C(A_1)=C_1$, and

$$(2.17)_N \qquad A_N(A; C_1, \ldots, C_N) = A_{N-1}(A_1; C'_1, \ldots, C'_{N-1})$$

where $\{C'_n\}_1^{N-1}$ is the reduced string of $\{C_n\}_1^N$ $(N=2, 3, \ldots)$. (Actually, one should write $A_N(A; T', T; U', U; C_1, \ldots, C_N)$ instead of $A_N(A; C_1, \ldots, C_N)$ since this operator depends also on T, T' and the concrete constructions of the isometric dilations U, U' of T, T', respectively; thus $(2.17)_N$ should be written in the form

$$A_N(A; T', T; U', U; C_1, \ldots, C_N) = A_{N-1}(A_1; T'_1, T_1; U', U; C'_1, \ldots, C'_{N-1}).$$

However, when no confusion seems possible, we shall not complicate the notations with this preciseness.)

The consistence of the definitions $(2.17)_N$ $(N=2, 3, \ldots)$ is a direct consequence of Lemma 2.2 and the fact that any $(N-1)$-PCID of a 1-PCID of A is an N-PCID of A. Also by an obvious inductive argument it follows that

$$(2.18)_N \qquad P'_{N-1}A_N(A; C_1, \ldots, C_N) = A_{N-1}(A; C_1, \ldots, C_{N-1})(P_{N-1}|H_N)$$

$(N=2, 3, \ldots)$, i.e. $A_N(A; C_1, \ldots, C_N)$ is a 1-PCID of $A_{N-1}(A; C_1, \ldots, C_{N-1})$.

Proposition 2.1. *For $N=1, 2, \ldots$, and $T, T', A\in\mathscr{I}(T', T)$ fixed, the mapping*

$$(2.19)_N \qquad \{C_n\}_1^N \to A_N(A; C_1, \ldots, C_N)$$

establishes a one-to-one correspondence between the A-cascade strings (of length N) and the N-PCID's of A.

Proof. For $N=1$, the statement in the proposition reduces to the first statement in Lemma 1.2. Therefore we assume that the statement is also true for $N=m-1\geqq 1$. Let moreover A_m be an m-PCID of A and let A_1 be the 1-PCID of A defined by $(1.2)_1$ (with $\nu=m$). Then, A_m is an $(m-1)$-PCID of A_1, thus by the inductive assumption, there exists a uniquely determined A_1-cascade string $\{C'_n\}_1^{m-1}$ such that

$$(2.20)_m \qquad A_m = A_{m-1}(A; C'_1, \ldots, C'_{m-1}).$$

By virtue of Lemma 2.3 there exists a unique A-cascade string $\{C_n\}_1^N$ such that $\{C'_n\}_1^{m-1}$ is the (A_1-cascade) reduced string of $\{C_n\}_1^m$; moreover (see $(2.14)_1$) $C_1=C(A_1)$. Therefore, from $(2.20)_m$ and $(2.17)_m$ we infer that A_m is of the form $A_m=A_m(A; C_1, C_2, \dots, C_m)$, where $\{C_n\}_1^m$ is the above (uniquely determined) A-cascade string. This finishes the proof of the proposition.

Lemma 2.4. *Within the frame of Proposition 2.1 we have*

$$(2.21)_N \qquad \|D_{A_N}h\| = \|D_{C_N}D_{C_{N-1}}\dots D_{C_1}D_A h\| \quad (h\in\mathfrak{H}),$$

where $A_N=A_N(A; C_1, C_2, \dots, C_N)$ *and* $\{C_n\}_1^N$ *is an A-cascade string.*

Proof. Relation $(2.21)_1$ follows directly from Lemma 1.2.

Suppose relation $(2.21)_N$ be true for $N=m-1\geqq 0$. Then from $(2.17)_m$ we infer

$$(2.22) \qquad \|D_{A_m}h\| = \|D_{A'_{m-1}}h\| = \|D_{C'_{m-1}}\dots D_{C'_1}D_{A_1}h\| \quad (h\in\mathfrak{H})$$

where $A'_{m-1}=A_{m-1}(A_1; C'_1, \dots, C'_{m-1})$, A_1 is the 1-PCID of A defined by $(1.2)_1$ (with $\nu=m$) and $\{C'_n\}_1^{m-1}$ is the reduced string of $\{C_n\}_1^m$. But, by virtue of Lemma 1.2 we have $D_{A_1}h=X(A_1)D_{C_1}D_A h$ so that, if $\{X_n\}_1^N$ is the $\{C_n\}_1^N$-extension of $X(A_1)$, we obtain

$$D_{C'_{N-1}}\dots D_{C'_1}D_{A_1}h = D_{C'_{N-1}}\dots D_{C'_1}X_1D_{C_1}D_A h = D_{C'_{N-1}}\dots D_{C'_2}X_1D_{C_2}D_{C_1}D_A h =$$
$$= D_{C'_{N-1}}\dots D_{C'_2}X_2D_{C_2}D_{C_1}D_A h = \dots = D_{C'_{N-1}}X_{N-2}D_{C_{N-1}}\dots D_{C_1}D_A h =$$
$$= X_{N-1}D_{C_N}D_{C_{N-1}}\dots D_{C_1}D_A h \quad (h\in\mathfrak{H}),$$

where we used, in order, relations $(2.11)_1$, $(2.12)_1$, $(2.11)_2$, ..., $(2.11)_{N-1}$, $(2.12)_{N-1}$. Since X_{N-1} is unitary, from (2.22) it follows that $(2.21)_m$ is also valid. This completes the proof.

Proposition 2.2. *The mapping*

$$(2.23) \qquad \{C_n\}_1^\infty \to A_\infty(A; C_1, C_2, \dots) = \text{strong}\lim A_N(A; C_1, \dots, C_N)P_N$$

establishes a one-to-one correspondence between all the A-cascade sequences and all the CID's *of A. Moreover,* $A_\infty=A_\infty(A; C_1, C_2, \dots)$ *is an isometry if and only if*

$$(2.24) \qquad \|D_{C_N}D_{C_{N-1}}\dots D_{C_1}D_A h\| \to 0 \quad (h\in\mathfrak{H}; N\to\infty).$$

Proof. The first statement of the proposition follows at once from Remark 1.1, Proposition 2.1 and $(2.18)_N$ ($N=2, 3, \dots$). Concerning the second, we remark that (2.24) holds if and only if

$$(2.25) \qquad \|D_{A_N}h\| \to 0 \quad (h\in\mathfrak{H}; N\to\infty),$$

where $A_N = A_N(A; C_1, \dots, C_N)$ $(N=1, 2, \dots)$.

From the first statement it follows that

$$\|D_{A_\infty}h\|^2 = \|h\|^2 - \|A_\infty h\|^2 = \|h\|^2 - \lim_{N\to\infty} \|A_N h\|^2 = \lim_{N\to\infty} \|D_{A_N} h\|^2$$

and consequently (2.25) (or equivalently (2.24)) holds if and only if $D_{A_\infty}|\mathfrak{H}=O$, that is, if $A_\infty|\mathfrak{H}$ is isometric. Thus it remains only to prove that the last property implies that A_∞ is isometric. Or, since $A_\infty U=U'A_\infty$, it follows at once that $A_\infty|U^n\mathfrak{H}$ is isometric for $n=0, 1, 2, \ldots$; in its turn, this implies that $D_{A_\infty}|U^n\mathfrak{H}=O$ $(n=0, 1, 2, \ldots)$.

Since the spaces $U^n\mathfrak{H}$ $(n=0, 1, 2, \ldots)$ span $\mathfrak{K}$, we conclude that $D_{A_\infty}=O$, i.e. A_∞ is isometric.

3. Propositions 2.1 and 2.2 reduce the study of all PCID's and CID's of an $A\in\mathscr{I}(T', T)$ (where T, T' and A are some given contractions) to that of A-cascade strings and sequences. However an A-cascade string or sequence is a rather involved concept. Therefore we shall show that the study can be actually confined to more transparent concepts, one of which is defined in the following.

Definition 3.1. A string (or sequence) $\{\Gamma_n\}_{1\le n<\nu}$ of operators will be called an *A-choice string (or sequence)* if each Γ_n $(1\le n<\nu)$ is a contraction acting from $\mathfrak{R}_A$ to $\mathfrak{R}'_A$ (if $n=1$) and from $\mathfrak{D}_{\Gamma_{n-1}}$ to $\mathfrak{D}_{\Gamma^*_{n-1}}$ (if $n\ge 2$). (Thus if $\{\Gamma_n\}_1^N$ is an A-choice string, then for *any* contraction $\Gamma_{N+1}\in L(\mathfrak{D}_{\Gamma_N}, \mathfrak{D}_{\Gamma^*_N})$, $\{\Gamma_n\}_1^{N+1}$ is also an A-choice string; this is the justification of the terminology.)

In this section we shall establish a natural connection between the A-cascade strings (or sequences) and the A-choice strings (or sequences). To this aim we need some simple facts, rather known, which, for the sake of completeness, will be collected in the following:

Lemma 3.1. *$\mathfrak{G}$, $\mathfrak{G}'$ and $\mathfrak{G}_0$ be some Hilbert spaces, $\mathfrak{G}_0$ being a subspace of $\mathfrak{G}$, and let $C_0\colon\mathfrak{G}_0\to\mathfrak{G}'$ be a contraction. Then the formulas*

$$D_{C_0^*}\Gamma(C_0, C) = C|\mathfrak{G}\ominus\mathfrak{G}_0, \quad C(C_0, \Gamma) = C_0Q + D_{C_0^*}\Gamma(I-Q) \tag{3.1}$$

(where Q denotes the orthogonal projection of $\mathfrak{G}$ onto $\mathfrak{G}_0$), establishes a one-to-one correspondence between all the contractions $C\colon\mathfrak{G}\to\mathfrak{G}'$ such that

$$C|\mathfrak{G}_0 = C_0, \tag{3.2}$$

and all the contractions $\Gamma\colon\mathfrak{G}\ominus\mathfrak{G}_0\to\mathfrak{D}_{C_0^*}$. Moreover, the formulas

$$ZD_\Gamma = RD_C|\mathfrak{G}\ominus\mathfrak{G}_0, \quad Z_*D_{\Gamma^*}D_{C_0^*} = D_{C^*}, \quad Z'D_{C_0} = D_C|\mathfrak{G}_0 \tag{3.3}$$

(where R denotes the orthogonal projection of $\mathfrak{D}_C$ onto $\mathfrak{D}_C\ominus(D_C\mathfrak{G}_0)^-$ define unitary operators $Z=Z(C_0, C)$ from $\mathfrak{D}_\Gamma$ to $\mathfrak{D}_C\ominus(D_C\mathfrak{G}_0)^-$, $Z_*=Z_*(C_0, C)$ from $\mathfrak{D}_{\Gamma^*}$ to $\mathfrak{D}_{C^*}$ and $Z'=Z'(C_0C)$ from $\mathfrak{D}_{C_0}$ to $(D_C\mathfrak{G}_0)^-$; also

$$Z\mathfrak{D}_\Gamma = \mathfrak{D}_C\ominus Z'\mathfrak{D}_{C_0}. \tag{3.4}$$

Proof. Let $C:\mathfrak{G}\to\mathfrak{G}'$ be a contraction with the property. (3.2). Then,

$$\|QC^*g'\|^2+\|(I-Q)C^*g'\|^2=\|C^*g'\|^2\leqq\|g'\|^2\quad(g'\in\mathfrak{G}'),\tag{3.5}$$

$$(QC^*g',g_0)=(C^*g',g_0)=(g',Cg_0)=(g',C_0g_0)=(C_0^*g',g_0)\quad(g'\in\mathfrak{G}',\ g_0\in\mathfrak{G}_0)$$

whence

$$QC^*=C_0^*\tag{3.6}$$

and therefore, by (3.5),

$$\|(I-Q)C^*g'\|\leqq\|D_{C_0^*}g'\|\quad(g\in\mathfrak{G}').\tag{3.7}$$

It follows that there exists a unique contraction $\Gamma^*:\mathfrak{D}_{C_0^*}\to\mathfrak{G}\ominus\mathfrak{G}_0$ such that

$$\Gamma^*D_{C_0^*}=(I-Q)C^*.\tag{3.8}$$

Consequently, setting $\Gamma(C_0,C)=\Gamma\in L(\mathfrak{G}\ominus\mathfrak{G}_0,\mathfrak{D}_{C_0^*})$ we obtain the first relation (3.1). Conversely, if we are given a contraction $\Gamma:\mathfrak{G}\ominus\mathfrak{G}_0\to\mathfrak{D}_{C^*}$ and if we define $C=C(C_0,\Gamma)$ by the second relation (3.1), then (3.2) and (3.8) are plainly satisfied; consequently we obtain (3.6) with the same argument as above. It follows

$$\begin{aligned}\|C^*g'\|^2&=\|QC^*g'\|^2+\|(I-Q)C^*g'\|^2=\|C_0^*g'\|^2+\|\Gamma^*D_{C_0^*}g'\|^2\leqq\\&\leqq\|C_0^*g'\|^2+\|D_{C_0^*}g'\|^2=\|g'\|^2\quad(g'\in\mathfrak{G}')\end{aligned}\tag{3.9}$$

hence C is a contraction; finally, (3.8) shows that $\Gamma(C_0,C)=\Gamma$.

This completes the proof of the first statement in the lemma. The statements on Z_* and Z' follow readily from (3.9) and (3.2), respectively. Concerning the statement on Z, we note that

$$\begin{aligned}\|D_Cg\|^2&=\|g\|^2-\|Cg\|^2=\|(I-Q)g\|^2+\|Qg\|^2-\|C_0Qg+D_{C_0^*}\Gamma(I-Q)g\|^2=\\&=\|(I-Q)g\|^2+\|Qg\|^2-\|C_0Qg\|^2-2\mathrm{Re}\,(C_0Qg,D_{C_0^*}\Gamma(I-Q)g)-\|D_{C_0^*}\Gamma(I-Q)g\|^2=\\&=\|(I-Q)g\|^2+\|D_{C_0}Qg\|^2-2\mathrm{Re}\,(D_{C_0^*}C_0Qg,\Gamma(I-Q)g)-\|D_{C_0^*}\Gamma(I-Q)\|^2=\\&=\|(I-Q)g\|^2+\|D_{C_0}Qg\|^2-2\mathrm{Re}\,(C_0D_{C_0}Qg,\Gamma(I-Q)g)-\|D_{C_0^*}\Gamma(I-Q)g\|^2=\\&=\|D_\Gamma(I-Q)g\|^2+\|C_0^*\Gamma(I-Q)g\|^2+\|D_{C_0}Qg\|^2-2\mathrm{Re}\,(D_{C_0}Qg,C_0^*\Gamma(I-Q)g)=\\&=\|D_\Gamma(I-Q)g\|^2+\|D_{C_0}Qg-C_0^*\Gamma(I-Q)g\|^2\quad(g\in\mathfrak{G})\end{aligned}$$

whence

$$\|D_Cg+D_Cg_0\|^2=\|D_\Gamma g\|^2+\|D_{C_0}g_0-C_0^*\Gamma g\|^2\quad(g\in\mathfrak{G}\ominus\mathfrak{G}_0,\ g_0\in\mathfrak{G}_0).\tag{3.10}$$

But since $C_0^*\Gamma(\mathfrak{G}\ominus\mathfrak{G}_0)\subset C_0^*\mathfrak{D}_{C_0^*}\subset\mathfrak{D}_{C_0}$, from relation (3.10) it follows that

$$\|RD_Cg\|^2=\inf_{g_0\in\mathfrak{G}_0}\|D_Cg+D_Cg_0\|^2=\|D_\Gamma g\|^2\quad(g\in\mathfrak{G}\ominus\mathfrak{G}_0).$$

This shows that the definition of Z is meaningful and that Z is unitary. Since (3.4) is now obvious, the proof is completed.

We now return to the aim of this section, stated before Lemma 3.1, by considering an A-cascade string $\{C_n\}_1^N$ (where T, T' and A are as in Sec. 2). We set

$$(3.11)\qquad \mathfrak{G}_{01} = \mathfrak{F}_A,\quad \mathfrak{G}_1 = \mathfrak{D}_A + \mathfrak{L},\quad \mathfrak{G}_1' = \mathfrak{L}',$$

$$\mathfrak{G}_{0n} = Y_{n-1}\mathfrak{D}_{n-2} + U^{n-1}\mathfrak{L},\quad \mathfrak{G}_n = \mathfrak{D}_{n-1} + U^{n-1}\mathfrak{L},\quad \mathfrak{G}_n' = U'^{n-1}\mathfrak{L}' \quad (n > 1)$$

and we define the contractions $C_{0n}: \mathfrak{G}_{0n} \to \mathfrak{G}_n'$ $(n = 1, 2, \dots, N)$ by

$$(3.12)\qquad C_{01} = C_A,\quad C_{0n}Y_{n-1}|\mathfrak{D}_{n-2} =$$

$$= U'C_{n-1}|\mathfrak{D}_{n-2},\ C_{0n}U|U^{n-2}\mathfrak{L} = U'C_{n-1}|U^{n-2}\mathfrak{L} \quad (n > 1).$$

By virtue of $(2.3)_1$ and $(2.6)_n$ (for $n > 1$) we have

$$(3.13)\qquad C_n|\mathfrak{G}_{0n} = C_{0n} \quad (n = 1, 2, \dots, N).$$

Therefore, Lemma 3.1 yields the operators

$$(3.14)_n\quad \Gamma_n' = \Gamma(C_{0n}, C_n),\quad Z_n = Z(C_{0n}, C_n),\quad Z_{*n} = Z_*(C_{0n}, C_n),\quad Z_n' = Z'(C_{0n}, C_n)$$

for $n = 1, 2, \dots, N$. (In the sequel, when a more precise notation will seem necessary, we shall write $\Gamma_n' = \Gamma_n'(C_1, \dots, C_n)$, $Z_n = Z_n(C_1, \dots, C_n)$, ... instead of Γ_n', Z_n, ...).

Lemma 3.2. *For $2 \leqq n \leqq N$ the range of Z_{n-1}' is $Y_{n-1}\mathfrak{D}_{n-2}$ and Γ_n' is a contraction from $Z_{n-1}\mathfrak{D}_{\Gamma_{n-1}'}$ to $U'\mathfrak{D}_{C_{n-1}^*}$.*

Proof. For proving

$$(3.15)_n\qquad Z_{n-1}'\mathfrak{D}_{C_{0,n-1}} = Y_{n-1}\mathfrak{D}_{n-2}$$

for $n = 2, \dots, N$, we note firstly that $(3.15)_2$ follows from (3.3) and $(2.5)_1$, by the relations

$$Z_1'D_{C_A}(D_ATh + (U-T)h) = D_{C_1}(D_ATh + (U-T)h) = Y_1D_Ah \quad (h \in \mathfrak{H}).$$

For $n > 2$ we have, by virtue of (3.3), (3.12) and $(2.5)_{n-1}$,

$$Z_{n-1}'D_{C_{0,n-1}}(Y_{n-2}d + U^{n-2}l) = D_{C_{n-1}}(Y_{n-2}d + U^{n-2}l) =$$
$$= Y_{n-1}D_{C_{n-2}}(d + U^{n-3}l) \quad (d \in \mathfrak{D}_{n-3},\ l \in \mathfrak{L}),$$

from which $(3.15)_n$ follows at once. Concerning the second statement in the lemma, we first notice that Lemma 3.1 yields

$$\Gamma_n' \in L(\mathfrak{G}_n \ominus \mathfrak{G}_{0n}, \mathfrak{D}_{C_{0n}^*}).$$

But, by virtue of (3.11), $(3.15)_n$ and (3.4) we have firstly

$$(3.16)_n\quad \mathfrak{G}_n \ominus \mathfrak{G}_{0n} = \mathfrak{D}_{n-1} \ominus Y_{n-1}\mathfrak{D}_{n-2} = \mathfrak{D}_{C_{n-1}} \ominus Z_{n-1}'\mathfrak{D}_{C_{0,n-1}} = Z_{n-1}\mathfrak{D}_{\Gamma_{n-1}'},$$

while, by virtue of (3.12), we have $C_{0n}C_{0n}^* = U'C_{n-1}C_{n-1}^*U'^*|U'^{n-1}\mathfrak{L}'$ whence

$$D_{C_{0n}^*}^2 = U'D_{C_{n-1}^*}^2U'^*|U'^{n-1}\mathfrak{L}',\quad D_{C_{0n}^*} = U'D_{C_{n-1}^*}U'^*|U'^{n-1}\mathfrak{L}'$$

and hence

$$\mathfrak{D}_{C_{0n}^*} = U'\mathfrak{D}_{C_{n-1}^*}. \tag{3.17$_n$}$$

We can thus conclude that $\Gamma'_n \in L(Z_{n-1}\mathfrak{D}_{\Gamma'_{n-1}}, U'\mathfrak{D}_{C_{n-1}^*})$, completing the proof of the lemma.

We shall associate with our A-cascade string $\{C_n\}_1^N$ an A-choice string $\{\Gamma_n\}_1^N$ in the following manner. We set

$$\Gamma_1 = W_A \Gamma'_1, \tag{3.18$_1$}$$

and

$$W_1 = I_{\mathfrak{D}_{\Gamma_1}}, \quad W_{*1} = W_A^* | \mathfrak{D}_{\Gamma_1^*}. \tag{3.19$_1$}$$

Since $W_A \in \mathfrak{L}(\mathfrak{D}_{C_A^*}, \mathfrak{R}'_A)$ is unitary (see Lemma 1.1), we have obviously $D_{\Gamma_1} = D_{\Gamma'_1}$, $W_A D_{\Gamma'^*_1} = D_{\Gamma_1^*} W_A$ hence the operators

$$W_1 : \mathfrak{D}_{\Gamma_1} \to \mathfrak{D}_{\Gamma'_1}, \quad W_{*1} : \mathfrak{D}_{\Gamma_1^*} \to \mathfrak{D}_{\Gamma'^*_1}, \tag{3.20$_1$}$$

defined by formula $(3.19)_1$, are unitary; moreover, we also have

$$\Gamma_1 \in L(\mathfrak{R}_A, \mathfrak{R}'_A), \tag{3.21$_1$}$$

thus $\{\Gamma_1\}$ is an A-choice string (of length 1); this will be associated with our A-cascade string if $N=1$. If $N>1$, we appeal to the following

Lemma 3.3. *Let* $N>1$. *Then formulas* $(3.18)_1$, $(3.19)_1$, *and, for* $2 \leq n \leq N$,

$$\Gamma_n = W_{*n-1}^* Z_{*n-1}^* U'^* \Gamma'_n Z_{n-1} W_{n-1}, \tag{3.18$_n$}$$

$$W_n = Z_{n-1} W_{n-1} | \mathfrak{D}_{\Gamma_n}, \quad W_{*n} = U' Z_{*n-1} W_{*n-1} | \mathfrak{D}_{\Gamma_n^*} \tag{3.19$_n$}$$

define an A-*choice string* $\{\Gamma_n\}_1^N$ *and unitary operators*

$$W_n : \mathfrak{D}_{\Gamma_n} \to \mathfrak{D}_{\Gamma'_n}, \quad W_{*n} : \mathfrak{D}_{\Gamma_n^*} \to \mathfrak{D}_{\Gamma'^*_n} \quad (1 \leq n \leq N). \tag{3.20$_n$}$$

Proof. Proceeding by recurrence, we notice that the statements concerning Γ_1, W_1 and W_{*1} were already established above. Assuming that those concerning W_{m-1} and W_{*m-1} (where $m-1 \geq 1$, $m \leq M$) are also established we infer by virtue of Lemma 3.2 and $(3.20)_{m-1}$ that the relation

$$\Gamma_n \in L(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma_{n-1}^*}) \tag{3.21$_n$}$$

is valid for $n=m$. From this we obtain

$$Z_{m-1} W_{m-1} \Gamma_m^* \Gamma_m = \Gamma_m^* \Gamma_m Z_{m-1} W_{m-1},$$

$$U' Z_{*m-1} W_{*m-1} \Gamma_m \Gamma_m^* = \Gamma'_m \Gamma'^*_m U' Z_{*m-1} W_{*m-1},$$

whence

$$Z_{m-1} W_{m-1} D_{\Gamma_m} = D_{\Gamma'_m} Z_{m-1} W_{m-1}, \quad U' Z_{*m-1} W_{*m-1} D_{\Gamma_m^*} = D_{\Gamma'^*_m} U' Z_{*m-1} W_{*m-1}.$$

From these relations it follows readily that formula $(3.19)_m$ defines the unitary operators $(3.20)_m$. Thus the operators W_n and W_{*n} $(1 \leqq n \leqq N)$ are unitary and $(3.21)_n$ is true for all n, $1 \leqq n \leqq N$, which means that $\{\Gamma_n\}_1^N$ is an A-choice sequence. This finishes the proof.

It is plain that the operators Γ_n, W_n and W_{*n} $(1 \leqq n \leqq N)$ yielded by the preceding argument depend only on $C_1, C_2, \ldots, C_n$ (and of course on A, T, T' and U, U'). Therefore we shall denote them by $\Gamma_n(C_1, \ldots, C_n)$, $W_n(C_1, \ldots, C_n)$ and $W_{*n}(C_1, \ldots, C_n)$. (When a confusion seems possible we shall explicitate also the dependence on A, T, T', U, U', for instance $\Gamma_n(A; T, T'; U, U'; C_1, \ldots, C_n)$ for Γ_n etc.)

Proposition 3.1. *For* $\nu = 2, 3, \ldots, \infty$ *and* $T, T', A \in \mathscr{I}(T', T)$, *fixed, the mapping*

$$\{C_n\}_{1 \leqq n < \nu} \to \{\Gamma_n(C_1, \ldots, C_n)\}_{1 \leqq n < \nu} \tag{3.22$_\nu$}$$

establishes a one-to-one correspondence between all the A-cascade strings (if $\nu < \infty$*), respectively sequences (if* $\nu = \infty$*) and all the A-choice strings (of length* $\nu - 1$, *if* $\nu < \infty$*), respectively sequences (if* $\nu = \infty$*).*

Proof. The case $\nu = \infty$ follows immediately from the case $\nu < \infty$.

Since the case $\nu = 2$ is a direct consequence of Lemma 3.1, we shall assume now that the proposition is valid if $\nu = m \geqq 2$.

Let $\{\Gamma_n\}_{1 \leqq n < m+1}$ be any A-choice string. Then by our assumption there exists a unique A-cascade string $\{C_n\}_{1 \leqq n < m}$ such that

$$\Gamma_n = \Gamma_n(C_1, C_2, \ldots, C_m) \quad (1 \leqq n < m). \tag{3.23$_n$}$$

Therefore, by virtue of Lemmas 3.1 and 3.2, the operators

$$\Gamma' = \Gamma'_{m-1}(C_1, \ldots, C_{m-1}) \colon \mathfrak{G}_{m-1} \ominus \mathfrak{G}_{0, m-1} \to \mathfrak{D}_{C^*_{0, m-1}}$$

and

$$Z = Z_{m-1} = Z(C_1, \ldots, C_{m-1}) \colon \mathfrak{D}_{\Gamma'_{m-1}} \to \mathfrak{G}_m \ominus \mathfrak{G}_{0m} \quad \text{(see } (3.16)_m\text{)}$$

$$Z_* = Z_{*m-1} = Z_*(C_1, \ldots, C_{m-1}) \colon \mathfrak{D}_{\Gamma'^*_{m-1}} \to \mathfrak{D}_{C^*_{m-1}},$$

$$W = W_{m-1} = W(C_1, \ldots, C_{m-1}) \colon \mathfrak{D}_{\Gamma_{m-1}} \to \mathfrak{D}_{\Gamma'_{m-1}},$$

$$W_* = W_{*m-1} = W_*(C_1, \ldots, C_{m-1}) \colon \mathfrak{D}_{\Gamma^*_{m-1}} \to \mathfrak{D}_{\Gamma'^*_{m-1}}$$

are also uniquely determined, and Γ' is a contraction while Z, Z_*, W, W_* are unitary. Setting

$$\Gamma' = U' Z_* W_* \Gamma_m W^* Z^* \tag{3.24}$$

we obtain a contraction from $\mathfrak{G}_m \ominus \mathfrak{G}_{0m}$ to $\mathfrak{D}_{C^*_{0m}}$ (see (3.11), (3.12) and $(3.17)_m$). By virtue of Lemma 3.1, there exists a uniquely determined contraction $C_m \colon \mathfrak{G}_m \to \mathfrak{G}'_m$ such that $C_m = C(C_{0m}, \Gamma')$. Comparing (3.11), (3.12) and (3.13) (in the case $n = m$) with $(2.6)_m$ we see that $\{C_n\}_{1 \leqq n < m+1}$ is an A-cascade string.

Comparing (3.24) with $(3.18)_m$ we finally see that $(3.23)_m$ is also valid. Thus we verified that the mapping $(3.22)_{m+1}$ is surjective. Since the last term in $\{C_n\}_{1 \leq n < m+1}$ is necessarily of the form $C_m = C(C_{0m}, \Gamma')$, where Γ' is given by (3.24), the mapping is also injective.

Now the proposition is concluded by induction.

4. In this section we shall associate the CID's of an $A \in \mathscr{I}(T', T)$ with T, T' and A as in the preceding sections, to a more usual concept, namely to contractive analytic functions ([16], Ch. V.). As preparation, we shall now discuss the preceding sections in a very particular case, namely that of an arbitrary contraction Γ from $\mathfrak{R}$ to $\mathfrak{R}'$ (where $\mathfrak{R}$ and $\mathfrak{R}'$ are two Hilbert spaces), considered as intertwining the corresponding null operators $0_{\mathfrak{R}}$, $0_{\mathfrak{R}'}$, i.e. $\Gamma \in \mathscr{I}(0_{\mathfrak{R}'}, 0_{\mathfrak{R}})$.

On this purpose, for the operator $0_{\mathfrak{R}}$ we shall choose as minimal isometric dilation $V_{\mathfrak{R}'}$ the canonical multiplication shift

$$V_{\mathfrak{R}} f(z) = z f(z) \quad (|z| < 1)$$

on $H^2(\mathfrak{R})$, where $\mathfrak{R}$ is identified with the space of constant functions in $H^2(\mathfrak{R})$;[4]) the minimal isometric dilation $V_{\mathfrak{R}'}$ of $0_{\mathfrak{R}'}$ will be chosen in the obvious similar way. Since any CID of Γ is a contraction intertwining $V_{\mathfrak{R}}$ and $V_{\mathfrak{R}'}$, it is the multiplication operator by a contractive analytic function $\{\mathfrak{R}, \mathfrak{R}', \Gamma(z)\}$ (see [16], Ch. V, Sec. 3), which obviously must satisfy the condition $\Gamma(0) = \Gamma$. Since the converse fact is also obvious, we can state the following consequence of our previous results.

Lemma 4.1. *Let $\Gamma: \mathfrak{R} \to \mathfrak{R}'$ be an arbitrary fixed contraction. Then Propositions* 2.2 *and* 3.1 *with $T = 0_{\mathfrak{R}}$, $T' = 0_{\mathfrak{R}}$, and $A = \Gamma \in \mathscr{I}(T'; T)$ yield a one-to-one correspondence between all contractive analytic functions $\{\mathfrak{R}, \mathfrak{R}', \Gamma(z)\}$ such that $\Gamma(0) = \Gamma$ and all Γ-choice sequences.*

Remark 4.1. We recall that within the frame of the preceding discussion, (1.5) and (1.5)$'$ take the form

$$\mathfrak{F}_\Gamma = V_{\mathfrak{R}} \mathfrak{R}, \quad \mathfrak{R}_\Gamma = (\mathfrak{D}_\Gamma + V_{\mathfrak{R}} \mathfrak{R}) \ominus \mathfrak{F}_\Gamma = \mathfrak{D}_\Gamma \tag{4.1}$$

and

$$\begin{cases} \mathfrak{F}'_\Gamma = \{D_\Gamma r \oplus V_{\mathfrak{R}'} \Gamma r : r \in \mathfrak{R}\}, \\ \mathfrak{R}'_\Gamma = (\mathfrak{D}_\Gamma \oplus V_{\mathfrak{R}'} \mathfrak{R}') \ominus \mathfrak{F}'_\Gamma = \{r \oplus V_{\mathfrak{R}'} r' : D_\Gamma r + \Gamma^* r' = 0, \quad r \in \mathfrak{D}_\Gamma, r' \in \mathfrak{R}'\}. \end{cases} \tag{4.1$'$}$$

Lemma 4.2. *The formula*

$$\omega(\Gamma) r' = (-\Gamma^* r') \oplus V_{\mathfrak{R}'} D_{\Gamma^*} r' \quad (r' \in \mathfrak{D}_{\Gamma^*}) \tag{4.2}$$

defines a unitary operator from $\mathfrak{D}_{\Gamma^*}$ to $\mathfrak{R}'_\Gamma$.

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[4]) For the Hardy spaces $H^2(\mathfrak{R})$, where $\mathfrak{R}$ is a Hilbert space, see [16]. Ch. V.

Proof. It is obvious, by virtue of (4.1)′ and the relation $D_\Gamma \Gamma^* = \Gamma^* D_{\Gamma^*}$, that (4.2) defines an isometric operator $\omega(\Gamma)$ from $\mathfrak{D}_{\Gamma^*}$ to $\mathfrak{R}'_\Gamma$. Moreover, if we are given $r \oplus Vr' \in \mathfrak{R}'_\Gamma$, then setting $r'_1 = D_{\Gamma^*} r' - \Gamma r$ we obtain $r'_1 \in \mathfrak{D}_{\Gamma^*}$ and

$$\omega(\Gamma) r'_1 = (-\Gamma^* D_{\Gamma^*} r' + \Gamma^* \Gamma r) \oplus V_{\mathfrak{R}'}(D^2_{\Gamma^*} r' - D_{\Gamma^*} \Gamma r) =$$
$$= \big(r - D_\Gamma(\Gamma^* r' + D_\Gamma r)\big) \oplus V_{\mathfrak{R}'}\big(r' - \Gamma(\Gamma^* r' + D_\Gamma r)\big) = r \oplus V_{\mathfrak{R}'} r'.$$

This finishes the proof of the lemma.

Let now T', T, $A \in \mathscr{I}(T', T)$ be arbitrary contractions with some fixed minimal isometric dilations U, U' of T, T'. For an A-choice sequence $\{\Gamma_n\}_1^\infty$ we set

$$(4.3)_n \qquad \gamma_n(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \quad (=\gamma_n) = \omega(\Gamma_1)\Gamma_{n+1} \quad (1 \leqq n < \infty).$$

Since (see Definition 3.1)

$$\mathfrak{D}_{\Gamma_1^*} \supset \mathfrak{D}_{\Gamma_2^*} \supset \ldots \supset \mathfrak{D}_{\Gamma_n^*} \supset \ldots,$$

definition $(4.3)_n$ $(1 \leqq n < \infty)$ makes sense.

Lemma 4.3. *The mapping*

$$(4.4) \qquad \{\Gamma_n\}_1^\infty \to \big\{\Gamma_1, \{\gamma_n(\Gamma_1, \ldots, \Gamma_n)\}_1^\infty\big\}$$

establishes a one-to-one correspondence between all A-choice sequences and all pairs formed by a contraction $\Gamma : \mathfrak{R}_A \to \mathfrak{R}'_A$ (considered as belonging to $\mathscr{I}(O_{\mathfrak{R}'}, O_{\mathfrak{R}})$) and a Γ-choice sequence.

Proof. Let $\{\Gamma_n\}_1^\infty$ be an A-choice sequence and let $\{\gamma_n\}_1^\infty$ be the sequence yielded by $(4.3)_n$ $(1 \leqq n < \infty)$. It is obvious that, by virtue of Lemma 4.2, we have

$$(4.5)_n \qquad D_{\gamma_n} = D_{\Gamma_{n+1}} \quad (1 \leqq n < \infty)$$

and, using also (4.1),

$$(4.6)_1 \qquad \gamma_1 : \mathfrak{R}_{\Gamma_1} = \mathfrak{D}_{\Gamma_1} \to \omega(\Gamma_1)\mathfrak{D}_{\Gamma_1^*} = \mathfrak{R}'_{\Gamma_1},$$

where, as already indicated above, Γ_1 is regarded as belonging to to $\mathscr{I}(O_{\mathfrak{R}'}, O_{\mathfrak{R}})$; moreover, we also have

$$(4.7)_n \quad D^2_{\gamma_n^*} = \omega(\Gamma_1) D^2_{\Gamma_{n+1}^*} \omega(\Gamma_1)^*, \quad D_{\gamma_n^*}\omega(\Gamma_1) = \omega(\Gamma_1) D_{\Gamma_{n+1}^*} \quad (1 \leqq n < \infty).$$

From $(4.5)_n$ and $(4.7)_n$ we infer readily that

$$(4.8)_n \qquad \mathfrak{D}_{\gamma_n} = \mathfrak{D}_{\Gamma_{n+1}}, \quad \mathfrak{D}_{\gamma_n^*} = \omega(\Gamma_1)\mathfrak{D}_{\Gamma_{n+1}^*} \quad (1 \leqq n < \infty).$$

Consequently, γ_{n+1} is a contraction from $\mathfrak{D}_{\gamma_n}$ to $\mathfrak{D}_{\gamma_n^*}$ $(1 \leqq n < \infty)$. Together with $(4.6)_1$ this shows that $\{\gamma_n\}_1^\infty$ is a Γ_1-choice sequence. If we are given now a pair $\{\Gamma_1, \{\gamma_n\}_1^\infty\}$ formed by a contraction $\Gamma_1 : \mathfrak{R}_A \to \mathfrak{R}'_A$ (regarded as belonging to $\mathscr{I}(0_{\mathfrak{R}'}, 0_{\mathfrak{R}})$) and a Γ_1-choice sequence $\{\gamma_n\}_1^\infty$, then there may exists only one A-choice sequence $\{\Gamma_n\}_1^\infty$ which is mapped by (4.4) onto our given pair, namely that given by the formula

$$(4.9)_n \qquad \Gamma_{n+1} = \omega(\Gamma_1)^* \gamma_n.$$

It is now easy to infer that if Γ_{n+1} $(1 \leqq n < \infty)$ are actually defined by $(4.9)_n$, then $(4.5)_n$, $(4.7)_n$ and consequently $(4.8)_n$ are also satisfied for $n \geqq 1$; it obviously follows that $\{\Gamma_n\}_1^\infty$ is an A-choice sequence. This concludes the proof.

We are now in state to formulate the main result of this section. To this aim let $T, T', A \in \mathcal{I}(T', T)$ (as well as U and U') be as above. Let A_∞ be a CID of A and let

$$\Lambda_2 : A_\infty \to \{C_n\}_1^\infty, \quad \Lambda_3 : \{C_n\}_1^\infty \to \{\Gamma_n\}_1^\infty, \quad \Lambda_4 : \{\Gamma_n\}_1^\infty \to \{\Gamma_1, \{\gamma_n\}_1^\infty\},$$
$$\Lambda_5 : \{\Gamma_1, \{\gamma_n\}_1^\infty\} \to \{\mathfrak{R}_A, \mathfrak{R}'_A, \Gamma(z)\}$$

respectively be (Λ_2) the inverse mapping of that given in Proposition 2.2, $(\Lambda_3)(\Lambda_4)$ the mappings given by Proposition 3.1 and Lemma 4.3, (Λ_5) the inverse mapping of that given by Lemma 4.1. Then, the bijectivity property of these mappings directly yields the

Proposition 4.1. *The mapping* $\Lambda_1 = \Lambda_5 \circ \Lambda_4 \circ \Lambda_3 \circ \Lambda_2$ *establishes a one-to-one correspondence between the CID's of A and the contractive analytic* $L(\mathfrak{R}_A, \mathfrak{R}'_A)$-*valued functions.*

Remark 4.2. The uniqueness theorem for CID's given in [6] is a direct corollary of Proposition 4.1. Indeed, by virtue of this proposition, there exists a unique CID of A, if and only if there exists only one contractive analytic function $\{\mathfrak{R}_A, \mathfrak{R}'_A, \Gamma(z)\}$. Obviously this happens if and only if at least one of the spaces $\mathfrak{R}_A$ or $\mathfrak{R}'_A$ reduces to $\{0\}$, i.e. (see [16], Ch. VII) if at least one of the factorizations $A \cdot T$ and $T' \cdot A$ is regular.

Let us present a particular case which might be instructive. On this purpose, we shall denote by $i_{\mathfrak{L}'}$, the natural isometric identification of $\mathfrak{L}'$ with the subspace $\{0\} \oplus \mathfrak{L}'$ of $\mathfrak{D}_A \oplus \mathfrak{L}'$ and by $P_{\mathfrak{L}_*}$ the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{L}_* = ((I - UT^*)\mathfrak{H})^-$, where the notation is, as usual, that of Sec. 1. Also let us firstly give the following

Lemma 4.4. *The operators* $i^*_{\mathfrak{L}'} | \mathfrak{R}'_A : \mathfrak{R}'_A \to \mathfrak{L}'$, $P_{\mathfrak{L}_*} | \mathfrak{R}_A : \mathfrak{R}_A \to \mathfrak{L}_*$ *are injective.*

Proof. Let $P_{\mathfrak{L}_*} r = 0$, $r \in \mathfrak{R}_A$ or equivalently $r = Uh_1$ for some $h_1 \in \mathfrak{H}$, and $(T^* D_A P + U^*(I-P))r = 0$. Then

$$T^* D_A T h_1 + (I - T^* T) h_1 = 0, \quad h_1 = T^*(I - D_A) T h_1. \tag{4.10}$$

But $O \leqq I - D_A \leqq I$ implies

$$\|Th_1\|^2 \leqq \|h_1\|^2 = (T^*(I-D_A)Th_1, h_1) = \|(I-D_A)^{1/2} Th_1\|^2 \leqq \|Th_1\|^2,$$

whence

$$(I - D_A)^{1/2} T h_1 = T h_1 \quad \text{and} \quad D_A T h_1 = 0. \tag{4.11}$$

From (4.10) it follows $\|(U-T)h_1\|^2 = ((1 - T^*T)h_1, h_1) = 0$, whence

$$r = Uh_1 = Th_1 \in \mathfrak{H}. \tag{4.12}$$

Since $r \in \mathfrak{D}_A$, from (4.11) and (4.12) we infer that $r=0$. This proves the injectivity of $P_{\mathfrak{L}_*}|\mathfrak{R}_A$. Concerning the injectivity of $i^*_{\mathfrak{L}'}|\mathfrak{R}'_A$, we notice, firstly, that if $r' \in \mathfrak{R}'_A$, $i^*_{\mathfrak{L}'} r'=0$, then $r'=d \oplus 0$ with some $d \in \mathfrak{D}_A$, and secondly, that $(r', D_A h \oplus (U'-T')Ah)=0$ $(h \in H)$ implies $D_A d=0$, $d=0$, thus $i^*_{\mathfrak{L}'}|\mathfrak{R}'_A$ is also injective. Thus the lemma is proved.

By virtue of the preceding lemma and of [16], Ch. II, Sec. I we have

$$\dim \mathfrak{R}_A \leqq \delta_{T^*} \stackrel{\text{def}}{=} \operatorname{rank} D_{T^*} = \dim \mathfrak{L}_*, \quad \dim \mathfrak{R}'_A \leqq \delta_{T'} \stackrel{\text{def}}{=} \operatorname{rank} D_{T'} = \dim \mathfrak{L}'.$$

Therefore, from Lemma 4.4 and Proposition 4.1 we can now readily obtain the following

Corollary 4.1. *Assume that, within the frame of Proposition 4.1, we have $\delta_{T^*}=\delta_{T'}=1$. Then either the set of all CID's of A is a singleton or it is in a one-to-one correspondence (explicitly given by Λ_1) with the unit ball of H^∞* (i.e. the set of all complex-valued analytic functions $u(z)$ on the unit disk $D=\{z:|z|<1\}$ such that $|u(z)| \leqq 1$ for all $z \in D$).

It is plain that in this corollary the first case occurs if $\min(\dim \mathfrak{R}_A, \dim \mathfrak{R}'_A)=0$ (see Remark 4.2), while the second one if $\dim \mathfrak{R}_A = \dim \mathfrak{R}'_A = 1$.

5. We shall now apply Proposition 4.1 to the labelling of all classes of isomorphic Ando dilations. To be more precise, for a pair $\{T_1, T_2\}$ of some fixed commuting contractions on some Hilbert space $\mathfrak{H}$, there always exists (as shown in a celebrated short note by Ando [5]) a pair $\{U_1, U_2\}$ of commuting isometric operators on some Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a (closed linear) subspace and such that

$$PU_1^{n_1} U^{n_2} | \mathfrak{H} = T_1^{n_1} T_2^{n_2} \quad (n_1, n_2 = 0, 1, 2, \ldots), \tag{5.1}$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Obviously we can and shall also suppose that

$$\mathfrak{K} = \bigvee_{n_1, n_2 \geqq 0} U_1^{n_1} U_2^{n_2} \mathfrak{H}. \tag{5.2}$$

Any such pair $\{U_1, U_2\}$ will be called an *Ando dilation* of $\{T_1, T_2\}$. Two Ando dilations $\{U_1, U_2\}$, $\{U'_1, U'_2\}$ are called isomorphic if there exists a unitary operator W from the space $\mathfrak{K}$, on which operate U_1 and U_2, to the space $\mathfrak{K}'$ on which operate U'_1 and U'_2, such that

$$WU_j = U'_j W \quad (j=1, 2), \quad W|\mathfrak{H} = I_{\mathfrak{H}}. \tag{5.3}$$

Let now U on K be a fixed minimal isometric dilation of $T=T_1$. Obviously any Ando dilation $\{U''_1, U''_2\}$ is isomorphic with some Ando dilation $\{U_1, U_2\}$ operating on a space $\mathfrak{K}$ containing K (as closed linear subspace), and such that

$$U_1|\mathsf{K} = U. \tag{5.4}$$

Let $\{U_1', U_2'\}$ be another such Ando dilation, isomorphic "by W" to $\{U_1, U_2\}$. Then by virtue of (5.3) we have $WU^n h = WU_1^n h = U_1'^n Wh = U_1'^n h = U^n h$ for all $h \in \mathfrak{H}$, $n=0, 1, 2, \ldots$; therefore

$$W|\mathsf{K} = I_{\mathsf{K}}. \tag{5.5}$$

By virtue of this discussion, we can and shall consider from now on Ando dilations satisfying (5.4). With this convened, we state the following

Lemma 5.1. *For* $T=T_1$ *and* $A=T_2$, *the formula*

$$\hat{A} = P_{\mathsf{K}} U_2 | \mathsf{K} \tag{5.6}$$

(where P_{K} *denotes the orthogonal projection of* $\mathfrak{K}$ *onto* K*) establishes a one-to-one correspondence between all classes of isomorphic Ando dilations of* $\{T_1, T_2\}$ *and all CID's of* A.

Proof. First we remark that

$$U_1^* \mathsf{K} \subset \mathsf{K}, \tag{5.7}$$

i.e. that K is reducing U_1. This was, for instance, proven in [13]. For the sake of completeness let us sketch the proof. On this purpose we infer easily from (5.1) and (5.2) that

$$PU_1 = T_1 P = TP, \quad PU_2 = T_2 P = AP; \tag{5.8}$$

from the first relation (5.8) it follows that

$$U_1^* | \mathfrak{H} = T_1^* = T^*, \tag{5.9}$$

whence, for $h \in \mathfrak{H}$,

$$U_1^* U^n h = T^* h \quad \text{if} \quad n=0, \quad \text{and} \quad U_1^* U^n h = U^{n-1} h \quad \text{if} \quad n=1, 2, \ldots,$$

so that, since these $U^n h$'s span K, (5.7) is true. We conclude thus that

$$P_{\mathsf{K}} U_1 = U_1 P_{\mathsf{K}}. \tag{5.10}$$

Now the fact that for a given $\{U_1, U_2\}$ formula (5.6) defines a CID $\hat{A}$ of A can be easily obtained from (5.10) and the second relation (5.8). Moreover if $\{U_1', U_2'\}$ is another Ando dilation of $\{T_1, T_2\}$, isomorphic (by W) to $\{U_1, U_2\}$, then $WP_{\mathsf{K}} = P_{\mathsf{K}} = P_{\mathsf{K}} W$, so that

$$P_{\mathsf{K}} U_2' | \mathsf{K} = P_{\mathsf{K}} W U_2 | \mathsf{K} = P_{\mathsf{K}} U_2 | \mathsf{K} = \hat{A}.$$

Thus we can conclude that (5.6) defines a mapping from the classes of isomorphic Ando dilations of $\{T_1, T_2\}$ to the set of the CID's of A. Let now $\hat{A}$ be a CID of A. Let U_2 on $\mathfrak{K}$ be a minimal isometric dilation of $\hat{A}$. Since U is an isometric operator commuting with A it has a unique CID (as operator in $\mathfrak{K}$ commuting with U_2; namely consider in Remark 4.2 the case when A is isometric and observe that in this particular case we have $\mathfrak{N}_A=\{0\}$), which we shall denote by U_1. The pair $\{U_1, U_2\}$ is an Ando dilation of $\{T_1, T_2\}$ satisfying the property (5.6). Indeed, (5.6) is satisfied by the very definition of U_2, while

$$PU_1^{n_1}U_2^{n_2} = PP_{\mathsf{K}}U_1^{n_1}U_2^{n_2} = PU^{n_1}P_{\mathsf{K}}U_2^{n_2} =$$
$$= PU^{n_1}\hat{A}^{n_2}P_{\mathsf{K}} = T^{n_1}P\hat{A}^{n_2}P_{\mathsf{K}} = T^{n_1}A^{n_2}PP_{\mathsf{K}} = T^{n_1}A^{n_2}P = T_1^{n_1}T_2^{n_2}P$$

for $n_1, n_2=0, 1, 2, \ldots$. Moreover, since U is isometric, it follows directly that $U_1|\mathsf{K}=U$, whence

$$\left\|U_1\sum_{n=0}^{N}U_2^n k_n\right\|^2 = \left\|\sum_{n=0}^{N}U_2^n Uk_n\right\|^2 =$$
$$= \sum_{N\geqq n\geqq m\geqq 0}(U_2^{n-m}Uk_n, Uk_m)+\sum_{0\leqq n<m\leqq N}(Uk_n, U_2^{m-n}Uk_m) =$$
$$= \sum_{N\geqq n\geqq m\geqq 0}(\hat{A}^{n-m}Uk_n, Uk_m)+\sum_{0\leqq n<m\leqq N}(Uk_n, \hat{A}^{m-n}Uk_m) =$$
$$= \sum_{N\geqq n\geqq m\geqq 0}(U\hat{A}^{n-m}k_n, Uk_m)+\sum_{0\leqq n<m\leqq N}(Uk_n, U\hat{A}^{m-n}k_m) =$$
$$= \sum_{N\geqq n\geqq m\geqq 0}(\hat{A}^{n-m}k_n, k_m)+\sum_{0\leqq n<m\leqq N}(k_n, \hat{A}k_m) = \left\|\sum_{n=0}^{N}U_2^n k_n\right\|^2$$

for all $k_1, k_2, \ldots, k_N\in\mathsf{K}$, $N=0, 1, \ldots$. Therefore U_1 is indeed isometric.

Finally, the fact that relation (5.2) is also satisfied, follows from

$$\bigvee_{n_1,n_2\geqq 0} U_1^{n_1}U_2^{n_2}\mathfrak{H} = \bigvee_{n_2\geqq 0} U_2^{n_2}\bigvee_{n_1\geqq 0} U_1^{n_1}\mathfrak{H} = \bigvee_{n_2\geqq 0} U_2^{n_2}\bigvee_{n_1\geqq 0} U^{n_1}\mathfrak{H} = \bigvee_{n_2\geqq 0} U_2^{n_2}\mathsf{K} = \mathfrak{K}$$

because U and U_2 are minimal isometric dilations of $T(=T_1)$ and $\hat{A}$, respectively.

It remains to prove that the mapping yielded by (5.6), is one-to-one. But this follows at once from the preceding construction, since if $P_{\mathsf{K}}U_2'|\mathsf{K}=P_{\mathsf{K}}U_2|\mathsf{K}$ $(=\hat{A})$ for two Ando dilations $\{U_1, U_2\}$, $\{U_1', U_2'\}$, the isometries U_2 and U_2' are actually minimal isometric dilations of $\hat{A}$, thus isomorphic, say by the unitary operator W. But then

$$WU_1U_2^{n_2}k = WU_2^{n_2}Uk = U_2'^{n_2}WUk = U_2'^{n_2}Uk = U_1'U_2'^{n_2}k = U_1'WU_2^{n_2}k$$

for all the elements $U_2^{n_2}k$ $(k\in\mathsf{K},\ n_2=0, 1, 2, \ldots)$. Since these elements span the space on which operate U_1 and U_2, we infer that (5.3) is valid (of course in the special case satisfying (5.4)), thus $\{U_1, U_2\}$ and $\{U_1', U_2'\}$ are isomorphic.

Proposition 5.1. *Let $\{T_1, T_2\}$ be a pair of commuting contractions on $\mathfrak{H}$ and let, for $i=1, j=2$ or $i=2, j=1$,*

$$\mathfrak{R}_{ij} = (\mathfrak{D}_{T_i} \oplus \mathfrak{D}_{T_j}) \ominus \{D_{T_i} T_j h \oplus D_{T_j} h : h \in \mathfrak{H}\}. \tag{5.10}$$

There exists a one-to-one (explicit) correspondence between all classes of isomorphic Ando dilations of $\{T_1, T_2\}$ and all the contractive analytic $L(\mathfrak{R}_{21}, \mathfrak{R}_{12})$-valued functions.

Proof. We set $T=T_1$ and $A=T_2$. By virtue of Lemma 5.1 and Proposition 4.1 we have an explicit one-to-one correspondence from the classes of isomorphic Ando dilations of $\{T_1, T_2\}$ and all contractive analytic functions $\{\mathfrak{R}_A, \mathfrak{R}'_A, B(z)\}$. Or by virtue of [16], Ch. II, Sec. 1, there exists a unitary (canonical) identification $\varphi: D_T h \to (U-T)h$ of $\mathfrak{D}_T = \mathfrak{D}_{T_1}$ with $\mathfrak{L}$. Thus $\varphi_1 = [I_{D_{T_2}}, \varphi]$ identifies $\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}$ to $\mathfrak{D}_{T_2} + \mathfrak{L}$ and takes $\mathfrak{R}_{21}$ onto $\mathfrak{R}_A$, while $\varphi'_1 = I_{\mathfrak{D}_{T_2}} \oplus \varphi$ identifies $\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}$ to $\mathfrak{D}_{T_2} \oplus \mathfrak{L}$ and takes $\mathfrak{R}'_{21} = (\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}) \ominus \{D_{T_2} h \oplus D_{T_1} T_2 h : h \in \mathfrak{H}\}$ onto $\mathfrak{R}'_A$.

Denoting φ' the unitary operator from $\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}$ to $\mathfrak{D}_{T_1} \oplus \mathfrak{D}_{T_2}$ which intertwines the coordinates, we obtain by $A(z) = \psi \varphi'^* B(z) \varphi_1 | R_{21}$ ($|z|<1$) the mapping yielding the one-to-one correspondence between the set of all contractive analytic function $\{\mathfrak{R}_A, \mathfrak{R}'_A, B(z)\}$ and that of those of the form $\{\mathfrak{R}_{21}, \mathfrak{R}_{12}, A(z)\}$. This plainly concludes the proof.

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Classical approximation processes in connection with Lax equivalence theorems with orders

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1. Introduction

In this note we continue our previous investigations [8], [10] on Lax equivalence theorems with orders in the setting of linear operators in Banach spaces. There we were concerned (compare also with [18a] as well as with [6] and the literature cited there) with a quantitative description of the approximation of the exact solution $\{E(t); t \geqq 0\}$ of a properly posed initial value problem, being a (continuous) semigroup of class (C_0), by some difference scheme $\{E_\tau^n; n \in \mathbf{P}\}$ constituting a family $(0 \leqq \tau \leqq \delta)$ of discrete semigroups (with $\mathbf{P}$ the set of non-negative integers). According to the hierarchy of the various convergence theorems for families of semigroups as outlined by STRANG [20] (see also [2], [23]), one may then ask whether one can also equip more general theorems than the original Lax one with orders. To this problem Thm. 2 below will give a modest contribution inasmuch as the convergence of a family $\{E_\tau(t); t \geqq 0\}$, $0 \leqq \tau \leqq \delta$ of continuous semigroups towards $\{E(t); t \geqq 0\}$ is considered with orders, but still in the Lax framework.

There is another point which motivated the present studies. In [14] GROETSCH—KING outlined an interesting interconnection between Bernstein polynomials and the convergence of a certain difference scheme (see Sec. 3, Ex. A) which was then continued in [15] with respect to some quantitative results. The procedure, however, looks somewhat isolated so as to be particularly taylored to Bernstein polynomials. Thus the question arises whether there are further classical noncommutative processes in approximation theory of the type

$$\sum f(k/n) Q_{k,n}(x),$$

the convergence of which may be interpreted from this numerical point of view. This is indeed the case and will be worked out explicitly for the familiar Szász—Mirakyan and Baskakov operators. But also the general class of approximation processes as introduced in [18] via the powers of certain functions fit into this program. In fact, it turns out that the procedure and results of [14] may be considered as a genuine application to our previous Lax equivalence theorem with orders.

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In Sec. 2 we first treat two alternative forms of the (discrete) Lax equivalence theorem with orders, extending by the way those of [8], [10] slightly (cf. Thm. 1; 3). Correspondingly, the matter is considered in connection with a continuous version of the theorem of Lax on the convergence of families of semigroups (see Thm. 2; 4). The latter results are obtained by exploiting methods used in [13] to give an elementary proof of a weak (non-order) version of the Trotter theorem. In Sec. 3 the Lax theory for difference schemes (Thm. 1; 3 of Sec. 2) is applied to some examples of the form

$$E_\tau = \sum_{k=0}^{\infty} \varphi_k(\lambda) T_h^k \quad (\lambda := \tau/h),$$

where $T_h f(x) := f(x+h)$. For explicit difference schemes (Ex. A) the series is finite, whereas for implicit difference schemes (Ex. B and C) the series may be infinite (compare [5], [6]). Stability and consistency properties are given in terms of the (positive) functions $\varphi_k(\lambda)$. As mentioned above, special choices of the $\varphi_k(\lambda)$ lead to Bernstein polynomials, Baskakov operators, and the operators of Szász—Mirakyan. In Sec. 4 we consider the same examples from the point of view of the continuous semigroups $\{E_\tau^{t/\tau}; t \geqq 0\}$ which interpolate the discrete ones $\{E_\tau^n; n \in \mathbf{P}\}$ used so far at the grid points $n\tau$. In this situation the continuous versions of the Lax equivalence theorem with orders (cf. Thm. 2; 4 of Sec. 2) may be applied. Finally in Sec. 5, instead of reproducing the Ex. A—C via the interpolating semigroups $\{E_\tau^{t/\tau}; t \geqq 0\}$, one may consider the family of semigroups $\{\exp[t(E_\tau - I)]; t \geqq 0\}$ being a familiar construction in the course of the proof of the original Trotter theorem. In this case one obtains a comparison between a given difference scheme and the corresponding line method which in turn implies a comparison theorem between the Bernstein polynomials and the operators of Szász—Mirakyan.

Summarizing, the applications deliver pointwise direct approximation theorems for the Bernstein polynomials, the Baskakov, and the Szász—Mirakyan operators which are best possible, apart from constants. Though these direct theorems as such are of course well-known, they do not only show interesting interconnections between the Lax theorem in numerical analysis and the convergence of some classical approximation processes but they also indicate that the notions and results of the abstract theory in Sec. 2 seem to be adequate.

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2. General theory

Let X be a Banach space (with norm $\|\cdot\|_X$) and $[X]$ the space of bounded linear operators of X into itself. A (continuous) semigroup $\{E(t); t \geqq 0\} \subset [X]$ (of class (C_0)) is a one parameter family of operators satisfying: $E(0)=I$, the identity, $E(t_1+t_2)=E(t_1)E(t_2)$, and $\lim_{t\to 0+} \|E(t)f-f\|_X=0$ for each $f \in X$. For a semigroup of class (C_0) there exist constants $M \geqq 1$, $\omega \geqq 0$ such that (for the fundamentals of semigroup theory see [7])

$$(2.1) \qquad \|E(t)\|_{[X]} \leqq Me^{\omega t} \quad (t \geqq 0).$$

Consider the initial value problem

$$(2.2) \qquad d/dt\, u(t) = Au(t) \quad \text{for} \quad t \geqq 0; \quad u(0) = f \quad \text{for} \quad f \in X,$$

where A is a closed linear operator with domain $D(A)$ dense in X and range in X, the given element f describing the initial state. The problem (2.2) is said to be properly (or correctly) posed if there exists a (continuous) semigroup $\{E(t); t \geqq 0\}$ (of class (C_0)) such that each solution of (2.2) is of the form $u(t)=E(t)f$. In this case, A is the infinitesimal generator of the semigroup, i.e. the closed linear operator defined densely in X via

$$Af := \lim_{t\to 0+} t^{-1}[E(t)-I]f,$$

the domain $D(A)$ consisting of all elements $f \in X$ for which the limit exists.

In numerical analysis one is now interested in approximating the family of "exact" operators $\{E(t)\}$ by powers of some finite difference operators $\{E_\tau; 0 \leqq \tau \leqq \delta\} \subset [X]$, in particular to treat the error $\|E_\tau^n f - E(n\tau)f\|_X$ in dependence upon smoothness properties of $f \in X$. In this connection the most important properties of the difference scheme are stability and consistency for which the following definitions (with orders) were used in [8] (see also the literature cited there):

Definition 1. The difference scheme $\{E_\tau; 0 \leqq \tau \leqq \delta\} \subset [X]$ is said to be *consistent of order* $O(\varphi(\tau))$ on the linear manifold $U \subset X$ with respect to the semigroup $\{E(t); t \geqq 0\}$ if there is a constant $C > 0$ such that for all $f \in U$, $t \geqq 0$, $0 \leqq \tau \leqq \delta$

$$(2.3) \qquad \|[E_\tau - E(\tau)]E(t)f\|_X \leqq C\tau\varphi(\tau)e^{\omega t}|f|_U$$

where $|f|_U$ denotes a suitable seminorm on U. If U is dense in X and $\varphi(\tau)$ in (2.3) is replaced by $o(1)$, $\tau \to 0+$, the difference scheme is said to be (ordinarily) *consistent.*

Definition 2. The difference scheme $\{E_\tau; 0 \leqq \tau \leqq \delta\} \subset [X]$ is said to be *stable of order* $O(\psi(\tau, 1/n))$ if there is a constant $S > 0$ such that for all $n \in \mathbf{N}$ (=set of natural numbers), $0 \leqq \tau \leqq \delta$

$$(2.4) \qquad \|E_\tau^n\|_{[X]} \leqq S/\psi(\tau, 1/n).$$

and (ordinarily) *stable* if the right-hand side of (2.4) is replaced by $O(1)$, $\tau \to 0+$.

Here $\varphi(\tau)$ is some non-negative function on $[0, \delta]$ and $\psi(\tau, y)$ a positive bounded function on $[0, \delta]\times[0, 1]$ monotonely increasing in y and normalized via (cf. (2.1))

(2.5) $$Me^{\omega n\tau} \leqq S/\psi(\tau, 1/n) \quad (0 \leqq \tau \leqq \delta, n\in\mathbf{N}).$$

In this terminology the Lax theorem in its original form reads (see [17], [19])

Theorem L 1 (discrete version). *Given the properly posed initial value problem (2.2) in X and a finite difference scheme $\{E_\tau; 0\leqq\tau\leqq\delta\}$ satisfying the (ordinary) consistency condition, then (ordinary) stability is necessary and sufficient for (ordinary) convergence, i.e. for each $f\in X$*

$$\lim_{j\to\infty} \|E_{\tau_j}^{n_j}f - E(t)f\|_X = 0$$

for each sequence $\{(n_j, \tau_j)\}_{j\in\mathbf{N}}$ with $\tau_j\to 0+$, $n_j\tau_j\to t<\infty$ as $j\to\infty$.

Following [8], [10] one can equip this equivalence theorem with orders, smoothness properties of the element $f\in X$ being measured in terms of the so-called modified K-functional $(t\geqq 0)$

(2.6) $$K(t,f) := K(t,f; X, U) := \inf_{g\in U}\{\|f-g\|_X + t|g|_U\}.$$

This is known to be a continuous and monotonely increasing function of t with $\lim_{t\to 0+} K(t, f)=0$ for all $f\in X$ if U is dense in X. One also has, in view of the definition,

(2.7) $$K(t, f) \leqq \begin{cases} \|f\|_X, & f\in X \\ t|f|_U, & f\in U. \end{cases}$$

Theorem 1. *Let the finite difference scheme $\{E_\tau; 0\leqq\tau\leqq\delta\}$ be consistent of order $O(\varphi(\tau))$ on $U\subset X$ with respect to the semigroup $\{E(t); t\geqq 0\}$. Then the following assertions are equivalent:*

(a) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau, 1/n)} K((C/2)n\tau e^{\omega n\tau}\varphi(\tau), f),$$

(b) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau, 1/n)} \begin{cases} M_f, & f\in X \\ (C/2)n\tau e^{\omega n\tau}\varphi(\tau)|f|_U, & f\in U, \end{cases}$$

(c) $$\|E_\tau^n\|_{[X]} \leqq S/\psi(\tau, 1/n),$$

where M_f is a constant only depending on f (there is a slight abuse of the constants S).

Proof. The implication (a)⇒(b) follows by (2.7). Moreover, by the uniform boundedness principle one may replace the constant M_f by $C_1\|f\|_X$ for some $C_1>0$. Together with (2.1) and (2.5) this shows (b)⇒(c). Concerning the proof (c)⇒(a), in view of the identity

(2.8) $$E_\tau^n g - E(n\tau)g = \sum_{j=0}^{n-1} E_\tau^{n-j-1}[E_\tau - E(\tau)]E(j\tau)g$$

one has for any $g\in U$

$$\|E_\tau^n g - E(n\tau)g\|_X \leqq \sum_{j=0}^{n-1} \big(S/\psi(\tau,(n-j-1)^{-1})\big) C\tau\varphi(\tau)e^{\omega j\tau}|g|_U \leqq \frac{SCn\tau}{\psi(\tau,1/n)} e^{\omega n\tau}\varphi(\tau)|g|_U,$$

using stability and consistency with orders. Hence for any $f\in X$ it follows by (2.4), (2.5) that for any $g\in U$

$$\|E_\tau^n f - E(n\tau)f\|_X \leqq \|E_\tau^n(f-g)\|_X + \|E(n\tau)(f-g)\|_X + \|E_\tau^n g - E(n\tau)g\|_X \leqq$$

$$\leqq \left[\frac{S}{\psi(\tau,1/n)} + Me^{\omega n\tau}\right]\|f-g\|_X + \frac{SCn\tau}{\psi(\tau,1/n)} e^{\omega n\tau}\varphi(\tau)|g|_U \leqq$$

$$\leqq \frac{2S}{\psi(\tau,1/n)}\{\|f-g\|_X + (C/2)n\tau e^{\omega n\tau}\varphi(\tau)|g|_U\}.$$

Taking the infimum over all $g\in U$ yields (a). This completes the proof.

Up to this stage we approximated the exact solution $\{E(t);t\geqq 0\}$ by some difference scheme, thus by some family $(0\leqq\tau\leqq\delta)$ of *discrete* semigroups $\{E_\tau^n;n\in\mathbf{P}\}$. Now we want to approximate the "exact" operators by a family of *continuous* semigroups $\{E_\tau(t);t\geqq 0, 0\leqq\tau\leqq\delta\}\subset[X]$ of class (C_0). Indeed, the most important properties determining the approximation error $\|E_\tau(t)f-E(t)f\|_X$ are very similar to those given in Def. 1; 2. So one may formulate

Definition 3. The semigroup scheme $\{E_\tau(t);t\geqq 0, 0\leqq\tau\leqq\delta\}$ with infinitesimal generators A_τ is said to be *consistent of order* $O(\varphi(\tau))$ on the linear manifold $U\subset X$ with respect to the semigroup $\{E(t);t\geqq 0\}$ with generator A if $E(t)U\subset\subset D(A)\cap D(A_\tau)$ and there exists a constant $C>0$ such that for all $f\in U$, $t\geqq 0$, $0\leqq\tau\leqq\delta$

$$\|[A_\tau - A]E(t)f\|_X \leqq Ce^{\omega t}\varphi(\tau)|f|_U. \tag{2.9}$$

It is said to be (ordinarily) *consistent* if U is dense in X and $\varphi(\tau)$ in (2.9) is replaced by $o(1)$.

Definition 4. The semigroup scheme $\{E_\tau(t);t\geqq 0, 0\leqq\tau\leqq\delta\}$ is said to be *stable of order* $O(M_\tau e^{\omega_\tau t})$ if there are constants M_τ and ω_τ with $M\leqq M_\tau$ and $\omega\leqq\omega_\tau$ (cf. (2.1)) such that for all $t\geqq 0$, $0\leqq\tau\leqq\delta$

$$\|E_\tau(t)\|_{[X]} \leqq M_\tau e^{\omega_\tau t}. \tag{2.10}$$

It is said to be (ordinarily) *stable* if $M_\tau\leqq M_0<\infty$ and $\omega_\tau\leqq\omega_0<\infty$.

Of course, since $\{E_\tau(t);t\geqq 0\}$ is assumed to be a semigroup of class (C_0) for each $0\leqq\tau\leqq\delta$, property (2.1) always ensures the existence of constants M_τ, ω_τ such that (2.10) holds. So Def. 4 just states that it is appropriate to take $M_\tau e^{\omega_\tau t}$ as a substitute for $S/\psi(\tau,1/n)$ in (2.4).

In the above terminology one has the following continuous counterpart to Theorem L 1 (cf. [2], [20]):

Theorem L 2 (continuous version). *Let* $\{E_\tau(t);\, t\geqq 0,\, 0\leqq\tau\leqq\delta\}$ *be a semigroup scheme (ordinarily) consistent with respect to* $\{E(t);\, t\geqq 0\}$. *Then (ordinary) stability is necessary and sufficient for (ordinary) convergence, i.e. for each* $f\in X$, $t\geqq 0$

$$\lim_{\tau\to 0+} \|E_\tau(t)f-E(t)f\|_X = 0.$$

Again this convergence theorem can be equipped with orders.

