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On the Riesz summability of eigenfunction expansions

Š. A. ALIMOV and I. JOÓ

Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

Let Ω be an arbitrary bounded domain in $\mathbf{R}^N$ $(N\geqq 3)$ having C^∞-smooth boundary, and q an arbitrary non-negative function from the class $L_2(\Omega)$. Consider the Schrödinger operator

$$L = L(x, D) = -\Delta + q(x)\cdot.$$

Denote $\hat{L}$ an arbitrary positive selfadjoint extension of the operator L from the domain $C_0^\infty(\Omega)$ with discrete spectrum. According to a theorem of K. O. FRIEDRICHS [1, 2] there exists such a selfadjoint extension. Let $0<\lambda_1\leqq\lambda_2\leqq\dots$ denote the sequence of the eigenvalues of the operator $\hat{L}$ and let $\{u_n\}_1^\infty$ be the complete orthonormal system of the corresponding eigenfunctions in $L_2(\Omega)$. For any $s\geqq 0$ and $f\in L_2(\Omega)$, consider the s-th Riesz means of the spectral expansion of f:

$$E_\lambda^s f(x) = \sum_{\lambda_n<\lambda} \left(1-\frac{\lambda_n}{\lambda}\right)^s (f, u_n)\, u_n(x).$$

It is assumed in this work that the potential q is spherically symmetric. Namely, let $a\in C^\infty(0, \infty)$ be a non-negative function satisfying

$$(1) \qquad t^k|a^{(k)}(t)| \leqq C_\tau t^{\tau-1} \quad (t>0;\ k=0, 1 \dots, [N/2])$$

for some $\tau>0$. If $N=3$, then it is assumed that $\tau>1/2$. In particular, we have

$$(2) \qquad a(t) \leqq C_\tau t^{\tau-1} \quad (t>0).$$

The constant C_τ depends only on τ. Now assume that the potential q has the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|}, \quad x\in\Omega,$$

where $x_0\in\Omega$ is an arbitrary but fixed point.

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Denote by $\mathring{L}_p^l(\Omega)$ the set of those elements of $L_p^l(\mathbf{R}^N)$ for which $\operatorname{supp} f \subset \bar{\Omega}$. It is well known that $C_0^\infty(\Omega)$ is dense in $\mathring{L}_p^l(\Omega)$ with respect to the norm of $L_p^l(\mathbf{R}^N)$ (cf. [14], 4.3.2/1(b)).

We shall prove the following theorems.

Theorem 1. *Let* $p \geqq 1, s \geqq 0, l > 0, l+s \geqq (N-1)/2, pl > N$. *Then for any* $f \in \mathring{L}_p^l(\Omega)$,

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x), \quad x \in \Omega. \tag{3}$$

Theorem 2. *Let* $s \geqq 0, l \geqq 0, l+s \geqq (N-1)/2$. *Then for any* $f \in \mathring{L}_2^l(\Omega)$,

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = 0, \quad x \in \Omega \setminus \operatorname{supp} f. \tag{4}$$

Remark. For $q \equiv 0$, Theorem 1 was proved in [10] and it was extended in [5], for an arbitrary elliptic operator with smooth coefficients. Earlier, the case of $q \equiv 0, s = 0$, when l is an integer was settled by V. A. Il'in [8]. In [10] it is proved that if $l+s < (N-1)/2, p = \infty$ and $q \equiv 0$, then (3) does not hold for any $f \in \mathring{L}_p^l(\Omega)$.

For the proof of the Theorems we have to estimate the eigenfunctions u_n. We use the method of V. A. Il'in [9] and to this we need a mean value formula for the functions u_n. Thereafter, the theorems follow by applying Hörmander's Tauber type theorem [7]. To this it is necessary to estimate the Fourier coefficients of functions from Liouville classes using interpolation theorems and to estimate the resolvent of the operator $\hat{L}$ outside of an angular domain, which contains the spectrum $\{\lambda_n\}$.

1. The mean value formula and its application

Define

$$W(t, r) = \frac{1}{4} \pi^{1-N/2} \Gamma\left(\frac{N}{2}\right) [J_{N/2-1}(t) Y_{N/2-1}(r) - Y_{N/2-1}(t) J_{N/2-1}(r)],$$

$$\omega(t) = \min(1, t^{(1-N)/2}),$$

where J_ν and Y_ν denote the ν-th Bessel and Neumann function [6], respectively.

Lemma 1.1. *The estimate*

$$r^{1-N/2} \int_0^r \omega(t\mu) |W(t\mu, r\mu)| t^{N/2-1} a(t)\, dt \leqq C_1 \mu^{-\tau} \omega(\mu r) \tag{1.1}$$

holds for every $r > 0$ *and* $\mu > 0$.

Proof. Using well known asymptotic formulas (cf. [6], 7.2.1 (2), (4); 7.13.1 (3), (4)) it follows

$$|W(t, r)| \leqq C_0 \frac{1}{\sqrt{tr}}, \quad \text{if} \quad t \geqq 1, \tag{1.2}$$

$$|W(t, r)| \leqq C_0 \frac{1}{t^{N/2-1}\sqrt{r}}, \quad \text{if} \quad 0 < t \leqq 1 \leqq r, \tag{1.3}$$

$$|W(t, r)| \leqq C_0 \left(\frac{r}{t}\right)^{N/2-1}, \quad \text{if} \quad 0 < t \leqq r \leqq 1. \tag{1.4}$$

If $r\mu \leqq 1$ then, by (1.4), it follows

$$I \leqq C_0 \int_0^r a(t)\, dt \leqq C_0 C_\tau \int_0^r t^{\tau-1}\, dt \leqq \frac{C_0 C_\tau}{\tau} r^\tau \leqq \frac{C_0 C_\tau}{\tau} \mu^{-\tau} \omega(r\mu),$$

where I denotes the left hand side of (1.1). If $r\mu > 1$, then we use the decomposition $I = \int_0^{1/\mu} + \int_{1/\mu}^{r} = I_1 + I_2$ and apply to the estimation of I_1 and I_2, (1.3) and (1.2), respectively. It follows

$$I_1 \leqq C_0 r^{1-N/2} \int_0^{1/\mu} (t\mu)^{1-N/2} \frac{1}{\sqrt{r\mu}} t^{N/2-1} a(t)\, dt = C_0 (r\mu)^{(1-N)/2} \int_0^{1/\mu} a(t)\, dt \leqq$$

$$\leqq \frac{C_0 C_\tau}{\tau} \mu^{-\tau} \omega(r\mu),$$

$$I_2 \leqq C_0 r^{1-N/2} \int_{1/\mu}^{r} (t\mu)^{1-N/2} \frac{1}{\sqrt{t\mu}} \frac{1}{\sqrt{\mu r}} t^{N/2-1} a(t)\, dt =$$

$$= C_0 (r\mu)^{(1-N)/2} (1/\mu) \int_{1/\mu}^{r} \frac{a(t)}{t}\, dt \leqq \frac{C_0 C_\tau}{\tau} \omega(r\mu) \int_{1/\mu}^{r} t^{-2+\tau}\, dt \leqq \frac{C_0 C_\tau}{1-\tau} \mu^{-\tau} \omega(r\mu).$$

Lemma 1.2. *For every $r > 0$, $\mu_n > C_2$,*

$$\int_\theta u_n(x_0 + r\theta)\, d\theta = u_n(x_0)[C_N (r\mu_n)^{1-N/2} J_{N/2-1}(r\mu_n) + \alpha(r, \mu_n)], \tag{1.5}$$

$$|\alpha(r, \mu_n)| \leqq \text{const}\, \mu_n^{-\tau} \omega(r\mu_n), \tag{1.6}$$

where $\mu_n = \sqrt{\lambda_n}$, $C_N = 2^{N/2-1}\Gamma(N/2)$. Here $\int_\theta f(x_0 + r\theta) d\theta$ denotes the integration with respect to the normalized Lebesgue measure over the sphere of radius r and centred in x_0.

Proof. Recall the mean value formula of E. C. TITCHMARSH (cf. [13]; p. 232) stating

$$\int_\theta u_n(x_0+r\theta)\,d\theta = u_n(x_0)C_N(r\mu_n)^{1-N/2}J_{N/2-1}(r\mu_n)+$$

$$+\int_0^r\Big(\int_\theta q(x_0+t\theta)u_n(x_0+t\theta)\,d\theta\Big)(t/r)^{N/2-1}tW(t\mu_n, r\mu_n)\,dt.$$

In the case of our spherically symmetrical potential we get

$$\int_\theta u_n(x_0+r\theta)\,d\theta = u_n(x_0)C_N(r\mu_n)^{1-N/2}J_{N/2-1}(r\mu_n)+$$

$$+r^{1-N/2}\int_0^r\Big(\int_\theta u_n(x_0+t\theta)\,d\theta\Big)W(t\mu_n, r\mu_n)t^{N/2-1}a(t)\,dt,$$

i.e., the function $v^*(r, \mu_n)=\int_\theta u_n(x_0+r\theta)d\theta$ satisfies the integral equation

$$v^*(r, \mu_n) = u_n(x_0)C_N(r\mu_n)^{1-N/2}J_{N/2-1}(r\mu_n)+$$

$$+r^{1-N/2}\int_0^r v^*(t, \mu_n)W(t\mu_n, r\mu_n)t^{N/2-1}a(t)\,dt$$

of Volterra type (cf. [2]). Define

$$v_0(r, \mu_n) = u_n(x_0)C_N(r\mu)^{1-N/2}J_{N/2-1}(r\mu),$$

$$v_k(r, \mu) = r^{1-N/2}\int_0^r v_{k-1}(t, \mu)W(t\mu, r\mu)t^{N/2-1}a(t)\,dt.$$

The estimates $|v_k(r; \mu)|\leqq \text{const}\,\omega(r\mu)[c_1/\mu^\tau]^k$ $(r>0,\ \mu>0)$ follow by induction on k. On the other hand, it is easy to see that

$$v^*(r, \mu_n) = u_n(x_0)v_0(r, \mu_n)+u_n(x_0)\sum_{k=1}^\infty v_k(r, \mu_n).$$

Hence (1.5) and (1.6) follow for the function

$$\alpha(r, \mu)\overset{\text{def}}{=}\sum_{k=1}^\infty v_k(r, \mu).$$

Lemma 1.3. *We have*

$$\sum_{\substack{n\\ |\mu_n-\mu|\leqq 1}} |u_n(x_0)|^2 \leqq C_3\mu^{N-1} \quad (\mu\geqq 1). \tag{1.7}$$

The constant C_3 does not depend on μ.

Proof. We use the method of V. A. Il'in [8, 9]. Consider the function

$$d(r,\mu)=\begin{cases}\mu^{N/2}\dfrac{J_{N/2-1}(r\mu)}{r^{N/2-1}} & \text{if} \quad R<r<2R\\ 0 & \text{if} \quad r\notin(R,2R),\end{cases}$$

where $0<2R<\operatorname{dist}(x_0,\partial\Omega)$, $r=|x-x_0|$, $\mu>0$. Calculate the Fourier coefficients of $d(r,\mu)$ with respect to the system $\{u_n\}$. Taking into consideration (1.5), we obtain

$$d_n=d_n(\mu)=\int_\Omega d(|x-x_0|,\mu)\,u_n(x)\,dx=$$

$$=\mu^{N/2}\int_R^{2R}\frac{J_{N/2-1}(r\mu)}{r^{N/2-1}}\Big(\int_\theta u_n(x_0+r\theta)\,d\theta\Big)r^{N-1}\,dr=$$

$$=\mu^{N/2}u_n(x_0)\Big[C_N\int_R^{2R}J_{N/2-1}(r\mu)\frac{J_{N/2-1}(r\mu_n)}{\mu_n^{N/2-1}}\,r\,dr+\int_R^{2R}J_{N/2-1}(r\mu)\alpha(r,\mu_n)r^{N/2}\,dr\Big].$$

It is proved in [8] that

$$\Big|\int_R^{2R}J_{N/2-1}(r\mu)J_{N/2-1}(r\mu_n)\,r\,dr\Big|\ge c\cdot\frac{1}{\mu},$$

if $|\mu-\mu_n|\le 1$ and μ is large enough, where the constant c does not depend on μ. Hence

$$\Big|\mu^{N/2}u_n(x_0)\int_R^{2R}J_{N/2-1}(r\mu)\frac{J_{N/2-1}(r\mu_n)}{\mu_n^{N/2-1}}\,r\,dr\Big|\ge c\cdot|u_n(x_0)|,$$

if $|\mu-\mu_n|\le 1$ and $\mu\ge\mu_0$. On the other hand, using (1.6), we obtain

$$\Big|\int_R^{2R}J_{N/2-1}(r\mu)\alpha(r,\mu_n)r^{N/2}\,dr\Big|\le\text{const}\,\mu_n^{-\tau}\Big|\int_R^{2R}J_{N/2-1}(r\mu)\,\omega(r\mu_n)r^{N/2}\,dr\Big|\le$$

$$\le\text{const}\,\mu_n^{-\tau}\mu_n^{(1-N)/2}\Big|\int_R^{2R}J_{N/2-1}(r\mu)\sqrt{r}\,dr\Big|=O\Big(\mu_n^{-\tau}\mu_n^{(1-N)/2}\cdot\frac{1}{\sqrt{\mu}}\Big).$$

Summarising our estimates, we get

$$|d_n(\mu)|\ge\text{const}\,|u_n(x_0)|\Big[1+O\Big(\frac{1}{\mu^\tau}\Big)\Big]\ge\text{const}\,|u_n(x_0)|,$$

if $|\mu_n-\mu|\le 1$ and μ is large enough. Hence the desired estimate (1.7) follows by applying the Parseval equality and the relation

$$\int_\Omega|d(|x-x_0|,\mu)|^2\,dx=O(\mu^{N-1}).$$

2. Estimates for the Fourier coefficients of functions from Liouville spaces

Lemma 2.1. *Let k be a natural number and $\alpha=(\alpha_1, \dots, \alpha_N)$ a multiindex such that $0\leqq|\alpha|<k\leqq N/2$. Then for every $\varepsilon>0$, there exists a constant $C_4=C_4(\varepsilon)$ for which*

$$\||x-x_0|^{\varepsilon-(k-|\alpha|)}D^\alpha f(x)\|_{L_2(\Omega)} \leqq C_4\|f\|_{W_2^k(\Omega)} \tag{2.1}$$

holds for all $f\in W_2^k(\Omega)$.

Proof. The estimate follows immediately using classical imbedding theorems and the Hölder inequality.

Lemma 2.2. *For every natural number k, $0\leqq k\leqq N/2$, the estimate*

$$\|\hat{L}^{k/2}f\|_{L_2} \leqq C_5\|f\|_{W_2^k} \tag{2.2}$$

holds for every $f\in \mathring{W}_2^k(\Omega)$. The constant C_5 does not depend on f.