Theorem 2. *Let the semigroup scheme* $\{E_\tau(t);\, t\geqq 0,\ 0\leqq\tau\leqq\delta\}$ *be consistent of order* $O(\varphi(\tau))$ *on* $U\subset X$ *with respect to the semigroup* $\{E(t);\, t\geqq 0\}$. *Then the following assertions are equivalent:*

$$\text{(a)} \qquad \|E_\tau(t)f-E(t)f\|_X \leqq 2M_\tau e^{\omega_\tau t} K((C/2)t\varphi(\tau), f),$$

$$\text{(b)} \qquad \|E_\tau(t)f-E(t)f\|_X \leqq 2M_\tau e^{\omega_\tau t}\begin{cases} M_f, & f\in X\\ (C/2)t\varphi(\tau)|f|_U, & f\in U,\end{cases}$$

$$\text{(c)} \qquad \|E_\tau(t)\|_{[X]} \leqq M_\tau e^{\omega_\tau t}.$$

Proof. By (2.7) we immediately obtain (a)⇒(b). Then by the uniform boundedness principle one may replace the constant M_f by $C_1\|f\|_X$ which together with (2.1) implies (b)⇒(c). For the proof of (c)⇒(a) it follows that for arbitrary $g\in U\subset$ $\subset D(A)\cap D(A_\tau)$

$$E_\tau(t)g-E(t)g = -\int_0^t \frac{d}{ds}E_\tau(t-s)E(s)g\,ds =$$

$$= \int_0^t [A_\tau E_\tau(t-s)E(s)g - E_\tau(t-s)AE(s)g]\,ds = \int_0^t E_\tau(t-s)[A_\tau - A]E(s)g\,ds$$

which should be compared with (2.8), thus with (2.3) and (2.9), respectively. Hence

$$\|E_\tau(t)g-E(t)g\|_X \leqq \int_0^t M_\tau e^{\omega_\tau(t-s)}Ce^{\omega s}\varphi(\tau)|g|_U\,ds \leqq M_\tau Cte^{\omega_\tau t}\varphi(\tau)|g|_U.$$

As in the proof of Thm. 1 we proceed for $f\in X$, $g\in U$

$$\begin{aligned}\|E_\tau(t)f-E(t)f\|_X &\leqq \|E_\tau(t)(f-g)\|_X + \|E(t)(f-g)\|_X + \|E_\tau(t)g-E(t)g\|_X \leqq\\ &\leqq (M_\tau e^{\omega_\tau t}+Me^{\omega t})\|f-g\|_X + M_\tau Cte^{\omega_\tau t}\varphi(\tau)|g|_U \leqq\\ &\leqq 2M_\tau e^{\omega_\tau t}\{\|f-g\|_X + (C/2)t\varphi(\tau)|g|_U\}.\end{aligned}$$

Taking the infimum over all $g\in U$ completes the proof.

So far Thms. 1, 2 do have the structure of the original Lax equivalence theorem, stating that stability is equivalent to convergence, provided the scheme is consistent. The adequacy of the notions with orders used above may also be illustrated by the fact that the alternative form is valid as well, namely that convergence is equivalent to stability plus consistency, provided some weak additional assumptions are made. First we claim the commutativity of seminorm and semigroup, more specifically, we suppose that $E(t)U \subset U$ and (cf. (2.1))

$$|E(t)g|_U \leqq Me^{\omega t}|g|_U \quad (t \geqq 0) \tag{2.11}$$

for each $g \in U$ (in [18a] problem (2.2) is then said to be strongly correctly posed). For example, inequality (2.11) obviously holds if $|g|_U := \|A^r g\|_X$, $U = D(A^r)$.

Theorem 3. *Given the finite difference scheme $\{E_\tau; 0 \leqq \tau \leqq \delta\}$ and the semigroup $\{E(t); t \geqq 0\}$, suppose that* (2.11) *be valid and $\psi(\tau, 1) \geqq C_2 > 0$ for all $0 \leqq \tau \leqq \delta$. Then the following assertions are equivalent:*

(a) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau, 1/n)} K((C/2)n\tau e^{\omega n\tau}\varphi(\tau), f),$$

(b) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau, 1/n)} \begin{cases} M_f, & f \in X \\ (C/2)n\tau e^{\omega n\tau}\varphi(\tau)|f|_U, & f \in U, \end{cases}$$

(c) (i) $\|E_\tau^n\|_{[X]} \leqq S/\psi(\tau, 1/n)$,

(ii) $\|[E_\tau - E(\tau)]E(t)f\|_X \leqq Ce^{\omega t}\tau\varphi(\tau)|f|_U$ *for all* $f \in U, t \geqq 0, 0 \leqq \tau \leqq \delta$.

For a proof one may consult [8].

Theorem 4. *Given the semigroup scheme $\{E_\tau(t);\ t \geqq 0,\ 0 \leqq \tau \leqq \delta\}$ and the semigroup $\{E(t); t \geqq 0\}$, suppose that inequality* (2.11) *be valid and $M_\tau \leqq M_0 < \infty$ for all $0 \leqq \tau \leqq \delta$. Then the following assertions are equivalent:*

(a) $$\|E_\tau(t)f - E(t)f\|_X \leqq 2M_\tau e^{\omega_\tau t} K((C/2)t\varphi(\tau), f),$$

(b) $$\|E_\tau(t)f - E(t)f\|_X \leqq 2M_\tau e^{\omega_\tau t} \begin{cases} M_f, & f \in X \\ (C/2)t\varphi(\tau)|f|_U, & f \in U, \end{cases}$$

(c) (i) $\|E_\tau(t)\|_{[X]} \leqq M_\tau e^{\omega_\tau t}$,

(ii) $\|[A_\tau - A]E(t)f\|_X \leqq Ce^{\omega t}\varphi(\tau)|f|_U$ *for all* $f \in U, t \geqq 0, 0 \leqq \tau \leqq \delta$.

Proof. In view of the proof of Thm. 2 we only need to show (b)$\Rightarrow$(c, ii). Let $f = E(s)g$ for some $g \in U$, $s \geqq 0$. Then (b) and (2.11) imply

$$\|E_\tau(t)f - E(t)f\|_X \leqq M_\tau e^{\omega_\tau t} Ct\varphi(\tau)|E(s)g|_U \leqq MM_\tau e^{\omega_\tau t} Ct\varphi(\tau)e^{\omega s}|g|_U.$$

Therefore one has

$$\|[A_\tau - A]E(s)g\|_X = \lim_{t \to 0+} \left\|[t^{-1}[E_\tau(t) - I] - t^{-1}[E(t) - I]]E(s)g\right\|_X \leqq$$

$$\leqq MM_\tau C\varphi(\tau)e^{\omega s}|g|_U \leqq C^*\varphi(\tau)e^{\omega s}|g|_U$$

which completes the proof.

3. Applications to specific difference schemes

An example of an initial value problem (2.2) is supplied by the hyperbolic differential equation

$$(3.1)\qquad d/dt\ u(x,t)=d/dx\ u(x,t),\quad x,t\geqq 0;\quad u(x,0)=f(x),\quad x\geqq 0,$$

where f is an element of $X:=UCB(\mathbf{R}^+)$, the Banach space of all bounded, uniformly continuous functions on $[0,\infty)$ with $\|f\|_X:=\sup_{x\geqq 0}|f(x)|$. This problem is properly posed, the solution operators $E(t)$ being given via

$$(3.2)\qquad u(x,t)=E(t)f(x)=f(x+t),\quad x,t\geqq 0.$$

Let us consider some examples of difference schemes applied in numerical analysis to approximate the exact solution (3.2). We use the notations

$$(3.3)\qquad (T_hf)(x):=f(x+h),\quad (E_\tau u)(x,t):=u(x,t+\tau),\quad \lambda:=\tau/h$$

for the translation operator T_h, $h\geqq 0$, the step operator E_τ, $\tau\geqq 0$, and the ratio $\lambda\geqq 0$ of the step sizes, respectively.

Example A. Instead of (3.1) we regard the problem

$$\tau^{-1}[u(x,t+\tau)-u(x,t)]=h^{-1}[u(x+h,t)-u(x,t)].$$

This defines an explicit difference scheme with step operator

$$(3.4)\qquad E_\tau=(1-\lambda)I+\lambda T_h.$$

Example B. If we replace (3.1) by

$$\tau^{-1}[u(x,t+\tau)-u(x,t)]=h^{-1}[u(x+h,t+\tau)-u(x,t+\tau)],$$

the step operator is defined via

$$E_\tau-I=\lambda[T_h-I]E_\tau.$$

This leads to the implicit difference scheme

$$(3.5)\qquad E_\tau=\frac{1}{1+\lambda}\left[I-\frac{\lambda}{1+\lambda}T_h\right]^{-1}=\frac{1}{1+\lambda}\sum_{k=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^k T_h^k.$$

Example C. Replacing only d/dx in (3.1) by the corresponding difference quotient, one has to solve the initial value problem

$$(3.6)\qquad d/dt\,u(x,t)=h^{-1}[u(x+h,t)-u(x,t)].$$

This line or semi-discrete method (cf. [21, p. 545] or [6, p 55]) leads to the step operator (see also Sec. 4)

(3.7) $$E_\tau = \exp(\tau/h)[T_h - I] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} T_h^k.$$

Obviously, each of these operators E_τ is of the form

(3.8) $$E_\tau = \sum_{k=0}^{\infty} \varphi_k(\lambda) T_h^k$$

with certain real-valued functions $\varphi_k(\lambda)$ defined on $[0, \infty)$. To discuss stability and consistency, let us suppose that there exists an interval $J \subset [0, \infty)$ such that for all $\lambda \in J$, $k \in \mathbf{P}$

(3.9) $$\text{(i)}\quad \varphi_k(\lambda) \geqq 0, \quad \text{(ii)}\quad \sum_{k=0}^{\infty} \varphi_k(\lambda) = 1, \quad \text{(iii)}\quad \sum_{k=0}^{\infty} k\varphi_k(\lambda) = \lambda \quad (\lambda \in J).$$

In particular, (i) assumes the positivity of the operators E_τ which together with (ii) leads to stability since

$$\|E_\tau^n\|_{[X]} \leqq \|E_\tau\|_{[X]}^n = \|E_\tau 1\|_X^n = 1.$$

Concerning consistency let

$$U := UCB^{(2)} := \{f \in X;\ f', f'' \in X\}, \quad |f|_U := \|f''\|_X.$$

Then one has by (3.2), (3.8), (3.9) that for every $f \in U$

(3.10) $$|E_\tau u(x, t) - E(\tau)u(x, t)| = \left|\sum_{k=0}^{\infty} \varphi_k(\lambda)[u(x+kh, t) - u(x, t+\tau)]\right| =$$

$$= \left|\sum_{k=0}^{\infty} \varphi_k(\lambda)\big[f\big(x+t+\tau+(kh-\tau)\big) - f(x+t+\tau)\big]\right| =$$

$$= \left|\sum_{k=0}^{\infty} \varphi_k(\lambda)\Big[(kh-\tau)f'(x+t+\tau) + \int_0^{kh-\tau}\int_0^{s} f''(v+x+t+\tau)\,dv\,ds\Big]\right| \leqq$$

$$\leqq \|f''\|_X \sum_{k=0}^{\infty} \varphi_k(\lambda)(kh-\tau)^2/2 = (h^2/2)\sigma(\lambda)\|f''\|_X$$

with second moment $\sigma(\lambda)$ given via

$$\sigma(\lambda) := \sum_{k=0}^{\infty} \varphi_k(\lambda)(k-\lambda)^2 = \sum_{k=0}^{\infty} k^2\varphi_k(\lambda) - \lambda^2.$$

Before giving an application of Thm. 1, let us recall that for the present choices of spaces X, U one may express the K-functional $K(t, f; UCB, UCB^{(2)})$ equivalently in terms of moduli of continuity. Indeed, one has for any $t \geqq 0$ (cf. [7, p. 192; 258], [9, p. 316])

$$c_1\omega_2(t^{1/2}, f) \leqq K(t, f) \leqq c_2\omega_2(t^{1/2}, f) \tag{3.11}$$

where the (second) modulus of continuity is defined by

$$\omega_2(t, f) := \sup_{0 \leqq h \leqq t} \|f(x+2h)-2f(x+h)+f(x)\|_X.$$

Thus it follows by Thm. 1, (c)⇒(a), that

Corollary 1. *Concerning the convergence of the difference scheme* (3.8) *towards the exact solution* (3.2) *of the initial value problem* (3.1) *one has*

$$|E_\tau^n f(x)-f(x+n\tau)| \leqq 2c_2\omega_2((h/2)[n\sigma(\lambda)]^{1/2}, f) \tag{3.12}$$

for any $f \in UCB(\mathbf{R}^+)$, $x \geqq 0$, $n \in \mathbf{N}$, $\tau \geqq 0$, *and* $\lambda \in J$.

More specifically, this yields for the examples mentioned above:

Example A: In view of (3.4) we see that

$$\varphi_0(\lambda) = 1-\lambda, \quad \varphi_1(\lambda) = \lambda, \quad \varphi_k(\lambda) = 0 \quad \text{for} \quad k \geqq 2.$$

Thus (cf. (3.9) (i)) one has $J=[0, 1]$ and $\sigma(\lambda)=\lambda(1-\lambda)$. Since for $x=0$, $h=1/n$

$$[E_\tau^n f](0) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^{n-k}\lambda^k T_{1/n}^k f(0) = \sum_{k=0}^{\infty} \binom{n}{k}(1-\lambda)^{n-k}\lambda^k f(k/n) = B_n(f, \lambda),$$

Cor. 1 implies the following (pointwise) direct theorem for the Bernstein polynomials.

Corollary 2. *For any function* f, *continuous on* $[0, 1]$, *one has for each* $\lambda \in [0, 1]$, $n \in \mathbf{N}$

$$|B_n(f, \lambda)-f(\lambda)| \leqq c\omega_2([\lambda(1-\lambda)/n]^{1/2}, f).$$

The present procedure to prove this well-known direct estimate (cf. [4, p. 698], [12], and the literature cited there) is essentially contained in [14] (explicitly they prove the Weierstrass convergence theorem for twice differentiable functions, the domain $x, t \in [0, \infty)$ (cf. (3.1)) being replaced by $x, t, x+t \in [0, 1]$). The argument was then refined in [15] in order to obtain an error estimate involving the first modulus of continuity of the first derivative f'.

Example B. In view of (3.5) we see that

$$\varphi_k(\lambda) = \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^k \quad (k \geqq 0).$$

Consequently, one has $J=[0, \infty)$ and $\sigma(\lambda)=\lambda(1+\lambda)$. Since for $x=0$, $h=1/n$

$$[E_\tau^n f](0) = \frac{1}{(1+\lambda)^n}\left[\left(I-\frac{\lambda}{1+\lambda}T_h\right)^{-n} f\right](0) =$$

$$= \frac{1}{(1+\lambda)^n}\sum_{k=0}^{n}\binom{n+k-1}{k}\left(\frac{\lambda}{1+\lambda}\right)^k f(k/n) = M_n(f, \lambda),$$

Cor. 1 yields the following direct estimate for the Baskakov operators $M_n(f, \lambda)$.

Corollary 3. *If $f\in UCB(\mathbf{R}^+)$, then for any $n\in\mathbf{N}$, $\lambda \geqq 0$*

$$|M_n(f, \lambda)-f(\lambda)| \leqq c\omega_2([\lambda(1+\lambda)/n]^{1/2}, f).$$

As is well-known (cf. [1], [11, p. 39]), this is the correct estimate, apart from constants.

Example C. Here we see from (3.7) that

$$\varphi_k(\lambda) = e^{-\lambda}\lambda^k/k! \quad (k \geqq 0).$$

Thus $J=[0, \infty)$ and $\sigma(\lambda)=\lambda$. Since for $x=0$, $h=1/n$

$$[E_\tau^n f](0) = \exp[n\lambda(T_h-I)]f(0) = e^{-n\lambda}\sum_{k=0}^{\infty}\frac{(n\lambda)^k}{k!}f(k/n) = S_n(f, \lambda),$$

Cor. 1 delivers the following (pointwise) direct estimate for the operators of Szász—Mirakyan.

Corollary 4. *For any $f\in UCB(\mathbf{R}^+)$, $\lambda\geqq 0$, $n\in\mathbf{N}$ one has*

$$|S_n(f, \lambda)-f(\lambda)| \leqq c\omega_2([\lambda/n]^{1/2}, f).$$

Again this is the correct estimate apart from constants.

Regarding Cor. 2—4, let us again point out that these (pointwise) direct approximation theorems for the Bernstein polynomials, the Baskakov, and Szász—Mirakyan operators, respectively, are of course well-known. In fact, these results may be obtained even more directly and elementarily exploiting (cf. (3.10)) the second moment of the kernel (cf. [11, p. 39; 244], see also [3] for more intricate results in polynomial weight spaces). Concerning this note, however, they do not only show interesting interconnections between the Lax theorem in numerical analysis and the convergence of some classical approximation processes but also indicate that the notions and results of the abstract theory in Sec. 2 seem to be adequate.

4. Applications to semigroup schemes

Let us regard Ex. C from another point of view. In order to obtain the difference scheme (3.7) one has to solve the initial value problem (3.6) for one time step $t=\tau$. Looking at the solution of (3.6) for any $t\geqq 0$, however, delivers (continuous) semigroups $\{E_\tau(t); t\geqq 0\}$, $\tau:=\lambda h\geqq 0$. Then (3.6) takes the form

$$d/dt\, E_\tau(t)f = h^{-1}[T_h-I]E_\tau(t)f.$$

Thus the infinitesimal generators A_τ of these semigroups are given via the bounded linear operators

(4.1) $$A_\tau = h^{-1}[T_h-I]$$

so that one has

(4.2) $$E_\tau(t) = \exp(tA_\tau) = e^{-t/h}\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!}T_h^k.$$

Obviously, for any $f\in U\,(:=UCB^{(2)}(\mathbf{R}^+))$

(4.3) $$A_\tau f(x) = f'(x)+\frac{1}{h}\int_0^h\int_0^s f''(x+v)\,dv\,ds.$$

Of course the infinitesimal generator of the solution semigroup $\{E(t); t\geqq 0\}$ (cf. (3.2)) is given via $Af(x)=f'(x)$. Moreover, since the present generators A_τ commute with $E(t)$, one has

(4.4) $$\|(A_\tau-A)E(t)f\|_X \leqq \|E(t)\|_{[X]}\|A_\tau f-Af\|_X = \|A_\tau f-Af\|_X.$$

Therefore the error of consistency (2.9) for the semigroup scheme (4.2) may be estimated by

$$\|(A_\tau-A)E(t)f\|_X \leqq (h/2)\|f''\|_X$$

for any $f\in U$ whereas stability follows from

$$\|E_\tau(t)\|_{[X]} \leqq e^{-t/h}\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!} = 1.$$

Thus an application of Thm. 2, (c)⇒(a), regains Cor. 1; 4, namely (with $K(t,f):=$ $=K(t,f;\,UCB,\,UCB^{(2)})$)

Corollary 5. *For the semigroup scheme* (4.2) *one has*

$$|E_\tau(t)f(x)-f(x+t)| \leqq 2K(ht/4,f) \quad \textit{for any}\quad f\in UCB,\, x, t, \tau, h\geqq 0.$$

More generally, given a difference scheme $\{E_\tau; 0\leqq\tau\leqq\delta\}\subset[X]$, to each discrete semigroup $\{E_\tau^n; n\in\mathbf{P}\}$ one may associate a continuous one $\{E_\tau(t); t\geqq 0\}$ according to the formula (cf. (3.7), (4.2))

(4.5) $$E_\tau(t) := E_\tau^{t/\tau}$$

in case the right-hand side can be interpreted suitably. The resulting continuous semigroup then has the interpolation property

$$E_\tau(n\tau) = E_\tau^n. \tag{4.6}$$

Let us continue with considering the matter in connection with Ex. *A*, *B*. First recall that for each $\alpha \in \mathbf{R}$

$$(1+u)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} u^k \tag{4.7}$$

absolutely and uniformly for $|u|<1$ (and even for $|u|\leqq 1$ in case $\alpha>0$).

Example A. It follows from (3.4) that

$$E_\tau(t) := E_\tau^{t/\tau} := (1-\lambda)^{t/\tau}\left[I+\frac{\lambda}{1-\lambda}T_h\right]^{t/\tau} = (1-\lambda)^{t/\tau}\sum_{k=0}^{\infty}\binom{t/\tau}{k}\left(\frac{\lambda}{1-\lambda}\right)^k T_h^k, \tag{4.8}$$

the series being convergent in the uniform operator topology for $\lambda\in[0,1/2]$. For the corresponding infinitesimal generator A_τ one has

$$A_\tau f(x) = f'(x) - \frac{1}{\tau}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{-\lambda}{1-\lambda}\right)^k \int_0^{kh}\int_0^s f''(x+v)\,dv\,ds$$

for any $f\in U$. Therefore (cf. (4.4)) for any $f\in U$, $\lambda\in[0,1/2)$

$$\|[A_\tau - A]E(t)f\|_X \leqq \frac{h}{2\lambda}\sum_{k=0}^{\infty} k\left|\frac{-\lambda}{1-\lambda}\right|^k \|f''\|_X \leqq \frac{h}{2}\frac{1-\lambda}{1-2\lambda}\|f''\|_X.$$

Concerning stability, for some given $t\geqq 0$ let $m\in\mathbf{P}$ be such that $m\tau\leqq t = m\tau+\eta< <m\tau+\tau$. Then in view of the stability of the explicit difference scheme we see that (cf. (3.9), (4.5), (4.8))

$$\|E_\tau(t)\|_{[X]} \leqq \|E_\tau(m\tau)\|_{[X]}\|E_\tau(\eta)\|_{[X]} \leqq \|E_\tau(\eta)\|_{[X]} =$$
$$= (1-\lambda)^{\eta/\tau}\sum_{k=0}^{\infty}\left|\binom{\eta/\tau}{k}\left(\frac{\lambda}{1-\lambda}\right)^k\right| = 2(1-\lambda)^{\eta/\tau}-(1-2\lambda)^{\eta/t} \leqq 2.$$

Application of Thm. 2, (c)$\Rightarrow$(a), therefore gives

$$|E_\tau(t)f(x) - f(x+t)| \leqq 4K\left(\frac{ht(1-\lambda)}{4(1-2\lambda)}, f\right) \quad (\lambda\in[0,1/2))$$

which is worse than Cor. 1 or 2, respectively.

Example B. The interpolating semigroups (4.5) for the difference operators E_τ of (3.5) are given by

$$E_\tau(t) := E_\tau^{t/\tau} := \left(\frac{1}{1+\lambda}\right)^{t/\tau}\left(I-\frac{\lambda}{1+\lambda}T_h\right)^{-t/\tau} = \tag{4.9}$$
$$= \left(\frac{1}{1+\lambda}\right)^{t/\tau}\sum_{k=0}^{\infty}\binom{-t/\tau}{k}\left(\frac{-\lambda}{1+\lambda}\right)^k T_h^k$$

the series being convergent in $[X]$ for each $\lambda \geqq 0$. For the infinitesimal generators A_τ one has

$$A_\tau f(x) = f'(x) + \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\lambda}{1+\lambda}\right)^k \int_0^{kh} \int_0^s f''(x+v)\,dv\,ds$$

for any $f \in U$. It follows that for any $f \in U$, $\lambda \geqq 0$

$$\|[A_\tau - A]E(t)f\|_X \leqq \frac{h}{2\lambda} \sum_{k=0}^{\infty} k \left(\frac{\lambda}{1+\lambda}\right)^k \|f''\|_X = \frac{h}{2}(1+\lambda)\|f''\|_X.$$

Since $\|E_\tau(t)\|_{[X]} \leqq 1$, one has for the semigroup scheme (4.9) that

$$|E_\tau(t)f(x) - f(x+t)| \leqq 2K\big(ht(1+\lambda)/4, f\big)$$

which reproduces the results of Cor. 1; 3 (upon setting $t = n\tau = nh\lambda$).

Summarizing, Thm. 2; 4 seem to be more appropriate for line methods (cf. treatment of Ex. C) whereas Thm. 1; 3 seem to be more suitable for genuine difference schemes.

5. A comparison theorem

In the course of the proof of the familiar Trotter theorem (cf. [22], [16, p. 507 ff]) one makes use of just another method (than (4.5)) to associate a semigroup scheme $\{\tilde{E}_\tau(t); t \geqq 0\}$ to some given difference scheme $\{E_\tau\}$, $0 < \tau \leqq \delta$. Indeed, with the step operator $E_\tau \in [X]$ one also has $B_\tau := (E_\tau - I)/\tau \in [X]$ so that via $(t \geqq 0)$

$$\tilde{E}_\tau(t) := \exp(tB_\tau) := \sum_{k=1}^{\infty} (t^k/k!) B_\tau^k \tag{5.1}$$

there is defined a (continuous) semigroup of class (C_0) for each $0 < \tau \leqq \delta$. Though $\{\tilde{E}_\tau(t); t \geqq 0\}$ does not have the interpolation property (4.6), one has (cf. [16, p. 508]):

Lemma 1. *With $\{E_\tau\} \subset [X]$, $0 < \tau \leqq \delta$, let $\tilde{E}_\tau(t)$ be given via (5.1). If there exist constants M_τ such that $\|E_\tau^n\|_{[X]} \leqq M_\tau$ uniformly for $n \in \mathbf{P}$, then also $\|\tilde{E}_\tau(t)\|_{[X]} \leqq M_\tau$ uniformly for $t \geqq 0$ and*

$$\|E_\tau^n f - \tilde{E}_\tau(n\tau)f\|_X \leqq (1/2) M_\tau n\tau^2 \|B_\tau^2 f\|_X \quad \text{for every} \quad f \in X, n \in \mathbf{P}, 0 < \tau \leqq \delta.$$

This may be interpreted as a comparison theorem between a given difference scheme and the corresponding line method. Whenever a discretization of (2.2) is given via

$$\frac{1}{\tau}[E_\tau - I]u(t) = B_\tau u(t)$$

with difference scheme $\{E_\tau;\, 0<\tau\leqq\delta\}$ satisfying the stability condition $\|E_\tau^n\|_{[X]}\leqq M_\tau$, then the approximation error can be estimated according to

(5.2) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \|\tilde{E}_\tau(n\tau)f - E(n\tau)f\|_X + (1/2)M_\tau n\tau^2\|B_\tau^2 f\|_X$$

where $\tilde{E}_\tau(t):=\exp(tB_\tau)$ denotes the line method defined by

(5.3) $$d/dt\, u(t) = B_\tau u(t).$$

Concerning Ex. A, the operators B_τ are given by

$$B_\tau := \frac{1}{\tau}[E_\tau - I] = \frac{1}{h}[T_h - I]$$

which are just the infinitesimal generators of the semigroup scheme in Ex. C. Thus $\tilde{E}_\tau(t)$ is equal to $E_\tau(t)$ from (4.2). Since $\|B_\tau^2 f\|_X\leqq\|f''\|_X$ for any $f\in U$, in view of the stability and Lemma 1 this leads to (cf. proof of Thm. 1, (c)$\Rightarrow$(a))

$$\|E_\tau^n f - E_\tau(n\tau)f\|_X \leqq 2K(n\tau^2/4, f)$$

with E_τ from (3.4) and $E_\tau(t)$ from (4.2). Therefore, proceeding as in the previous sections, one obtains

Corollary 6. *For $f\in UCB$ one has the following comparison estimate between the Bernstein polynomials and operators of Szász—Mirakyan:*

$$|B_n(f,\lambda) - S_n(f,\lambda)| \leqq 2K(\lambda^2/4n, f) \quad \textit{for all} \quad \lambda\in[0,1],\, n\in\mathbf{N}.$$

Thus, though the individual operators behave like $O(\lambda)$ at $\lambda=0+$, their difference behaves like $O(\lambda^2)$.

Added in proof: For a (parallel to [14]) concrete discussion of the pure convergence of the Bernstein and Baskakov operators in connection with the explicit and implicit difference scheme of Ex. A, B, respectively, see also G. C. Papanicolau, *Amer. Math. Monthly,* **82** (1975), 674—676.

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On automorphisms of the subalgebra lattice induced by automorphisms of the algebra

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1. Introduction. We are going to prove the following result:

Theorem. *Let G be a group, L an algebraic lattice with more than one element, and let φ be a homomorphism of G into* $\operatorname{Aut} L$. *Then there exists an algebra $\mathfrak{A}$ such that there are isomorphisms $\alpha: G \to \operatorname{Aut} \mathfrak{A}$ and $\beta: L \to \operatorname{Sub} \mathfrak{A}$ satisfying* (see Figure) $\alpha\varphi_{\mathfrak{A}} = \varphi \operatorname{Aut} \beta$, *where* $\operatorname{Aut} \beta$ *is the isomorphism of* $\operatorname{Aut} L$ *and* $\operatorname{Aut} \operatorname{Sub} \mathfrak{A}$ *induced by β.*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & \operatorname{Aut} L \arrow[d, "\operatorname{Aut} \beta"] \\ \operatorname{Aut} \mathfrak{A} \arrow[r, "\varphi_{\mathfrak{A}}"'] & \operatorname{Aut} \operatorname{Sub} \mathfrak{A} \end{tikzcd}$$

To put it simply, $\langle \operatorname{Aut} \mathfrak{A}, \operatorname{Sub} \mathfrak{A}, \varphi_{\mathfrak{A}} \rangle$ is characterized as $\langle G, L, \varphi \rangle$. The exception is that we have to assume that $|L| > 1$. Indeed, if $|L| = 1$, then A is the only subalgebra of $\mathfrak{A}$, that is, every element is an algebraic constant. In this case, $|G| = 1$. Thus $\langle \operatorname{Aut} \mathfrak{A}, \operatorname{Sub} \mathfrak{A}, \varphi_{\mathfrak{A}} \rangle$ is just as independent as $\langle \operatorname{Aut} \mathfrak{A}, \operatorname{Sub} \mathfrak{A} \rangle$ is.

Corollary. (E. T. SCHMIDT [7]) *Given a group G and an algebraic lattice L with more than one element, there exists an algebra $\mathfrak{A}$ satisfying $G \cong \operatorname{Aut} \mathfrak{A}$ and* $L \cong \operatorname{Sub} \mathfrak{A}$.

Proof. Let φ map all of G into the identity element of $\operatorname{Aut} L$. Then the algebra $\mathfrak{A}$ we obtain from the Theorem yields the Corollary.

This Corollary contains earlier results of G. BIRKHOFF [1] characterizing automorphis groups of algebras and of G. BIRKHOFF and O. FRINK [2] characterizing the subalgebra lattices of algebras.

It may be of some interest to note that in Schmidt's construction φ is indeed the constant map. If in our proof φ is the constant map, we obtain a somewhat simplified proof of Schmidt's result.

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2. The construction. Let G, L, and φ be given as in the Theorem. Let C be the set of all compact elements of L. Then C is a join-semilattice with zero, and the ideal lattice, $\operatorname{Id} C$, of C is isomorphic to L (see, for instance, [5]). It is also trivial that $\operatorname{Aut} C$ and $\operatorname{Aut} L$ are isomorphic, hence we can assume that φ is a homomorphism of G into $\operatorname{Aut} C$.

Set $A=(G\times(C-\{0\}))\cup\{0\}$. We define some operations on A ($\alpha,\beta\in G$, $a,b\in C-\{0\}$):

k is a constant operation with value 0;

$\vee$ is a binary operation defined by

$$0\vee 0=0,\quad 0\vee\langle\alpha,a\rangle=\langle\alpha,a\rangle\vee 0=\langle\alpha,a\rangle,\quad \langle\alpha,a\rangle\vee\langle\beta,b\rangle=\langle\alpha,a\vee b\rangle;$$

$f_{\alpha,a}$ is a unary operation: $f_{\alpha,a}(0)=0$ and

$$f_{\alpha,a}(\langle\beta,b\rangle)=\begin{cases}\langle\alpha\beta,a(\beta\varphi)\rangle & \text{if } a(\beta\varphi)\leqq b,\\ \langle\alpha\beta,b\rangle & \text{if } b\leqq a(\beta\varphi),\\ 0 & \text{otherwise.}\end{cases}$$

Observe that if $a\neq 0$, then $a(\beta\varphi)$ is the image of a under the automorphism $\beta\varphi$ of C, hence $a(\beta\varphi)\neq 0$. Thus $f_{\alpha,a}$ is an operation on A.

Let F consist of k, $\vee$, and all the $f_{\alpha,a}$, $\alpha\in G$, $a\in C-\{0\}$ and set $\mathfrak{A}=\langle A;F\rangle$.

3. Verification. Now we prove that $\mathfrak{A}$ satisfies the conditions of the Theorem.

Claim 1. Let $B\subseteqq A$. B is closed under all the operations in F iff $B=(G\times(I-\{0\}))\cup\{0\}$, where $I\in\operatorname{Id} C$.

Proof. Checking the definition of the operations, it is clear that, for $I\in\operatorname{Id} C$,

$$(G\times(I-\{0\}))\cup\{0\}$$

is closed under all the operations in F.

Now let $B\subseteqq A$ and let B be closed under all the operations in F. Since $k\in F$, we obtain $0\in B$. Define

$$I=\{a\mid a\in C \text{ and } \langle\alpha,a\rangle\in B \text{ for some } \alpha\in G\}\cup\{0\}.$$

If $B=\{0\}$, then $I=\{0\}$ is an ideal. Now let $B\neq\{0\}$. Obviously, if $a,b\in I$, then $a\vee b\in I$. Let $b\in I$ and $c\leqq b$; we wish to prove that $c\in I$. If $c=0$, then $0\in I$ by definition. If $c\neq 0$, then $b\neq 0$, hence we can choose a $\beta\in G$ such that $\langle\beta,b\rangle\in B$ by the definition of I. Thus, for any $\alpha\in G$,

$$f_{\alpha\beta^{-1},c(\beta\varphi)^{-1}}(\langle\beta,b\rangle)=\langle\alpha,c\rangle,$$

since $c(\beta\varphi)^{-1}(\beta\varphi)=c\leqq b$. We conclude that $\langle\alpha,c\rangle\in B$, since $c\in I$. Therefore, $I\in\operatorname{Id} C$. Since we have $\langle\alpha,c\rangle\in B$ for all $\alpha\in G$, we also conclude that $B=(G\times(I-\{0\}))\cup\{0\}$, verifying the claim.

Claim 2. Sub $\mathfrak{A} \cong L$.

Proof. It is clear from Claim 1 that $I \to (G \times (I - \{0\})) \cup \{0\}$ is an isomorphism between Id C and Sub $\mathfrak{A}$. Since Id $C \cong L$, the claim follows.

Claim 3. For every $\gamma \in G$, the map T_γ: $\langle \beta, b \rangle \to \langle \beta\gamma, b(\gamma\varphi) \rangle$, $0 \to 0$ is an automorphism of $\mathfrak{A}$.

Proof. It is trivial that $0T_\gamma = 0$, $(x \vee y)T_\gamma = xT_\gamma \vee yT_\gamma$, for $x, y \in A$. Since right-multiplication of G and $\gamma\varphi$ on C are permutations, so is T_γ. It remains to prove that $f_{\alpha,a}(xT_\gamma) = f_{\alpha,a}(x)T_\gamma$. This is obvious for $x=0$. Now let $x = \langle \beta, b \rangle$. If $a(\beta\varphi)$ and b are not comparable, then $(a(\beta\varphi))(\gamma\varphi)$ and $b(\gamma\varphi)$ are not comparable, that is, $a((\beta\gamma)\varphi)$ and $b(\gamma\varphi)$ are not comparable, hence

$$f_{\alpha,a}(\langle \beta, b \rangle)T_\gamma = 0T_\gamma = 0 = f_{\alpha,a}(\langle \beta\gamma, b(\gamma\varphi) \rangle) = f_{\alpha,a}(\langle \beta, b \rangle T_\gamma).$$

The other two cases $(a(\beta\varphi) \leqq b$ and $b \leqq a(\beta\varphi))$ are similar.

Claim 4. Every automorphism of $\mathfrak{A}$ is of the form T_γ for a unique $\gamma \in G$.

Proof. Let T be an automorphism of $\mathfrak{A}$. Define the functions f and g on $C - \{0\}$ by

$$\langle 1, c \rangle T = \langle f(c), g(c) \rangle,$$

where 1 is the identity of G. Then, for $c, d \in C - \{0\}$,

$$\langle f(c \vee d), g(c \vee d) \rangle = \langle 1, c \vee d \rangle T = (\langle 1, c \rangle \vee \langle 1, d \rangle)T =$$
$$= \langle 1, c \rangle T \vee \langle 1, d \rangle T = \langle f(c), g(c) \rangle \vee \langle f(d), g(d) \rangle = \langle f(c), g(c) \vee g(d) \rangle.$$

Thus, for any $c, d \in C - \{0\}$,

$$f(c) = f(c \vee d) = f(d),$$

that is, $f(c)$ is a constant function, $f(c) = f \in C - \{0\}$. Thus $\langle 1, c \rangle T = \langle f, g(c) \rangle$ and $g(c \vee d) = g(c) \vee g(d)$, implying that g is an automorphism of $C - \{0\}$. Set $c = a \vee g^{-1}(a(f\varphi))$. Since $a \leqq c$ the first clause of the definition of $f_{\alpha,a}$ applies so we have

$$\langle \alpha, a \rangle T = f_{\alpha,a}(\langle 1, c \rangle)T = f_{\alpha,a}(\langle 1, c \rangle T) = f_{\alpha,a}(\langle f, g(c) \rangle) = \langle \alpha f, a(f\varphi) \rangle,$$

where, in the last step, the first clause of the definition of $f_{\alpha,a}$ again applies since $a(f\varphi) \leqq g(c)$.

This proves that $T = T_f$ since they agree on $A - \{0\}$, and obviously agree at 0. The uniqueness of f is obvious.

Claim 5. $G \cong$ Aut $\mathfrak{A}$.

Proof. $f \to T_f$ is the required isomorphism by Claims 3 and 4.

We have verified all but the last statement of the Theorem. Let $\alpha: G \to$ Aut $\mathfrak{A}$ and $\beta: L \to$ Sub $\mathfrak{A}$ be defined as in Claim 5 and Claim 2. Let $\gamma \in G$. Then $\gamma\varphi$ is an automorphism of C. An ideal I of C is carried to $(G\times(I-\{0\}))\cup\{0\}$ by Aut β and thus $(\gamma\varphi)$Aut β is an automorphism of Sub $\mathfrak{A}$ mapping $(G\times(I-\{0\}))\cup\{0\}$ to $(G\times(I(\gamma\varphi)-\{0\}))\cup\{0\}$. Now $\gamma\alpha$ is an automorphism of $\mathfrak{A}$, namely, T_γ. Thus $(\gamma\alpha)\varphi_{\mathfrak{A}}$ is an automorphism of Sub $\mathfrak{A}$ carrying a subalgebra B to BT_γ, that is, $(G\times(I-\{0\}))\cup\{0\}$ to $((G\times(I-\{0\}))\cup\{0\})T_\gamma=(G\times(I(\gamma\varphi)-\{0\}))\cup\{0\}$ (this equality follows from the definition of T_γ). This completes the proof of the Theorem.

4. Concluding remarks. Let $\mathfrak{m}$ be an infinite regular cardinal. The finitary concepts ($\mathfrak{m}=\aleph_0$) of the Theorem generalize naturally (see G. GRÄTZER [3] and [4]) to the concepts: $\mathfrak{m}$-algebraic lattice and algebra of characteristic $\mathfrak{m}$. Subalgebra lattices of algebras of characteristic $\mathfrak{m}$ can be characterized, up to isomorphism, as $\mathfrak{m}$-algebraic lattices. The Theorem of this note generalizes to $\mathfrak{m}$-algebraic lattices and algebras of characteristic $\mathfrak{m}$. In the proof, it is only necessary to replace the binary operation $\vee$ by infinitary joins of less than $\mathfrak{m}$ elements.

It is a curious fact that the algebra $\mathfrak{A}$ constructed has no endomorphisms other than the automorphisms.

Similarly to the definition of $\varphi_{\mathfrak{A}}$, we can define $\psi_{\mathfrak{A}}$: Aut $\mathfrak{A}\to$ Aut Con $\mathfrak{A}$, where Con $\mathfrak{A}$ is the congruence lattice of $\mathfrak{A}$ and we can ask for a characterization of $\langle \text{Aut}\,\mathfrak{A}, \text{Con}\,\mathfrak{A}, \psi_{\mathfrak{A}}\rangle$. (For the most recent accounting of the characterization problems connected with Con $\mathfrak{A}$, see G. GRÄTZER and W. A. LAMPE [6].) Even harder is the characterization problem of

$$\langle \text{Aut}\,\mathfrak{A}, \text{Sub}\,\mathfrak{A}, \text{Con}\,\mathfrak{A}, \varphi_{\mathfrak{A}}, \psi_{\mathfrak{A}}\rangle.$$

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Convexoid operators and generalized growth conditions associated with unitary ϱ-dilations of Sz.-Nagy and Foiaş

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Dedicated to the memory of the late Professor H. Hiruta

An operator on a complex Hilbert space is said to be convexoid if the closure of its numerical range coincides with the convex hull of its spectrum. We shall consider some generalized growth conditions associated with unitary ϱ-dilations defined by B. Sz.-Nagy and C. Foiaş and as an application of these generalized growth conditions we shall give some characterization of convexoid operators which is an improvement form of the already known criterions due to G. H. Orland, C.-S. Lin and S. M. Patel.

Subsequently we shall give some generalizations of both theorems of S. K. Berberian and S. M. Patel for operators implying the equation $\operatorname{Re}\sigma(T)=\sigma(\operatorname{Re} T)$ and we shall give some characterization of the class R introduced by G. R. Luecke.

1. Introduction

In this paper an operator T means a bounded linear operator on a complex Hilbert space $\mathfrak{H}$. The class $C_\varrho(\varrho>0)$ denotes the set of all operators with unitary ϱ-dilation [20]: there exist a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace and a unitary operator U on $\mathfrak{K}$ such that

(1) $$T^n x = \varrho P U^n x \quad \text{for} \quad x \in \mathfrak{H}\ (n = 1, 2, \ldots)$$

where P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$.

It is well known that $C_1=\{T:\|T\|\leqq 1\}$ [21] and $C_2=\{T:w(T)\leqq 1\}$ [2], where $w(T)$ indicates the numerical radius of T, i.e. $w(T)=\sup\{|\lambda|:\lambda\in W(T)\}$ and $W(T)$ denotes the numerical range of T defined by $W(T)=\{(Tx, x):\|x\|=1, x\in\mathfrak{H}\}$. In [21] there are given several characterizations of the operators belonging to C_ϱ and one of them is as follows:

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Theorem A [21]. *In order that T belong to the class C_ϱ it is necessary and sufficient that the condition*

$$(2) \qquad (\varrho-2)\|(I-zT)x\|^2+2\,\mathrm{Re}\,((I-zT)x, x) \geqq 0$$

be satisfied for all $x\in\mathfrak{H}$ and $|z|\leqq 1$.

In [9] an operator radius $w_\varrho(T)$ is defined by

$$(3) \qquad w_\varrho(T) = \inf\{u : u > 0, u^{-1}T\in C_\varrho\}.$$

$w_\varrho(T)$ is non-increasing function of ϱ, in particular $w_1(T)=\|T\|$, $w_2(T)=w(T)$ and $w_\infty(T)=r(T)$ ($r(T)$ denotes the spectral radius of T) [9]. Moreover,

$$(4) \qquad \text{if} \quad 0<\beta<\varrho\leqq\infty \quad \text{and} \quad w_\varrho(T)=w_\beta(T), \quad \text{then} \quad w_\alpha(T)=w_\beta(T)$$

whenever $\beta\leqq\alpha\leqq\infty$ [9, Theorem 5.3], [10, (e)].

In [9] the following characterization of C_ϱ is given in term of operator radii:

$$(5) \qquad C_\varrho = \{T: w_\varrho(T)\leqq 1\}.$$

An operator T is called to be ϱ-oid [4], [5] if

$$(6) \qquad w_\varrho(T^k) = \big(w_\varrho(T)\big)^k \quad (k=1,2,\ldots).$$

For each $\varrho\geqq 1$, $w_\varrho(T)=r(T)$ if and only if T is ϱ-oid and for each $0<\varrho<1$ there exists no non-zero ϱ-oid which is included in the class of normaloids [4]. Clearly 1-oid is normaloid and 2-oid is spectraloid (recall that an operator T is said to be normaloid if $\|T\|=r(T)$ and spectraloid if $w(T)=r(T)$).

We shall define generalized growth conditions associated with unitary ϱ-dilations as follows.

Definition 1. An operator T is called to satisfy the condition $(\varrho-G_1)$ for (M,N), in symbol $T\in(\varrho-G_1)$ for (M,N), if T satisfies the following inequality:

$$(7) \qquad w_\varrho\big((T-\mu)^{-1}\big)\leqq\frac{1}{d(\mu, M)} \quad \text{for all complex} \quad \mu\notin N,$$

where M and N are two closed and bounded sets satisfying $N\supset M\supset\sigma(T)$.

Definition 2. An operator T is called to satisfy the condition $E-(\varrho-G_1)$ for (M,N), in symbol $T\in E-(\varrho-G_1)$ for (M,N), if there is equality in (7).

$T\in(\varrho-G_1)$ for M (resp. $T\in E-(\varrho-G_1)$ for M) means $T\in(\varrho-G_1)$ for (M,M) (resp. $T\in E-(\varrho-G_1)$ for (M,M)).

Remark 1. Since $r(T)\leqq w_\varrho(T)$ holds for any $\varrho>0$ [9] and $1/d(\mu,\sigma(T))= =r\big((T-\mu)^{-1}\big)$ is always valid for all $\mu\notin\sigma(T)$, so that we remark that $T\in(\varrho-G_1)$ for $(\sigma(T),N)$ is equivalent to $T\in E-(\varrho-G_1)$ for $(\sigma(T),N)$, namely $(T-\mu)^{-1}$ is ϱ-oid for all complex $\mu\notin N$.

Remark 2. T is called an operator of class $M_\varrho(\varrho\geqq 1)$ [14] if $(T-\mu)^{-1}$ is ϱ-oid for all $\mu\notin\sigma(T)$, so that we remark $T\in M_\varrho(\varrho\geqq 1)$ coincides with $T\in E-(\varrho-G_1)$ for $\sigma(T)$. $T\in(G_1)$ for M [18] means $T\in(1-G_1)$ for (M, M) and $T\in(G_1)$ means $T\in(1-G_1)$ for $\sigma(T)$, equivalently, $T\in E-(1-G_1)$ for $\sigma(T)$.

An operator T is said to be convexoid [8] if $\overline{W(T)}=\mathrm{co}\,\sigma(T)$, where $\overline{M}$ denotes the closure of a set M in the complex plane and co M means the convex hull of M. It is well known [12] that T is convexoid if and only if $T\in(G_1)$ for co $\sigma(T)$. A new class designed by R of convexoid operators was introduced in [11] as follows: $T\in R$ if

$$\|(T-\mu)^{-1}\| = \frac{1}{d(\mu, \overline{W(T)})} \quad \text{for all} \quad \mu\notin\overline{W(T)}, \tag{8}$$

that is, $T\in R$ if and only if $T\in E-(1-G_1)$ for $\overline{W(T)}$.

Generalized numerical ranges $W_\alpha(T)$ $(\alpha\geqq 1)$ is defined in [10] as follows:

$$W_\alpha(T) = \bigcap_\mu \{\lambda : |\lambda-\mu| \leqq w_\alpha(T-\mu)\}. \tag{9}$$

$W_\alpha(T)$ is a compact convex set containing co $\sigma(T)$. In case $1\leqq\alpha\leqq 2$ $W_\alpha(T)$ coincides with $\overline{W(T)}$ [10] and $W_\infty(T)=\mathrm{co}\,\sigma(T)$ [6], [7], [10]. Since $w_\alpha(T-\mu)$ is a non-increasing function of α [9], $W_\alpha(T)\supset W_\beta(T)$ if $1\leqq\alpha<\beta$. The function $w_\varrho^0(T)$ is defined by $w_\varrho^0(T)=\sup\{|\lambda|:\lambda\in W_\varrho(T)\}$ for $1\leqq\varrho\leqq\infty$. $w_\varrho^0(T)$ satisfies the following properties [10];

$$r(T) \leqq w_\varrho^0(T) \leqq w_\varrho(T), \quad w_\infty^0(T) = r(T),$$
$$w_\varrho^0(\mu T) = |\mu|\, w_\varrho^0(T) \quad \text{for all complex } \mu,$$
$$w_2(T) = w_\varrho^0(T) \quad \text{for} \quad 1 \leqq \varrho \leqq 2.$$

The hen-spectrum $\tilde\sigma(T)$ is defined by $\tilde\sigma(T)=[[\sigma(T)^c]_\infty]^c$ in [3], where M^c is the complement of M, and $[M]_\infty$ is the unbounded component of M. $\tilde\sigma(T)$ is a compact set containing $\sigma(T)$ in the complex plane [3]. Using this notion of $\tilde\sigma(T)$, another new class denoted by (H_1) of convexoid operators was introduced in [3]: $T\in(H_1)$ if

$$\|(T-\mu)^{-1}\| \leqq \frac{1}{d(\mu, \tilde\sigma(T))} \quad \text{for all} \quad \mu\notin\tilde\sigma(T), \tag{10}$$

i.e. $T\in(H_1)$ if and only if $T\in(G_1)$ for $\tilde\sigma(T)$. (H_1) properly contains both (G_1) and R [3].

Theorem B [14]. *T is convexoid if and only if there exists $\varrho\geqq 1$ such that*

$$w_\varrho((T-\mu)^{-1}) \leqq \frac{1}{d(\mu, \mathrm{co}\,\sigma(T))} \quad \textit{for all} \quad \mu\notin\mathrm{co}\,\sigma(T). \tag{11}$$

Theorem B is an improvement of the well-known criterion for convexoidity due to [12].

C. R. PUTNAM considered conditions on an operator T implying

$$\operatorname{Re}\sigma(T)=\sigma(\operatorname{Re}T). \tag{$*$}$$

This equation $(*)$ holds for normal and also seminormal operators [16] and moreover $(*)$ has played a role in the proofs in [16], [17] which state that a seminormal operator whose spectrum has zero area is normal. S. K. BERBERIAN has not only given a simple proof of this Putnam's result, but he also has proved the following theorem.

Theorem C [1]. *If* $T\in(G_1)$ *and* $\sigma(T)$ *is connected, then* $(*)$ *holds.*

Related to Theorem C, S. M. PATEL [13] has established that the equation $(*)$ also holds for operations in the class R without any restriction on the spectrum as follows:

Theorem D [13]. *If* $T\in R$, *then* $(*)$ *holds.*

S. M. PATEL shows the following characterization of operators in the class R.

Theorem E [15]. $T\in R$ *if and only if there exist* $\varrho\geqq 1$ *and* $\alpha\geqq 1$ *such that*

$$w_\varrho\big((T-\mu)^{-1}\big)=\frac{1}{d\big(\mu,W_\alpha(T)\big)}\quad \text{for all}\quad \mu\notin W_\alpha(T). \tag{12}$$

Our Theorem 1 below is an improvement of Theorem B. Theorem 2 implies Corollary 2 which is a generalization of Theorem C and Theorem D. Finally, Theorem 3 is an improvement of Theorem E.

2. Statement of the results

Theorem 1. *Any one of the following conditions is necessary and sufficient in order that T be convexoid:*

(i) $T-\mu$ *is spectraloid for all complex* μ ([6], [7], [10]),

(ii) $T-\mu$ *is spectraloid for all complex* μ *whose absolute values are sufficiently large,*

(iii) *there exist* $\varrho\geqq 1$ *and* $2<\alpha\leqq\infty$ *such that* $T\in(\varrho-G_1)$ *for* $\big(W_\alpha(T),N\big)$, *where* N *runs over the closed and bounded sets containing* $W_\alpha(T)$.

Theorem 2. *If there exists* $\varrho\geqq 1$ *such that* $T\in(\varrho-G_1)$ *for* $\big(\sigma(T),\tilde\sigma(T)\big)$ *and* $\operatorname{Re}\sigma(T)$ *is connected, then* $(*)$ *holds.*

Corollary 1. *If* $T\in M_\varrho$ *and* $\operatorname{Re}\sigma(T)$ *is connected, then* $(*)$ *holds.*

Corollary 2. *If* $T\in(H_1)$ *and* $\operatorname{Re}\sigma(T)$ *is connected, then* $(*)$ *holds.*

Theorem 3. *$T \in R$ if and only if there exist $\varrho \geqq 1$ and $1 \leqq \beta \leqq \alpha \leqq \infty$ such that $T \in E-(\varrho-G_1)$ for $(W_\alpha(T), W_\beta(T))$.*

Take $N=W_\alpha(T)$ and $\alpha=\infty$ in (iii) of Theorem 1. Since $W_\infty(T)=\operatorname{co}\sigma(T)$, Theorem 1 implies Theorem B. The class (H_1) properly contains (G_1) [3], consequently Corollary 2 contains Theorem C.

$T \in R$ if and only $\partial W(T) \subset \sigma(T)$ by [11] (that is, $\overline{W(T)}=\tilde{\sigma}(T)$ [3]). The convex set $\overline{W(T)}$ contains $\sigma(T)$, consequently $T \in R$ implies that $\operatorname{Re}\sigma(T)=\operatorname{Re}\overline{W(T)}$ is connected. The class (H_1) properly contains R [3], so Corollary 2 contains Theorem D.

Corollary 1 easily implies Theorem C. Take $\alpha=\beta$ in Theorem 3, then $T \in R$ if and only if $T \in E-(\varrho-G_1)$ for $W_\alpha(T)$ for $1 \leqq \varrho$ and $1 \leqq \alpha$, that is, (12) holds, so Theorem 3 contains Theorem E.

3. Proofs of the theorems

In order to prove Theorem 1 we need the following Lemma.

Lemma 1. *If X is a closed convex subset of the complex plane, then $X \supset \overline{W(T)}$ if and only if there exists $\varrho \geqq 1$ such that $T \in (\varrho-G_1)$ for (X, Y), where Y runs over the closed and bounded sets containing X.*

Proof. The proof is along the same lines as the argument in [14, Theorem 4] and we shall state it for the sake of convenience in the subsequent discussion. If $X \supset \overline{W(T)}$, then there exists $\varrho \geqq 1$ such that $T \in (\varrho-G_1)$ for X by [14, Theorem 4] so there exists $\varrho \leqq 1$ such that $T \in (\varrho-G_1)$ for (X, Y).

Conversely, assuming that there exists $\varrho \leqq 1$ such that $T \in (\varrho-G_1)$ for (X, Y), we have only to show that every half plane M containing X also contains $\overline{W(T)}$. Without loss of generality we may assume $M=\{\lambda: \operatorname{Re}\lambda \geqq 0\}$. Since $M \supset X$ and the hypothesis holds we have

$$w_\varrho((\mu^{-1}T+I)^{-1}) = w_\varrho(\mu(T+\mu)^{-1}) \leqq \frac{\mu}{d(-\mu, X)} \leqq 1$$

for all positive μ whose absolute values are sufficiently large. Therefore, by (5), we have $(\mu^{-1}T+I)^{-1} \in C_\varrho$ for all positive μ whose absolute values are sufficiently large. By Theorem A we have

$$(\varrho-2)\|(I-(\mu^{-1}T+I)^{-1})x\|^2+2\operatorname{Re}\big((I-(\mu^{-1}T+I)^{-1})x, x\big) \geqq 0,$$

that is,

$$(\varrho-2)\|\mu^{-1}T(\mu^{-1}T+I)^{-1}x\|^2+2\operatorname{Re}\big(\mu^{-1}T(\mu^{-1}T+I)^{-1}x, x\big) \geqq 0$$

for all x in H. Multiply this above inequality by μ and transfering μ to ∞, we obtain $\operatorname{Re}(Tx, x) \geqq 0$ for all x in H, whence $\overline{W(T)} \subset X$, so the proof is complete.

Proof of Theorem 1. The proof of (i) was shown in [6], [7] and thereafter in [10], so that we have only to show the sufficiency of (ii). If X is any bounded closed set in the complex plane, then $\mathrm{co}\, X$ coincides with the intersection of all the circles with sufficiently large radii which contain the set X, so that

(13) $\mathrm{co}\, X = \bigcap_{\mu} \{\lambda : |\lambda - \mu| \leqq \sup_{x \in X} |x - \mu|$ for all complex μ whose absolute values are sufficiently large$\}$.

Taking $X = \overline{W(T)}$ and $\sigma(T)$ in (13) respectively, we have the following formulas since $\overline{W(T)}$ is convex [8],

(14) $\overline{W(T)} = \bigcap_{\mu} \{\lambda : |\lambda - \mu| \leqq w(T - \mu)$ for all complex μ whose absolute values are sufficiently large$\}$,

(15) $\mathrm{co}\, \sigma(T) = \bigcap_{\mu} \{\lambda : |\lambda - \mu| \leqq r(T - \mu)$ for all complex μ whose absolute values are sufficiently large$\}$.

The sufficiency of (ii) follows from (14) and (15).

(iii) Assume the hypothesis in (iii), then by Lemma 1 we have $W_\alpha(T) \supset \overline{W(T)}$ for $2 < \alpha \leqq \infty$. On the other hand $\overline{W(T)} \supset W_\alpha(T)$ holds in general for $2 < \alpha \leqq \infty$, so that $W_\alpha(T) = \overline{W(T)}$ for $2 < \alpha \leqq \infty$. This is equivalent to $w(T-\mu) = w_\alpha^0(T-\mu)$ for $2 < \alpha \leqq \infty$ [10, Corollary 1] and this implies $w(T-\mu) = w_2(T-\mu) = w_\alpha(T-\mu)$ for $2 < \alpha \leqq \infty$ and for all complex μ since $r(T) \leqq w_\alpha^0(T) \leqq w_\alpha(T) \leqq w(T)$ always holds for $2 < \alpha \leqq \infty$ [10]. So by (4) we have $w(T-\mu) = w_\infty(T-\mu) = r(T-\mu)$ for all complex μ, hence T is convexoid [6], [7], [10].

Conversely, if T is convexoid, then there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $\mathrm{co}\, \sigma(T)$, by Theorem B and therefore $T \in (\varrho - G_1)$ for $W_\alpha(T)$ $(2 < \alpha \leqq \infty)$ since $W_\alpha(T) \supset \mathrm{co}\, \sigma(T)$. Hence there exist $\varrho \geqq 1$ and $2 < \alpha \leqq \infty$ such that $T \in (\varrho - G_1)$ for $(W_\alpha(T), N)$, so the proof is complete.

To give the proof of Theorem 2, we shall show the following Lemmas.

Lemma 2. *If there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$ and λ is a semibare point of henspectrum $\tilde{\sigma}(T)$, then*

(i) *λ is a normal approximate eigenvalue of T, i.e. $\emptyset \neq A_\lambda(T) = A_{\lambda^*}(T^*)$*

(ii) *if in addition λ is an eigenvalue of T, then λ is a normal eigenvalue of T, i.e. $N_\lambda(T) = N_{\lambda^*}(T^*)$ where $A_\lambda(T) = \{\{x_n\} : \|x_n\| = 1,\ \|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty\}$ and $N_\lambda(T)$ denotes the kernel of $T - \lambda$.*

Lemma 3. *If there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, then* $\mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T)$ *holds.*

Lemma 4. *If T is convexoid, then*

(i) *if* $\mathrm{Re}\, \sigma(T) \subset \sigma(\mathrm{Re}\, T)$ *and* $\mathrm{Re}\, \sigma(T)$ *is connected, then* $(*)$ *holds,*

(ii) *if* $\sigma(\mathrm{Re}\, T) \subset \mathrm{Re}\, \sigma(T)$ *and* $\sigma(\mathrm{Re}\, T)$ *is connected, then* $(*)$ *holds,*

(iii) *if both* $\mathrm{Re}\, \sigma(T)$ *and* $\sigma(\mathrm{Re}\, T)$ *are connected, then* $(*)$ *holds* [1], [6].

Proof of Lemma 2. If there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, then $T-\lambda$ also belongs to the same class since $\tilde{\sigma}(T+\lambda I)=\tilde{\sigma}(T)+\lambda$ holds for every complex λ, so that we can assume $\lambda=0$. As $\lambda=0$ is a semibare point of $\tilde{\sigma}(T)$, we can choose a nonzero complex number $\lambda_0 \notin \tilde{\sigma}(T)$ such that $\{\lambda : |\lambda-\lambda_0| \leqq |\lambda_0|\}$ meets $\tilde{\sigma}(T)$ only at 0. As $\partial\tilde{\sigma}(T)=\tilde{\sigma}(T)\cup\overline{[\tilde{\sigma}(T)^c]}\subset\sigma(T)$ and $\sigma(T)\subset\tilde{\sigma}(T)$, it follows that $d(\lambda_0, \sigma(T))=d(\lambda_0, \tilde{\sigma}(T))=|\lambda_0|$. By the assumption there exists $\varrho \geqq 1$ such that $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, consequently we have the following equality by Remark 1:

$$w_\varrho((T-\lambda_0)^{-1}) = \frac{1}{|\lambda_0|}. \tag{16}$$

As $0 \in \partial\tilde{\sigma}(T)$ (that is, $\partial\sigma(T)$), then $\lambda=0$ is an approximate eigenvalue of T [8, Problem 63], [19, Theorem 66-B] i.e. there exists a sequence $\{x_n\}$ of unit vectors such that $Tx_n \to 0$. Then

$$\left\|(T-\lambda_0)^{-1}x_n+\frac{1}{\lambda_0}x_n\right\| \leqq \|(T-\lambda_0)^{-1}\| \left\|x_n+(T-\lambda_0)\frac{1}{\lambda_0}x_n\right\| =$$
$$= \|(T-\lambda_0)^{-1}\| \left\|\frac{1}{\lambda_0}Tx_n\right\| \to 0,$$

i.e. $(T-\lambda_0)^{-1}x_n+\frac{1}{\lambda_0}x_n \to 0$ and this convergence implies that $(T^*-\lambda_0^*)^{-1}x_n+ +\frac{1}{\lambda_0^*}x_n \to 0$ by S. M. Patel's result [14, Theorem 1] since (16) holds. Whence $T^*x_n \to 0$ by an easy calculation and this means that 0 is an approximate eigenvalue of T^* also. When we replace T by T^* and λ by λ^*, then the above argument is reversible, so we have (i). If we replace x_n by a vector x in the proof of (i), then we have (ii) so the proof is complete.

Proof of Lemma 3. Let $\alpha_0 \in \operatorname{Re}\sigma(T)$. Then there exists $\lambda_0 \in \partial\tilde{\sigma}(T)$ such that $\operatorname{Re}\lambda_0=\alpha_0$ and λ_0 is an approximate eigenvalue of T by the definition of hen-spectrum $\tilde{\sigma}(T)$. Let $D_n=\left\{\lambda : |\lambda-\lambda_0| \leqq \frac{1}{n}\right\}$ for $n=1, 2, \ldots,$ then D_n contains a point $\mu_n \notin \tilde{\sigma}(T)$ such that $|\mu_n-\lambda_0|<\frac{1}{2n}$. Clearly it is possible to choose λ_n with the following properties: $\lambda_n \in \tilde{\sigma}(T)$ and $d(\mu_n, \tilde{\sigma}(T))=d(\mu_n, \sigma(T))=|\mu_n-\lambda_n|$.

Now $\lambda_n \in \partial\tilde{\sigma}(T)$ lies on the circumference of a closed disc centered at μ_n whose interior contains no point of $\tilde{\sigma}(T)$, whence λ_n is a semibare point of $\tilde{\sigma}(T)$. Since $T \in (\varrho - G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$, λ_n is a normal approximate eigenvalue of T by Lemma 2, consequently there exists a sequence of unit vectors $\{x_n\}$ such that

$$Tx_n-\lambda_n x_n \to 0 \quad \text{and} \quad T^*x_n-\lambda_n^* x_n \to 0 \quad \text{as} \quad n \to 0.$$

Then we have $Tx_n-\lambda_0 x_n \to 0$ as $n \to \infty$ because

$$\|Tx_n-\lambda_0 x_n\| \leqq \|Tx_n-\lambda_n x_n\|+\|(\lambda_n-\lambda_0)x_n\| \to 0$$

as $n\to\infty$. Similarly $T^*x_n-\lambda_0^*x_n\to 0$ as $n\to\infty$, so that

$$\|(\operatorname{Re}T-\operatorname{Re}\lambda_0)x_n\| \leqq \tfrac{1}{2}\|Tx_n-\lambda_0x_n\|+\tfrac{1}{2}\|T^*x_n-\lambda_0^*x_n\|\to 0$$

as $n\to\infty$, whence $\operatorname{Re}\lambda_0\in\sigma(\operatorname{Re}T)$ and this is the desired relation, so the proof is complete.

We remark that S. K. Berberian has shown Lemma 3 in the case if T satisfies (G_1) for $\sigma(T)$ [1], here we have given the proof of Lemma 3 which is based on (i) of Lemma 2.

Proof of Lemma 4. It is known that T is convexoid if and only if

$$(\Sigma-\theta)\qquad \operatorname{Re}\Sigma(e^{i\theta}T)=\Sigma(\operatorname{Re}e^{i\theta}T)\quad\text{for all}\quad 0\leqq\theta\leqq 2\pi,$$

where $\Sigma(T)$ denotes $\operatorname{co}\sigma(T)$, and this $(\Sigma-\theta)$ is equivalent to $\operatorname{co}\operatorname{Re}\sigma(e^{i\theta}T)=$ $=\operatorname{co}\sigma(\operatorname{Re}e^{i\theta}T)$ for all $0\leqq\theta\leqq 2\pi$ [6].

If T is convexoid, then we have the following property by $(\Sigma-\theta)$

$$\operatorname{co}\operatorname{Re}\sigma(T)=\operatorname{co}\sigma(\operatorname{Re}T). \tag{17}$$

On the other hand, by the hypothesis of (i) we have

$$\operatorname{co}\operatorname{Re}\sigma(T)=\operatorname{Re}\sigma(T)\subset\sigma(\operatorname{Re}T)\subset\operatorname{co}\sigma(\operatorname{Re}T) \tag{18}$$

hence we have $(*)$ by (17) and (18). Similarly we have (ii). By (17) and the hypothesis of (iii), we have (iii).

In order to prove Theorem 2 we shall use only (i) of Lemma 4, but here we state (ii) and (iii) for the sake of completeness as some related results.

Proof of Theorem 2. If there exists $\varrho\geqq 1$ such that $T\in(\varrho-G_1)$ for $(\sigma(T),\tilde\sigma(T))$, then $\operatorname{Re}\sigma(T)\subset\sigma(\operatorname{Re}T)$ holds by Lemma 3 and T is convexoid by Theorem B. So we have $(*)$ by the hypothesis and (i) of Lemma 4 and we have finished the proof.

Corollary 1 easily follows from Theorem 2 by the definition of M_ϱ.

Proof of Corollary 2. As stated in the proof of Lemma 2, for all $\mu\notin\tilde\sigma(T)$, $d(\mu,\tilde\sigma(T))=d(\mu,\sigma(T))$ holds, consequently $T\in(\varrho-G_1)$ for $\tilde\sigma(T)$ if and only if $T\in(\varrho-G_1)$ for $(\sigma(T),\tilde\sigma(T))$.

Specially $T\in(H_1)$ if and only if $T\in(G_1)$ for $(\sigma(T),\tilde\sigma(T))$. So Theorem 2 implies Corollary 2.

Proof of Theorem 3. If $T\in R$, then there exist $\varrho\geqq 1$ and $\alpha\geqq 1$ such that $T\in E-(\varrho-G_1)$ for $W_\alpha(T)$ by Theorem E, consequently there exist $\varrho\geqq 1$ and $1\leqq\beta\leqq\alpha\leqq\infty$ such that $T\in E-(\varrho-G_1)$ for $(W_\alpha(T),W_\beta(T))$.

Conversely, suppose that there exist $\varrho\geqq 1$ and $1\leqq\beta\leqq\alpha\leqq\infty$ such that $T\in E-(\varrho-G_1)$ for $(W_\alpha(T),W_\beta(T))$. We remark that the condition $1\leqq\beta\leqq\alpha\leqq\infty$ can be replaced by $2\leqq\beta\leqq\alpha\leqq\infty$ since $W_\alpha(T)=\overline{W(T)}$ for $1\leqq\alpha\leqq 2$ [10]. When $\alpha=\beta=2$, the hypothesis implies $w_\varrho((T-\mu)^{-1})=1/d(\mu,W(T)$ for all $\mu\notin\overline{W(T)}$ and

$\varrho \geqq 1$. On the other hand $w_\varrho((T-\mu)^{-1}) \leqq \|(T-\mu)^{-1}\|$ for $\varrho \geqq 1$ [9] and $\|(T-\mu)^{-1}\| \leqq \leqq 1/d(\mu, W(T))$ always holds for all $\mu \notin \overline{W(T)}$ [22, Theorem 6.2-A]. So we have $\|(T-\mu)^{-1} = 1/d(\mu, W(T))$ for all $\mu \notin \overline{W(T)}$, i.e. $T \in R$, consequently we have only to prove Theorem 3 in case $2 < \alpha$. We can apply (iii) of Theorem 1 in this case, then T turns out to be convexoid, hence $\overline{W(T)} = W_\alpha(T) = W_\beta(T) = \mathrm{co}\,\sigma(T)$ so that the proof can be reduced to the case $\alpha = \beta = 2$ in which the theorem is already proved, so the proof is complete.

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Second order Briot-Bouquet differential equations

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1. Introduction. A Briot-Bouquet equation of order k is a DE of the form

$$P[w, w^{(k)}] = 0 \tag{1.1}$$

where $P(x, y)$ is a polynomial in x and y with constant coefficients. In the study of such equations the main problem is to find necessary and, if possible, sufficient conditions in order that the solutions be single-valued functions, holomorphic save for poles in the finite plane.

In 1887 PICARD [7] proved that an algebraic curve

$$P(x, y) = 0 \tag{1.2}$$

admits of a parametric representation

$$x = S(t), \quad y = T(t) \tag{1.3}$$

where S and T are transcendental entire or meromorphic functions of t iff the curve is of *genus* 0 or 1. Since $w(z)$ and $w^{(k)}(z)$ are either both entire or meromorphic or neither has this property we have

Theorem 1. *A necessary condition that* (1.1) *have a single-valued solution, holomorphic save for poles in the finite plane, is that the genus of the curve* (1.2) *be zero or one.*

The condition is not sufficient. Thus the second order DE

$$w'' = w^4 \quad \text{with e.g.} \quad w(z) = \left[w_0^{-3/2} - \frac{3}{2}\sqrt{\frac{2}{5}}(z - z_0)\right]^{-2/3} \tag{1.4}$$

has movable branch-points. The general solution is obtained by inverting a hyper-elliptic integral and has of course infinitely many branch-points.

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For $k=1$ the investigations of FUCHS [1]. POINCARÉ [8] and SCHLESINGER [9] have determined the limitations which are put on the polynomial coefficients of the powers of w' by the existence of meromorphic solutions that are non-rational.

Suppose that

$$P(x, y) = P_0(x)y^n + P_1(x)y^{n-1} + \ldots + P_n(x) \tag{1.5}$$

and let δ_j be the degree of $P_j(x)$. Fuchs showed that the existence of solutions of the described type requires that $P_0(x)$ be a constant, say $P_0(x)=1$, and that

$$\delta_j = 2j, \quad j = 1, 2, \ldots, n. \tag{1.6}$$

These conditions apply to first order BB equations. If they are satisfied and the genus is 0 or 1, then (1.1) has single-valued solutions which are rational functions of z or of e^{az} for some constant a, or of the Weierstrass $\wp$-function and its first derivative. Thus the solutions belong to the class of functions for which Weierstrass has shown the existence of algebraic addition theorems.