Proof. According to the spectral theorem, we have to prove the estimate

$$\sum_{n=1}^{\infty} \lambda_n^k|(f, u_n)|^2 \leqq \text{const}\, \|f\|_{W_2^k}. \tag{2.3}$$

By definition, $\mathring{W}_2^k(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W_2^k(\Omega)$. Hence it is enough to prove (2.3) for functions from the class $C_0^\infty(\Omega)$. If $k=2m$, then we have

$$\sum_{n=1}^{\infty} \lambda_n^k|(f, u_n)|^2 = \|(\Delta-q)^m f\|_{L_2}^2$$

for every $f\in C_0^\infty(\Omega)$. In this case we use Lemma 2.1 and the following simple facts: for any natural number $1\leqq m\leqq N/2$, we can write

$$(\Delta-q)^m = \Delta^m + \sum_{|\alpha|\leqq 2m-2} C_{m,\alpha}(x)D^\alpha, \tag{2.4}$$

where the functions $C_{m,\alpha}(x)$ belong to $C^\infty(\Omega\setminus\{x_0\})$ and (for each multiindex β) we have

$$|D^\beta C_{m,\alpha}(x)| \leqq \text{const}\, |x-x_0|^{|\alpha|+\tau-2m-|\beta|}. \tag{2.5}$$

These facts follow easily by induction on m. For the sake of simplicity, in this section we assume $x_0=0$. In the case $k=2m$, (2.3) follows immediately. If $k=2m+1$, then we have

$$\sum_{n=1}^{\infty} \lambda_n^k|(f, u_n)|^2 = \|\nabla(\Delta-q)^m f\|_{L_2}^2 - \|\sqrt{q}\,(\Delta-q)^m f\|_{L_2}^2. \tag{2.6}$$

For the estimation of the first term on the right hand side we use (2.4):

$$\nabla(\Delta-q)^m f = \nabla\Delta^m f + \sum_{|\alpha|\leqq 2m-2} [(\nabla C_{m,\alpha})D^\alpha f + C_{m,\alpha}\nabla D^\alpha f],$$

so it is enough to prove the estimate

$$\| |x|^{|\alpha|-k+\tau} D^\alpha f\|_{L_2} \leqq \text{const} \|f\|_{W_2^k},$$

for every natural number $k, 0=k\leqq N/2$, but this is the statement of Lemma 2.1. To estimate the second term on the right hand side of (2.6), we use (2.4). It follows:

$$\sqrt{q}(\Delta-q)^m f = \sqrt{q}\Delta^m f + \sum_{|\alpha|\leqq 2m-2} \sqrt{q}\, C_{m,\alpha} D^\alpha f.$$

Hence, taking into account the trivial estimates

$$\left|(\sqrt{q}\,\Delta^m f)(x)\right| \leqq \text{const}\, |x|^{-1+\tau/2} |\Delta^m f(x)| \leqq \text{const} \sum_{|\alpha|=2m} |x|^{\tau/2-(k-|\alpha|)} |D^\alpha f(x)|$$

and

$$\left|(\sqrt{q}\, C_{m,\alpha} D^\alpha f)(x)\right| \leqq \text{const}\, |x|^{-1+\tau/2} |x|^{|\alpha|+\tau-2m} |D^\alpha f(x)| \leqq \text{const}\, |x|^{3\tau/2-(k-|\alpha|)},$$

the desired result follows by applying Lemma 2.1.

Lemma 2.3. *For any real number* $s, 0\leqq s\leqq N/2$, *we have*

$$\|\hat{L}^{s/2} f\|_{L_2} \leqq C_6 \|f\|_{\mathring{L}_2^s} \tag{2.7}$$

for every $f\in \mathring{L}_2^s(\Omega)$. *The constant* C_6 *does not depend on* f.

Proof. We use Lemma 2.2 and apply Theorem 4.3.2/2 of [14] for $\mathscr{H}=L_2(\Omega)$; $W=\mathring{W}_2^{[N/2]}(\Omega)$, $A=\hat{L}^{[N/2]/2}$. Using the notations of [14], we obtain $(L_2; \mathring{W}_2^l)_{\theta,2}=\mathring{L}_2^{\theta l}$ for $l=[N/2]$, $s=\theta l$. (Here $A^\theta=\hat{L}^{s/2}$, $\mathring{L}_2^{\theta l}=\mathring{L}_2^s$.) Hence (2.7) follows.

Lemma 2.4. *There exists* $\sigma>N/4$ *such that*

$$C_0^\infty(\Omega)\subset \operatorname{dom}(\hat{L}^\sigma). \tag{2.8}$$

Proof. Let $m=[N/4]$. Applying (2.4) we have

$$\hat{L}^m f = \Delta^m f + \sum_{|\alpha|\leqq 2m-2} C_{m,\alpha} D^\alpha f, \quad f\in C_0^\infty(\Omega),$$

and

$$|\hat{L}^m f(x)| \leqq \text{const}\, |x|^{\tau-2m}, \quad |\nabla \hat{L}^m f(x)| \leqq \text{const}\, |x|^{\tau-2m-1}.$$

Hence, $\nabla\hat{L}^m f\in L_p$ if $(\tau-2m-1)p>-N$, i.e., $N/p>2m-\tau+1$. It follows $\hat{L}^m f\in W_p^1$ if $f\in C_0^\infty(\Omega)$. By the classical imbedding theorem $W_p^1 \hookrightarrow W_2^\delta$ if $\delta-N/2=1-N/p$ (cf. [12], Ch. 6). It follows: $\hat{L}^m f\in \mathring{L}_2^\delta$ for every $\delta<N/2-2m+\tau$. Thus, using Lemma 2.3, we get $\hat{L}^{m+\delta/2} f\in L_2$, i.e., $f\in\operatorname{dom}(\hat{L}^{m+\delta/2})$ for every $f\in C_0^\infty(\Omega)$. Choose $\delta= N/2-2m+\tau-\varepsilon$, where $\varepsilon>0$ is small enough; then $m+\delta/2=m+N/4-m+$ $+\tau/2-\varepsilon/2=N/4+(\tau-\varepsilon)/2$, and if $\varepsilon<\tau$, then we have $\sigma\overset{\text{def}}{=}m+\delta/2>N/4$.

3. Estimation of the Green function

Let $R(\lambda, \hat{L})$ denote the resolvent of the operator $\hat{L}$, i.e., $R(\lambda, \hat{L})=(\hat{L}-\lambda I)^{-1}$ and $G_\mu=R(\mu^2, \hat{L})=(\hat{L}-\mu^2 I)^{-1}$. Let $0<\varepsilon_0<\pi/2$ be an arbitrary small real number and define

$$Z_0 = \{z\in\mathbf{C}:\ \varepsilon_0 \leqq \arg z \leqq \pi-\varepsilon_0\}.$$

Set $\mu=\sqrt{\lambda}$ with $\operatorname{Im}\mu\geqq 0$, i.e., $0\leqq\arg\mu\leqq\pi$.

The aim of the present paragraph is to investigate (estimate) the Green function of the operator $\hat{L}-\lambda I$, i.e., the kernel function of the resolvent $(\hat{L}-\lambda I)^{-1}$. Using the method of E. E. Levi (cf. [7], [13]) first we construct a fundamental solution $E(x, y, \mu)$ of the operator $\hat{L}-\lambda I$, i.e., a function for which

$$(-\Delta+q(x)\cdot-\mu^2 I)E(x, y, \mu) = \delta(x-y) \quad (x, y\in\Omega).$$

In case of $q\equiv 0$, the fundamental solution $E_0(x, y, \mu)$ which decreases exponentially for $\operatorname{Im}\mu>0$ is the following:

$$E_0(x, y, \mu) = C_N(\mu/r)^{N/2-1}H^{(1)}_{N/2-1}(r\mu)$$

(cf. [13], (13.7.2)). Here $H^{(1)}_\nu(z)$ denotes the ν-th Hankel function of first order. Obviously, the exponentially decreasing fundamental solution E is the solution of the integral equation

$$E(x, y, \mu) = E_0(x, y, \mu)-\int_\Omega E_0(x, u, \mu)E(u, y, \mu)\,q(u)\,du.$$

Now define

$$E_k(x, y, \mu)\overset{\text{def}}{=} E_0(x, y, \mu)-\int_\Omega E_0(x, u, \mu)E_{k-1}(u, y, \mu)\,q(u)\,du,$$

$$F_0(x, y, \mu)\overset{\text{def}}{=} E_0(x, y, \mu), \quad F_k(x, y, \mu)\overset{\text{def}}{=} E_k(x, y, \mu)-E_{k-1}(x, y, \mu) \quad (k = 1, 2, \ldots),$$

and $k_0\overset{\text{def}}{=}[(N-2)/\tau]$. Obviously,

$$E(x, y, \mu) = \sum_{k=0}^{\infty} F_k(x, y, \mu) \tag{3.1}$$

if the series is uniformly convergent. Furthermore,

$$F_k(x, y, \mu) = -\int_\Omega E_0(x, u, \mu)F_{k-1}(u, y, \mu)\,q(u)\,du. \tag{3.2}$$

Our first aim is to prove that the series (3.1) has good convergence properties. To this we must estimate the functions F_k. If $p>N/(N-2+\tau)$, then $p'<N/(2-\tau)$ $(1/p+1/p'=1)$ and hence, taking into account that, according to our assumption on q,

$$q(x) = a(|x-x_0|)|x-x_0|^{-1} \leqq c_\tau|x-x_0|^{-2+\tau}\in L_{p'}(\Omega),$$

we obtain the following estimate for F_k:

(3.3) $$|F_k(x, y, \mu)|^p \leqq \|q\|_{L_{p'}}^p \int_\Omega |E_0(x, u, \mu)|^p |F_{k-1}(u, y, \mu)|^p \, du$$

for every $p > N/(N-2+\tau)$.

Lemma 3.1. *If $k \leqq k_0$, then for any $\delta \in (0, \tau)$,*

(3.4) $$|F_k(x, y, \mu)| \leqq C_7 |x-y|^{2-N+k\delta} e^{-\alpha|x-y||\mu|} \quad (x, y \in \Omega, \ \mu \in Z_0).$$

Here C_7 and α are positive constants not depending on x, y and μ.

Proof. For $k=0$ we have $F_0 = E_0$, and (3.4) follows from

(3.5) $$|E_0(x, y, \mu)| \leqq C_7 |\mu|^{-\varkappa} |x-y|^{2-\varkappa-N} e^{-\alpha|x-y||y|} \quad (x, y \in \Omega; \ 0 \leqq \varkappa \leqq 2).$$

Here C_7 and α are positive constants, $\mu \in Z_0$. This estimate is immediate from [6] (7.2.1 (2) and (5); 7.3.1 (1)). Suppose, (3.4) is fulfilled for $k-1$ in place of k. Using (3.3) and the fact that $|x-y| \leqq |x-u| + |u-y|$ implies $e^{-\alpha|x-u||\mu|} e^{-\alpha|u-y||\mu|} \leqq$ $\leqq e^{-\alpha|x-y||\mu|}$, we obtain by the induction hypothesis that

$$|F_k(x, y, \mu)|^p \leqq \text{const}\, e^{-\alpha|x-y||\mu|} \int_\Omega |x-u|^{(2-N)p} |u-y|^{(2-N-(k-1)\delta)\delta} \, du \leqq$$

$$\leqq \text{const}\, e^{-\alpha|x-y||\mu|p} |x-y|^{(2-N-k\delta)p}.$$

Thus (3.4) follows by induction.

Lemma 3.2. *If $k > k_0$, then for every $\varkappa \in (0, \tau)$,*

(3.6) $$|F_k(x, y, \mu)| \leqq C_8 e^{-\alpha|x-y||\mu|} (C^* |\mu|^{-\varkappa})^{k-k_0-1} \quad (x, y \in \Omega; \ \mu \in Z_0).$$

The constants C_8 and C^* do not depend on x, y and μ.

Proof. Let $k = k_0 + 1$. Choose $\delta \in (0, \tau)$ such that $N-2 < (k_0+1)\delta$. According to the definition of k_0, this is possible. Then choose $p\ (> N/(N-2+\tau))$ so that $1/p = (N-2)/N + \delta/N$, apply (3.3) and (3.5) for $\varkappa = 0$ and (3.4) for $k = k_0$, respectively. Using the notation $(N-2-k_0\delta)p = \varepsilon$ it follows that

$$|F_k(x, y, \mu)|^p \leqq \text{const}\, e^{-\alpha|x-y||\mu|p} \int_\Omega |x-u|^{p\delta-N} |u-y|^{-\varepsilon} \, du.$$

According to the definition of k_0, we have $k_0\tau \leqq N-2 < (k_0+1)\tau$ and hence $k_0\delta < N-2$, i.e., $\varepsilon > 0$. Furthermore, according to the choice of δ we have $\varepsilon = (N-2-k_0\delta)p = p\delta((N-2)/\delta - k_0) < p\delta(k_0+1-k_0) = p\delta$. A result of Titchmarsh's book ([13], 22.1) states that, in this case,

$$\int_\Omega |x-u|^{p\delta-N} |u-y|^{-\varepsilon} \, du < \infty.$$

Thus (3.6) is proved for $k = k_0 + 1$. Then (3.6) follows by induction on k.

It follows from (3.6) that the series in (3.1) converges for every $x, y \in \Omega$, if μ is large enough, and hence shifting the spectrum of the operator $\hat{L}$ we obtain

Lemma 3.3. *For any* $x, y \in \Omega$ *and* $\mu \in Z_0$, *we have the estimate*

$$|E(x, y, \mu)| \leqq C_9 |x-y|^{2-N} e^{-\alpha |x-y| |\mu|}. \tag{3.7}$$

Let $E^t(x, y, \mu) \overset{\text{def}}{=} E(y, x, \mu)$ (the formal adjoint of E). A standard calculation shows that for any $f \in W_p^2(\Omega)$ with $p > N/2$ and $\operatorname{supp} f \subset \Omega$, the equality

$$\int_\Omega E^t(x, y, \mu)[L(y, D) - \mu^2] f(y)\, dy = f(x), \quad x \in \Omega$$

holds. Let Ω_0 and Ω_1 be two domains in $\mathbf{R}^N$ for which $x_0 \in \Omega_0$, $\bar{\Omega}_0 \subset \Omega_1$ and $\bar{\Omega}_1 \subset \Omega$. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta(x) = 1$ if $x \in \Omega_1$. Define

$$H(x, y, \mu) \overset{\text{def}}{=} E^t(x, y, \mu) \cdot \eta(y)$$

and

$$K(x, y, \mu) \overset{\text{def}}{=} 2(\nabla_y \eta(y)) \nabla_y E^t(x, y, \mu) + (\Delta_y \eta(y)) E^t(x, y, \mu).$$

Obviously,

$$\big(L(y, D) - \mu^2\big) H(x, y, \mu) = \eta(y)\big(L(y, D) - \mu^2\big) E^t(x, y, \mu) -$$

$$-2(\nabla \eta) \nabla_y E^t(x, y, \mu) - (\Delta \eta) E^t(x, y, \mu).$$

Furthermore, $K(x, y, \mu) = 0$ if $y \in \Omega$ and hence, using (3.7), we get

$$|K(x, y, \mu)| \leqq \text{const}\, e^{-c|\mu|} \quad (x \in \Omega_0,\ y \in \Omega,\ \mu \in Z_0). \tag{3.8}$$

It is easy to verify that for every $f \in W_p^{2, \text{loc}}(\Omega)$ $(p > N/2)$ the equality

$$[\hat{H}(L - \mu^2 I) f](x) = [f - \hat{K} f](x), \quad x \in \Omega_0 \tag{3.9}$$

holds, where the operation $\hat{}$ is defined by

$$(\hat{\varphi} f)(x) = \int_\Omega \varphi(x, y) f(y)\, dy.$$

On the other hand, for any $f \in C_0^\infty(\Omega)$ we have $(\hat{L} - \lambda I)^{-1} f \in W_p^{2, \text{loc}}(\Omega)$ if $p > N/2$ and $q \in L_p(\Omega)$. Now apply (3.9) for $(\hat{L} - \lambda I)^{-1} f$ $\big(f \in C_0^\infty(\Omega)\big)$ instead of f. It follows

$$\hat{H} f(x) = \hat{G}_\mu f(x) - \hat{K} \hat{G}_\mu f(x) \quad \big(f \in C_0^\infty(\Omega),\ x \in \Omega_0\big)$$

and hence, by continuous extension we get

Lemma 3.4. *For every* $f \in L_2(\Omega)$,

$$\hat{G}_\mu f(x) - \hat{H} f(x) = \hat{K} \hat{G}_\mu f(x), \quad x \in \Omega_0. \tag{3.10}$$

Using the equality (3.10) and the estimates (3.7) and (3.11) we obtain by an easy calculation

Lemma 3.5. *Let* $0 \leqq l \leqq (N-1)/2$ *and* $0 \leqq \varepsilon < 1/2$. *Then for any* $f \in \mathring{L}_2^l(\Omega)$ *for which* $f(x)=0$ *whenever* $|x-x_0| \leqq r$ *we have*

$$(3.11) \qquad |\hat{G}_\mu f(x_0)| \leqq \text{const}\, \frac{r^{l-N/2}}{|\mu|^2} e^{-\alpha r|\mu|} (r|\mu|)^\varepsilon \|f\|_{L_2^l} \quad (\mu \in Z_0).$$

Lemma 3.6. *Let* $\sigma > N/4$. *Then for any* $f \in L_2(\Omega)$ *and on each compact set* $K \subset \Omega$ *we have the estimate*

$$(3.12) \qquad \|\hat{L}^{-\sigma} f\|_{L_\infty(K)} \leqq C_{10}(K) \|f\|_{L_2(\Omega)}.$$

Proof. First remark that the following fact is easily proved by induction on k: Let $N/4 = m+\delta$, where m is a natural number and $0 \leqq \delta < 1$ (i.e. $\delta = 0$, $1/4$, $2/4$, $3/4$). Then for every $f \in L_2(\Omega)$ and $0 \leqq k \leqq m$ (k is a natural number) we have

$$(3.13) \qquad \|\hat{L}^{-k} f\|_{L_{p_k}(\Omega')} \leqq \text{const}\, \|f\|_{L_2(\Omega)} \quad (\Omega' \subset\subset \Omega,\ 1/p_k = 1/2 - 2k/N).$$

Thereafter, we prove (3.12) for some $\delta < \varepsilon < 1$ if $\sigma = m + \varepsilon$. Define

$$\hat{H}_\varepsilon \overset{\text{def}}{=} \frac{\sin \pi\varepsilon}{\pi} \int_0^\infty t^{-\varepsilon} \hat{H}_t \, dt;$$

then, using

$$\hat{L}^{-\sigma} = \hat{L}^{-m-\varepsilon} = \frac{\sin \pi\varepsilon}{\pi} \int_0^\infty t^{-\varepsilon} (\hat{L} + tI)^{-1} \hat{L}^{-m} \, dt$$

([14], 1.15.1 (1)) we obtain $\hat{L}^{-\sigma} = \hat{H}_\varepsilon \hat{L}^{-m} + \hat{K}_\varepsilon \hat{L}^{-m}$. Obviously,

$$\|\hat{K}_\varepsilon \hat{L}^{-m} f\|_{L_\infty(K)} \leqq \text{const}\, \|f\|_{L_2(\Omega)}$$

and

$$|H(x, y, \lambda)| \leqq \text{const}\, |x-y|^{2-N} e^{-\alpha|x-y|\,|\sqrt{\lambda}|},$$

hence $|H_\varepsilon(x, y)| \leqq \text{const}\, |x-y|^{2\varepsilon - N}$. By Hölder's inequality

$$\|\hat{H}_\varepsilon g\|_{L_\infty(K)} \leqq \text{const}\, \|g\|_{L_p(\Omega')} \quad (K \subset \Omega' \subset\subset \Omega,\ g = \hat{L}^{-m} f)$$

if $1/p < 2\varepsilon/N$, i.e., $\varepsilon > N/2p$; on the other hand, by (3.13),

$$\|\hat{L}^{-m} f\|_{L_p(\Omega')} \leqq \text{const}\, \|f\|_{L_2(\Omega)}, \quad 1/p = 1/2 - 2m/N.$$

But $\varepsilon > \delta = N/4 - m = (N/2)(1/2 - 2m/N) = (N/2)(1/p)$, i.e., $\varepsilon > N/2p$.

Corollary 1. *For every* $\sigma > N/4$ *the estimate*

$$(3.14) \qquad \sum_{n=1}^\infty |u_n(x)|^2 \lambda_n^{-2\sigma} \leqq C_{11}(K), \quad x \in K$$

holds uniformly on every compact subset $K \subset \Omega$.

Proof. By the spectral theorem, for any $f \in L_2(\Omega)$ we have

$$\hat{L}^{-\sigma} f \stackrel{L_2}{=} \sum_{n=1}^{\infty} (f, u_n) u_n(x) \lambda_n^{-\sigma},$$

and hence, using Lemma 3.6, the estimate

$$\left| \sum_{n=1}^{\infty} (f, u_n) u_n(x) \lambda_n^{-\sigma} \right| \leqq \text{const} \left(\sum_{n=1}^{\infty} |(f, u_n)|^2 \right)^{1/2}$$

follows, which implies the statement.

Corollary 2. *For any* $\sigma > N/4$ *and* $f \in \operatorname{dom}(\hat{L}^{\sigma})$, *the series*

$$\sum_{n=1}^{\infty} (f, u_n) u_n(x) \tag{3.15}$$

converges absolutely and uniformly on every compact set $K \subset \Omega$.

Proof. Using (3.14) the statement of Corollary 2 follows from the estimates

$$\sum_{k=n}^{n+p} |(f, u_k) u_k(x)| \leqq \left(\sum_{k=n}^{n+p} |(f, u_k)|^2 \lambda_k^{2\sigma} \right)^{1/2} \left(\sum_{k=n}^{n+p} |u_k(x)|^2 \lambda_k^{-2\sigma} \right)^{1/2} \leqq$$

$$\leqq \text{const} \left(\sum_{k=n}^{n+p} |(f, u_k)|^2 \lambda_k^{2\sigma} \right)^{1/2} \to 0 \quad (n, p \to \infty).$$

Corollary 3. *For every* $f \in C_0^{\infty}(\Omega)$ *the spectral expansion* $E_\lambda f(x)$ *tends to* $f(x)$ *as* $\lambda \to \infty$, *uniformly on every compact set* $K \subset \Omega$.

Proof. This follows immediately from Corollary 2, using Lemma 2.4.

4. Localization and convergence of the Riesz means

In this section we prove Theorem 1 only, because the proof of Theorem 2 goes on by the same argument.

Lemma 4.1. *Suppose* $0 \leqq l \leqq [N/2]$, $h > 0$, $t > 0$. *Then we have*

$$|\varphi[(t+h)^2] - \varphi(t^2)| \leqq C_{12} \|f\|_{L_2^l} (1 + \sqrt{h})(t+h)^{(N-1)/2-l} \tag{4.1}$$

where

$$\varphi(t) \stackrel{\text{def}}{=} E_t f(x_0) = \sum_{\lambda_n < t} (f, u_n) u_n(x_0), \quad f \in \mathring{L}_2^l(\Omega).$$

Proof. (4.1) follows from Lemma 1.3 immediately.

Using the method of [4] and the estimate (4.1), the following statement follows by applying Hörmander's Tauber type theorem (cf. [7]).

Lemma 4.2. *Suppose* $0 \leqq l < N/2$, $f \in \mathring{L}_2^l(\Omega)$ *and* $f(x)=0$ *if* $|x-x_0| \leqq r$. *Then we have the following estimate for every* $s \geqq 0$:

$$(4.2) \qquad |E_\lambda^s f(x_0)| \leqq \text{const}\, \|f\|_{L_2^l} \lambda^{(1/2)(N/2-l)} (1+r\sqrt{\lambda})^{-1/2-s}.$$

Lemma 4.3. *Let* $s \geqq 0$, $l > 0$, $p > 1$ *and*

$$(4.3) \qquad s+l = (N-1)/2, \quad 0 < l - N/p < 1.$$

Then for every $f \in \mathring{L}_p^l(\Omega)$,

$$(4.4) \qquad |E_\lambda^s f(x_0)| \leqq \text{const}\, \|f\|_{L_p^l}.$$

Proof. 1°. First suppose that $f \in \mathring{L}_p^l(\Omega)$ is such that $f(x_0)=0$. Let $0 \leqq \varphi \in C_0^\infty(\Omega)$ be a function for which $\operatorname{supp} \varphi \subset (1/4, 1)$ and $\varphi(t/2)+\varphi(t)=1$ $(1/2<t<1)$. Taking into consideration that Ω is bounded, there exists a natural number k^* such that for any $f \in \mathring{L}_p^l(\Omega)$ and $x \in \Omega$ we have

$$f(x) = f(x) \sum_{k=-k^*}^{\infty} \varphi(2^k r), \quad r = |x-x_0|.$$

Denote

$$f_k(x) = f(x)\varphi(2^k r) \in \mathring{L}_p^l(\Omega) \quad (k \geqq -k^*).$$

Obviously $f_k(x)=0$ if $|x-x_0| \leqq c2^{-k}$, and by Lemma (4.2) it follows

$$|E_\lambda^s f_k(x_0)| \leqq \text{const}\, \|f_k\|_{L_p^l} \lambda^{(1/2)(N/2-l)} \cdot (1+2^{-k}\sqrt{\lambda})^{-1/2-s}.$$

Hence, using (4.3) and the estimate $\|f_k\|_{L_2^l} \leqq \text{const}\, 2^{-kN(1/2-1/p)} \|f\|_{L_p^l}$ (cf. [4], Lemma 1.1) we obtain

$$|E_\lambda^s f_k(x_0)| \leqq \text{const}\, 2^{-k(N/2-N/p-1/2-s)} (\sqrt{\lambda})^{(N-1)/2-l-s} \|f\|_{L_p^l} \leqq \text{const}\, 2^{-k(l-N/p)} \|f\|_{L_p^l}.$$

Consequently

$$|E_\lambda^s f(x_0)| \leqq \text{const}\, \|f\|_{L_p^l} \cdot \sum_{k=-k^*}^{\infty} 2^{-k(l-N/p)} \leqq \text{const}\, \|f\|_{L_p^l}.$$

2°. Now suppose $f(x_0) \neq 0$. Let $g \in C_0^\infty(\Omega)$ such that $g(x_0)=1$. Denote $f_1(x) = f(x) - f(x_0)g(x)$. According to 1° we have

$$(4.5) \qquad |E_\lambda^s f_1(x_0)| \leqq \text{const}\, \|f\|_{L_p^l}.$$

By Corollary 3 of Lemma 3.6, the expansion $E_\lambda^s g(x_0)$ is bounded, and hence $|E_\lambda^s f(x_0) g(x)| \leqq \text{const}\, |f(x_0)|$. Using the imbedding theorem $L_p^l \hookrightarrow L_\infty$ if $l - N/p > 0$ we get

$$(4.6) \qquad |E_\lambda^s(f(x_0)g(x))| \leqq \text{const}\, \|f\|_{L_p^l}.$$

From (4.5) and (4.6) we obtain (4.4). Thus Lemma 4.3 is proved.

Proof of Theorem 1. Theorem 1 follows from Lemma 4.3, using the facts that, according to Corollary 3 of Lemma 3.6, it is true for every $f\in C_0^\infty(\Omega)$, and the set $C_0^\infty(\Omega)$ is dense in $\mathring{L}_p^l$.

Proof of Theorem 2. This theorem follows from Lemma 4.2 by the same argument as that of the proof of Theorem 1.

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An inequality between symmetric function means of positive operators

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Dedicated to Professor B. Szőkefalvi-Nagy on his seventieth birthday

1. There are various methods of averaging of an n-tuple $\vec{A}=(A_1, \ldots, A_n)$ of bounded *positive* (semi-definite) operators on a Hilbert space. The most basic are the *arithmetic mean* $(A_1+\ldots+A_n)/n$ and the *harmonic mean* $n(A_1^{-1}+\ldots+A_n^{-1})^{-1}$ (provided all A_i are invertible). ANDERSON and TRAPP [2] called $(A_1^{-1}+\ldots+A_n^{-1})^{-1}$ the *parallel sum* of the n-tuple $\vec{A}$, and denoted it by $A_1: \ldots :A_n$, or in short $\prod_{i=1}^{n}: A_i$. Further they gave a variational description for parallel sum;

$$\left\langle x, \left(\prod_{i=1}^{n}: A_i\right) x\right\rangle = \inf\left\{\sum_{i=1}^{n} \langle x_i, A_i x_i\rangle \,\middle|\, x = \sum_{i=1}^{n} x_i\right\}, \tag{1}$$

where $\langle x, y\rangle$ denotes the inner product of the vectors x and y. Formula (1) was then used to define the parallel sum for a general n-tuple of positive operators.