The present note is devoted to the case $k=2$. Here the analogue of the conditions of Fuchs read (see [3] Theorem 3)

$$P_0(w) \equiv 1, \quad \delta_j \leqq 3j, \quad j = 1, 2, \ldots, n. \tag{1.7}$$

If the solutions are to be entire functions of z, the inequalities become more restrictive:

$$\delta_j \leqq j \tag{1.8}$$

and this inequality holds for all values of k when the solutions are entire functions. Cf. [3] Theorem 4.

In the present note we use the method of Fuchs as presented by Schlesinger to the case $k=2$. We also lean heavily on the results of Painlevé and Gambier concerning second order DE's with fixed critical points. It will be found that the solutions are either of the same three types as for $k=1$ or reducible to such types by a change of variables.

2. Euqations of genus zero. I. Suppose now that $k=2$, conditions (1.7) hold and the curve (1.2) is of genus zero. Then a rational function of x and y exists such that

$$t = R(x, y) \tag{2.1}$$

leads to

$$x = R_1(t), \quad y = R_2(t) \tag{2.2}$$

where R_1 and R_2 are rational functions of t. The DE then becomes

$$w''(z) = R_2(t) \quad \text{with} \quad w(z) = R_1(t). \tag{2.3}$$

Differentiation of the second equation with respect to z gives

$$w'(z) = R_1'(t)\frac{dt}{dz}, \quad w''(z) = R_1''(t)\left(\frac{dt}{dz}\right)^2 + R_1'(t)\frac{d^2t}{dz^2}. \tag{2.4}$$

Thus t as a function of z satisfies the DE

$$\frac{d^2t}{dz^2} + \frac{R_1''(t)}{R_1'(t)}\left(\frac{dt}{dz}\right)^2 = \frac{R_2(t)}{R_1'(t)}, \tag{2.5}$$

where R_1 and R_2 are rational in t. There are two distinct possibilities according as $R_1''(t)\equiv 0$ or not. The first case is by far the simpler one.

I. $R_1''(t)\equiv 0$. We may assume $R_1'(t)\equiv 1$. The DE (2.5) now reduces to

$$\frac{d^2t}{dz^2} = R_2(t). \tag{2.6}$$

Since in this case $w(z)=t(z)+t_0$, the requirement that $w(z)$ shall have no branch-points implies that $R_2(t)$ is a polynomial in t of degree $\leqq 3$ by (1.7). It is necessary to distinguish between a number of subcases.

I:1. $\delta[R_2]=3$. A first integral of (2.6) takes the form

$$\left(\frac{dt}{dz}\right)^2 = A(t-a_1)(t-a_2)(t-a_3)(t-a_4). \tag{2.7}$$

Here there are essentially five different possibilities.

I:11. The a_j's are distinct. Then there exists an affine transformation $z=as$, $t=bv+c$ which takes (2.7) into the Jacobi normal form

$$\left(\frac{dv}{dz}\right)^2 = (1-v^2)(1-k^2v^2) \tag{2.8}$$

where the modulus k is determined by the a_j's. The solutions of (1.1) are thus elliptic functions of z.

I:12. $a_1=a_2=a$, $a_3\neq a_4$, $(a-a_3)(a-a_4)\neq 0$. Set

$$t = a+\frac{1}{v} \tag{2.9}$$

which reduces (2.7) to the form

$$\left(\frac{dv}{dz}\right)^2 = B(v-v_1)(v-v_2). \tag{2.10}$$

The corresponding solution $w(z)$ is a rational function of e^{az} for some a. It is simply periodic.

I:13. $a_1=a_2=a,\ a_3=a_4=b,\ a\neq b.$ Thus

$$\frac{dt}{dz} = B(t-a)(t-b). \tag{2.11}$$

Here also $w(z)$ is simply-periodic and a rational function of $\exp[(a-b)Bz]$.

I:14. $a_1=a_2=a_3=a,\ a_4=b\neq a$ so that

$$\left(\frac{dt}{dz}\right)^2 = A(t-a)^3(t-b). \tag{2.11}$$

The substitution (2.9) leads to a DE of the form

$$\left(\frac{dv}{dz}\right)^2 = B(v-c) \tag{2.12}$$

which is satisfied by a quadratic polynomial so that $w(z)$ is a rational function of z.

I:15. All the a_j's are equal to a. The equation may be reduced to the form

$$\frac{dv}{dz} = v^2 \quad \text{with} \quad v(z) = v_0-(z-z_0)^{-1} \tag{2.13}$$

so that $w(z)$ is also in this case a rational function of z.

This exhausts the possibilities when $\delta(R_2)=3$, $R_1(t)\equiv 1$ and $p=0$.

I:2. $\delta(R_2)=2$ gives the first integral

$$\left(\frac{dt}{dz}\right)^2 = A(t-a_1)(t-a_2)(t-a_3). \tag{2.14}$$

Here we have the following subcases:

I:21. The a's are distinct. An affine transformation leads to Weierstrass's normal form

$$\left(\frac{dv}{ds}\right)^2 = 4v^3-g_2v-g_3 \tag{2.15}$$

so that $w(z)$ is of the form

$$w(z) = b\wp(cz-s_0;\, g_2, g_3)+v_0 \tag{2.16}$$

with s_0 and v_0 as arbitrary constants.

I:22. $a_1=a_2=a,\ a_3=b\neq a.$ The substitution (2.9) reduces the DE to the form (2.10).

I:23. All the a_j's are equal to a so that the solution $w(z)$ is a rational function of z.

The only remaining case is that where $\delta(R_2)=1$ so that

$$\frac{d^2t}{dz^2}=c^2t-a, \quad c\neq 0 \tag{2.17}$$

with

$$t(z)=K_1e^{cz}+K_2e^{-cz}+ac^{-2} \tag{2.18}$$

so that $w(z)$ is a rational function of e^{cz}. This ends the case $R_1''(t)\equiv 0$.

3. General case with $p=0$. We have now equation (2.5) with $R_1''(t)\not\equiv 0$. Suppose that $w(z)$ has a pole at a finite point z_0. Since $w(z)=R_1(t)$ is a rational function of t, it is seen that $R_1(t)\to\infty$ as $t\to\infty$ and this says that at $t=\infty$

$$R_1(t)=a_0t^\mu+o(t^\mu), \quad a_0\neq 0, \tag{3.1}$$

where μ is a positive integer. The cases $\mu=1$ and $\mu>1$ require separate treatment. If $\mu=1$, then

$$R_1(t)=a_0t+a_1+a_2t^{-\lambda}+O(t^{-\lambda-1}), \quad a_0a_2\neq 0 \tag{3.2}$$

where λ is a positive integer. Hence

$$\frac{R_1''(t)}{R_1'(t)}=\frac{a_2}{a_0}\lambda(\lambda+1)t^{-\lambda-2}+O(t^{-\lambda-3}). \tag{3.3}$$

In the second case $\mu>1$ the ratio equals

$$(\mu-1)t^{-1}+O(t^{-2}). \tag{3.4}$$

Further

$$R_2(t)=b_0t^\nu+O(t^{\nu-1}). \tag{3.5}$$

Since $R_2(t)=w''(z)$ and $w''(z)/w(z)$ becomes infinite as z approaches a pole, one concludes that

$$\nu\geqq\mu+1 \tag{3.6}$$

and in

$$\frac{R_2(t)}{R_1'(t)}=\frac{b_0}{\mu a_0}t^{\nu+1-\mu}+O(t^{\nu-\mu}) \tag{3.7}$$

the leading exponent is at least 2.

We can start to whittle down the exponent. Some of this work is elementary but ultimately we have to fall back on the results of Painlevé and Gambier. Suppose that $w(z)$ has a pole of order α at $z=z_0$ where

$$w(z)=a(z-z_0)^{-\alpha}[1+o(1)] \tag{3.8}$$

so that

$$w'(z)=-\alpha a(z-z_0)^{-\alpha-1}[1+o(1)], \quad w''(z)=\alpha(\alpha+1)a(z-z_0)^{-\alpha-2}[1+o(1)]. \tag{3.9}$$

Set

$$\frac{R_1''(t)}{R_1'(t)} = Q_1(t), \quad \frac{R_1(t)}{R_1'(t)} = Q_2(t) \tag{3.10}$$

so that (2.5) becomes

$$t''(z) + Q_1(t)[t'(z)]^2 = Q_2(t). \tag{3.11}$$

Now

$$w(z) = R_1[t(z)] = a_0[t(z)]^\mu[1+o(1)]$$

in a neighborhood of a pole and $t(z)$ is a rational function of z, $w(z)$ and $w''(z)$ by (2.1) so any infinitude of $t(z)$ must be a pole, say of z order β at $z=z_0$ and here

$$\beta = \frac{\alpha}{\mu} \tag{3.12}$$

so that μ is a divisor of α. At $z=z_0$ the three terms of (3.11) have poles of order

$$\beta+2, \quad \beta+2 \quad \text{or} \quad 2-\lambda\beta, \quad \text{and} \quad (\nu+1-\mu)\beta,$$

respectively. Since the infinitary terms must balance in the equation, it is seen that $\beta+2 \geqq (\nu+1-\mu)\beta$ or $(\nu-\mu)\beta \leqq 2$. Here both factors on the left are positive integers, at least equal to 1. It follows that

$$1 \leqq \beta \leqq 2, \quad 1 \leqq \nu-\mu \leqq 2. \tag{3.13}$$

Since $\beta=\alpha/\mu$, it is seen that

$$\alpha = \frac{2\mu}{\nu-\mu} \quad \text{and} \quad \nu \leqq 3\mu. \tag{3.14}$$

This is as far as we can get with elementary methods.

P. Painlevé [6] and R. Gambier [2] have determined the DE's of the form

$$v''(z) = L(z, v)[v'(z)]^2 + M(z, v)v'(z) + N(z, v) \tag{3.15}$$

which have fixed critical points. Here L, M, N are analytic functions of z and rational in v. An excellent presentation of the theory is given in Ince [5, Chapter XIV]. Painlevé and Gambier found that the equations of type (3.15) with fixed critical points (branch points and essential singular points) could be reduced, possibly by change of variables, to one of 50 different normal forms. We shall apply these results to equation (3.11). This equation does not involve z explicitly, further $M(v)$ is identically zero while $N(v)$ is definitely not. This reduces the types that have to be considered from 50 to 15. These are listed by Ince [5, pp. 337—343] under the headings XII, XVI, XVIII, XIX, XXI—XXIII, XXVI, XXIX, XXX, XXXII, XXXIII, XXXVIII, XLIV and XLIX. We refer to Ince for details.

On the face of it his equation XIV should also be considered, but this equation contains a misprint: a factor $\frac{dw}{dZ}$ is missing so $M(v)$ is not identically zero.

One can set the arbitrary functions $q(Z)$ and $r(Z)$ equal to zero but this also makes $N(v)\equiv 0$ so the equation does not qualify.

The reduction to a normal form may involve a change of variables but in the case of (3.11) and $p=0$ only affine transformations

$$Z = az+b, \quad V = cv+d, \tag{3.16}$$

need be considered.

The function $v\mapsto L(v)$ has at most 3 poles and for XLIX this number is reached and the normal form is

$$L(v) = \frac{1}{2}\left\{\frac{1}{v}+\frac{1}{v-1}+\frac{1}{v-a}\right\}, \quad a\neq 0,1. \tag{3.17}$$

Two poles occur in XXXVIII and XLIV with $L(v)$ equal to

$$\frac{1}{2v}+\frac{1}{v-1} \quad \text{and} \quad \frac{3}{4}\left\{\frac{1}{v}+\frac{1}{v-1}\right\}, \tag{3.18}$$

respectively. All the other L's are of the form Cv^{-1} where the constant C has only three possible values $\frac{1}{2}\cdot\frac{3}{4}\cdot 1$. At infinity $L(v)$ has a simple zero. Comparison with (3.3) and (3.4) shows that $\mu>1$.

The rational function $N(v)$ is normally of degree $\nu=3$; it is 2 for XVIII, XIX, XXI, XXIII, XXXIII. For XXII $\nu=0$ and -1 for XXXII. The latter two equations are excentional in as much as the solutions are polynomials in z and thus have no finite poles. There is no contradiction with (3.13) and (3.14) since these relations presuppose the existence of poles.

The solutions of (3.11) are elliptic functions when $\nu=2$ or 3. Combining the results of this section with those of the preceding one leads to

Theorem 2. *If the curve* (1.1) *is of genus zero and if* (1.7) *holds then the solutions of the DE*

$$[w''(z)]^n + \sum_{j=1}^{n} P_j[w(z)][w''(z)]^{n-j} = 0 \tag{3.19}$$

are rational functions of z or of e^{az} for some a, or finally of $\wp(az+b;g_2,g_3)$ and its derivative with respect to z for some choice of the parameters a, b, g_2, g_3.

4. The case $p=1$. Here we can find four rational functions, each of two arguments, such that

$$s = R_1(x,y), \quad t = R_2(x,y), \quad x = R_3(s,t), \quad y = R_4(s,t) \tag{4.1}$$

with

$$t^2 = 4s^3-g_2s-g_3 \tag{4.2}$$

where the parameters g_2 and g_3 may be determined from the coefficients p_{jk} of (1.2).

Since t^2 is a polynomial in s, the functions R_3 and R_4 may be written as follows

(4.3) $$R_3(s,t) = R_{31}(s)+tR_{32}(s), \quad R_4(s,t) = R_{41}(s)+tR_{42}(s)$$

where the R_{jk} are rational functions of s.

Since by definition $\dfrac{d^2x}{dz^2}=y$ we get

(4.4) $$\frac{d^2x}{dz^2} = \left\{R'_{31}+tR'_{32}+R_{32}\frac{6s^2-\frac{1}{2}g_2}{t}\right\}\frac{d^2s}{dz^2}+$$

$$+\left\{R''_{31}+tR''_{32}+\frac{12s^2-g_2}{t}R'_{32}+R_{32}\frac{1}{t}\left[12s-\left(\frac{12s^2-g_2}{2t}\right)^2\right]\right\}\left(\frac{ds}{dz}\right)^2 = y = R_{41}+tR_{42}.$$

It follows that $z\mapsto s(z)$ satisfies a DE of the form

(4.5) $$\frac{d^2s}{dz^2} = [Q_{11}(s)+tQ_{12}(s)]\left(\frac{ds}{dz}\right)^2+Q_{21}(s)+tQ_{22}(s)$$

where the Q_{jk} are rational functions of s.

Here there are various possibilities.

I. $Q_{12}(s)=Q_{22}(s)\equiv 0$. The equation (4.5) is then essentially of the same nature as (2.5) and the previous results apply. The solution is normally an elliptic function of z but it may degenerate to a rational function of z or of e^{az} for some constant a. This case gives nothing new.

II. At least one of the functions $Q_{12}(s)$ and $Q_{22}(s)$ is not identically zero. We note that at least one of the functions Q_{21} and Q_{22} cannot vanish identically save for the trivial DE $[w''(z)]^n=0$.

Suppose that at $s=\infty$

(4.6) $$Q_{jk}(s) = a_{jk}s^{\delta_{jk}}[1+o(1)]$$

and suppose that a solution $s(z)$ of (4.5) has an infinitude of order β at $z=z_0$ so that

(4.7) $$s(z) = b(z-z_0)^{-\beta}[1+o(1)].$$

Equation (4.5) involves five terms that may become infinite as a negative power of $(z-z_0)$. The orders are respectively

(4.8) $$\beta+2, \quad (\delta_{11}+2)\beta+2, \quad \left(\delta_{12}+\frac{7}{2}\right)\beta+2, \quad \delta_{21}\beta, \quad \left(\delta_{22}+\frac{3}{2}\right)\beta$$

provided the corresponding $a_{jk}\neq 0$. Here the δ_{jk} are integers, >0 or 0 or <0. Since no term can dominate the first term and at least one of the other terms must be of the same order of magnitude we get a set of inequalities which must be satisfied by the δ_{jk}'s:

(4.9) $$\delta_{11}\leqq -1, \quad \delta_{12}\leqq -3, \quad \delta_{21}\leqq 3, \quad \delta_{22}\leqq 1.$$

Let us now bring the known facts to bear on our problem. Painlevé also examined the case where the coefficients L, M, N are algebraic functions of v so that L, M, N are rational functions of the variables v and W where

$$C(w, W) = 0 \tag{4.10}$$

and C is a polynomial with constant coefficients and the curve (4.10) is of genus 0 or 1.

Besides the 50 types found in the rational case $p=0$ Painlevé found only 3 additional types free from movable critical points. If in these equations the arbitrary functions are replaced by arbitrary constants and the conditions $M(v) \equiv \equiv 0, N(v) \not\equiv 0$ are imposed, only two types are found to qualify. These equations may be written

$$s''(z) = \frac{1}{2}\frac{T'(s)}{T(s)}[s'(z)]^2 + r[T(s)]^{1/2} \tag{4.11}$$

and

$$s''(z) = \left\{\frac{1}{2}\frac{T'(s)}{T(s)} - \frac{\pi i}{\omega}[T(s)]^{-1/2}\right\}[s'(z)]^2 + r[T(s)]^{\frac{1}{2}}. \tag{4.12}$$

Here

$$T(s) = 4s^3 - g_2 s - g_3 \tag{4.13}$$

and 2ω is an arbitrary period of $\wp(s; g_2, g_3)$.

Equation (4.11) is equivalent to the system

$$\begin{cases} s'(z) = u(z)\{T[s(z)]\}^{1/2} \\ u'(z) = r \end{cases}, \tag{4.14}$$

with solutions

$$s(z) = \wp\left(\tfrac{1}{2} rz^2 + C_1 z + C_2; g_2, g_3\right). \tag{4.15}$$

By (4.1) the solution $w(z)$ of (1.2) is a rational function of $s(z)$ and $T[s(z)]$, that is expressible in terms of elliptic functions of a quadratic polynomial. Such an elliptic function would necessarily have Nevanlinna order 4. But this contradicts Theorem 6 of [4] according to which the Nevanlinna order of a meromorphic solution of a Briot-Bouquet DE is at most 2. We conclude that an equation of type (4.11) can not arise when the birational transformation (4.1) is applied to a BB equation.

As we shall see in a moment, equation (4.12) can also be dismissed. This equation also leads to a simple system

$$\begin{cases} s'(z) = u(z)T[s(z)]^{1/2} \\ u'(z) = i\dfrac{\pi}{\omega}[u(z)]^2 + r. \end{cases} \tag{4.16}$$

Here $r \neq 0$ since $N(v) \not\equiv 0$. We set $r = i\dfrac{\pi}{\omega}a^2$ so that

$$u(z) = ai \tanh\left[a\frac{\pi}{\omega}(z-z_0)\right], \tag{4.17}$$

$$\int^z u(s)\,ds = i\frac{\omega}{\pi}\log\sinh\left[a\frac{\pi}{\omega}(z-z_0)\right] \tag{4.18}$$

so that

$$s(z) = \wp\left\{i\frac{\omega}{\pi}\log\sinh\left[a\frac{\pi}{\omega}(z-z_0)\right]; g_2, g_3\right\}. \tag{4.19}$$

This solution has singularities at all the points $z_k = z_0 + k\dfrac{\omega}{a}i$. If z describes a positive circuit around one of these points, the logarithm is increased by $2\pi i$ and the argument of the $\wp$-function decreases by 2ω which is a period so the solution returns to its original value. Thus the solution is single-valued but it is not a meromorphic function. In fact, each of the points z_k is a point of accumulation of poles. Thus $z \mapsto s(z)$ takes on every value infinitely often in an arbitrarily small neighborhood of z_k. Now a rational function of $s(z)$ and $T[s(z)]$ inherits these properties of $s(z)$.

According to Theorem 4 of [4] the determinateness theorem of Painlevé holds also for second order BB-equations. This shows that an equation of type (4.11) cannot be obtained as a transform of a BB-equation. Thus we have proved

Theorem 3. *If the curve* (1.1) *is of genus* 1 *and if* (1.7) *holds, then the solutions of* (3.19) *are rational functions of z or of e^{az} for some a or, finally, of $\wp[L(z); g_2, g_3]$ and its z-derivative where $L(z)$ is a linear function of z.*

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Compact Operator Ranges and Reductive Algebras

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1. Introduction. Let $\mathscr{A}$ be an arbitrary subalgebra of $\mathscr{B}(\mathfrak{H})$ — the algebra of all (bounded) operators on the (complex) Hilbert space $\mathfrak{H}$. A sufficient condition that $\mathscr{A}$ be strongly dense in $\mathscr{B}(\mathfrak{H})$ was found by FOIAŞ [4]; it requires that $\mathscr{A}$ have no invariant operator ranges other than $\{0\}$ and $\mathfrak{H}$. This requirement is stronger than that of (topological) transitivity for $\mathscr{A}$, i.e., the hypothesis that $\mathscr{A}$ has no non-trivial invariant (closed) subspaces. This result was generalized in [5] to the theorem that if every proper operator range invariant under the transitive algebra $\mathscr{A}$ is the range of a compact operator, then $\mathscr{A}$ is strongly dense. One of the purposes of the present paper is to demonstrate that this generalization is not vacuous, and that in fact there exist proper, dense subalgebras of $\mathscr{B}(\mathfrak{H})$ leaving invariant an abundance of compact operator ranges but no other operator ranges.

The second purpose of this paper is to give an extension of the above result to reductive subalgebras of $\mathscr{B}(\mathfrak{H})$, i.e., those algebras whose invariant subspaces are all reducing. The new result, also shown to be non-vacuous, states that if the invariant operator ranges of a reductive algebra $\mathscr{A}$ are all "compact perturbations" of its invariant subspaces, in a certain sense, then $\mathscr{A}$ is strongly dense in a self-adjoint algebra. This also strengthens the theorem in [2] with the same conclusion but requiring that all invariant operator ranges be closed.

Algebras considered will be assumed to contain the identity, although this is not at all essential; the trivial modification necessary for the general case will be obvious to the reader.

2. Algebras with Invariant Compact Operator Ranges. We start with the following lemma whose proof can be found in [4].

Lemma 1. *Let $\mathscr{B}$ be a uniformly closed subalgebra of $\mathscr{B}(\mathfrak{H})$ which leaves the range of an injective operator S invariant. Then there exists $M>0$ such that $\|S^{-1}BS\|\leqq M\|B\|$ for every $B\in\mathscr{B}$.*

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Theorem 2. *Let K be a compact operator with dense range. Let $\mathscr{A}$ be the (transitive) algebra of all operators leaving $K\mathfrak{H}$ invariant. Then every proper operator range invariant under $\mathscr{A}$ is the range of a compact operator.*

(We remark that every $\mathscr{A}$-invariant operator range has to contain $K\mathfrak{H}$ by a result of [7].)

Proof. Assume, with no loss of generality, that $0 \leqq K \leqq 1$. Fix λ with $0<\lambda<1$, and let P_i be the finite-dimensional spectral projection of K corresponding to all the eigenvalues in the interval $(\lambda^i, \lambda^{i-1}]$. Let $\mathscr{T}$ denote the algebra of all upper-block-triangular operators relative to the decomposition $\sum_{i=1}^{\infty} \oplus P_i \mathfrak{H}$ of $\mathfrak{H}$. It follows from the characterization of $\mathscr{A}$ given in [7] that $\mathscr{T} \subseteqq \mathscr{A}$. We must prove that if S is an operator such that $S\mathfrak{H}$ is invariant under $\mathscr{A}$ and $S\mathfrak{H} \neq \mathfrak{H}$, then S is compact. Again we assume, with no loss of generality, that S is positive. This implies, since $S\mathfrak{H}$ is dense in $\mathfrak{H}$, that S is also injective.

Assume S is not compact. Then there is $\varepsilon>0$ and an infinite-dimensional spectral subspace $\mathfrak{M}$ for S such that $S|\mathfrak{M} \geqq \varepsilon$ (and thus $S\mathfrak{M}=\mathfrak{M}$). Now, since the subspace $\sum_{i=n+1}^{\infty} \oplus P_i \mathfrak{H}$ has finite codimension, it intersects $\mathfrak{M}$ nontrivially for every n. Pick a unit vector x in this intersection. Observe that if y is an arbitrary unit vector in $\sum_{i=1}^{n} \oplus P_i \mathfrak{H}$ for any n, then there exists $T \in \mathscr{T}$ with $\|T\|=1$ such that $Tx=y$. This is so because the subset

$$\left(\sum_{i=1}^{n} \oplus P_i\right) \mathscr{T} \left(\sum_{i=n+1}^{\infty} \oplus P_i\right)$$

of $\mathscr{T}$ contains all bounded linar transformations from $\sum_{i=n+1}^{\infty} P_i \mathfrak{H}$ into $\sum_{i=1}^{n} \oplus P_i \mathfrak{H}$, and, in particular, the rank-one operator that sends x to y and $\{x\}^{\perp}$ to $\{0\}$. Hence $y \in TS\mathfrak{H} \subseteqq S\mathfrak{H}$ and

$$\|S^{-1}y\| = \|S^{-1}TS(S^{-1}x\| \leqq \|S^{-1}TS\|/\varepsilon.$$

Since $\|S^{-1}TS\|$ is bounded on the unit ball of $\mathscr{A}$ (Lemma 1), we conclude that S^{-1} is bounded on the dense linear manifold $\bigcup_{n=1}^{\infty} \sum_{i=1}^{n} \oplus P_i \mathfrak{H}$. Thus S^{-1} is bounded and $S\mathfrak{H}=\mathfrak{H}$, which is a contradiction. This completes the proof.

We note here that the algebra $\mathscr{A}$ of the above theorem has many invariant operator ranges, e.g., the range of K^r for $0<r<1$. It also has mutually non-comparable invariant operator ranges (See [4] and [7].)

3. Reductive Algebras. A natural first question on reductive algebras suggested by the above-mentioned result of [5] is: What happens if it is assumed that every proper operator range invariant under the (infinite-dimensional) reductive algebra

$\mathscr{A}$ is a compact operator range? It is very easy to see that such an $\mathscr{A}$ is actually transitive and thus strongly dense by [5]. The next question is: What if we replace "proper" by "non-closed" in the above question? The answer is as expected: Such an algebra will have to be strongly dense in a self-adjoint algebra. But we shall prove a stronger result.

In what follows, by a *compact perturbation* of a subspace $\mathfrak{M}$ of $\mathfrak{H}$ we shall mean the range of any operator of the form $P+K$, where P is the orthogonal projection on $\mathfrak{M}$ and K is a compact operator with $KP=PK=0$. We allow P or K to be trivial. If $P\mathfrak{H}$ and $K\mathfrak{H}$ are both infinite-dimensional, this type of operator ranges are called *class 2b* ranges by Dixmier [1]. (See also [3].) An invariant subspace $\mathfrak{M}$ of an algebra $\mathscr{A}$ is an *atom* if $\mathscr{A}|\mathfrak{M}$ is transitive.

Theorem 3. *Let $\mathscr{A}$ be a reductive algebra on $\mathfrak{H}$ such that every invariant operator range of $\mathscr{A}$ is a compact perturbation of a subspace of $\mathfrak{H}$ (not necessarily invariant under $\mathscr{A}$). Then $\mathfrak{H}$ is a finite direct sum of atoms for $\mathscr{A}$.*

Proof. Any infinite chain (under inclusion) of subspaces of $\mathfrak{H}$ contains either a subchain isomorphic to the integers or one anti-isomorphic to the integers. Now pick a maximal chain C of invariant subspaces of $\mathscr{A}$. If C is infinite, then, by the above remark and by the reductivity of $\mathscr{A}$, we obtain infinitely many, mutually orthogonal invariant subspaces for $\mathscr{A}$. If some or all of these subspaces are finite-dimensional, we rearrange them in a double sequence $\mathfrak{N}_{ij}$ and let $\mathfrak{M}_i=\sum_j\oplus\mathfrak{N}_{ij}$. Thus we can assume $\mathfrak{H}=\sum_{i=1}^{\infty}\oplus\mathfrak{M}_i$, where each $\mathfrak{M}_i$ is an infinite-dimensional invariant subspace for $\mathscr{A}$. Then the operator $\sum_{i=1}^{\infty}\oplus(1/i)I_i$, where I_i is the identity on $\mathfrak{M}_i$ commutes with (every member of) $\mathscr{A}$, and thus its range is invariant under $\mathscr{A}$. But this range is not closed and is easily seen not to be a compact perturbation of any subspace, because every eigenvalue $1/i$ has infinite multiplicity. It follows from the hypotheses that C is finite. By the reductivity of $\mathscr{A}$ this chain gives rise to a finite number of atoms $\mathfrak{H}_i$ for $\mathscr{A}$ with $\mathfrak{H}=\sum\oplus\mathfrak{H}_i$.

In the rest of the paper we shall freely use the notation and terminology of [8] with one exception: we do not assume, as part of definition, that a reductive algebra is closed under any topology. The symbol $\mathscr{A}$ will consistently denote a reductive algebra.

We need the following lemmas in the proof of the main result of this section.

Lemma 4. *Let $\mathfrak{H}=\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m$, where each $\mathfrak{H}_i$ is an atom for $\mathscr{A}$. Let $X\in\operatorname{Lat}\mathscr{A}^{(k)}$ and assume $\mathfrak{X}$ is (a graph subspace) of the form*

$$\mathfrak{X}=\{(C_{11}x\oplus\ldots\oplus C_{1m}x)\oplus\ldots\oplus(C_{k1}x\oplus\ldots\oplus C_{km}x)\colon x\in\mathfrak{D}\},$$

where $\mathfrak{D}$ is a nonzero linear manifold in $\mathfrak{H}_1$, each C_{ij} is a (not necessarily bounded) linear transformation with the common domain $\mathfrak{D}$ and range in $\mathfrak{H}_j$, and C_{11} is the identity on $\mathfrak{D}$. Then there exists bounded linear transformations D_{ij} from $\mathfrak{H}_1$ into $\mathfrak{H}_j$ such that

$$\mathfrak{X} = \{(D_{11}y\oplus\ldots\oplus D_{1m}y)\oplus\ldots\oplus(D_{k1}y\oplus\ldots\oplus D_{km}y): y\in\mathfrak{H}_1\}.$$

Proof. Since $\mathfrak{D}$ is the domain of the closed operator

$$T:\mathfrak{D}\to(\mathfrak{H}_2\oplus\ldots\oplus\mathfrak{H}_m)\oplus[\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m]^{(k-1)}$$

defined by $Tx=C_{12}x\oplus\ldots\oplus C_{km}x$, it is also the range of a bounded injective operator $S:\mathfrak{H}_1\to\mathfrak{H}_1$ (Theorem 1.1 of [3]). Thus TS is a closed operator defined on $\mathfrak{H}_1$ and hence bounded by the Closed Graph Theorem. It follows that the transformations $D_{ij}=C_{ij}S$ are all bounded on $\mathfrak{H}_1$ and satisfy the requirements of the lemma.

The following lemma is easily verified; its proof is also given, e.g., in [8, Proof of Theorem 9.11].

Lemma 5. *Let $\mathscr{A}$ be a reductive algebra on $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ and let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be atoms for $\mathscr{A}$. Assume T is a bounded linear transformation from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ whose graph is in* Lat $\mathscr{A}$. *Then T is a scalar multiple of an isometry U of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ and, consequently, $A|\mathfrak{H}_2=U(A|\mathfrak{H}_1)U^*$ for all $A\in\mathscr{A}$.*

It is convenient to introduce another ad-hoc definition: a subspace N of $(\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m)^{(k)}$ will be called *special* if there exist scalars $\alpha_1, \ldots, \alpha_k$, and an integer i, $1\leqq i\leqq m$ such that

$$\mathfrak{N} = \{(0\oplus\ldots\oplus0\oplus\alpha_1x\oplus0\oplus\ldots\oplus0)\oplus\ldots\oplus(0\oplus\ldots\oplus0\oplus\alpha_kx\oplus0\oplus\ldots\oplus0): x\in\mathfrak{H}_i\},$$

where in each pair of parantheses the component α_jx occurs at the i-th place.

Lemma 6. *Let $\mathscr{A}$ be a reductive algebra on $\mathfrak{H}=\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m$, where each $\mathfrak{H}_i$, $1\leqq i\leqq m$, is an atom for $\mathscr{A}$, and for no pair (i, j), $i\neq j$, there is an isometry U from $\mathfrak{H}_i$ onto $\mathfrak{H}_j$ such that*

$$A|\mathfrak{H}_j=U(A|\mathfrak{H}_i)U^*, \quad \textit{for all} \quad A\in\mathscr{A}.$$

Suppose also that the only proper operator ranges invariant under $\mathscr{A}|\mathfrak{H}_i$ are ranges of compact operators. Let $\mathfrak{N}\in\mathrm{Lat}\,\mathscr{A}^{(k)}$, so that every $y\in\mathfrak{N}$ has mk components y_j relative to $(\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m)^{(k)}$. Assume that there is a subset J of the integers $1, \ldots, mk$ such that $y\in\mathfrak{N}$, $y\neq0$ implies $y_j\neq0$ for $j\in J$ and $y_j=0$ otherwise. Then $\mathfrak{N}$ is a special subspace.

Proof. Suppose $\mathfrak{N}\neq\{0\}$; otherwise the conclusion holds. From the hypothesis on $\mathfrak{N}$ one can easily conclude that for any $y\in\mathfrak{N}$, $y\neq0$, all the mk components of y can be uniquely (and hence linearly) determined by any nonzero one. Assume, with no loss of generality, that $y_{11}\neq0$ for all $y\in\mathfrak{N}$. Then Lemma 4 yields

$$\mathfrak{N} = \{(D_{11}x\oplus\ldots\oplus D_{1m}x)\oplus\ldots\oplus(D_{k1}x\oplus\ldots\oplus D_{km}x): x\in\mathfrak{H}_1\},$$

where D_{11} and hence all the nonzero D_{ij} are injective bounded linear transformations, by hypothesis. Since the range of each D_{ij} is invariant under $\mathscr{A}|\mathfrak{H}_j$, it follows that every nonzero D_{ij} is either compact or bijective.

If every D_{ij} is compact, then the operator

$$R: x \to (D_{11}x \oplus \ldots \oplus D_{1m}x) \oplus \ldots \oplus (D_{k1}x \oplus \ldots \oplus D_{km}x)$$

is also compact and thus $\mathfrak{N}$ is the range of a compact operator; since it is closed it must be finite-dimensional, and thus the nonzero D_{ij} are surjective. If $\mathfrak{N}$ is infinite-dimensional, then at least one D_{ij} should be surjective. Hence in all cases we can assume, with no loss of generality again, that D_{11} is surjective. Replacing x by $D_{11}^{-1}y$ and putting $E_{ij} = D_{ij}D_{11}^{-1}$ yields

$$\mathfrak{N} = \{(y \oplus \ldots \oplus E_{1m}y) \oplus \ldots \oplus (E_{k1}y \oplus \ldots \oplus E_{km}y) : y \in \mathfrak{H}_1\}.$$

The proof will be complete if we show that

(a) $E_{ij} = 0$ for $j \neq 1$, and (b) $E_{i1} = \alpha_i I$, $i = 1, 2, \ldots, k$.

To prove (a) we note that if $E_{ij} \neq 0$ for some i and j with $j \neq 1$, then $\{x \oplus E_{ij}x : x \in \mathfrak{H}_1\}$ will be an invariant graph subspace for the reductive algebra $\mathscr{A}|(\mathfrak{H}_1 \oplus \mathfrak{H}_j)$ (cf. [8, Lemma 9.1]). Hence, by Lemma 5 there is an isometry U of $\mathfrak{H}_1$ onto $\mathfrak{H}_j$ with $A|\mathfrak{H}_j = U(A|\mathfrak{H}_1)U^*$, contradicting the hypotheses.

To show (b) we observe that $\mathfrak{N} \in \operatorname{Lat} \mathscr{A}^{(k)}$ implies $E_{i1}A_1 = A_1 E_{i1}$ for all A_1 in $\mathscr{A}|\mathfrak{H}_1$. Since $\mathscr{A}|\mathfrak{H}_1$ is strongly dense in $\mathscr{B}(\mathfrak{H}_1)$ (by Theorem 2 of [5]) E_{i1} must be scalar.

Theorem 7. *Let $\mathscr{A}$ be as in the above lemma and k an arbitrary positive integer. Then every invariant subspace of $\mathscr{A}^{(k)}$ is the orthogonal direct sum of (at most mk) special subspaces, and $\mathscr{A}$ is strongly dense in $\mathscr{B}(\mathfrak{H}_1) \oplus \ldots \oplus \mathscr{B}(\mathfrak{H}_m)$.*

Proof. Let $\mathfrak{N} \in \operatorname{Lat} \mathscr{A}^{(k)}$, $\mathfrak{N} \neq \{0\}$. Let y_1 be a nonzero element of $\mathfrak{N}$ with maximal number of zeros among its mk components. Let $\mathfrak{N}_1$ be the invariant subspace of $\mathscr{A}^{(k)}$ generated by y_1, i.e.,

$$\mathfrak{N}_1 = \{A^{(k)}y_1 : A \in \mathscr{A}\}^-.$$

Since $\mathfrak{N}_1 \subseteqq \mathfrak{N}$, every nonzero member of $\mathfrak{N}_1$ has the same nonzero components as y_1. Lemma 6 implies that $\mathfrak{N}_1$ is a special subspace of $(\mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m)^{(k)}$. But every special subspace is invariant under the algebra $[\mathscr{B}(\mathfrak{H}_1) \oplus \ldots \oplus \mathscr{B}(\mathfrak{H}_m)]^{(k)}$ and, hence, so is its orthogonal complement, because this algebra is self-adjoint. Thus $\mathfrak{N}_1^{\perp} \in \operatorname{Lat} \mathscr{A}^{(k)}$ and consequently $\mathfrak{N} \ominus \mathfrak{N}_1 \in \operatorname{Lat} \mathscr{A}^{(k)}$.

If $\mathfrak{N} \ominus \mathfrak{N}_1 \neq \{0\}$, we repeat the process and find a special subspace $\mathfrak{N}_2 \subseteqq \mathfrak{N} \ominus \mathfrak{N}_1$ and so on. Since $(\mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m)^{(k)}$ is the orthogonal direct sum of at most mk special subspaces, it follows that this process terminates after a finite number of steps and we obtain

$$\mathfrak{N} = \mathfrak{N}_1 \oplus \ldots \oplus \mathfrak{N}_r,$$

where each $\mathfrak{N}_i$ is a special subspace invariant not only under $\mathscr{A}^{(k)}$ but actually under $[\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_m)]^{(k)}$. Thus we have shown that

$$\operatorname{Lat}\mathscr{A}^{(k)} \subseteq \operatorname{Lat}[\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_m)]^{(k)}$$

for every integer k; it follows from a result of [9] that the strong closure of $\mathscr{A}$ is $\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_m)$ as asserted.

We now consider the most general reductive algebra whose invariant operator ranges are compact perturbations of its invariant subspaces. This can be done, in view of Theorem 3, by allowing isometries of the sort excluded in Lemma 6. Then $\mathscr{A}$ is easily seen to be unitarily equivalent to an algebra (denoted by $\mathscr{A}$ again) of the following form: The underlying space $\mathfrak{H}$ is expressed as $\mathfrak{H}_1^{(p_1)}\oplus\ldots\oplus\mathfrak{H}_r^{(p_r)}$; for each i, $\mathscr{A}|\mathfrak{H}_i^{(p_i)}=\mathscr{A}_i^{(p_i)}$, where $\mathscr{A}_i$ is a transitive algebra on $\mathfrak{H}_i$ whose proper invariant operator ranges are all compact operator ranges. Furthermore for no pair i, j with $i\neq j$, there is an isometry U such that $A_j=UA_iU^*$, $A_i\in\mathscr{A}_i$, $A_j\in\mathscr{A}_j$. (All such unitarily equivalent summands of $\mathscr{A}$ have already been put in the r "bunches".)

Theorem 8. *If all the invariant operator ranges of a reductive algebra $\mathscr{A}$ are compact perturbations of its invariant subspaces, then its strong closure is self-adjoint. More precisely, $\mathscr{A}$ is strongly dense in an algebra of the form*

$$[\mathscr{B}(\mathfrak{H}_1)]^{(p_1)}\oplus\ldots\oplus[\mathscr{B}(\mathfrak{H}_r)]^{(p_r)}$$

modulo a suitable unitary equivalence.

Proof. Let $\mathfrak{H}_i$ and p_i be as in the paragraph preceding the theorem, after a suitable unitarily equivalent form of $\mathscr{A}$ is chosen. Take a "representative of each bunch" and form the subspace

$$\mathfrak{K} = \mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_r.$$

Then $\mathscr{A}|\mathfrak{K}$ and $\mathfrak{K}$ satisfy all the hypotheses of theorem 7 and, hence, $\mathscr{A}|\mathfrak{K}$ is strongly dense in $\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_r)$. It follows that $\mathscr{A}$ is strongly dense in an algebra of the form exhibited above.

We can use Theorem 2 to construct non-trivial examples of reductive algebras satisfying the hypotheses of Theorem 8.

Example. For each i, $1\leq i\leq n$, let $\mathscr{A}_i$ be the algebra of all operators on $\mathfrak{H}$ leaving the range of an injective compact operator K_i invariant. Let $\mathscr{A}$ be the algebra

$$\{A_1\oplus\ldots\oplus A_n : A_i\in\mathscr{A}_i;\quad i=1,\ldots,n\}.$$

It can be verified that $\mathscr{A}$ is reductive and that if $\mathfrak{X}$ is an operator range invariant under $\mathscr{A}$, then $\mathfrak{X}$ is the range of an operator of the form $B_1\oplus\ldots\oplus B_n$, where each B_i is O, I, or a non-zero compact operator (by Theorem 2).

We conclude the paper with a question: can one get a density result for the reductive algebra $\mathscr{A}$ by merely assuming that its invariant operator ranges are all compact perturbations of arbitrary (not necessarily $\mathscr{A}$-invariant) subspaces? We observe that, even in the special case where $\mathscr{A}$ is transitive, the question seems to be non-trivial.

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Counting additive spaces of sets

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1. Introduction. In this paper we consider an asymptotic counting problem which occurs in a number of forms.

Definition 1. A family Ω of subsets of $\{1, 2, \dots, n\}$ is an *additive space* if $\emptyset \in \Omega$ and $AB \in \Omega$ whenever $A, B \in \Omega$. Two such families are isomorphic iff they are isomorphic as semigroups under union.

Definition 2. Let V_n be the set of all n-tuples from the two-element Boolean algebra $\{0, 1\}$. A subset U of V_k is called a *Boolean subspace* iff the vector $(0, 0, \dots, 0)$ belongs to the subspace, and whenever $u, v \in U$, the vector $u+v= =(\sup \{u_1, v_1\}, \dots, \sup \{u_n, v_n\})$ also belongs to U. Two subspaces are isomorphic iff they are isomorphic as semigroups under $+$.

Definition 3. A lattice is of *type*-(n, m) iff it has exactly m nonzero join irreducible elements and exactly n meet irreducible elements other than its highest element.

Remark. Every Boolean subspace of V_n has a partial order given by $v \leqq w$ iff $v+w=w$. This makes the subspace into a lattice, with the join operation being Boolean sum, and the meet operation on v, w being the sum of all Boolean vectors less than or equal to both v, w.

Definition 4. By a *Boolean matrix* of order n is meant an $n \times n$ matrix over the two-element Boolean algebra $\{0, 1\}$. Let B_n denote the set of all such matrices. We consider the sum and product of members of B_n to be the sum and product over the two-element Boolean algebra $\{0, 1\}$. Then B_n is a monoid under multiplication.

Definition 5. Two Boolean matrices A, B are $\mathcal{R}$*-equivalent* iff there exist Boolean matrices X, Y such that $AX=B$, $BY=A$. They are $\mathcal{L}$*-equivalent* iff there

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exist matrices U, V such that $UA=B$, $VB=A$. They are *$\mathscr{D}$-equivalent* iff there exists a matrix C such that $A\mathscr{R}C$ and $C\mathscr{L}B$. They are $\mathscr{H}$-equivalent iff they are both $\mathscr{R}$-equivalent and $\mathscr{L}$-equivalent.

Remark. $\mathscr{R}, \mathscr{L}, \mathscr{H}$ are equivalence relations, by a quick computation. As a relation, $\mathscr{D}$ is the composition $\mathscr{R}\circ\mathscr{L}$. It can be shown that $\mathscr{R}\circ\mathscr{L}=\mathscr{L}\circ\mathscr{R}$, and this implies $\mathscr{D}$ is also an equivalence relation.

Definition 6. An *ideal* of B_n is a subset I of B_n such that for all $x\in I$, $a, b\in B_n$, the element axb belongs to I. Principal ideals, principal left and right ideals are defined in a similar way.

Questions.

1. What is the asymptotic number of isomorphism classes of additive spaces of subsets of $\{1, 2, \ldots, n\}$ which have m generators other than the empty set?
2. What is the asymptotic number of isomorphism classes of Boolean subspaces of V_n with m generators other than $(0, 0, \ldots, 0)$?
3. What is the asymptotic number of isomorphism classes of lattices of type-(n, m)?
4. What is the asymptotic number of $\mathscr{D}$-classes of $n\times m$ Boolean matrices?
5. What is the asymptotic number of principal ideals in B_n?

The answers to 1—4 coincide, and for $m=n$ the fifth also has the same answer. We prove that if $n, m\to\infty$ in such a way that $\frac{n}{m}$ approaches a nonzero constant, the answer to 1—4 is $\frac{2^{nm}}{n!m!}$.

We also obtain information about related questions: the number of subspaces of V_n with m generators (not just isomorphism classes), the number of $\mathscr{R}, \mathscr{L}, \mathscr{H}$-classes. Also on the number of matrices X such that for some non-identity permutation matrices P, Q, $PXQ=X$ (for instance if X were a *projective plane*, such P, Q would give a collineation, the existence of P, Q is an unsolved problem [2], [5]).

2. Facts about Boolean matrices; lemmas. Equivalence of questions 1 and 2 is by an isomorphism of semigroups. Equivalence to question 3 follows by results about lattices involving duality, regarding lattices as idempotent abelian semigroups [1].

The *row space* of an $m\times n$ Boolean matrix is the subspace of V_n generated by its rows, with $(0, 0, \ldots, 0)$. Likewise there is a column space. It is known that the row space (as a subset of V_n) determines the $\mathscr{L}$-class of a matrix and the column space determines the $\mathscr{R}$-class [3]. Every subspace of V_n has a unique smallest generating set excluding $(0, 0, \ldots, 0)$. Such a set is called a *basis*. A basis for the row space of a matrix is called a *row basis*, and a basis for the column space of a matrix is called a *column basis*. It is known [3] that the isomorphism class of the row space determines the $\mathscr{L}$-class showing that questions 2, 4 have the same answer. It follows by semigroup theory [4] that for $n=m$ questions 4, 5 have the same answer.

We will begin to answer question 4. The *row rank* of a Boolean matrix is the number of elements in a row basis; likewise for the column rank. For any two Boolean matrices A, B we say $A \leqq B$ if $a_{ij}=1$ implies $b_{ij}=1$ for all i, j.

Lemma 1. *Let n, m tend to infinity in such a way that*

$$\frac{\log n}{m} \to 0, \quad \frac{\log m}{n} \to 0.$$

Then the proportion of $m \times n$ Boolean matrices which have both row rank m and column rank n tends to 1.

Proof. For a Boolean matrix A, let A_{i*} be its i^{th} row, and A_{*j} be its j^{th} column. Let N_{ij} denote the number of $m \times n$ Boolean matrices with $A_{i*} \geqq A_{j*}$ and M_{ij} the number with $A_{*i} \geqq A_{*j}$. Let N denote 2^{mn}, the number of all $m \times n$ Boolean matrices. Then for fixed $i \neq j$ we have

$$\frac{N_{ij}}{N} = \left(\frac{3}{4}\right)^n \quad \text{and} \quad \frac{M_{ij}}{N} = \left(\frac{3}{4}\right)^m.$$

Thus the number of matrices having no row greater than or equal to any other, and no column greater than or equal to any other is at least

$$\left(1-(n^2-n)\left(\frac{3}{4}\right)^m-(m^2-m)\left(\frac{3}{4}\right)^n\right)2^{mn}.$$

All these matrices have row rank m and column rank n. Under the given hypotheses this number divided by 2^{mn} will tend to 1. The proof of Lemma 1 is completed.

If two matrices of row rank m are $\mathscr{L}$-equivalent their rows must be permutations of each other by the uniqueness of a row basis. So $A=PB$. Likewise for $\mathscr{R}$-equivalence if the column rank is n. So for X of the type of this lemma, the only matrices $\mathscr{D}$-equivalent to it will be of the form PXQ. Thus such $\mathscr{D}$-classes have at most $n!m!$ members, and asymptotically the number of $\mathscr{D}$-classes is at least $\frac{2^{nm}}{n!m!}$. The proof of the reverse inequality will be based on a study of the equation $PXQ=X$.

Lemma 2. *If P or Q have no more than k cycles the number of solutions X of $PXQ=X$ is no more than 2^{kn} or 2^{km}, respectively.*

Proof. Let P have no more than k cycles. Choose one row from each cycle, and specify it. This can be done in 2^{kn} ways, and these rows determine the rest. Similarly for Q.

Lemma 3. *If a permutation P has at least k cycles, it will fix at least $m-2(m-k)$ numbers from $\{1, 2, \dots, m\}$.*

Proof. Immediate.

Lemma 4. *Let a permutation group G act on a set T of letters. If for any element g of G, g fixes at least $|T|-a$ letters with $a>0$, then there is a set of $|T|-2a+1$ letters fixed by every element of G.*

Proof. The action of G on T gives a linear representation R of G by permutation matrices. Let $o_1, \dots, o_f, o_{f+1}, \dots, o_{f+t}$ be the G-orbits contained in T, where $o_1, \dots, o_f$ contain only one element each, and the rest contain more than one element. Corresponding to this orbit decomposition we have a direct sum decomposition $R=R_1\oplus\dots\oplus R_f\oplus R_{f+1}\oplus\dots\oplus R_{f+t}$. A theorem in group representation theory (see [6], p. 280) states that

$$\sum_{g\in G} Tr(g) = (f+t)\,|G|.$$

But $Tr(g) \geqq |T|-a$ for any $g\in G$, and assuming $a>0$, $Tr(I) > |T|-a$. Therefore $|T|-a<f+t$. Yet $|T| \geqq f+2t$. Therefore

$$|T|-a<f+\frac{|T|-f}{2}$$

which yields the desired inequality on f.

3. Main results

Theorem 5. *Let n, m tend to infinity such that $\frac{n}{m}$ tends to a nonzero constant. Then the number of $\mathscr{D}$-classes of $m\times n$ matrices is asymptotically equal to $\frac{2^{mn}}{m!\,n!}$.*

Proof. By Lemma 1 and the considerations after its proof we need only prove this formula gives an asymptotic upper bound. Let $k=\sup\left\{\lim \frac{n}{m}, \lim \frac{m}{n}\right\}$.

Case 1. $\mathscr{D}$-classes containing some X such that $PXQ=X$ for some P, Q such that P has no more than $m-(4k+1)\log m$ cycles. (All logarithms are base 2.) For fixed P, Q with P satisfying the hypothesis of this case, there are at most

$$2^{(m-(4k+1)\log m)\,n}$$

matrices X such that $PXQ=X$, by Lemma 2. The number of possibilities for P, Q cannot exceed $n!m!$. Thus the number of possibilities for X in the present case is at most

$$2^{(m-(4k+1)\log m)\,n}\,n!\,m!.$$

Therefore also the number of $\mathscr{D}$-classes containing at least one such X is at most

$$2^{(m-(4k+1)\log m)n}n!m!.$$

The ratio of this number to $\dfrac{2^{nm}}{n!m!}$ will approach zero.

Case 2. $\mathscr{D}$-classes containing some matrix X such that $PXQ=X$ for some P, Q such that Q has no more than $n-(4k+1)\log n$ cycles. This case is treated like Case 1.

Case 3. $\mathscr{D}$-classes containing a matrix X such that $PXQ=X$ for some P, Q not both the identity, but such that $PXQ=X$ does not hold for any P, Q with P having no more than $m-(4k+1)\log m$ cycles or Q having no more than $n-(4k+1)\log n$ cycles. For such an X, choose a pair P, Q satisfying $PXQ=X$ such that $\sup\{m-\text{number of cycles in } P, n-\text{number of cycles in } Q\}$ is a maximum. Let s denote this maximum. We have $0<s<(4k+1)\sup\{\log m, \log n\}$. For a given X the set $\{P: PXQ=X \text{ for some } Q\}$ forms a group [2]. Each element of this group will fix at least $m-2s$ letters by Lemma 3. Therefore by Lemma 1 the whole group will fix at least $m-4s$ letters. There is a similar group of Q's which fixes at least $n-4s$ letters.

Fix s. We first choose a set of $4s$ letters which is to contain the set of all non-fixed letters under $\{P: PXQ=X \text{ for some } Q\}$. There are $\binom{m}{4s}$ such choices. There are $\binom{n}{4s}$ choices for a similar set for $\{Q: PXQ=X \text{ for some } P\}$. Provided these sets are chosen, we can choose P in $(4s)!$ ways to act on its set and Q in $(4s)!$ ways to act on its set. Once P, Q are chosen we can choose X in at most

$$2^{nm-s\min\{n,m\}}$$

ways by Lemma 2. Thus for a given s, there are at most

$$\binom{m}{4s}\binom{m}{4s}(4s)!(4s)!\,2^{nm-s\min\{n,m\}}$$

choices of X having the required value of s. However these X's do not all lie in different $\mathscr{D}$-classes. For any permutation matrices R, S, RXS will lie in the same $\mathscr{D}$-class and have the same value of s.

How many different matrices RXS are there for a given X? We have a group action of the product of two symmetric groups on such matrices, sending Y to RYS^{-1}. The isotropy group of X has order at most $((4s)!)^2$ by the remarks above about choosing P, Q such that $PXQ=X$. Thus a $\mathscr{D}$-class containing one X also contains at least

$$\frac{n!m!}{(4s)!(4s)!}$$

other matrices with the same s. Therefore the number of $\mathcal{D}$-classes containing matrices of this type for a given s is at most

$$\frac{m^{4s}n^{4s}2^{nm-s\min\{n,m\}}((4s)!)^2}{n!m!}.$$

Allowing any value of s we have at most

$$\max_{1\leq s\leq(4k+1)n_1}\frac{m^{4s}n^{4s}2^{nm-sn_2}((4s)!)^2(4k+1)\log n_1}{n!m!}$$

where $n_1=\max\{n, m\}$ and $n_2=\min\{n, m\}$. The ratio of this quantity to $\frac{2^{nm}}{n!m!}$ tends to zero.

Case 4. All PXQ are distinct so the $\mathcal{D}$-classes have at least $n!m!$ elements. There are at most $\frac{2^{nm}}{n!m!}$ $\mathcal{D}$-classes of this type. This proves the theorem.

Corollary 6. *Let N be the number of matrices X such that $PXQ=X$ for some P, Q not both the identity. Then if $n, m\to\infty$ in such a way that $\frac{n}{m}$ approaches a nonzero constant, $\frac{N}{2^{nm}}$ approaches* 0.

Theorem 7. *Under the hypotheses of Lemma* 1, *the number of $\mathcal{R}$ and $\mathcal{L}$-classes of $m\times n$ matrices are asymptotically equal to $\frac{2^{nm}}{n!}, \frac{2^{nm}}{m!}$ respectively. The number of $\mathcal{H}$-classes is asymptotically equal to 2^{nm}.*

Proof. For an upper bound, for instance for $\mathcal{R}$-classes, we have

$$\binom{2^m}{n}+\binom{2^m}{n-1}+\dots+\binom{2^m}{1}$$

by, for column rank k, choosing a set of k column vectors to be a column basis. This is less than or equal to

$$\binom{2^m}{n}\sum_{i=1}^{\infty}\left(\frac{n}{2^m-n}\right)^i$$

which gives the theorem. Similar methods apply in the other cases.

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Quasi-varieties of binary relations

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A *quasi-variety* of groups is the class of all groups satisfying a certain set of laws of the form $\wedge u_i = v_i \Rightarrow u = v$ for all values of the variables in the groups, where u, v, u_i, v_i are terms. Likewise quasi-varieties of semigroups, groupoids, etc. have been studied. Here we define an analogous concept for binary relations.

Definition. A *quasi-variety* of binary relations is the class of all binary relations R for which a class of laws of the following forms holds:

I. $$\big(\forall (i, j) \in K, x_i \mathrm{R} x_j\big) \Rightarrow x_a \mathrm{R} x_b,$$

II. $$\big(\wedge (i, j) \in K, x_i \mathrm{R} x_j\big) \Rightarrow x_a \overline{\mathrm{R}} x_b,$$

III. $$\big(\forall (i, j) \in K, x_i \mathrm{R} x_j\big) \Rightarrow x_a = x_b.$$

Such a law is specified by giving K, a, b. Here K can be any subset of the Cartesian product of any set with itself. The notation $x_a \overline{\mathrm{R}} x_b$ means "$x_a \mathrm{R} x_b$ is false".

Definition. Let S_a be a family of sets and let R_a be a binary relation defined on S_a for each a. Then the *direct product* of the R_a is the relation R on ΠS_a such that $x\mathrm{R}y$ if and only if $x_a \mathrm{R}_a y_a$ for each a, where x_a and y_a are the components in factor a of x and y.

Definition. Two binary relations $\mathrm{R}_1, \mathrm{R}_2$ on sets S_1, S_2 are *isomorphic* if and only if there exists an isomorphism f from S_1 to S_2 such that $x \mathrm{R}_1 y$ iff $f(x) \mathrm{R}_2 f(y)$.

Theorem 1. *A non-empty class of binary relations is a quasi-variety if and only if it is closed under direct products, restrictions, and isomorphisms.*

Proof. It follows from the form of the laws that quasi-varieties are in fact closed under direct products, restrictions, and isomorphisms.

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Let V be a non-empty class of binary relations closed under direct products, restrictions, and isomorphisms. Let T be a binary relation which satisfies every law of the forms I, II, III which holds for every member of V. Let S be the set on which T is defined. We will show there exists a member Q of V and a mapping $s \to x_s$ of S into the set on which Q is defined, such that $i\mathrm{T}j$ implies $x_i \mathrm{Q} x_j$ for all $i, j \in S$. Suppose not. Then $a\mathrm{T}b$ for some a, b. Let $K = \{(i, j): i\mathrm{T}j, i, j \in S$ and $(i, j) \neq (a, b)\}$.

$$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_a \overline{\mathrm{R}} x_b$$

is a law in V but is not true for T. This is contrary to assumption.

Let A be the set of all relations Q defined on subsets of S, and having the above property; let, furthermore, elements x_s be of the set on which Q is defined, such that $i\mathrm{T}j$ implies $x_i \mathrm{Q} x_j$ for all $i, j \in S$. Given a Q defined on some set as above, by restriction and isomorphism we obtain a Q′ defined on a subset of S. Thus A is non-empty. Let π be the direct product of all the relations of A. Let a_s for $s \in S$ be the element of π which is x_s in each factor.

Suppose $a_c = a_d$ for $c \neq d$. Then put $K = \{(i, j): i\mathrm{T}j\}$.

$$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_c = x_d$$

is a law which holds for all relations of V defined on subsets of S. This means that it is a law of V. But it does not hold for T. This is contrary to assumption. Therefore the a_s are distinct.

Suppose $c\overline{\mathrm{T}}d$ but $a_c \pi a_d$. Then with the same K

$$(\forall (i, j) \in K, x_i \mathrm{R} x_j) \Rightarrow x_c \mathrm{R} x_d$$

is a law in V but not for T. This is contrary to assumption. Therefore $c\overline{\mathrm{T}}d$ implies $a_c \overline{\pi} a_d$. It follows by construction that $c\mathrm{T}d$ implies $a_c \pi a_d$. Therefore the restriction of π to the a_s is isomorphic to T. Therefore T is isomorphic to a member of V. Therefore $\mathrm{T} \in V$. This shows V is a quasi-variety and proves the theorem.

Remark. Theorem 1 can be proved for any relational structure.

In the following we represent a binary relation R by the *Boolean matrix* $A = (a_{ij})$ such that $a_{ij} = 1$ if and only if $(i, j) \in \mathrm{R}$; otherwise $a_{ij} = 0$.

Theorem 2. *Every* 3×3 *Boolean matrix belongs to a proper sub-quasi-variety of the quasi-variety of all binary relations.*

Proof. Suppose A is a 3×3 Boolean matrix which does not belong to any proper sub-quasi-variety of the quasi-variety of all binary relations.

The set of binary relations R such that $x\mathrm{R}x$ holds for at most one x is a quasi-variety. So A has at least two 1's on its diagonal. Since reflexive matrices are a quasi-variety, A has exactly two ones on its diagonal.

Then if A does not belong to the quasi-variety given by the law

$$(i\mathrm{R}i \text{ and } j\mathrm{R}j \text{ and } i\mathrm{R}j \text{ and } j\mathrm{R}i) \quad \Rightarrow \quad i=j,$$

it will belong to the quasi-variety given by

$$(i\mathrm{R}i \text{ and } j\mathrm{R}j \text{ and } i\mathrm{R}j) \quad \Rightarrow \quad j\mathrm{R}i.$$

Theorem 3. *The binary relation corresponding to a* 4×4 *Boolean matrix of the form*

$$\begin{bmatrix} 0 & 1 & * & * \\ 1 & 1 & 1 & 0 \\ * & 1 & 1 & 1 \\ * & 1 & 1 & 1 \end{bmatrix}$$

does not belong to any proper sub-quasi-variety of the quasi-variety of all binary relations.

Proof. It suffices to show no non-trivial law of the form I, II or III can hold for this relation. Since 2R2, no law of type II can hold. Since 2R2, 3R3, 2R3, 3R2, no law of type III can hold. Suppose a law of type I holds, $a\neq b$, $(a, b)\notin K$. Then set $x_a=2$, $x_b=4$, all other $x_i=3$. Then for all $(i, j)\in K$, $x_i\mathrm{R}x_j$ is true but $x_a\mathrm{R}x_b$ is false. So the law does not hold. If a law of type I with $a=b$ is given, $(a, b)\notin K$, let $x_a=1$, all other $x_i=2$. The law will not hold.

Proposition 4. *Any quasi-variety containing all idempotent binary relations also contains all transitive binary relations.*

Proof. Any transitive relation is a restriction of an idempotent relation, by the following construction. Let T be transitive. For each x, y such that $x\mathrm{T}y$ add an element $z(x, y)$ to the set on which T is a relation. Define T_1 on the new set by $x\mathrm{T}_1y$ if and only if $x\mathrm{T}y$ or $x=y=z(u, v)$ for some u, v or $x=z(u, v)$, $y=v$ or $x=u$, $y=z(u, v)$. Let T_2 be the transitive relation generated by T_1, i.e. $x\mathrm{T}_2y$ if and only if $x\mathrm{T}_1x_1$, $x_1\mathrm{T}_1x_2, \dots, x_k\mathrm{T}_1y$ for some sequence $x_1, \dots, x_k$. Then T_2 is idempotent and T is a restriction of T_2.

Proposition 5. *Any quasi-variety containing only idempotent binary relations is contained in one of the following two quasi-varieties:* (i) *all transitive relations such that* $x\mathrm{R}y$ *implies* $x\mathrm{R}x$, *and* (ii) *all transitive relations such that* $x\mathrm{R}y$ *implies* $y\mathrm{R}y$.

Proof. Suppose a quasi-variety V contains relations R_1 and R_2 which are transitive but such that (i) fails for R_1 and (ii) fails for R_2. Let $a\mathrm{R}_1b$ and $a\overline{\mathrm{R}}_1a$ and $c\mathrm{R}_2d$ and $d\overline{\mathrm{R}}_2d$. Then if we restrict $\mathrm{R}_1\times\mathrm{R}_2$ to (a, c), (b, d) we have a relation whose matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Neither is idempotent.

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On the strong summability of Fourier series and the classes H^ω

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1. Let f be a 2π-periodic integrable function and let $\{s_\varkappa\}$ be the sequence of the partial sums of the Fourier series of this function.

FREUD [1] proved that if $1<p<\infty$ and

$$\left\| \sum_{\varkappa=0}^{\infty} |f-s_\varkappa|^p \right\| < \infty \text{ [1]} \tag{1}$$

then $f \in \operatorname{Lip} \frac{1}{p}$. LEINDLER and NIKIŠIN [3] proved that under the condition (1) with $p=1$,

$$\omega(x,f) = O\left(x \log \frac{1}{x}\right) \quad \text{as} \quad x \to 0,$$

but no estimate better than this can be given. OSKOLKOV [7] and SZABADOS [9] (independently) proved that condition (1) with $0<p<1$ implies $f \in \operatorname{Lip} 1$. This is an answer to a problem of LEINDLER [4] in connection with the above result of LEINDLER and NIKIŠIN.

In this paper we investigate the problem to find a necessary and sufficient condition for a monotonic sequence $\{\lambda_\varkappa\}$ such that the condition

$$\left\| \sum_{\varkappa=0}^{\infty} \lambda_\varkappa |f-s_\varkappa|^p \right\| < \infty, \quad 0<p<\infty$$

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[1]) $\|f\| = \sup |f(x)|, \quad 0 \leq x \leq 2\pi.$

should imply $f\in H^\omega$, where ω is a fixed modulus of continuity and H^ω denotes the set of functions f having modulus of continuity $\omega(f,\delta)$ with $\omega(f,\delta)=O(\omega(\delta))$. For a monotonic sequence $\{\lambda_\varkappa\}$ and $0<p<\infty$ we denote

$$S_p\{\lambda_\varkappa\}=\left\{f:\left\|\sum_{\varkappa=0}^{\infty}\lambda_\varkappa|f-s_\varkappa|^p\right\|<\infty\right\}.$$

We prove the following

Theorem. *Let $\{\lambda_\varkappa\}$ be a positive monotonic (nondecreasing or nonincreasing) sequence, furthermore let ω be a modulus of continuity and $0<p<\infty$. Then*

i) *condition*

$$\sum_{\varkappa=1}^{n}(\varkappa\lambda_\varkappa)^{-1/p}=O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2}$$

implies

$$S_p\{\lambda_\varkappa\}\subset H^\omega; \tag{3}$$

ii) *if there exists a number θ such that $0\leqq\theta<1$ and*

$$\varkappa^\theta\lambda_\varkappa\uparrow \tag{4}$$

then, conversely, (3) *implies* (2).

Obviously, this Theorem includes all the results mentioned above and, hereby, we give an answer to a problem raised in [6]. Furthermore, our Theorem includes some results of Leindler [2].

2. To prove our Theorem we require the following lemmas.

Lemma 1. *If $\{a_m\}$ is a nonincreasing positive sequence and if $q>0$, then there exists a constant $C_q>0$ not depending on n such that*

$$\sum_{m=0}^{n}2^m a_m\leqq C_q\sum_{m=0}^{n}2^m a_m\left(\frac{a_{m+1}}{a_m}\right)^q\quad(n=1,2,\ldots).$$

Proof. Let $\{m_i\}$ and $\{M_i\}$ $(i=1,2,\ldots)$ be two sequences of natural numbers such that

$$a_{m+1}>\frac{1}{4}a_m\quad\text{for}\quad M_i\leqq m<m_{i+1} \tag{5}$$

and

$$a_{m+1}\leqq\frac{1}{4}a_m\quad\text{for}\quad m_i\leqq m<M_i. \tag{6}$$

By (6) we obtain

$$a_{m_i+r}\leqq 4^{-r}a_{m_i}\quad(r=0,\ldots,M_i-m_i-1;\ i\geqq 2),$$

therefore, if $i \geqq 2$, then

$$\sum_{m=m_i}^{M_i-1} 2^m a_m = \sum_{r=0}^{M_i-m_i-1} 2^{m_i+r} a_{m_i+r} \leqq 2^{m_i} a_{m_i} \sum_{r=0}^{\infty} 2^{-r} \leqq$$

$$\leqq 4^{1+q} 2^{m_i-1} a_{m_i-1} \left(\frac{a_{m_i}}{a_{m_i-1}}\right)^q.$$

Furthermore, (5) implies

$$\sum_{m=M_i}^{m_{i+1}-1} 2^m a_m \leqq 4^q \sum_{m=M_i}^{m_{i+1}-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q$$

and the last two inequalities give for $i \geqq 2$

$$\sum_{m=m_i}^{m_{i+1}-1} 2^m a_m \leqq 4^{1+q} \sum_{m=m_i-1}^{m_{i+1}-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q. \tag{6}$$

If $m_i \leqq n < m_{i+1}$ and $i \geqq 2$, then

$$\sum_{m=m_i}^{n} 2^m a_m \leqq 4^{1+q} \sum_{m=m_i-1}^{n} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q. \tag{7}$$

The proof runs exactly as before.

Finally, we set

$$C = \max_{1 \leqq n \leqq m_2} \sum_{m=0}^{n} 2^m a_m \Big/ \sum_{m=0}^{n-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q;$$

then

$$\sum_{m=0}^{n} 2^m a_m \leqq C \sum_{m=0}^{n-1} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q$$

for $n=1, \dots, m_2$ and (6) and (7) imply

$$\sum_{m=0}^{n} 2^m a_m = \sum_{m=0}^{m_2-1} + \sum_{i=2}^{\varkappa-1} \sum_{m=m_i}^{m_{i+1}-1} + \sum_{m=m_k}^{n} \leqq$$

$$\leqq C \sum_{m=0}^{m_2-2} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q + 8 \cdot 4^q \sum_{m=m_2-1}^{n} 2^m a_m \left(\frac{a_{m+1}}{a_m}\right)^q$$

for $m_\varkappa \leqq n < m_{\varkappa+1}$ $(\varkappa \geqq 2)$. Therefore, our inequality is true with

$$C_q = \max (C, 8 \cdot 4^q).$$

and Lemma 1 is proved.

Lemma 2. *Let $\{\lambda_\varkappa\}$ and p be as in the Theorem. Then $f \in S_p\{\lambda_\varkappa\}$ implies*

$$\sum_{m=0}^{n} 2^m E_{2^m}(f) \leqq C_{p,\lambda}(f) \sum_{m=0}^{n} 2^m (2^m \lambda_{2^m})^{-1/p} \quad (n = 1, 2, \dots), \tag{8}$$

where $C_{p,\lambda}(f)$ is a positive constant and $E_\varkappa(f)$ is the best approximation of f by trigonometric polynomials of degree at most $\varkappa$.

Proof. First we assume $p \geqq 1$. Then by Hölder's inequality we have

$$E_{2n}(f) \leqq \left\| \frac{1}{n+1} \sum_{\varkappa=n}^{2n} s_\varkappa - f \right\| \leqq \frac{1}{n+1} \left(\sum_{\varkappa=n}^{2n} 1 \right)^{1-(1/p)} \left\| \left\{ \sum_{\varkappa=n}^{2n} |f-s_\varkappa|^p \right\}^{1/p} \right\| \leqq$$

$$\leqq \left\| \left\{ \frac{1}{n} \sum_{\varkappa=n}^{2n} |f-s_\varkappa|^p \right\}^{1/p} \right\| \leqq C(f)(n\lambda_n^*)^{1/p} \quad (n=1,2,\ldots),$$

where $\lambda_n^* = \min(\lambda_n, \lambda_{2n})$. This implies (8) for $p \geqq 1$.