For an n-tuple of positive numbers, $\vec{\alpha}=(\alpha_1, \ldots, \alpha_n)$, MARCUS and LOPES [5] defined symmetric function means (or Marcus—Lopes means) $E_{k,n}(\vec{\alpha})$ by

$$E_{k,n}(\vec{\alpha}) \equiv \frac{e_{k,n}(\vec{\alpha})}{e_{k-1,n}(\vec{\alpha})}, \quad k = 1, 2, \ldots, n \tag{2}$$

where $e_{k,n}(\vec{\alpha})$ is the normalized k-th elementary symmetric function of $\vec{\alpha}= =(\alpha_1, \ldots, \alpha_n)$, that is,

$$e_{0,n}(\vec{\alpha}) \equiv 1 \quad \text{and} \quad e_{k,n}(\vec{\alpha}) \equiv \frac{\sum\limits_{1\le i_1<\ldots<i_k\le n} \prod\limits_{j=1}^{k} \alpha_{i_j}}{\binom{n}{k}}.$$

Using an equivalent version of definition (2), ANDERSON, MORLEY, and TRAPP [3] introduced two kinds of *symmetric function means* for an n-tuple $\vec{A}=(A_1, \ldots, A_n)$

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of positive operators;

$$\mathfrak{S}_{1,n}(\vec{A}) \equiv \left(\sum_{i=1}^{n} A_i\right)/n \quad \text{(arithmetic mean)},$$

$$\mathfrak{s}_{n,n}(\vec{A}) \equiv n\left(\prod_{i=1}^{n}{}: A_i\right) \quad \text{(harmonic mean)},$$

and

$$\mathfrak{S}_{k,n}(\vec{A}) \equiv \sum_{i=1}^{n}\left\{\left(\frac{1}{n-k+1}A_i\right): \left(\frac{1}{k-1}\mathfrak{S}_{k-1,n-1}(\vec{A}_{(i)})\right)\right\}, \quad k = 2, \ldots, n$$

$$\mathfrak{s}_{k,n}(\vec{A}) \equiv \prod_{i=1}^{n}{}: \{kA_i+(n-k)\mathfrak{s}_{k,n-1}(\vec{A}_{(i)})\}, \quad k = 1, \ldots, n-1,$$

where $\vec{A}_{(i)}$ denotes the $(n-1)$-tuple $(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$. By definition both $\mathfrak{S}_{k,n}(\vec{A})$ and $\mathfrak{s}_{k,n}(\vec{A})$ are invariant under permutations of $A_1, \ldots, A_n$, and the maps $\vec{A}\mapsto\mathfrak{S}_{k,n}(\vec{A})$ and $\vec{A}\mapsto\mathfrak{s}_{k,n}(\vec{A})$ are positively homogeneous and monotone with respect to coordinatewise ordering. If all A_i are invertible, then

$$\mathfrak{S}_{k,n}(\vec{A}^{-1})^{-1} = \mathfrak{s}_{n-k+1,n}(\vec{A}), \quad k = 1, \ldots, n,$$

where $\vec{A}^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$. For any n-tuple $\vec{A}$

$$\mathfrak{s}_{1,n}(\vec{A}) = \mathfrak{S}_{1,n}(\vec{A}) \quad \text{and} \quad \mathfrak{S}_{n,n}(\vec{A}) = \mathfrak{s}_{n,n}(\vec{A}).$$

Besides the easily proved inequalities

$$n\left(\prod_{i=1}^{n}{}: A_i\right) \leqq \begin{Bmatrix}\mathfrak{S}_{k,n}(\vec{A})\\ \mathfrak{s}_{k,n}(\vec{A})\end{Bmatrix} \leqq \left(\sum_{i=1}^{n} A_i\right)/n, \quad k = 1, \ldots, n$$

not much is known about the order relation among $\mathfrak{S}_{j,n}(\vec{A})$ and $\mathfrak{s}_{k,n}(\vec{A})$, $j, k=2, \ldots, n-1$. If all A_i are scalars, that is, $\vec{A}=\vec{\alpha}$, then both $\mathfrak{S}_{k,n}(\vec{\alpha})$ and $\mathfrak{s}_{k,n}(\vec{\alpha})$ coincide with the Marcus—Lopes mean $E_{k,n}(\vec{\alpha})$. Therefore it follows via spectral theory that if $\vec{A}$ is a commuting n-tuple then

$$\mathfrak{S}_{k,n}(\vec{A}) = \mathfrak{s}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k+1,n}(\vec{A}) = \mathfrak{S}_{k+1,n}(\vec{A}), \quad k = 2, \ldots, n-2.$$

The equality $\mathfrak{S}_{k,n}(\vec{A})=\mathfrak{s}_{k,n}(\vec{A})$, $k=2, \ldots, n-1$ is not valid in general for a non-commuting n-tuple.

ANDERSON, MORLEY, and TRAPP [3] asked if the inequalities

$$\mathfrak{S}_{k,n}(\vec{A}) \geqq \mathfrak{S}_{k+1,n}(\vec{A}), \quad k = 2, \ldots, n-2$$

(or equivalently

$$\mathfrak{s}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k+1,n}(\vec{A}), \quad k = 2, \ldots, n-2)$$

and

$$\mathfrak{S}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k,n}(\vec{A}), \quad k = 2, \ldots, n-1$$

are valid for every n-tuple $\vec{A}$. They mentioned, without proof, that in case $n=3$ the inequality $\mathfrak{S}_{2,3}(\vec{A})\geqq\mathfrak{s}_{2,3}(\vec{A})$ could be derived via electrical network consideration.

The purpose of the present paper is to give a mathematical proof to the inequality $\mathfrak{S}_{2,3}(\vec{A}) \geqq \mathfrak{s}_{2,3}(\vec{A})$.

2. Our proof is based on a solution of an extremal problem, due to FLANDERS [4].

Lemma. *Given two set of vectors $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, define a functional $\psi(A)$ for an invertible positive operator A by*

$$\psi(A) \equiv \sum_{i=1}^{m} \langle x_i, Ax_i\rangle + \sum_{j=1}^{n} \langle y_j, A^{-1}y_j\rangle.$$

Then $\inf_A \psi(A) = 2\|[\langle x_i, y_j\rangle]\|_1$, where $\|[\langle x_i; y_j\rangle]\|_1$ is the trace norm of the $m\times n$ matrix $[\langle x_i, y_j\rangle]$.

See [1] and [4] for a proof.

Theorem. *For any triple $\vec{A}=(A_1, A_2, A_3)$ of positive operators*

$$\mathfrak{S}_{2,3}(\vec{A}) \geqq \mathfrak{s}_{2,3}(\vec{A}). \tag{3}$$

Proof. All A_i can be assumed invertible. Since $\mathfrak{s}_{2,3}(\vec{A}) = \mathfrak{S}_{2,3}(\vec{A}^{-1})^{-1}$, and

$$\langle x, \mathfrak{S}_{2,3}(\vec{A}^{-1})^{-1}x\rangle = \sup_y \frac{|\langle y, x\rangle|^2}{\langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1})y\rangle},$$

operator inequality (3) is equivalent to

$$\langle x, \mathfrak{S}_{2,3}(\vec{A})x\rangle^{1/2} \cdot \langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1})y\rangle^{1/2} \geqq |\langle x, y\rangle| \quad \text{for all} \quad x, y,$$

which is equivalent, in view of the arithmetic-geometric means inequality, to

$$\langle x, \mathfrak{S}_{2,3}(\vec{A})x\rangle + \langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1})y\rangle \geqq 2|\langle x, y\rangle| \quad \text{for all} \quad x, y. \tag{4}$$

Since

$$\mathfrak{S}_{2,3}(\vec{A}) = \frac{1}{2}\{A_1\colon (A_2+A_3) + A_2\colon (A_3+A_1) + A_3\colon (A_1+A_2)\},$$

formula (1) gives

$$\langle x, \mathfrak{S}_{2,3}(\vec{A})x\rangle =$$
$$= \inf_{x_1, x_2, x_3} \frac{1}{2} \sum_{i=1}^{3} \{\langle x+x_i, A_i(x+x_i)\rangle + \langle x_{i+1}, A_i x_{i+1}\rangle + \langle x_{i+2}, A_i x_{i+2}\rangle\}, \tag{5}$$

where $x_j \equiv x_{j-3}$ for $j=4, 5$, and similarly

$$\langle y, \mathfrak{S}_{2,3}(\vec{A}^{-1})y\rangle =$$
$$= \inf_{y_1, y_2, y_3} \frac{1}{2} \sum_{i=3}^{3} \{\langle y+y_i, A_i^{-1}(y+y_i)\rangle + \langle y_{i+1}, A_i^{-1} y_{i+1}\rangle + \langle y_{i+2}, A_i^{-1} y_{i+2}\rangle\}, \tag{6}$$

where $y_j \equiv y_{j-3}$ for $i=4, 5$. Then Lemma yields, for fixed $x, x_1, x_2, x_3, y, y_1, y_2$, and y_3,

$$\begin{aligned} \langle x+x_i, A_i(x+x_i)\rangle + \langle x_{i+1}, A_i x_{i+1}\rangle + \langle x_{i+2}, A_i x_{i+2}\rangle + \langle y+y_i, A_i^{-1}(y+y_i)\rangle + \\ + \langle y_{i+1}, A_i^{-1} y_{i+1}\rangle + \langle y_{i+2}, A_i^{-1} y_{i+2}\rangle \geqq 2\|S_i\|_1, \quad i = 1, 2, 3, \end{aligned} \tag{7}$$

where

$$S_i \equiv \begin{bmatrix} \langle x+x_i, y+y_i\rangle & \langle x+x_i, y_{i+1}\rangle & \langle x+x_i, y_{i+2}\rangle \\ \langle x_{i+1}, y+y_i\rangle & \langle x_{i+1}, y_{i+1}\rangle & \langle x_{i+1}, y_{i+2}\rangle \\ \langle x_{i+2}, y+y_i\rangle & \langle x_{i+2}, y_{i+1}\rangle & \langle x_{i+2}, y_{i+2}\rangle \end{bmatrix}.$$

Now according to (5), (6) and (7), the inequality (4) will follow from

$$\sum_{i=1}^{3} \|S_i\|_1 \geqq 2|\langle x, y\rangle|. \tag{8}$$

To see (8), consider a 3×3 Hermitian matrix

$$T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

Since T has -3, 0 and 3 as its eigenvalues, $\|T\|_\infty$, the operator norm of T, is equal to 3. Easy computation shows $\sum_{i=1}^{3} \operatorname{tr}(S_i T) = 6\langle x, y\rangle$. Then

$$\sum_{i=1}^{3} \|S_i\|_1 = \frac{1}{3}\sum_{i=1}^{3} \|S_i\|_1 \cdot \|T\|_\infty \geqq \frac{1}{3}\left|\sum_{i=1}^{3} \operatorname{tr}(S_i T)\right| = 2|\langle x, y\rangle|.$$

This completes the proof.

The method in the above proof can be used to prove $\mathfrak{S}_{2,n}(\vec{A}) \geqq \mathfrak{s}_{n-1,n}(\vec{A})$ for every n-tuple $\vec{A}$. But the inequality $\mathfrak{S}_{k,n}(\vec{A}) \geqq \mathfrak{s}_{k,n}(\vec{A})$ stands still open.

Added in proof. In the revised version of [3] a different proof is presented.

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Round-off error propagation in the integration of ordinary differential equations by one-step methods

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70-th birthday

The connection between round-off errors in the integration of a system of ordinary differential equations by one-step methods and stochastic differential equations with respect to a wide sense Wiener process is examined. The generalization of RADEMACHER's [8] and HENRICI's theorems [5] is given using Ito's integral. Under some weak conditions on the behavior of local round-off errors one can calculate the mean value and variance of the propagated round-off error. It turns out that to any system of differential equations a stochastic system of equations is related which describes the round-off error propagation.

The distribution of the propagated round-off error, $\mathbf{r}_x$; depends on the distribution of local errors; the conditions of Gaussiannes are also given. This is a sharpening of a special result of Henrici. We take advantage of the optimal filtering equations to give the expected value and variance of round-off error. This problem was studied earlier for the best approximate solutions of linear algebraic systems (see TIHONOV [9], [10], LIPTSER and SHIRYAEV [6]). The natural question on the distribution of $\max \|\mathbf{r}_x\|$, in $0 \leqq x \leqq b$, is also examined and using some recent results of NOVIKOV [7] we answer it in the one-dimensional case. The description of stochastic equations in multistep methods and especially in predictor-corrector methods remains open.

1. Introduction. Let us consider the following first order vector initial value problem

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}(x)), \ \mathbf{y}(x_0) = \mathbf{y}(0), \quad 0 < x < b, \tag{1}$$

where $\mathbf{y}$ and $\mathbf{f}$ are column vectors

$$\mathbf{y} = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^k \end{pmatrix}, \quad f = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^k \end{pmatrix}.$$

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(The superscripts always denote indices and asterisk indicates the transposition of a matrix or vector; e.g. $\mathbf{y}^*$ means a row vector $\mathbf{y}^*=(y^1, ..., y^k)$.) If $\mathbf{v}$ is a vector with real or complex components the norm is given by $\|\mathbf{v}\|=|v^1|+...+|v^k|$.

A one-step method for the solution of the initial value problem is defined by the formulas

(2) $$\mathbf{y}_0 = \mathbf{y}(0), \quad \mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{\Phi}(x, \mathbf{y}_n;\ h), \quad h > 0,$$

where $\mathbf{\Phi}(x, \mathbf{y}; h)$ is called the increment function and is chosen so as to approximate $(y(x+h)-y(x))/h$ as well as possible. We assume that $\mathbf{\Phi}(x, \mathbf{y}; h)$ is continuous and that there exists a constant L_1 such that

(3) $$\|\mathbf{\Phi}(x, \tilde{\mathbf{y}};\ h)-\mathbf{\Phi}(x, \mathbf{y};\ h)\| \leqq L_1\|\tilde{\mathbf{y}}-\mathbf{y}\|,$$

for all points $(x, \tilde{\mathbf{y}}; h)$ and $(x, \mathbf{y}; h)$, $h<h_0$ (h_0 is fixed).

The discretization error $\mathbf{e}_n$ is defined as

(4) $$\mathbf{e}_n = \mathbf{y}_n - \mathbf{y}(x_n)$$

where $\mathbf{y}(x_n)$ denotes the solution of the initial value problem at x_n.

The round-off error $\mathbf{r}_n$ is defined as the difference between $\mathbf{y}_n$ and its numerical approximation $\hat{\mathbf{y}}_n$, i.e.,

$$\mathbf{r}_n = \hat{\mathbf{y}}_n - \mathbf{y}_n,$$

and it depends on the local errors and the kind of arithmetic used in computer. $\mathbf{r}_n$ fulfils a stochastic difference equation, depending on $\mathbf{\Phi}(x, \mathbf{y}; h)$ (see (12)). Under some weak conditions on $\mathbf{\Phi}$ the continuous function $\mathbf{r}_x$ of x is a solution of the stochastic differential equation (19) (see below), which we call the related stochastic equation to (1). The investigation of equation (19) is the main goal of this paper as in earlier papers only approximations and estimations were given (see HENRICI [5]).

As a new and natural aspect we exercise the distribution of $\max\limits_{a\leqq x\leqq b}\|\mathbf{r}_x\|$, which is an effective measure of the error behavior. Theorem 4 gives the answer in the one dimensional case, with sharp bounds instead of the estimation of the mean, as it is used in the literature.

2. Derivation of the related stochastic equation. Using Euler's method, i.e., $\mathbf{\Phi}(x, \mathbf{y})=\mathbf{f}(x, \mathbf{y})$, it is easy to prove that

(5) $$\mathbf{e}_{n+1} = \mathbf{e}_n + h[\mathbf{f}(x_n, \mathbf{y}(x_n))-\mathbf{f}(x_n, \mathbf{y}_n)]+\frac{h^2}{2}\mathbf{y}''(\xi),$$

$$\|e_{n+1}\| \leqq (1+hL)\|e_n\|+\frac{h^2}{2}k,$$

and this gives, assuming $\|\mathbf{y}''\|<K$ and $\|\mathbf{f}(x, \mathbf{y})-\mathbf{f}(x, \tilde{\mathbf{y}})\|<L\|\mathbf{y}-\tilde{\mathbf{y}}\|$, the following estimation (see e.g. HENRICI [5])

$$\|e_n\| \leqq \frac{hK}{L}[e^{L(x_n-x_0)}-1]. \tag{6}$$

Let $\hat{\mathbf{y}}_n$ denote the numerical approximation of $\mathbf{y}_n$. The local error $\boldsymbol{\varepsilon}_n$ at step n is induced by computer round-off (or chopping) and the inherent error by inaccuracy of evaluation of function $\boldsymbol{\Phi}(x_n, \mathbf{y}_n; h)$.

Instead of equation (2) we have

$$\hat{\mathbf{y}}_{n+1} = \hat{\mathbf{y}}_n + h\boldsymbol{\Phi}(x_n, \hat{\mathbf{y}}_n;\ h)+\tilde{\boldsymbol{\varepsilon}}_{n+1}, \tag{7}$$

and the accumulated round-off error $\mathbf{r}_n=\hat{\mathbf{y}}_n-\mathbf{y}_n$ fulfils the equation, subtracting (7) and (2)

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h[\boldsymbol{\Phi}(x_n, \hat{\mathbf{y}}_n;\ h)-\boldsymbol{\Phi}(x_n, \mathbf{y}_n;\ h)]+\tilde{\boldsymbol{\varepsilon}}_{n+1}. \tag{8}$$

This means that the accumulated round-off error is not simply the sum of local round-off errors. It depends on the kind of arithmetic used in computer, the way in which the machine rounds, the order in which the arithmetic operations are performed and on the numerical procedures being used. As over an extended interval the loss of accuracy may be serious, it is desirable to obtain estimates by making some statistical assumptions on the behaviour of local round-off errors $\tilde{\boldsymbol{\varepsilon}}_n$.

It is known that by double precision the possible gain in accuracy can be very significant, but we have a loss in performance and efficiency.

A crude bound for the accumulated round-off error $\mathbf{r}_n$ can be obtained from (8) if we assume that

$$\|\tilde{\boldsymbol{\varepsilon}}_n\| \leqq \varepsilon, \quad n = 1, 2, \ldots; \tag{9}$$

namely, using (3) we get

$$\|\mathbf{r}_n\| \leqq \frac{\varepsilon}{hL_1}[e^{L_1(x_n-x_0)}-1]. \tag{10}$$

Comparing (10) and (6) we see, as the accuracy of numerical integration depends upon the discretization error and the accumulated rounding error, that it is impossible to keep both of the errors small. To keep the discretization error small, we will normally choose the stepsize h small. On the other hand, the smaller h is taken, the more integration steps we shall have to perform, and the greater the rounding error is likely to be. An optimum value of the stepsize h must exist but it seems difficult to find it in practice.

In order to obtain realistic statements concerning the behaviour of the propagated round-off errors, from here we shall assume that the local round-off errors $\tilde{\boldsymbol{\varepsilon}}_n$ are random variables. In the simplest case $\tilde{\boldsymbol{\varepsilon}}_n$ is a white noise process, i.e., $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_n, \tilde{\boldsymbol{\varepsilon}}_m)=0$, if $n\neq m$.

Let us assume further that $\boldsymbol{\mu}_n = E\tilde{\boldsymbol{\varepsilon}}_n$ for which

(11) $$\begin{pmatrix} \mu_n^1 \\ \vdots \\ \mu_n^k \end{pmatrix} = \mu(h)\mathbf{p}(x_n),$$

where $\mu(h)/h \to \mu$, $h \downarrow 0$, and μ is a constant and $\mathbf{p}(x)$ is a known vector function with components which are smooth functions of x. Let

(12) $$\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_n, \tilde{\boldsymbol{\varepsilon}}_m) = E(\tilde{\boldsymbol{\varepsilon}}_n - \mu_n)(\tilde{\boldsymbol{\varepsilon}}_m - \boldsymbol{\mu}_m)^* = \begin{cases} B_{\tilde{\varepsilon}}(x_n)h & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

And assuming the smoothness of $\boldsymbol{\Phi}$ let

(13) $$\boldsymbol{\Phi}(x_n, \hat{\mathbf{y}}_n;\ h) - \boldsymbol{\Phi}(x_n, \mathbf{y}_n;\ h) = G(x_n)\mathbf{r}_n + \boldsymbol{\theta}_n h,$$

where the matrix $G(x_n)$ can be expressed by the derivatives of $\boldsymbol{\Phi}$ (see [5]). Then (8) can be rewritten in the form

(14) $$\mathbf{r}_{n+1} = (I + hG(x_n))\mathbf{r}_n + \boldsymbol{\varepsilon}_{n+1}$$

where $\boldsymbol{\varepsilon}_n = \tilde{\boldsymbol{\varepsilon}}_n + \boldsymbol{\theta}_n h$, or

(14′) $$\mathbf{r}_{n+1} - \mathbf{r}_n = hG(x_n)\mathbf{r}_n + \mu(h)\mathbf{p}(x_{n+1}) + B_\varepsilon^{1/2}(x_{n+1})\boldsymbol{\varepsilon}_{n+1}^0,$$

where $\boldsymbol{\varepsilon}_{n+1} - \mu(h)\mathbf{p}(x_{n+1}) = B_\varepsilon^{1/2}(x_{n+1})\boldsymbol{\varepsilon}_{n+1}^0$, with $\operatorname{cov}(\boldsymbol{\varepsilon}_n^0, \boldsymbol{\varepsilon}_n^0) = hI$. Now approximating the process $\boldsymbol{\varepsilon}_{n+1}^0$ by a wide sense Wiener process (see the definition in Liptser—Siryaev [6] Section 15) we get that $\mathbf{r}_n = \mathbf{r}(x_n)$ has the stochastic differential

(15) $$d\mathbf{r}_x = G(x)\mathbf{r}_x\,dx + \mu\mathbf{p}(x)\,dx + B_w^{1/2}(x)\,dw(x), \quad \mathbf{r}_0 = \mathbf{0},$$

where $\mathbf{w}(0) = \mathbf{0}$, $E\mathbf{w}(x) = \mathbf{0}$, $E\mathbf{w}(x_1)\mathbf{w}^*(x_2) = I\min(x_1, x_2)$. It is clear that any Wiener process is a wide sense Wiener process at the same time.

Equation (15) can be considered as the linear equation

(16) $$\mathbf{r}_x = \int_0^x [\mu\mathbf{p}(u) + G(u)\mathbf{r}_u]\,du + \int_0^x B_w^{1/2}(u)\,d\mathbf{w}(u), \quad \mathbf{r}_0 = \mathbf{0},$$

with the unique continuous solution (in mean square)

(17) $$\mathbf{r}_x = \Phi_0(x)\left\{\int_0^x (\Phi_0(s))^{-1}\mu\mathbf{p}(s)\,ds + \int_0^x (\Phi_0(s))^{-1}B_w^{1/2}(s)\,d\mathbf{w}(s)\right\}, \quad \mathbf{r}_0 = 0,$$

where $\Phi_0(x)$ is the fundamental matrix

(18) $$\frac{d\Phi_0(x)}{dx} = G(x)\Phi_0(x), \quad \Phi_0(0) = I_{k\times k},$$

i.e.,

(18′) $$\Phi_0(x) = \exp\left\{\int_0^x G(u)\,du\right\}.$$

From the smoothness assumption on $\mathbf{p}(x)$ and $G(x)$ it follows that

$$\int_0^b |p^i(x)|\,dx < \infty, \quad \int_0^b |g_{ij}(x)|\,dx < \infty, \quad \int_0^b b_{ij}(x)\,dx < \infty, \tag{19}$$

$$i, j = 1, 2, \ldots, k.$$

The following statement immediately follows from Theorem 15.1 in [6], where

$$E\mathbf{r}_x = \mathbf{m}(x), \quad B(x, u) = E(\mathbf{r}_x - \mathbf{m}(x))(\mathbf{r}_u - \mathbf{m}(u))^*. \tag{20}$$

Theorem 1 (see HENRICI [5]). *Suppose that the conditions* (19) *hold and* $\mathbf{r}_x$ *fulfils the stochastic differential equation* (15) *with a wide sense Wiener process* $\mathbf{w}(x)$. *Then the vector* $\mathbf{m}(x)$ *and the matrix* $B(x)$ *are solutions of the differential equations*

$$\frac{d\mathbf{m}(x)}{dx} = \mu\mathbf{p}(x) + G(x)\mathbf{m}(x), \tag{21}$$

$$\frac{dB(x)}{dx} = G(x)B(x) + B(x)G^*(x) + B_w(x). \tag{22}$$

The matrix $B(x, u)$ *is given by the formula*

$$B(x, u) = \begin{cases} \Phi(u, x)B(u), & u \leqq x, \\ B(x)(\Phi(x, u))^*, & u \geqq x, \end{cases} \tag{23}$$

and

$$\Phi(u, x) = \Phi_0(x)(\Phi_0(u))^{-1}, \quad u \leqq x.$$

Proof. Taking expectations of both sides in (16) we get (21) and from (17) it follows that

$$\mathbf{m}(x) = \Phi_0(x)\Big\{\int_0^x (\Phi_0(u))^{-1}\mu p(u)\,du\Big\}, \quad \mathbf{m}(0) = \mathbf{0}. \tag{24}$$

Let $\mathbf{r}_x - \mathbf{m}(x) = \tilde{\mathbf{r}}_x$, then from (17) and (24) one can get

$$\tilde{\mathbf{r}}_x = \Phi_0(x)\Big\{\int_0^x (\Phi_0(u))^{-1}B_w^{1/2}(u)\,d\mathbf{w}(u)\Big\}, \tag{25}$$

and

$$E(\tilde{\mathbf{r}}_x\tilde{\mathbf{r}}_x^*) = \Phi_0(x)E\Big\{\int_0^x (\Phi_0(u))^{-1}B_w^{1/2}(u)\,d\mathbf{w}(u)\cdot \tag{26}$$

$$\cdot\Big[\int_0^x (\Phi_0(u))^{-1}B_w^{1/2}(u)\,d\mathbf{w}(u)\Big]^*\Big\}\Phi_0^*(x) = \Phi_0(x)\int_0^x (\Phi_0(u))^{-1}B_w(u)(\Phi_0^{-1}(u))^*\,du\,\Phi_0^*(u).$$

By differentiating and taking into account (18) one can get (22).

To establish (23) let $x \geqq u$. Then

$$E\tilde{\mathbf{r}}_x\tilde{\mathbf{r}}_n^* = \Phi_0(x)E\Big(\int_0^x \Phi_0^{-1}(u)B_w^{1/2}(u)\,d\mathbf{w}(u)\Big[\int_0^x \chi(u \geqq s)\Phi_0^{-1}(s)B_w^{1/2}(s)\,d\mathbf{w}(s)\Big]\Big)^* \cdot \tag{27}$$

$$\cdot(\Phi_0(u))^* = \Phi_0(x)\Phi_0(u)\Big\{\int_0^u \Phi_0^{-1}(s)B_w(s)(\Phi_0^{-1}(s))^*\,ds\Big\}\Phi_0^*(u) = \Phi(u, x)B(u),$$

which proves the theorem.

The following reverse statement is also true.

Theorem 2. *Let* $\mathbf{r}_x=(r_x^1, \ldots, r_x^k)$, $0 \leqq x \leqq b$, *be a random vector process with given first two moments*

$$\mathbf{m}(x) = E\mathbf{r}_x, \quad B(x, u) = E(\mathbf{r}_x-\mathbf{m}(x))(\mathbf{r}_u-\mathbf{m}(u))^*. \tag{28}$$

Assume that $B_w(x)$ *is nonnegative definite and the following assumptions are satisfied:*

a) *The elements of the vector* $\mathbf{p}(x)$ *and the matrices* $B(x), B_w(x)$ *are Lebesgue integrable.*

b) *The matrix* $B(x)=B(x, x)$ *has continuous elements and*

$$B(x) = \int_0^x [G(u)B(u)+B(u)G^*(u)]\,du+\int_0^x B_w(u)\,du, \quad B(0) = 0, \tag{29}$$

c) $\mathbf{m}(x)$ *has continuous components and*

$$\mathbf{m}(x) = \int_0^x [\mu\mathbf{p}(u)+G(u)\mathbf{m}(u)]\,du. \tag{30}$$

Then there exists a wide sense Wiener process $\tilde{\mathbf{w}}^*(x)=(\tilde{w}^1(x), \ldots, \tilde{w}^k(x))$ *such that for all* x, $0 \leqq x \leqq b$,

$$\mathbf{r}_x = \int_0^x [\mu\mathbf{p}(u)+G(u)\mathbf{r}_u]\,du+\int_0^x B_w^{1/2}(u)\,d\tilde{\mathbf{w}}(u) \tag{31}$$

The proof immediately follows from Theorem 15.2 in [6].

3. The distribution of the maximum of round-off error. The interpretation of the results in Section 2 is the following. If $E\boldsymbol{\varepsilon}_n=\boldsymbol{\mu}_n=\mu h\mathbf{p}(x_n)$, then we have

$$E\mathbf{r}_n = (\mathbf{m}(x_n)+O(h)), \tag{32}$$

where $\mathbf{m}(x)$ is the solution of the initial value problem

$$\mathbf{m}'(x) = G(x)\mathbf{m}(x)+\mu\mathbf{p}(x), \tag{33}$$

with the assumption that the matrix $G(x)$ is given by

$$\mathbf{\Phi}(x_n, \mathbf{y}_n;\ h)-\mathbf{\Phi}(x_n, \tilde{\mathbf{y}}_n;\ h) = G(x_n)(\mathbf{y}_n-\tilde{\mathbf{y}}_n)+\varepsilon\mathbf{\Theta}_n, \quad \varepsilon > 0, \quad \|\mathbf{\Theta}_n\| < 1. \tag{34}$$

The process $\mathbf{r}_x$ is stationary if $G(x)=A$ and in this case $B'_x=0$ and $B_x=B_0$ is the solution of the equation (see [1], [2])

$$AB_0+B_0A=-B_w, \tag{35}$$

i.e., $\mathbf{r}_x$ has a normal distribution with parameters $(0, B_0)$. Note that if h is small, $B_\varepsilon \cong B_w h$ and from (35) we see that

$$B_0 \sim \frac{1}{h}(B_\varepsilon + O(h)), \quad (\text{see}\,(10)).$$

In many cases we are interested in the behavior of the round-off error on the whole interval, i.e., in the values

$$P\{\sup_{0\leqq x\leqq b} \|\mathbf{r}_x\| \leqq \varepsilon\}, \quad P\{\|\mathbf{r}_x\| \leqq g(x),\ 0\leqq x\leqq b\},$$

which gives a better estimation than (32). For simplicity let us consider the one dimensional case and we assume $p(x)=0$. Let $G(x)$ be given by

$$G(x)=\frac{m'(x)}{m(x)}, \tag{36}$$

where $m(x)$ is a positive continuous function for $x\geqq 0$.

We prove the following statements.