In the case $0<p<1$ we require the following result of [5]:

$$E_n(f) \left[\frac{E_{2n}(f)}{E_n(f)} \right]^{1/p_2} \leqq C_p \left\| \left\{ \frac{1}{n} \sum_{\varkappa=n}^{2n} |f-s_\varkappa|^p \right\}^{1/p} \right\| \quad (n=1,2,\ldots),$$

where C_p depends only on p. Using this inequality, by Lemma 1 we obtain (8).

Lemma 3. *If $a_\varkappa \geqq 0$ and the function*

$$f \sim \sum_{\varkappa=1}^{\infty} a_\varkappa \sin \varkappa x$$

belongs to the class H^ω, then

$$\sum_{\varkappa=1}^{n} \varkappa a_\varkappa = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Proof. Since $f(0)=0$, $f \in H^\omega$ implies

$$\max_{0<t\leqq x} |f(x)| \leqq C\omega(x), \quad 0<x<\pi.$$

Therefore,

$$2 \sum_{\varkappa=1}^{\infty} \frac{a_\varkappa}{\varkappa} \sin^2 \frac{\varkappa x}{2} = \int_0^x f(t)\,dt \leqq Cx\omega(x).$$

If we take $x=\frac{\pi}{n}$, then

$$n^{-2} \sum_{\varkappa=0}^{n} \varkappa a_\varkappa = \sum_{\varkappa=1}^{n} \frac{a_\varkappa}{\varkappa} \left(\frac{\varkappa}{n}\right)^2 \leqq \sum_{\varkappa=1}^{n} \frac{a_\varkappa}{\varkappa} \sin^2 \frac{\varkappa\pi}{2n} \leqq \frac{C}{n} \omega\left(\frac{1}{n}\right)$$

for $n=1,2,\ldots$ and Lemma 3 is proved.

Lemma 4. *If $\lambda_\varkappa \uparrow$ or $\lambda_\varkappa \downarrow$ and if there exists a number θ, $0 \leqq \theta < 1$, such that $\varkappa^\theta \lambda_\varkappa \uparrow$, then the function*

$$f \sim \sum_{\varkappa=1}^{\infty} \frac{1}{\varkappa} (\varkappa\lambda_\varkappa)^{-1/p} \sin \varkappa x \tag{9}$$

belongs to the class $S_p(\lambda_\varkappa\}$, $0<p<\infty$.

Proof. To prove that $f \in S_p\{\lambda_\varkappa\}$ we fix $0<x<\pi$ and choose N such that

$$\frac{1}{N+1} < x \leqq \frac{1}{N}.$$

We consider the series

$$\sum_{\varkappa=1}^{\infty} \lambda_\varkappa |f(x)-S_\varkappa(x)|^p \leqq C_p \sum_{\varkappa=1}^{N} \lambda_\varkappa \left| \sum_{n=\varkappa+1}^{N+1} \frac{1}{n}(n\lambda_n)^{-1/p} \sin nx \right|^p +$$

$$+\sum_{\varkappa=1}^{N} \lambda_\varkappa \left| \sum_{n=N+2}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p} \sin nx \right|^p + \sum_{\varkappa=N+1}^{\infty} \lambda_\varkappa \left| \sum_{n=\varkappa+1}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p} \sin nx \right|^p \equiv$$

$$\equiv c_p(\textstyle\sum_1+\sum_2+\sum_3).$$

First we assume that $\lambda_\varkappa\downarrow$. Then $\varkappa^\theta\lambda_\varkappa\uparrow$ with some $\theta>1-p$. Hence, $\dfrac{\theta-1}{p}>-1$, and we have

$$\textstyle\sum_1 \leqq x^p \displaystyle\sum_{\varkappa=1}^{N} \lambda_\varkappa \left[\sum_{n=\varkappa+1}^{N+1} (n\lambda_n)^{-1/p}\right]^p \leqq x^p \sum_{\varkappa=1}^{N} \varkappa^{-\theta} \left[\sum_{n=1}^{N} n^{(\theta-1)/p}\right]^p =$$

$$= O(x^p N^{1-\theta} N^{\theta-1+p}) = O(1).$$

Furthermore,

$$\textstyle\sum_2 \leqq \displaystyle\sum_{\varkappa=1}^{N} \lambda_\varkappa \left[\sum_{n=N+2}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p}\right]^p \leqq$$

$$\leqq N^{-\theta}\lambda_N^{-1}\left(\sum_{n=N+2}^{\infty} n^{-1-(1-\theta)/p}\right)^p \sum_{\varkappa=1}^{N}\lambda_\varkappa = O(N^{-\theta}\cdot N\cdot N^{-(1-\theta)}) = O(1).$$

In order to estimate $\sum_3$ we make use of the inequality

$$\left|\sum_{n=\varkappa+1}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p} \sin nx\right| \leqq \frac{c}{\varkappa x}(\varkappa\lambda_\varkappa)^{-1/p}$$

for $0<x<\pi$. Hence,

$$\textstyle\sum_3 \leqq Cx^{-p} \displaystyle\sum_{\varkappa=N+1}^{\infty} \varkappa^{-1-p} = O(x^{-p}N^{-p}) = O(1).$$

The proof in the case $\lambda_\varkappa\uparrow$ is almost the same as for $\lambda_\varkappa\downarrow$, we only have to replace condition (4) by $\lambda_\varkappa\uparrow$. Therefore, we can omit the details.

The proof is completed.

3. Proof of the Theorem. i) If $f\in S_p\{\lambda_\varkappa\}$ then using (2), (8) and the following inequality of Stečkin [8]:

$$\omega(2^{-n}, f) \leqq C2^{-n}\sum_{m=0}^{n-1} 2^m E_{2^m}(f) \quad (n=1,2,\ldots)$$

we obtain $\omega(2^{-n},f)=O(\omega(2^{-n}))$ and $f\in H^\omega$.

ii) If condition (2) is not fulfilled, then, by Lemma 3, the function given in (9) does not belong to H^ω, but, by Lemma 4, it belongs to the class $S_p\{\lambda_\varkappa\}$.

Thus the Theorem is proved.

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Entropy of states of a gage space

ARTHUR LIEBERMAN

Let (H, A, m) be a regular gage space. Let ϱ, σ, and $\psi=\lambda\varrho+(1-\lambda)\sigma$, $0<\lambda<1$, be regular states. The density operator D_ϱ of a regular state is a non-negative (possibly unbounded) self-adjoint measurable operator. Let F be a continuous convex function on $[0, \infty)$ and define the entropy of ϱ by $e(\varrho)=m(F(D_\varrho))$. Conditions are obtained, in terms of $e(\varrho)$ and $e(\sigma)$, for $e(\psi)$ to be $-\infty$, finite, ∞, or undefined. If both ϱ and σ have finite entropy, then ψ has finite entropy and $e(\psi)\geqq\lambda e(\varrho)+(1-\lambda)e(\sigma)$; if $A=B(H)$, F is strictly convex, and $\varrho\neq\sigma$, then strict inequality is obtained. These results are restated as inequalities concerning the trace of a convex function of an operator.

1. Introduction

We work in the context of a regular gage space (H, A, m); H is a Hilbert space, A is a von Neumann algebra on H, and m is a faithful semi-finite normal trace on A. (See [4] for definitions and notation.) A regular state of A is a positive linear functional ϱ on A with $\varrho(I)=1$, where I is the identity operator on H, which is strongly continuous on the unit ball of A. If ϱ is a regular state of A, then by [4] Theorem 14 there is a unique operator $D_\varrho\in L^1(H, A, m)$ with $D_\varrho\geqq 0$, $m(D_\varrho)=1$, and $\varrho(T)=m(D_\varrho T)$ for all $T\in A$; D_ϱ is called the density operator of ϱ.

The entropy of a regular state ϱ is usually defined by $e(\varrho)=m(-D_\varrho \ln D_\varrho)$, cf. [3] Chapter V and [5]. Both von Neumann and Segal suggested defining the entropy by $e(\varrho)=m(F(D_\varrho))$, where F is an arbitrary continuous convex function on $[0, \infty)$; we use this definition for the remainder of this paper. The results basically say that the mixing of states cannot reduce entropy.

Bendat and Sherman [1] determined when a continuous convex function defined on an interval is operator convex; i.e., when $F(\lambda K+(1-\lambda)L)\geqq\lambda F(K)+(1-\lambda)F(L)$ holds for bounded self-adjoint operators K and L whose spectra

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are contained in the domain of F. Below we show that $m(F(\lambda K+(1-\lambda)L))\geqq \geqq\lambda m(F(K))+(1-\lambda)m(F(L))$ holds under suitable hypotheses for self-adjoint measurable operators K and L; this is merely a restatement of the fact that mixing of states cannot reduce entropy.

2. Statement of the results

Theorem 1. *Let (H, A, m) be a gage space with regular states ϱ and σ. Let $0<\lambda<1$, and $\psi=\lambda\varrho+(1-\lambda)\sigma$. Assume* $\liminf_{x\to\infty} F(x)/F(kx)>0$ *for each $k>1$. Then:*

A. *$e(\psi)$ is defined iff both $e(\varrho)$ and $e(\sigma)$ are defined and $\{e(\varrho), e(\sigma)\}\neq\{-\infty, \infty\}$.*

B. *$e(\psi)$ is finite iff both $e(\varrho)$ and $e(\sigma)$ are finite.*

C. *$e(\psi)=\infty$ iff $\{\infty\}\subseteqq\{e(\varrho), e(\sigma)\}\subseteqq R\cup\{\infty\}$, where R is the set of real numbers.*

D. *$e(\psi)=-\infty$ iff $\{-\infty\}\subseteqq\{e(\varrho), e(\sigma)\}\subseteqq\{-\infty\}\cup R$.*

Corollary 1. *Let (H, A, m) be a gage space with regular states ϱ and σ. Let $0<\lambda<1$, and $\psi=\lambda\varrho+(1-\lambda)\sigma$. Then*

A. *$e(\psi)$ is defined if both $e(\varrho)$ and $e(\sigma)$ are defined and $\{-\infty, \infty\}\neq\{e(\varrho), e(\sigma)\}$.*

B. *$e(\psi)$ is finite if both $e(\varrho)$ and $e(\sigma)$ are finite.*

C. *$e(\psi)=\infty$ if $\{\infty\}\subseteqq\{e(\varrho), e(\sigma)\}\subseteqq R\cup\{\infty\}$, and* $\lim_{x\to\infty} F(x)=-\infty$.

Theorem 2. *Let (H, A, m) be a gage space with regular states ϱ and σ. Let $0<\lambda<1$ and $\psi=\lambda\varrho+(1-\lambda)\sigma$. If $e(\varrho)$ and $e(\sigma)$ are finite, then $e(\psi)$ is finite and $e(\psi)\geqq\lambda e(\varrho)+(1-\lambda)e(\sigma)$. If $A=B(H)=$all bounded operators on H, $\varrho\neq\sigma$, and the function F is strictly convex, then $e(\psi)>\lambda e(\varrho)+(1-\lambda)e(\sigma)$.*

Corollary 2. *Let (H, A, m) be a gage space. Let $K, L\in L^1(H, A, m)$. Assume that either $K\geqq 0$ and $L\geqq 0$ or $m(I)<\infty$ and K and L are both bounded from below (or from above). Let F be a continuous convex function defined on an interval which includes the spectra of K and L and let $0<\lambda<1$. If $F(K), F(L)\in L^1(H, A, m)$, then $F(\lambda K+(1-\lambda)L)\in L^1(H, A, m)$, and $m(F(\lambda K+(1-\lambda)L))\geqq\lambda m(F(K))+ +(1-\lambda)m(F(L))$. If $A=B(H)$, $K\neq L$, and F is strictly convex, then $m(F(\lambda K+(1-\lambda)L))>\lambda m(F(K))+(1-\lambda)m(F(L))$.*

Remark. In Theorem 2 and Corollary 2, the restriction that $A=B(H)$ in order to have strict inequality seems unnecessary; this was first suggested by SEGAL [5]. We know of no example which requires this extra hypothesis, but are unable to prove strict inequality without it.

3. Proof of the results

Corollaries 1 and 2 are restatements of Theorems 1 and 2 and require no proof.

We now introduce some notation. The self-adjoint operator T has spectral decomposition $T=\int_{-\infty}^{\infty} \alpha dP_T(\alpha)$; the function P_T is continuous from the left. If S is a Borel measurable set of real numbers, then $P_T(S)$ is the spectral projection of T for the set S. The spectral distribution function Λ_T is defined by $\Lambda_T(x)=\sup\{\lambda : m(P_T[\lambda,\infty))\geqq x\}$; the domain of Λ_T is $(0, m(I)]$ if $m(I)<\infty$ and $(0,\infty)$ if $m(I)=\infty$. $\Lambda_T(x)$ is a nonincreasing function of x and is continuous from the left. $m(P_T(\Lambda_T(x),\infty))=x$ if P has no point mass at $\Lambda_T(x)$ and $T\in L^1(H, A, m)$. The properties of the spectral distribution function are developed in [2]. To simplify the notation, we will frequently write P_ϱ for P_{D_ϱ} and Λ_ϱ for Λ_{D_ϱ}.

Lemma 1. *Let (H, A, m) be a gage space, let $K\in L^1(H, A, m)$ with $K\geqq 0$, and let F be a continuous function on (r,∞), where $P_K\{r\}=0$. Then $\int_r^\infty F(\lambda)\,dm(P_K(\lambda))=$ $=\int_0^{m(P_K[r,\infty))} F(\Lambda_K(x))dx$ in the sense that if either integral is defined, then both integrals are defined and are equal. In addition, if F is continuous on $[0,\infty)$, then*

$$\int_{[0,\infty)} F(\lambda)\,dm(P_K(\lambda))=\int_0^{m(I)} F(\Lambda_K(x))\,dx.$$

Proof. Let $s>r$ with $P_K\{s\}=0$. We will show below that $\int_r^s F(\lambda)dm(P_K(\lambda))=$ $=\int_{m(P_K[s,\infty))}^{m(P_K[r,\infty))} F(\Lambda_K(x))dx$. The first conclusion of the theorem will follow by taking the limit as $s\to\infty$. The second conclusion then follows by taking the limit as $r\to 0$.

Let $P=\{x_1, x_2, \ldots, x_{n+1}\}$ be a partition of $[r,s]$ with $m(P_K\{x_i\})=0$ for $1\leqq i\leqq n+1$. Then $\int_r^s F(\lambda)dm(P_K(\lambda))\sim\sum_{i=1}^n F(\Lambda_K(m(P_K[x_i,\infty))))\,m(P_K[x_i,x_{i+1}])=$

$$=\sum_{i=1}^n F(\Lambda_K(m(P_K[x_i,\infty))))(m(P_K[x_i,\infty))-m(P_K[x_{i+1},\infty)))\sim\int_{m(P_K[s,\infty))}^{m(P_K[r,\infty))} F(\Lambda_K(x))\,dx.$$

Note that, although $m(P_K[x_i,\infty))-m(P_K[x_{i+1},\infty))$ may be large due to the spectrum of K having point masses in the interval (x_i, x_{i+1}), $F(\Lambda_K(\alpha))$ is nearly constant on the interval $m(P_K[x_{i+1},\infty))<\alpha\leqq m(P_K[x_i,\infty))$ since for α in this interval, $x_i\leqq\Lambda_K(\alpha)<x_{i+1}$.

Proof of Theorem 1. There are essentially four different non-trivial possibilities for F:

A. $F'(0)>0, \lim_{x\to\infty} F(x)=-\infty.$

B. $F'(0)>0, \lim_{x\to\infty} F(x)=\infty.$

C. $F'(0)>0, \lim_{x\to\infty} F(x)=k$, where $0<k<\infty$.

D. $F'(0)\leqq 0, \lim_{x\to\infty} F(x)=-\infty.$

Theorem 1 will be proved for case A since this is the most difficult case; the proofs for the other cases are trivial modifications and parts of the results are vacuous in the other cases. For the sake of simplicity, we assume $F(0)=0$; if $F(0)\neq 0$, little change is needed if $m(I)$ is finite and the results become essentially vacuous if $m(I)=\infty$. We further assume that F has a relative maximum at $x=1$, $F(1)=1$, and that $F(2)=0$. We will prove the "if" parts of B, C and D. The remainder of the proof is essentially redundant.

Assume now that $e(\varrho)$ and $e(\sigma)$ are both finite. $e(\psi)$ can be infinite in two ways: ψ can be highly concentrated so that D_ψ is unbounded and $e(\psi)=-\infty$, or ψ can be so spread out that D_ψ has very large support and $e(\psi)=\infty$.

Let $\alpha>0$ and $x\in H, x\neq 0$. If $P_\psi[\alpha,\infty)x=x$, then $\lambda(D_\varrho x,x)+(1-\lambda)(D_\sigma x,x)\geqq \geqq\alpha\|x\|^2$, so that either $P_\varrho[\alpha,\infty)x\neq 0$ or $P_\sigma[\alpha,\infty)x\neq 0$. By [2, lemma 2], $m(P_\psi[\alpha,\infty))\leqq m(P_\varrho[\alpha,\infty))+m(P_\sigma[\alpha,\infty))$. Then

$$\int_2^\infty F(\alpha)\,dm(P_\psi(\alpha))\geqq\int_2^\infty F(\alpha)\,dm(P_\varrho(\alpha))+\int_2^\infty F(\alpha)\,dm(P_\sigma(\alpha))>-\infty,$$

have finite entropy.

Now let $0<\alpha<1$. $m(P_\psi(\alpha,1])=m(P_\psi(\alpha,\infty))-m(P_\psi(1,\infty))\leqq m(P_\varrho(\alpha,\infty))+ +m(P_\sigma(\alpha,\infty))-m(P_\psi(1,\infty))=m(P_\varrho(\alpha,1])+m(P_\sigma(\alpha,1])+m(P_\varrho(1,\infty))+m(P_\sigma(1,\infty))- -m(P_\psi(1,\infty))$. Let $c=m(P_\varrho(1,\infty))+m(P_\sigma(1,\infty))-m(P_\psi(1,\infty))$. Then $0\leqq c<\infty$, and $m(P_\psi(\alpha,1])\leqq m(P_\varrho(\alpha,1])+m(P_\sigma(\alpha,1])+c$. Let M be the unique Borel measure on $(0,1]$ such that $M(\alpha,1]=m(P_\varrho(\alpha,1])+m(P_\sigma(\alpha,1])+c$. Then $\int_{(0,1]} F(\alpha)\,dm(P_\psi(\alpha))\leqq \leqq\int_{(0,1]} F(\alpha)\,dM(\alpha)$, since F is non-negative and non-decreasing on $(0,1]$, and $\int_{(0,1]} F(\alpha)\,dM(\alpha)<\infty$ since ϱ and σ have finite entropy.

We now prove part C. Assume $e(\varrho)=\infty$. Then $\int_0^1 F(\alpha)\,dm(P_\varrho(\alpha))=\infty$, and by lemma 1, $\int_c^\infty F(\Lambda_\varrho(\alpha))\,d\alpha=\infty$ for some c such that $\Lambda_\varrho(c)\leqq 1$ and $\Lambda_\psi(c)\leqq 1$. Note that F is non-negative and non-decreasing on $[0,1]$. Since $\psi=\lambda\varrho+(1-\lambda)\sigma$,

$D_\psi \geqq \lambda D_\varrho$, so by [2] Corollary 1, $\Lambda_\psi(\alpha) \geqq \Lambda_{\lambda D_\varrho}(\alpha) = \lambda \Lambda_\varrho(\alpha)$. Then for $\alpha \geqq c$, $F(\Lambda_\psi(\alpha)) \geqq F(\lambda \Lambda_\varrho(\alpha)) \geqq \lambda F(\Lambda_\varrho(\alpha))$ by convexity. Then

$$\int_c^\infty F(\Lambda_\psi(\alpha))\,d\alpha \geqq \lambda \int_c^\infty F(\Lambda_\varrho(\alpha))\,d\alpha = \infty.$$

We now prove part D. Assume $e(\varrho) = -\infty$, so that $\int_0^\infty F(\alpha)\,dm(P_\varrho(\alpha)) = -\infty$. Choose $\varepsilon > 0$ and $q > 0$ so that $F(x)/F(x/\lambda) \geqq \varepsilon$ for $x \geqq \lambda q$. Since $D_\psi \geqq \lambda D_\varrho$, $m(P_\psi[\alpha, \infty)) \geqq m(P_{\lambda D_\varrho}[\alpha, \infty)) = m(P_\varrho[\alpha/\lambda, \infty))$. Then $\int_0^\infty F(\alpha)\,dm(P_\varrho(\alpha)) = -\infty$ implies $\int_q^\infty F(\alpha)\,dm(P_\varrho(\alpha)) = -\infty$, so that $\int_{\lambda q}^\infty F(\alpha/\lambda)\,dm(P_\varrho(\alpha/\lambda)) = -\infty$. Then $\int_{\lambda q}^\infty F(\alpha/\lambda)\,dm(P_\psi(\alpha)) = -\infty$ so $\int_{\lambda q}^\infty F(\alpha)\,dm(P_\psi(\alpha)) = -\infty$ and $e(\psi) = -\infty$.

Lemma 2. *Let R and S be either finite sequences with the same number of members or countable sequences. Assume $r_k \geqq r_{k+1} \geqq 0$, $s_k \geqq s_{k+1} \geqq 0$, $\sum_k r_k = \sum_k s_k$, and $\sum_{k=1}^{j} r_k \geqq \sum_{k=1}^{j} s_k$ for $j \geqq 1$. Then there is a doubly stochastic matrix M with $s_j = \sum_k m_{jk} r_k$ for $j \geqq 1$.*

Proof. If R and S are finite sequences the result is well known; our proof will contain this case if R and S are extended to countable sequences by adding a string of zeroes at the end. Let R and S be countable sequences and assume $r_k \neq 0$ for all k. M will be constructed one row at a time; each row of M will have finitely many non-zero entries. Let $w(1)$ be the smallest integer such that $s_1 \geqq r_{w(1)}$. Express s_1 as a convex combination of $\{r_i : 1 \leqq i \leqq w(1)\}$ to obtain the first row of M.

Assume $k-1$ rows of M have been obtained. If $s_k \geqq r_{w(k-1)}$, let $w(k) = 1 + w(k-1)$; otherwise, let $w(k)$ be the smallest integer such that $s_k \geqq r_{w(k)}$. We will show that s_k can be expressed as a convex combination of $\{r_i : 1 \leqq i \leqq w(k)\}$ such that $\sum_{i=1}^{k} m_{ij} \leqq 1$ for $1 \leqq j \leqq w(k)$ by showing that there is such a convex combination which is $\geqq s_k$ and that there is such a convex combination (namely, $\sum_{i=1}^{w(k)-1} 0 r_i + 1 r_{w(k)}$) which is $\leqq s_k$.

When $\sum_i m_{ij} = 1$, we will say r_j is "used up". Let the number $c = \sum_{i=1}^{k} c_i r_i$ be formed as follows: c_1 is chosen so that r_1 is used up; i.e., $c_1 = 1 - \sum_{i=1}^{k-1} m_{i1}$. Choose c_2 so that $c_1 + c_2 \leqq 1$ and r_2 is used up if possible; $c_2 = \min\left(1 - c_1, 1 - \sum_{i=1}^{k-1} m_{i2}\right)$. Continue this process until c_k is chosen. Then $c \geqq s_k$ follows from the hypothesis that $\sum_{i=1}^{k} r_i \geqq \sum_{i=1}^{k} s_i$.

This completes the construction of the matrix M. Clearly $s_j=\sum_k m_{jk}r_k$ for all j, $m_{jk}\geqq 0$ for all j, k, and the sum of the elements of any row of M is 1. It remains to show that the sum of the elements of any column of M is 1. $1=\sum_i s_i=\sum_i\sum_j m_{ij}r_j=$ $=\sum_j\sum_i m_{ij}r_j$; since all terms are non-negative the interchange of order of summation is valid. Since $1=\sum_j r_j$, $0=\sum_j(1-\sum_i m_{ij})r_j$. By the construction of M, $(1-\sum_i m_{ij})\geqq 0$ for each j. Since $r_j\neq 0$ for all j, $1=\sum_i m_{ij}$.

If $r_j=0$ for some j, then $r_k=0$ for all $k\geqq j$. The construction of M must then be modified so that, for $k\geqq j$, r_k is used up before one begins to use r_{k+1}.

Lemma 3. *Let (H, A, m) be a gage space, let $T\in L^1(H, A, m)$ with $T\geqq 0$, let $\gamma>0$, and let $q=m(P_T(\gamma,\infty))$. Let P be any projection in A with $m(P)=q$. Then $m(PT)\leqq m(P_T(\gamma,\infty)T)$.*

Proof. By lemma 1, $m(PT)=m(PTP)=\int_0^{m(I)}\Lambda_{PTP}(x)\,dx=\int_0^{q}\Lambda_{PTP}(x)\,dx$. By [2] Theorem 4, $\Lambda_{PTP}(x)\leqq\Lambda_T(x)$ for $0<x\leqq m(I)$. Note that $\Lambda_{P_T(\gamma,\infty)T}(x)=$ $=\Lambda_T(x)$ for $0<x\leqq q$ so that $\Lambda_{PTP}(x)\leqq\Lambda_{P_T(\gamma,\infty)T}(x)$ for $0<x\leqq q$. Then

$$\int_0^q \Lambda_{PTP}(x)\,dx\leqq\int_0^q\Lambda_{P_T(\gamma,\infty)T}(x)\,dx=\int_0^{m(I)}\Lambda_{P_T(\gamma,\infty)T}(x)\,dx=m\big(P_T(\gamma,\infty)T\big).$$

Proof of Theorem 2. Assume first that $A=B(H)$. Let ϱ_i be the i^{th} eigenvalue of D_ϱ, where the eigenvalues of D_ϱ are arranged in decreasing order and are counted according to multiplicity.

Define a sequence A by $a_i=\lambda\varrho_i+(1-\lambda)\sigma_i$ and a sequence B by $b_i=\psi_i$. The first three hypotheses of lemma 2 are clearly satisfied. The last hypothesis of lemma 2 follows from lemma 3; a trivial modification of lemma 3 is needed if D_ϱ or D_σ has a repeated eigenvalue. By lemma 2, there is a doubly stochastic matrix M with $\psi_i=\sum_j m_{ij}(\lambda\varrho_j+(1-\lambda)\sigma_j)$. Then $F(\psi_i)\geqq\sum_j m_{ij}(\lambda F(\varrho_j)+(1-\lambda)\,F(\sigma_j))$. Summing this relation yields $m(F(D_\psi))=\sum_i F(\psi_i)\geqq\sum_i\sum_j m_{ij}(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))=$ $=\sum_j\sum_i m_{ij}(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))=\sum_j(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))=\lambda m(F(D_\varrho))+(1-\lambda)\cdot$ $\cdot m(F(D_\sigma))$; the interchange of the order of summation is valid since ϱ and σ each have finite entropy by hypothesis. If $\varrho\neq\sigma$, then $\psi_{i_0}\neq\lambda\varrho_{i_0}+(1-\lambda)\sigma_{i_0}$ for some i_0, so that M is not the identity matrix. If F is then strictly convex, then $F(\psi_{i_0})>$ $>\sum_j m_{i_0 j}(\lambda F(\varrho_j)+(1-\lambda)F(\sigma_j))$.

We now prove the general case when $m(I)=\infty$; the proof when $m(I)<\infty$ is virtually identical. Let ε be an arbitrary positive number. For n a natural number, let

$$\psi_n=\frac{1}{\varepsilon}\int_{(n-1)\varepsilon}^{n\varepsilon}\Lambda_\psi(x)\,dx;$$

define sequences ϱ_n and σ_n similarly. Assume that D_ψ, D_ϱ, D_σ have no point masses at $\Lambda_\psi(k\varepsilon)$, $\Lambda_\varrho(k\varepsilon)$, $\Lambda_\sigma(k\varepsilon)$ respectively, for all natural numbers k; arbitrarily small ε can always be found so that this holds. By lemma 1 and lemma 3,

$$\varepsilon\sum_{n=1}^{k}\psi_n=\int_0^{k\varepsilon}\Lambda_\psi(x)\,dx=\int_{\Lambda_\psi(k\varepsilon)}^{\infty}\alpha dm(P_\psi(\alpha))=m\big(D_\psi P_\psi(\Lambda_\psi(k\varepsilon,\infty))\big)=$$
$$=\lambda m\big(D_\varrho P_\psi(\Lambda_\psi(k\varepsilon,\infty))\big)+(1-\lambda)m\big(D_\sigma P_\psi(\Lambda_\psi(k\varepsilon,\infty))\big)\leqq$$
$$\leqq\lambda m\big(D_\varrho P_\varrho(\Lambda_\varrho(k\varepsilon,\infty))\big)+(1-\lambda)m\big(D_\sigma P_\sigma(\Lambda_\sigma(k\varepsilon,\infty))\big)=\varepsilon\lambda\sum_{n=1}^{k}\varrho_n+\varepsilon(1-\lambda)\sum_{n=1}^{k}\sigma_n.$$

By the first part of the proof of this theorem,

$$\sum_n F(\psi_n)\geqq\lambda\sum_{n=1}^{\infty}F(\varrho_n)+(1-\lambda)\sum_{n=1}^{\infty}F(\sigma_n).$$

To complete the proof, it suffices to show that $\varepsilon\sum_{i=1}^{n}F(\psi_n)$ approximates $\int_0^\infty F(\Lambda_\psi(x))\,dx$ for ε small. This is immediate since Λ_ψ is a non-increasing function implies $\Lambda_\psi((n-1)\varepsilon)\geqq\psi_n\geqq\Lambda_\psi(n\varepsilon)$ and $e(\psi)$ is finite by the hypotheses and Corollary 1.

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Uniformly distributed sequences on compact, separable, non metrizable groups

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Let G be a compact topological group with normalized Haar measure λ. G^∞ shall denote the product of denumerably many copies of G, equipped with the product measure λ_∞. If G is metrizable, a well-known theorem ([5] Th. 2.2) asserts that the set of uniformly distributed (u.d.) sequences has measure one in G. If G is separable (i.e. it contains a countable dense subset) but non-metrizable, it follows from a result of W. A. VEECH [9] that u.d. sequences still exist (a shorter proof of this result has been given by H. RINDLER [7], the abelian case has been treated earlier in [1]). Let S be the set of all u.d. sequences in G. In this paper we show that for a compact, separable, non-metrizable group G, S is not measurable in G, its outer measure is one, the inner measure is zero. By the way of proving this result, we extend a result of [6]: if G is a compact, separable group, G/H a metrizable quotient, then any u.d. sequence in G/H can be lifted to G.

For arbitrary separable compact spaces the situation is different. We give an example of a class of spaces for which u.d. sequences exist in trivial cases only, namely if the measure is concentrated on a countable set.

I want to thank W. Maxones and H. Rindler who led my interest to these problems.

For basic notions concerning uniformly distributed sequences see [5].

Lemma 1. *Let G be a compact topological group with normalized Haar-measure λ, A a measurable subset of G. Then there exists a closed normal subgroup H of G and H-periodic measurable subsets B, C of G* (i.e. $B=BH$, $C=CH$) *such that $B\subseteqq A\subseteqq C$, $\lambda(C\setminus B)=0$ and G/H is metrizable.*

Proof. It suffices to prove this for open subsets A. By the regularity of λ, there exists an open Baire-set $B\subseteqq A$ for which $\lambda(A\setminus B)=0$. B has the form $\{x\in G: f(x)>0$ for some continuous, real valued function f on G. Now let H be a closed normal

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subgroup of G such that G/H is metrizable and f is H-periodic, i.e. $f(yx)=f(y)$ for $x\in H$. For $\dot{x}\in G/H$ put

$$T_H(c_A)(\dot{x}) = \int_H c_A(xy)\,dy \quad\text{and}\quad T_H(c_B)(\dot{x}) = \int_H c_B(xy)\,dx.$$

$$T_H(c_B) = c_{\dot{B}}$$

since B is H-periodic. Define $\dot{C}=\{\dot{x}\in G/H: T_H(c_A)(\dot{x})>0\}$ and $C=\{x\in G: \dot{x}\in\dot{C}\}$. Then C is open and H-periodic, $C\supseteq A$. Since $\dot{C}=\dot{B}$ a.e. on G/H we have $\lambda(C\setminus B)=0$.

Remark. It is essential that the inclusion $B\subseteq A\subseteq C$ and the equations $BH=B$, $CH=C$ hold in the set-theoretical sense and not only λ-a.e..

One easily concludes that an arbitrary subset A of G has outer measure one if and only if the canonical image of A has outer measure one in any metrizable quotient group G/H.

Theorem 1. *Let G be a compact topological group which is separable but non-metrizable, λ the normalized Haar measure and S the set of all u.d. sequences in G. Then S has interior measure zero in G^∞.*

Proof. We want to show that $M=G^\infty\setminus S$ has outer measure one. Let $p_n: G^\infty\to G$ be the n-th coordinate function. If H_1 is a closed normal subgroup of $G_1=G^\infty$ for which G_1/H_1 is metrizable, there exists a countable set $\{f_n\}$ of continuous functions on G_1/H_1 which is dense in $C(G_1/H_1)\subseteq C(G_1)$ (the space of all continuous functions on G_1/H_1 resp. G_1 with the topology of uniform convergence). By the Stone Weierstrass theorem there exists a countable set $\{g_n\}$ of continuous functions on G such that the closed subalgebra spanned by $\{g_n\circ p_m\}_{m,n=1}^\infty$ contains the f_n. Now take a closed normal subgroup H of G for which all g_n are H-periodic and such that G/H is metrizable. Then $H^\infty\subseteq H_1$ and $(G/H)^\infty=G^\infty/H^\infty$ is metrizable too. It suffices therefore to prove that the image of M has outer measure zero in G^∞/H^∞ for all closed normal subgroups H of G, for which G/H is metrizable. Since G is not metrizable, H must be a non-trivial subgroup. Take a symmetric neighbourhood U of the unit element in G such that H is not contained in U^2. If (y_n) is an arbitrary sequence in G/H there exists a sequence (x_n) in $G\setminus U$ such that $\pi(x_n)=y_n$ ($\pi: G\to G/H$ denotes the canonical projection). (x_n) belongs to M since it is not dense in G. Therefore the image of M comprises all of $(G/H)^\infty$.

Lemma 2. *Let G be a compact topological group, H a closed normal subgroup, $\pi: G\to G/H$ the canonical projection. If (y_n) is a sequence in G/H which converges to identity, there exists a sequence (x_n) in G, which converges to identity and satisfies $\pi(x_n)=y_n$.*

Proof. Let $(U_\alpha)_{\alpha<\gamma}$ be a well-ordering for the set of all non-equivalent irreducible unitary representations of G. For $\alpha<\gamma$ put

$$H_\alpha = H \cap \bigcap_{\beta<\alpha} \ker U_\beta, \quad H_0 = H, \quad G_\alpha = G/H_\alpha = (G/H_{\alpha+1})/(H_\alpha/H_{\alpha+1}).$$

If α is a limit-ordinal, then G_α is the projective limit of the groups $\{G_\beta : \beta<\alpha\}$. It suffices therefore to show that we can lift the sequence from G_α to $G_{\alpha+1}$. Since U_α seperates the points of $H_\alpha/H_{\alpha+1}$, this group is metrizable (in fact a Lie group). This means that we have reduced the proof of the lemma to the case that H is metrizable.

By induction we can define a sequence $\{U_m\}$ of open neighborhoods of the unit element in G with the following properties: $U_{m+1} \subseteqq U_m$, $\{U_m \cap H\}$ is a neighborhood base of the unit element in H, if $F_m = H \setminus U_m$, then $U_{m+1} \cap U_{m+1} F_m = \emptyset$. Now we choose elements $x_n \in G$ such that $\pi(x_n) = y_n$ and almost all x_n belong to U_m $(m=1, 2, \ldots)$. We want to show that $\{x_n\}$ tends to zero. Let V be an arbitrary neighborhood of e in G and W an open neighborhood for which $W^2 \subseteqq V$. Put $F = H \setminus W$ then there exists an index m such that $U_m \cap U_m F = \emptyset$. If $x \in U_m$ and $\pi(x) \in \pi(W \cap U_m)$ there exists $y \in W \cap U_m$ such that $\pi(y) = \pi(x)$, i.e. $y^{-1}x \in H$. If $y^{-1}x$ would belong to F then $x \in y \subseteqq U_m F$, a contradiction. Therefore $y^{-1}x \in W$ and it follows that $x \in yW \subseteqq V$.

Theorem 2. *Let G be a compact, separable group, H a closed normal subgroup, G/H metrizable, (y_n) a u.d. sequence in G/H. Then there exists a u.d. sequence (x_n) in G such that $\pi(x_n) = y_n$ (where π denotes the canonical quotient map).*

Proof. This result was already claimed without proof in [6], but we were informed by the author of that paper that his proof contains a gap. The methods of [6] enable us to show the following: If G is a compact group, H a separable subgroup, G/H metrizable, (y_n) u.d. in G/H, then there exists a u.d. sequence (x_n) in G such that $y_n^{-1}\pi(x_n)$ tends to identity. Now it follows from the previous lemma that there exists a sequence (z_n) in G such that (z_n) converges to identity and $\pi(z_n^{-1}) = y_n^{-1}\pi(x_n)$. $(x_n z_n)$ is u.d. in G and $\pi(x_n z_n) = y_n$. According to [4] a compact topological group is separable if and only if it has an open base for its topology of cardinality $\leqq c$ (the power of the continuum). It follows that a closed subgroup of a separable compact group is separable and the proof is finished.

Corollary. *Let G be a separable compact group, λ the normalized Haar measure and S the set of all u.d. sequences in G. Then S has outer measure one in G.*

Now let X be an arbitrary compact Hausdorff space. We write $C(X)$ for the space of continuous scalar-valued functions on X with supremum norm, $M(X)$ for the dual and $M(X)'$ for the bidual of $C(X)$. According to [8] we call X a *G-space* if $\sigma(M(X), C(X))$-convergent sequences are $\sigma(M(X), M(X)')$ convergent.

Proposition. *If X is a G-space and μ is a probability measure on X, which is not concentrated on a countable subset, then there exist no μ-u.d. sequences in X.*

Proof. Assume that (x_n) is u.d.. Put $\mu_N = N^{-1} \sum_{n \leq N} \varepsilon_{x_n}$, where ε_x denotes the point measure of mass one concentrated in x. μ_N converges to μ in the topology $\sigma(M(X), C(X))$ and consequently also for $\sigma(M(X), M(X)')$. It follows that $\mu_N(A)$ converges to $\mu(A)$ for any Borel-subset A of X, in particular that μ is concentrated on the set $\{x_n\}_{n=1}^{\infty}$.

It was first proved in [3] that any extremely disconnected space is a G-space (X is called extremely disconnected, if the closure of any open subset is open). For example $\beta\mathbf{N}$ the Stone—Čech compactification of $\mathbf{N}$ is extremely disconnected and clearly separable. The last proposition shows that there exists no u.d. sequence for a measure μ on $\beta\mathbf{N}$ which is not concentrated on a countable set.

More generally it has been shown in [8] that any F-space is a G-space (a compact space X is an F-space, if disjoint open F-sets have disjoint closures. See [2] for further properties and examples of F-spaces).

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Построение полугрупповой амальгамы, независимо вложимой в полугруппу

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Полугрупповая амальгама *вкладывается независимо* в полугруппу, если в этой полугруппе независимы исходные полугруппы, составляющие данную амальгаму.

В настоящей работе изучаются свойства таких амальгам и в некоторых частных случаях (например, для некоторого гомоморфного образа всех упомянутых амальгам) дается полное описание.

Эта статья по существу есть продолжение статьи [6]. Мы предполагаем, что читатель знаком с [6] и используем определения и обозначения этой работы.

§ 1. Построение полугрупп, содержащих M-элементы

Пусть A слабо ассоциативная амальгама полугрупп A_ξ ($\xi\in\mathcal{I}$, где $\mathcal{I}$ — некоторое множество индексов), которая может быть вложена в некоторую полугруппу B таким образом чтобы A_ξ ($\xi\in\mathcal{I}$) в ней являлись независимыми подполугруппами. Другими словами, амальгама A удовлетворяет условиям теоремы 3 в статье [6] и по теореме 2 (в [6]) обладает свойствами $\alpha-\vartheta$. Этими свойствами мы будем в дальнейшем неоднократно пользоваться.

Отбросим из $\mathcal{I}$ такие ундексы α, для которых A_α не обладает L-элементами (T-элементами, M-элементами). Полученные три множества обозначим соответственно через $\mathcal{I}_L$, $\mathcal{I}_T$, $\mathcal{I}_M$. В этих множествах определим *отношения* $\varrho_L, \varrho_T, \varrho_M$ следующим образом:

ϱ_L: $\alpha\varrho_L\beta$ ($\alpha, \beta\in\mathcal{I}_L$) $\Leftrightarrow$ существует L-элемент x, для которого $\alpha, \beta\in\bar{x}$,

ϱ_T: $\alpha\varrho_T\beta$ ($\alpha, \beta\in\mathcal{I}_T$) $\Leftrightarrow$ существует T-элемент t, для которого $\alpha, \beta\in\bar{t}$,

ϱ_M: $\alpha\varrho_M\beta$ ($\alpha, \beta\in\mathcal{I}_M$) $\Leftrightarrow$ существует такая (конечная) последовательность $\gamma_0=\alpha, \gamma_1, \gamma_2, \dots, \gamma_n=\beta$ ($\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n\in\mathcal{I}_M$) для которой найдутся M-элементы $y_1, y_2, \dots, y_n$ со свойством: $\bar{y}_1=\langle\gamma_0, \gamma_1\rangle$, $\bar{y}_2=\langle\gamma_1, \gamma_2\rangle, \dots, \bar{y}_n=\langle\gamma_{n-1}, \gamma_n\rangle$.

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Очевидно, что ϱ_L, ϱ_T рефлексивны и симметричны. Транзитивность отношения ϱ_L следует из свойства β, а транзитивность отношения ϱ_T следует из ϑ. Таким образом, ϱ_L, ϱ_T и ϱ_M являются отношениями эквивалентности (для ϱ_M это очевидно). Следовательно, ϱ_L, ϱ_T и ϱ_M определяют разбиения. Обозначим ϱ_L-классы через P_i $(i\in\mathscr{I}_P)$, ϱ_T-классы через Q_j $(j\in\mathscr{I}_Q)$, ϱ_M-классы через R_k $(k\in\mathscr{I}_R)$ $(\mathscr{I}_P, \mathscr{I}_Q, \mathscr{I}_R$ — некоторые (подходящие) множества индексов). Мощность множеств P_i, Q_j, R_k записываем так $|P_i|$, $|Q_j|$, $|R_k|$. В дальнейшем пусть L_α, M_α, T_α обозначают соответственно множества всех L-, M-, T-элементов из A_α $(\alpha\in\mathscr{I})$ и $D_\alpha=A_\alpha\setminus(L_\alpha\cup T_\alpha)$.

Далее, пусть $K_{\alpha\beta}=A_\alpha\cap A_\beta$ $(\alpha,\beta\in\mathscr{I})$, M^l- и M^r-компоненты $K_{\alpha\beta}$ в A_α будем называть M-компонентами и ее L-, M^l-, M^r-, T-компоненты в A_α обозначать через L, $M_{\alpha\beta}$, $M_{\beta\alpha}$, T. Значит $M_{\alpha\beta}=\langle y\,|\,y$ — M-элемент и $\hat{y}=(\alpha,\beta)\rangle$, $M_{\beta\alpha}=\langle y\,|\,y$ — M-элемент, $\hat{y}=(\beta,\alpha)\rangle$.

Лемма 1. *Если ϱ_M-класс R_k содержит такое α, что A_α обладает T-элементом, то R_k содержится в некотором ϱ_T-классе.*

Доказательство. Достаточно показать следующее: если y — M-элемент и $\bar{y}=\langle\alpha,\beta\rangle$ и A_α содержит T-элемент то содержит и A_β. А это утверждение следует из свойства η.

Лемма 2. *Пусть R_k ϱ_M-класс; $\alpha,\beta\in R_k$.*

1) Если $M_{\alpha\beta}\neq\emptyset$ то существует разбиение $F_{\alpha\beta}=U^{(\alpha)}_{\alpha\beta}\cup V^{(\alpha)}_{\alpha\beta}$ такое, что $ux=x$, $vx\in T$, $uM_{\alpha\beta}\subseteqq M_{\alpha\beta}$, $vM_{\alpha\beta}\subseteqq T$ при всех $u\in U^{(\alpha)}_{\alpha\beta}$, $v\in V^{(\alpha)}_{\alpha\beta}$, $x\in F_{\alpha\beta}$ и элементы множества $N^{(\alpha)}_{\alpha\beta}=U^{(\alpha)}_{\alpha\beta}M_{\alpha\beta}$ являются правыми нулями в полугруппе $L\cup M_{\alpha\beta}\cup U^{(\alpha)}_{\alpha\beta}$.

2) Если $M_{\beta\alpha}\neq\emptyset$, то существует разбиение $F_{\alpha\beta}=U^{(\alpha)}_{\beta\alpha}\cup V^{(\alpha)}_{\beta\alpha}$ такое, что $xu=x$, $xv\in T$, $M_{\beta\alpha}u\subseteqq M_{\beta\alpha}$, $M_{\beta\alpha}v\subseteqq T$ при всех $u\in U^{(\alpha)}_{\beta\alpha}$, $v\in V^{(\alpha)}_{\beta\alpha}$, $x\in F_{\alpha\beta}$ и элементы множества $N^{(\alpha)}_{\beta\alpha}=M_{\beta\alpha}U^{(\alpha)}_{\beta\alpha}$ являются левыми нулями в полугруппе $L\cup M_{\beta\alpha}\cup U^{(\alpha)}_{\beta\alpha}$.

3) Если $U^{(\alpha)}_{\alpha\beta}\neq\emptyset$ и $U^{(\beta)}_{\alpha\beta}\neq\emptyset$ (аналогично $U^{(\alpha)}_{\beta\alpha}\neq\emptyset$ и $U^{(\beta)}_{\beta\alpha}\neq\emptyset$) то $N^{(\alpha)}_{\alpha\beta}=N^{(\beta)}_{\alpha\beta}$ (соответственно, $N^{(\alpha)}_{\beta\alpha}=N^{(\beta)}_{\beta\alpha}$) состоит из одного элемента, который является нулем в $L\cup M_{\alpha\beta}$ (соответственно, в $L\cup M_{\beta\alpha}$).

Доказательство. Пусть R_k — ϱ_M-класс, и $\alpha,\beta\in R_k$. Предположим, что $M_{\alpha\beta}\neq\emptyset$. Доказательство в случае $M_{\beta\alpha}\neq\emptyset$ аналогично. Из теоремы 2.8 в [3] следует, что $uy\in T\cup M_{\alpha\beta}$ для всех $u\in F_{\alpha\beta}$, $y\in M_{\alpha\beta}$. Так как при всех $x\in F_{\alpha\beta}$

$$ux=u(yx)=(uy)x=\begin{cases}x & \text{если}\quad uy\in M_{\alpha\beta}\\ \in T & \text{если}\quad uy\in T,\end{cases}\tag{1}$$

то для элемента $u\in F_{\alpha\beta}$ или $uM_{\alpha\beta}\subseteqq M_{\alpha\beta}$ и тогда $ux=x$ при всех $x\in F_{\alpha\beta}$ (обозначим множество таких элементов u через $U^{(\alpha)}_{\alpha\beta}$) или $uM_{\alpha\beta}\subseteqq T$ и тогда $ux\in T$ при всех $x\in F_{\alpha\beta}$ (такие элементы u образуют множества $V^{(\alpha)}_{\alpha\beta}$). Если $uy\in M_{\alpha\beta}$ т. е. $u\in U^{(\alpha)}_{\alpha\beta}$, $y\in M_{\alpha\beta}$, то $uy=(wu)y=w(uy)$ при любых $w\in L\cup M_{\alpha\beta}\cup U^{(\alpha)}_{\alpha\beta}$. Отсюда следует, что элементы множества $N_{\alpha\beta}^{(\alpha)} \in U_{\alpha\beta}^{(\alpha)} M_{\alpha\beta}$ являются правыми нулями в $L \cup M_{\alpha\beta} \cup U_{\alpha\beta}^{(\alpha)}$. Легко проверяется, что $L \cup M_{\alpha\beta} \cup U_{\alpha\beta}^{(\alpha)}$ является полугруппой.

Аналогично доказывается, что элементы $N_{\alpha\beta}^{(\beta)} = M_{\alpha\beta} U_{\alpha\beta}^{(\beta)}$ являются левыми нулями в полугруппе $L \cup M_{\alpha\beta} \cup U_{\alpha\beta}^{(\beta)}$. Отсюда следует, что в случае $U_{\alpha\beta}^{(\alpha)} \neq \emptyset$ $U_{\alpha\beta}^{(\beta)} \neq \emptyset$ множества $N_{\alpha\beta}^{(\alpha)}$ и $N_{\alpha\beta}^{(\beta)}$ совпадают и состоят только из одного элемента, который является нулем в $L \cup M_{\alpha\beta}$.

Замечание. В случае $M_{\alpha\beta} \neq \emptyset$ $M_{\beta\alpha} \neq \emptyset$ из леммы 2 следует, что либо $F_{\alpha\beta} = V_{\alpha\beta}^{(\alpha)} = V_{\beta\alpha}^{(\alpha)}$ (т. е. $U_{\alpha\beta}^{(\alpha)} = U_{\beta\alpha}^{(\alpha)} = \emptyset$), либо $F_{\alpha\beta} = U_{\alpha\beta}^{(\alpha)} = U_{\beta\alpha}^{(\alpha)} = \langle a \rangle$, т. е. $F_{\alpha\beta}$ состоит из одного элемента a. Этот последний случай возможен только тогда, когда $K_{\alpha\beta}$ является особой р. е. и. подполугруппой в A_α (это доказано в 2.1 в статье [4]). (Определение особой р. е. и. подполугруппы см. в §2.) Таким образом, этот случай не может иметь места, если A_α содержит только один T-элемент. Из леммы 2 вытекает следующая

Теорема 1. *Пусть A — слабо ассоциативная амальгама двух полугрупп A_α, A_β; независимо вложимая в некоторую полугруппу. Если A содержит M-элементы и не более чем один T-элемент то возможны следующие случаи:*

1) *В A нет T-элемента. $M_{\beta\alpha} = \emptyset$ и $L \cup M_{\alpha\beta}$ полугруппа с нулем O_M, $F_{\alpha\beta}$ полугруппа правых нулей, $F_{\beta\alpha}$ полугруппа левых нулей и имеют место соотношения: $xz = O_M$, $zy = O_M$ при всех $z \in M_{\alpha\beta}$, $x \in F_{\alpha\beta}$, $y \in F_{\beta\alpha}$.*

2) *A содержит один T-элемент. Этот элемент: 0 является нулем и в A_α и в A_β. Возможны следующие случаи:*

а) $(F_{\alpha\beta} \cup M_{\beta\alpha})(F_{\alpha\beta} \cup M_{\alpha\beta}) = 0$, $(F_{\beta\alpha} \cup M_{\alpha\beta})(F_{\beta\alpha} \cup M_{\beta\alpha}) = 0$.

б) $M_{\alpha\beta} \neq \emptyset$, $M_{\beta\alpha} = \emptyset$, $(F_{\beta\alpha} \cup M_{\alpha\beta}) F_{\beta\alpha} = 0$, $F_{\alpha\beta} = U_{\alpha\beta}^{(\alpha)} \cup V_{\alpha\beta}^{(\alpha)}$ $(U_{\alpha\beta}^{(\alpha)} \cap V_{\alpha\beta}^{(\alpha)} = \emptyset)$ где $ux = x$, $vx = 0$, $uM_{\alpha\beta} \subseteq M_{\alpha\beta}$, $vM_{\alpha\beta} = 0$ при всех $u \in U_{\alpha\beta}^{(\alpha)}$, $v \in V_{\alpha\beta}^{(\alpha)}$, $x \in F_{\alpha\beta}$ и элементы множества $N_{\alpha\beta}^{(\alpha)} = U_{\alpha\beta}^{(\alpha)} M_{\alpha\beta}$ являются правыми нулями в $L \cup M_{\alpha\beta} \cup U_{\alpha\beta}^{(\alpha)}$.

в) $M_{\alpha\beta} \neq \emptyset$, $M_{\beta\alpha} = \emptyset$. Имеют место разбиения $F_{\alpha\beta} = U_{\alpha\beta}^{(\alpha)} \cup V_{\alpha\beta}^{(\alpha)}$, $F_{\beta\alpha} = U_{\alpha\beta}^{(\beta)} \cup V_{\alpha\beta}^{(\beta)}$ такие, что $ux = x$, $vx = 0$, $vM_{\alpha\beta} = 0$ при всех $x \in F_{\alpha\beta}$, $u \in U_{\alpha\beta}^{(\alpha)}$, $v \in V_{\alpha\beta}^{(\alpha)}$; и $yu = y$, $yv = 0$, $M_{\alpha\beta} v = 0$ при всех $y \in F_{\beta\alpha}$, $u \in U_{\alpha\beta}^{(\beta)}$, $v \in V_{\alpha\beta}^{(\beta)}$. Далее, $L \cup M_{\alpha\beta}$ обладает нулем O_M и $U_{\alpha\beta}^{(\alpha)} M_{\alpha\beta} = O_M$, $M_{\alpha\beta} U_{\alpha\beta}^{(\beta)} = O_M$.

Дальнейшие свойства L-, M-элементов во всех случаях следуют из их определений, или из свойств $*$-связки. Остальные возможности получаются из приведенных выше из соображений двойственности.

Лемма 3. *Если ϱ_M-класс R_k состоит не только из двух индексов, то каждая $K_{\alpha\beta}$ $(\alpha, \beta \in R_k)$ может содержать не более чем одну M-компоненту и существуют подмножества $R_k^{(l)}$, $R_k^{(r)}$ класса R_k $(R_k = R_k^{(l)} \cup R_k^{(r)}$; $R_k^{(l)} \cap R_k^{(r)} = \emptyset)$ такие, что для всех $M_{\alpha\beta}$ $(\alpha, \beta \in R_k)$ имеет место $\alpha \in R_k^{(l)}$, $\beta \in R_k^{(r)}$.*

Доказательство. Пусть $\alpha, \beta \in R_k$. Если в $K_{\alpha\beta}$ две M-компоненты $M_{\alpha\beta}$, $M_{\beta\alpha}$ непусты, то R_k состоит только из индексов α, β. Действительно, если $M_{\alpha\gamma} \neq \emptyset$

($\gamma \in R_k$), то существование элементов $y \in M_{\beta\alpha}$ и $z \in M_{\alpha\gamma}$, для которых по определению $\hat{y}=(\beta, \alpha)$ $\hat{z}=(\alpha, \gamma)$, противоречит свойству γ. Предположим, что A_α содержит M-компоненту в $K_{\alpha\beta}$ и в $K_{\alpha\xi_1}, K_{\alpha\xi_2}, \ldots$ ($\alpha, \beta, \xi_1, \xi_2, \ldots \in R_k$). Если в $K_{\alpha\beta}$ подполугруппа $M_{\alpha\beta} \neq \emptyset$, то другими M-компонентами в A_α будут $M_{\alpha\xi_1}, M_{\alpha\xi_2}, \ldots$ (согласно γ) и в этом случае $\alpha \in R_k^{(l)}$; если же $M_{\beta\alpha} \neq \emptyset$ в $K_{\alpha\beta}$, то остальные M-компоненты в A_α — $M_{\xi_1\alpha}, M_{\xi_2\alpha}, \ldots$ т. е. $\alpha \in R_k^{(r)}$.

Лемма 4. *Пусть R_k — ϱ_M-класс, для которого $|R_k| \geqq 3$ и пусть $A_\alpha (\alpha \in R_k)$ такая полугруппа, которая содержит более одной M-компоненты.*

1) *Множество D_α обладает*

а) или разбиением таким $D_\alpha = U_\alpha^{(l)} \cup V_\alpha^{(l)}$ (если $\alpha \in R_k^{(l)}$), что $M_\alpha \subseteqq U_\alpha^{(l)}$, $ux=x, vx \in T, lx=x, yl=y$ для всех $u \in U_\alpha^{(l)}, v \in V_\alpha^{(l)}, l \in L_\alpha, x \in D_\alpha, y \in V_\alpha^{(l)} \cup (U_\alpha^{(l)} \setminus M_\alpha)$;

б) или таким разбиением $D_\alpha = U_\alpha^{(r)} \cup V_\alpha^{(r)}$ (если $\alpha \in R_k^{(r)}$), что $M_\alpha \subseteqq U_\alpha^{(r)}$, $xu=x, xv \in T, xl=x, ly=l$ для всех $u \in U_\alpha^{(r)}, v \in V_\alpha^{(r)}, l \in L_\alpha, x \in D_\alpha, y \in V_\alpha^{(r)} \cup (U_\alpha^{(r)} \setminus M_\alpha)$.

2) *Множество $M_{\alpha\beta}$ (или $M_{\beta\alpha}$) только в том случае состоит не более чем из одного элемента если A_β содержит только одну M-компоненту (именно $M_{\alpha\beta}$ или $M_{\beta\alpha}$) и одновременно $F_{\beta\alpha} F_{\beta\alpha} \subseteqq T$, (т. е. $U_{\alpha\beta}^{(\beta)} = U_{\beta\alpha}^{(\beta)} = \emptyset$).*

3) *L-компонента для всех пересечений $K_{\varrho\sigma} = A_\varrho \cap A_\sigma$ ($\varrho, \sigma \in R_k$) одна и та же L_k ($=L_\alpha$). L-компонента пересечений $K_{\varrho\xi}$ ($\varrho \in R_k$, $\xi \in \mathscr{I}$, $\xi \bar{\in} R_k$) является подмножеством L_k. Далее, $M_{\alpha\beta} L_k \subseteqq M_{\alpha\beta}$ (и аналогично $L_k M_{\beta\alpha} \subseteqq M_{\beta\alpha}$) ($\beta \in R_k$) для всех непустых M-компонент $M_{\alpha\beta}$ ($M_{\beta\alpha}$) в A_α.*

Доказательство. Пусть $\alpha, \beta \in R_k$; $|R_k| \geqq 3$. Пусть A_α полугруппа, которая содержит более одной M-компоненты. Можно предположить, что $\alpha \in R_k^{(l)}$ и в силу леммы 3 можно считать что в A_α существуют M-компоненты $M_{\alpha\beta}$, $M_{\alpha\gamma}$ ($\beta, \gamma \in R_k$). Применим лемму 2. Так как $za=a$ при $z \in M_{\alpha\beta} \cup M_{\alpha\gamma}$ и $a \in A_\alpha \setminus (K_{\alpha\beta} \cup K_{\alpha\gamma})$, то $M_{\alpha\gamma} \subset U_{\alpha\beta}^{(\alpha)}$, $M_{\alpha\beta} \subset U_{\alpha\gamma}^{(\alpha)}$. Следовательно $M_{\alpha\beta}$, $M_{\alpha\gamma}$ является полугруппой правых нулей, и $N_{\alpha\beta}^{(\alpha)} = M_{\alpha\beta}$, $N_{\alpha\gamma}^{(\alpha)} = M_{\alpha\gamma}$. Пусть $U_\alpha^{(l)} = M_{\alpha\beta} \cup U_{\alpha\beta}^{(\alpha)} = M_{\alpha\gamma} \cup U_{\alpha\gamma}^{(\alpha)}$ и $V_\alpha^{(l)} = V_{\alpha\beta}^{(\alpha)}$, тогда согласно лемме 2 и определению L-компоненты выполняются все требования утверждения 1. (Равенства $lx=x, yl=y$ ($x \in D_\alpha$, $y \in V_\alpha^{(l)} \cup (U_\alpha^{(l)} \setminus M_\alpha)$) получаются только при $x \in L$, но из утверждения 3 будет следовать $L=L_\alpha$.)

Рассмотрим теперь те условия, при которых $M_{\alpha\beta}$ состоит более чем из одного элемента. В этом случае $U_{\alpha\beta}^{(\beta)} = \emptyset$ так как согласно утверждению 3 леммы 2 из $U_{\alpha\beta}^{(\beta)} \neq \emptyset$ следует, что $M_{\alpha\beta} = N_{\alpha\beta}^{(\alpha)}$ состоит только из одного элемента. Как мы доказали выше, если в A_β есть не только одна M-компонента то $U_{\alpha\beta}^{(\beta)} \neq \emptyset$. Отсюда вытекает второе утверждение леммы.

Для 3) мы покажем, что L-компонента пересечения $K_{\varrho\tau}$ является подмножеством L-компоненты $K_{\varrho\sigma}$, если $M_{\varrho\sigma} \neq \emptyset$ ($\varrho, \sigma \in R_k$, $\tau \in \mathscr{I}$, $\tau \neq \sigma$). Пусть x — L-элемент, $x \in K_{\varrho\tau}$. Если $x \bar{\in} K_{\varrho\sigma}$, то для всякого $z \in M_{\varrho\sigma}$ имеем $zx=x$ (так как z — M-элемент) и $zx=z$ (так как x — L-элемент и $z \bar{\in} K_{\varrho\tau}$). Отсюда следует, что L-компонента $K_{\alpha\beta}$ и $K_{\alpha\gamma}$ если $M_{\alpha\beta} \neq \emptyset$, $M_{\alpha\gamma} \neq \emptyset$ ($\alpha, \beta \in R_k$), одна и та же полугруппа: L_k, и L-компонента $K_{\beta\gamma}$ является подмножеством L_k. С другой стороны, $L_k \subseteqq K_{\alpha\beta} \cap K_{\alpha\gamma}$, т. е. L_k — подмножество L-компоненты $K_{\beta\gamma}$. Значит L_k совпадает с L-компонентой $K_{\beta\gamma}$ (независимо от того, что имеет $K_{\beta\gamma}$ M-компоненту или нет). Из этих рассуждений вытекает утверждение 3. ($M_{\alpha\beta} L_k \subseteqq M_{\alpha\beta}$ следует из свойств $*$-связки.)

§ 2. Особые р. е. и. подполугруппы

Определение. Р. е. и. подполугруппа K полугруппы S называется *особой* р. е. и. подполугруппой в S, если T-компонента полугруппы K не является двусторонним идеалом в S.

Доказанная в статье [4] теорема 1.4. дает необходимое и достаточное условие для того, чтобы р. е. и. подполугруппа K являлась в S особой. В частности, доказывается, что если K — особая р. е. и. подполугруппа в S, то M^l- и M^r-компонента подполугруппы K непусты, и $S \setminus K$ состоит только из одного элемента. Если $S \setminus K = a$, то возможны два случая: или $a^2 = a$ или a^2 содержится в T-компоненте K (см. § 5. в [4]).

Далее, из того, что р. е. и. подполугруппа $K_{\alpha\beta} = A_\alpha \cap A_\beta$ в A_α является особой, не следует что $K_{\alpha\beta}$ особая и в A_β (пример 5 в [5]).

Если $K_{\alpha\beta}$ особая и в A_α, и в A_β, то возможен каждый из следующих случаев: $(A_\alpha \setminus K_{\alpha\beta} = F_{\alpha\beta} = a,\ A_\beta \setminus K_{\alpha\beta} = F_{\beta\alpha} = b)$

1) $a^2 = a, \quad b^2 = b,$
2) $a^2 = a, \quad b^2 \in T,$
3) $a^2 \in T, \quad b^2 = b,$
4) $a^2 \in T, \quad b^2 \in T$ (примеры 2, 3, 4 в [5]).

Определение. Пусть $K_{\alpha\beta} = A_\alpha \cap A_\beta$ особая р. е. и. подполугруппа в A_α и пусть $a = F_{\alpha\beta}$. Назовем *T^*-компонентой* (и обозначим через T^*) множество всех элементов t из T-компоненты подполугруппы $K_{\alpha\beta}$, для которых $w_1 t w_2 \in T$ для всех слов w_1, w_2 из элементов $a \cup F_{\beta\alpha}$ (одно из слов w_1 или w_2 может быть пустым). Множество $T \setminus T^*$ будем называть $(T \setminus T^*)$-компонентой полугруппы $K_{\alpha\beta}$ в A_α.

Теорема 2. *Пусть A слабо ассоциативная амальгама двух полугрупп A_α, A_β, независимо вложимая в некоторую полугруппу и пусть $K_{\alpha\beta} = A_\alpha \cap A_\beta$ особая р. е. и. подполугруппа в A_α. Если T^*-компонента подполугруппы $K_{\alpha\beta}$ непуста, то она является двусторонним идеалом и в A_α, и в A_β.*

Доказательство. Достаточно показать, что если $t \in T^*$, $x \in A_\alpha \cup A_\beta$ то $xt \in T^*$. Очевидно, что $xt \in T^*$, если $x = a = F_{\alpha\beta}$ или $x \in F_{\beta\alpha}$. Пусть $x \in K_{\alpha\beta}$, и рассмотрим w_1xtw_2, где w_1, w_2 слова элементов $a\cup F_{\beta\alpha}$. Если w_1 пустое, то утверждение следует из свойств $*$-связки. Пусть c — последний элемент в слове w_1. Так как $K_{\alpha\beta}$ р. е. и. подполугруппа, то или $cx=c$, или $cx\in K_{\alpha\beta}$. Продолжая этот процесс, получаем: или w_1x является первой частью w_0 слова w_1 и в этом случае очевидно, что $w_1xtw_2=w_0tw_2\in T$, или $w_1x\in K_{\alpha\beta}$, и так как $tw_2\in T$, то из свойств $*$-связки следует, что $w_1xtw_2\in T$.

Лемма 5. *Если $K_{\alpha\beta}$ особая р. е. и. подполугруппа в A_α, то T-элементы пересечения $K_{\alpha\beta}\cap K_{\alpha\xi}$ $(\alpha, \beta, \xi\in\mathscr{I})$ необходимо принадлежат T^*-компоненте полугруппы $K_{\alpha\beta}$.*

Доказательство. Пусть t — T-элемент, для которого $t\in K_{\alpha\beta}\cap K_{\alpha\xi}$, и $a=F_{\alpha\beta}$, $b\in F_{\beta\alpha}$. Из определения амальгамы следует, что $a\overline{\in} K_{\alpha\xi}$ и поэтому $at\in K_{\alpha\xi}$. Очевидно, что $at\in K_{\alpha\beta}$, тогда $at\in K_{\alpha\beta}\cap K_{\alpha\xi}$. Если $K_{\alpha\beta}$ и в A_β особая р. е. и. подполугруппа, то аналогично получается $bt\in K_{\alpha\beta}\cap K_{\alpha\xi}$. Предположим, что $K_{\alpha\beta}$ не является особой в A_β. Если $b\overline{\in} K_{\alpha\xi}$, то по определению T-компоненты $bt\in K_{\alpha\xi}$. Если $b\in K_{\alpha\xi}$, то из свойств $*$-связки следует, что $bt\in K_{\alpha\xi}$. Отсюда $bt\in K_{\alpha\beta}\cap K_{\alpha\xi}$. Лемма 3 показывает, что в $K_{\alpha\xi}$ могут существовать только L- и T-элементы, и очевидно, что at, bt — T-элементы. Аналогично доказывается, что ta, tb — T-элементы в $K_{\alpha\beta}\cap K_{\alpha\xi}$. Повторение этого процесса дает нам следующее: для любых слов w_1 и w_2 из элементов $a\cup F_{\beta\alpha}$ имеем: w_1tw_2 — T-элемент в $K_{\alpha\beta}\cap K_{\alpha\xi}$. Следовательно, $t\in T^*$.

Замечание. В статье [5] приводятся примеры 2 и 3, в которых T^* пусто, и примеры 4 и 5, в которых T^* непусто.

§ 3. Построение полугруппы всех L-элементов полугрупп $A_\xi(\xi\in P_l)$

Следуя работе [1], введем следующее:

Определение. Будем говорить что полугруппа S разлагается в *последовательно аннулирующее объединение* подполугрупп $S_\xi(\xi\in\mathscr{I})$, если эта совокупность $S_\xi(\xi\in\mathscr{I})$ линейно упорядочена при помощи индексов, причем выполнены следующие условия:

1) $\bigcup\limits_{\xi\in\mathscr{I}} S_\xi=S$,

2) для любых $x\in S_\varrho$, $y\in S_\sigma$ $(\varrho<\sigma;\ \varrho,\sigma\in\mathscr{I})$ имеет место $xy=yx=x$,

3) если $\varrho<\sigma$ и $S_\varrho\cap S_\sigma\neq\emptyset$, то ϱ,σ соседние элементы упорядоченного множества $\mathscr{I}$, далее пересечение $S_\varrho\cap S_\sigma$ может состоить только из одного элемента, который является в S_ϱ единицей, а в S_σ нулем.

Введем еще следующее:

Определение. Пусть S полугруппа с единицей e. Будем говорить, что единица e *неприсоединена* (к полугруппе S), если существуют отличные от e элементы $a, b \in S$, для которых $ab = e$.

Замечание. Если $xy = e$ и $x \neq e$, $y \neq e$, где $x, y \in S$ и e — единица в S, то элементы x, y порождают бициклическую полугруппу в S или нетривиальный гомоморфный образ бициклической полугруппы, которая, как известно, может быть только циклической группой (см. леммы 1.31, 1.32 в [2]).