Lemma. *Let $w(x)$ be the standard Wiener process, $w(0)=0$, $Ew(x)=0$, $Ew^2(x)=x$, and let $m(x)$ be a positive continuous function, and let $G(x)$ be defined by* (36). *Then*

$$r_x = m(x)\int_0^x m^{-1}(u)B_w^{1/2}(u)\,dw(u), \tag{37}$$

where

$$dr_x = G(x)r_x\,dx + B_w^{1/2}(x)\,dw(x). \tag{16'}$$

Proof. By Ito's formula it is easy to get from (37) that

$$dr_x = \frac{m'(x)}{m(x)}r_x\,dx + B_w^{1/2}(x)\,dw(x), \tag{38}$$

and comparing with (16′) one can get the statement.

Theorem 3. *Let r_x be the process defined by* (16), *where $p(x)=0$, and further let $m(x)$ be a positive function with continuous $m''(x)$ $(x\geqq 0)$. Then for all $0\leqq b<\infty$*

$$\frac{8}{3\pi} \leqq P\{|r_x| \leqq km(x),\ 0\leqq x\leqq b\}\exp\left\{\frac{\pi^2}{8k^2}\int_0^b m^{-2}(x)B_w(x)\,dx\right\} \leqq \frac{4}{\pi}. \tag{39}$$

Proof. From the Lemma it follows that

$$P\{|r_x|\leqq km(x),\ 0\leqq x\leqq b\}=P\Big\{\Big|\int_0^x m^{-1}(u)B_w^{1/2}(u)\,dw(u)\Big|\leqq k,\ 0\leqq x\leqq b\Big\}=$$
$$= P\Big\{|\tilde{w}(u)|\leqq k,\ 0\leqq u\leqq \int_0^b m^{-2}(u)B_w(u)\,du\Big\}, \tag{40}$$

where $\tilde{w}(u)$ is a new Wiener process obtained by the "time" change

$$u=\int_0^x m^{-2}(s)B_w(s)\,ds \tag{41}$$

from the stochastic integral $\int_0^x m^{-1}(s)B_w^{1/2}(s)\,dw(s)$, (see [4]). But for the Wiener process the following representation is well known ([3], p. 330)

$$P\{|w(u)|\leqq k,\ 0\leqq u\leqq c\}=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left[-(2n+1)^2\frac{\pi^2 c}{8k^2}\right]. \tag{42}$$

In (42) on the right side there is an alternating series and we have the following estimates

$$\frac{4}{\pi}\left[\exp\left(-\frac{\pi^2}{8k^2}c\right)-\frac{1}{3}\exp\left(-\frac{9\pi^2}{8k^2}c\right)\right]\leqq P\{|w(u)|\leqq k,\ 0\leqq u\leqq c\}\leqq$$
$$\leqq\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8k^2}c\right), \tag{43}$$

which together with (40) gives (39), and this proves the statement.

Remark 1. In the case $m(x)=ax+\tilde{b}$, $a>0$, $\tilde{b}\geqq 0$, and $B_w=\sigma^2$ we have

$$\frac{8}{3\pi}\exp\left(-\frac{\pi^2}{8k^2}\cdot\frac{\sigma^2}{a}\left[\frac{1}{\tilde{b}}-\frac{1}{ab+\tilde{b}}\right]\right)\leqq P\{|r_x|\leqq km(x),\ 0\leqq x\leqq b\}\leqq$$
$$\leqq\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8k^2}\cdot\frac{\sigma^2}{a}\left[\frac{1}{\tilde{b}}-\frac{1}{ab+\tilde{b}}\right]\right). \tag{44}$$

Remark 2. Let $m(x)=a(x+1)^{1/2}$; $a>0$ and $B_w=\sigma^2$, then

$$\frac{8}{3\pi}(1+b)^{-\frac{\pi^2}{8}\frac{\sigma^2}{ka^2}}\leqq P\{|r_x|\leqq ka(1+x)^{1/2},\ 0\leqq x\leqq b\}\leqq$$
$$\leqq\frac{4}{\pi}(1+b)^{-\frac{\pi^2}{8}\frac{\sigma^2}{ka^2}}. \tag{45}$$

This formula gives the following asymptotic result for the stopping time

$$\tau = \inf\{x\colon\ |r_x| \geqq km(x)\},$$

$$P(\tau > b) = P\{|r_x| \leqq km(x), 0 \leqq x \leqq b\} \sim c_0(1+b)^{-\frac{c_1}{a^2}}, \quad c_0, c_1 > 0,$$

where $a \sim 0$.

Remark 3. Estimates for the probability that the process r_x will not exit to a one-sided moving boundary can be handled in the same way (see NOVIKOV [7]).

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О вычислении энтропийных функционалов и их минимумов в неопределённых проблемах продолжения

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Посвящается академику Б. Секефальви-Надь к его семидесятилетию в знак глубокого уважения и самых лучших чувств

Введение

Для измеримых при $|\zeta|=1$ матриц-функций (сокращённо — м.-ф.) $f(\zeta)$ порядка $m\times n$, таких, что $f^*(\zeta)f(\zeta)\leqq I_n$ п. в. $(f\in K^{m\times n})$ в настоящей работе рассматриваются функционалы

$$(1)\qquad i(f;\ z) = -\frac{1}{4\pi}\int\limits_{|\zeta|=1} \ln\det[I_n - f^*(\zeta)f(\zeta)]\frac{1-|z|^2}{|\zeta-z|^2}|d\zeta| \quad (|z|<1).$$

Функционал $i(f)\overset{\text{def}}{=}i(f;0)$ имеет следующий энтропийный смысл. Условие $f^*f\leqq I_n$ равносильно неравенству

$$\begin{pmatrix} I_m & f \\ f^* & I_n \end{pmatrix} \geqq 0.$$

Поэтому [1] для f из $K^{m\times n}$ м.-ф. $f(e^{i\mu})$ можно интерпретировать как смешанную спектральную плотность $f_{\xi,\eta}(\mu)$ двух стационарных и стационарно-связанных гауссовских случайных процессов $\xi=\{\xi_i(k)\}_{i=1}^m$ и $\eta=\{\eta_j(k)\}_{j=1}^n$ с дискретным временем $k(\in\mathbf{Z})$ (размерностей m и n со спектральными плотностями $f_{\xi,\xi}(\mu)=$ $=I_m$ и $f_{\eta,\eta}(\mu)=I_n$*)). Согласно формуле Пинскера [2] величина $i(f)$ может быть проинтерпретирована как скорость передачи информации $i(\xi,\eta)$ одним из процессов ξ и η о другом. В силу этого функционалы $i(f;\ z)$ мы называем энтропийными.

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*) Последнее условие означает, что гауссовы случайные величины $\xi_i(k)$ (и $\eta_j(k)$) независимы при различных i (j) или k и имеют единичную дисперсию.

В настоящей статье вычисляются значения $i(f;z)$ и минимума $i(f;z)$ для решений $f(\zeta)$ следующей проблемы продолжения и других проблем, сводящихся к ней (задача Шура, би-касательная задача Неванлинны—Пика и др.).

Задача $N(m;n)$. Найти м.-ф. $f(\zeta)$ из $K^{m\times n}$ с заданной «главной частью» её разложения в ряд Фурье: $f(\zeta)\sim\sum_{1}^{\infty}\gamma_k\zeta^{-k}+\dots$.

Ещё в 1957 г. Нехари [3] в скалярном случае $(m=n=1)$ получил критерий разрешимости такой проблемы продолжения. Позже авторы совместно с В. М. Адамяном [4а] получили критерий того, когда множество $\mathfrak{A}$ решений этой проблемы содержит более одной функции и формулу (2), дающую в этом случае параметрическое описание множества $\mathfrak{A}$. Эти и другие результаты в дальнейшем были обобщены на случай, когда значениями $f(\zeta)$ являются матрицы-операторы, действующие из $\mathbf{C}^n$ в $\mathbf{C}^m$ [4c] (определённая часть результатов сохранила силу и на случай операторов в бесконечномерных гильбертовых пространствах).

В этот же период стало ясно, что к задаче $N(m;n)$ легко сводится ряд интерполяционных задач таких, как задача Неванлинны—Пика, Шура, Каратеодори—Фейера и др. и их матричные обобщения, а также появившаяся к тому моменту задача Сарасона [5]. Один из авторов (М. Г. Крейн) совместно с Ф. Мелик-Адамяном выполнил исследование [6], в котором был рассмотрен матрично-континуальный аналог задачи $N(n;n)$ и установлена связь такой задачи с задачей рассеяния для канонических систем.

Таким образом, имеется довольно широкий круг задач, в которых в так называемом вполне неопределённом случае описание решений получается по формуле (2). В основу вычисления энтропийных функционалов $i(f;z)$ и их минимума для решений $f(\zeta)$ задачи $N(m;n)$ в настоящей работе положена именно эта формула.

В нашей заметке [7] сформулированы полученные на таком же пути аналогичные результаты для родственных задач продолжения в классе голоморфных при $|z|<1$ м.-ф. $f(z)$ с $\operatorname{Re} f(z)\geqq 0$, где вместо $I_n-f^*(\zeta)f(\zeta)$ рассматривается $\operatorname{Re} f(\zeta)$ $(|\zeta|=1)$. После опубликования этой заметки авторам стали известны статьи P. Dewild'а и H. Dym'а [8], где, в частности, на том же пути применения формулы пописания всех решений задачи указано решение с минимальной энтропией для специальной задачи продолжения. Статья [8b] передана в печать в один и тот же день с нашей заметкой [7] и в ней рассмотрена задача вычисления минимума энтропии для матрично-значной касательной проблемы Неванлинны—Пика с конечным числом узлов интерполяции. Она явилась дальнейшим развитием предыдущей работы тех же авторов [8a], где была рассмотрена такая задача в скалярном случае.

За недостатком места мы не имеем возможности подробно остановиться на связях, существующих между результатами настоящей статьи и нашей заметки [7], где, кстати, указана дополнительная литература, имеющая отношение к рассматриваемому циклу исследований. Отметим только, что настоящая статья является подробным раскрытием содержания п. 5 заметки [7]. Развёрнутое изложение остальной части заметки [7] будет нами дано в другом месте.

1. Некоторые положения об энтропийных функционалах

1. Напомним, что через $K^{m\times n}$ обозначается класс измеримых при $|\zeta|=1$ матриц-функций $f(\zeta)$ порядка $m\times n$ с $\|f(\zeta)\|\leqq 1$ $(f^*(\zeta)f|\zeta|\leqq I_n)$ п. в. Через $\mathbf{B}^{m\times n}$ обозначим класс голоморфных при $|z|<1$ м.-ф. $\mathscr{E}(z)$ порядка $m\times n$ с $\|\mathscr{E}(z)\|\leqq 1$. По теореме Фату для $\mathscr{E}(z)$ из $\mathbf{B}^{m\times n}$ существует п.в. при $|\zeta|=1$ граничное значение $\mathscr{E}(\zeta)(=\lim\limits_{r\uparrow 1}\mathscr{E}(r\zeta))$. Для м.-ф. $\mathscr{E}(\zeta)$ имеем $\mathscr{E}(\zeta)\in K^{m\times n}$ и

$$\operatorname*{ess\,sup}_{|\zeta|=1}\|\mathscr{E}(\zeta)\| \doteq \sup_{|z|<1}\|\mathscr{E}(z)\|\,(=\|\mathscr{E}\|_\infty).$$

Переходом от $\mathscr{E}(z)$ к $\mathscr{E}(\zeta)$ осуществляется естественное вложение $\mathbf{B}^{m\times n}$ в $K^{m\times n}$.

В этом параграфе центральным объектом будет семейство $\mathfrak{A}(A)=$ $=\{f_{\mathscr{E}}\colon \mathscr{E}\in\mathbf{B}^{m\times n}\}$ м.-ф. $f_{\mathscr{E}}(\zeta)$ из $K^{m\times n}$, являющееся образом $\mathbf{B}^{m\times n}$ при инъективном дробно-линейном преобразовании

$$f_{\mathscr{E}}(\zeta)=[a_{11}(\zeta)\mathscr{E}(\zeta)+a_{12}(\zeta)][a_{21}(\zeta)\mathscr{E}(\zeta)+a_{22}(\zeta)]^{-1},\qquad \mathscr{E}\in\mathbf{B}^{m\times n}, \tag{2}$$

переводящем сжимающие матрицы $\mathscr{E}$ $(\|\mathscr{E}\|\leqq 1)$ в м.-ф. $f_{\mathscr{E}}(\zeta)$ из $K^{m\times n}$ и изометрические матрицы $\mathscr{E}$ $(\mathscr{E}^*\mathscr{E}=I_n)$ в изометрическизначные м.-ф. $f_{\mathscr{E}}(\zeta)$ (предполагается, что $n\leqq m$). Известно [9, 10], что дробно-линейное преобразование является таковым, когда его измеримая м.-ф. коэффициентов $A(\zeta)=[a_{ik}(\zeta)]_1^2$ умножением на некоторую измеримую скалярную функцию может быть сделана j-унитарной с

$$j=\begin{pmatrix} I_m & 0\\ 0 & -I_n\end{pmatrix}.$$

В дальнейшем будем предполагать, что уже сама матрица $A(\zeta)$ является j-унитарной, т. е. $A^*(\zeta)jA(\zeta)=j$ п. в., $|\zeta|=1$. Поблочная запись этого равенства означает, что

$$\begin{gathered} a_{11}^*(\zeta)a_{11}(\zeta)-a_{21}^*(\zeta)a_{21}(\zeta)=I_m,\quad a_{11}^*(\zeta)a_{12}(\zeta)-a_{21}^*(\zeta)a_{22}(\zeta)=0,\\ a_{12}^*(\zeta)a_{12}(\zeta)-a_{22}^*(\zeta)a_{22}(\zeta)=-I_n. \end{gathered} \tag{3}$$

Так как вместе с $A(\zeta)$ является j-унитарной и матрица $A^*(\zeta)$, то из системы равенств (3) вытекает:

$$a_{21}(\zeta)a_{21}^*(\zeta)-a_{22}(\zeta)a_{22}^*(\zeta)=-I_n.$$

Отсюда видно, что матрица $a_{22}(\zeta)$ обратима, так что можно определить

$$\chi(\zeta) \stackrel{\text{def}}{=\!=} -a_{22}^{-1}(\zeta)a_{21}(\zeta), \tag{4}$$

и что для м.-ф. $\chi(\zeta)$ имеем

$$a_{22}^{-1}(\zeta)[a_{22}^{-1}(\zeta)]^* = I_n - \chi(\zeta)\chi^*(\zeta), \tag{5}$$

$$i(\chi) = \frac{1}{2\pi}\int\limits_{|\zeta|=1} \ln|\det a_{22}(\zeta)|\,|d\zeta|. \tag{6}$$

Будем в дальнейшем предполагать, что м.-ф. $\chi(\zeta)$ удовлетворяет следующим дополнительным условиям:

$$1)\quad \chi\in\mathbf{B}^{n\times m}, \qquad 2)\quad i(\chi)<\infty. \tag{!}$$

Во всех представляющих интерес конкретных задачах, приводящих к семейству $\mathfrak{A}(A)$, м.-ф. $\chi(\zeta)$ удовлетворяет условиям (!). Напоминаем, что включение $\chi\in\mathbf{B}^{n\times m}$ означает, что $\chi(\zeta)$ — граничное значение м.-ф. $\chi(z)$ из $\mathbf{B}^{n\times m}$.

2. Теорема 1. *Пусть* $A(\zeta)=(a_{ik}(\zeta))_1^2$ — *м.-ф., принимающая j-унитарные значения п.в. при* $|\zeta|=1$, *и пусть определённая по формуле* (4) *м.-ф.* $\chi(\zeta)$ *удовлетворяет условиям* (!). *Тогда для м.-ф.* $f_{\mathscr{E}}$ *из семейства* $\mathfrak{A}(A)$, *определённых по формуле* (2), *имеем:*

$$i(f_{\mathscr{E}}) = i(\chi)+i(\mathscr{E})+\ln|\det[I-\chi(0)\mathscr{E}(0)]|. \tag{7}$$

В $\mathfrak{A}(A)$ *существует и притом единственная м.-ф.* $f_{0;\min}(\zeta)$, *на которой функционал* $i(f)$ *принимает наименьшее значение. М.-ф.* $f_{0;\min}(\zeta)$ *получается по формуле* (2) *при постоянной м.-ф.* $\mathscr{E}(\zeta)=\chi^*(0)$, *так что*

$$i(f_{0;\min}) = i(\chi)+(1/2)\ln\det[I_n-\chi(0)\chi^*(0)]. \tag{8}$$

Если, в частности, $\chi(0)=0_{m\times n}$, то: $f_{0;\min}=a_{12}\,a_{22}^{-1}$,

$$i(f_{\mathscr{E}}) = i(\chi)+i(\mathscr{E}), \quad i(f_{0;\min}) = i(\chi). \tag{9}$$

Доказательство. Пользуясь системой равенств (3), получаем

$$\begin{aligned} &I_n - f_{\mathscr{E}}^*(\zeta)f_{\mathscr{E}}(\zeta) = \\ &= [a_{22}^{-1}(\zeta)]^*\{[I_n-\chi(\zeta)\mathscr{E}(\zeta)]^{-1}\}^*[I_n-\mathscr{E}^*(\zeta)\mathscr{E}(\zeta)](I_n-\chi(\zeta)\mathscr{E}(\zeta)]^{-1}a_{22}^{-1}(\zeta). \end{aligned} \tag{10}$$

Для $\chi(\in B^{n\times m})$ условие $i(\chi)<\infty$ равносильно следующему

$$\int\limits_{|\zeta|=1} \ln(1-\|\chi(\zeta)\|)|d\zeta| > -\infty.$$

Поэтому $\|\chi(\zeta)\|<1$ п. в. Из принципа максимума для м.-ф. из $\mathbf{B}^{n\times m}$ вытекает, что $\|\chi(z)\|<1$ при $|z|<1$. Поэтому для любой м.-ф. $\mathscr{E}(z)$ из $\mathbf{B}^{m\times n}$ определена и голоморфна при $|z|<1$ м.-ф. $[I_n-\chi(z)\mathscr{E}(z)]^{-1}$. Более того, $I_n-\chi(z)\mathscr{E}(z)$ — внешняя м.-ф. (см., например, [11], лемма 3.1), т. е.

$$\ln|\det[I_n-\chi(0)\mathscr{E}(0)]| = \frac{1}{2\pi}\int_{|\zeta|=1}\ln|\det[I_n-\chi(\zeta)\mathscr{E}(\zeta)]|\,|d\zeta|.$$

Это равенство вместе с (6) и (10) дают (7). Остаётся показать, что величина

$$i(\mathscr{E})+\ln|\det[I_n-\chi(0)\mathscr{E}(0)]| \quad (\mathscr{E}\in\mathbf{B}^{m\times n})$$

принимает наименьшее значение тогда и только тогда, когда $\mathscr{E}(\zeta)\equiv\chi^*(0)$. Это очевидно, когда $\chi(0)=0_{n\times m}$, ибо $i(\mathscr{E})\geqq 0$ и $i(\mathscr{E})=0$, если $\mathscr{E}(\zeta)\equiv 0_{m\times n}$ и только в этом случае. Пусть $\chi(0)\neq 0_{n\times m}$. Тогда рассмотрим дробно-линейное преобразование

$$\mathscr{E}_1(\zeta) = [\mathring{a}_{11}\mathscr{E}(\zeta)+\mathring{a}_{12}][\mathring{a}_{21}\mathscr{E}(\zeta)+\mathring{a}_{22}]^{-1}$$

с постоянной j-унитарной матрицей коэффициентов $\mathring{A}=(\mathring{a}_{ik})_1^2$, отображающее $\mathscr{E}(\zeta)\equiv\chi^*(0)$ в $\mathscr{E}_1(\zeta)\equiv 0_{m\times n}$,

$$\mathring{A} = \begin{pmatrix} (I_m-\chi_0^*\chi_0)^{-1/2} & -(I_m-\chi_0^*\chi_0)^{-1/2}\chi_0^* \\ -(I_n-\chi_0\chi_0^*)^{1/2}\chi_0 & (I_n-\chi_0\chi_0^*)^{-1/2} \end{pmatrix}, \quad \chi_0=\chi(0).$$

Применяя для семейства $\mathfrak{A}(\mathring{A})$ уже доказанную формулу (7), получаем:

$$i(\mathscr{E}_1) = i(\chi_0)+i(\mathscr{E})+\ln|\det[I_n-\chi(0)\mathscr{E}(0)]|,$$

где $\chi_0(\zeta)\equiv\chi_0=\chi(0)$. Таким образом, равенство (7) можно переписать в виде

$$i(f_{\mathscr{E}}) = i(\chi)+i(\mathscr{E}_1)-i(\chi_0).$$

Остаётся заметить, что $i(\mathscr{E}_1)\geqq 0$ и $i(\mathscr{E}_1)=0$ тогда и только тогда, когда $\mathscr{E}_1(\zeta)\equiv 0$, т. е. когда $\mathscr{E}(\zeta)\equiv\chi^*(0)$. При такой м.-ф. $\mathscr{E}(\zeta)$ получается $f_{0;\min}(\zeta)$ с $i(f_{0;\min})=i(\chi)-i(\chi_0)$, что равносильно формуле (8). Теорема доказана.

Замечание 1. Нетрудно показать, что каково бы ни было число c, большее, чем $i(f_{0;\min})$, существует постоянная м.-ф. $\mathscr{E}(\zeta)$ такая, что $i(f_{\mathscr{E}})=c$.

3. Для $f\in K^{m\times n}$ и $|z|<1$ рассмотрим функционал $i(f;z)$, определённый по формуле (1). Очевидно, что если $f\in K^{m\times n}$, то $f_z(\zeta)\overset{\text{def}}{=}f\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right)$ — м.-ф. из $K^{m\times n}$ и $i(f_z)(=i(f_z;\,0))=i(f;\,z)$. Положим:

$$A_z(\zeta) = A\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right), \quad \chi_z(\zeta)=\chi\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right), \quad f_{\mathscr{E},z}(\zeta)=f_{\mathscr{E}}\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right).$$

Тогда $\chi_z(0)=\chi(z)$, $i(\chi_z)=i(\chi;z)$, $i(f_{\mathscr{E},z})=i(f_{\mathscr{E}};z)$ и, применяя теорему 1 к семейству $\mathfrak{A}(A_z)=\{f_{\mathscr{E},z}\}$, получаем, что справедлива

Теорема 2. *Пусть м.-ф.* $A(\zeta)$ *удовлетворяет условиям теоремы* 1. *Тогда при каждом фиксированном* z $(|z|<1)$:

1) *для м.-ф.* $f_{\mathscr{E}}$ *из* $\mathfrak{A}(A)$ *имеем*

$$i(f_{\mathscr{E}};\ z)=i(\chi;\ z)+i(\mathscr{E};\ z)+\ln|\det[I_n-\chi(z)\mathscr{E}(z)]|; \tag{11}$$

2) *в* $\mathfrak{A}(A)$ *существует единственная м.-ф.* $f_{z,\min}(\zeta)$, *на которой функционал* $i(f;z)$ *принимает наименьшее значение;*

3) *м.-ф.* $f_{z,\min}(\zeta)$ *получается по формуле* (2) *при постоянной м.-ф.* $\mathscr{E}(\zeta)\equiv\equiv\chi^*(z)$, *так что*

$$i(f_{z,\min};\ z)=i(\chi;\ z)+(1/2)\ln\det[I_n-\chi(z)\chi^*(z)]. \tag{12}$$

Замечание 2. Если $\chi(z)$ — произвольная м.-ф. из $\mathbf{B}^{n\times m}$ такая, что $I_m-\chi^*(\zeta)\chi(\zeta)>0$ п. в. $(|\zeta|=1)$, то существует измеримая при $|\zeta|=1$ м.-ф. $A(\zeta)=[(a_{ik}(\zeta)]_1^2$, принимающая j-унитарные значения п. в. и такая, что $a_{22}^{-1}(\zeta)a_{21}(\zeta)=-\chi(\zeta)$. М.-ф. $A(\zeta)$ можно определить по $\chi(\zeta)$ точно так же, как при доказательстве основной теоремы 1 была определена $\mathring{A}$ по χ_0. Из теоремы 2 следует, что при $|z|<1$

$$-\frac{1}{2}\ln\det[I_m-\chi^*(z)\chi(z)]\leqq-\frac{1}{4\pi}\int_{|\zeta|=1}\ln\det[I_m-\chi^*(\zeta)\chi(\zeta)]\frac{1-|z|^2}{|\zeta-z|^2}|d\zeta|.$$

Это неравенство означает, что в левой его части стоит субгармоническая функция (впрочем, в этом можно убедиться и непосредственно). В правой части неравенства стоит её наилучшая гармоническая мажоранта. Согласно теореме 2, разность между наилучшей гармонической мажорантой субгармонической при $|z|<1$ функции $-(1/2)\ln\det[I_m-\chi^*(z)\chi(z)]$ и самой этой функцией имеет теоретико-информационный смысл: она равна наименьшему значению величины $i(f;z)$ $((=i(f_z))$, когда f пробегает множество $\mathfrak{A}(A)$.

2. Матричное обобщение энтропийного неравенства

1. Как известно, для f из $K^{m\times n}$ условие $i(f)<\infty$ является необходимым и достаточным для существования внешней м.-ф. ψ_f такой, что

$$\psi_f^*(\zeta)\psi_f(\zeta)=I_n-f^*(\zeta)f(\zeta)\quad \text{п. в.}\quad (|\zeta|=1)$$

$$\left(\psi_f\in\mathbf{B}^{m\times n},\ \ln|\det\psi_f(0)|=\frac{1}{2\pi}\int_{|\zeta|=1}\ln|\det\psi_f(\zeta)||d\zeta|\right).$$

М.-ф. ψ_f определяется по f с точностью до постоянного левого унитарного множителя, а при нормировке $\psi_f(0)>0$ — однозначно. Очевидно, что $i(f)=$

$= -\ln|\det\psi_f(0)|$. Поскольку предполагается, что $i(\chi)<\infty$, то существует $\psi_\chi(z)$. Из равенства (5) следует, что

(13) $$a_{22}(\zeta) = b(\zeta)\varphi^{-1}(\zeta),$$

где $b(\zeta)$ — унитарно-значная м.-ф., а $\psi(z)$ — внешняя м.-ф. такая, что

(14) $$\varphi(\zeta)\varphi^*(\zeta) = I_n - \chi(\zeta)\chi^*(\zeta).$$

Имеем: $i(\chi) = -\ln|\det\psi_\chi(0)| = -\ln|\det\psi(0)|$. Совершая замену переменной $\zeta\mapsto(\zeta+z)/(1+\bar{z}\zeta)$, получаем:

(15) $$i(f;\, z) = -\ln|\det\psi_f(z)|, \quad i(\chi;\, z) = -\ln|\det\varphi(z)|$$

Введём в рассмотрение м.-ф. $B_0(z)$:

(16) $$B_0(z) \stackrel{\text{def}}{=} \varphi^*(z)[I_n-\chi(z)\chi^*(z)]^{-1}\varphi(z).$$

Тогда равенство (12) можно записать в виде

$$-\ln|\det\psi_{f_z,\min}(z)| = -(1/2)\ln\det B_0(z)$$

так что для $f_{\mathscr{E}}$ из $\mathfrak{A}(A)$ имеем:

(17) $$\ln\det[\psi^*_{f_{\mathscr{E}}}(z)\psi_{f_{\mathscr{E}}}(z)] \leqq \ln\det B_0(z).$$

В этой оценке достигается равенство точно тогда, когда $\mathscr{E}(\zeta)\equiv\chi^*(z)$.

2. Если предположить, что в представлении (13) функция $b(\zeta)$ является скалярной, то получается значительно более сильное утверждение, чем неравенство (17).

Теорема 3 (матричное неравенство). *Пусть для м.-ф.* $A(\zeta)=[a_{ik}(\zeta)]_1^2$ *выполняются условия теоремы* 1 *и её блок* $a_{22}(\zeta)$ *допускает факторизацию* (13) *со скалярной унимодулярной функцией* $b(\zeta)$. *Тогда при каждом фиксированном* z ($|z|<1$) *для семейства м.-ф.* $f_{\mathscr{E}}(\zeta)$, *определённых по формуле* (2), *выполняется неравенство*

(18) $$\psi^*_{f_{\mathscr{E}}}(z)\psi_{f_{\mathscr{E}}}(z) \leqq B_0(z),$$

где $B_0(z)$ *определяется по формуле* (16). *Знак* $=$ *здесь достигается точно тогда, когда* $\mathscr{E}(\zeta)\equiv\chi^*(z)$.