Теорема 3. *Пусть A слабо ассоциативная амальгама полугрупп A_ξ ($\xi \in \mathcal{J}$), независимо вложимая в некоторую полугруппу. Пусть P_i — ϱ_L-класс индексов, и L^* — множество всех L-элементов $\bigcup_{\eta \in P_i} A_\eta$. Тогда L^* является последовательно аннулирующим объединением полугрупп S_ϱ ($\varrho \in \mathcal{J}$. где $\mathcal{J}$ некоторое линейно упорядоченное множество), и имеют место следующие утверждения:*

1) *Для всякого $\mu \in \mathcal{J}$ существует по крайней мере одна такая полугруппа $A_\alpha (\alpha \in P_i)$, что $L_\alpha = \bigcup_{\varrho \geqq \mu} S_\varrho$ и обратно, в каждой полугруппе A_ξ множество L_ξ представимо в виде $L_\xi = \bigcup_{\varrho \geqq \tau} S_\varrho$ для некоторого $\tau \in \mathcal{J}$.*

2) *Если S_ϱ, S_σ содержит общий элемент: O_σ ($\varrho < \sigma$; $\varrho, \sigma \in \mathcal{J}$ и ϱ, σ соседние), то O_σ является неприсоединенной к S_ϱ единицей.*

3) *Если в A_α ($\alpha \in P_i$) полугруппа $\{B_\alpha\}$, порожденная множеством $B_\alpha = A_\alpha \setminus \bigcup_{\xi \in \mathcal{J} - \langle\alpha\rangle} K_{\alpha\xi}$ содержит и L-элежент (в этом сучае $M_\alpha = \emptyset$), то $\{B_\alpha\}$ может содержать только один L-элемент, а именно — нуль множества $L_\alpha = \bigcup_{\varrho \geqq \mu} S_\varrho$ (который содержитя в S_μ) и является неприсоединенной к $\{B_\alpha\}$ единицей.*

Доказательство. Пусть P_i — ϱ_L-класс, L^* — множество всех L-элементов $\bigcup_{\xi \in P_i} A_\xi$. Из свойства β следует, что для $x_1, x_2 \in L^*$ либо $\bar{x}_1 \subseteqq \bar{x}_2$, либо $\bar{x}_2 \subseteqq \bar{x}_1$. Поэтому можно разбить множество L^* на такие подмножества S_ϱ^* ($\varrho \in \mathcal{J}$), что каждое S_ϱ^* может иметь только такие L-элементы, которые содержится в точно тех же полугруппах из $A_\xi (\xi \in \mathcal{J})$ и совокупность $S_\varrho^* (\varrho \in \mathcal{J})$ линейно упорядочена при помощи индексов, таким образом, что в случае $\varrho < \tau$ ($\varrho, \tau \in \mathcal{J}$) $\bar{x} \subset \bar{y}$ (но $\bar{x} \neq \bar{y}$) при всех $x \in S_\varrho^*$, $y \in S_\tau^*$. Из β следует, что $xy = yx = x$ если $x \in S_\varrho^*$, $y \in S_\tau^*$ ($\varrho < \tau$; $\varrho, \tau \in \mathcal{J}$). Это же свойство β показывает, что для всех $x_1, x_2 \in S_\varrho^*$ произведение $x_1 x_2 \in S_\sigma$ где $\sigma \geqq \varrho$ ($\sigma \in \mathcal{J}$). Предположим, что существует η, для которого $\varrho < \eta < \sigma$. Так как $\varrho < \eta$, то $zx_1 = x_1$, $x_2 z = x_2$ при всех $z \in S_\eta^*$, значит $zx_1x_2 = x_1x_2z = x_1x_2$ далее, из $\eta < \sigma$ следует $zx_1x_2 = x_1x_2z = z$. Это противоречие показывает, что либо $\sigma = \varrho$, либо σ и ϱ соседние элементы из $\mathcal{J}$. Если $x_1, x_2 \in S_\varrho^*$ и $x_1 x_2 = O_\sigma \in S_\sigma^*$ ($\varrho < \sigma$ и ϱ, σ соседние в $\mathcal{J}$), то O_σ является нулем в S_σ^*, так как для всякого $y \in S_\sigma^*$ имеет место $yx_1 = x_1$, $x_2 y = x_2$, т. е. $yx_1x_2 = x_1x_2y = x_1x_2$ и одновременно $O_\sigma x = xO_\sigma = x$ при всех $x \in S_\varrho^*$ (в силу $\varrho < \sigma$).

Теперь определим полугруппу S_ϱ ($\varrho \in \mathscr{I}$) следующим образом:

а) $S_\varrho = S_\varrho^*$ если S_ϱ^* полугруппа,

б) $S_\varrho = S_\varrho^* \cup O_\sigma$ (где ϱ, σ соседние в $\mathscr{I}$; $\varrho < \sigma$) и O_σ является нулем в S_σ^* и неприсоединенной единицей в полугруппе $S_\varrho = S_\varrho^* \cup O_\sigma$.

Легко проверяется, что L^* разлагается в последовательно аннулирующее объединение подполугрупп S_ϱ ($\varrho \in \mathscr{I}$) и все требования утверждения 1 и 2 выполняются.

Пусть теперь $A_\alpha (\alpha \in P_i)$ — полугруппа и $B_\alpha = A_\alpha \setminus \bigcup\limits_{\xi \in \mathscr{I} - \langle \alpha \rangle} K_{\alpha\xi}$. Из (1) следует, что в случае $M_\alpha \neq \emptyset$ полугруппа $\{B_\alpha\}$, порожденная множеством B_α, не содержит L-элементов. Предположим, что $M_\alpha = \emptyset$ и существуют элементы a, b из B_α, для которых произведение $ab \in L_\alpha$. Очевидно, что ab является нулем в L_α (так как $la = a$, $bl = b$, следовательно, $lab = abl = ab$ при всех $l \in L_\alpha$). Если $L_\alpha = \bigcup\limits_{\varrho \geqq \mu} S_\varrho$ (для некоторого $\mu \in \mathscr{I}$), то $ab = O_\mu \in S_\mu$. Так как O_μ L-элемент, поэтому $ab = O_\mu$ является единицей в $\{B_\alpha\}$. Очевидно, что O_μ — неприсоединена к $\{B_\alpha\}$.

§ 4. Построение амальгамы полугрупп, содержащих не более одного T-элемента

Определение. Пусть A — амальгама полугрупп A_ξ ($\xi \in \mathscr{I}$). Амальгаму A^* будем называть *сокращенной* амальгамой данной амальгамы A, если A^* содержит:

1) все L- и M-элементы амальгамы A;

2) все элементы, входящие только в одну из полугрупп A_ξ ($\xi \in \mathscr{I}$),

3) все T-элементы, которые в особой р. е. и. подполугруппе $K_{\alpha\beta}$ в A_α принадлежат $(T \setminus T^*)$-компоненте ($\alpha, \beta \in \mathscr{I}$),

(при этом действия для этих элементов сохраняются),

4) по одному новому элементу, который является общим нулем для всех A_ξ ($\xi \in Q_j$) для каждого ϱ_T-класса Q_j.

Очевидно, что сокращенная амальгама действительно является амальгамой. Из теоремы 2, леммы 5 и из теоремы 1.4 в [2] следует:

Теорема 4. *Сокращенная амальгама является гомоморфным образом исходной амальгамы. Если слабо ассоциативная амальгама удовлетворяет условиям теоремы 3 в* [6], *то и ее сокращенная амальгама также удовлетворяет этим условиям.*

Рассмотрим теперь амальгамы, в которых каждая полугруппа может содержать не более одного T-элемента. Легко показать, что эти амальгамы совпадают с такими сокращенными амальгамами, в которых ни одно пересечение $K_{\alpha\beta}$ не является особой р. е. и. подполугруппой ни в A_α, ни в A_β.

Приведем две теоремы, которые вытекают из вышеуказанной части работы.

Теорема 5. *Пусть A — слабо ассоциативная амальгама полугрупп $A_\xi (\xi \in \mathscr{I})$, независимо вложимая в некоторую полугруппу, и которая в пересечениях полугрупп $A_\xi (\xi \in \mathscr{I})$ содержит только M- и L-элементы. Тогда для произвольной полугруппы A_α $(\alpha \in \mathscr{I})$ имеет место одно из следующих утверждений:*

1) *A_α не содержит M-элементов (т. е. $M_\alpha = \emptyset$). (Описание построения полугруппы A_α следует из теоремы 3)*

2) *A_α содержит M-элементы, которые все входят в $K_{\alpha\beta} = A_\alpha \cap A_\beta$ (т. е. $\alpha, \beta \in R_k$, $|R_k| = 2$) (Описание построения полугрупп A_α, A_β см. в пункте 1 теоремы 1. Непустые пересечения $K_{\alpha\xi}$, $K_{\beta\xi}$ $(\xi \in \mathscr{I}, \xi \neq \alpha, \beta)$ содержат только L-элементы (см. теорему 3)).*

3) *$\alpha \in R_k$, для которого $|R_k| \geqq 3$. $A_\alpha \setminus L_\alpha$ (допускается и $L_\alpha = \emptyset$) является полугруппой правых (или левых) нулей, и имеет место $la = al = a$ при всех $a \in A_\alpha \setminus L_\alpha$, $l \in L_\alpha$. Каждое пересечение $K_{\alpha\xi} (\xi \in \mathscr{I})$ имеет вид: или $K_{\alpha\xi} = a_\xi \cup L_\alpha$ (где $a_\xi \in A_\alpha \setminus L_\alpha$) и в этом случае $A_\xi \setminus L_\alpha$ является полугруппой левых (соответственно правых) нулей, или $K_{\alpha\xi} = L_\xi$, где $L_\xi \subseteqq L_\alpha$.*

Теорема 6. *Пусть A — слабо ассоциативная амальгама полугрупп A_ξ $(\xi \in \mathscr{I})$, независимо вложимая в некоторую полугруппу, каждая полугруппа A_ξ которой содержит не более одного T-элемента. Тогда для произвольной полугруппы A_α $(\alpha \in \mathscr{I})$ имеет место одно из следующих утверждений:*

1) *Подмножество $\bigcup_{\xi \in \mathscr{I} - \langle\alpha\rangle} K_{\alpha\xi}$ полугруппы A_α может содержать*

а) *только L-элементы,*

б) *один T-элемент и L-элементы,*

в) *M- и L-элементы.*

(Описание см. в теоремах 3,5.)

2) *$\alpha \in R_k$ где $|R_k| = 2$, $R_k = \langle \alpha, \beta \rangle$; A_α, A_β содержит один T-элемент: 0. Описание построений полугрупп A_α, A_β см. в теореме 1 в пункте 2. (Непустые пересечения $K_{\alpha\xi}$, $K_{\beta\xi}$ $(\xi \in \mathscr{I}, \xi \neq \alpha, \beta)$ содержат 0 и L-элементы (см. теорему 3).)*

3) *$\alpha \in R_k$, где $|R_k| \geqq 3$. В A_α находится T-элемент 0, который является нулем. A_α содержит одну M-компоненту $M_{\alpha\beta}$, $(M_{\beta\alpha})$ $(\beta \in R_k)$, которая является полугруппой левых (правых) нулей. Имеет место $xx_1 = 0$, $lx = xl = x$, $xy = 0$, $yx = x$, $yl = y$, $L_\alpha M_{\alpha\beta} \subseteqq M_{\alpha\beta}$ при всех $x, x_1 \in F_{\alpha\beta} = A_\alpha \setminus K_{\alpha\beta}$, $y \in M_{\alpha\beta}$, $l \in L_\alpha$, (или соответственно $xx_1 = 0$, $lx = xl = x$, $yx = 0$, $xy = x$, $ly = y$, $M_{\beta\alpha} L_\alpha \subseteqq M_{\beta\alpha}$ при всех $x, x_1 \in F_{\alpha\beta}$, $y \in M_{\beta\alpha}$, $l \in L_\alpha$). (Допускается и $L_\alpha = \emptyset$.)*

$K_{\alpha\beta} = L_\alpha \cup M_{\alpha\beta} \cup 0$ *(или соответственно* $K_{\alpha\beta} = L_\alpha \cup M_{\beta\alpha} \cup 0$*) и* A_β *— такая полугруппа, построение которой описано в пункте 4.*

Другие пересечения: $K_{\alpha\xi} = L_\alpha \cup 0$, *если* $\xi \in R_k$; *и* $K_{\alpha\xi} \subseteq L_\alpha \cup 0$ *если* $\xi \in \mathcal{I}$, $\xi \bar{\in} R_k$.

4) $\alpha \in R_k$, *где* $|R_k| \geqq 3$, *В* A_α *находится* T*-элемент* 0, *который является нулем. Существует разбиение множества* $D_\alpha = U_\alpha^{(l)} \cup V_\alpha^{(l)}$ *где* $D_\alpha = A_\alpha \setminus (L_\alpha \cup 0)$ *такое, что* $M_\alpha \subseteq U_\alpha^{(l)}$ *и* $ux = x$, $lx = x$, $vx = 0$, $yl = y$ *при всех* $u \in U_\alpha^{(l)}$, $v \in V_\alpha^{(l)}$, $x \in D_\alpha$, $l \in L_\alpha$, $y \in V_\alpha^{(l)} \cup (U_\alpha^{(l)} \setminus M_\alpha)$ *(или разбиение* $D_\alpha = U_\alpha^{(r)} \cup V_\alpha^{(r)}$ *такое, что* $M_\alpha \subseteq U_\alpha^{(r)}$, $xu = x$, $xl = x$, $xv = 0$, $ly = y$ *при всех* $u \in U_\alpha^{(r)}$, $v \in V_\alpha^{(r)}$, $x \in D_\alpha$, $l \in L_\alpha$, $y \in V_\alpha^{(r)} \cup (U_\alpha^{(r)} \setminus M_\alpha)$*).*

Если $\beta \in R_k$, *то либо*

а) $K_{\alpha\beta} = L_\alpha \cup M_{\alpha\beta} \cup 0$ *(или соответственно* $K_{\alpha\beta} = L_\alpha \cup M_{\alpha\beta} \cup 0$*) имеет место включение* $M_{\alpha\beta} L_\alpha \subseteq M_{\alpha\beta}$ $(L_\alpha M_{\beta\alpha} \subseteq M_{\beta\alpha})$ *и если* $M_{\alpha\beta}$ $(M_{\beta\alpha})$ *состоит не только из одного элемента, то* A_β *— такая полугруппа, построение которой описано в пункте 3, либо*

б) $K_{\alpha\beta} = L_\alpha \cup 0$.

Если $\xi \in \mathcal{I}$, $\xi \bar{\in} R_k$, *то* $K_{\alpha\xi} \subseteq L_\alpha \cup 0$.

Замечание 1. Во всех случаях для L-элементов некоторого пересечения $K_{\alpha\beta}$ нужно иметь в виду теорему 3, дающую соответствующие построения.

Замечание 2. Теоремы 1, 3, 5, 6 дают метод, позволяющий строить амальгамы удовлетворяющие условиям теоремы 3 в [6] в многочисленных случаях.

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On a special decomposition of regular semigroups

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In [1] a general disjoint decomposition of semigroups was given, which can be applied for the case of regular semigroups. The aim of the present paper is to obtain a characteristic decomposition of regular semigroups based on the decomposition studied in [1]. We shall investigate the components of this decomposition and the interrelations between them. By making use of [2] we study the cases of regular semigroups with or without a left or right identity element.

Notation. For two sets A, B we write $A \subset B$ if A is a proper subset of B. By a magnifying element we mean a left magnifying element.

1. Let S be a semigroup without nonzero annihilator. This is not a proper restriction because every semigroup can be reduced to this case. Then S has the following disjoint decomposition:

$$S = \bigcup_{i=0}^{5} S_i, \tag{1}$$

where

$$\begin{aligned}
S_0 &= \{a \in S \mid aS \subset S \text{ and } \exists x \in S;\ x \neq 0 \text{ and } ax = 0\},\\
S_1 &= \{a \in S \mid aS = S \text{ and } \exists y \in S,\ y \neq 0 \text{ and } ay = 0\},\\
S_2 &= \{a \in S \setminus (S_0 \cup S_1) \mid aS \subset S \text{ and } \exists x_1, x_2 \in S,\ x_1 \neq x_2 \text{ and } ax_1 = ax_2\},\\
S_3 &= \{a \in S \setminus (S_0 \cup S_1) \mid aS = S \text{ and } \exists y_1, y_2 \in S,\ y_1 \neq y_2 \text{ and } ay_1 = ay_2\},\\
S_4 &= \{a \in S \setminus (S_0 \cup S_1 \cup S_2 \cup S_3) \mid aS \subset S\},\\
S_5 &= \{a \in S \setminus (S_0 \cup S_1 \cup S_2 \cup S_3) \mid aS = S\}.
\end{aligned}$$

It is easy to see that the components S_i $(i=0, 1, \ldots, 5)$ are semigroups, $S_i \cap S_j = \emptyset$ $(i \neq j)$ and the following relations hold:

$$\begin{aligned}
&S_5 S_i \subseteq S_i, \quad S_i S_5 \subseteq S_i \quad (0 \leqq i \leqq 5),\\
&S_4 S_3 \subseteq S_2, \quad S_4 S_2 \subseteq S_2, \quad S_4 S_1 \subseteq S_0, \quad S_4 S_0 \subseteq S_0,\\
&S_2 S_3 \subseteq S_2, \quad S_0 S_1 \subseteq S_0.
\end{aligned} \tag{2}$$

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It is obvious that there exists an analogous decomposition

$$S=\bigcup_{i=0}^{5} T_i, \tag{1'}$$

where T_i $(0\leqq i\leqq 5)$ is the dual of S_i.

Remark. The above decomposition is in fact "group oriented". That is, we select consecutively the elements of S having a property that is very far from that of an element of a group. So we consecutively select the annihilators, the (left) zero divisors, the elements for which the products are not left cancellative, and what remains is a right group.

Our theorems concern the decomposition (1), but analogous results can be formulated for the decomposition (1').

Theorem 1.1. *S_5 is a right group.*

Proof. It is easy to see that S_5 is right simple and left cancellative, whence the assertion follows.

Set $S_0\cup S_2=\bar{S}_2$ and $S_1\cup S_3=\bar{S}_3$.

Theorem 1.2. *$\bar{S}_2$ is a subsemigroup of S.*

Proof. If $s_0\in S_0$ and $s_2\in S_2$, then $s_0s_2\in\bar{S}_2$. There are elements $x, y\in S$, $x\neq y$ such that $s_2x=s_2y$. We have $s_0s_2\notin \bar{S}_2$ and $s_0s_2\notin S_5$ because $s_0s_2S=s_0(s_2S)\subset S$. If $s_0s_2\neq 0$, then $(s_0s_2)x=(s_0s_2)y$ $(x\neq y)$, whence $s_0s_2\in S_2\subseteqq\bar{S}_2$. Similarly, $s_2s_0\in\bar{S}_2$. If $s_0\neq 0$ then $s_2s_0\neq 0$ because $s_2\in S_2$. Since $s_0\in S_0$, there is an element $z\neq 0$ such that $s_0z=0$, hence $(s_2s_0)z=0$. Therefore $s_2s_0\in S_0$. Q.E.D.

Theorem 1.3. *$\bar{S}_3$ contains all the magnifying elements of S and only them.*

Proof. Let $a\in S_1\cup S_3$. If $a\in S$ and $aS=S$, and if furthermore, there is an $y\neq 0$ such that $ay=0$, then $S'=S\setminus\{0\}\subset S$ and $aS'=S$, whence a is a magnifying element. If $a\in S_3$, $aS=S$ and if, furthermore, there exist $x, y\in S$ $(x\neq y)$ such that $ax=ay$, then $a(S-\{x\})=S$ and a is a magnifying element.

Conversely, if $a\in S$ is a magnifying element, then $a\notin S_0\cup S_2\cup S_4$ and $aM=S$ $(M\subset S)$. Thus there exist $m\in M$ and $s\in S\setminus M$ such that $am=as$. Hence it follows that $a\in S_1\cup S_3$. Q.E.D.

Remark. Theorems 1.2 and 1.3 imply

$$S_0S_2\subseteqq S_0\cup S_2,\quad S_2S_0\subseteqq S_0\cup S_2,\quad S_1S_3\subseteqq S_1\cup S_3,\quad S_3S_1\subseteqq S_1\cup S_3. \tag{3}$$

In what follows we assume that S is a regular semigroup, i.e. for every $a\in S$ there is an $x\in S$ such that $a=axa$ and $x=xax$ (x is an inverse of a). The elements

ax, xa are idempotent and $aS \supseteqq axS \supseteqq axaS = aS$ implies $axS = aS$, and similarly, $xaS = xS$. The regular semigroup S can contain a zero element hence the components S_0 and S_1 can exist in the decomposition (1).

Theorem 1.4. *The inverses of the elements of $\bar{S}_3$ are in S_4 and the inverses of the elements of S_4 are in $\bar{S}_3$.*

Proof. Let $a \in \bar{S}_3$ and let $x \in S$ be an inverse of a, that is, let $axa = a$ and $xax = x$. First we show that $xS \subset S$. Suppose that $xS = S$, then there is a subset $S' \subset S$ such that $aS' = S$ because a is a magnifying element. Hence it follows that $xaS' = xS = S$. But we have $(xa)S = xS = S$ and xa is idempotent, that is, xa is a left identity of S. Therefore, $(xa)S' = S' \neq S$, which is a contradiction. Thus $xS \subset S$, whence x is contained in S_0, S_2 or S_4. If $x \in S_2$, then $xs_1 = xs_2$ $(s_1 \neq s_2)$ and $(ax)s_1 = (ax)s_2$. Since $(ax)S = aS = S$ and ax is idempotent we obtain that ax is a left identity of S, i.e. $(ax)s_1 = (ax)s_2$ implies $s_1 = s_2$, which is a contradiction. It can be proved similarly that $x \notin S_0$. It remains the case $x \in S_4$.

Conversely, let $b \in S_4$, that is, $bS = S' \subset S$. Let y be an inverse of b in S. Hence $byS = bS = S'$. Suppose that $yS \subset S$. Let $yS = S''$ $(\neq S)$. Hence $bS'' = byS = bS$. Thus there are elements $s \notin S''$, and $s'' \in S''$ such that $bs'' = bs$. But every element a of S for which $ax_1 = ax_2$ $(x_1 \neq x_2)$, is contained in $S_0 \cup S_1$ or $S_2 \cup S_3$, which contradicts the fact that $b \in S_4$. Thus necessarily $yS = S$, that is, $y \notin S_0 \cup S_2 \cup S_4$. If $y \in S_5$, then $(yb)S = yS = y(bS) = yS' = S$ $(S' \neq S)$, i.e. $y \in S_1 \cup S_3$, which is a contradiction. It remains the only case $y \in S_1 \cup S_3 = \bar{S}_3$. Q.E.D.

It is easy to see that the set of inverses of the elements of $\bar{S}_3$ is equal to S_4 and the set of inverses of the elements of S_4 is equal to $\bar{S}_3$.

Corollary 1.5. *If a regular semigroup S does not contain a magnifying element $(\bar{S}_3 = \emptyset)$, then $S_4 = \emptyset$ and conversely, $S_4 = \emptyset$ implies $\bar{S}_3 = \emptyset$.*

Corollary 1.6. *If a regular semigroup S does not contain a left identity, then $\bar{S}_3 = \emptyset$; and hence $S_4 = \emptyset$.*

For if $a \in \bar{S}_3$ and $x \in S_4$ is an inverse of a, then ax is a left identity of S.

Theorem 1.7. *$\bar{S}_2$ is a regular semigroup and the inverses of an element of $\bar{S}_2$ are contained in $\bar{S}_2$.*

Proof. Let $a \in \bar{S}_2$ and x an inverse of a in S. Since $a \in S_0 \cup S_2$, we have $aS \subset S$. Assume that $xS = S$. Then $(xa)S = x(aS) = xS = S$, whence x is a magnifying element, i.e., $x \in \bar{S}_3$. But every inverse of an element of $\bar{S}_3$ is (by Theorem 1.4) in S_4, thus $a \in S_4$, which is a contradiction. Therefore $xS \subset S$. But $x \notin S_4$ because $a \in \bar{S}_2$. We conclude that $x \in S_0 \cup S_2 = \bar{S}_2$. Q.E.D.

The above results yield:

Theorem 1.8. *A semigroup S is regular if and only if it has a decomposition (1) where*

a) $\bar{S}_2 = S_0 \cup S_2$ *is regular;*

b) *the inverses of the elements of* $\bar{S}_3 = S_1 \cup S_3$ *are contained in* S_4 *and conversely;*

c) S_5 *is a right group.*

Proof. Necessity follows from Theorems 1.1, 1.4, 1.7. Sufficiency follows from the fact that a right group is regular.

2. In this section we shall deepen our kowledge concerning the decomposition (1) of a regular semigroup S as well as the components $\bar{S}_2, \bar{S}_3$ and S_4.

Theorem 2.1. *Let S be a regular semigroup without (left) magnifying elements. Using the notations* $\bar{S}_2 = \bar{S}_2^1$, $S_5 = S_5^1$ *we obtain the following decompositions:*

$S = \bar{S}_2^1 \cup S_5^1$ *and if* $\bar{S}_2^1$ *has no magnifying element,*

$\bar{S}_2^1 = \bar{S}_2^2 \cup S_5^2$ *and if* $\bar{S}_2^1$ *has no magnifying element,*

$\vdots$

$\bar{S}_2^k = \bar{S}_2^{k+1} \cup S_5^{k+1}$,

$\vdots$

where every $\bar{S}_2^k$ *is a regular semigroup, every* S_5^k *is a right group and the following inclusions hold:*

$$S_5^k S_5^j \subseteqq S_5^k, \quad S_5^j S_5^k = S_5^k \quad \text{for} \quad k \geqq j, \tag{4}$$
$$S_5^k \bar{S}_2^j = \bar{S}_2^j, \quad \bar{S}_2^j S_5^k \subseteqq \bar{S}_2^j \quad \text{for} \quad k \leqq j.$$

Proof. It is enough to give a proof for the cases:

$$S_5^1 S_5^k, \quad S_5^k S_5^1, \quad S_5^1 \bar{S}_2^j, \quad \bar{S}_2^j S_5^1$$

because the proof for the semigroups $\bar{S}_2^i$ is similar.

The proof is by induction on k and j. It is trivial that

$$S_5^1 S_5^1 = S_5^1, \quad s_5^1 \bar{S}_2^1 = \bar{S}_2^1, \quad s_5^2 \bar{S}_2^1 = \bar{S}_2^1 \quad (s_5^k \in S_5^k).$$

Hence, $s_5^1 s_5^2 \bar{S}_2^1 = \bar{S}_2^1$, i.e., $s_5^1 s_5^2 \in S_5^2$ for all $s_5^1 \in S_5^1$ and $s_5^2 \in S_5^2$. Since $s_5^1 \bar{S}_2^1 = \bar{S}_2^1$ and, furthermore, $s_5^1 S_5^2 \subseteqq S_5^2$ and $s_5^1 (s_2^2 \bar{S}_2^1) \subset \bar{S}_2^1$, that is, $s_5^1 s_2^2 \in \bar{S}_2^2$, we conclude that $s_5^1 S_5^2 = S_5^2$ and $s_5^1 \bar{S}_2^2 = \bar{S}_2^2$, whence $S_5^1 S_5^2 = S_5^2$, $S_5^1 \bar{S}_2^2 = \bar{S}_2^2$. Thus we have $S_5^1 S_5^1 = S_5^1$, $S_5^1 \bar{S}_2^1 = \bar{S}_2^1$, $S_5^1 S_5^2 = S_5^2$, $S_5^1 \bar{S}_2^2 = \bar{S}_2^2$, $S_5^2 S_5^1 \subseteqq S_5^2$ because $s_5^2 s_5^1 S_5^2 = s_5^2 S_5^2 = S_5^2$, and thus $s_5^2 s_5^1 \in S_5^2$. The first step of the proof is complete.

Now suppose that the following conditions hold:

$$S_5^1 S_5^k = S_5^k, \quad S_5^k S_5^1 \subseteqq S_5^k, \quad S_5^1 \bar{S}_2^j = \bar{S}_2^j, \quad \bar{S}_2^j S_5^1 \subseteqq \bar{S}_2^j.$$

By definition, we have $s_5^{k+1} \bar{S}_2^k = \bar{S}_2^k$. Hence, $(s_5^1 s_5^{k+1}) \bar{S}_2^k = s_5^1 \bar{S}_2^k = \bar{S}_2^k$, whence $s_5^1 s_5^{k+1} \in S_5^{k+1}$. Thus we obtain $S_5^{k+1} = (s_5^1 s_5^{k+1}) S_5^{k+1} = s_5^1 S_5^{k+1}$, whence $S_5^1 S_5^{k+1} = S_5^{k+1}$.

We have $(s_5^{k+1} s_5^1) S_5^{k+1} = S_5^{k+1}$ and, furthermore, $s_5^{k+1} s_5^1 \in \bar{S}_2^k$; thus $s_5^{k+1} s_5^1 \in S_5^{k+1}$ implies $S_5^{k+1} S_5^1 \subseteqq S_5^{k+1}$. We also have $(s_5^1 s_2^{j+1}) \bar{S}_2^j \subset s_5^1 \bar{S}_2^j = \bar{S}_2^j$, whence $s_5^1 s_2^{j+1} \in$

$\in \bar{S}_2^{j+1}$, and $s_5^1 \bar{S}_2^{j+1} = \bar{S}_2^{j+1}$ implies $S_5^1 \bar{S}_2^{j+1} = \bar{S}_2^{j+1}$. Finally, we have $s_2^{j+1} s_5^1 \in \bar{S}_2^j$ and $s_2^{j+1} s_5^1 \bar{S}_2^j = s_2^{j+1} S_2^j \subset S_2^j$, whence it follows that $s_2^{j+1} s_5^1 \in \bar{S}_2^{j+1}$ and $\bar{S}_2^{j+1} S_5^1 \subseteq \subseteq \bar{S}_2^{j+1}$. Q.E.D.

Corollary 2.2. *If S and $\bar{S}_2^k$ ($k \geqq 1$) are regular semigroups without magnifying elements, then S has one of the following four types of decompositions:*

a) $S = ((((\ldots) \cup S_5^4) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *with an infinite number of components;*

b) $S = \bar{S}_2 \cup ((((\ldots) \cup S_5^4) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *where $\bar{S}_2$ is a semigroup of type $\bar{S}_2$ and there are infinitely many components;*

c) $S = (((S_5^n \cup \ldots) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *where the number of components equals n;*

d) $S = ((((\bar{S}_2^m \cup S_5^m) \cup \ldots) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *where the number of components is $m+1$.*

We shall treat some properties of the semigroups $\bar{S}_3$ and S_4.

Theorem 2.3. *Let $a, b \in \bar{S}_3$, and let x be an inverse of a, and y an inverse of b ($x, y \in S_4$). Then xy is an inverse of ba.*

Proof. Since ax and by are left identities of S, we have $baxyba = b(axy)ba = byba = ba$, and $xybaxy = xyb(axy) = xyby = xy$. Q.E.D.

Theorem 2.4. *If $a, b \in S_4$ and if x is an inverse of a and y is an inverse of b, then yx and ab are inverses of each other.*

Proof. By Theorem 2.3, $(yby)(xax)$ is an inverse of ab. Then we get $ab = ab(yby)(xax)ab = a(byb)yx(axa)b = abyxab$, $yxabyx = ybyx = yx$, since xa, yb are left identities of S. Q.E.D.

By Theorem 1.4, $\bar{S}_3 \cup S_4$ is a regular subset of S, but it fails to be a subsemigroup, because, e.g., $S_4 S_3 \subseteq S_2$ (cf. (2)). Set

$$X_1 = \{x \in S_4 \mid x \text{ is an inverse of some } a \in S_1\},$$

$$X_3 = \{y \in S_4 \mid y \text{ is an inverse of some } b \in S_3\}.$$

Then $S_4 = X_1 \cup X_3$.

Corollary 2.5. *X_1 and X_3 are subsemigroups of S_4. In general, if $A \subseteq \bar{S}_3$ is a subsemigroup, then the inverses of the elements of A form a subsemigroup in S_4.*

Proof. This is an easy consequence of Theorem 2.3.

Corollary 2.6. *$\bar{S}_3$ and S_4 have no idempotent elements.*

Proof. Every element of $\bar{S}_3$ is magnifying, thus $a \neq a^2$ ($a \in \bar{S}_3$). Assume that $e \in S_4$ is idempotent. Since e is an inverse of e, $e \in \bar{S}_3$ (by Theorem 1.4), which is a contradiction.

Theorem 2.7. *Every element of $\bar{S}_3$ and S_4 generates an infinite cyclic semigroup.*

Proof. In the opposite case, $\bar{S}_3$ or S_4 contains an idempotent element which contradicts Corollary 2.6.

Theorem 2.8. 1) *$\bar{S}_3$ has no (proper) right magnifying element.* 2) *S_4 has no left magnifying element.* 3) *If $1 \in S$ (i.e. S is a monoid), then $S_0 \cup S_2 \cup S_5$ has no left or right magnifying element.* 4) *S_5 has no left magnifying element.*

Proof. 1) is a consequence of [4], Chap. III. 5.6 (β). Since in the product $s_4 S$ ($s_4 \in S_4$) the representation of each element is unique, thus the same holds for $s_4 S_4$, and 2) is true. 3) follows from [4], Chap. III. 5.6 (γ), because the union $S_0 \cup S_2 \cup S_5$ does not contain left or right magnifying element of S. Finally, S_5 is a right group, and hence has no left magnifying element, cf. [4], Chap. III. 5.3 (γ).

3. In this section the results of [2] will be applied to the decomposition (1) of regular semigroups. For a regular semigroup S we shall investigate the following cases based on Theorem 4 in [2]:

1) S has neither a left nor a right identity element;

2) S has an identity element;

3) S has either a single left or a single right identity element.

In the case 3) we may assume that S has only a left identity element. In the opposite case we have to study the decomposition (1′) instead of (1). As it is well known, an idempotent element e is $\mathscr{D}$-primitive if it is minimal among the idempotents D_e, where D_e is the $\mathscr{D}$-class of e ($\mathscr{D}$ is one of Green's relations).

In the case 1) S has no left magnifying element (cf. Corollary 1.6), that is, $S_1 \cup S_3 = \emptyset$ and $S_4 = \emptyset$, furthermore, $S_5 = \emptyset$, because in the opposite case S would have a left identity element. Hence $S = S_0 \cup S_2 = \bar{S}_2$.

In the case 2) suppose that $1 \in S$ is the identity element. If 1 is $\mathscr{D}$-primitive then we have $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ (e.g. $1 \in S_5$). In this subcase we obtain that $S = S_0 \cup S_2 \cup S_5$. If 1 is not $\mathscr{D}$-primitive, then there are magnifying elements, that is, $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, S_5 is equal to the subsemigroup of all invertable elements and thus it is nonempty. Since $S_4 S_3 \subseteq S_2$ and $S_4 S_1 \subseteq S_0$, at least one of the subsemigroups S_0, S_2 is nonempty. Hence we obtain $S = \bar{S}_2 \cup \cup \bar{S}_3 \cup S_4 \cup S_5$, where all the components are nonvoid.

In the case 3) suppose that e is the only left identity element of S. If e is $\mathscr{D}$-primitive, then $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ (for example, $e \in S_5$). Therefore $S = S_0 \cup S_2 \cup S_5$. If e fails to be $\mathscr{D}$-primitive, then there are magnifying elements, that is, $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5 \neq \emptyset$ and, similarly to the second subcase of 2), we have $S_0 \cup S_2 \neq \emptyset$. Hence $S = \bar{S}_2 \cup \bar{S}_3 \cup S_4 \cup S_5$, where all the components are nonempty.

Summing up:

Theorem 3.1. *Let S be a regular semigroup. Then:*
1) *If S has no left identity element then* $S_1\cup S_3=\emptyset$, $S_4=\emptyset$, $S_5=\emptyset$.
2) *If S has an identity element and*
a) *if 1 is $\mathcal{D}$-primitive then* $S_1\cup S_3=\emptyset$, $S_4=\emptyset$, $S_5\neq\emptyset$,
b) *if 1 is not $\mathcal{D}$-primitive then* $S_1\cup S_3\neq\emptyset$, $S_4\neq\emptyset$, $S_5\neq\emptyset$, $S_0\cup S_2\neq\emptyset$.
3) *If e is the unique left identity of S and*
a) *if e is $\mathcal{D}$-primitive then* $S_1\cup S_3=\emptyset$, $S_4=\emptyset$, $S_5\neq\emptyset$,
b) *if e is not $\mathcal{D}$-primitive then* $S_1\cup S_3\neq\emptyset$, $S_4\neq\emptyset$, $S_5\neq\emptyset$, $S_0\cup S_2\neq\emptyset$.

4. Finally, we make some remarks concerning the decomposition (1). For $x\in S_4$, $a\in\bar{S}_3$ let $B_x=\{b\in S|b$ is an inverse of $x\}$, $C_a=\{y\in S|y$ is an inverse of $a\}$. If $x\in S_4$ and $b\in B_x$ $(b\in\bar{S}_3)$, then bx is a left identity of S. Analogonsly, ay $(a\in\bar{S}_3,\ y\in C_a)$ is also a left identity of S.

Theorem 4.1. *If $x\in S_4$ then B_x fails to be a subsemigroup. If $a\in\bar{S}_3$, then C_a fails to be a subsemigroup.*

Proof. Suppose that B_x is a semigroup and $a,b\in B_x$. Then $axa=a$, $bxb=b$ and $ba\in B_x$. Hence $baxba=ba$. Since ax is a left identity element, hence $b(ba)=ba$. On the other hand, $ba\in\bar{S}_3$, thus $baS=S$, whence $bs=s$ for all $s\in S$, which is a contradiction (b is a left magnifying element!).

Let $x,y\in C_a$. If C_a is a semigroup, then $a(xy)a=(ax)ya=ya$. But $ya\neq a$, because ya is idempotent, while the element $a\in\bar{S}_3$ is not. Thus $xy\notin C_a$. Q.E.D.

Let $M\subset S$ be a subset of S such that $aM=S$. Then the set M is *left increasable* by a. Such a set M *is not* uniquely determined by a.

Theorem 4.2. *If $a\in\bar{S}_3$ then* $a(S_0\cup S_2\cup S_4)=S$.

Proof. Let $a\in\bar{S}_3$ and $x\in S_4$ an inverse of a. Then we have $axS=aS=S$ and $xS\subset S$. On the other hand, $xS\subseteq S_4S$, furthermore, by making use of the relations (2) we get

$$S_4S=S_4(S_0\cup S_1\cup S_2\cup S_3\cup S_4\cup S_5)\subseteq S_0\cup S_2\cup S_4.$$

Hence $xS\subseteq S_0\cup S_2\cup S_4$ and thus $a(S_0\cup S_2\cup S_4)=S$. Q.E.D.

Theorem 4.2 implies for every $a\in\bar{S}_3$ the existence of an element $y_a\in S_0\cup S_2\cup S_4$ such that $ay_a=a$.

Theorem 4.3. a) *If $a\in S_3$, then $y_a\notin S_0$.* b) *The elements $a\in\bar{S}_3$ for which $y_a\in S_4$ $(ay_a=a)$, have a two-sided identity element in S.*

Proof. a) If $y_a\in S_0$, then there is an $x\neq0$ such that $y_ax=0$. Thus $ax=(ay_a)x=a(y_ax)=a0=0$, whence $a\in S_0\cup S_1$, which is a contradiction.

b) If $y_a\in S_4$, then there exists $b\in\bar{S}_3$, such that $by_ab=b$ and $y_aby_a=y_a$. Then $ay_ab=ab$, $ay_aby_a=aby_a$, that is, $ay_a=aby_a$, whence it follows that $a=a(by_a)$. On the other hand, by_a is a left identity element of S, whence $by_aa=a=aby_a$. Q.E.D.

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Otsukische Übertragung mit rekurrentem Maßtensor

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§ 1. Einleitung

Im Aufsatz [3] begründete T. Otsuki eine Übertragungstheorie in den n-dimensionalen Punkträumen, die in der lokalen Schreibweise auf die folgende Weise charakterisiert werden kann:

Der invariante Differentialquotient eines Tensors $V^{i_1 \dots i_p}_{j_1 \dots j_q}$ ist längs einer Kurve $C: x^i = x^i(t)$ durch die Formeln

$$\frac{D}{dt} V^{i_1 \dots i_p}_{j_1 \dots j_q} \overset{\text{def}}{=} P^{i_1}_{r_1} \dots P^{i_p}_{r_p} V^{r_1 \dots r_p}_{s_1 \dots s_q | k} \frac{dx^k}{dt} P^{s_1}_{j_1} \dots P^{s_q}_{j_q} \tag{1.1}$$

$$V^{r_1 \dots r_p}_{s_1 \dots s_q | k} \overset{\text{def}}{=} \partial_k V^{r_1 \dots r_p}_{s_1 \dots s_q} + \sum_{t=1}^{p} {'\Gamma_h{}^{r_t}{}_k} V^{r_1 \dots r_{t-1} h r_{t+1} \dots r_p}_{s_1 s_2 \dots \dots s_q} - \sum_{t=1}^{q} {''\Gamma_{s_t}{}^h{}_k} V^{r_1 r_2 \dots \dots r_p}_{s_1 \dots s_{t-1} h s_{t+1} \dots s_q} \tag{1.2}$$

definiert, wo die Indizes jetzt und im folgenden immer die Werte $1, 2, \dots, n$ annehmen werden, und $'\Gamma_a{}^b{}_c$, $''\Gamma_a{}^b{}_c$ gewöhnliche affine Übertragunsgparameter bedeuten. P^i_j bedeutet in diesen Formeln und auch im folgenden einen gemischten Tensor (vgl. [3], Formeln (4.9) und (4.10), wo aber die — von uns im folgenden nicht zu benützende — Bezeichnung:

$$\frac{\bar{D} V^{r_1 \dots r_p}_{s_1 \dots s_q}}{dt} \overset{\text{def}}{=} V^{r_1 \dots r_p}_{s_1 \dots s_q | k} \frac{dx^k}{dt}$$

verwendet wurde). Die Übertragungsparameter $'\Gamma_a{}^b{}_c$ bzw. $''\Gamma_a{}^b{}_c$, die bei der Bildung des invarianten Differentialquotienten bei den kontra- bzw. kovarianten Indizes verwendet sind, brauchen nicht übereinstimmen.

Neben dem Tensor P^i_j soll auch der inverse Tensor Q^i_j eindeutig bestimmt sein, d. h. es ist $det(P^i_j) \neq 0$ und

(1.3a) $\quad P^i_j Q^j_k = \delta^i_k, \qquad$ (1.3b) $\quad P^i_r Q^s_i = \delta^s_r.$

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Satt (1.2) benützt man in der Otsukischen Theorie lieber die kovariante Ableitung:

(1.4) $$\nabla_k V^{i_1\dots i_p}_{j_1\dots j_q} \overset{\text{def}}{=} P^{i_1}_{r_1}\dots P^{i_p}_{r_p} V^{r_1\dots r_p}_{s_1\dots s_q|k} P^{s_1}_{j_1}\dots P^{s_q}_{j_q}$$

(vgl. [3], (3.8)), wodurch der fundamentale Differentialquotient (1.1) die Form:

(1.5) $$\frac{D}{dt} V^{i_1\dots i_p}_{j_1\dots j_q} = \nabla_k V^{i_1\dots i_p}_{j_1\dots j_q} \frac{dx^k}{dt} = P^{i_1}_{r_1}\dots P^{i_p}_{r_q}\left(\frac{\bar D}{dt} V^{r_1\dots r_p}_{s_1\dots s_p}\right) P^{s_1}_{j_1}\dots P^{s_q}_{r_p}$$

haben wird. Mit den Bezeichnungen

(1.6a) $\Gamma_j{}^i{}_k \overset{\text{def}}{=} P^i_m {}'\Gamma_j{}^m{}_k,$ (1.6b) $\Lambda_j{}^i{}_k \overset{\text{def}}{=} P^m_j {}''\Gamma_m{}^i{}_k$

kann leicht verifiziert werden, daß unsere Formeln (1.4) und (1.5) — auf Grund von (1.2) und (1.3a), (1.3b) — eben in die Otsukischen Formeln (2.15) und (2.14) von [3] übergehen, wo ${}'\Gamma_j{}^i{}_k$ und ${}''\Gamma_j{}^i{}_k$ der Bedingung

$${}''\Gamma_j{}^i{}_k = P^i_m {}'\Gamma_l{}^m{}_k Q^l_j - Q^m_j \frac{\partial P^i_m}{\partial x^k}$$

unterworfen sind.

Im folgenden wollen wir eine solche Otsukische Übertragung bestimmen, in der die Übertragungsparameter durch einen, in seinen Indizes symmetrischen, Fundamentaltensor $g_{ij}(x)$ bestimmt sind, der den Relationen

(1.7) $$\nabla_k g_{ij} = \gamma_k(x) g_{ij}$$

genügt, wo $\gamma_k(x)$ einen kovarianten Vektor bedeutet. Die Formel (1.7) drückt aus, daß diese Übertragungstheorie, die wir im folgenden entwickeln werden, die Weylsche [5] und Otsukische [4] Übertragungstehorien in sich vereinigen wird, wobei sie als ein Spezialfall von [3] betrachtet werden kann. Vom metrischen Fundamentaltensor g_{ij} soll noch angenommen werden, daß $det(g_{ij}) \neq 0$ ist, d. h. der inverse Tensor g^{ij} eindeutig bestimmt ist.

Die Grundgrößen des Raumes sind also der metrische, symmetrische Grundtensor g_{ij}, der Rekurrenzvektor γ_k und der gemischte Tensor P^i_j, der im folgenden der Symmetriebedingung

(1.8) $$P_{ij} \overset{\text{def}}{=} g_{jr} P^r_i = g_{ir} P^r_j = P_{ji}$$

genügen soll.

Mit Hilfe von g_{ij} bzw. mit Hilfe des inversen Tensors g^{rk} können die Indizes in der gewöhnlichen Weise herauf- bzw. heruntergezogen werden.

Das Ziel unserer Arbeit ist die Bestimmung der Form der Übertragungsparameter und die Untersuchung des invarianten Differentials der Eigenvektoren bei einer Kontraktion mit g_{ij}; ferner wollen wir im Satz 5 von § 4 die notwendigen und hinreichenden Bedingungen bestimmen dafür, daß Dg_{ij} ein Eigentensor längs einer Kurve des Raumes sei. Diesen Satz mit dem Satz 7 zusammen betrachten wir als Hauptsätze dieser Arbeit.

§ 2. Bestimmung der Übertragungsparameter

In diesem Paragraphen wollen wir die der Relation (1.7) genügenden Übertragungsparameter bestimmen. Aus (1.7) folgt auf Grund von (1.4):

$$P_i^r P_j^s g_{rs|k} = \gamma_k g_{ij}. \tag{2.1a}$$

Wir wollen hier bemerken, daß für den kontravarianten metrischen Fundamentaltensor g_{ij} die analoge Forderung:

$$P_r^i P_s^j g^{rs}{}_{|k} = -\gamma_k g^{ij} \tag{2.1b}$$

wäre (vgl. etwa [1], Formel (7.1)a)). Doch wäre die Formel (2.1b) im allgemeinen in der Otsukischen Übertragungstheorie nicht eine Folgerung von (2.1a), sondern eine neue Forderung, die — abgesehen von gewissen Spezialfällen — nicht mit (2.1a) gleichzeitig gelten konnte, da nach (1.2) $\delta^r_{i|k} \neq 0$ ist. Gelten aber die Identitäten: $P^i_j = \delta^i_j$ ($\doteq$Kronecker-δ), ferner $'\Gamma_a{}^b{}_c = ''\Gamma_a{}^b{}_c$, so wäre (2.1b) eine Folgerung von (2.1a). In diesem Fall wäre aber die Otsukische Übertragungstheorie mit der gewöhnlichen affinen Übertragungstheorie identisch.

Auf Grund des durch (1.3a) und (1.3b) definierten inversen Tensors Q^i_j von P^i_j, kann (2.1a) bzw. (2.1b) in Hinsicht auf (1.2) in der Form:

$$\partial_k g_{rs} - ''\Gamma_r{}^t{}_k g_{ts} - ''\Gamma_s{}^t{}_k g_{rt} = \gamma_k g_{bt} Q^b_r Q^t_s \tag{2.2a}$$

$$\partial_k g^{rs} + '\Gamma_t{}^r{}_k g^{ts} + '\Gamma_t{}^s{}_k g^{rt} = -\gamma_k g^{ab} Q^r_a Q^s_b \tag{2.2b}$$

geschrieben werden.

Sind die Übertragungsparameter $''\Gamma_r{}^t{}_k$ in den unteren Indizes symmetrisch — was wir im folgenden immer annehmen wollen —, so erhält man diese Übertragungsparameter in der gewöhnlichen Weise, durch zyklische Permutation der Indizes r, s, k in (2.2a), bei der letzten Permutation mit einer Vorzeichenveränderung und dann nach einer Addition, in der Form:

$$''\Gamma_r{}^t{}_k = \tfrac{1}{2} g^{ts} \{\partial_k g_{rs} + \partial_r g_{sk} - \partial_s g_{rk} - (\gamma_k m_{rs} + \gamma_r m_{sk} - \gamma_s m_{rk})\} \tag{2.3}$$

mit

$$m_{rs} \stackrel{\text{def}}{=} g_{ij} Q^i_r Q^j_s.$$

Bemerkung. Nach den Bezeichnungen von H. Weyl: [5] ist $\gamma_k = -\varphi_k$ und $''\Gamma_{rsk} = \Gamma_{s,rk}$. —

Etwas komplizierter wäre die Bestimmung von $'\Gamma_t{}^r{}_k$ aus (2.2b), die wir nur skizzieren wollen. Eine Überschiebung von (2.2b) mit $g_{ri} g_{sj}$ gibt eine Identität für $'\Gamma_{jik}$. Angenommen, daß $'\Gamma_{jik}$ in (j, k) symmetrisch ist, erhält man $'\Gamma_{jik}$ in analoger Weise — abgesehen von Vorzeichenveränderungen — wie $''\Gamma_{jik}$. Eine Überschiebung mit g^{it} ergibt die gewünschte Größe: $'\Gamma_j{}^t{}_k$.

Wir beweisen den folgenden:

Satz 1. *Ist $''\Gamma$ die durch $'\Gamma$ induzierte Übertragung (vgl.* [2], *S. 161, oder* [3] *S. 109), so ist*

$$(2.4)\qquad (\partial_k P^m_t)(Q^r_m g^{ts}+Q^s_m g^{tr})+\gamma_k Q^r_a Q^s_b g^{ab}+\partial_k g^{rs}+Q^r_m {''\Gamma_t}^m{}_k P^{ts}+$$
$$+Q^s_m {''\Gamma_t}^m{}_k P^{tr}=0,\quad P^{tr}\overset{\mathrm{def}}{=}P^t_j g^{jr}.$$

Gilt die Relation:

$$(2.5)\qquad P^i_j=\varrho\delta^i_j\quad(\varrho=\mathit{Konst.}\neq 0),$$

so ist $'\Gamma=''\Gamma$, und (2.2a) und (2.2b) sind gleichzeitig erfüllt.

Beweis. Bezüglich der erste Behauptung des Satzes beachte man, daß wenn $''\Gamma$ die induzierte Übertragung von $'\Gamma$ ist, so gilt nach der Formel (3.13) von [3]:

$$(2.6)\qquad \partial_k P^i_j+{''\Gamma_r}^i{}_k P^r_j-{'\Gamma_j}^r{}_k P^i_r=0.$$

Aus dieser Formel folgt nun nach einer Überschiebung mit Q^l_i nach (1.3b), daß

$${'\Gamma_j}^l{}_k=Q^l_m\partial_k P^m_j+Q^l_t {''\Gamma_m}^t{}_k P^m_j.$$

Substituieren wir das in (2.2b), so erhält man unmittelbar (2.4); die Identität (2.4) entspricht der Identität (2.6) im metrischen Fall.

Bezüglich der zweite Behauptung des Satzes beachte man, daß aus (1.3a) folgt, daß neben (2.5) auch

$$(2.7)\qquad Q^i_j=\varrho^{-1}\delta^i_j\quad(\varrho=\mathit{Konst.}\neq 0)$$

besteht. Auf Grund der Form (2.5) von P^i_j folgt noch nach (2.6), daß die affinen Übertragungen $'\Gamma$ und $''\Gamma$ übereinstimmen, d. h. es ist ${'\Gamma_i}^j{}_k={''\Gamma_i}^j{}_k$. (2.2a) geht somit im Hinblick auf $'\Gamma=''\Gamma$ in

$$(2.8)\qquad \partial_k g_{rs}-{'\Gamma_r}^t{}_k g_{ts}-{'\Gamma_s}^t{}_k g_{rt}=\varrho^{-2}\gamma_k g_{rs}$$

über. Es ist nun $g_{jm}g^{ms}=\delta^s_j$, woraus nach einer partiellen Ableitung nach x^k, auf Grund von (2.8), die Formel

$$(\partial_k g^{ms})g_{jm}\equiv -g^{ms}\partial_k g_{jm}=-g^{ms}(\varrho^{-2}\gamma_k g_{jm}+{'\Gamma_j}^t{}_k g_{tm}+{'\Gamma_m}^t{}_k g_{jt})$$

folgt. Eine Überschiebung mit g^{jr} gibt nun — unter einer nochmaligen Beachtung von (2.7) — unmittelbar (2.2b). Die Formeln (2.2a) und (2.2b) sind also gleichzeitig gültig, wie behauptet wurde.

§ 3. Eigenvektoren und ihre Kontraktionen

Ein kontravarianter Eigenvektor $V^i(x)$ ist durch die Definitionsformel

$$(3.1)\qquad P^i_j(x)V^j(x)=\tau(x)V^i(x)\quad(\tau\neq 0)$$

festgelegt (vgl. [3], Formel (5.2)); für einen kovarianten Eigenvektor $V_k(x)$ lautet die analoge Formel:

$$(3.2)\qquad P^k_i(x)V_k(x)=\tau(x)V_i(x),$$

wo $\tau(x)$ eine im Raum definierte Eigenfunktion bedeutet. Für die folgenden wird es hinreichend sein, wenn V^i bzw. V_k und τ nur längs einer Kurve $C: x^i = x^i(t)$ definiert sind. Es kann sehr einfach der folgende Satz bewiesen werden:

Satz 2. *Ist P_{ij} in (i, j) symmetrisch, so folgt aus (3.1) die Relation (3.2) mit $V_i = g_{ir} V^r$.*

Ist P_{ij} in (i, j) nicht symmetrisch, so folgt aus (3.1):

$$P_i^{*k} V_k = \tau V_i, \quad P_i^{*k} \stackrel{\text{def}}{=} g_{it} P^t_j g^{jk}. \tag{3.3}$$

Beweis. Eine Kontraktion von (3.1) mit g_{im} führt im Hinblick auf

$$V^j = g^{jk} V_k, \quad V_k \stackrel{\text{def}}{=} g_{kr} V^r$$

nach gewissen Indexveränderungen auf die Relation:

$$P_i^{*k} V_k \equiv g_{it} P^t_j g^{jk} V_k = \tau V_i,$$

womit wir schon gezeigt haben, daß aus (3.1) die Relation (3.3) folgt. Ist nun P_{ij} symmetrisch, so folgt aus (1.8), daß $P_i^{*k} = P_i^k$ ist, wodurch aus (3.3) die Formel (3.2) entsteht, w. z. b. w.

Im folgenden wollen wir eine wichtige Formel von Otsuki, die wir auch verwenden wollen, durch eine einfachere Methode ableiten (vgl. [3], Formel (5.8)).

Nehmen wir an, daß für den kontravarianten Vektor V^j, (3.1) besteht. Da auf beiden Seiten von (3.1) je ein kontravarianter Vektor steht, bekommt man nach invarianter Ableitung von beiden Seiten auf Grund von (1.1) und (1.2):

$$\begin{aligned} P^i_k \left(\frac{dP^k_j}{dt} V^j + P^k_j \frac{dV^j}{dt} + {'\Gamma_r}^k{}_s P^r_j V^j \frac{dx^s}{dt} \right) &= \\ = P^i_k \left(\frac{d\tau}{dt} V^k + \tau \left(\frac{dV^k}{dt} + {'\Gamma_r}^k{}_s V^r \frac{dx^s}{dt} \right) \right)&. \end{aligned} \tag{3.4}$$

Beachten wir nun die Formel (3.13) von [3], die offenbar mit unserer Formel (2.6) äquivalent ist, so wird:

$$\frac{dP^k_j}{dt} \equiv \frac{\partial P^k_j}{\partial x^s} \frac{dx^s}{dt} = (P^k_r {'\Gamma_j}^r{}_s - P^r_j {''\Gamma_r}^k{}_s) \frac{dx^s}{dt}. \tag{3.5}$$

Substituiert man das in (3.4), beachten wir ferner auf der rechten Seite die Formel des invarianten Differentials (1.1) für den kontravarianten Vektor V^i, so wird nach entsprechenden Vertauschungen der Indizes:

$$P^i_k \left(\frac{DV^k}{dt} + P^r_j ({'\Gamma_r}^k{}_s - {''\Gamma_r}^k{}_s) V^j \frac{dx^s}{dt} \right) = \tau \frac{DV^i}{dt} + \frac{d\tau}{dt} P^i_k V^k. \tag{3.6}$$

Auf Grund von (1.1) und (1.2) hat man

$$\frac{D}{dt}\delta^i_j \equiv P^i_k P^r_j ({}'\Gamma_r{}^k{}_s - {}''\Gamma_r{}^k{}_s)\frac{dx^s}{dt}. \tag{3.7}$$

Beachten wir in (3.6) diese Identität, ferner (3.1), so wird:

$$P^i_k \frac{DV^k}{dt} + \frac{D\delta^i_j}{dt}V^j = \tau\left(\frac{DV^i}{dt} + \frac{d\tau}{dt}V^i\right), \tag{3.8}$$

was mit der Formel (5.8) von [3] übereinstimmt.

Wir gehen nun zur Untersuchung des invarianten Differentials von $V_k \equiv g_{ik}V^i$ über, falls für den kontravarianten Vektor V^i die Bedingung (3.1) besteht, der metrische Fundamentaltensor g_{ij} der Relation (1.7) bzw. (2.1a) genügt, und endlich für den Tensor P^i_j die Symmetriebedingung (1.8) gültig ist.

Aus den Formeln (1.4) und (1.2) folgt, daß

$$\nabla_k(g_{is}V^s) = P^r_i(V^s\partial_k g_{rs} + g_{rs}\partial_k V^s - {}''\Gamma_r{}^t{}_k g_{ts}V^s) \tag{3.9}$$

besteht, da $g_{is}V^s$ ein kovarianter Vektor ist. Beachten wir nun (2.2a), die eine Folgerung von (1.7) ist, so wird durch die Elimination von $\partial_k g_{rs}$:

$$\nabla_k(g_{is}V^s) = P^r_i(\gamma_k g_{jm}Q^j_r Q^m_s V^s + g_{rs}\partial_k V^s + g_{rs}{}''\Gamma_t{}^s{}_k V^t).$$

Auf Grund der Symmetriebedingung (1.8) ist nun nach gewissen geeigneten Veränderungen der Indizes, und im Hinblick auf (1.3b):

$$\nabla_k(g_{is}V^s) = \gamma_k g_{ir}Q^r_s V^s + g_{ir}P^r_s(\partial_k V^s + {}''\Gamma_t{}^s{}_k V^t).$$

Eine weitere Umformung — d. h. die Eliminierung von ${}''\Gamma_t{}^s{}_k$ — mittels der Identität (vgl. [3], Formel (3.10)):

$$\delta^s_{t|k} = {}'\Gamma_t{}^s{}_k - {}''\Gamma_t{}^s{}_k, \tag{3.10}$$

gibt nach den Grundformeln (1.1) und (1.2)

$$\nabla_k(g_{is}V^s) = \gamma_k g_{ir}Q^r_s V^s + g_{ir}\nabla_k V^r - g_{ir}P^r_s\delta^s_{t|k}V^t. \tag{3.11}$$

Nun ist nach (1.4):

$$\nabla_k \delta^r_m = P^r_s P^j_m \delta^s_{j|k},$$

woraus nach einer Überschiebung mit Q^m_t die Relation

$$P^r_s\delta^s_{t|k} = Q^m_t\nabla_k\delta^r_m \tag{3.12}$$

folgt, und das führt die Formel (3.11) in

$$\nabla_k(g_{is}V^s) = g_{ir}\nabla_k V^r + g_{ir}(\gamma_k Q^r_t - Q^m_t\nabla_k\delta^r_m)V^t$$

über, woraus nach einer Überschiebung mit dx^k/dt die Formel

$$\frac{D}{dt}(g_{is}V^s) = g_{ir}\left(\frac{DV^r}{dt} + Q^r_t V^t \gamma_k \frac{dx^k}{dt} - Q^m_t V^t \frac{D\delta^r_m}{dt}\right) \tag{3.13}$$

folgt.

Nehmen wir nun an, daß V^i ein Eigenvektor ist, d. h. (3.1) ist gültig. Zieht man mit g_{ir} den Index „i" ab, so folgt wieder nach gewissen Veränderungen der Summationsindizes und im Hinblick auf die Symmetriebedingung (1.8):

$$P^r_i g_{rj} V^j = \tau g_{ir} V^r. \tag{3.14}$$

Bilden wir jetzt die Operation $\frac{D}{dt} \equiv \frac{dx^k}{dt} \nabla_k$ auf beide Seiten von (3.14), so erhalten wir nach den Definitionsformeln (1.4) und (1.2)

$$P^s_i \left\{ g_{rj} V^j \frac{dP^r_s}{dt} + (P^r_s V^j \partial_k g_{rj} + P^r_s g_{rj} \partial_k V^j - {}''\Gamma^{\,t}_{s\,k} P^r_t g_{rj} V^j) \frac{dx^k}{dt} \right\} = \tag{3.15}$$

$$= P^s_i \left(\frac{d\tau}{dt} g_{sr} V^r + \tau(\partial_k g_{sr}) V^r \frac{dx^k}{dt} + \tau g_{sr} (\partial_k V^r) \frac{dx^k}{dt} - \tau\, {}''\Gamma^{\,t}_{s\,k} g_{tr} V^r \frac{dx^k}{dt} \right).$$

Eliminieren wir von der linken Seite $\frac{dP^r_s}{dt}$ mittels der Formel (3.5), beachten wir dann (3.9) und (3.10), so entsteht auf der linken Seite der Ausdruck:

$$P^s_i \{ \nabla_k (g_{sj} V^j) + P^r_t \delta^t_{s|k} g_{rj} V^j \} \frac{dx^k}{dt}.$$

Auf der rechten Seite von (3.15) erhält man wieder unter Beachtung von (3.9)

$$\frac{d\tau}{dt} P^s_i g_{sr} V^r + \tau (\nabla_k g_{is} V^s) \frac{dx^k}{dt},$$

und somit wird aus (3.15) im Hinblick auf (3.12) und (3.14)

$$P^s_i \frac{D}{dt}(g_{sj} V^j) + \frac{D\delta^r_i}{dt} g_{rj} V^j = \tau \left(\frac{d\tau}{dt} g_{ir} V^r + \frac{D}{dt}(g_{is} V^s) \right). \tag{3.16}$$

Vergleicht man (3.8) und (3.16), so folgt der

Satz 3. *Ist V^i ein Eigenvektor, die der Formel (3.1) genügt, ist ferner P_{ij} in (i, j) symmetrisch, so verhält sich der Vektor $V_i \equiv g_{ir} V^r$ bezüglich der invarianten Differentiation (1.1) ebenso, wie der Vektor V^i.*

In ähnlicher Weise folgt unmittelbar aus den Gleichungen (3.1), (3.8) und (3.16), unter Beachtung von Satz 2 der folgende

Satz 4. *Gilt für einen Vektor V^i die Relation (3.1) längs einer Kurve $C: x^i = x^i(t)$, ist P_{ij} in (i, j) symmetrisch, ist endlich $\frac{D\delta^i_j}{dt} = 0$, so gehören V^i, $\frac{DV^i}{dt}$, $V_i \equiv g_{ir} V^r$, $\frac{DV^i}{dt}$ zu demselben Eigenraum von τ.*

Für den Fall, daß $\frac{D\delta^i_j}{dt} \neq 0$ besteht, kann noch eine Formel für die Eigenvektoren mittels (3.16) und (3.13) abgeleitet werden. Eliminiert man aus (3.16) $\frac{D}{dt} g_{sj} V^j$ mittels der Formel (3.13), so wird unter Beachtung von (1.8):

$$g_{is} P^s_r \left(\frac{DV^r}{dt} + Q^r_h V^h \gamma_k \frac{dx^k}{dt} - Q^m_h V^h \frac{D\delta^r_m}{dt} \right) + g_{rj} V^j \frac{D\delta^r_i}{dt} =$$

$$= \tau g_{ir} \left(\frac{d\tau}{dt} V^r + \frac{DV^r}{dt} + Q^r_h V^h \gamma_k \frac{dx^k}{dt} - Q^m_h V^h \frac{D\delta^r_m}{dt} \right).$$

Beachten wir jetzt die aus (3.1) folgende Rleation

$$V^r = \tau Q^r_h V^h$$

und (1.3a), so wird:

$$g_{is} \left(P^s_r \frac{DV^r}{dt} + V^m \frac{D\delta^s_m}{dt} \right) + g_{rj} V^j \frac{D\delta^r_i}{dt} - \tau^{-1} g_{is} P^s_r V^m \frac{D\delta^r_m}{dt} = \tag{3.17}$$

$$= g_{ir} \tau \left(\frac{d\tau}{dt} V^r + \frac{DV^r}{dt} \right).$$

Auf Grund der Formel (3.8) wird aus dieser Identität:

$$g_{is} P^s_r V^m \frac{D\delta^r_m}{dt} = \tau g_{rj} V^j \frac{D\delta^r_i}{dt}. \tag{3.18}$$

Offenbar muß (3.18) längs der Kurve $x^i = x^i(t)$, längs der unsere Tensoren genommen wurden, eine Identität sein. (3.17) wird somit auch eine Identität, nämlich eben die mit g_{ir} kontrahierte Formel (3.8) (abgesehen von gewissen Indizes-Veränderungen).

§ 4. Der metrische Fundamentaltensor als Eigentensor

In diesem Paragraphen werden wir die Eigenschaften von g_{ij} und $\frac{Dg_{ij}}{dt}$ untersuchen, falls der metrische Grundtensor ein Eigentensor ist, d. h. längs einer Kurve $C: x^i = x^i(t)$ der Identität

$$P^r_i P^s_j g_{rs} = \tau g_{ij} \quad (\tau \neq 0), \tag{4.1}$$

genügt. In unseren Untersuchungen werden wir aber meist nur die Symmetrie von g_{ij} in seinen Indizes benützen und die kennzeichnende Identität (1.7) bzw. (2.2a) außer Acht lassen. Wenn also nicht nachdrücklich betont wird, sind unsere Resultate auch für allgemeine symmetrische rein kovariante Tensoren zweiter Stufe gültig.

Bilden wir den invarianten Differentialquotient auf beiden Seiten von (4.1), so wird:

$$\frac{D}{dt}(P_i^r P_j^s g_{rs}) = P_i^r P_s^j g_{rs}\frac{d\tau}{dt}+\tau\frac{Dg_{ij}}{dt}.$$

Bemerkung. Die Leibnizsche Regel besteht für die Operation (1.1) im allgemeinen nicht.

Auf Grund von (1.1) und (1.2) wird unter Beachtung (auf der rechten Seite) der Formel (4.1):

$$P_i^a P_j^b\left\{\frac{dP_a^r}{dt}P_b^s g_{rs}+P_a^r\frac{dP_b^s}{dt}g_{rs}+P_a^r P_b^s\frac{dg_{rs}}{dt}-({}''\Gamma_a{}^p{}_k P_p^r P_b^s+{}''\Gamma_b{}^p{}_k P_a^r P_p^s)g_{rs}\frac{dx^k}{dt}\right\}=$$

$$=\tau\left(g_{ij}\frac{d\tau}{dt}+\frac{Dg_{ij}}{dt}\right).$$

Wir eliminieren nun die Glieder $\frac{DP_m^j}{dt}$ mittels (3.5) und beachten dann noch die Formel (3.10); im Hinblick auf (1.2) und (1.4) wird somit:

$$P_i^a P_j^b(\delta_{a|k}^p P_p^r P_b^s g_{rs}+\delta_{b|k}^p P_a^r P_p^s g_{rs}+\nabla_k g_{ab})\frac{dx^k}{dt}=\tau\left(\frac{Dg_{ij}}{dt}+g_{ij}\frac{d\tau}{dt}\right).$$

Auf Grund der Formeln (1.1) und (1.5) wird nun das folgende Lemma bestehen:

Lemma 1. *Ist für den in (i, j) symmetrischen Tensor g_{ij} die Relation (4.1) gültig, so besteht:*

$$P_i^a P_j^b\frac{Dg_{ab}}{dt}+\left(P_j^b P_b^s\frac{D\delta_i^r}{dt}+P_i^b P_b^r\frac{D\delta_j^s}{dt}\right)g_{rs}=\tau\left(\frac{Dg_{ij}}{dt}+\frac{d\tau}{dt}g_{ij}\right). \tag{4.2}$$

Wir wollen betonen, daß in der Formel (4.2) der Tensor g_{ij} nicht unbedingt der rekurrente metrische Fundamentaltensor sein muß, da bei der Herleitung von (4.2) nur (4.1), d. h. die Annahme, daß g_{ij} ein Eigentensor der Eigenfunktion τ ist, benützt wurde. Es gilt aber das

Lemma 2. *Ist g_{ij} der rekurrente metrische Fundamentaltensor, der ein Eigentensor der Eigenfunktion τ ist, so ist auch $\frac{Dg_{ij}}{dt}$ ein Eigentensor von τ, d. h. es gilt*

$$P_i^a P_j^b\frac{Dg_{ab}}{dt}=\tau\frac{Dg_{ij}}{dt}. \tag{4.3}$$

Beweis. Da für g_{ij} die Annahme (1.7) gilt, d. h. g_{ij} rekurrente kovariante Ableitung hat, bekommt man nach der Formel (1.5):

$$\frac{Dg_{ij}}{dt} \equiv \nabla_k g_{ij} \frac{dx^k}{dt} = g_{ij}\gamma_k \frac{dx^k}{dt}.$$

Beachten wir nun außer dieser Relation noch (4.1), so wird:

$$P_i^a P_j^b \frac{Dg_{ab}}{dt} = P_i^a P_j^b g_{ab}\gamma_k \frac{dx^k}{dt} = \tau g_{ij}\gamma_k \frac{dx^k}{dt} = \tau \frac{Dg_{ij}}{dt},$$

und das beweist das Lemma.

Mit Hilfe des Lemmas 1 kann der folgende Satz, die wir, mit dem späteren Satz 7 zusammen, als Hauptsätze unserer Arbeit betrachten wollen, bewiesen werden:

Satz 5. *Ist der Tensor g_{ij} ein symmetrischer Eigentensor der Eigenfunktion $\tau \not\equiv 0$, die längs einer Kurve $C: x^i = x^i(t)$ definiert ist, und gilt für P_j^i die Symmetriebedingung (1.8), so ist die Relation*

$$g_{rj}\frac{D\delta_i^r}{dt} + g_{ri}\frac{D\delta_j^r}{dt} = g_{ij}\frac{d\tau}{dt} \tag{4.4}$$

notwendig und hinreichend dafür, daß $\frac{Dg_{ij}}{dt}$ längs C auch ein Eigentensor der Eigenfunktion τ sei.

Die Eigenfunktion τ ist eine Konstante dann und nur dann, falls längs C

$$g_{rj}\frac{D\delta_i^r}{dt} + g_{ri}\frac{D\delta_j^r}{dt} = 0. \tag{4.5}$$

Beweis. Nehmen wir erstens an, daß g_{ij} ein Eigentensor ist, d. h. längs C besteht (4.1), und nach Lemma 1 gilt auch (4.2). Ist nun neben g_{ij} auch $\frac{Dg_{ij}}{dt}$ ein Eigentensor, d. h. ist auch (4.3) gültig, so reduziert sich (4.2) auf Grund von (4.3) auf

$$\left(P_j^b P_b^s \frac{D\delta_i^r}{dt} + P_i^b P_b^r \frac{D\delta_j^s}{dt}\right) g_{rs} = \tau \frac{d\tau}{dt} g_{ij}. \tag{4.6}$$

Eine Kontraktion von (4.1) mit Q_m^i gibt:

$$P_j^s g_{ms} = \tau Q_m^s g_{js}.$$

Beachten wir diese Relation zweimal auf der linken Seite von (4.6), so wird:

$$P_j^b g_{bs} Q_r^s \frac{D\delta_i^r}{dt} + P_i^b g_{br} Q_s^r \frac{D\delta_j^s}{dt} = \frac{d\tau}{dt} g_{ij}.$$

Diese Identität geht nun nach der Beachtung der Symmetrieforderung (1.8) im Hinblick auf (1.3a) unmittelbar in (4.4) über, womit die Notwendigkeit von (4.4) bewiesen ist.

Zweitens beweisen wir, daß (4.4) hinreichend ist. Eine Multiplikation von (4.4) mit τ gibt im Hinblick auf (4.1)

$$\left(P^s_r P^b_j \frac{D\delta^r_i}{dt} + P^s_r P^b_i \frac{D\delta^r_j}{dt}\right) g_{sb} = \tau g_{ij} \frac{d\tau}{dt},$$

woraus auf Grund von (1.8) unmittelbar die Relation

$$\left(P^b_j P^s_b \frac{D\delta^r_i}{dt} + P^b_i P^r_b \frac{D\delta^s_j}{dt}\right) g_{rs} = \tau \frac{d\tau}{dt} g_{ij} \tag{4.6*}$$

folgt. Das reduziert aber die aus (4.1) entstandene Identität (4.2) eben auf (4.3), d. h. $\dfrac{Dg_{ij}}{dt}$ ist ein Eigentensor. Damit ist bewiesen, daß die Bedingungsgleichung (4.4) hireichend ist.

Die letzte Behauptung des Satzes ist eine triviale Folgerung von (4.4).

Aus dem Lemma 2 folgt nach dem Satz 5 das

Korollar. *Ist der rekurrente metrische Fundamentaltensor g_{ij} ein Eigentensor längs einer Kurve C und genügt P^i_j (1.8), so ist längs C die Rleation (4.4) immer gültig.*

Aus den Formeln (4.2) und (4.5) folgt noch der

Satz 6. *Ist g_{ij} ein symmetrischer Tensor, der, längs einer Kurve C, ein Eigentensor der längs C definierten Funktion τ ist, und gilt längs C (4.5), besteht ferner für P^i_j die Symmetriebedingung (1.8), so gehören g_{ij} und $\dfrac{Dg_{ij}}{dt}$ zu demselben Eigenraum von τ.*

Beweis. Ebenso, wie vorher, im Beweis des Satzes 5, von (4.4) die Formel (4.6*) abgeleitet werden konnte, bekommen wir aus (4.5) nach einer Multiplikation mit τ, und dann unter Beachtung von (4.1) und (1.8) die Relation:

$$\left(P^b_j P^s_b \frac{D\delta^r_i}{dt} + P^b_i P^r_b \frac{D\delta^s_j}{dt}\right) g_{rs} = 0,$$

wodurch (4.2) sich auf

$$P^a_i P^b_j \frac{Dg_{ab}}{dt} = \tau \left(\frac{Dg_{ij}}{dt} + \frac{d\tau}{dt} g_{ij}\right) \tag{4.7}$$

reduziert; das beweist schon die Behauptung des Satzes.

Wir beweisen jetzt den

Satz 7. *Ist längs einer Kurve: C*

$$\frac{D\delta_i^r}{dt} = 0, \tag{4.8}$$

und ist ferner längs der Kurve: C der symmetrische Tensor g_{ij} Eigentensor der Eigenfunktion τ, so gehören längs C

$$g_{ij}, \quad \frac{Dg_{ij}}{dt}, \quad \frac{D^2 g_{ij}}{dt^2}, \ldots$$

zu demselben Eigenraum von τ.

Bemerkung. Der Satz 7 ist im wesentlichen ein Analogon eines Otsukischen Satzes (vgl. [3], Satz 5.7 auf S. 120) auf die symmetrischen kovarianten Tensoren zweiter Stufe. —

Beweis des Satzes 7. Die Behauptung des Satzes können wir in der folgenden Form ausdrücken:

$$P_i^r P_j^s \frac{D^m g_{rs}}{dt^m} = \tau \frac{D^m g_{ij}}{dt^m} + \sum_{k=1}^{m} \psi_k \frac{D^{m-k} g_{ij}}{dt^{m-k}}, \quad (m = 1, 2, \ldots), \tag{4.9}$$

wo die Funktionen ψ_k Polynome von

$$\tau, \quad \frac{d\tau}{dt}, \quad \frac{d^2\tau}{dt^2}, \ldots, \frac{d^m\tau}{dt^m}$$

sind.

Die über g_{ij} gestellte Bedingung ist die Gültigkeit von (4.1). Auf Grund des Lemmas 1 ist aber auch (4.2) gültig; diese Identität geht aber nach der Annahme (4.8) in (4.7) über. Die Identität (4.7) drückt schon aus, daß (4.9) für $m=1$ besteht.

Der Beweis werden wir nun durch vollständige Induktion durchführen. Nehmen wir also an, daß (4.9) bis irgendein $m \geqq 1$ gilt.

Nach der Bildung des invarianten Differentialquotienten beider Seiten und unter Beachtung, daß die ψ_k $(k=1, 2, \ldots, m)$ Skalare sind, bekommt man auf Grund von (1.5):

$$\frac{D}{dt}\left(P_i^r P_j^s \frac{D^m g_{rs}}{dt^m}\right) = \frac{d\tau}{dt} P_i^r P_j^s \frac{D^m g_{rs}}{dt^m} + \tau \frac{D^{m+1} g_{ij}}{dt^{m+1}} +$$
$$+ \sum_{k=1}^{m} \left(\frac{d\psi_k}{dt} P_i^r P_j^s \frac{D^{m-k} g_{rs}}{dt^{m-k}} + \psi_k \frac{D^{m+1-k} g_{ij}}{dt^{m+1-k}}\right).$$

Man sieht sofort, daß auf der rechten Seite die Glieder, die $P_i^r P_j^s$ enthalten mittels (4.9) eliminiert werden können, somit erhält man auf der rechten Seite solche Glieder von $\dfrac{D^h g_{ij}}{dt^h}$ $(h=0, 1, 2, \ldots, m+1)$, wie in (4.9). Es gilt also:

$$\frac{D}{dt}\left(P_i^r P_j^s \frac{D^m g_{rs}}{dt^m}\right) = \tau \frac{D^{m+1} g_{ij}}{dt^{m+1}} + \sum_{k=1}^{m+1} \psi_k^* \frac{D^{m+1-k} g_{ij}}{dt^{m+1-k}}, \tag{4.10}$$

wo die ψ_k^* $(k=1, \ldots, m+1)$ Polynome von

$$\tau, \quad \frac{d\tau}{dt}, \quad \frac{d^2\tau}{dt^2}, \ldots, \frac{d^{m+1}\tau}{dt^{m+1}}$$

sind.

Wir berechnen nun die linke Seite von (4.10). Es ist:

$$\frac{D}{dt}\left(P_i^r P_j^s \frac{D^m g_{rs}}{dt^m}\right) = P_i^a P_j^b \left\{\left(\frac{dP_a^r}{dt} P_b^s + P_a^r \frac{dP_b^s}{dt}\right) \frac{D^m g_{rs}}{dt^m} + \right.$$

$$\left. + P_a^r P_b^s \frac{d}{dt} \frac{D^m g_{rs}}{dt^m} - ({}''\Gamma_a{}^p{}_k P_p^r P_b^s + {}''\Gamma_b{}^p{}_k P_a^r P_p^s) \frac{dx^k}{dt} \frac{D^m g_{rs}}{dt^m}\right\}.$$

Wir eliminieren aus dieser Formel die Gleider $\dfrac{dP_e^h}{dt}$ mittels der Identität (3.5). Beachten wir noch die Bedingung (4.8), d. h.

$$P_p^r P_i^h ({}'\Gamma_h{}^p{}_k - {}''\Gamma_h{}^p{}_k) \frac{dx^k}{dt} = 0,$$

so wird:

$$\frac{D}{dt}\left(P_i^r P_j^s \frac{D^m g_{rs}}{dt^m}\right) =$$

$$= P_i^a P_j^b \left\{P_a^r P_b^s \left[\frac{d}{dt} \frac{D^m g_{rs}}{dt^m} - \left({}''\Gamma_r{}^p{}_k \frac{D^m g_{ps}}{dt^m} + {}''\Gamma_s{}^p{}_k \frac{D^m g_{rp}}{dt^m}\right) \frac{dx^k}{dt}\right]\right\} \equiv P_i^a P_j^b \frac{D^{m+1} g_{ab}}{dt^{m+1}}.$$

Die Formel (4.10) geht demnach eben in die gewünschte Formel

$$P_i^a P_j^b \frac{D^{m+1} g_{ab}}{dt^{m+1}} = \tau \frac{D^{m+1} g_{ij}}{dt^{m+1}} + \sum_{k=1}^{m+1} \psi_k^* \frac{D^{m+1-k} g_{ij}}{dt^{m+1-k}} \tag{4.11}$$

über, wo die ψ_k^* Polynome von $\tau, \dfrac{d\tau}{dt}, \ldots, \dfrac{d^{m+1}\tau}{dt^{m+1}}$ sind. (4.11) beweist aber, daß die Relation (4.9) auch für $(m+1)$ besteht, womit die vollständige Induktion beendet ist.

Endlich wollen wir den Eigentensor g_{ij} untersuchen, falls g_{ij} rekurrent ist, d. h. es besteht (1.7) bzw. die mit (1.7) äquivalente Relation:

$$\frac{Dg_{ij}}{dt} = \left(\gamma_k \frac{dx^k}{dt}\right) g_{ij}. \tag{4.12}$$

Wir beweisen nun den

Satz 8. *Gelten für den in (i, j) symmetrischen Tensor g_{ij} (4.1) und (4.12), so ist*

$$\frac{D^m g_{ij}}{dt^m} = \omega g_{ij}, \quad (m = 1, 2, \ldots), \tag{4.13}$$

wo ω eine skalare Funktion von τ, $\frac{dx^k}{dt}$, γ_k und deren Ableitungen bis höchstens m-ter Ordnung nach dem Parameter t ist.

Beweis. Für $m=1$ ist der Satz nach (4.12) gültig. Nehmen wir an, daß der Satz für irgendein $m \geqq 1$ gültig ist. Nach der Formel (4.13) hat man:

$$\frac{D^{m+1} g_{ij}}{dt^{m+1}} = P^a_i P^b_j \left\{ \frac{d}{dt} \frac{D^m g_{ab}}{dt^m} - {}''\Gamma_a{}^r{}_h \frac{D^m g_{rb}}{dt^m} \frac{dx^h}{dt} - {}''\Gamma_b{}^r{}_h \frac{D^m g_{ar}}{dt^m} \frac{dx^h}{dt} \right\} =$$

$$= \frac{d\omega}{dt} P^a_i P^b_j g_{ab} + \omega \frac{D g_{ij}}{dt}.$$

Beachten wir jetzt (4.1) und (4.12), so folgt unmittelbar

$$\frac{D^{m+1} g_{ij}}{dt^{m+1}} = \omega^* g_{ij}, \quad \omega^* \overset{\text{def}}{=} \frac{d\omega}{dt} \tau + \omega \gamma_k \frac{dx^k}{dt}.$$

Die Formel (4.13) gilt also auch für $(m+1)$, nur die skalare Funktion ω geht in ω^* über. Nach der vollständigen Induktion ist der Satz bewiesen.

Aus (4.1) und (4.13) folgt noch, daß wenn die Bedingungen von Satz 8 bestehen, dann auch

$\frac{D^m g_{ij}}{dt^m}$ *ein Eigentensor mit dem Eigenfunktion τ ist.*

Zum Schluß bemerken wir noch, daß unsere Untersuchungen — *nach unserer Vermutung* — in ähnlicher Weise auf Grund von (2.2b) auch für den kontravarianten Tensor g^{ij} durchgeführt werden könnten.

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Multiparameter strong laws of large numbers. I (Second order moment restrictions)

F. MÓRICZ

Dedicated to Professor Béla Szőkefalvi-Nagy on his 65th birthday

§ 1. Notations and a preliminary result

Let Z^d be the set of d-tuples $\mathbf{k}=(k_1, k_2, \ldots, k_d)$ with non-negative integers for coordinates, where $d \geqq 1$ is a fixed integer. If the coordinates k_j are positive integers, we write $\mathbf{k} \in Z_+^d$. Two tuples $\mathbf{k}=(k_1, k_2, \ldots, k_d)$ and $\mathbf{m}=(m_1, m_2, \ldots, m_d)$ are said to be distinct if for at least one j we have $k_j \neq m_j$. Z^d is partially ordered by agreeing that $\mathbf{k} \leqq \mathbf{m}$ iff $k_j \leqq m_j$ for each j. If $\mathbf{k} \leqq \mathbf{m}$ and $\mathbf{k} \neq \mathbf{m}$, then write $\mathbf{k}<\mathbf{m}$.

Let $\mathbf{k}+\mathbf{m}$ and $\mathbf{km}$ denote the usual coordinatewise sums and products, respectively. Let $2^{\mathbf{k}}=(2^{k_1}, 2^{k_2}, \ldots, 2^{k_d})$ and let $|\mathbf{k}|$ stand for the product $k_1 k_2 \ldots k_d$. Further, let us write $\mathbf{0}$ and $\mathbf{1}$ for the points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in Z^d, respectively.

Let $(X, \mathscr{A}, \mu)$ be a (not necessarily σ-finite) positive measure space. Let $\{\zeta_{\mathbf{k}}\}=\{\zeta_{\mathbf{k}}: \mathbf{k} \in Z_+^d\}$ be a set of measurable functions defined on $(X, \mathscr{A}, \mu)$ and having finite second moments:

$$\sigma_{\mathbf{k}}^2 = \int \zeta_{\mathbf{k}}^2 \, d\mu < \infty$$

for all $\mathbf{k} \in Z_+^d$, where for the sake of simplicity we write $\int \cdot \, d\mu$ instead of $\int_X \cdot \, d\mu$. Consider the d-multiple series

$$\sum_{\mathbf{k} \geqq \mathbf{1}} \zeta_{\mathbf{k}} = \sum_{j=1}^{d} \sum_{k_j=1}^{\infty} \zeta_{k_1, k_2, \ldots, k_d}. \text{[1]} \tag{1.1}$$

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[1]) Here $\sum_{j=1}^{d} \sum_{k_j=1}^{\infty}$ means the d-fold summation $\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \ldots \sum_{k_d=1}^{\infty}$.

For $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$ set

$$S(\mathbf{b}, \mathbf{m}) = \sum_{\mathbf{b}+\mathbf{1}\le\mathbf{k}\le\mathbf{b}+\mathbf{m}} \zeta_{\mathbf{k}} = \sum_{j=1}^{d} \sum_{k_j=b_j+1}^{b_j+m_j} \zeta_{k_1,k_2,\dots,k_d}. \text{ [2]}$$

In case $\mathbf{b}=\mathbf{0}$ the abbreviated notation $S(\mathbf{m})=S(\mathbf{0}, \mathbf{m})$ is used.

Convergence properties of the following types will be discussed:

(i) $S(\mathbf{m})$ converges a. e. as $\mathbf{m}\to\infty$, which expresses the convergence of the d-multiple series (1.1);

(ii) $S(\mathbf{m})/|\mathbf{m}|$ converges to 0 a.e. as $\mathbf{m}\to\infty$, which expresses a strong law of large numbers (SLLN) for the d-multiple sequence $\{\zeta_{\mathbf{k}}\}$.

We want to emphasize that the term "$\mathbf{m}\to\infty$" in (i) and (ii) has different meanings. By the limit $\mathbf{m}\to\infty$ in statements of type (i) we mean $\min_{1\le j\le d} m_j\to\infty$, while in statements of type (ii) we mean $\max_{1\le j\le d} m_j\to\infty$. In other words, the neighbourhood of ∞ defined by a positive number K in the first case is $\bigcap_{j=1}^{d} \{\mathbf{k}\in Z_+^d : k_j>K\}$, whereas in the second case is $\bigcup_{j=1}^{d} \{\mathbf{k}\in Z_+^d : k_j>K\}$.

As is well-known from the theory of multiple Fourier series, the notion "partial sum" is used in several ways. If there are no restrictions on the ratios m_i/m_j, then $S(\mathbf{m})$ is called *unrestricted rectangular* partial sum, while if there are positive constants C_1 and C_2 such that for each i and j we have $C_1\le m_i/m_j\le C_2$, then $S(\mathbf{m})$ is called *restricted rectangular* partial sum. If here $C_1=C_2=1$, that is if $m_1=m_2=\dots$ $\dots=m_d=m$, then $S(m, m, \dots, m)$ is called *square* partial sum. In this paper $S(\mathbf{m})$ always means unrestricted rectangular partial sum. It is obvious that the requirement of a.e. convergence for the rectangular partial sums is stronger than for the square partial sums. The same observation is true concerning a.e. convergence to 0 of $S(\mathbf{m})/|\mathbf{m}|$. Finally, the *spherical* partial sum $\tilde{S}(r)$ is defined by

$$\tilde{S}(r) = \sum_{k_1^2+k_2^2+\dots+k_d^2\le r^2} \zeta_{k_1,k_2,\dots,k_d},$$

where r is a positive integer. Clearly, the notions of rectangular, square, and spherical partial sums coincide only for $d=1$.

The asymptotic behaviour of both square and spherical partial sums will be studied in the following more general setting. Let $Q_1\subset Q_2\subset\dots$ be an arbitrary sequence of finite regions in Z_+^d such that either $\bigcup_{r=1}^{\infty} Q_r=Z_+^d$ in statements of

[2]) $\sum_{j=1}^{d} \sum_{k_j=b_j+1}^{b_j+m_j}$ also denotes a d-fold summation: $\sum_{k_1=b_1+1}^{b_1+m_1} \sum_{k_2=b_2+1}^{b_2+m_2} \dots \sum_{k_d=b_d+1}^{b_d+m_d}$.

type (i) or $\bigcup_{r=1}^{\infty} Q_r$ contains infinitely many points from Z_+^d in statements of type (ii). For $r=1,2,\ldots$ set

$$T(r) = \sum_{\mathbf{k}\in Q_r} \zeta_{\mathbf{k}}.$$

The next two particular choices of $\{Q_r\}$ provide both square and spherical partial sums.

Case 1. For each j, $1\leqq j\leqq d$, let $\{m_j(r)\}_{r=1}^{\infty}$ be a non-decreasing sequence of positive integers such that either $\min_{1\leqq j\leqq d} m_j(r)\to\infty$ in statements of type (i) or $\max_{1\leqq j\leqq d} m_j(r)\to\infty$ in statements of type (ii) as $r\to\infty$. Setting $Q_r=\{\mathbf{k}\in Z_+^d: k_j\leqq m_j(r)$ for each $j\}$ we have $T(r)=S(m_1(r), m_2(r), \ldots, m_d(r))$. In particular, if $m_j(r)=r$ for each j and r, then we get back the square partial sums.

Case 2. The choice $Q_r=\{\mathbf{k}\in Z_+^d: k_1^2+k_2^2+\ldots+k_d^2\leqq r^2\}$ provides the spherical partial sums: $T(r)=\tilde{S}(r)$.

Thus the sums $T(r)$ can be considered as generalized partial sums of the d-multiple series (1.1), although they form a set $\{T(r)\}_{r=1}^{\infty}$ depending only on one parameter.

Since Z_+^d is a partially ordered set, the main difficulties in studying convergence properties of $S(\mathbf{m})$ arise from the lack of linear ordering when $d\geqq 2$. On the other hand, Z_+^1 has a linear ordering and this explains the better convergence properties of $T(r)$.

Our results will be obtained by making use of a d-multiple maximal inequality of [2] which states bounds on the second moment of

$$M(\mathbf{b}, \mathbf{m}) = \max_{1\leqq \mathbf{k}\leqq \mathbf{m}} |S(\mathbf{b}, \mathbf{k})| = \max_{1\leqq j\leqq d}\ \max_{1\leqq k_j\leqq m_j} |S(\mathbf{b}, \mathbf{k})|\ ^{3)}$$

in terms of bounds on the second moment of $S(\mathbf{b}, \mathbf{m})$, whilst $\mathbf{b}$ and $\mathbf{m}$ run over Z^d and Z_+^d, respectively.

We obviously have

$$\int S^2(\mathbf{b}, \mathbf{m})\, d\mu \leqq \sum_{\mathbf{b}+1\leqq \mathbf{k},\mathbf{l}\leqq \mathbf{b}+\mathbf{m}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\right| \equiv f(\mathbf{b}, \mathbf{m})\ .^{4)}$$

[3]) Here $\max_{1\leqq j\leqq d}\ \max_{1\leqq k_j\leqq m_j}$ indicates that the maximum has to be taken for all possible integers $k_1, k_2, \ldots, k_d$ such that $1\leqq k_1\leqq d_1$, $1\leqq k_2\leqq d_2, \ldots$, and $1\leqq k_d\leqq m_d$.

[4]) $\sum_{\mathbf{b}+1\leqq \mathbf{k},\mathbf{l}\leqq \mathbf{b}+\mathbf{m}}$ abbreviates the following $2d$-fold summation:

$$\sum_{k_1=b_1+1}^{b_1+m_1}\ \sum_{l_1=b_1+1}^{b_1+m_1} \cdots \sum_{k_d=b_d+1}^{b_d+m_d}\ \sum_{l_d=b_d+1}^{b_d+m_d}.$$

The following lemma is the special case of [2, Theorem 8] when $\gamma=2$ and $\lambda_j(m_j)=1$, consequently $\Lambda_j(m_j)=\log 2m_j$ for each j. In this paper all logarithms are of base 2.

Lemma 1 (the Rademacher—Menšov inequality). *For all* $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$ *we have*

$$\int M^2(\mathbf{b},\mathbf{m})\,d\mu \leqq f(\mathbf{b},\mathbf{m})\prod_{j=1}^{d}(\log 2m_j)^2. \tag{1.2}$$

For the convenience of using "dyadic blocks" $S(2^{\mathbf{p}}, 2^{\mathbf{p}})$, $\mathbf{p}\in Z^d$, to represent the partial sums $S(\mathbf{m})$ during the proofs below, we may assume that $\zeta_{\mathbf{k}}\equiv 0$ if for at least one j we have $k_j=1$. It is clear that this assumption is of technical character and does not affect generality.

§ 2. A.e. convergence of the rectangular partial sums

On the basis of (1.2) we prove the following

Theorem 1 (the non-orthogonal Rademacher—Menšov theorem). *If*

$$\sum_{\mathbf{m}\geqq 0}|\mathbf{m}+\mathbf{1}|^2 \sum_{2^{\mathbf{m}}+1\leqq \mathbf{k},\,\mathbf{l}\leqq 2^{\mathbf{m}+1}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\,d\mu\right| < \infty, \tag{2.1}$$

then (1.1) *converges a.e. in the sense that* $S(\mathbf{m})$ *converges a.e. as* $\mathbf{m}\to\infty$.

If the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal, i.e., if for all distinct pairs $\mathbf{k}$ and $\mathbf{l}$ we have

$$\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\,d\mu = 0,$$

then the general term of (2.1) may be simplified as follows

$$|\mathbf{m}+\mathbf{1}|^2 f(2^{\mathbf{m}}, 2^{\mathbf{m}}) \leqq \sum_{2^{\mathbf{m}}+1\leqq \mathbf{k}\leqq 2^{\mathbf{m}+1}} \sigma_{\mathbf{k}}^2 \prod_{j=1}^{d}(\log 2k_j)^2.$$

Hence Theorem 1 yields

Corollary 1 (the Rademacher—Menšov theorem). *If the functions* $\zeta_{\mathbf{k}}$ *are mutually orthogonal and if*

$$\sum_{\mathbf{k}\geqq 1}\sigma_{\mathbf{k}}^2\prod_{j=1}^{d}(\log 2k_j)^2 < \infty, \tag{2.2}$$

then (1.1) *converges a.e.*

Condition (2.2) is satisfied if, for example,

$$\sigma_{\mathbf{k}}^2 = O\left\{\prod_{j=1}^{d} k_j^{-1}(\log 2k_j)^{-3}(\log\log 4k_j)^{-1-\varepsilon}\right\}$$

or

$$\sigma_{\mathbf{k}}^2 = O\{|\mathbf{k}|^{-1}(\log 2|\mathbf{k}|)^{-3d}(\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\} \tag{2.3}$$

with an $\varepsilon>0$. The fulfilment of (2.2) in the second case can be verified by repeated use of the estimation

$$\sum_{j=1}^{\infty} j^{-1}(\log 2aj)^{-i}(\log\log 4aj)^{-1-\varepsilon} = O\{(\log 2a)^{-i+1}(\log\log 4a)^{-1-\varepsilon}\},$$

where $a\geqq 1$ and $i\geqq 2$ are integers, and $\varepsilon>0$.

We remark that Theorem 1 for $d=1$ was essentially proved by SZÉP [6] (although it is stated there in a slightly weaker form), while Corollary 1 for $d=2$ was proved by AGNEW [1] (see also PANDZAKIDZE [3], where the proof of Step 2 is not complete).

Proof of Theorem 1. By the above remark it is enough to treat the case $d\geqq 2$.

Step 1. We begin with proving that $S(2^{\mathbf{p}})$ converges a.e. as $\mathbf{p}\to\infty$. By the Cauchy convergence criterion it is sufficient to show that

$$S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}}) \quad \text{tends to 0 a.e.} \quad \text{as} \quad \mathbf{p}\to\infty \quad \text{and} \quad \mathbf{q}>\mathbf{0}. \tag{2.4}$$

To this end let us represent the difference in (2.4) as follows

$$S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}}) = \Big\{\sum_{\mathbf{1}\leqq\mathbf{k}\leqq 2^{\mathbf{p}+\mathbf{q}}} - \sum_{\mathbf{1}\leqq\mathbf{k}\leqq 2^{\mathbf{p}}}\Big\}\zeta_{\mathbf{k}} = \Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\} S(2^{\mathbf{m}}, 2^{\mathbf{m}}),$$

where $\mathbf{p}\geqq\mathbf{1}$ and $\mathbf{q}>\mathbf{0}$. Applying the Cauchy inequality hence we get that

$$\big(S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}})\big)^2 \leqq \Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}|\mathbf{m}+\mathbf{1}|^2 S^2(2^{\mathbf{m}}, 2^{\mathbf{m}})\times$$

$$\times\Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}\frac{1}{|\mathbf{m}+\mathbf{1}|^2}.$$

Taking into account that the second factor on the right is uniformly bounded for all $\mathbf{p}\geqq\mathbf{1}$ and $\mathbf{q}>\mathbf{0}$,

$$\big(S(2^{\mathbf{p}+\mathbf{q}})-S(2^{\mathbf{p}})\big)^2 = O(1)\Big\{\sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}+\mathbf{q}-\mathbf{1}} - \sum_{\mathbf{0}\leqq\mathbf{m}\leqq\mathbf{p}-\mathbf{1}}\Big\}|\mathbf{m}+\mathbf{1}|^2 S^2(2^{\mathbf{m}}, 2^{\mathbf{m}}). \tag{2.5}$$

Since by (2.1)

$$\sum_{\mathbf{m}\geqq\mathbf{0}}|\mathbf{m}+\mathbf{1}|^2\int S^2(2^{\mathbf{m}}, 2^{\mathbf{m}})\,d\mu \leqq \sum_{\mathbf{m}\geqq\mathbf{0}}|\mathbf{m}+\mathbf{1}|^2 f(2^{\mathbf{m}}, 2^{\mathbf{m}}) < \infty,$$

the B. Levi theorem implies the a.e. convergence of the d-multiple series

$$\sum_{\mathbf{m}\geqq\mathbf{0}}|\mathbf{m}+\mathbf{1}|^2 S^2(2^{\mathbf{m}}, 2^{\mathbf{m}}).$$

Consequently, the right-hand side of (2.5) can be made as small as needed by choosing $\min_{1\leqq j\leqq d} p_j$ large enough. This proves (2.4).

Step 2. It has remained to prove that the maximal deviation

$$\max_{1 \leq \mathbf{m} \leq 2^{\mathbf{p}}} |S(2^{\mathbf{p}}+\mathbf{m})-S(2^{\mathbf{p}})| \quad \text{tends to 0 a.e. as} \quad \mathbf{p} \to \infty. \tag{2.6}$$

Let $\mathbf{p} \geq 0$ and $1 \leq \mathbf{m} \leq 2^{\mathbf{p}}$ be fixed. It is not hard to check that

$$S(2^{\mathbf{p}}+\mathbf{m})-S(2^{\mathbf{p}}) = \sum_{\boldsymbol{\varepsilon}} S(\boldsymbol{\varepsilon}2^{\mathbf{p}}, \boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{p}}),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is extended over all possible 2^d-1 choices of $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$, $\varepsilon_j=0$ or 1, the case $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_d=0$ excluded. From this representation it follows immediately that

$$\max_{1 \leq \mathbf{m} \leq 2^{\mathbf{p}}} |S(2^{\mathbf{p}}+\mathbf{m})-S(2^{\mathbf{p}})| \leq \tag{2.7}$$

$$\leq \sum_{\boldsymbol{\varepsilon}} \max_{\substack{1 \leq j \leq d \\ \varepsilon_j=1}} \max_{1 \leq m_j \leq 2^{p_j}} |S(\boldsymbol{\varepsilon}2^{\mathbf{p}}, \boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{p}})| \equiv \sum_{\boldsymbol{\varepsilon}} M_{\boldsymbol{\varepsilon}}(\mathbf{p}),$$

i.e. $M_{\boldsymbol{\varepsilon}}(\mathbf{p})$ is the maximum of all $|S(\boldsymbol{\varepsilon}2^{\mathbf{p}}, \boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{p}})|$, where those coordinates m_j run between 1 and 2^{p_j} whose subscript j is such that $\varepsilon_j=1$ in $\boldsymbol{\varepsilon}$.

Let us fix an $\boldsymbol{\varepsilon}$. If for each j we have $\varepsilon_j=1$, then the corresponding maximum on the right of (2.7) is

$$M_{\mathbf{1}}(\mathbf{p}) = \max_{1 \leq j \leq d} \max_{1 \leq m_j \leq 2^{p_j}} |S(2^{\mathbf{p}}, \mathbf{m})| = M(2^{\mathbf{p}}, 2^{\mathbf{p}}).$$

In virtue of Lemma 1 we have

$$\int M^2(2^{\mathbf{p}}, 2^{\mathbf{p}})\, d\mu \leq |\mathbf{p}+\mathbf{1}|^2 f(2^{\mathbf{p}}, 2^{\mathbf{p}}).$$

By (2.1) hence

$$\sum_{\mathbf{p} \geq 0} \int M^2(2^{\mathbf{p}}, 2^{\mathbf{p}})\, d\mu < \infty,$$

which implies via the B. Levi theorem that $M(2^{\mathbf{p}}, 2^{\mathbf{p}})$ tends to 0 a.e. as $\mathbf{p} \to \infty$.

Now consider an $\boldsymbol{\varepsilon}$ such that for at least one j we have $\varepsilon_j=0$. For the sake of simplicity we assume that $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_e=1$ and $\varepsilon_{e+1}=\ldots=\varepsilon_d=0$, where $1 \leq e < d$. Then for the corresponding maximum $M_{\boldsymbol{\varepsilon}}(\mathbf{p})$ we have

$$M_{\boldsymbol{\varepsilon}}(\mathbf{p}) = \max_{1 \leq j \leq e} \max_{1 \leq m_j \leq 2^{p_j}} |S(\boldsymbol{\varepsilon}2^{\mathbf{p}}, \boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{p}})| \leq$$

$$\leq \sum_{i=e+1}^{d} \sum_{n_i=0}^{p_i-1} \Big(\max_{1 \leq j \leq e} \max_{1 \leq m_j \leq 2^{p_j}} |S(\boldsymbol{\varepsilon}2^{\mathbf{p}}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{n}}, \boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{n}})|\Big), \text{[5]}$$

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[5]) We remind that $\max_{1 \leq j \leq e} \max_{1 \leq m_j \leq 2^{p_j}}$ abbreviates $\max_{1 \leq m_1 \leq 2^{p_1}} \max_{1 \leq m_2 \leq 2^{p_2}} \ldots \max_{1 \leq m_e \leq 2^{p_e}}$, and $\sum_{i=e+1}^{d} \sum_{n_i=0}^{p_i-1}$ abbreviates $\sum_{n_{e+1}=0}^{p_{e+1}-1} \ldots \sum_{n_d=0}^{p_d-1}$.

where $\mathbf{n}=(n_1, n_2, \ldots, n_d)$, although the first e coordinates $n_1, n_2, \ldots, n_e$ of $\mathbf{n}$ play no role on the right-hand side of the last inequality. By the Cauchy inequality,

$$M_\varepsilon^2(\mathbf{p}) = O(1) \sum_{i=e+1}^{d} \sum_{n_i=0}^{p_i-1} \Big\{ \prod_{i=e+1}^{d} (n_i+1)^2 \times \tag{2.8}$$
$$\times \max_{1\le j\le e} \max_{1\le m_j\le 2^{p_j}} S^2(\varepsilon 2^{\mathbf{p}}+(1-\varepsilon)2^{\mathbf{n}}, \varepsilon\mathbf{m}+(1-\varepsilon)2^{\mathbf{n}})\Big\}.$$

We have to apply the e-parameter version of Lemma 1 for all sets

$$\Big\{\xi_{k_1,\ldots,k_e} = \sum_{i=e+1}^{d} \sum_{k_i=2^{n_i}+1}^{2^{n_i+1}} \zeta_{k_1,\ldots,k_e,k_{e+1},\ldots,k_d} : 2^{p_j}+1 \le k_j \le 2^{p_j+1};\quad j=1,2,\ldots,e\Big\},$$

where n_i may take on the values $0, 1, \ldots, p_i-1$ for $i=e+1, \ldots, d$. By virtue of (1.2) we come to the inequality

$$\int \{\max_{1\le j\le e} \max_{1\le m_j\le 2^{p_j}} S^2(\varepsilon 2^{\mathbf{p}}+(1-\varepsilon)2^{\mathbf{n}}, \varepsilon\mathbf{m}+(1-\varepsilon)2^{\mathbf{n}})\}\, d\mu \le \tag{2.9}$$
$$\le \prod_{j=1}^{e} (p_j+1)^2 \sum_{j=1}^{e} \sum_{2^{p_j}+1\le k_j, l_j\le 2^{p_j+1}} \Big|\int \xi_{k_1,\ldots,k_e} \xi_{l_1,\ldots,l_e}\, d\mu\Big| \le$$
$$\le \prod_{j=1}^{e} (p_j+1)^2 \sum_{2^{\mathbf{q}}+1\le \mathbf{k},\mathbf{l}\le 2^{\mathbf{q}+1}} \Big|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\Big|$$

with $\mathbf{q}=\varepsilon\mathbf{p}+(1-\varepsilon)\mathbf{n}$. [6])

Combining inequalities (2.8) and (2.9), we obtain that

$$\int \{\max_{e+1\le i\le d} \sup_{p_i\ge 0} M_\varepsilon^2(\mathbf{p})\}\, d\mu = O(1) \sum_{i=e+1}^{d} \sum_{n_i=0}^{\infty} |\mathbf{q}+\mathbf{1}|^2 f(2^{\mathbf{q}}, 2^{\mathbf{q}}).\ ^{7)}$$

By (2.1) we can establish that

$$\sum_{j=1}^{e} \sum_{p_j=0}^{\infty} \int \{\max_{e+1\le i\le d} \sup_{p_i\ge 0} M_\varepsilon^2(\mathbf{p})\}\, d\mu = O(1) \sum_{\mathbf{q}\ge \mathbf{0}} |\mathbf{q}+\mathbf{1}|^2 f(2^{\mathbf{q}}, 2^{\mathbf{q}}) < \infty,$$

whence via the B. Levi theorem it follows that $M_\varepsilon(\mathbf{p})$ tends to 0 a.e. as $\mathbf{p}\to\infty$. Since this is true for each $M_\varepsilon(\mathbf{p})$ on the right-hand side of (2.7), statement (2.6) holds true.

To put (2.4) and (2.6) together, we can conclude the assertion of Theorem 1.

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[6]) We remind that $\sum_{j=1}^{e} \sum_{2^{p_j}+1\le k_j, l_j\le 2^{p_j+1}}$ abbreviates $\sum_{k_1=2^{p_1}+1}^{2^{p_1+1}} \sum_{l_1=2^{p_1}+1}^{2^{p_1+1}} \ldots \sum_{k_e=2^{p_e}+1}^{2^{p_e+1}} \sum_{l_e=2^{p_e}+1}^{2^{p_e+1}}$ (a $2e$-fold summation).

[7]) $\max_{e+1\le i\le d} \sup_{p_i\ge 0} M_\varepsilon^2(\mathbf{p})$ is understood as the supremum of all $M_\varepsilon^2(\mathbf{p})$, when the last $d-e$ coordinates $p_{e+1}, \ldots, p_d$ of $\mathbf{p}\in Z^d$ run, independently of each other, over the non-negative integers.

§ 3. A.e. convergence of the square and the spherical partial sums

Let $Q_1 \subset Q_2 \subset \dots$ be an arbitrary sequence of finite regions in Z_+^d such that $\bigcup_{r=1}^{\infty} Q_r = Z_+^d$, and let $Q_0 = \emptyset$. Set

$$T(r) = \sum_{\mathbf{k} \in Q_r} \zeta_{\mathbf{k}} \quad (r = 1, 2, \dots).$$

The one-parameter versions of Theorem 1 and Corollary 1 read as follows.

Theorem 2. *If*

$$\sum_{t=0}^{\infty} (t+1)^2 \sum_{\mathbf{k}, \mathbf{l} \in Q_{2^{t+1}-1} \setminus Q_{2^t-1}} \left| \int \zeta_{\mathbf{k}} \zeta_{\mathbf{l}} \, d\mu \right| < \infty,$$

then $T(r)$ *converges a.e. as* $r \to \infty$.

Corollary 2. *If the functions* $\zeta_{\mathbf{k}}$ *are mutually orthogonal and if*

$$\sum_{r=1}^{\infty} \Big(\sum_{\mathbf{k} \in Q_r \setminus Q_{r-1}} \sigma_{\mathbf{k}}^2 \Big) \log^2 2r < \infty, \tag{3.1}$$

then $T(r)$ *converges a.e. as* $r \to \infty$.

By setting $\xi_r = \sum_{\mathbf{k} \in Q_r \setminus Q_{r-1}} \zeta_{\mathbf{k}}$ for $r = 1, 2, \dots$, Theorem 2 follows from Theorem 1 in the case $d=1$, while Corollary 2 is a consequence of Theorem 2.

It is worth going into details in connection with the square partial sums, i.e., when $Q_r = \{\mathbf{k} \in Z_+^d : k_j \leqq r \text{ for each } j\}$. Then $\mathbf{k} \in Q_r \setminus Q_{r-1}$ iff $\max(k_1, k_2, \dots, k_d) = r$, further, $|Q_r \setminus Q_{r-1}| = O(r^{d-1})$. Here $|Q|$ denotes the number of the points of Z_+^d contained in Q. Condition (3.1) is satisfied if, e.g., for $\mathbf{k} \in Q_r \setminus Q_{r-1}$ we have

$$\sigma_{\mathbf{k}}^2 = O\{r^{-d} (\log 2r)^{-3} (\log\log 4r)^{-1-\varepsilon}\}$$

or

$$\begin{aligned} \sigma_{\mathbf{k}}^2 &= O\{|\mathbf{k}|^{-1} (\log 2r)^{-d-2} (\log\log 4r)^{-1-\varepsilon}\} = \\ &= O\{|\mathbf{k}|^{-1} (\log 2|\mathbf{k}|)^{-d-2} (\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\} \end{aligned} \tag{3.2}$$

with an $\varepsilon > 0$. The first relation in (3.2) ensures the fulfilment of (3.1) since

$$\sum_{\mathbf{k} \in Q_r \setminus Q_{r-1}} |\mathbf{k}|^{-1} = O\{r^{-1} (\log 2r)^{d-1}\}.$$

The second relation in (3.2) follows from

$$r = \max(k_1, k_2, \dots, k_d) \leqq |\mathbf{k}| \leqq r^d.$$

Condition (3.2) is clearly weaker than (2.3) for $d \geqq 2$.

We note that in the more general situation when e coordinates of $\mathbf{m} \in Z_+^d$ depend on a parameter r, while the other $d-e$ coordinates vary independently of each other where $1 \leqq e < d$, then the following result can be achieved. For the sake of simplicity we consider only the case when the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal. Let $\{m_j(r)\}_{r=1}^{\infty}$ be non-decreasing sequences of positive integers such that $m_j(1)=1$ and $m_j(r)\to\infty$ as $r\to\infty$ for each $j=1, 2, \ldots, e$. If

$$\sum_{\mathbf{k}\geqq 1} \sigma_{\mathbf{k}}^2 \lambda^2(k_1, k_2, \ldots, k_e) \prod_{i=e+1}^{d} (\log 2k_i)^2 < \infty,$$

where $\lambda(k_1, k_2, \ldots, k_e) = \log 2r$ if $m_j(r) \leqq k_j < m_j(r+1)$ for each $j=1, 2, \ldots, e$, then $S(m_1(r), \ldots, m_e(r), m_{e+1}, \ldots, m_d)$ converges a.e. as $r\to\infty$ and $m_i\to\infty$ for each $i=e+1, \ldots, d$.

§ 4. A d-parameter version of the SLLN

Application of the results of § 2 to the series $\sum_{\mathbf{k}\geqq 1} \zeta_{\mathbf{k}}/|\mathbf{k}|$ yields, via the d-parameter version of the Kronecker lemma (for $d=1$ see, e.g., [5, p. 35]), criteria for the a.e. convergence to 0 of $S(\mathbf{m})/|\mathbf{m}|$ as $\mathbf{m}\to\infty$. However, as we emphasized in § 1, the limit $\mathbf{m}\to\infty$ is used in different senses according as the a.e. convergence of a d-multiple series $(\min_{1\leqq j\leqq d} m_j\to\infty)$ or the a.e. convergence to 0 of $S(\mathbf{m})/|\mathbf{m}|$ $(\max_{1\leqq j\leqq d} m_j\to\infty)$ is studied. Since the convergence notion $\max_{1\leqq j\leqq d} m_j\to\infty$ induces a finer topology than the notion $\min_{1\leqq j\leqq d} m_j\to\infty$, the application of a generalized form of the widely used Kronecker lemma is not appropriate at present. Thus we follow another way to obtain the following SLLN.

Theorem 3. *If*

$$\sum_{\mathbf{m}\geqq 0} \frac{|\mathbf{m}+\mathbf{1}|^2}{|2^{\mathbf{m}}|^2} \sum_{2^{\mathbf{m}}+1\leqq \mathbf{k},\mathbf{l}\leqq 2^{\mathbf{m}+1}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\right| < \infty, \tag{4.1}$$

then

$$\lim_{\mathbf{m}\to\infty} S(\mathbf{m})/|\mathbf{m}| = 0 \quad a.e. \tag{4.2}$$

Corollary 3 (SLLN for orthogonal functions). *If the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal and if*

$$\sum_{\mathbf{k}\geqq 1} \frac{\sigma_{\mathbf{k}}^2}{|\mathbf{k}|^2} \prod_{j=1}^{d} (\log 2k_j)^2 < \infty, \tag{4.3}$$

then (4.2) *follows.*

Condition (4.3) is satisfied if, for example, we have

$$\sigma_{\mathbf{k}}^2 = O\left\{\prod_{j=1}^{d} k_j (\log 2k_j)^{-3} (\log\log 4k_j)^{-1-\varepsilon}\right\}$$

or

$$\sigma_{\mathbf{k}}^2 = O\{|\mathbf{k}| (\log 2|\mathbf{k}|)^{-3d} (\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\} \tag{4.4}$$

with an $\varepsilon>0$. We mention that Corollary 3 for $d=1$ was established by TANDORI [7] (see also PETROV [4]).

To prove Theorem 3 we begin with a generalization of the so-called Toeplitz lemma (for $d=1$ see, e.g., [5, p. 36]).

Lemma 2. *Let $\{w(\mathbf{m},\mathbf{k})\colon \mathbf{m},\mathbf{k}\in Z_+^d\}$ be a set of non-negative numbers with the following two properties:*

$$\sum_{\mathbf{k}\geqq 1} w(\mathbf{m},\mathbf{k}) \leqq C \tag{4.5}$$

for all $\mathbf{m}\in Z_+^d$ with a constant C, and

$$\lim_{\mathbf{m}\to\infty} w(\mathbf{m},\mathbf{k}) = 0 \tag{4.6}$$

for all $\mathbf{k}\in Z_+^d$. If $\{s(\mathbf{k})\colon \mathbf{k}\in Z_+^d\}$ is a d-multiple sequence of real numbers such that

$$s(\mathbf{k})\to 0 \quad as \quad \mathbf{k}\to\infty, \tag{4.7}$$

then

$$t(\mathbf{m}) = \sum_{\mathbf{k}\geqq 1} w(\mathbf{m},\mathbf{k})s(\mathbf{k})\to 0 \quad as \quad \mathbf{m}\to\infty. \tag{4.8}$$

Proof of Lemma 2. By (4.7) for any $\varepsilon>0$ there exists a $\mathbf{k}_0\in Z_+^d$ such that

$$|s(\mathbf{k})| \leqq \varepsilon \quad \text{if} \quad \mathbf{k} \nleqq \mathbf{k}_0. \tag{4.9}$$

Consider the decomposition of $t(\mathbf{m})$ into two summands:

$$t(\mathbf{m}) = \Big\{\sum_{1\leqq\mathbf{k}\leqq\mathbf{k}_0} + \sum_{\mathbf{k}\nleqq\mathbf{k}_0}\Big\} w(\mathbf{m},\mathbf{k})s(\mathbf{k}) \equiv t_1+t_2.$$

On account of (4.5) and (4.9), for all $\mathbf{m}\in Z_+^d$ we have that $|t_2|\leqq C\varepsilon$. By (4.6) we can choose an $\mathbf{m}_0\in Z_+^d$ such that for each $\mathbf{k}\in Z_+^d$ with $1\leqq\mathbf{k}\leqq\mathbf{k}_0$ we have

$$|w(\mathbf{m},\mathbf{k})| \leqq \varepsilon\Big/\sum_{1\leqq\mathbf{k}\leqq\mathbf{k}_0}|s(\mathbf{k})| \quad \text{if} \quad \mathbf{m}\nleqq\mathbf{m}_0.$$

Hence $|t_1|\leqq\varepsilon$.

Collecting the above reasonings we conclude that

$$|t(\mathbf{m})| \leqq (C+1)\varepsilon \quad \text{if} \quad \mathbf{m}\nleqq\mathbf{m}_0.$$

This is the wanted (4.8).

Lemma 2 just proved makes it possible to show the following simple assertion. Let $\{u_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ be a d-multiple sequence of numbers. Put

$$s(\mathbf{b},\mathbf{m}) = \sum_{\mathbf{b}+1\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} u_{\mathbf{k}} \quad \text{and} \quad s(\mathbf{m}) = s(\mathbf{0},\mathbf{m}),$$

where $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$.

Lemma 3. *The statements*

$$\lim_{\mathbf{m}\to\infty} s(2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \tag{4.10}$$

and

$$\lim_{\mathbf{m}\to\infty} s(2^{\mathbf{m}}, 2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \tag{4.11}$$

are equivalent.

Proof of Lemma 3. From the well-known representation

$$s(\mathbf{b}, \mathbf{m}) = \sum_{\boldsymbol{\varepsilon}} (-1)^{\sum_{j=1}^{d} \varepsilon_j} s(\mathbf{b}+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{m}),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is taken for all 2^d choices of $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$ with components $\varepsilon_j=0$ or 1, the implication (4.10)$\Rightarrow$(4.11) immediately follows.

To facilitate the use of dyadic blocks $s(2^{\mathbf{k}}, 2^{\mathbf{k}})$, we assume that $u_{\mathbf{k}}=0$ if fo at least one j we have $k_j=1$. Then

$$\frac{s(2^{\mathbf{m}})}{|2^{\mathbf{m}}|} = \sum_{\mathbf{1}\le\mathbf{k}\le\mathbf{m}-\mathbf{1}} w(\mathbf{m}, \mathbf{k}) \frac{s(2^{\mathbf{k}}, 2^{\mathbf{k}})}{|2^{\mathbf{k}}|}$$

with $w(\mathbf{m}, \mathbf{k})=|2^{\mathbf{k}}|/|2^{\mathbf{m}}|$ for $\mathbf{1}\le\mathbf{k}\le\mathbf{m}-\mathbf{1}$ and $w(\mathbf{m}, \mathbf{k})=0$ otherwise. The assumptions of Lemma 2 are clearly satisfied, the application of which gives the implication (4.11)$\Rightarrow$(4.10). This completes the proof.

Proof of Theorem 3. *Step* 1. First we prove that

$$\lim_{\mathbf{m}\to\infty} S(2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \quad \text{a.e.} \tag{4.12}$$

By Lemma 3 it suffices to show that

$$\lim_{\mathbf{m}\to\infty} S(2^{\mathbf{m}}, 2^{\mathbf{m}})/|2^{\mathbf{m}}| = 0 \quad \text{a.e.} \tag{4.13}$$

For convenience we again assume that $\zeta_{\mathbf{k}}\equiv 0$ if $k_j=1$ for at least one j. Since by (4.1)

$$\sum_{\mathbf{m}\ge\mathbf{0}} |2^{\mathbf{m}}|^{-2} \int S^2(2^{\mathbf{m}}, 2^{\mathbf{m}})\, d\mu \le \sum_{\mathbf{m}\ge\mathbf{0}} |2^{\mathbf{m}}|^{-2} f(2^{\mathbf{m}}, 2^{\mathbf{m}}) < \infty,$$

where, as before,

$$f(2^{\mathbf{m}}, 2^{\mathbf{m}}) \equiv \sum_{2^{\mathbf{m}}+\mathbf{1}\le\mathbf{k},\mathbf{l}\le 2^{\mathbf{m}+\mathbf{1}}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\right|,$$

the B. Levi theorem implies (4.13), and consequently (4.12).

Step 2. Now we turn to the proof of the relation

$$\lim_{\mathbf{m}\to\infty} |2^{\mathbf{m}}|^{-1} \max_{\mathbf{1}\le\mathbf{p}\le 2^{\mathbf{m}}} |S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}})| = 0 \quad \text{a.e.} \tag{4.14}$$

As in the proof of Theorem 1, we start with the representation

$$S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}}) = \sum_{\boldsymbol{\varepsilon}} S(\boldsymbol{\varepsilon}2^{\mathbf{m}}, \boldsymbol{\varepsilon}\mathbf{p}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{m}}),$$

where the summation $\sum_{\boldsymbol{\varepsilon}}$ is extended for all $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)$ such that the coordinates ε_j assume the values 0 and 1 independently of each other, excluding the case $\varepsilon_1=\varepsilon_2=\dots=\varepsilon_d=0$. Thus

$$\max_{\mathbf{1}\le\mathbf{p}\le 2^{\mathbf{m}}} |S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}})| \le \tag{4.15}$$

$$\le \sum_{\boldsymbol{\varepsilon}} \sum_{\substack{j=1\\ \varepsilon_j=0}}^{d} \sum_{q_j=0}^{m_j-1} M(\boldsymbol{\varepsilon}2^{\mathbf{m}}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{q}}, \boldsymbol{\varepsilon}2^{\mathbf{m}}+(\mathbf{1}-\boldsymbol{\varepsilon})2^{\mathbf{q}}),$$

where $\sum_{\substack{j=1\\ \varepsilon_j=0}}^{d} \sum_{q_j=0}^{m_j-1}$ has the following meaning. For given $\boldsymbol{\varepsilon}$, let $\varepsilon_j=0$ iff $j=j_1, j_2, \dots, j_e$, where $1\le j_1<j_2<\dots<j_e\le d$. Then we have to form the e-fold summation $\sum_{q_{j_1}=0}^{m_{j_1}-1} \dots \sum_{q_{j_e}=0}^{m_{j_e}-1}$.

By virtue of Lemma 1 and (4.1) we have

$$\sum_{\mathbf{k}\ge 0} |2^{\mathbf{k}}|^{-2} \int M^2(2^{\mathbf{k}}, 2^{\mathbf{k}})\, d\mu \le \sum_{\mathbf{k}\ge 0} \frac{|\mathbf{k}+\mathbf{1}|^2}{|2^{\mathbf{k}}|^2} f(2^{\mathbf{k}}, 2^{\mathbf{k}}) < \infty.$$

Hence the B. Levi theorem implies the a.e. convergence to 0 of $M(2^{\mathbf{k}}, 2^{\mathbf{k}})/|2^{\mathbf{k}}|$ as $\mathbf{k}\to\infty$.

Rewriting (4.15) into the form

$$|2^{\mathbf{m}}|^{-1} \max_{\mathbf{1}\le\mathbf{p}\le 2^{\mathbf{m}}} |S(2^{\mathbf{m}}+\mathbf{p})-S(2^{\mathbf{m}})| \le \sum_{\boldsymbol{\varepsilon}} \sum_{\substack{j=1\\ \varepsilon_j=0}}^{d} \sum_{q_j=0}^{m_j-1} w(\mathbf{m}, \mathbf{k}) \frac{M(2^{\mathbf{k}}, 2^{\mathbf{k}})}{|2^{\mathbf{k}}|}$$

with $\mathbf{k}=\boldsymbol{\varepsilon}\mathbf{m}+(\mathbf{1}-\boldsymbol{\varepsilon})\mathbf{q}$ and $w(\mathbf{m}, \mathbf{k})=|2^{\mathbf{k}}|/|2^{\mathbf{m}}|$ if for at least one j we have $k_j=m_j$ and $w(\mathbf{m}, \mathbf{k})=0$ otherwise, it is enough to apply Lemma 2 in order to get (4.14). This completes the proof of Theorem 3.

§ 5. A one-parameter version of the SLLN

Let $Q_1\subset Q_2\subset\dots$ be an arbitrary sequence of finite regions in Z_+^d such that $\bigcup_{r=1}^{\infty} Q_r$ contains infinitely many points of Z_+^d, and let $Q_0=\emptyset$.

Theorem 4. *If*

$$\sum_{t=0}^{\infty} \frac{(t+1)^2}{|Q_{2^t}|^2} \sum_{\mathbf{k},\mathbf{l}\in Q_{2^{t+1}-1}\setminus Q_{2^t-1}} \left|\int \zeta_{\mathbf{k}}\zeta_{\mathbf{l}}\, d\mu\right| < \infty, \tag{5.1}$$

then

$$|Q_r|^{-1} \sum_{\mathbf{k}\in Q_r} \zeta_{\mathbf{k}} \to 0 \quad a.e.\ as \quad r \to \infty. \tag{5.2}$$

Corollary 4. *If the functions $\zeta_{\mathbf{k}}$ are mutually orthogonal and if*

$$\sum_{r=1}^{\infty} \Big(\sum_{\mathbf{k}\in Q_r \setminus Q_{r-1}} \sigma_{\mathbf{k}}^2 \Big) \frac{\log^2 2r}{|Q_r|^2} < \infty, \tag{5.3}$$

then (5.2) *follows.*

In fact, set $\xi_t = \sum_{\mathbf{k}\in Q_t \setminus Q_{t-1}} \zeta_{\mathbf{k}}$ for $t=1, 2, \dots$. Condition (5.1) ensures, owing to Theorem 2 in the case $d=1$, that the series $\sum_{t=1}^{\infty} \xi_t/|Q_t|$ converges a.e. Hence (the usual one-parameter form of) the Kronecker lemma yields

$$|Q_r|^{-1} \sum_{t=1}^{r} \xi_t = |Q_r|^{-1} \sum_{\mathbf{k}\in Q_r} \zeta_{\mathbf{k}} \to 0 \quad \text{a.e. as} \quad r \to \infty,$$

as asserted in (5.2).

If $\{Q_r\}$ is chosen as in Case 1 of § 1, then we obtain criteria for the a.e. convergence to 0 of $S(m_1(r), m_2(r), \dots, m_d(r))/\prod_{j=1}^{d} m_j(r)$, while in Case 2 we obtain criteria for the a.e. convergence to 0 of $\tilde{S}(r)/r^d$ as $r \to \infty$.

It is instructive to specialize condition (5.3) for square partial sums $S(r, r, \dots, r)$, i.e., when $Q_r = \{\mathbf{k}\in Z_+^d : k_j \leqq r \text{ for each } j\}$. Since $|Q_r| = r^d$, (5.3) is surely satisfied if

$$\sum_{\mathbf{k}\in Q_r \setminus Q_{r-1}} \sigma_{\mathbf{k}}^2 = O\{r^{2d-1} (\log 2r)^{-3} (\log\log 4r)^{-1-\varepsilon}\}$$

with an $\varepsilon > 0$. Taking into consideration that $\mathbf{k} \in Q_r \setminus Q_{r-1}$ iff $\max(k_1, k_2, \dots, k_d) = r$ and that

$$\sum_{\mathbf{k}\in Q_r \setminus Q_{r-1}} |\mathbf{k}| = O(r^{2d-1}),$$

condition (5.3) is also satisfied if

$$\begin{aligned} \sigma_{\mathbf{k}}^2 &= O\{|\mathbf{k}| (\log 2r)^{-3} (\log\log 4r)^{-1-\varepsilon}\} = \\ &= O\{|\mathbf{k}| (\log 2|\mathbf{k}|)^{-3} (\log\log 4|\mathbf{k}|)^{-1-\varepsilon}\}, \end{aligned} \tag{5.4}$$

where $\varepsilon > 0$ (cf. (3.2)). In case $d \geqq 2$ condition (5.4) is essentially weaker than (4.4).

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Differentiations-Kompositionsringe

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In dieser Arbeit werden Differentiations-Kompositionsringe näher untersucht und einige grundlegende Eigenschaften dieser Algebren hergeleitet.

Um eine formale Differential-Rechnung für Ringe zu bekommen, führten Kolchin und Ritt (vgl. etwa [1]) den Begriff des „differential ring" ein. Das ist ein Ring R auf dem Derivationen δ definiert sind, d. h. Abbildungen $\delta: R \to R$, die für alle Elemente $a, b \in R$ die Bedingungen $\delta(a+b) = \delta(a) + \delta(b)$ und $\delta(a \cdot b) = \delta(a) \cdot b + a \cdot \delta(b)$ erfüllen. Da man jedoch in Ringen differenzierbarer Funktionen sehr viele Derivationen erhält, die nur sehr „weitläufig" mit der Differentiation von Funktionen in der Analysis verwandt sind, wurde in [3] ein schärferer Differentiations-Begriff für Kompositionsringe definiert. Es werden dabei Abbildungen von Kompositionsringen in sich betrachtet, die neben der Summen- und der Produktregel auch noch einer Abstraktion der aus der Analysis bekannten Kettenregel genügen.

Sei $\langle A, +, \cdot, \circ \rangle$ ein Kompositionsring im Sinne von Lausch-Nöbauer [2]. Es ist dann $\langle A, +, \cdot \rangle$ ein Ring, $\langle A, \circ \rangle$ eine Halbgruppe und es gelten für alle Elemente $f, g, h \in A$ die beiden Rechtsdistributivgesetze $(f+g) \circ h = (f \circ h) + (g \circ h)$ und $(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$.

Definition. Ein Kompositionsring $\langle A, +, \cdot, \circ \rangle$ zusammen mit einer Abbildung $D: A \to A$ heißt ein *Differentiations-Kompositionsring*, falls für alle Elemente $f, g \in A$ die folgenden Beziehungen erfüllt sind:

(S) $D(f+g) = D(f) + D(g)$,

(P) $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$,

(K) $D(f \circ g) = \big(D(f) \circ g\big) \cdot D(g)$.

Jede solche Abbildung D von A heißt dann eine Differentiation oder Derivation mit Kettenregel (kurz K-Derivation) von A.

Die Klasse aller Differentiations-Kompositionsringe bildet offenbar eine Varietät bezüglich der Operationenmenge $\{+, -, 0, \cdot, \circ, D\}$. Beispiele für

Differentiations-Kompositionsringe wurden in [3] gegeben. Dort wurde die Gesamtheit aller möglichen Differentiationen in einigen speziellen Kompositionsringen ermittelt. Wir leiten nun im folgenden Eigenschaften allgemeiner Differentiations-Kompositionsringe her.

Sei $\langle A, +, \cdot, \circ, D\rangle$ ein Differentiations-Kompositionsring. Eine nicht leere Teilmenge U von A heißt ein *Differentiations-Unterkompositionsring* von A, wenn $\langle U, +, \cdot, \circ, D\rangle$ wieder ein Differentiations-Kompositionsring ist.

Ein Element $c \in A$ heißt *Differentiationskonstante*, wenn $D(c)=0$.

Es gilt der folgende

Satz 1. *Die Differentiationskonstanten bilden einen Differentiations-Unterkompositionsring von A, dem das Einselement der Multiplikation angehört, sofern A ein Ring mit Einselement ist. Die Konstanten aus A bilden einen Differentiations-Unterkompositionsring des Differentiations-Kompositionsringes der Differentiationskonstanten.*

Beweis. Sei K_D die Menge der Differentiationskonstanten. Da $D(0)=0$ gilt, ist K_D nicht leer. Mit $f, g \in K_D$ folgt wegen $D(f-g)=D(f)-D(g)=0$, daß auch $f-g \in K_D$ ist. Wegen (P) ist auch $f \cdot g \in K_D$. Schließlich folgt $f \circ g \in K_D$ direkt aus (K). Klarerweise ist D aber auch eine Differentiation auf K_D. Besitzt A das Einselement 1, so folgt aus $D(1)=D(1\cdot 1)=D(1)+D(1)$, daß $D(1)=0$ gilt. Damit ist die erste Behauptung gezeigt.

Bezeichnet nun K die Menge der Konstanten aus A, das ist die Menge $\{a \in A \mid a \circ 0 = a\}$, so gilt für $a, b \in K$: $(a-b)\circ 0=(a\circ 0)-(b\circ 0)=a-b$, $(a\cdot b)\circ 0= =(a\circ 0)\cdot(b\circ 0)=a\cdot b$ und $(a\circ b)\circ 0=a\circ(b\circ 0)=a\circ b$. Da für alle $a \in K$ $D(a)= D(a\circ 0)=(D(a)\circ 0)\cdot D(0)=0 \in K$ folgt, bildet K einen Differentiations-Unterkompositionsring von A, und es gilt $K \subseteqq K_D$.

Wir zeigen nun, daß im allgemeinen K echt in K_D enthalten ist, es also Elemente in K_D gibt, die nicht in K liegen. Um alle Elemente von K_D zu bestimmen, muß man die Lösungen der „Differentialgleichung" $D(f)=0$ ermitteln. Dieses Problem dürfte jedoch selbst in den meisten Kompositionsringen, für die man die Gesamtheit aller möglichen Derivationen mit Kettenregel kennt (vgl. [3], [4]), sehr schwer zu lösen sein. Betrachten wir den Polynomring $R[x]$ in der Unbestimmten x über einem kommutativen Ring R mit Einselement. Durch Hinzunahme der Operation des Einsetzens von Polynomen wird $R[x]$ zu einem Kompositionsring. Nach [4] ist dann die Gesamtheit aller K-Derivationen in $R[x]$ durch die Abbildungen $\lambda\cdot\frac{d}{dx}$, λ Idempotente in R, gegeben, wobei $\frac{d}{dx}$ die Ableitung von Polynomen bezeichnet. Nun ist aber $\lambda\cdot\frac{d}{dx}f=0$ genau dann, wenn $\frac{d}{dx}f$ aus dem Annullator

$\alpha(\lambda)$ von λ ist. Da λ Idempotente aus R ist, gilt $R[x]=\lambda\cdot R[x]\oplus(1-\lambda)\cdot R[x]$, und für $a\in R[x]$ mit $a\cdot\lambda=0$ folgt $a=a_1+a_2$ mit $a_1\in\lambda\cdot R[x]$, $a_2\in(1-\lambda)\cdot R[x]$, also $a\cdot\lambda=a_1\cdot\lambda+a_2\cdot\lambda=a_1\cdot\lambda=a_1=0$. Daher gilt $\alpha(\lambda)=(1-\lambda)\cdot R[x]$. Es bleibt noch zu untersuchen, wann $\frac{d}{dx}f\in(1-\lambda)\cdot R[x]$ gilt. Für $\lambda=0$, D ist dann gleich der Nullabbildung, ist jedes $f\in R[x]$ in K_D. Für $\lambda=1$ und Charakteristik von R gleich n gilt $\frac{d}{dx}x^n=0$, wobei $x^n\notin K$. Ist die Charakteristik von R gleich ∞ (vgl. [5]), so gibt es ein $a\neq 0$ aus R mit $n\cdot a=0$. Es ist dann $\frac{d}{dx}a\cdot x^n=0$. Ist jedoch die Charakteristik von R gleich 0, so ist für $f(x)=a_nx^n+\ldots+a_1x+a_0$, $a_n\neq 0$, $n\geqq 1$, stets $\frac{d}{dx}f(x)=na_nx^{n-1}+\ldots+a_1\neq 0$. Also gilt in diesem Fall $K_D=K$. Für $\lambda\neq 0$ und $\lambda\neq 1$ gibt es stets Elemente in $(1-\lambda)\cdot R[x]$, die nicht in K liegen. Wir bekommen damit

Lemma 1. *Im Polynomring $\langle R[x], +, \cdot, \circ, D\rangle$ ist die Menge der Differentiationskonstanten genau dann gleich der Menge der Konstanten, wenn $D=\frac{d}{dx}$ ist und R die Charakteristik 0 hat.*

Nun zeigen wir

Lemma 2. *Ist $\langle A, +, \cdot, \circ, D\rangle$ ein Differentiations-Kompositionsring mit kommutativer Multiplikation, so sind alle Idempotenten $\lambda\in A$ Differentiationskonstante.*

Beweis. Sei $\lambda\in A$ mit $\lambda\cdot\lambda=\lambda$. Dann folgt $D(\lambda)=D(\lambda\cdot\lambda)=D(\lambda)\cdot\lambda+{}$ ${}+\lambda\cdot D(\lambda)=2\lambda\cdot D(\lambda)$, weiter $\lambda\cdot D(\lambda)=2\lambda\cdot D(\lambda)$, d. h. $\lambda\cdot D(\lambda)=0$ und damit auch $D(\lambda)=0$.

Als nächstes beweisen wir

Lemma 3. *Für die Differentiationskonstanten $f\in K_D$ gilt $D(f\cdot g)=f\cdot D(g)$ und $D(g\cdot f)=D(g)\cdot f$ für alle $g\in A$. Besitzt A ein (im Sinne von [5]) reguläres Element gegenüber der Multiplikation, so sind die Elemente von K_D durch jede dieser beiden Bedingungen charakterisiert.*

Beweis. Die erste Behauptung ist klar.

Gilt $D(f\cdot g)=f\cdot D(g)$, bzw. $D(g\cdot f)=D(g)\cdot f$, für alle $g\in A$, so folgt wegen (P) $D(f)\cdot g=0$, bzw. $g\cdot D(f)=0$. Ist nun g regulär, so gilt $D(f)=0$, also $f\in K_D$.

Unter dem *Differentiations-Linksannullator* N_A von A verstehen wir die Menge $N_A=\{f\in A\,|\,D(f\cdot g)=f\cdot D(g)$ für alle $g\in A\}$. Ganz analog wird der *Differentiations-Rechtsannullator* von A definiert.

Es gilt

Satz 2. *Der Differentiations-Linksannullator N_A ist ein Differentiations-Unterkompositionsring von A.*

Beweis. Wegen (P) ist $D(f\cdot g)=f\cdot D(g)$ für alle $g\in A$ gleichbedeutend damit, daß $D(f)\cdot g=0$ für alle $g\in A$, also $D(f)$ im Linksannullator von A liegt. Sicher ist $0\in N_A$. Wegen $D((f_1-f_2)\cdot g)=D(f_1\cdot g-f_2\cdot g)=D(f_1\cdot g)-D(f_2\cdot g)=f_1\cdot D(g)-$ $-f_2\cdot D(g)=(f_1-f_2)\cdot D(g)$ ist mit $f_1,f_2\in N_A$ auch $f_1-f_2\in N$. Ebenso liegt wegen $D((f_1\cdot f_2)\cdot g)=D(f_1\cdot(f_2\cdot g))=f_1\cdot D(f_2\cdot g)=(f_1\cdot f_2)\cdot D(g)$ auch $f_1\cdot f_2$ in N_A. Weiters gilt $D((f_1\circ f_2)\cdot g)=D(f_1\circ f_2)\cdot g+(f_1\circ f_2)\cdot D(g)=(D(f_1)\circ f_2)\cdot D(f_2)\cdot g+$ $+(f_1\circ f_2)\cdot D(g)=(f_1\circ f_2)\cdot D(g)$, also auch $f_1\circ f_2\in N_A$. Da $D(f)\cdot g=0$ für alle $f\in N_A$, folgt $D(D(f)\cdot g)=0$, und $D(f)\cdot D(g)=0$, gilt $D(D(f)\cdot g)=D(f)\cdot D(g)$, also ist mit f auch $D(f)$ aus N_A.

Bemerkung 1. Wie man ganz analog zeigt, bildet auch der Differentiations-Rechtsannullator von A einen Differentiations-Unterkompositionsring von A.

Bemerkung 2. Besitzt A ein reguläres Element gegenüber der Multiplikation, dann gilt $N_A=K_D$.

Das folgende Lemma benützt man bei der Ermittlung aller möglichen Differentiationen eines vorgegebenen Kompositionsringes.

Lemma 4. *Besitzt A ein neutrales Element i bezüglich der Komposition, so gilt:*

a) *$D(i)$ ist stets Idempotente in A.*

b) *Das Bild von A unter D liegt stets in $A\cdot D(i)$.*

c) *Gilt $D(i)=0$, so ist D die Nullabbildung.*

Beweis. Wegen $D(i)=D(i\circ i)=(D(i)\circ i)\cdot D(i)=D(i)\cdot D(i)$ ist $D(i)$ Idempotente aus A. Aus $D(f)=D(f\circ i)=D(f)\cdot D(i)$ für alle $f\in A$ folgt $D: A\to A\cdot D(i)$. Die Behauptung c) folgt sofort aus b).

Als nächstes zeigen wir

Satz 3. *Ist A ein Kompositionsring mit kommutativer Multiplikation und entsteht A durch Adjunktion der Elemente $a_1,\dots,a_n$ aus der Menge der Konstanten K, d. h. $A=K[a_1,\dots,a_n]$, so gibt es höchstens eine Differentiation D von A mit $D(a_1)=$ $=b_1,\dots,D(a_n)=b_n$, wobei die $b_1,\dots,b_n$ fest aus A vorgegeben sind.*

Beweis. Jedes Element $f\in A$ hat die Gestalt $f=\sum c_{j_1\dots j_n}a_1^{j_1}\dots a_n^{j_n}$ mit $c_{j_1\dots j_n}\in K$. Wegen (S) und (P) gilt

$$D(f)=\sum D(c_{j_1\dots j_n}\cdot a_1^{j_1}\dots a_n^{j_n})=\sum c_{j_1\dots j_n}\cdot D(a_1^{j_1}\dots a_n^{j_n})=$$

$$=\sum_{k=1}^{n}\sum c_{j_1\dots j_n}\cdot a_1^{j_1}\dots a_{k-1}^{j_{k-1}}a_{k+1}^{j_{k+1}}\dots a_n^{j_n}\cdot D(a_k^{j_k})=$$

$$=\sum_{k=1}^{n}\sum c_{j_1\dots j_n}\cdot a_1^{j_1}\dots a_{k-1}^{j_{k-1}}a_{k+1}^{j_{k+1}}\dots a_n^{j_n}\cdot b_k\cdot j_k\cdot a_k^{j_k-1},$$

womit D festgelegt ist.

Bemerkung 3. Mit Hilfe von Satz 3 und Lemma 4 lassen sich leicht sämtliche K-Derivationen vom schon erwähnten Polynomring $A[x]$ bestimmen. Da für jede K-Derivation von $A[x]$ $D(x)$ eine Idempotente aus $A[x]$ ist und damit sogar in A liegt, muß man nur überprüfen, welche der Abbildungen $A[x]\to A[x]$, $f\to\lambda\cdot\frac{d}{dx}f$, λ Idempotente aus A, die Gesetze (S), (P) und (K) erfüllen.

Ist $\langle A, +, \cdot\rangle$ ein Körper, so nennen wir $\langle A, +, \cdot, \circ, D\rangle$ einen *Differentiations-Kompositionskörper*.

Es gilt

Satz 4. *In einem endlichen Differentiations-Kompositionskörper sind alle Elemente Differentiationskonstanten.*

Beweis. Ist die Ordnung des Körpers gleich der Primzahlpotenz p^e, so gilt für alle Elemente $f\in A$, daß $f^{p^e-1}=1$ ist. Daraus folgt $D(f^{p^e-1})=(p^e-1)\cdot \cdot f^{p^e-2}\cdot D(f)=D(1)=0$, also $D(f)=0$.

Folgerung. In jedem endlichen Differentiations-Kompositionskörper ist die einzige Differentiation die Nullabbildung.

Bemerkung 4. Bekanntlich ist jeder Kompositionsring isomorph zu einem Unterkompositionsring eines vollen Funktionenringes (vgl. [2]). Zur Bestimmung aller möglichen Differentiations-Kompositionsringe über einem Kompositionsring genügt es also, alle möglichen Differentiationen für volle Funktionenringe und ihre Unterkompositionsringe zu bestimmen. Da der volle Funktionenring über einem kommutativen Ring mit Einselement nach [4] nur die triviale Differentiation besitzt, also nur mit der Nullabbildung zu einem Differentiations-Kompositionsring gemacht werden kann, stellt sich die Frage, den größten Unterkompositionsring zu finden, in dem eine nicht-triviale Differentiation existiert. Die Lösung dieser Aufgabe müssen wir leider offen lassen.

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On injections, intertwining operators of class C_0

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1. An operator T on a (complex, separable) Hilbert space $\mathfrak{H}$ is of class C_0 if it is a completely non-unitary contraction and if $m(T)=0$ for some inner function $m(\lambda)$ on the unit disc: $|\lambda|<1$. T is of class $C_0(N)$ with some integer $N\geqq 0$ if, moreover, its defect indices are $\leqq N$. For a first introduction to the study of these classes see [*H*]. These investigations have already lead to many new concepts and methods in the theory of Hilbert space operators, and in particular to generalizations of the "Jordan model" for finite matrices.

In the present Note we are going to make use of these models for establishing some further properties of class C_0 operators.

Let us recall some further definitions and facts.

An operator $X:\mathfrak{H}'\to\mathfrak{H}$ is called an *injection* if $\ker X=\{0\}$, and a *quasi-surjection* if $\overline{X\mathfrak{H}'}=\mathfrak{H}$ or, equivalently, if $\ker X^*=\{0\}$.

Given two operators, T on $\mathfrak{H}$ and T' on $\mathfrak{H}'$, we say that T' can be *injected in* T, or *quasi-surjected on* T if there exists an operator $X:\mathfrak{H}'\to\mathfrak{H}$ satisfying $TX=XT'$ and which is an injection, or a quasi-surjection, respectively.

An operator X which is both an injection and a quasi-surjection, is called a *quasi-affinity*, and if $TX=XT'$ holds with such an operator X then T' is called a *quasi-affine transform* of T, in notation $T\succ T'$. If both $T\succ T'$ and $T'\succ T$ hold then T and T' are called *quasi-similar*, $T\succ T'$.

Every operator $T\in C_0$ is quasi-similar to a unique "Jordan operator"

$$S(M)=S(m_1)\oplus S(m_2)\oplus\ldots \quad \text{on} \quad \mathfrak{H}(M)=\mathfrak{H}(m_1)\oplus\mathfrak{H}(m_2)\oplus\ldots \tag{1}$$

where $M=(m_1, m_2, \ldots)$ is a sequence if inner functions each of which is a divisor of the preceding one. Here $S(m)$ means, for any inner function $m(\lambda)$, the operator on the function space $\mathfrak{H}(m)=H^2\ominus mH^2$, defined by $S(m)=P_{\mathfrak{H}(m)}S|\mathfrak{H}(m)$, S denoting the unilateral shift $u(\lambda)\to\lambda u(\lambda)$ on the Hardy—Hilbert space H^2 for

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the disc. We have $\mathfrak{H}(m)=\{0\}$ if (and only if) m is constant, $m=1.$[1]) The number of non-constant functions m_k in (1) is equal to the multiplicity μ_T of the operator $T.$[2]) For $T\in C_0$ we have $\mu_T=\mu_{T^*}$. For these facts see [1], [2], [3].

2. In the sequel we shall be dealing with operators of class C_0.

Theorem 1. *Let us be given two operators of class C_0, say T on $\mathfrak{H}$ and T' on $\mathfrak{H}'$.* a) *If T and T' can be injected in one another, then they are quasi-similar.* b) *If T' can be injected in, and also quasi-surjected on T, then they are quasi-similar.*

Proof. We have to show that if there exist injections $X:\mathfrak{H}'\to\mathfrak{H}$ and $X':\mathfrak{H}\to\mathfrak{H}'$ such that

$$TX = XT' \tag{2}$$

and

$$\text{(a) } T'X' = X'T, \quad \text{or} \quad \text{(b) } T'^*X' = X'T^*; \tag{3}$$

then $T\sim T'$.

As it is easy to see, there is no loss in generality if we argue with the Jordan models of T and T' instead of T and T' themselves, i.e. if we assume that

$$\mathfrak{H}=\mathfrak{H}(M),\quad \mathfrak{H}'=\mathfrak{H}(M'),\quad T=S(M),\quad T'=S(M'), \tag{4}$$

where $M=(m_1, m_2, \ldots)$ and $M'=(m_1', m_2', \ldots)$. Following a standard argument (*cf.*, in particular, [4], proof of Theorem 4), we set, for any inner function w,

$$T^w = T\,|\,\overline{w(T)\mathfrak{H}},\quad T'^w = T\,|\,\overline{w(T')\mathfrak{H}'},\quad X^w = X\,|\,\overline{w(T')\mathfrak{H}'},$$

and first notice that by condition (2) we also have

$$T^wX^w = X^wT'^w;$$

clearly, X^w is also an injection. Now, T^w and T'^w are unitarily equivalent to $\oplus S(q_i)$ and $\oplus S(q_i')$, respectively, where $q_i=m_i/(m_i\wedge w)$ and $q_i'=m_i'/(m_i'\wedge w)$. Choosing $w=m_k$ for a fixed k we infer that

$$\bigoplus_{i=1}^{k-1} S\left(\frac{m_i}{m_k}\right)\cdot X^{(k)} = X^{(k)}\cdot \bigoplus_{i=1}^{\infty} S\left(\frac{m_i'}{m_i'\wedge m_k}\right),\text{[3])}$$

with some injection $X^{(k)}$. By virtue of [4], Theorem 4, the second direct sum cannot have more non-trivial terms than the first, so we must have $m_i'/(m_i'\wedge m_k)=1$ for $i\geqq k$, and in particular, $m_k'/(m_k'\wedge m_k)=1$, $m_k'|m_k$.

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[1]) Up to a constant factor of modulus one. It is convenient not to distinguish two inner functions which differ in such a factor only.

[2]) For any operator T, μ_T is defined as the smallest cardinal of a set of vectors which, together with its transforms by T, T^2, etc., span the whole space of T.

[3]) If $k=1$ the first direct sum should be meant as the trivial operator on the space $\{0\}$.

In case condition (3a) holds the same argument applies with the rôles of T and T' interchanged, and we have $m_k|m'_k$ for every k. We conclude that $m_k=m'_k$.

If it is condition (3b) which is assumed, we arrive at the same result as follows. It is well known that for any inner m, $S(m)^*$ is unitarily equivalent to $S(m^\sim)$, where $m^\sim(\lambda)=\overline{m(\bar\lambda)}$. So (4) implies that T^* and T'^* are unitarily equivalent to $S(M^\sim)$ and $S(M'^\sim)$, respectively, and then we deduce from (3b) that $m_k^\sim|m_k'^\sim$ for every k, in the same way as we deduced $m'_k|m_k$ from (2). But $m_k^\sim|m_k'^\sim$ obviously implies $m_k|m'_k$ and we conclude again that $m_k=m'_k$.

Thus in both cases T and T' have the same Jordan model so they are quasi-similar. This concludes the proof.

Remark. The quasi-similarity of T and T' is, in general, not effectuated by the operators X, X' figuring in (2), (3a) or (3b), since they need not be quasi-affinities. Example: $T=T'=0$ on an infinite dimensional Hilbert space $\mathfrak{H}$, and $X=X'=$(a unilateral shift on $\mathfrak{H}$).

However, such a phenomenon cannot occur if the operators T, T' are of finite multiplicity. This will be proved in the rest of this paper.

3. First we prove the following

Lemma. *Let T be an operator of class $C_0(N)$ on $\mathfrak{H}$, with some finite N. Then every injection X on $\mathfrak{H}$, commuting with T, is a quasi-affinity.*

Proof. Let T_1 be the restriction of T to the subspace $\mathfrak{H}_1=\overline{X\mathfrak{H}}$, which is invariant for T_1 because

$$T_1X = XT. \tag{5}$$

Let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of T with respect to the decomposition $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$. Let $\Theta_T=\Theta_2\Theta_1$ be the corresponding regular factorization of the characteristic function Θ_T of T; Θ_{T_1} and Θ_{T_2} coincide then with the purely contractive parts Θ_1^0 and Θ_2^0 of Θ_1 and Θ_2, respectively; *cf.* [*H*] Chapter VII. Since T is of class $C_0(N)$, all these functions are finite square-matrix valued so we have

$$\det\Theta_T = \det\Theta_2\cdot\det\Theta_1 = \det\Theta_2^0\cdot\det\Theta_1^0 = \det\Theta_{T_2}\cdot\det\Theta_{T_1}, \tag{6}$$

up to constant factors of modulus one.

Since X can be regarded as a quasi-affinity $\mathfrak{H}\to\mathfrak{H}_1$, from (5) it follows that T is a quasi-affine transform of T_1; $T_1\succ T$. Hence, T and T_1 are quasi-similar to the same Jordan operator $S(M)$, $M=(m_1, m_2, \ldots, m_k)$, $K\leqq N$; *cf.* [1] Theorem 2 (and also [2] Theorem 3). Hence we have, by the formula (1.7) of [1],

$$\det\Theta_T = m_1\ldots m_k = \det\Theta_{T_1};$$

comparing this with (6) we conclude that det Θ_{T_2} is a constant (of modulus one). The minimal function m_{T_2}, being a divisor of det Θ_{T_2} (*cf.* [*H*] Sec. VI. 5), is also constant, and therefore we have $\mathfrak{H}_2=\{0\}$, $\mathfrak{H}_1=\mathfrak{H}$, as asserted.

4. Every operator $T\in C_0(N)$ is of finite multiplicity, $\mu_T\leqq N$, but not every operator $T\in C_0$ with finite multiplicity belongs to some class $C_0(N)$. Therefore, the following theorem is an extension of the Lemma (even if $T=T'$).

Theorem 2. *Let T and T' be operators of class C_0 on the spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, and suppose T and T' are quasi-similar and have finite multiplicity, $\mu_T=\mu_{T'}=K$. Then every injection operator intertwining T and T' is a quasi-affinity.*[4])

Proof. Let X be an injection $\mathfrak{H}'\to\mathfrak{H}$ such that $TX=XT'$. Since $T\sim T'$, there exists a quasi-affinity $Q:\mathfrak{H}\to\mathfrak{H}'$ such that $T'Q=QT$, and hence $TX'=X'T$ with $X'=XQ$. Clearly X' is an injection; furthermore, X' is a quasi-surjection iff so is X. Hence it suffices to show that every injection X' on $\mathfrak{H}$ satisfying

$$TX' = X'T \tag{7}$$

is a quasi-surjection, i.e. such that $\overline{X'\mathfrak{H}}=\mathfrak{H}$.

Setting $\mathfrak{H}_1=\overline{X'\mathfrak{H}}$ and $T_1=T|\mathfrak{H}_1$ we deduce from (7), as in the proof of the Lemma, that $T_1\succ T$, $T^*\succ T_1^*$. Since, on the other hand, $T^*\sim S(M)^*$ with some $M=(m_1, m_2, \ldots, m_K)$, we conclude that there exist quasi-affinities

$$A:\mathfrak{H}(M)\to\mathfrak{H},\quad B:\mathfrak{H}\to\mathfrak{H}(M),\quad B_1:\mathfrak{H}_1\to\mathfrak{H}(M)$$

such that

$$T^*A = A\,S(M)^*,\quad S(M)^*B = BT^*,\quad S(M)^*B_1 = B_1T_1^*. \tag{8}$$

Set $Y=B_1P_1A$, where P_1 denotes the orthogonal projection of $\mathfrak{H}$ onto its subspace $\mathfrak{H}_1$. Then $P_1T^*=T_1^*P_1$ and by (8):

$$YS(M)^* = B_1P_1A\,S(M)^* = B_1P_1T^*A = B_1T_1P_1A = S(M)^*B_1P_1A = S(M)^*Y.$$

Furthermore, by the quasi-surjectivity of A and B_1,

$$\overline{Y\mathfrak{H}(M)} = \overline{B_1P_1A\mathfrak{H}(M)} = \overline{B_1P_1\mathfrak{H}} = \overline{B_1\mathfrak{H}_1} = \mathfrak{H}(M).$$

It follows that Y^* is an injective operator on $\mathfrak{H}(M)$, commuting with $S(M)$. As we have $S(M)\in C_0(K)$ it follows from the Lemma that Y^* is quasi-surjective. Hence, Y $(=B_1P_1A)$ is injective. This implies that P_1A is injective also.

Let us now assume that $\mathfrak{H}_1\neq\mathfrak{H}$, and consider in $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$ a cyclic subspace for T_2^* $(=T^*|\mathfrak{H}_2)$. The restriction of T_2^* to this subspace is then quasi-similar to an operator $S(n)$ associated with a non-constant inner function n (*cf.* [1], Theorem 2 applied to a C_0-class operator of multiplicity 1). Since $S(n)^*$ is unitarily equivalent

[4]) This was conjectured by P. Y. Wu in connection with his investigations [5] on commutants of class C_0 operators (communication to the first author on October 1, 1975). As far as we know the following one is the first proof.

to $S(m)$, where $m=n^{\sim}$, there exists in particular an *injection* $C\colon \mathfrak{H}(m)\to\mathfrak{H}_2$ such that

$$CS(m)^* = T^*C. \tag{9}$$

Next consider the operator Z on $\mathfrak{H}(M)\oplus\mathfrak{H}(m)$ defined by

$$Z(h_M\oplus h_m) = B(Ah_M+Ch_m)\oplus 0 \quad (h_M\in\mathfrak{H}(M),\ h_m\in\mathfrak{H}(m)). \tag{10}$$

From (8), (9), (10) we obtain

$$B(A\,S(M)^*h_M + C\,S(m)^*h_m) = B(T^*Ah_M+T^*Ch_m) =$$
$$= BT^*(Ah_M+Ch_m) = S(M)^*B(Ah_M+Ch_m),$$

and hence,

$$Z(S(M)\oplus S(m)^*)(h_M\oplus h_m) = (S(M)^*\oplus S(m)^*)Z(h_M\oplus h_m),$$

i.e. Z commutes with the operator $S(M)^*\oplus S(m)^*$, which clearly belongs to $C_0(K+1)$. Furthermore, Z is injective. Indeed, $Z(h_M\oplus h_m)=0$ implies $Ah_m=-Ch_m\in\mathfrak{H}_2$, and hence $P_1Ah_M=0$, $h_M=0$ because P_1A is an injection. Since C is also injective, $Ch_m=-Ah_M=0$ implies $h_m=0$.

Applying the Lemma we get that Z is also quasi-surjective. This contradicts the fact that, by (10), its range lies in $\mathfrak{H}(M)\oplus\{0\}$.

This contradiction proves that $\mathfrak{H}_1=\mathfrak{H}$ so X is a quasi-affinity.

5. To end let us venture the following conjecture, which would largely generalize Theorem 2 in case $T=T'$:

Conjecture. For any contraction T on $\mathfrak{H}$ of class C_0 and of finite multiplicity, and for any operator X on $\mathfrak{H}$ such that $TX=XT$, the operators

$$T|\ker X \quad\text{and}\quad (T^*|\ker X^*)^*$$

are quasi-similar.

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On extendibility of $*$-representations from $*$-ideals

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In this note we give necessary and sufficient conditions for the extension of a $*$-representation from a $*$-ideal in a complex involutory algebra, briefly $*$-algebra, to the whole algebra. One of our conditions, given in the Theorem below, is the same as the one we required in [3], Corollary 2. 8, and [4], Corollary 6, for the existence of such a $*$-representation of the whole algebra on some Hilbert space for which the norms of the representing operators of the elements in the $*$-ideal are equal to the norms of the corresponding representing operators concerning the given $*$-representation. We also give a new proof of our previous result (see [3], Theorem 2. 6; [4], Theorem 4) concerning the extension of C^*-semi-norms from $*$-ideals in $*$-algebras to the whole algebras.

Let us be given a $*$-ideal J of a $*$-algebra A over the complex number field $\mathbf{C}$ and a $*$-representation T of J on a Hilbert space H, i.e. a $*$-preserving algebra homomorphism $T:J\to B(H)$ of J into the C^*-algebra $B(H)$ of all bounded linear operators on H. It is natural to ask: when does a $*$-representation $\overline{T}:A\to B(H)$ of A on the same Hilbert space exist, which is an extension of the given $*$-representation T, i.e. for which $\overline{T}_b=T_b$ holds whenever b is in J.

We answer this question in three ways, giving necessary and sufficient conditions, the first of which is simple enough still it needs no restriction on the algebra unlike in PALMER [2], Theorem 3.1.

Lemma. *Let $T:J\to B(H)$ be a $*$-representation of a $*$-ideal J in the complex $*$-algebra A on the Hilbert space H. Then there exists a $*$-representation $\overline{T}:A\to B(H)$ of A on the same Hilbert space H extending T, if and only if*

$$(1)\qquad \sup\Big\{\Big\|\sum_n T_{ab_n}x_n\Big\| : b_n\in J,\ x_n\in H,\ \Big\|\sum_n T_{b_n}x_n\Big\|\leqq 1\Big\}<\infty$$

holds for every a in A; here and further on $\sum_n$ denotes a finite sum.

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Proof. Assume first that such a $*$-representation $\bar{T}$ of A exists. We have then for every $a\in A$, $b_n\in J$ and $x_n\in H$, since $ab_n\in J$,

$$\Big\|\sum_n T_{ab_n}x_n\Big\| = \Big\|\sum_n \bar{T}_{ab_n}x_n\Big\| = \Big\|\bar{T}_a\Big(\sum_n \bar{T}_{b_n}x_n\Big)\Big\| = \Big\|\bar{T}_a\Big(\sum_n T_{b_n}x_n\Big)\Big\| \leqq |\bar{T}_a|\Big\|\sum_n T_{b_n}x_n\Big\|;$$

hence (1) is satisfied; here $|\bar{T}_a|$ is the norm of the operator $\bar{T}_a$ on the Hilbert space H.

Let

$$H_0 = \{x\in H: T_b x = 0 \text{ for all } b\in J\} = \bigcap\{\operatorname{Ker} T_b : b\in J\}$$

be the maximal closed linear subspace in H, the restriction of T to which is a zero $*$-representation T^0 of J on H_0. Denote by $\bar{T}^0$ the zero $*$-representation of A on H_0, the trivial extension of T^0 from J to A. Denote further

$$H_1 = \{x\in H: (x, y) = 0 \text{ for all } y\in H_0\}$$

the orthogonal complement of H_0 in H, and T^1 the restriction of T to H_1 (which is clearly an invariant subspace for T). In this way T is the direct sum of a zero and an essential $*$-representation T^0 and T^1 respectively: $T=T^0\oplus T^1$. Indeed, $G=\big\{\sum_n T_{b_n}x_n : x_n\in H_1, b_n\in J\big\}$ is a dense linear manifold in H_1, which is invariant for T also, because if an element x in H_1 is orthogonal to G, then for all $c\in J$

$$\|T_c x\|^2 = (T_c x, T_c x) = (x, T_c^* T_c x) = (x, T_{c^*c}x) = 0,$$

hence $x\in H_0$ and thus $x=0$.

Define for an element a in A a linear operator S_a in H_1 given on G by

$$S_a\Big(\sum_n T_{b_n}x_n\Big) = \sum_n T_{ab_n}x_n \quad (b_n\in J, x_n\in H_1),$$

We have now to show that S_a is well defined on G, that is, $\sum_n T_{b_n}x_n=0$ implies $\sum_n T_{ab_n}x_n=0$. For $y=\sum_n T_{ab_n}x_n$ we have

$$\|y\|^2 = \Big(\sum_m T_{ab_m}x_m, \sum_n T_{ab_n}x_n\Big) = \sum_{m,n}(T_{ab_m}x_m, T_{ab_n}x_n) =$$

$$= \sum_{m,n}(T_{b_n^*a^*ab_m}x_m, x_n) = \sum_{m,n}(T_{a^*ab_m}x_m, T_{b_n}x_n) = \Big(\sum_m T_{a^*ab_m}x_m, \sum_n T_{b_n}x_n\Big),$$

hence $y=0$ indeed. But our assumption (1) means then that S_a is a densely defined bounded linear operator in H_1, thus it has a unique extension $\bar{T}_a^1$ to H_1 in a standard way. For an $a\in J$ we have $S_a x=T_a x$ $(x\in G)$, hence that $\bar{T}_a^1=T_a$. It is now easy to check that $\bar{T}^1: A\to B(H_1)$ is a $*$-representation of A on H_1. The linearity and multiplicativity of $\bar{T}^1$ is immediate. On the other hand, if $a\in A$; $b_n, c_m\in J$; $x_n, y_m\in H_1$, then we

$$\Big(S_a\Big(\sum_n T_{b_n}x_n\Big), \sum_m T_{c_m}y_m\Big) = \Big(\sum_{m,n} T_{ab_n}x_n, T_{c_m}y_m\Big) =$$

$$= \sum_{m,n}(x_n, T_{b_n^*a^*c_m}y_m) = \sum_{m,n}(T_{b_n}x_n, T_{a^*c_m}y_m) = \Big(\sum_n T_{b_n}x_n, S_{a^*}\Big(\sum_m T_{c_m}y_m\Big)\Big)$$

and thus $(\bar{T}_a^1)^* = \bar{T}_{a*}^1$ as well. Moreover, the extension of T^1 to A is unique. Indeed, if $\hat{T}^1$ is an arbitrary $*$-representation of A that is an extension of T^1, then for $a \in A$, $b_n \in J$, $x_n \in H_1$ we have

$$\hat{T}_a^1\Big(\sum_n T_{b_n}^1 x_n\Big) = \hat{T}_a^1\Big(\sum_n \hat{T}_{b_n}^1 x_n\Big) = \sum_n \hat{T}_{ab_n}^1 x_n = \sum_n T_{ab_n}^1 x_n = S_a\Big(\sum_n T_{b_n}^1 x_n\Big).$$

We have finally that $\bar{T} = \bar{T}^0 \oplus \bar{T}^1$ is a $*$-representation of A on the Hilbert space H which extends T and the lemma is proved.

We are now able to improve the above result as follows.

Proposition. *The $*$-representation $T: J \to B(H)$ has an extension, a $*$-representation $\bar{T}: A \to B(H)$ if and only if*

(2) $$\sup\{\|T_{ab}x\| : b \in J,\ x \in H,\ \|T_b x\| \leqq 1\} < \infty \quad \textit{for every } a \in A.$$

Proof. The necessity part is obvious from Lemma as (2) is a specialization of (1).

Suppose now that (2) holds. We are going to prove that T has an extension $\bar{T}: A \to B(H)$. Let T^1 and H^1 be as before and write T^1 in the form of direct sum of topologically cyclic sub-$*$-representations (see [2]):

$$T^1 = \oplus\{T^\lambda : \lambda \in \Lambda\};\ T^\lambda = T^1|_{H_\lambda},\ T^\lambda : J \to B(H_\lambda)$$

for every index λ in Λ; on a maximal family of pairwise orthogonal T^1-invariant subspaces $\{H_\lambda : \lambda \in \Lambda\}$ in H_1 (and thus spanning H_1) with topological cyclic vectors $x_\lambda \in H_\lambda$ such that $G_\lambda = \{T_b x_\lambda : b \in J\}$ is a dense linear manifold in H_λ for each $\lambda \in \Lambda$. An argument similar to that used in Lemma shows, by (2), that for $a \in A$ the linear operator S_a^λ in H_λ given on G_λ by $S_a^\lambda(T_b x_\lambda) = T_{ab} x_\lambda$ $(b \in J)$ is a densely defined bounded linear operator on H_λ. This has a unique extension $\bar{T}_a^\lambda$ to H_λ with norm

$$|\bar{T}_a^\lambda| = \sup\{\|S_a^\lambda(T_b x_\lambda)\| : b \in J,\ \|T_b x_\lambda\| \leqq 1\} =$$
$$= \sup\{\|T_{ab} x_\lambda\| : b \in J,\ \|T_b x_\lambda\| \leqq 1\} \leqq \sup\{\|T_{ab}x\| : b \in J,\ x \in H,\ \|T_b x\| \leqq 1\} < \infty.$$

We thus have the $*$-representations $\bar{T}^\lambda : A \to B(H_\lambda)$ $(\lambda \in \Lambda)$. $\bar{T}^\lambda$ extends T^λ in a unique way for all λ in Λ and thus

$$\bar{T}^1 = \oplus\{\bar{T}^\lambda : \lambda \in \Lambda\},$$

the direct sum of $\bar{T}^\lambda$-s, is a $*$-representation of A on H_1 extending T^1 uniquely and such that for each $a \in A$

$$|\bar{T}_a^1| = \sup\{|\bar{T}_a^\lambda| : \lambda \in \Lambda\} = \sup\{\|T_{ab} x_\lambda\| : b \in J,\ \lambda \in \Lambda,\ \|T_b x_\lambda\| \leqq 1\}$$
$$\leqq \sup\{\|T_{ab}x\| : b \in J,\ x \in H,\ \|T_b x\| \leqq 1\} < \infty.$$

$\bar{T} = \bar{T}^0 \oplus \bar{T}^1$ is then a $*$-representation of A on H which extends T as well and the proof is complete.

The main result of this note is the following.

Theorem. *Let $T:J\to B(H)$ be a $*$-representation of a $*$-ideal in the complex $*$-algebra A on the Hilbert space H. There exists then a $*$-representation $\bar{T}:A\to B(H)$, which is an extension of T, if and only if*

$$q(a):=\sup\{|T_{ab}|: b\in J, |T_b|\leqq 1\}<\infty \quad \textit{for all } a \textit{ in } A. \tag{3}$$

Proof. Assume first that such a $*$-representation $\bar{T}$ of A exists. Then we have for all $a\in A, b\in J$

$$|T_{ab}|=|\bar{T}_{ab}|=|\bar{T}_a\bar{T}_b|\leqq|\bar{T}_a|\cdot|\bar{T}_b|=|\bar{T}_a|\cdot|T_b|$$

whence $q(a)\leqq|\bar{T}_a|$ follows and (3) is satisfied.

Suppose now that (3) holds for all a in A. The quantity q is obviously a semi-norm on A. Moreover, if $|T_b|=0$ for some $b\in J$, then $|T_{ab}|=0$ for all $a\in A$ since

$$|T_{ab}|^2=|T_{ab}^*T_{ab}|=|T_{b^*a^*}T_{ab}|=|T_{b^*a^*ab}|=|T_{b^*a^*a}T_b|\leqq|T_{b^*a^*a}|\,|T_b|$$

and thus

$$|T_{ab}|\leqq q(a)|T_b| \quad \text{holds for} \quad b\in J, a\in A.$$

We are now going to show that q has the C^*-property:

$$q(a^*a)=\big(q(a)\big)^2 \quad (a\in A), \tag{4}$$

in other words, q is a C^*-semi-norm, and that (3) implies (2) whence our statement follows by the Proposition. For $a\in A$, $b\in J$ we have

$$|T_{ab}|^2=|T_{b^*a^*ab}|=|T_{b^*}T_{a^*ab}|\leqq|T_{b^*}|\,|T_{a^*ab}|=|T_b|\,|T_{a^*ab}|\leqq q(a^*a)\,|T_b|^2$$

and thus

$$\big(q(a)\big)^2\leqq q(a^*a) \quad (a\in A). \tag{4'}$$

On the other hand, for $a\in A$, $b\in J$ we have

$$|T_{a^*ab}|\leqq q(a^*)|T_{ab}|\leqq q(a^*)q(a)\,|T_b|.$$

Hence

$$q(a^*a)\leqq q(a^*)q(a) \quad (a\in A). \tag{4''}$$

But (4′) and (4″) together give $q(a)\leqq q(a^*)$ for each $a\in A$; whence by interchanging the rôles of a and a^* we obtain that $q(a)=q(a^*)$ for all a in A. We have then by (4′) and (4″)

$$\big(q(a)\big)^2\leqq q(a^*a)\leqq q(a^*)q(a)=\big(q(a)\big)^2$$

whence (4) follows.

We are now able to prove that (2) holds. For if $a\in A$, $b\in J$ and $x\in H$, then we have

$$\|T_{ab}x\|^2=(T_{ab}x, T_{ab}x)=(T_{ab}^*T_{ab}x, x)=(T_{b^*a^*ab}x, x)=$$
$$=(T_{b^*}T_{a^*ab}x, x)=(T_{a^*ab}x, T_bx)\leqq\|T_{a^*ab}x\|\,\|T_bx\|.$$

Replacing a by a^*a we obtain

$$\|T_{a^*ab}x\|^2\leqq\|T_{(a^*a)^2b}x\|\,\|T_bx\|,$$

and by recurrence,

$$\|T_{ab}x\|^{2^{n+1}} \leqq \|T_{(a^*a)^{2^n}b}x\| \, \|T_bx\|^{2^{n+1}-1} \quad (n = 0, 1, 2, \ldots)$$

But then we have by (4)

$$\|T_{ab}x\|^{2^{n+1}} \leqq |T_{(a^*a)^{2^n}b}| \, \|x\| \, \|T_bx\|^{2^{n+1}-1} \leqq q\big((a^*a)^{2^n}\big)|T_b| \, \|x\| \, \|T_bx\|^{2^{n+1}-1} =$$
$$= \big(q(a^*a)\big)^{2^n} |T_b| \, \|x\| \, \|T_bx\|^{2^{n+1}-1} = \big(q(a)\big)^{2^{n+1}} |T_b| \, \|x\| \, \|T_bx\|^{2^{n+1}-1}.$$

By letting $n \to \infty$ we obtain

$$\|T_{ab}x\| \leqq q(a) \|T_bx\| \quad \text{for all} \quad a \in A, \, b \in J, \, x \in H;$$

hence by (3)

$$\sup \{\|T_{ab}x\| : b \in J, \, x \in H, \, \|T_bx\| \leqq 1\} \leqq q(a) \quad (a \in A),$$

which finishes the proof.

We get finally a new proof of Theorem 2. 6 in [3] (or Theorem 4 in [4]).

Corollary. *Let p be a C^*-semi-norm on a $*$-ideal J of the complex $*$-algebra A. There exists then a C^*-semi-norm on A which is equal to p on J if and only if*

$$\sup \{p(ab) : b \in J, \, p(b) \leqq 1\} < \infty \tag{5}$$

holds for all a in A.

Proof. Let q be such a C^*-semi-norm. q is also submultiplicative in consequence of our previous result (see [3], Theorem 2. 3 or [4], Theorem 2). We have for every $a \in A$, $b \in J$ that

$$p(ab) = q(ab) \leqq q(a) q(b) = q(a) p(b),$$

proving that (5) is necessary.

To show that (5) is sufficient consider $J_p = \{b \in J : p(b) = 0\}$. Since p is automatically submultiplicative and the $*$-operation is isometric with respect to p also, J_p is in fact a $*$-ideal in J such that the completion B_p of the quotient algebra J/J_p with respect to the quotient norm and with natural involution is a C^*-algebra. The classical Gelfand—Naimark theorem assures a canonical isometrical $*$-representation of B_p on some Hilbertspace H: $\bar{T}^p : B_p \to B(H)$. From the restriction T^p of $\bar{T}^p$ to J/J_p we get in a standard way a $*$-representation $T: J \to B(H)$ of J on H such that $T_b = T^p_{b+J_p}$ and $|T_b| = p(b + J_p) = p(b)$ holds for all $b \in J$. As a consequence (5) implies (3) for this $*$-representation T and the statement follows from the Theorem.

Remark. Our result on automatic submultiplicativity of a C^*-semi-norm, mentioned above, is an improvement of a recent result due to Araki and Elliott concerning the definition of C^*-algebras (see Theorem 1 in [1]).

For the following two remarks we are indebted to Dr. J. Szűcs. First, in the proof of the Lemma S_a is well defined simply because (1) implies for $a \in A$, $b \in J$, $x \in H$

$$\sum_n T_{ab_n} x_n = 0 \quad \text{provided} \quad \sum_n T_{b_n} x_n = 0$$

since for each $t > 0$, $0 = \sum_n T_{tb_n} x_n = t \cdot \sum T_{b_n} x_n$ so that by (1)

$$\sup \{t \|\sum_n T_{ab_n} x_n\| : t > 0\} = \sup \{\|\sum_n T_{atb_n}\| : t > 0\} < \infty.$$

Secondly, (3) implies (2) as an easy application of Kaplansky's Density Theorem shows. For if $a \in A$, $b \in J$, $x \in H_1$ and $\|T_b x\| \leqq 1$ then it is enough to show $\|T_{ab} x\| \leqq q(a^*)$ or equivalently

$$|(T_{ab} x, y)| \leqq q(a^*) \|y\| \quad \text{for each} \quad y \in H.$$

$\{T_b^1 : b \in J\} \subset B(H_1)$ is a $*$-algebra of bounded linear operators on H_1 such that $\{T_b^1 x : b \in J, x \in H_1\}$ spans H_1, its double commutant N is a von Neumann algebra containing the identity operator on H_1. But $\{T_b^1 : b \in J\}$ is strongly dense in N hence by Kaplansky's theorem the strong closure of the unit ball in $\{T_b^1 : b \in J\}$ contains N_1, the unit ball of N, especially the identity operator. Hence for a fixed $y \in H_1$ there exists $\{b_n\}_{n=1}^\infty \subset J$ with $|T_{b_n}| \leqq 1$ such that $\|T_{b_n} y - y\| \to 0$. We then obtain for $a \in A$, $b \in J$, $x \in H_1$ that

$$\begin{aligned} |(T_{ab} x, y)| &\leqq \sup_n |(T_{ab} x, T_{b_n} y)| = \sup_n |(T_{b_n^* ab} x, y)| = \\ &= \sup_n |(T_{(a^* b_n)^*} T_b x, y)| = \sup_n |(T_b x, T_{a^* b_n} y)| \leqq \sup_n \|T_{a^* b_n} y\| \leqq \\ &\leqq \sup_n |T_{a^* b_n}| \|y\| \leqq \|y\| q(a^*) \sup_n |T_{b_n}| \leqq \|y\| q(a^*), \end{aligned}$$

and thus the required inequality follows.

The author is indebted to Professor Béla Sz.-Nagy for having called his attention to the problem dealt with in this paper.

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Concrete representation of related structures of universal algebras. I

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In his recent book [6], I. I. VALUCE quotes without proof a result of A. V. Kuznecov, unpublished up to now. Trying to re-establish the proof, we observed some general facts concerning mutual properties of relations and operations. This enables us to solve several concrete representation problems for related structures of algebras in a uniform way.

The basic propositions of this article are Lemmas 1—5 preceeded by a survey of notions we shall need. Using them we give a simultaneous characterization for related structures of universal algebras (Theorem 6). As special cases of Theorem 6 we get characterizations for the systems of subalgebras of finite direct powers of algebras (G. Grätzer's Problem 19 in [3]; Theorem 7 and 9) and the endomorphism semigroups of algebras (Grätzer's Problem 3 in [3]; Theorem 15; for another solution of this problem, see N. SAUER and M. G. STONE [5]). As corollaries we get Jürgen Schmidt's concrete representation theorem for the subalgebra systems of algebras (see, e.g. [2]) and the Bodnarčuk—Kalužnin—Kotov—Romov theorem for the subalgebra systems of all finite direct powers of finite algebras [1]. Moreover, we characterize the bicentralizers of sets of operations in arbitrary algebras. Then Kuznecov's above mentioned result appears as a special case.

In a forthcoming Part II, we shall apply the method developed here for the representation of other related structures.

Let A be a nonempty set which will be fixed in the sequel. Let O_n $(n=0, 1, 2, \dots)$ and O denote the set of all n-ary and all finitary operations of A, respectively; furthermore, let $\mathcal{R}_n$ $(n=1, 2, \dots)$ and $\mathcal{R}$ denote the set of all n-ary and all finitary relations of A, respectively. In general, we shall not distinguish between an operation and the associated relation, i.e., an n-ary operation may be considered as a mapping $f\colon A^n \to A$ and as an $(n+1)$-ary relation $\{(a_1, \dots, a_n, f(a_1, \dots, a_n)) \mid (a_1, \dots, a_n) \in A^n\}$ as well. Thus we have $O \subseteq \mathcal{R}$ and $O_n \subseteq \mathcal{R}_{n+1}$, $n=0, 1, 2, \dots$. If R is an n-ary relation, we shall often write $R(a_1, \dots, a_n)$ instead of $(a_1, \dots, a_n) \in R$.

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We say that an n-ary operation f *preserves* an m-ary relation R, if $R(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$ holds whenever $R(a_{1k}, \ldots, a_{mk})$, $k=1, \ldots, n$, i.e., (R, f) is a subalgebra of the algebra $(A, f)^m$ (the m-th direct power of (A, f)). Remark that the empty set is an n-ary relation for every $n \geqq 1$, and it is preserved by every m-ary operation where $m \geqq 1$. Let f and g be operations of arity n and m, respectively. If M is an $m \times n$ matrix of elements of A, we can apply f [g] to each row [column] of M. Thus we get a column [row] consisting of m [n] elements, which will be denoted by $f(M)$ [$(M)g$]. If for any $m \times n$ matrix M of elements of A, $f((M)g) = (f(M))g$ holds then we say that f and g *commute*. Clearly, two operations commute if and only if any of them preserves the other as a relation. For any set of relations Γ, denote by Γ^* the set of all operations preserving every member of Γ. We call Γ^* the *centralizer* of Γ. If $\Gamma = \Omega$ is a set of operations, then Ω^{**} is called the *bicentralizer* of Γ. The symbol Ω° will denote the set of all relations preserved by every member of Ω. Remark that $\Omega^* = \Omega^\circ \cap O$ for any set of operations Ω.

Let Π be a set of relations of A, i.e., $\Pi \subseteqq \mathcal{R}$. If a relation belongs to Π, we shall call it a *Π-relation*. Let (A, Ω) be an algebra. By the *related structure of type Π* of (A, Ω) (in symbol: $\mathrm{Rel}_\Pi(A, \Omega)$) we mean the set of all Π-relation preserved by every operation of Ω, i.e., $\mathrm{Rel}_\Pi(A, \Omega) = \Omega^\circ \cap \Pi$. Observe that if Π_1 is the set of all n-ary relations of A, Π_2 is the set of all equivalences of A, Π_3 is the set of all unary operations of A, and Π_4 is the set of all bijective unary operations of A, then $\mathrm{Rel}_{\Pi_1}(A, \Omega) = \mathrm{Sub}\,((A, \Omega)^n)$, $\mathrm{Rel}_{\Pi_2}(A, \Omega) = \mathrm{Con}\,(A, \Omega)$, $\mathrm{Rel}_{\pi_3}(A, \Omega) = \mathrm{End}\,(A, \Omega)$ and $\mathrm{Rel}_{\Pi_4}(A, \Omega) = \mathrm{Aut}\,(A, \Omega)$.

Let $X = \{x_i \mid i \in I\}$ be a set of variables indexed by an arbitrary set I and let Γ be a set of relations of A. If R is a symbol of an n-ary relation in Γ and f, g are symbols of operations of arity m, s that denote a projection or an operation belonging to Γ, respectively, then $R(x_{i_1}, \ldots, x_{i_n})$ and $f(x_{j_1}, \ldots, x_{j_m}) = g(x_{t_1}, \ldots, x_{t_s})$ are said to be formulas of the variable set X over Γ provided $x_{i_1}, \ldots, x_{i_n}, x_{j_1}, \ldots, x_{j_m}, x_{t_1}, \ldots, x_{t_s} \in X$. (Note that we might have formulas of the first kind only, but introducing these two kinds of formulas our considerations became somewhat simpler.) We say that a family $(a_i \mid i \in I) \in A^I$ satisfies the above formulas if $R(a_{i_1}, \ldots, a_{i_n})$, resp. $f(a_{j_1}, \ldots, a_{j_m}) = g(a_{t_1}, \ldots, a_{t_s})$ holds. Consider a triple $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ where $X = \{x_i \mid i \in I\}$ is a set of variables indexed by I, $(x_{i_1}, \ldots, x_{i_n}) \in X^n$, and Σ is a set of formulas of variable set X over Γ. Such a triple will be referred to as a *formula scheme* over Γ. We say that Ψ is finite if both Σ and X are finite. If $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ $(X = \{x_i \mid i \in I\})$ is a formula scheme then we associate with Ψ an n-ary relation R_Ψ defined as follows: $R_\Psi = \{(a_{i_1}, \ldots, a_{i_n}) \mid (a_i \mid i \in I) \in A^I$ and $(a_i \mid i \in I)$ satisfies (every member of) $\Sigma\}$. Then we say that R_Ψ is defined by the formula scheme Ψ.

We say that a formula scheme $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}))$ $(X = \{x_i \mid i \in I\})$

defines the n-ary operation f on $B\subseteq A^n$ if for any $(a_1, \dots, a_n)\in B$, $f(a_1, \dots, a_n)= =a_{n+1}$ for some $a_{n+1}\in A$ if and only if $R_\Psi(a_1, \dots, a_n, a_{n+1})$ holds. For $B=A^n$ we say that Ψ defines f. An n-ary operation f is said to be locally definable by a set of relations Γ, if for every finite $B\subseteq A^n$ there exists a formula scheme over Γ defining f on B.

The following lemmas describe the connection between the notions "relations preserved by operations" and "relations defined by formula schemes".

Lemma 1. *Let Γ be a set of relations of A. If a relation R can be defined by a formula scheme over Γ, then $R\in\Gamma^{*0}$.*

Proof. Let $\Psi=(\Sigma, X, (x_{i_1}, \dots, x_{i_n}))$ $(X=\{x_i|i\in I\})$ be a formula scheme over Γ and let f be an m-ary operation preserving all members of Γ. If $R_\Psi=\emptyset$ then f preserves R_Ψ trivially, unless $m=0$. However if $m=0$, i.e., f is a nullary operation then $R(f, \dots, f)$ holds for every $R\in\Gamma$, whence it follows that Σ is satisfied by $(a_i|i\in I)$ where $a_i=f$ for all $i\in I$. Then $R_\Psi(f, \dots, f)$ holds, a contradiction.

Now suppose $R_\Psi\neq\emptyset$ and let $R_\Psi(a_1^k, \dots, a_n^k)$, $k=1, \dots, m$. Then there exist families $(b_i^k|i\in I)$ satisfying Σ such that $(a_1^k, \dots, a_n^k)=(b_{i_1}^k, \dots, b_{i_n}^k)$, $k=1, \dots, m$. Using the fact that f preserves all relations and commutes with all operations whose symbols occur in Σ, one can observe by routine that $\big(f(b_i^1, \dots, b_i^m)\,|\,i\in I\big)$ satisfies Σ. Hence it follows

$$\big(f(a_1^1, \dots, a_1^m), \dots, f(a_n^1, \dots, a_n^m)\big) = \big(f(b_{i_1}^1, \dots, b_{i_1}^m), \dots, f(b_{i_n}^1, \dots, b_{i_n}^m)\big)\in R_\Psi$$

showing that f preserves R_Ψ. Q.E.D.

Lemma 2. *Let Γ be a set of relations of A. Then for every positive integer n, every finitely generated subalgebra of the algebra $(A, \Gamma^*)^n$ can be defined by a formula scheme over Γ. Moreover, if A is a finite set, then we can choose these formula schemes to be finite.*

Proof. Let T be a finitely generated subalgebra of $(A, \Gamma^*)^n$. If $T=\emptyset$ then Γ^* has no nullary operation. Consider the set of formulas $\Sigma=\{R(x_1, \dots, x_1)|R\in\Gamma\}$. Then there is no element of A satisfying Σ. For if $a\in A$ satisfies Σ then we get $R(a, \dots, a)$ for all $R\in\Gamma$ which implies that $a\in\Gamma^*$, i.e., Γ^* has a nullary operation, a contradiction. Thus the formula scheme $\Psi=(\Sigma, \{x_1\}, (x_1))$ defines $T=\emptyset$, i.e., $R_\Psi=\emptyset=T$. Furthermore, as $R_\Psi=\emptyset$, i.e., there is no element of A satisfying Σ, for any $a\in A$ there is a formula $R_a(x_1, \dots, x_1)\in\Sigma$ such that $R_a(a, \dots, a)$ does not hold. Then the formula scheme $\Psi'=(\Sigma', \{x_1\}, (x_1))$ with $\Sigma'=\{R_a(x_1, \dots, x_1)|a\in A\}$ defines $T=\emptyset$, too. Moreover, if A is a finite set then Ψ' is a finite formula scheme.

Now suppose $T\neq\emptyset$ and the set $\{t_i=(t_{1i}, \dots, t_{ni})|t_i\in A^n, i=1, \dots, s\}$ generates T. Since Γ^* is a clone (i.e., it contains all projections and is closed under superposition), $T=\{f(t_1, \ldots, t_s)|f\in\Gamma^*\cap O_s\}$. We construct a formula scheme Ψ which defines T.

Let X be a set of variables indexed by A^s, i.e., $X=\{x_i|i\in A^s\}$. Consider an arbitrary relation Q from Γ. Let m be the arity of Q. Considering every element of Q as a column vector of length m, every element of Q^s is an $m\times s$ matrix of elements of A. With Q and any matrix $M\in Q^s$ we associate a formula $Q(x_{M_1}, \ldots, x_{M_m})$ of the variable set X, where M_k is the k-th row of M, $k=1, \ldots, m$. Now consider the formula scheme $\Psi=(\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ where $X=\{x_i|i\in A^s\}$, $\Sigma=\{Q(x_{M_1}, \ldots, x_{M_m})|Q\in\Gamma$ and $M\in Q^s\}$, and $(i_1, \ldots, i_n)=((t_{11}, \ldots, t_{1s}), \ldots, (t_{n1}, \ldots, t_{ns}))$. We show that T is defined by Ψ, i.e., $T=R_\Psi$. Clearly $R_\Psi=\{(a_{i_1}, \ldots, a_{i_n})|(a_i|i\in A^s)\in A^{A^s}$ and $(a_i|i\in A^s)$ satisfies $\Sigma\}$. Remark, however, that $A^{A^s}=O_s$, and thus we can write $f\in O_s$ instead of $(a_i|i\in A^s)\in A^{A^s}$. Using this notation we get

$$R_\Psi = \{(f(i_1), \ldots, f(i_n))|f\in O_s \text{ and } f \text{ satisfies } \Sigma\} =$$
$$= \{(f(t_{11}, \ldots, t_{1s}), \ldots, f(t_{n1}, \ldots, t_{ns}))|f\in O_s \text{ and } f \text{ satisfies } \Sigma\} =$$
$$= \{f(t_1, \ldots, t_s)|f\in O_s \text{ and } f \text{ satisfies } \Sigma\}.$$

Furthermore, an s-ary operation f satisfies Σ if and only if $f\in\Gamma^*$. To show this first suppose that $f\in O_s$ satisfies Σ. Let Q be an arbitrary m-ary relation from Γ, and let $q_j=(q_{1j}, \ldots, q_{mj})\in Q$, $j=1, \ldots, s$. Then from $M=(q_1, \ldots, q_s)\in Q^s$ we get $Q(x_{M_1}, \ldots, x_{M_m})\in\Sigma$, which implies $Q(f(M_1), \ldots, f(M_m))$, i.e., $Q(f(q_{11}, \ldots, q_{1s}), \ldots, f(q_{m1}, \ldots, q_{ms}))$ proving that f preserves Q. Hence $f\in\Gamma^*$. Conversely suppose that $f\in O_s\cap\Gamma^*$ and $Q(x_{j_1}, \ldots, x_{j_m})$ is an arbitrary formula from Σ, where $j_k=(j_{k1}, \ldots, j_{ks})$, $k=1, \ldots, m$. Then the matrix $(j_{kl})_{m\times s}$ is an element of Q^s, i.e., $(j_{1l}, \ldots, j_{ml})\in Q$, $l=1, \ldots, s$. Taking into account that f preserves Q we get that $Q(f(j_{11}, \ldots, j_{1s}), \ldots, f(j_{m1}, \ldots, j_{ms}))$, i.e., $Q(f(j_1), \ldots, f(j_m))$ proving that f satisfies the formula $Q(x_{j_1}, \ldots, x_{j_m})$. Hence f satisfies Σ. This implies $R_\Psi=(f(t_1, \ldots, t_s)|f\in\Gamma^*\cap O_s\}$, and the right side is the same as T.

Now let A be a finite set, and consider the formula scheme Ψ constructed above. For every s-ary operation f that does not satisfy Σ there exists a formula $\mathcal{T}_f\in\Sigma$ such that f does not satisfy $\mathcal{T}_f$. Consider the set of formulas $\Sigma'=\{\mathcal{T}_f|f\in O_s$ and f does not satisfy $\Sigma\}$. It is evident that an s-ary operation satisfies Σ if and only if it satisfies Σ'. Therefore, the formula scheme $\Psi'=(\Sigma', X, (x_{i_1}, \ldots, x_{i_n}))$ where X and $(x_{i_1}, \ldots, x_{i_n})$ are the same as above, defines the relation T. Namely,

$$T = R_\Psi = \{(f(i_1), \ldots, f(i_n))|f\in O_s \quad\text{and}\quad f \text{ satisfies } \Sigma\} =$$
$$= \{(f(i_1), \ldots, f(i_n))|f\in O_s \text{ and } f \text{ satisfies } \Sigma'\} = R_{\Psi'}.$$

Furthermore, from $|X|=|A^s|$ and $|\Sigma'|\leqq|O_s|=|A^{A^s}|$ it follows that X and Σ' are finite. Hence Ψ' is a finite formula scheme. Q.E.D.

Lemma 3. *If A is a finite set and a relation can be defined by a formula scheme over a set of relations Γ, then it can be defined by a finite formula scheme over Γ.*

Proof. Suppose an n-ary relation R can be defined by a formula scheme over Γ. From Lemma 1 it follows $R \in \mathrm{Sub}((A, \Gamma^*)^n)$. Applying Lemma 2 we get that R can be defined by a finite formula scheme over Γ. Q.E.D.

Lemma 4. *Let Γ be a set of relations of A. Then a relation R belongs to Γ^{*0} if and only if R is the union of a directed system of relations defined by formula schemes over Γ.*

Proof. First let $R = \bigcup_{i \in I} R_i$ where $(R_i | i \in I)$ is a directed system of relations defined by formula schemes over Γ. Therefore, by Lemma 1, we get that $R_i \in \Gamma^{*0}$, $i \in I$. Furthermore, one can see easily that the union of a directed system of elements of Γ^{*0} belongs to Γ^{*0}.

Now suppose that $R \in \Gamma^{*0}$ is an n-ary relation. Then R is a subalgebra of the algebra $(A, \Gamma^*)^n$. Therefore $R = \bigcup_{i \in I} R_i$ where $(R_i | i \in I)$ is the directed system of the finitely generated subalgebras of $(A, \Gamma^*)^n$ contained in R. In view of Lemma 2, we have that R_i, $i \in I$, can be defined by a formula scheme over Γ. Q.E.D.

Lemma 5. *Let Γ be a set of relations of A. Then an operation f belongs to Γ^{**} if and only if f can be defined by Γ locally.*

Proof. First suppose that f is an n-ary operation which is defined by Γ locally. Choose an m-ary operation g from Γ^* and let $M = (a_{kl})_{m \times n}$ be an $m \times n$ matrix of elements of A. According to our assumption, there is a formula scheme Ψ that defines f on

$$B = \{(a_{k1}, \ldots, a_{kn}) | k = 1, \ldots, m\} \cup \{(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn}))\}.$$

Then $R_\Psi(a_{k1}, \ldots, a_{kn}, f(a_{k1}, \ldots, a_{kn}))$ holds, $k = 1, \ldots, m$. Using Lemma 1 we get that $R_\Psi(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn}), g(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$ holds, too, whence

$$f(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn})) = g(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$$

follows, i.e., $f((M)g) = (f(M))g$. Hence f commutes with g showing that $f \in \Gamma^{**}$.

Now suppose that $f \in \Gamma^{**}$ is an n-ary operation and let $B \subseteq A^n$ be a finite set. Considering f as an $(n+1)$-ary relation we have $f \in \Gamma^{*0}$. Therefore, by Lemma 4, we get $f = \bigcup_{i \in I} R_i$ where $(R_i | i \in I)$ is a directed system of $((n+1)$-ary$)$ relations defined by formula schemes over Γ. As $(R_i | i \in I)$ is a directed system and B is a finite set, $f = \bigcup_{i \in I} R_i$ implies $f | B \subseteq R_{i_0}$ for some $i_0 \in I$. Now let Ψ be a formula scheme over Γ defining R_{i_0}. Then $f | B \subseteq R_{i_0} \subseteq f$ implies

$$f | B = \{(a_1, \ldots, a_n, a_{n+1}) | (a_1, \ldots, a_n) \in B \quad \text{and} \quad (a_1, \ldots, a_n, a_{n+1}) \in R_{i_0} = R_\Psi\}$$

and this means exactly that Ψ defines f on B. Q.E.D.

Theorem 6. *Let $\Gamma_i \subseteq \Pi_i (\subseteq \mathscr{R})$, $i \in I$, be sets of relations of A; furthermore, let $\Omega_j \subseteq \Pi_j (\subseteq \mathscr{R})$, $j \in J$, be sets of such relations which are operations of A. Put $\Gamma = (\bigcup_{i \in I} \Gamma_i) \cup (\bigcup_{j \in J} \Omega_j)$. Then the following two statements are equivalent:*

I. *There exists an algebra (A, Ω) such that $\Gamma_i = \mathrm{Rel}_{\Pi_i}(A, \Omega)$ and $\Omega_j = \mathrm{Rel}_{\Pi_j}(A, \Omega)$ for every $i \in I$ and $j \in J$.*

II. (α) *For every $i \in I$, if a Π_i-relation is the union of a directed system of relations defined by formula schemes over Γ, then it belongs to Γ_i.*

(β) *For every $j \in J$, if a Π_j-relation (operation) can be defined by Γ locally then it belongs to Ω_j.*

Proof. I$\Rightarrow$II. Suppose that $\Gamma_i = \mathrm{Rel}_{\Pi_i}(A, \Omega)$ and $\Omega_j = \mathrm{Rel}_{\Pi_j}(A, \Omega)$ for some algebra (A, Ω) for every $i \in I$ and $j \in J$. First let $i_0 \in I$ and suppose a Π_{i_0}-relation R to be the union of a directed system of relations defined by formula schemes over Γ. Taking into account Lemma 4 and $\Gamma^* \supseteq \Omega$ we have that $R \in \Gamma^{*0} \subseteq \Omega^0$. This fact together with R being a Π_{i_0}-relation shows that $R \in \mathrm{Rel}_{\Pi_{i_0}}(A, \Omega)$. Hence ($\alpha$) holds.

Now let $j_0 \in J$ and suppose a Π_{j_0}-operation f can be defined by Γ locally. Then, by Lemma 5, we have $f \in \Gamma^{**} \subseteq \Omega^* \subseteq \Omega^0$. Hence $f \in \mathrm{Rel}_{\Pi_{j_0}}(A, \Omega)$, i.e., ($\beta$) holds.

II$\Rightarrow$I. Let $\Omega = \Gamma^*$. We shall prove that $\Gamma_i = \mathrm{Rel}_{\Pi_i}(A, \Omega)$ and $\Omega_j = \mathrm{Rel}_{\Pi_j}(A, \Omega)$ for every $i \in I$ and $j \in J$. First choose an arbitrary $i_0 \in I$. The inclusion $\Gamma_{i_0} \subseteq \mathrm{Rel}_{\Pi_{i_0}}(A, \Omega)$ is obvious. Let $R \in \mathrm{Rel}_{\Pi_{i_0}}(A, \Omega)$. Then $R \in \Omega^0 = \Gamma^{*0}$. Therefore, by Lemma 4, we have that R is the union of a directed system of relations defined by formula schemes over Γ. Thus, by the condition (α), $R \in \Gamma_{j_0}$.

Now choose an arbitrary $j_0 \in J$. Again, $\Omega_{j_0} \subseteq \mathrm{Rel}_{\Pi_{j_0}}(A, \Omega)$ is obvious. Let $f \in \mathrm{Rel}_{\Pi_{j_0}}(A, \Omega)$ be a Π_{j_0}-operation. Then $f \in \Omega^* = \Gamma^{**}$. Therefore, by Lemma 5, we get that f can be defined by Γ locally. Thus, by the condition (β), $f \in \Omega_{j_0}$. Q.E.D.

Theorem 7. *Let $(\Gamma_n | n = 1, 2, \ldots)$ be a family of sets of relations of A such that Γ_n has n-ary relations only, $n = 1, 2, \ldots$. Then the following two statements are equivalent:*

I. *There exists an algebra (A, Ω) such that $\Gamma_n = \mathrm{Sub}((A, \Omega)^n)$, $n = 1, 2, \ldots$.*

II. (α) *For every n, if an n-ary relation can be defined by a formula scheme over $\bigcup_{k=1}^{\infty} \Gamma_k$ then it belongs to Γ_n.*

(β) *For every n, Γ_n is closed under union of directed systems.*

Proof. Put $I = \{1, 2, \ldots\}$, $J = \emptyset$ and, as Π_n, the set of all n-ary relations of A in Theorem 6.

Corollary 8. *If A is a finite set then statement II in Theorem 6 can be replaced by*

II'. *For every n, if an n-ary relation can be defined by a finite formula scheme over* $\bigcup_{k=1}^{\infty} \Gamma_k$ *then it belongs to* Γ_n.

Proof. As A is a finite set, the assumption (β) in Theorem 6 is superfluous and we can apply Lemma 3.

Theorem 9. *Let Γ be a set of n-ary relations of A. Then there exists an algebra* (A, Ω) *such that* $\Gamma=\mathrm{Sub}\left((A, \Omega)^n\right)$ *if and only if Γ is closed under union of directed systems and Γ contains every n-ary relation defined by a formula scheme over Γ.*

Proof. Put $I=\{1\}$, $\Gamma_1=\Gamma$, $J=\emptyset$ and, as Π_1, the set of all n-ary relations of A in Theorem 6.

Corollary 10. *Let A be finite and let Γ be a set of n-ary relations of A. Then there exists an algebra* (A, Ω) *such that* $\Gamma=\mathrm{Sub}\left((A, \Omega)^n\right)$ *if and only if Γ contains every n-ary relation defined by a finite formula scheme over Γ.*

Corollary 11. (J. Schmidt) *For a set Γ of unary relations of A, there is an algebra* (A, Ω) *such that* $\Gamma=\mathrm{Sub}\,(A, \Omega)$ *if and only if Γ is an algebraic closure system.*

Proof. Suppose that $\Gamma=\mathrm{Sub}\,(A, \Omega)$ for some algebra (A, Ω). Let $\{R_j \mid j\in J\}$ be a subset of Γ. Then the formula scheme $\left(\Sigma, \{x_1\}, (x_1)\right)$ with $\Sigma=\{R_j(x_1) \mid j\in J\}$ defines $\bigcap_{j\in J} R_j$. Applying Theorem 9, we get that $\bigcap_{j\in J} R_j\in\Gamma$, i.e., Γ is closed under intersections. This fact together with the conditions of Theorem 9 proves that Γ is an algebraic closure system.

Conversely, suppose that Γ is an algebraic closure system. Then Γ is closed under union of directed systems. Now consider a formula scheme $\Psi=(\Sigma, X, (x_1))$ $(X=\{x_i \mid i\in I\})$ over Γ. If $R_\Psi=\emptyset$ then $R_\Psi=\emptyset=\bigcap_{R\in\Gamma} R$. Otherwise, $a\in\bigcap_{R\in\Gamma} R$ implies that $(a_i \mid i\in I)$ where $a_i=a$ for all $i\in I$, satisfies Σ showing $R_\Psi(a)$, a contradiction. Thus $R_\Psi=\emptyset\in\Gamma$. If $R_\Psi\neq\emptyset$, then it is a routine to check that $R_\Psi=\bigcap_{R(x_1)\in\Sigma} R$, i.e., $R_\Psi\in\Gamma$. Thus we get that Γ satisfies the condition of Theorem 9. Q.E.D.

In [1], Kalužnin and his co-workers have given a characterization for the subalgebra system $\bigcup_{n=1}^{\infty}\mathrm{Sub}\left((A, \Omega)^n\right)$ of a finite algebra (A, Ω). Now we derive their result from Corollary 8. We need some additional notions and notations.

For an m-ary relation R of A and a permutation τ of the set $\{1, \dots, m\}$ the *τ-translate* of R is an m-ary relation R^τ of A defined by $R^\tau=\{a_{1\tau}, \dots, a_{m\tau}) \mid R(a_1, \dots, a_m)\}$. For any two relations R and T of arity m and n, respectively, the *direct product* of R and T is an $(m+n)$-ary relation $R\times T$ defined by $R\times T=\{(a_1, \dots, a_{m+n}) \mid R(a_1, \dots, a_m)$ and $T(a_{m+1}, \dots, a_{m+n})\}$. If R is an m-ary relation and $1\leqq i_1<\dots<i_t\leqq m$, then

the *projection* of R to the coordinates $i_1, \ldots, i_t$ is a t-ary relation $R_{i_1, \ldots, i_t}$ defined by $R_{i_1, \ldots, i_t} = \{(a_{i_1}, \ldots, a_{i_t}) | R(a_1, \ldots, a_m)\}$. If R is an m-ary relation and Θ is an equivalence relation of the set $\{1, \ldots, m\}$, then the *Θ-diagonal* of R is an m-ary relation R_Θ defined by $R_\Theta = \{(a_1, \ldots, a_m) | R(a_1, \ldots, a_m) \text{ and } (i\Theta j \Rightarrow a_i = a_j)\}$. Finally, the *$n$-ary diagonal* D_n is defined by $D_n = \{(a, \ldots, a) | a \in A\}$ for any n.

Corollary 12. (V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, V. A. Romov) *If A is a finite set and Γ is a set of relations of A then there exists an algebra (A, Ω) such that $\Gamma = \bigcup_{n=1}^{\infty} \operatorname{Sub}((A, \Omega)^n)$ if and only if all diagonals belong to Γ, and Γ is closed under formation of direct products, as well as arbitrary τ-translates, projections, and Θ-diagonals.*

Proof. By Corollary 8 we have to prove only that a set of relations Γ fulfils the assumptions of the corollary if and only if every relation defined by a finite formula scheme over Γ belongs to Γ.

First suppose that all relations defined by finite formula schemes belong to Γ. Then for any n the formula scheme $(\emptyset, \{x_1\}, (x_1, \ldots, x_1))$ defines D_n. If R and T are relations from Γ of arity m and n, respectively, τ is a permutation and Θ is an equivalence relation of the set $\{1, \ldots, m\}$ and $1 \leqq i_1 < \ldots < i_t \leqq m$, then the formula schemes

$$(\{R(x_1, \ldots, x_m), T(x_{m+1}, \ldots, x_{m+n})\}, \{x_1, \ldots, x_{m+n}\}, (x_1, \ldots, x_{m+n})),$$

$$(\{R(x_1, \ldots, x_m)\}, \{x_1, \ldots, x_m\}, (x_{1\tau}, \ldots, x_{m\tau})),$$

$$(\{R(x_1, \ldots, x_m)\}, \{x_1, \ldots, x_m\}, (x_{i_1}, \ldots, x_{i_t})),$$

and

$$(\{R(x_1, \ldots, x_m)\} \cup \{D_2(x_k, x_l) | k\Theta l\}, \{x_1, \ldots, x_m\}, (x_1, \ldots, x_m))$$

define $R \times T$, R^τ, $R_{i_1, \ldots, i_t}$ and R_Θ, respectively.

Conversely, suppose that Γ satisfies the assumptions of the corollary and let $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ $(X = \{x_i | i \in I\})$ be a finite formula scheme over Γ. We have to prove that R_Ψ can be got from Γ in a finite number of steps by formation of directed products, τ-translates, projections, and Θ-diagonals. Concerning Ψ, we can assume w.l.o.g. that every component of $(x_{i_1}, \ldots, x_{i_n})$ occurs in some formula of Σ, otherwise we can add the formulas $D_2(x_{i_1}, x_{i_1}), \ldots, D_2(x_{i_n}, x_{i_n})$ to Σ. Furthermore, we can assume that $(x_{i_1}, \ldots, x_{i_n})$ has pairwise distinct components, otherwise we can consider the formula scheme $\Psi = (\Sigma', X', (y_1, \ldots, y_n))$ where $X' = X \cup \{y_1, \ldots, y_n\} (X \cap \{y_1, \ldots, y_n\} \neq \emptyset)$ and $\Sigma' = \Sigma \cup \{D_2(x_{i_1}, y_1), \ldots, D_2(x_{i_n}, y_n)\}$. Clearly $R_\Psi = R_{\Psi'}$. Finally, we can also assume that Σ has formulas of the form $R(x_{j_1}, \ldots, x_{j_m})$ $(R \in \Gamma)$ only. Otherwise, if a formula ε of the form $f(x_{t_1}, \ldots, x_{t_s}) = g(x_{k_1}, \ldots, x_{k_r})$ belongs to Σ, then replace ε by the formulas

$f(x_{t_1}, \dots, x_{t_s}) = y_\varepsilon$ and $g(x_{k_1}, \dots, x_{k_r}) = y_\varepsilon$. Considering f and g as $(s+1)$-ary and $(r+1)$-ary relations, respectively, these formulas have the form we required. Thus we get a set of formulas Σ''. Then the formula scheme $\Psi'' = (\Sigma'', X'', (x_{i_1}, \dots, x_{i_n}))$ with $X'' = X \cup \{y_\varepsilon \mid \varepsilon \in \Sigma$ and ε is of the form $f=g\}$ defines R_{Ψ}.

Now suppose that Ψ has these properties. Then let

$$\Sigma = \{R_1(y_1^1, \dots, y_{n_1}^1), \dots, R_s(y_1^s, \dots, y_{n_s}^s)\}, \quad y_k^l \in X, \quad l = 1, \dots, s, \quad k = 1, \dots, n_l.$$

Consider the formula scheme $\Phi = (\Sigma, X, (y_1^1, \dots, y_{n_1}^1, \dots, y_1^s, \dots, y_{n_s}^s))$. Observe that R_Ψ can be got from R_Φ by formation of a suitable projection and τ-translate. Furthermore, let Θ be an equivalence of the set $\left\{1, \dots, \sum_{k=1}^{s} n_k\right\}$ defined as follows: $j\Theta l$ if and only if the j-th and l-th components of $(y_1^1, \dots, y_{n_1}^1, \dots, y_1^s, \dots, y_{n_s}^s)$ are equal, $j, l = 1, \dots, \sum_{k=1}^{s} n_k$. Now it is a routine to verify that R_Φ equals the Θ-diagonal of $R_1 \times \dots \times R_s$. Q.E.D.

Theorem 13. *If Ω is a set of operations of A, then $\Omega = \Omega^{**}$ if and only if Ω contains every operation defined by Ω locally.*

It follows from Lemma 5 immediately.

Corollary 14. (A. V. Kuznecov) *If A is a finite set, then $\Omega = \Omega^{**}$ for some set of operations Ω if and only if every operation defined by a finite formula scheme over Ω belongs to Ω.*

Proof. If A is a finite set, an operation f locally definable by Ω can be defined by a formula scheme over Ω. Lemma 3 shows that we can restrict ourselves to finite formulas. It remains to apply Theorem 13.

Theorem 15. *For a set E of transformations of A there exists an algebra (A, Ω) such that $E = \mathrm{End}\,(A, \Omega)$ if and only if E contains every transformation defined by E locally.*

Proof. Put $I = \emptyset$, $J = \{1\}$, $\Omega_1 = E$ and, as Π_1, the set of all unary operations in Theorem 6.

Corollary 16. *If A is a finite set, then for a set E of transformations of A there exists an algebra (A, Ω) such that $E = \mathrm{End}\,(A, \Omega)$ if and only if E contains every transformation defined by a finite formula scheme over E.*

Proof. We can proceed similarly as it was done in the proof of Corollary 12.

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A quasimilarity model for algebraic operators

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In this note, all Hilbert spaces $\mathfrak{H}$ will be understood to be complex. We denote by $\mathcal{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. For A in $\mathcal{L}(\mathfrak{H})$, $\sigma(A)$ denotes the spectrum of A, $\mathcal{K}(A)$ the kernel of A, and $\mathcal{R}(A)$ the range of A. An operator A in $\mathcal{L}(\mathfrak{H})$ is said to be *algebraic* if there exists a nonzero polynomial $p(z)$ with complex coefficients such that $p(A)=0$. If $A^m=0$ for some positive integer m, then we say that A is *nilpotent*. If n is a positive integer, then the nilpotent operator acting on the direct sum of n copies of $\mathfrak{H}$ and defined by the $n\times n$ matrix $[A_{ij}]$, where

$$A_{i,i+1} = 1_{\mathfrak{H}} \quad \text{for} \quad i = 1, 2, \dots, n-1, \quad \text{and} \quad A_{ij}=0 \text{ for all other entries,}$$

is called a *Jordan block* operator of order n. (By definition, the zero operator on $\mathfrak{H}$ is a Jordan block operator of order one.) Suppose that $\mathfrak{H}_1, \dots, \mathfrak{H}_m$ are Hilbert spaces and $n_1, \dots, n_m$ are positive integers. Let $\tilde{\mathfrak{H}}_k$ be the direct sum of n_k copies of $\mathfrak{H}_k$ and J_k be the Jordan block operator of order n_k acting on $\tilde{\mathfrak{H}}_k$, $k=1, 2, \dots, m$. An operator of the form $J_1\oplus\dots\oplus J_m$ acting on $\tilde{\mathfrak{H}}_1\oplus\dots\oplus\tilde{\mathfrak{H}}_m$ is called a *Jordan operator*.

We recall that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are Hilbert spaces and $X: \mathcal{K}_1\to\mathcal{K}_2$ is a bounded linear transformation such that $\mathcal{K}(X)=\mathcal{K}(X^*)=\{0\}$, then X is called a *quasiaffinity*. If $A_1\in\mathcal{L}(\mathcal{K}_1)$ and $A_2\in\mathcal{L}(\mathcal{K}_2)$ and there exist quasiaffinities $X: \mathcal{K}_1\to\mathcal{K}_2$ and $Y: \mathcal{K}_2\to\mathcal{K}_1$ such that $XA_1=A_2X$ and $A_1Y=YA_2$, then A_1 and A_2 are said to be *quasisimilar*. In case that there exists an invertible bounded linear transformation $Z: \mathcal{K}_1\to\mathcal{K}_2$ such that $ZA_1=A_2Z$, then A_1 and A_2 are said to be *similar*.

It is well-known that every operator on a finite dimensional Hilbert space is algebraic and similar to its Jordan canonical form. Hence it is natural to ask

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whether there exists an analogous model for the class of algebraic operators on an infinite dimensional Hilbert space. APOSTOL, DOUGLAS, and FOIAŞ proved in [1] that every nilpotent operator on a Hilbert space is quasisimilar to a Jordan operator. (This author provided a different proof of this theorem in [2].) The first purpose of this note is to show that there exists such a model for the class of algebraic operators also.

Necessary and sufficient conditions that a nilpotent operator is similar to a Jordan operator were also presented in [2]. We proved that a nilpotent operator A is similar to a Jordan operator if and only if the range of A^i is closed, $i=1, 2, \ldots$. The following theorem generalizes this result also.

Theorem. (a) *Suppose that A is an algebraic operator on a Hilbert space $\mathfrak{H}$ and $\sigma(A)=\{\lambda_1, \ldots, \lambda_n\}$. Then there exist Jordan operators $J_1, \ldots, J_n$ acting on Hilbert spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$, respectively, such that A is quasisimilar to $B=\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$.*

(b) *A is similar to B if and only if the range of $(A-\lambda_j)^i$ is closed ($i=1, 2, \ldots$, $j=1, 2, \ldots, n$).*

Note that as a result of the spectral mapping theorem, the spectrum of every algebraic operator is a finite set. Thus the operator A in the Theorem is the most general algebraic operator. (Of course, in the Theorem and throughout this note, we assume that if $i\neq j$, then $\lambda_i\neq\lambda_j$.)

We begin with the following lemma.

Lemma 1. *Suppose that A is an algebraic operator on a Hilbert space $\mathfrak{H}$, say $p(A)=0$, and let $\sigma(A)=\{\lambda_1, \ldots, \lambda_n\}$. Then there are operators A_k with $p(A_k)=0$ and $\sigma(A_k)=\{\lambda_k\}$ ($k=1, 2, \ldots, n$), such that A is similar to $A_1\oplus\ldots\oplus A_n$.*

Proof. We prove the lemma by induction on the number of points n in $\sigma(A)$. If $n=1$, the lemma is obviously true. Suppose that $n>1$ and that the lemma is true for every algebraic operator which has $n-1$ points in its spectrum. Let f_1 be an analytic function which is identically one in a neighborhood of $\{\lambda_1, \ldots, \lambda_{n-1}\}$ and identically zero in a neighborhood of $\{\lambda_n\}$. Let $f_2(z)=1-f_1(z)$ for each z where f_1 is defined. The idempotent operators $f_1(A)$ and $f_2(A)$ are defined by the Riesz functional calculus. Let $\mathfrak{M}=\mathfrak{R}(f_1(A))$ and $\mathfrak{N}=\mathfrak{R}(f_2(A))$. According to the theory of the Riesz functional calculus, $\mathfrak{M}$ and $\mathfrak{N}$ are hyperinvariant subspaces for A, $\sigma(A|\mathfrak{M})=\{\lambda_1, \ldots, \lambda_{n-1}\}$, and $\sigma(A|\mathfrak{N})=\{\lambda_n\}$. The matrices of A, $f_1(A)$, and $f_2(A)$ with respect to the decomposition $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{M}^\perp$ are respectively

$$\begin{bmatrix} A_0 & B \\ 0 & C \end{bmatrix}, \quad \begin{bmatrix} 1_{\mathfrak{M}} & D \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & -D \\ 0 & 1_{\mathfrak{M}^\perp} \end{bmatrix},$$

where $A_0=A|\mathfrak{M}$. Since the operators A and $f_1(A)$ commute, we have $A_0D=$ $=B+DC$. Now we have

$$\begin{bmatrix} 1_{\mathfrak{M}} & D \\ 0 & 1_{\mathfrak{M}^\perp} \end{bmatrix} \begin{bmatrix} A_0 & B \\ 0 & C \end{bmatrix} \begin{bmatrix} 1_{\mathfrak{M}} & -D \\ 0 & 1_{\mathfrak{M}^\perp} \end{bmatrix} = \begin{bmatrix} A_0 & -A_0D+B+DC \\ 0 & C \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & C \end{bmatrix}.$$

Thus A is similar to $A_0\oplus C$. It follows that A_0 and C are algebraic operators; indeed $p(A_0)=p(C)=0$.

We now show that C is similar to $A|\mathfrak{N}$, and thus $\sigma(C)=\sigma(A|\mathfrak{N})=\{\lambda_n\}$. Indeed, from the matrix of $f_2(A)$ we have $\mathfrak{N}=\left\{\begin{bmatrix} -Dy \\ y \end{bmatrix}\in\mathfrak{M}\oplus\mathfrak{M}^\perp : y\in\mathfrak{M}^\perp\right\}$. Define a linear transformation $S\colon \mathfrak{M}^\perp\to\mathfrak{N}$ by setting $Sy=\begin{bmatrix} -Dy \\ y \end{bmatrix}$ for each y in $\mathfrak{M}^\perp$. Then S is invertible and

$$(A|\mathfrak{N})Sy = \begin{bmatrix} A_0 & B \\ 0 & C \end{bmatrix} \begin{bmatrix} -Dy \\ y \end{bmatrix} = \begin{bmatrix} (-A_0D+B)y \\ Cy \end{bmatrix} = \begin{bmatrix} -DCy \\ Cy \end{bmatrix} = SCy$$

for each y in $\mathfrak{M}^\perp$. Hence $(A|\mathfrak{N})S=SC$, and thus C is similar to $A|\mathfrak{N}$.

We observe that $p(A_0)=0$ and $\sigma(A_0)=\{\lambda_1, \dots, \lambda_{n-1}\}$. By the induction hypothesis, there exist operators A_k with $p(A_k)=0$ and $\sigma(A_k)=\{\lambda_k\}$, $k=1, 2, \dots, n-1$, such that A_0 is similar to $A_1\oplus\dots\oplus A_{n-1}$. Hence A is similar to $A_1\oplus\dots\oplus A_n$ where $A_n=C$. The proof is complete since $p(A_n)=0$ and $\sigma(A_n)=\{\lambda_n\}$.

Lemma 2. *Suppose that A is an algebraic operator on $\mathfrak{H}$ and $\sigma(A)=\{\lambda\}$. Then there exists a Jordan operator J acting on a Hilbert space $\mathfrak{H}_0$ such that A is quasisimilar to $\lambda 1_{\mathfrak{H}_0}+J$.*

Proof. Apply Theorem 1 of [2] to the operator $T=A-\lambda$ to get that T is quasisimilar to a Jordan operator J acting on a Hilbert space $\mathfrak{H}_0$. Hence $A=\lambda+T$ is quasisimilar to $\lambda 1_{\mathfrak{H}_0}+J$.

Proof of the Theorem. (a) This follows immediately from Lemma 1 and Lemma 2.

(b) Suppose that there exist Jordan operators $J_1, \dots, J_n$ acting on Hilbert spaces $\mathfrak{H}_1, \dots, \mathfrak{H}_n$, respectively, such that A is similar to $\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$. Then, for positive integers i and j, $1\leqq j\leqq n$, the operator $(A-\lambda_j)^i$ is similar to $\sum_{k=1}^{n}\oplus((\lambda_k-\lambda_j)1_{\mathfrak{H}_k}+J_k)^i$, which has closed range. Thus the range of $(A-\lambda_j)^i$ is also closed. On the other hand, suppose that the range of each $(A-\lambda_j)^i$ is closed. According to Lemma 1, there exist algebraic operators A_k with $\sigma(A_k)=\{\lambda_k\}$ such that A is similar to $\sum_{k=1}^{n}\oplus A_k$; and hence $(A-\lambda_j)^i$ is similar to $\sum_{k=1}^{n}\oplus(A_k-\lambda_j)^i$. So for each positive integer i and for each integer k, $1\leqq k\leqq n$, the range of $(A_k-\lambda_j)^i$

is closed. In particular, the operator $A_j-\lambda_j$ is nilpotent and the range of $(A_j-\lambda_j)^i$ is closed. Thus, by Theorem 2 of [2], there exists a Jordan operator J_j acting on a Hilbert space $\mathfrak{H}_j$ such that $A_j-\lambda_j$ is similar to J_j. Hence it follows that A_j is similar to $\lambda_j 1_{\mathfrak{H}_j}+J_j$, $j=1, 2, \ldots, n$. Thus $\sum_{k=1}^{n}\oplus A_k$ is similar to $\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$. Therefore, A is similar to $\sum_{k=1}^{n}\oplus(\lambda_k 1_{\mathfrak{H}_k}+J_k)$.

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Jordan model for weak contractions

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SZ.-NAGY and FOIAŞ defined in [10] a class of multiplicity-free operators among C_0 contractions (also *cf.* [8]). Later on in [1] they developed a "Jordan model" for C_0 contractions, which resembles in some respects the usual canonical model of a finite matrix. In the present paper we extend both concepts from the context of C_0 contractions to that of weak contractions.

1. Preliminaries. Let T be a contraction defined on a complex, separable Hilbert space H. Denote by $d_T=\operatorname{rank}(I-T^*T)^{1/2}$, $d_{T^*}=\operatorname{rank}(I-TT^*)^{1/2}$ the defect indices of T.

Recall that T is called a weak contraction if (i) its spectrum $\sigma(T)$ does not fill the open unit disk D, and (ii) $I-T^*T$ is of finite trace. Thus in particular $C_0(N)$ contractions and C_{11} contractions with finite defect indices are weak contractions. For the theory of $C_0(N)$ contractions and C_{11} contractions, we refer the reader to [9]. If T is a completely non-unitary (c.n.u.) weak contraction on H, then $d_T=d_{T^*}$ and we can consider its C_0-C_{11} decomposition. Let H_0 and H_1 be the invariant subspaces for T such that $T_0\equiv T|H_0$ and $T_1\equiv T|H_1$ are the C_0 part and C_{11} part of T. Note that T_0 and T_1 are the operators appearing in the triangulations

$$T=\begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix} \quad \text{and} \quad T=\begin{bmatrix} T_1 & Y \\ 0 & T_0' \end{bmatrix}$$

of type

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix},$$

respectively. These triangulations, in term, correspond to the *-canonical factorization and canonical factorization

$$\Theta(\lambda)=\Theta_{*e}(\lambda)\Theta_{*i}(\lambda), \quad \Theta(\lambda)=\Theta_i(\lambda)\Theta_e(\lambda) \quad (\lambda\in D)$$

of the characteristic function $\Theta(\lambda)$ of T, *cf.* [9], Chap. VIII.

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Let H^2 denote the Hardy space of analytic functions on D. For each inner function φ, S_φ denotes the operator on $H^2\ominus\varphi H^2$ defined by $(S_\varphi f)(\lambda)=P(\lambda f(\lambda))$ for $\lambda\in D$, where P denotes the (orthogonal) projection of H^2 onto $H^2\ominus\varphi H^2$. For inner functions φ_1 and φ_2, $\varphi_1=\varphi_2$ means that φ_1 and φ_2 differ by a constant factor of modulus one; $\varphi_1|\varphi_2$ means that φ_1 is a divisor of φ_2. $H^2\ominus\varphi H^2$ reduces to $\{0\}$ if and only if φ is a constant inner function. For a measurable subset E of the unit circle C, M_E denotes the operator of multiplication by e^{it} on the space $L^2(E)$ of square-integrable functions on E, where the measure considered is the (normalized) Lebesgue measure. For measurable subsets E_1 and E_2 of C, $E_1=E_2$ means that $(E_1\setminus E_2)\cup(E_2\setminus E_1)$ is of Lebesgue measure zero. If $E=\emptyset$ then $L^2(E)$ reduces to $\{0\}$.

For arbitrary operators T_1, T_2 on H_1, H_2, respectively, $T_1\prec T_2$ denotes that T_1 is a quasi-affine transform of T_2, that is, there exists a one-to-one, continuous linear transformation X from H_1 onto a dense linear manifold of H_2 (called quasi-affinity) such that $XT_1=T_2X$. T_1 and T_2 are quasi-similar if $T_1\prec T_2$ and $T_2\prec T_1$. For an arbitrary operator T on H, let μ_T denote the multiplicity of T, that is, the least cardinal number of a subset K of vectors in H for which $H=\bigvee\limits_{n=0}^{\infty} T^nK$. In particular, if $\mu_T=1$ then T is cyclic and the vector in K is a cyclic vector for T. Note that both S and M_E are cyclic and that quasi-similar operators have equal multiplicities.

2. Jordan model. The following theorem, gives the Jordan model for C_0 contractions (*cf.* [1] and [10]).

Theorem 1. *Let T be a C_0 contraction on a separable Hilbert space, with defect indices $d_T=d_{T^*}$. Then T is quasi-similar to a uniquely determined operator of the form*

$$S_{\varphi_1}\oplus S_{\varphi_2}\oplus\ldots\oplus S_{\varphi_m}\oplus\ldots,$$

where the φ_j's are inner functions satisfying $\varphi_{j+1}|\varphi_j$ $(j=1,2,\ldots)$. Moreover, φ_1 is the minimal function of T, and if there are just m $(0\leqq m\leqq\infty)$ non-constant φ_j's, then $m=\mu_T=\mu_{T^*}\leqq d_T=d_{T^*}$.

Next we consider C_{11} contractions. In this case a "Jordan model" can also be given.

Theorem 2. *Let T be a c.n.u. C_{11} contraction on a separable Hilbert space, with defect indices $d_T=d_{T^*}$. Then T is quasi-similar to a uniquely determined operator of the form*

$$M_{E_1}\oplus M_{E_2}\oplus\ldots\oplus M_{E_n}\oplus\ldots,\tag{1}$$

where the E_k's are measurable subsets of C satisfying $E_{k+1} \subseteqq E_k$ $(k=1, 2, \ldots)$. If there are just n $(0 \leqq n \leqq \infty)$ E_k's with nonzero measure, then $n=\mu_T=\mu_{T^*} \leqq \leqq d_T = d_{T^*}$.

We start the proof with the following

Lemma 1. *Let T_1 and T_2 be operators on H_1 and H_2, respectively. Then* $\max\{\mu_{T_1}, \mu_{T_2}\} \leqq \mu_{T_1 \oplus T_2} \leqq \mu_{T_1} + \mu_{T_2}$.

Proof. Let $K=\{x_\alpha \oplus y_\alpha\}_{\alpha \in A}$ be a subset of vectors in $H_1 \oplus H_2$ such that $H_1 \oplus H_2 = \bigvee_{n=0}^{\infty} (T_1 \oplus T_2)^n K$. Then $K_1 \equiv \{x_\alpha\}_{\alpha \in A}$ is a subset of H_1 satisfying $H_1 = \bigvee_{n=0}^{\infty} T_1^n K_1$. It follows that $\mu_{T_1} \leqq \mu_{T_1 \oplus T_2}$. By symmetry we have $\mu_{T_2} \leqq \mu_{T_1 \oplus T_2}$, and hence $\max\{\mu_{T_1}, \mu_{T_2}\} \leqq \mu_{T_1 \oplus T_2}$.

To prove the second inequality, let $K_1 = \{x_\alpha\}_{\alpha \in A} \subseteqq H_1$ and $K_2 = \{y_\beta\}_{\beta \in \Omega} \subseteqq H_2$ be such that $H_1 = \bigvee_{n=0}^{\infty} T_1^n K_1$ and $H_2 = \bigvee_{n=0}^{\infty} T_2^n K_2$, respectively. Then $K = \{x_\alpha \oplus 0, 0 \oplus y_\beta\}_{\alpha \in A, \beta \in \Omega}$ is a subset of $H_1 \oplus H_2$ satisfying $H_1 \oplus H_2 = \bigvee_{n=0}^{\infty} (T_1 \oplus T_2)^n K$. It follows that $\mu_{T_1 \oplus T_2} \leqq \mu_{T_1} + \mu_{T_2}$.

Note that the inequalities in Lemma 1 actually occur. For example, if $T_1 = T_2$ is a simple unilateral shift then $\mu_{T_1} = \mu_{T_2} = 1$ and $\mu_{T_1 \oplus T_2} = 2$ (*cf.* [15], p. 308); if $T_1 = T_2$ is the adjoint of a simple unilateral shift then $\mu_{T_1} = \mu_{T_2} = 1$ and $\mu_{T_1 \oplus T_2} = 1$ (*cf.* [6], Problem 126).

Lemma 2. *If there are just n $(0 \leqq n \leqq \infty)$ E_k's with nonzero measure in the operator $T = M_{E_1} \oplus M_{E_2} \oplus \ldots \oplus M_{E_n} \oplus \ldots$, where $E_{k+1} \subseteqq E_k$ $(k=1, 2, \ldots)$, then $n = \mu_T = \mu_{T^*}$.*

Proof. By the first inequality in Lemma 1, it suffices to consider the case $n < \infty$, that is, we have to show that if $T = M_{E_1} \oplus \ldots \oplus M_{E_n}$, $n < \infty$, then $n = \mu_T$. Inequality $\mu_T \leqq n$ is obvious. To prove that $\mu_T \geqq n$, let us make use of the direct integral representation of the Hilbert-space $H = L^2(E_1) \oplus \ldots \oplus L^2(E_n)$, associated with the unitary operator T, that is, let

$$H = \int_{\sigma(T)}^{\oplus} H_\lambda \, dm \quad \text{with} \quad T^{*k} T^h \{x(\lambda)\} = \{\bar{\lambda}^k \lambda^h x(\lambda)\},$$

where m denotes the Lebesgue measure on $\sigma(T) \subseteqq C$. Let $N = \mu_T$. If $K = \{x_1, \ldots, x_N\}$ satisfies $H = \bigvee_{m=0}^{\infty} T^m K$ then $K_\lambda = \{x_1(\lambda), \ldots, x_N(\lambda)\}$ is a set of vectors in H_λ such that $H_\lambda = \bigvee_{m=0}^{\infty} \{\lambda^m x_1(\lambda), \ldots, \lambda^m x_N(\lambda)\}$ for almost all λ in $\sigma(T)$. But for λ in E_n, H_λ is an n-dimensional space. Hence we have $\mu_T = N \geqq n$, completing the proof.

Proof of Theorem 2. Part of this theorem is implicitly contained in the work of Sz.-Nagy and Foiaş [9]. Indeed, since T is quasi-similar to the dual residual part of its minimal unitary dilation ([9], p. 72), by [9], pp. 88—89 we can infer that T is quasi-similar to an operator of the form (1) and $n \leqq d_{T^*}$ (also *cf.* [9], pp. 271—273 for the case $d_T = d_{T^*} < \infty$). The uniqueness follows from the multiplicity theory of normal operators [5] and the fact that quasi-similar normal operators are unitarily equivalent. Lemma 2 furnishes the proof of the remaining part.

In light of these results we can generalize the notion of Jordan operators to the following

Definition. An operator T is called a *Jordan operator* if it is of the form

$$S_{\varphi_1} \oplus \ldots \oplus S_{\varphi_m} \oplus \ldots \oplus M_{E_1} \oplus \ldots \oplus M_{E_n} \oplus \ldots,$$

where the φ_j's are inner functions satisfying $\varphi_{j+1} | \varphi_j$ $(j = 1, 2, \ldots)$, and the E_k's are measurable subsets of C satisfying $E_{k+1} \subseteq E_k$ $(k = 1, 2, \ldots)$.

Combining Theorems 1 and 2 we obtain

Theorem 3. *Let T be a weak contraction on a separable Hilbert space, with defect indices $d_T = d_{T^*}$. Then T is quasi-similar to a uniquely determined Jordan operator*

$$S_{\varphi_1} \oplus \ldots \oplus S_{\varphi_m} \oplus \ldots \oplus M_{E_1} \oplus \ldots \oplus M_{E_n} \oplus \ldots. \tag{2}$$

If there are m $(0 \leqq m \leqq \infty)$ non-constant φ_j's and n $(0 \leqq n \leqq \infty)$ E_k's with nonzero measure, then $\mu_T = \mu_{T^*} = \max\{m, n\}$. Moreover, if T is c.n.u., then its corresponding Jordan operator is also a weak contraction and $\mu_T = \mu_{T^*} = \max\{m, n\} \leqq d_T = d_{T^*}$ hold.

We will call the uniquely determined Jordan operator the *Jordan model* for T.

We start the proof of Theorem 3 with the following

Lemma 3. *Let T_1, T_1' be C_0 contractions on H_1, H_1' and let T_2, T_2' be unitary operators on H_2, H_2', respectively. If $T_1 \oplus T_2$ is a quasi-affine transform of $T_1' \oplus T_2'$, then T_1 is quasi-similar to T_1' and T_2 is unitarily equivalent to T_2'.*

Proof. Let $X: H_1 \oplus H_2 \to H_1' \oplus H_2'$ be a quasi-affinity such that $X(T_1 \oplus T_2) = (T_1' \oplus T_2')X$. For any $h \in H_1$, let $h_1 \oplus h_2 = X(h \oplus 0)$, where $h_1 \in H_1'$ and $h_2 \in H_2'$. Since T_1', being a C_0 contraction, is of class $C_{0\cdot}$, we have $(T_1'^n h_1) \oplus (T_2'^n h_2) = (T_1' \oplus T_2')^n X(h \oplus 0) = X(T_1 \oplus T_2)^n (h \oplus 0) = X(T_1^n h \oplus 0) \to 0$ as $n \to \infty$. Thus $T_2'^n h_2 \to 0$ as $n \to \infty$. Since T_2' is of class $C_{1\cdot}$, this implies that $h_2 = 0$, and hence that $X(h \oplus 0) \in H_1'$. Thus with respect to the decompositions $H_1 \oplus H_2$ and $H_1' \oplus H_2'$, X is triangulated as

$$X = \begin{bmatrix} X_1 & Z \\ 0 & X_2 \end{bmatrix}.$$

By considering the adjoint, a similar argument as above shows that $Z=0$. Hence we obtain quasi-affinities $X_1\colon H_1\to H_1'$ and $X_2\colon H_2\to H_2'$ such that $X_1T_1=T_1'X_1$ and $X_2T_2=T_2'X_2$, that is, $T_1\prec T_1'$ and $T_2\prec T_2'$. By the uniqueness of the Jordan model for C_0 contractions, we infer that T_1 is quasi-similar to T_1' (*cf.* [1], Theorem 1). On the other hand, that T_2 is unitarily equivalent to T_2' follows from [4], Lemma 4.1.

Lemma 4. *The operator $S_\varphi\oplus M_E$ on the space $(H^2\ominus\varphi H^2)\oplus L^2(E)$ is cyclic.*

Proof. Let f be an essentially bounded function in $L^2(E)$, which is cyclic for M_E. If $E\neq C$, such is the identity function $1(e^{it})=e^{it}$ on E. If $E=C$ then it is well known that the cyclic vectors for the bilateral shift are those functions $f\in L^2$ for which $|f|>0$ a.e. and $\int\log|f|=-\infty$; we may assume that f is essentially bounded, for otherwise let $F=\{e^{it}\colon |f(e^{it})|\geqq 1\}$. Consider $\chi_{C\setminus F}f+\chi_F$. Let P be the (orthogonal) projection of H^2 onto $H^2\ominus\varphi H^2$, and let 1 also denote the identity function in H^2. We want to show that $P(1)\oplus f$ is a cyclic vector for $S_\varphi\oplus M_E$.

Let $K=\bigvee_{n=0}^{\infty}(S_\varphi\oplus M_E)^n(P(1)\oplus f)$. For each $h\in H^2$, let $\{p_n\}$ be a sequence of polynomials such that $p_n\to\varphi h$ in L^2-norm. Since f is essentially bounded, we have $p_nf\to\varphi hf$, and hence $P(p_n)\oplus p_nf\to P(\varphi h)\oplus\varphi hf=0\oplus\varphi hf$. This shows that $0\oplus\varphi hf$ is in K for any $h\in H^2$.

Now let g be an arbitrary function in $L^2(E)$. Since f is a cyclic vector for M_E, there exists a sequence of polynomials $\{q_n\}$ such that $q_nf\to\bar\varphi g$ in L^2-norm. Then $\varphi q_nf\to\varphi\bar\varphi g=g$. By what we proved before, we conclude that $0\oplus g\in K$ for any $g\in L^2(E)$. On the other hand, since it is clear that $P(h)\oplus hf\in K$ for any $h\in H^2$, we have $P(h)\oplus 0=(P(h)\oplus hf)-(0\oplus hf)\in K$. Hence we obtain $(H^2\ominus\varphi H^2)\oplus L^2(E)=K$, which completes the proof.

Proof of Theorem 3. Let $T=U\oplus T'$ be the decomposition of T into the direct sum of its unitary part U and its c.n.u. part T'. Since T' is also a weak contraction, we may consider its C_0 part T_0 and C_{11} part T_1. It was proved in [16] that T' is quasi-similar to $T_0\oplus T_1$. Hence T is quasi-similar to $T_0\oplus T_1\oplus U$. By Theorem 1, Lemma 3 and the multiplicity theory of normal operators [5], we conclude that T is quasi-similar to a uniquely determined Jordan operator (2).

If T is quasi-similar to (2), then T^* is quasi-similar to

$$S_{\varphi_1^\sim}\oplus\ldots\oplus S_{\varphi_m^\sim}\oplus\ldots\oplus M_{E_1^\sim}\oplus\ldots\oplus M_{E_n^\sim}\oplus\ldots,$$

where $\varphi_j^\sim(\lambda)=\overline{\varphi_j(\bar\lambda)}$ $(j=1,2,\ldots)$ and $E_k^\sim=\{e^{it}\colon e^{-it}\in E_k\}$ $(k=1,2,\ldots)$. Hence it is clear that to show that $\mu_T=\mu_{T^*}=\max\{m,n\}$, we have only to show that $\mu_T=\max\{m,n\}$. For convenience, we assume that $n\leqq m$. Let d_{T_0} and d_{T_1} denote the defect indices of T_0 and T_1, respectively. By Theorem 1 and Lemma 1 we have $m=\mu_{T_0}\leqq\mu_{T_0\oplus T_1\oplus U}=\mu_T$. If $m=\infty$ then we have already had the result. Hence we may assume that $m<\infty$. Since (2) is unitarily equivalent to

$$(S_{\varphi_1}\oplus M_{E_1})\oplus\dots\oplus(S_{\varphi_n}\oplus M_{E_n})\oplus M_{E_{n+1}}\oplus\dots\oplus M_{E_m},$$

using Lemmas 1 and 4 we have $\mu_T\leqq\underbrace{1+\dots+1}_{n}+\underbrace{1+\dots+1}_{m-n}=m$. Thus $\mu_T=m=\max\{m,n\}$. The case $m<n$ is similarly proved.

Now we assume that T is c.n.u., that is, $T=T_0\oplus T_1$. We show that the Jordan model (2) is a weak contraction. Indeed, it is enough to show that $S\equiv S_{\varphi_1}\oplus\dots\oplus S_{\varphi_m}\oplus\dots$ is weak. But S is the Jordan model of T_0, which is a weak C_0 contraction, so the assertion follows from the results of §8 of [2]. By Theorems 1 and 2 we have $m=\mu_{T_0}\leqq d_{T_0}$ and $n=\mu_{T_1}\leqq d_{T_1}$. Since $d_{T_0}\leqq d_T$ and $d_{T_1}\leqq d_T$ (cf. [9] p. 302) we obtain $\max\{m,n\}\leqq d_T=d_{T^*}$, completing the proof.

We make some remarks to conclude this section.

By Theorem 3 and Lemma 3 we infer that for weak contractions T_1, T_2, if T_1 is a quasi-affine transform of T_2 then T_1 and T_2 are quasi-similar to each other.

For c.n.u. weak contractions the unitary part of the Jordan model has an absolutely continuous spectrum.

If T is a c.n.u. weak contraction with finite defect indices, then in the Jordan model of T we have $E_k=\{e^{it}:\operatorname{rank}\Delta(e^{it})\geqq k\}$ $(k=1,2,\dots,n)$, where $\Delta(e^{it})=[I-\Theta(e^{it})^*\Theta(e^{it})]^{1/2}$ and $\Theta(\lambda)$ denotes the characteristic function of T. Indeed, since the characteristic function $\Theta_1(\lambda)$ of the C_{11} part T_1 is the purely contractive part of the outer factor $\Theta_e(\lambda)$ of $\Theta(\lambda)$, if $\Delta_1(e^{it})=[I-\Theta_1(e^{it})^*\Theta_1(e^{it})]^{1/2}$ then $\operatorname{rank}\Delta(e^{it})=\operatorname{rank}\Delta_1(e^{it})$ a.e.. Thus the assertion follows from [9] Theorem VI. 6.1. In particular, $E_1=\{e^{it}:\Theta(e^{it})$ is not isometric$\}$ and $n=\operatorname{ess\,sup}\operatorname{rank}\Delta(e^{it})$.

3. Multiplicity-free operators. A C_0 contraction T is called multiplicity-free if $\mu_T=1$, or equivalently, T has a cyclic vector. Some of the equivalent conditions for multiplicity-free C_0 contractions are gathered in the next theorem (*cf.* [10] and [13]).

Theorem 4. *Let T be a C_0 contraction on a separable Hilbert space. Then the following conditions are equivalent to each other:*

(i) *T is multiplicity-free;*

(ii) *T is quasi-similar to S_φ for some inner function φ;*

(iii) *$\{T\}'$ is commutative.*

Here $\{T\}'$ denotes the commutant of T.

We generalize this to the following

Theorem 5. *Let T be a c.n.u. weak contraction on a separable Hilbert space. Let T_0 and T_1 denote the C_0 and C_{11} part of T, respectively. Then the following conditions are equivalent to each other:*

(i) *T admits a cyclic vector;*

(ii) *T_0 and T_1 admit cyclic vectors;*

(iii) *T is quasi-similar to $S_\varphi \oplus M_E$ for some inner function φ and some measurable subset E of C (here φ may be constant and E may have measure zero);*

(iv) *$\{T\}'$ is commutative;*

(v) *$\{T_0\}'$ and $\{T_1\}'$ are commutative.*

This theorem suggests the following

Definition. A c.n.u. weak contraction T is called *multiplicity-free* if it satisfies the equivalent conditions (i)—(v) in Theorem 5.

Note that CLARK [3] also defined multiplicity-free operators among operators of class $[C_{0\cdot} \cup C_{1\cdot}] \cap [C_{\cdot 0} \cup C_{\cdot 1}]$. It is clear that our definition is consistent with his.

Proof of Theorem 5. The equivalence of (i), (ii) and (iii) follows from Theorems 1, 2 and 3. The implication (i)$\Rightarrow$(iv) and the equivalence (ii)$\Leftrightarrow$(v) were proved by SZ.-NAGY and FOIAŞ (*cf.* [11], [12] or [7], [13]). Thus to complete the proof we have only to show that (iv) implies one of the other conditions. Let us prove the implication (iv)$\Rightarrow$(iii). Let $S \oplus M$ denote the Jordan model of T, where $S = S_{\varphi_1} \oplus S_{\varphi_2} \oplus \dots$ and $M = M_{E_1} \oplus M_{E_2} \oplus \dots$, and let X, Y be two quasi-affinities such that

$$TX = X(S \oplus M) \quad \text{and} \quad (S \oplus M)Y = YT.$$

Then, from (iv) it follows that the relation

$$(XAY)(XBY) = (XBY)(XAY)$$

holds whenever $A, B \in \{S \oplus M\}'$ and hence

$$A(YX)B = B(YX)A.$$

Now by Lemma 3 it follows that $YX = Z \oplus V$ where $Z \in \{S\}'$, $V \in \{M\}'$ and we have

$$AZB = BZA, \quad A'VB' = B'VA' \tag{3}$$

for any $A, B \in \{S\}'$, $A', B' \in \{M\}'$. Taking $B = I$, $B' = I$ in (3), it follows that $Z \in \{S\}''$ and $V \in \{M\}''$ such that, again by (3), we infer that $\{S\}'$ and $\{M\}'$ are commutative. From the implication (v)$\Rightarrow$(ii) it follows that $S = S_{\varphi_1}$ and $M = M_{E_1}$ and (iii) follows.

We remark that conditions (i)—(v) in Theorem 5 are equivalent to the corresponding conditions for T^*. (This follows from Theorem 3 that $\mu_T = \mu_{T^*}$.) Also note that if the defect indices of T are finite, then these conditions are equivalent to:

(vi) The minors of order $d_{T_0} - 1$ of the matrix of $\Theta_{*i}(\lambda)$ have no common inner divisor, and rank $\Delta(e^{it}) \leqq 1$ a.e. (*cf.* [9], pp. 267 and 271). In particular, we have

Corollary 1. *If T is a c.n.u. contraction with scalar-valued characteristic function $\varphi(\lambda)\not\equiv 0$, then T is cyclic and $\{T\}'$ is commutative.*

Proof. T is certainly a c.n.u. weak contraction which satisfies condition (vi). The assertion follows from the remark we made above.

Part of the previous result was obtained earlier by Sz.-Nagy and Foiaş [14].

Corollary 2. *Let T be a c.n.u. multiplicity-free weak contraction on H. If K is an invariant subspace for T such that $T|K$ is also a weak contraction, then $T|K$ is multiplicity-free.*

Proof. Since T is multiplicity-free, we have $\mu_{T^*}=1$, so that $\mu_{(T|K)^*}=1$. Therefore, if $T|K$ is a weak contraction, it follows that it is multiplicity-free.

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K. B. Athreya—P. E. Ney, Branching processes (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 196), XI+287 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1972.

This book can be regarded as a continuation of T. E. Harris' book (Springer-Verlag, 1963). Chapter I gives a brief summary of classical results on Galton—Watson processes, and a more detailed treatise of its modern refinements, e.g. the decomposition of the supercritical branching process.

Chapter II develops the potential theoretical tools and their application to the discrete time case.

One of the most interesting theorems here is the sharp limit law due to the authors. Chapter III deals with the Markovian case using Kolmogorov's equations and martingale convergence theorems, and investigates the problem of imbedding a Galton—Watson process into a continuous time Markov branching process.

Chapter IV is devoted to the so called age-dependent process. It is not Markovian and the methods of the previous chapters do not apply. Such models were introduced and first studied by R. Bellman and T. E. Harris (1952). The key tool here is the renewal equation.

Chapter V generalizes results proved in previous chapters for the multitype case.

Chapter VI examines special processes, e.g. the branching diffusion, branching processes with random environments and the continuous state branching processes and processes with immigrations. The authors cite an abundance of literature. After every chapter there are complements and problems, including open ones.

The treatment is thorough, precise and easy to follow.

A. Krámli (Budapest)

N. L. Biggs, Interaction models, London Math. Soc. Lecture Notes Series 30, 101 pp., Cambridge University Press, Cambridge—London—New York—Melbourne, 1977.

Have you ever thought that the Four Colour Problem is in any connection with ferromagnetism? Well, this book shows such a connection!

To be less sensational but more specific, it has been the Four Colour Problem which has inspired most of the research done in connection with the so-called *chromatic polynomial.* This polynomial $p_G(x)$ expresses the number of colorations of the graph G in x colors. Graph theorists, primarily G. D. Birkhoff, H. Whitney and W. T. Tutte have developed several expansion formulas, generalizations, and other properties of this polynomial.

Meanwhile physicists, among others Ising, Mayer, Lieb etc. have studied "interaction models", which can be described in terms of a graph: the vertices are particles, and interaction exists between adjacent vertices only. Each particle has a finite number of possible states, and each interacting pair contributes to the Hamiltonian H of the system depending on their states. Important physical phenomena, like phase transitions, can be described in terms of the expression $\Sigma \exp H$ (summed over all states of the particles). This expression has properties very similar to those of the chromatic polynomial, in fact, the chromatic polynomial is a special case of it. Various properties and expansions of the chromatic polynomial have their analogues and generalizations in physics; most of them have been discovered independently but some discoveries have been inspired by this analogy.

"The lectures on which this book is based were intended for a 'mixed audience' ... some of the audience were basically physicists, and others were basically mathematicians... The desire to be intelligible to two classes of students has been my main preoccupation in preparing the lectures and writing the book." So the book may serve as a bridge between theoretical physicists and mathematicians working on two ends of the same problem, and who have now caught sight of each other.

The author also points out: "It would be idle to pretend that the material treated in this book is in its final form... If some of my errors stimulate other mathematicians to put things right, then the book will have served its purpose." Indeed, one feels the air of openness and the temptation to join the research when reading the book. And this, I think, is a good reccommendation.

L. Lovász (Szeged)

R. P. Burn, Deductive Transformation Geometry, XI+121 pages, Cambridge University Press, 1975.

There are two usual types of descriptions of the relationship between the Euclidean plane and the real number system. A geometric description was made by Hilbert under which the real number system emerged from postulated properties of points and lines. Algebraic descriptions explicitly start from the real number system, and construct points as ordered pairs, lines as linear subsets of ordered pairs of real numbers. The main aim of this book to show which properties of the real number system are required to establish particular theorems in geometry. The approach will be geometric in that the axioms will all be axioms of or about incidence, but the method will be algebraic, in that the existence and properties of a coordinate system will be obtained by exploring groups of transformations.

In Chapter 1 the affine incidence axioms in the plane are assumed, only to see what kind of geometry can be done without any algebraic properties of real numbers. The study of finite affine geometries gives rise to a great number of easily stated, but as yet unsolved problems. At the end of this chapter, by adjoining a line of new points to the affine plane, the projective plane is constructed. Chapter 2 goes back to the affine plane, defines parallel projections of lines, and affinities of lines as products of parallel projections. The affinities of a line onto itself which are products of two parallel projections are called (affine) permutations. Addition and multiplication of permutations can be defined in the usual way. By using the permutations, coordinate systems on lines can be introduced. Postulating that the permutations of a line form a group, the coordinate elements form an Abelian group under addition, and for the (not necessarily commutative) multiplication a one-sided distributive law follows. Chapter 3 is devoted to the study of those transformations of the plane which map each line onto a parallel line. There are two classes of these transformations; those with no fixed points, called translations, and those with just one fixed point, called enlargements. In Chapter 4 those planes are considered which admit all possible translations and enlargements. These are the Desarguesian planes. In these planes the underlying algebra of coordinates satifies both distributive laws. Chapter 5 deals with collineations of the plane, in particular with collineations having a line of fixed points, called axial collineations. The theorems obtained here are strikingly analogous to those obtained in Chapter 3 for translations and enlargements. In Chapter 6 a geometric description is given for those planes which have a field as their underlying algebra. It turns out that these are the Pappian planes. In Chapter 7 reflections are studied. Chapter 8 compares different systems of axioms given by Hilbert, Birkhoff, Moise and others.

The book is written in a well readable way.

L. Gehér (Szeged)

Combinatorial Surveys, Proceedings of the Sixth British Combinatorial Conference, Edited by P. J. Cameron, 226 pages, Academic Press, London—New York—San Francisco, 1977

The book consists of lectures of invited speakers of the Combinatorial Conference held at Egham (London), 1977. The chapters, as the title shows, are survey-type, therefore the reader receives a good cross-section of advanced combinatorics. The index of the book shows both the unity of the subject (with designs, graphs, matroids, projective spaces as prevailing themes) and its diversity (campanology, the golden ration, parallelism, Ramanujan numbers, root systems, etc.). This Proceedings is a useful reference book.

Chapter 1 (F. Buekenhout, What is a subspace?) investigates the abstract concept of subspace and exhibits how to use it for graphs, matroids and block designs.

Chapter 2 (P. J. Cameron, Extensions of design: variations of a theme). It is an old result of the theory of designs that a projective plane can be extended by one point to a 3-design only if its order is 2, 4 or (possibly) 10. Generalizations in three different directions are given.

Chapter 3 (L. Lovász, Flats in matroids and geometric graphs) is devoted to show how to use the concept of a geometric graph to unify the theory of τ-critical graphs and to prove a conjecture of Gallai and some new Helly-type theorems on flats in geometries.

Chapter 4 (D. K. Ray-Chaudhuri, Combinatorial characterization theorems for geometric incidence structures) provides a thorough and deep survey of the theory of geometric incidence structures. It contains 29 theorems and embraces a great part of this theory. The emphasis is on theorems asserting that certain incidence structures are "coordinatizable", i. e. can be derived from geometries over finite fields.

Chapter 5 (N. J. A. Sloane, Binary codes, lattices, and sphere-packings) surveys a surprising connection between binary codes on the one hand, and sphere-packings and lattices in R^n on the other hand.

Chapter 6 (A. T. White, Graphs of groups on surfaces) deals with Cayley graphs of groups, embedding of them in a surface, voltage and current graphs, genus etc.

Chapter 7 (D. R. Woodall, Zeros of chromatic polynomials) provides an introduction to the theory of chromatic polynomials of graphs. The emphasis is on the location of real zeros, and on the multiplicities of the integer zeros.

A. Frank (Budapest)

R. D. Driver, Ordinary and delay differential equations (Applied Mathematical Sciences, Vol. 20), IX+501 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

The so-called functional differential equations arose a long time ago in the history of mathematics. These are differential equations, in which the unknown function and its derivatives occur with various different arguments. Nowadays such equations play a particularly important role in applications; they are motivated by problems in control theory, physics, biology, ecology, economics and the theory of nuclear reactors. During the past two or three decades a number of valuable monographs on this subject were published, but none of them can be considered as an introductory text.

This book is an excellent first course on *delay* differential equations (this means an equation expressing the highest order derivative of the unknown function x at time t in terms of x and its lower order derivatives at t and at *earlier* instants). One of its advantages is that the author gives simultaneously an introduction to ordinary differential equations also. The book is especially well-organized from the point of view of didactics. This is mostly due to the great number of examples worked out. They prepare the reader excellently for understanding the theorems and convince him of their applicability. Some of the examples are very interesting in themselves, e.g. population growth, electrical circuits, two-body problem of electrodynamics, control systems, and numerous mechanical examples. At the end of the chapters several problems of various difficulty are listed, with answers or hints.

The structure of the book is the following. First comes the uniqueness of the solution of ordinary differential equations satisfying a Lipschitz condition. The properties of the solutions of ordinary linear systems are treated. Further on existence and stability problems are discussed for ordinary and delay equations simultaneously. The final chapter is devoted to autonomous differential equations.

The book is recommended to lecturers wishing to introduce delay differential equations in the senior and beginning graduate level curricula. It is suitable also for students and users of mathematics interested in differential equations.

L. Hatvani—L. Pintér (Szeged)

Lars Gårding, Encounter with Mathematics, X+270 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

This book is meant mainly for people on the level of a junior student and aims to give a general but comparatively comprehensive picture of some of the central topics of mathematics. Roughly speaking, these are the branches initiated before the middle of the last century, but their development is mostly followed up to quite recent results. It is by no way an easy piece of reading for a beginner so that it seems to be more apt to inform students already interested in mathematics than to intrigue those maintaining a lukewarm relation to it.

After an introductory chapter on the interrelation between mathematics and reality and a short account on the basic facts and problems of number theory, chapter 3 deals with algebra (equations, groups, rings, Galois theory). It includes Hilbert's Nullstellensatz with proof. The next section (Geometry and linear algebra) gives among others the foundations of Banach and Hilbert spaces, including the contraction theorem for Banach spaces with full proof and the spectral theorem for compact linear operators (proved for finite dimensions only). Speaking about continuity, the author presents Dedekind's theory of real numbers, uniform continuity and uniform convergence, and the notion of a topological space. After an interlude on the history of mathematics in the seventeenth century, chapters 7 and 8 deal with differentiation and integration and give a rather thorough picture of both fields. However to work with the Grassmann algebra is perhaps too formal at this level. — Fourier transform and the inversion formula are proved, to be used in the chapter treating probabilities. The section on series deals, among others, with the Weierstrass approximation theorems. The section on probability presents the basic classical results, a bit of statistics and of physical applications. In chapter 11 (Applications) mathematical modeling is illustrated on acoustics. The last section deals with sociology, psychology, and teaching of mathematics.

All in all, the material the book ranges over is rather large, even too large. This is partly compensated by a clear and well-considered treatment. It was also a good idea to give a few selected passages of classical works in the corresponding fields. There are a lot of misprints, several embarrassing ones.

G. Pollák (Szeged)

D. Gilbarg—N. S. Trudinger, Elliptic partial differential equations of second order (Grundlehren der mathematischen Wissenschaften 224), X+401 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

The book grew out of graduate courses held by the authors at Stanford University.

The theory of partial differential equations is now so large that a comprehensive treatment is impossible. Every book contains a small part of the theory only. The tools also vary considerably, there are books written in classical style with several applications while others are making use of concepts of modern mathematics. The present authors' aim was to write a book on second order elliptic partial differential equations for a broad spectrum of readers, interested in the various concepts, methods and techniques which have been developed in this theory.

The book consists of two parts. In Part One, Chapters 2—8, the linear theory is developed. Naturally, Laplace's and Poisson's equations are the starting points for the study of classical solutions. The Dirichlet problem for harmonic functions with continuous boundary conditions is investigated through the Perron method of subharmonic functions, emphasizing the maximum principle and the barrier concept for studying boundary behavior. The Hölder estimates for the solution of the Poisson equation are derived from an analysis of the Newton potential. Ch. 6 develops an extension of potential theory based on the fundamental observation that equations with Hölder continuous coefficients can be treated locally as a perturbation of constant coefficient equations. Ch. 8 on "Generalized solutions and regularity" shows that by Hilbert space methods a more general approach can be achieved to linear problems. Throughout the book — whenever it is possible — the authors emphasize the connections and applications to the nonlinear theory; thus the regularity theory and Hölder estimates of generalized solutions are fundamental to the nonlinear theory.

Part Two deals with the Dirichlet problem and related estimates for quasilinear equations. In Ch. 9 maximum and comparison principles are given for the solutions of quasilinear equations. Ch. 10 contains fixed point theorems of Schauder, Leray—Schauder and Brouwer, and some of their applications. Chapters 12, 13 and 14 present gradient estimates. Here we find the fundamental results of Ladyzhenskaya and Ural'tseva, the Jenkins and Serrin criterion for solvability of the Dirichlet problem for the minimal surface equation.

The work is almost entirely self-contained, some basic facts of real analysis and linear algebra are supposed only. Much of the material appears in a single volume for the first time. A number of interesting problems, historical material, and bibliographical references are added to each chapter.

Summing up, if one wishes to see a variety of modern methods and their applications to some classical problems of elliptic partial differential equations, examples to illustrate the developments a few up-to-date results, and references to further study, all gathered in a book which is written carefully and in an enjoyable style, then he should read this book.

L. Hatvani—L. Pintér (Szeged)

Computing Methods in Applied Sciences and Engineering, Second International Symposium, December 15—19, 1975, Iria Laboria, Institut de Recherche d'Informatique et d'Automatique: Edited by R. Glowinski and J. L. Lions (Lecture Notes in Economics and Mathematical Systems 134), VIII+390 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

This book consists of a selected part of the lectures which were presented during the symposium indicated in the title. It has five parts: Numerical Algebra, Finite Elements, Dynamical Problems, Identification and Inverse Problems, and Integral Methods.

Part I treats the solution of large linear sparse systems of linear algebraic equations arising at the application of the finite element method (by Alan George), hypermatrix algorithms in connection with the solution of linear equations or eigenreduction (by K. A. Braun, G. Dietrich, G. Frik, Th. L. Johnson, K. Straub, and G. Vallianos), iterative methods for the solution of non-compatible systems of linear equations, occurring in connection with the solution of systems of difference equations approximating the Neumann boundary problem for elliptic differential equations (by Y. A. Kuznetsov), a generalized conjugate gradient method for nonsymmetric systems of linear equations (by P. Concus and G. H. Golub), and spline approximation in Euclidean spaces (by V. A. Vasilenko).

Part II presents the mathematical foundations of hybrid and mixed finite element methods for plate bending problems described by fourth order elliptic equations (by F. Brezzi); the result that the strain energy of a shell is elliptic, using W. T. Koiter's linear model (by M. Bernadou and P. G. Ciarlet); the homogenization approach in engineering, where homogenization is meant as a method which studies the macrobehaviour of a medium by its microproperties (by Ivo Babuška); and finite element approximations for solving elastic problems (by J. Nitsche).

Part III deals with a finite element approximation for parabolic equations via an operator theoretical approach (by Hiroshi Fujita): a variational method for increasing the accuracy of the difference scheme in a close relation to the Richardson extrapolation (by G. I. Marchouk and V. V. Shaydourov); Runge—Kutta methods for the approximation of the evolution problem (by M. Crouzeix); and continuous and discontinuous finite element methods for solving the neutron transport equation (by P. Lesaint).

Part IV contains the survey papers of J. Cea (on domain identification problems), J. M. Boisserie and R. Glowinski (on optimization of rotational membranes), R. Glowinski and O. Pironneau (on optimal control), Masaya Yamaguti (on solidification), J. Galligani (on numerical problems of earth science), T. Dupont and H. H. Rachford Jr. (on a Galerkin method for liquid pipelines).

In Part V two papers give a glimpse into the integral equation methods applied to elasticity problems (by J. C. Lachet and J. O. Watson) and to fluid mechanics (by T. S. Luu), and a paper reviews curved finite element methods for the solution of singular integral equations on surfaces in R^3 (by J. C. Nedelec).

Of the above papers 13 are written in English, 9 in French. Each paper is followed by a rich and up-to-date bibliography.

The book is warmly recommended for use to research workers in numerical analysis as well as to experts in theoretical physics and engineering.

F. Móricz (Szeged)

H. Grauert—R. Rennert, Theorie der Steinschen Räume (Grundlehren der mathematischen Wissenschaften, 227), XX+249 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1977.

The central results of complex function theory show that the fundamental difficulty in generalizing the classical theorems for a single variable to several variables is that in $\mathbf{C}^n$ there exist domains which are not domains of holomorphy. ($G\subset\mathbf{C}^n$ is called a domain of holomorphy if there is a holomorphic function on G which is singular in every boundary point of G). Since the main problems of complex analysis are solvable for domains of holomorphy, the natural question was raised how can one axiomatize intrinsically the complex spaces which are generalizations of domains of holomorphy and for which the classical results of complex analysis can be extended. A Stein space X is such a generalization of a domain of holomorphy, which in the singularity-free case (that is when X is a complex analytical manifold) can be characterized by the following properties:

(i) X is holomorphically separable, i.e. for every $x_0\in X$ there are holomorphic functions $f_1, \dots, f_m$ on X such that x_0 is isolated in the set $\{x\in X: f_1(x)=\dots=f_m(x)=0\}$.

(ii) X is holomorphically convex, that is, for every compact subset $K \subset X$ its holomorphically convex hull $\hat{K}$ is compact, where $\hat{K}$ is defined by

$$\hat{K} = \{x \in X: |f(x)| \leqq \sup_K |f|, \ f \text{ holomorphic on } X\}.$$

The theory of Stein spaces is developed using the methods of sheaf theoretic cohomology theory. A breaf survey on this subject is given in Chapters A and B. In Chapter I the coherence theorems on finite holomorphic maps are treated. Chapter II is devoted to the de Rham and Dolbeault cohomology theory. In Chapters III—IV the main theorems on Stein spaces (Theorems A and B) are proved, which are the generalizations of the Cartan—Serre theorems on singularity-free Stein mainifolds. Chapter V contains the fundamental applications of the main theorems to Cousin and Poincaré problems and to characterizations of Stein spaces. In Chapter VI the finite dimensionality theorem of Cartan and Serre is generalized to the complex spaces with singularities. Chapter VII treats the theory of compact Riemann surfaces, applying the preceding general results.

The book is a fundamental monography on the subject, it is well organized, the presentation is always clear. The reader is assumed to have a certain knowledge of complex analysis and sheaf theory.

P. T. Nagy (Szeged)

H. Grauert—K. Fritzsche, Several Complex Variables (Granduate Texts in Mathematics), VIII+207 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1976.

This textbook is an excellent introduction to complex analysis in several variables, suitable for undergraduate students. The reader is only supposed to be familiar with elementary calculus, with the theory of complex functions of a single variable and with a few elements of algebra and general topology. In accord with its introductory character the book treats many examples and special cases in full detail and with numerous illustrative figures. At the end of the chapters the authors give a survey of the fundamental results of the theory, tempting the readers to further study.

In Chapter I the notion and basic properties of holomorphic functions of several variables are introduced. In contrast to the one-variable theory, there exist domains G in $\mathbf{C}^n$ such that every holomorphic function on G has a holomorphic continuation beyond G. Domains $G \subset \mathbf{C}^n$ for which such a continuation of holomorphic functions on G do not exist are called domains of holomorphy. In Chapter II some characterizations of domains of holomorphy are given. Chapter III is devoted to the algebraic treatment of the ring of convergent power series and to its applications to the theory of analytic sets, which are locally the sets of zeros of holomorphic functions. Chapter IV is a brief introduction to sheaf theory. In Chapter V the notion of complex manifolds and Stein manifolds are introduced. (The latter is a natural generalization of the domain of holomorphy.) Here a lot of examples of complex manifolds are discussed. In Chapter VI the cohomology theory of sheaves is treated. It is a useful generalization of the Čech cohomology theory and provides a frame to express the main results for domains of holomorphy and Stein manifolds. Chapter VII is devoted to the analysis of real differentiability in complex manifolds. The notions of tangent spaces, differential forms and exterior derivation are introduced and the theorems of Dolbeault and de Rham are proved.

P. T. Nagy (Szeged)

James E. Humphreys, Linear Algebraic Groups (Graduate Texts in Mathematics), XIV+247 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1975.

The theory of linear algebraic groups has been studied extensively during the past twenty years following the fundamental work of A. Borel, Chevalley, Steinberg, Tits and others, and has made a significant progress in a number of areas: semisimple Lie groups and arithmetic subgroups,

p-adic groups, classical linear groups, finite simple groups, invariant theory, etc. This theory is hardly accessible for beginners, because in the fundamental monographs on the subject a substantial familiarity with the abstract methods of algebraic geometry is assumed. This book is to serve as a detailed textbook for graduate students on affine algebraic groups over an algebraically closed field K, containing a very useful introduction to algebraic geometry.

An affine algebraic group G is defined as an affine algebraic variety (i. e. a set of common zeros of a finite collection of polynomials in an affine space K^n), endowed with a structure of a group such that the group operations $(x, y) \to xy$ and $x \to x^{-1}$ are polynomial functions on G. The standard examples of affine algebraic groups are the classical linear groups: the general linear group $GL(n, K)$, the special linear group $SL(n, K)$, the symplectic group $Sp(n, K)$, the special orthogonal group $SO(n, K)$. It is true that any affine algebraic group is "linear" in the sense that it is isomorphic with an algebraic subgroup of some $GL(n, K)$.

The building of the theory of affine algebraic groups is considerably parallel to the theory of Lie groups, only the differential topological terms and methods in Lie group theory are replaced by the terms and methods of algebraic geometry. One can define an intrinsic algebraic notion of tangent space to an algebraic variety at a point, which in the case of an algebraic group can be endowed with an additional Lie algebra structure. This way a functor can be defined from the category of affine algebraic groups to the category of Lie algebras, and with the help of this functor the structure theory of Lie algebras can be applied to the theory of affine algebraic groups.

The basic concepts of algebraic geometry are introduced in Chapter I. The treatment is detailed only according to necessity and is not scheme-oriented. In Chapters II—V the basic facts about algebraic groups, their Lie algebras and homogeneous spaces are treated. In Chapters IV—IX special questions, essential tools for the structure theory are discussed: the Jordan—Chevalley decomposition, diagonizable groups, nilpotent and solvable groups, Borel subgroups, maximal tori etc. Chapter X is devoted to the study of structure theory of reductive groups, especially properties of the root systems, normal and parabolic subgroups. In Chapter XI the representations and classification of semisimple groups are treated. Chapter XII contains a survey, without proofs, of the rationality properties of algebraic groups.

The reader is supposed to be conversant with the standard results of commutative algebra and the structure theory of semisimple Lie algebras. The treatise provides a rich and up-to-date account of the theory of linear algebraic groups.

P. T. Nagy (Szeged)

J. Lindenstrauss—L. Tzafriri, Classical Banach Spaces. I. Sequence Spaces, 188 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1977.

The present volume deals with sequence spaces; the notion of a Schauder basis plays a central role here. The text is divided into four chapters. Chapter 1 contains a quite complete account of the main results on Schauder bases in general Banach spaces. Some notions related to Schauder bases, e.g. approximation properties, biorthogonal systems, Schauder decompositions are discussed in detail. Chapter 2 is devoted to the study of the classical sequence spaces $l^p (1 \leq p \leq \infty)$ and c_0. Subspaces and characterizations of these spaces among Banach spaces are studied. This chapter contains also a discussion of general results related to the approximation property. The last section deals with extension properties of c_0 and l^∞, the lifting property of l^1 and the automorphisms of these spaces. In Chapter 3 the special properties of symmetric bases and the relation between symmetric bases and unconditional bases are discussed. The final chapter deals with the study of the structure of some particular classes of spaces with symmetric bases, mainly Orlicz sequence spaces, and gives a detailed discussion of such spaces.

Familiarity with the basic results of real analysis and functional analysis is assumed. The book is highly recommended to anyone who is interested in Banach space theory.

L. Gehér (Szeged)

P. Medgyessy, Decomposition of superpositions of density functions and discrete distributions (Disquisitiones Mathematicae Hungaricae 8), 308 pages, Akadémiai Kiadó, Budapest, 1977.

The main problem discussed in this book reads as follows. Given the graph of the superposition of an unknown number, say N, of components (of unimodal density functions or discrete distributions) of a given type, how is it possible to determine N and obtain approximate values of some of the unknown parameters of the superposition? Problems of this nature arise e.g. in analysing spectra in spectroscopy or nuclear physics, in biology, in mathematical statistics etc. The book is a successful attempt to treat these problems in a unified way making use of rigorous mathematical tools.

A little monograph "Decomposition of superpositions of distribution functions" (Akadémiai Kiadó, Budapest, 1961) was already published by the present author. As far as we know, this was the first systematic elaboration on such problems. Unfortunately, several problems that needed to be treated were numerically incorrect (ill-posed) thus their treatment was unsatisfactory there. A systematic investigation of handling ill-posed problems started in 1962. On the other hand, the author also found new ideas and methods in connection with the decomposition problems, as a result of which the whole discipline has taken a more coherent form.

The present book is not a revised or enlarged edition of the earlier work. Naturally, the main problem and certain results are the same in both books. However, they are restricted here to a narrower area: to superpositions of density functions and of discrete distributions, while the treatment in the earlier book was excessively general. As to the rest, however, this work is thoroughly new. The fundamental scope of the present book essentially belongs to numerical analysis, and not to probability theory or mathematical statistics. Only methods that can be realized numerically are considered, and several former procedures analytically elegant in themselves but useless in practice are omitted. In spite of this the majority of tools come from probability theory.

The book consists of five chapters, divided and subdivided into paragraphs, sections and subsections. A Postscript summarizes the possible tasks of future research. Remarks, historical comments, unanswered questions etc. are collected at the end of each paragraph under the title Supplements and problems. They may also point out the future tasks in this field.

Chapter I is an introduction. It provides the basic concepts and formulates the so-called decomposition problem. For density functions this reads as follows. Let $f(x; \alpha, \beta)$ be a two-parameter density function of known analytical form, and let a superposition

$$g(x) = \sum_{k=1}^{N} p_k f(x; \alpha_k, \beta_k)$$

be given, where $N, p_k, \alpha_k, \beta_k$ $(k=1, 2, \ldots, N)$ are unknown parameters; there are no identical pairs (α_k, β_k) and $p_k > 0$. We have to estimate these parameters or a part of them on the basis of the knowledge of $g(x)$.

Chapter II summarizes the mathematical tools applied in the book. Many of them are due to the author, e.g., a new characterization of the shape (of the "narrowness") of the graph of a density function or discrete distribution (§§ 4 and 5).

Chapter III is devoted to the decomposition of superpositions of density functions essentially by means of the unimodality preserving, narrowness-increasing transformations. Historically the first decomposition problem: the decomposition of superpositions of normal density functions, investigated by N. Sen in 1922, appears here as a particular case within a group of problems. This chapter also includes the decomposition of superpositions of exponential density funcions, which is more difficult than the former one. This problem is important in many fields of natural sciences, too.

Chapter IV deals with the decomposition of superpositions of discrete distributions. The methods here differ from those in the previous chapter essentially in their discrete character. From the viewpoint of numerical analysis the situation is much easier here; in the most important cases ill-posed problems do not appear and every theoretically good method can be adopted in practice.

Chapter V surveys those numerical methods which are of use in decomposition problems, e.g. the solution of integral equations of the first kind, numerical computation of convolution transforms, etc. The most important part of this chapter deals with the numerical treatment of ill-posed problems. Among others, the so-called regularization method invented by A. N. Tihonov in 1963 is given in detail, as well as several methods of the author, which have been proved useful in certain special cases.

The reading of the book requires only a few notions from probability theory and numerical analysis. A great number of figures helps the reader to understand the text. There is an abundance of numerical examples taken from practice.

At the end References, complete up to 1975, list the papers comprising nearly 400 items. Certain items from the References were picked out to compose a Chronological Bibliography.

To sum up, this well-written book fills in a gap in the literature. It provides a rich and up-to-date material of the fast-growing discipline indicated in its title, whose significance is becoming crucial for practice.

It is no exaggeration to say that the book is indispensable for everyone, either mathematician or specialist in a field of sciences, dealing with decomposition problems. It is also very useful for all those mathematicians whose interest is in probability theory, mathematical statistics or numerical analysis.

F. Móricz (Szeged)

N. Rouche—P. Habets—M. Laloy, Stability theory by Liapunov's direct method (Applied Mathematical Sciences, Vol. 22), XII+396 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1977.

In 1892 A. M. Ljapunov invented a new method — called by himself a direct method — for the study of stability and asymptotic properties of solutions of ordinary differential equations. By the aid of this method, based upon the study of the behaviour of certain scalar auxiliary "energy-like" functions along the solutions, he solved numerous important problems in theoretical mechanics and in the qualitative theory of differential equations. The direct method was further developed by N. G. Cetajev and his school in the 1930—40's. In the 1950's the development was even more rapid since the method proved to be most useful in the study of stability problems in control systems and in biological, physical, technical and económical systems described by ordinary differential equations.

Although this development is still going on, no book wholly devoted to the subject was published since 1967, when W. Hahn's book appeared. So an up-to-date monograph was needed to synthetize modern results of the theory, to describe its present state, and to make users of mathematics acquainted with the latest interesting applications taken from various fields. This excellent book answers these purposes in every respect. It is a collective work based on the material of a seminar held at the University of Louvain during the academic year 1971—72. Besides the three authors, C. Risito, K. Peiffer, R. J. Ballieu, Dang Chan Phien and J. L. Corne also worked on some chapters.

In the first two chapters the authors give a compact but intelligible introduction to the basic concepts, theorems and topics of stability theory, which are already considered classical.

Chapter III is a pearl in the book. As Lagrange stated and Dirichlet proved, a mechanical equilibrium of a conservative system is stable at each point where the potential function has a strict minimum. In this chapter, the authors first study some versions of this theorem. Next, they consider the inversion of this theorem, a classical incompletely solved problem of theoretical mechanics. In the two final sections the effect of dissipative and gyroscopic forces on stability properties of an equilibrium position is treated. The chapter is very valuable also because this topic was not considered in earlier monographs.

The following three chapters are: IV. Stability in the presence of first integrals; V. Instability; VI. A survey of qualitative concepts.

A set of the phase space is called attractive if every solution starting near the set tends to the set as $t\to\infty$. Chapters VII and VIII deal with this concept for autonomous and nonautonomous equations, which have been recently studied in addition to asymptotic stability.

Chapter IX is a splendid review of the comparison method, which can be considered as the combination of the classical Ljapunov method with the theory of differential inequalities.

The subject-matter of the book is fortunately selected so that the reader is informed of what happened in the Ljapunov theory of stability during the last decade. The style is always clear, precise, but not too abstract even for users of mathematics. For them the latest significant and characteristic applications of Ljapunov's direct method will especially be useful, e.g. stability and instability of the betatron, nonlinear electrical networks, the ecological problem of interacting populations, stability of composite systems.

This book is indispensable for specialists in stability problems, or more generally, in the qualitative theory of differential equations, but it is also useful for students and for everybody interested in applications of differenctial equations.

L. Hatvani—L. Pintér (Szeged)

The State of the Art in Numerical Analysis, Proceedings of the conference held at The University of York, England, April 12—15, 1976; organized by The Institute of Mathematics and its Applications; edited by D. A. H. Jacobs; XIX+978 pages, Academic Press, London—New York—San Francisco, 1977.

The book surveys those areas of numerical analysis in which considerable advance has been achieved during the last ten years (1965—1975). It provides descriptions of theories, comparisons of methods, computational techniques and even algorithms, while indicating in some cases where current research efforts are being concentrated, and in others where future research might profitably be directed.

The book is divided into seven Parts, each of which comprises several chapters. Alternatively each Part of the book can be consulted to obtain a survey of one branch of numerical analysis, or all the Parts together to obtain an up-to-date overview of the many different branches and topics of numerical analysis.

Part I: *Linear Algebra.* Very valuable contributions are made here by J. H. Wilkinson (*Some Recent Advances in Numerical Linear Algebra* giving, among others, a concise account of the square-root-free Givens transformations, the *QZ* algorithm), P. E. Gill and W. Murray *(Modification of Matrix Factorizations after a Rank-one Change)*, and J. K. Reid *(Sparse Matrices)*.

Part II: *Error analysis.* It begins with the explanation of the ideas of error analysis made by C. G. Broyden, then follow contributions by N. Metropolis *(Methods of Significance Arithmetic)* and K. Nickel *(Interval-Analysis)*.

Part III: *Optimization and Non-Linear Systems.* Here the most remarkable chapters are by K. W. Brodlie *(Unconstrained Minimization)* and J. E. Dennis, Jr. *(Non-Linear Least Squares and*

Equations). The first gives a rather comprehensive account of the quasi-Newton methods from the original conception to modern implementations and new ideas, while the second treats unconstrained minimization, non-linear squares, and simultaneous solution of non-linear equations as a trilogy of problems. There follow contributions by P. E. Gill and W. Murray *(Linearly-Constrained Problems including Linear and Quadratic Programming),* R. Fletcher *(Methods for Solving Non-Linearly Constrained Optimization Problems),* and E. M. L. Beale *(Integer Programming).*

Part IV: *Ordinary Differential Equations and Quadrature.* The initial value problem (including the problem of stiff systems) is discussed by J. D. Lambert and the boundary value problem by J. Walsh. J. E. Lyness *(Quid, Quo, Quadrature)* deals in detail with the principles of automatic quadrature routines and multidimensional quadrature.

Part V: *Approximation Theory.* It contains three chapters: *Numerical Methods for Fitting Functions of Two Variables* by M. J. D. Powell, *Recent Results in Approximation Theory* by D. Kershaw, and *A Survey of Numerical Methods for Data and Function Approximation* by M. G. Cox.

Part VI: *Parabolic and Hyperbolic Problems* consisting of the following chapters: *Finite Element Methods in Time Dependent Problems* by A. R. Mitchell, *Initial-Value Problems by Finite Difference and Other Methods* by K. W. Morton with special emphasis on stability problems and convergence, *Splitting Methods for Time Dependent Partial Differential Equations* by A. R. Gourlay.

Part VII: *Elliptic Problems and Integral Equations.* An exhaustive study is given by L. Fox *(Finite-Difference Methods for Elliptic Boundary-Value Problems),* presenting new developments in economic direct methods, strongly implicit iterative and factorization methods, mechanization of some acceleration devices, etc. The further chapters are by R. Wait *(Finite Element Methods for Elliptic Problems)* and Ben Noble *(The Numerical Solution of Integral Equations).*

It may be anticipated that a large number of those practicing numerical analysis in industry or at universities and technical colleges will find great value in reading this book. Their knowledge and appreciation of the different aspects of numerical analysis should be greatly increased. It will also be of great value for teachers, as a source book of up-to-date information. Useful references for further study and a rich bibliography are added to this valuable work.

F. Móricz (Szeged)

André Weil, Elliptic functions according to Eisenstein and Kronecker (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 88), 93 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1976.

In the first part of the book the author successfully presents Eisenstein's approach to elliptic functions. One of the most important virtues of this approach is that it supplies directly (without recourse to function theory) many formulas on elliptic functions, in the explicit form, appropriate for their use in number theory.

The principal chapters of this part are: Trigonometric functions, The basic elliptic functions. Basic relations and infinite products.

The second part gives a systematic exposition of applications of Eisenstein's approach. It also reinterpretes some of the results of Kronecker using the theory of distributions.

The principal chapters of this part are: Kronecker's double series, Pell's equation and the Chowla—Selberg formula.

J. Németh (Szeged)

Alan I. Weir, General Integration and Measure, vol. 2, XI+298 pages, Cambridge University Press, 1974.

The present volume (comprising Chapters 8—17 of the work) is written as "self-contained" as possible. Chapter 8 introduces the notion of the Daniell integral. The fundamental Monotone and Dominated Convergence Theorems are established. Measurable functions are defined in terms of integrable functions by using the idea of truncation of a function by another function. The notion of measure follows naturally. Stone's theorem relating measure and integration is proved. In Chapter 9 Lebesgue—Stieltjes integrals are defined as Daniell integrals on spaces $L^1(R^k)$, which contain all the step functions (or equivalently all the continuous functions on R^k which vanish outside a compact set). Various forms of Riesz's Representation Theorem for bounded linear functionals on the space of continuous functions on a compact topological space provide the subject matter of Chapter 10. In Chapter 11 the general notion of measure is introduced, the extension of a measure on a ring to a measure on a σ-algebra is done by means of the Daniell integral. Chapter 12 contains a classical approach of the problem of integration with respect to a measure on a σ-algebra. Chapter 13 is devoted to the study of uniqueness of extensions of measures in the case where the universal space is σ-finite with respect to the measure. In Chapter 14 product measures are defined as extensions of a measure on elementary product sets to a measure on a σ-algebra. Chapter 16 introduces the notions of real and complex measures. The Jordan Decomposition Theorem shows the connection between positive and real or complex measures. In Chapter 17 the Radon—Nikodým theorem is proved both in measure theoretic form and for Daniell integrals, and used for a study of dual spaces of L^p spaces. A short Appendix gives a summary of the most important topological notions used in the text. All the chapters end with exercises; the solutions can be found at the end of the book.

An elementary knowledge of topological spaces is assumed. The book is offered for students.

L. Gehér (Szeged)

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