Доказательство. Из равенства (10), учитывая (в существенном) единственность решения задачи факторизации неотрицательно-значной м.-ф. в классе внешних м.-ф. и то, что согласно условию теоремы в представлении (13) функция $b(\zeta)$ является скалярной, получаем:

$$\psi_{f_{\mathscr{E}}}(z) = u_{\mathscr{E}}\psi_{\mathscr{E}}(z)[I_n-\chi(z)\mathscr{E}(z)]^{-1}\varphi(z),$$

где $u_{\mathscr{E}}$ — постоянная унитарная матрица. Докажем сначала утверждение теоремы для $z=0$. Если $\chi(0)=0_{n\times m}$, то имеем $\psi_{f_{\mathscr{E}}}(0)=u_{\mathscr{E}}\psi_{\mathscr{E}}(0)\psi(0)$, $B_0(0)=\varphi^*(0)\varphi(0)$ и остаётся учесть, что $\psi_{\mathscr{E}}^*(0)\psi_{\mathscr{E}}(0)\leqq I_n$ и $\psi_{\mathscr{E}}^*(0)\psi_{\mathscr{E}}(0)=I_n$ лишь когда $\mathscr{E}(\zeta)\equiv 0_{m\times n}(=\chi^*(0))$. Случай, когда $\chi(0)\neq 0_{n\times m}$, рассматривается так же, как в аналогичной ситуации в доказательстве теоремы 1. Для семейства $\mathfrak{A}(\mathring{A})$ м.-ф. $\mathscr{E}_1(\zeta)$ имеем

$$\psi_{\mathscr{E}_1}(z) = v_{\mathscr{E}}\psi_{\mathscr{E}}(z)[I_n-\chi(0)\mathscr{E}(z)]^{-1}[I_n-\chi(0)\chi^*(0)]^{1/2},$$

где $v_{\mathscr{E}}$ — постоянная унитарная матрица. Следовательно,

$$\psi_{f_{\mathscr{E}}}(0) = u_{\mathscr{E}}v_{\mathscr{E}}^*\psi_{\mathscr{E}_1}(0)[I_n-\chi(0)\chi^*(0)]^{-1/2}\varphi(0).$$

Остаётся здесь учесть, что $u_{\mathscr{E}}v_{\mathscr{E}}^*$ — унитарная матрица, $\psi_{\mathscr{E}_1}^*(0)\psi_{\mathscr{E}_1}(0)\leqq I_n$ и $\psi_{\mathscr{E}_1}^*(0)\psi_{\mathscr{E}_1}(0)=I_n$ тогда и только тогда, когда $\mathscr{E}_1(\zeta)\equiv 0_{m\times n}$, т.е. когда $\mathscr{E}(\zeta)\equiv\chi^*(0)$. Утверждение теоремы для произвольной точки z ($|z|<1$) получается из уже доказанного для $z=0$ путём замены переменной: $\zeta\mapsto(\zeta+z)/(1+\bar{z}\zeta)$.

Замечание 3. Можно показать, что при выполнении условий теоремы 3 при каждом фиксированном z ($|z|<1$) для любой матрицы $c(z)$ такой, что $0<c(z)\leqq B_0(z)$ существует постоянная м.-ф. $\mathscr{E}(\zeta)\equiv\mathscr{E}_c$, при которой $\psi_{f_{\mathscr{E}_c}}^*(z)\psi_{f_{\mathscr{E}_c}}(z)=c(z)$.

3. Важным для приложений является случай, когда $a_{22}(\zeta)$ является граничным значением мероморфной при $|z|<1$ м.-ф. При этом такими же будут $a_{21}(\zeta)$ и м.-ф. $b(\zeta)$ в представлении (13), так что

$$\begin{aligned}\varphi(z) &= a_{22}^{-1}(z)b(z), \quad \chi(z) = -a_{22}^{-1}(z)a_{21}(z),\\ B_0(z) &= b^*(z)[a_{22}(z)a_{22}^*(z)-a_{21}(z)a_{21}^*(z)]b(z).\end{aligned} \tag{19}$$

Пусть $A(z)=[a_{ik}(z)]_1^2$ — произвольная j-внутренняя м.-ф., т.е.: 1) $A(z)$ — мероморфная при $|z|<1$ м.-ф., 2) в каждой точке голоморфности при $|z|<1$ она принимает j-сжимающие значения $(A^*(z)jA(z)\leqq j)$, 3) она имеет j-унитарные граничные значения $A(\zeta)$ п. в.

Такой м.-ф. $A(z)$ отвечает дробно-линейное преобразование (2), инъективно отображающее $\mathbf{B}^{m\times n}$ в себя так, что внутренним м.-ф. $\mathscr{E}(z)$ $(\mathscr{E}\in\mathbf{B}^{m\times n}, \mathscr{E}^*(\zeta)\mathscr{E}(\zeta)=I_n$ п. в.$)$ отвечают внутренние м.-ф. $f_{\mathscr{E}}(z)$. Это свойство является характеристическим: произвольная мероморфная при $|z|<1$ м.-ф. $A(z)$, обладающая этим свойством, лишь мероморфным скалярным множителем отличается от j-внутренней м.-ф. [10].

Для j-внутренней м.-ф. $A(z)$ условия (!) выполняются автоматически. Действительно, вместе с $A(z)$ матрица $A^*(z)$ также является j-сжимающей. Поэтому:

$$-a_{22}(z)a_{22}^*(z)+a_{21}(z)a_{21}^*(z)\leqq -I_n.$$

Отсюда вытекает, что: $\chi \in \mathbf{B}^{n\times m}$, $a_{22}^{-1} \in \mathbf{B}^{n\times m}$. Из последнего включения следует, что правая часть в формуле (5) является конечной, т. е. $i(\chi) < \infty$. Из него также следует представление $a_{22}(\zeta)$ в виде (13), где $b^{-1}(z)$ — внутренняя м.-ф. Поэтому из теорем 2 и 3 вытекает

Теорема 4. *Пусть* $A(z) = [a_{ik}(z)]_1^2$ — *произвольная j-внутреняя м.-ф. Тогда для семейства* $\mathfrak{A}(A)$ *м.-ф.* $f_\varepsilon(z)$, *определяемых по формуле* (2), *справедливы утверждения теоремы* 2. *Если при этом в факторизации* (13) *блока* $a_{22}(z)$ *функция* $b(z)$ *является скалярной, то для* $\mathfrak{A}(A)$ *справедливы утверждения теоремы* 3.

3. Применение к задаче $N(m; n)$

1. Остановимся на применении результатов § 1—2 к задаче $N(m; n)$. Напомним некоторые положения, полученные в [4с] при исследовании задачи $N(m; n)$.

По коэффициентам γ_k $(k=1, 2, \ldots)$ заданной «главной части» $\sum_1^\infty \gamma_k \zeta^{-k}$ м.-ф. $f(\zeta)$ строится блочно-ганкелева матрица $\Gamma = [\gamma_{j+k-1}]_1^\infty$ и рассматривается определяемый ею ганкелев оператор $\mathbf{\Gamma}$, действующий из $l^2(\mathbf{C}^n)$ в $l^2(\mathbf{C}^m)$ по формуле:

$$\mathbf{\Gamma}\xi = \eta = \{\eta_j\}_1^\infty, \quad \eta_j = \sum_1^\infty \gamma_{j+k-1}\xi_k, \quad \xi = \{\xi_k\}_1^\infty.$$

Задача $N(m; n)$ имеет решение тогда и только тогда, когда $I - \mathbf{\Gamma}^*\mathbf{\Gamma} \geqq 0$ $(\|\mathbf{\Gamma}\| \leqq 1)$. Если $\|\mathbf{\Gamma}\| < 1$, то множество $\mathfrak{A}$ описывается формулой (2), т. е. $\mathfrak{A} = \mathfrak{A}(A)$, где $A(\zeta) = [a_{ik}(\zeta)]_1^2$ удовлетворяет условиям теоремы 2 и $a_{22}(\zeta)$ — внешняя м.-ф., так что в представлении $a_{22}(\zeta)$ в виде (13) имеем $b(\zeta) = I$, $a_{22}(\zeta) = \varphi^{-1}(\zeta)$. Такое же описание решений получается и в более общем, так называемом вполне неопределённом случае, когда для подпространства $\mathfrak{N}_1$ векторов $\xi = \{\xi_k\}_1^\infty$ с $\xi_k = 0$ при $k > 1$ имеем

(20) $$\mathfrak{N}_1 \subset (I - \mathbf{\Gamma}^*\mathbf{\Gamma})^{1/2} l^2(\mathbf{C}^n).$$

Это включение равносильно следующему

$$\mathfrak{N}_2 \subset (I - \mathbf{\Gamma}\mathbf{\Gamma}^*)^{1/2} l^2(\mathbf{C}^m),$$

где $\mathfrak{N}_2$ — подпространство в $l^2(\mathbf{C}^m)$ векторов $\eta = \{\eta_k\}_1^\infty$ с $\eta_k = 0$ при $k > 1$.

Для $A(\zeta) = [a_{ik}(\zeta)]_1^2$ имеем:

(21) $$a_{11}(\zeta) = \mathscr{P}_-(\zeta), \quad a_{12}(\zeta) = \mathscr{Q}_-(\zeta), \quad a_{21}(\zeta) = \mathscr{Q}_+(\zeta), \quad a_{22}(\zeta) = \mathscr{P}_+(\zeta),$$

где $\mathscr{P}_\pm(\zeta)$ и $\mathscr{Q}_\pm(\zeta)$ вычисляются с помощью следующих процедур.

Пусть сначала $\|\mathbf{\Gamma}\| < 1$. Обозначим через a положительную матрицу порядка $n \times n$ такую, что

$$((I - \mathbf{\Gamma}^*\mathbf{\Gamma})^{-1}\xi, \xi) = (a^{-1}\xi_1, \xi_1), \quad \xi = \{\xi_k\}_1^\infty \in \mathfrak{N}_1.$$

Соответствующий этой матрице оператор a в $\mathfrak{N}_1$ определяется по формуле

$$a = [P_{\mathfrak{N}_1}(I - \mathbf{\Gamma}^*\mathbf{\Gamma})^{-1}|\mathfrak{N}_1]^{-1/2}.$$

Рассмотрим оператор $\mathscr{P} = (I - \mathbf{\Gamma}^*\mathbf{\Gamma})^{-1}a$, действующий из $\mathbf{C}^n (= \mathfrak{N}_1)$ в $l^2(\mathbf{C}^n)$. Он определяется последовательностью матриц $\{p_k\}_1^\infty$ порядка $n \times n$: $\mathscr{P}\xi_1 = \{p_k\xi_1\}_1^\infty$, $\xi_1 \in \mathbf{C}^n$. Точно так же определяется оператор $\mathscr{Q} = \mathbf{\Gamma}(I - \mathbf{\Gamma}^*\mathbf{\Gamma})^{-1}a$ и вводится соответствующая последовательность матриц $\{q_k\}_1^\infty$ порядка $m \times n$. М.-ф. $\mathscr{P}_+(z)$ и $\mathscr{Q}_+(z)$ определяются по формулам

$$\mathscr{P}_+(\zeta) = \sum_1^\infty p_k\zeta^{k-1}, \quad \mathscr{Q}_+(\zeta) = \sum_1^\infty q_k\zeta^k.$$

М.-ф. $\mathscr{P}_-(\zeta)$ и $\mathscr{Q}_-(\zeta)$ вычисляются аналогично: следует в формулах, записанных для получения $\mathscr{P}_+(\zeta)$ и $\mathscr{Q}_+(\zeta)$, заменить $\mathbf{\Gamma}$ на $\mathbf{\Gamma}^*$, $\mathbf{\Gamma}^*$ — на $\mathbf{\Gamma}$ и ζ — на ζ^{-1}.

Если $\|\mathbf{\Gamma}\| = 1$, то во вполне неопределённом случае м.-ф. $\mathscr{P}_\pm(\zeta)$ и $\mathscr{Q}_\pm(\zeta)$ получаются как пределы при $\varrho \uparrow 1$ м.-ф. $\mathscr{P}_\pm^{(\varrho)}(\zeta)$ и $\mathscr{Q}_\pm^{(\varrho)}(\zeta)$, отвечающих $\Gamma^{(\varrho)} = \varrho\Gamma$ $(0 < \varrho < 1)$ (см. [12]).

Итак, для семейства $\mathfrak{A}$ решений задачи $N(m; n)$ во вполне неопределённом случае применима теорема 2. Учитывая, что $a_{22}(\zeta) = \mathscr{P}_+(\zeta)$ — внешняя м.-ф., для $B_0(\zeta)$, определённой по формуле (19), получаем

$$B_0(z) = [\mathscr{P}_+(z)\mathscr{P}_+^*(z) - \mathscr{Q}_+(z)\mathscr{Q}_+^*(z)]^{-1}. \tag{22}$$

Для м.-ф. $\chi(z)$ имеем

$$\chi(z) = -\mathscr{P}_+^{-1}(z)\mathscr{Q}_+(z) \tag{23}$$

и, так как $\mathscr{Q}_+(0) = 0$, то $\chi(0) = 0$. Легко видеть, что $\mathscr{P}_+(0) = a^{-1}$, так что $B_0(0) = a^2$, где a — положительная матрица порядка $n \times n$, определяемая уже равенством

$$\|(I - \mathbf{\Gamma}^*\mathbf{\Gamma})^{-1/2}\xi\|^2 = \|a^{1/2}\xi_1\|^2, \quad \xi = \{\xi_k\}_1^\infty \in \mathfrak{N}_1.$$

Предложение. *Пусть задача $N(m; n)$ является вполне неопределённой, т. е. для неё выполняется условие* (20), *и, значит множество всех её решений описывается формулой* (2), *где м.-ф. $A(\zeta) = [a_{ik}(\zeta)]_1^2$ вычисляется по формулам* (21). *Тогда для всех решений $f_{\mathscr{E}}(\zeta)$ этой задачи справедливо матричное неравенство* (18), *где $B_0(z)$ и $\chi(z)$ определяются по формулам* (22) *и* (23) *и в* (18) *имеем место знак равенства тогда и только тогда, когда $\mathscr{E}(\zeta) \equiv \chi^*(z)$. При нормировке $\psi_f(0) > 0$, в частности, для всех решений $f_{\mathscr{E}}(\zeta)$:*

$$\psi_{f_{\mathscr{E}}}(0) \leqq a, \quad i(f_{\mathscr{E}}) \geqq -\ln\det a,$$

причём равенства здесь имеют место тогда и только тогда, когда $\mathscr{E}(\zeta)\equiv 0$, *т. е. когда* $f_{\mathscr{E}}(\zeta)=\mathscr{Q}_-(\zeta)\mathscr{P}_+^{-1}(\zeta)$.

2. К задаче $N(m;\ n)$ сводятся задачи Шура, Неванлинны—Пика и другие. Напомним, что в задаче Шура требуется описать множество $\mathfrak{A}$ всех м.-ф. $f(z)$ класса $\mathbf{B}^{m\times n}$, имеющих заданные первые p коэффициентов $\{a_k\}_0^{p-1}$ разложения $f(z)$ в ряд Маклорена

$$f(z) = a_0+a_1z+\ldots+a_{p-1}z^{p-1}+\ldots.$$

Задача Шура сводится к задаче $N(m;n)$ рассмотрением граничных значений м.-ф. $z^{-p}f(z)$. При этом получаем $\gamma_k=a_{p-k}$ при $1\leqq k\leqq p$ и $\gamma_k=0$ при $k>p$. Для такой задачи $N(m;n)$ будем иметь $p_k=0$ и $q_k=0$ при $k>p$, так что $\mathscr{P}_+(z)$ и $\mathscr{Q}_+(z)$ — многочлены с матричными коэффициентами степени не выше $p-1$ и p соответственно. В рассматриваемом случае вместо бесконечной блочно-ганкелевой матрицы Γ в формулах для $\mathscr{P}_\pm(z)$ и $\mathscr{Q}_\pm(z)$ можно писать конечные блочно-ганкелевые матрицы Γ_p,

$$\Gamma_p = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_p \\ \gamma_2 & \gamma_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_p & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a_{p-1} & a_{p-2} & \cdots & a_0 \\ a_{p-2} & a_{p-3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_0 & 0 & \cdots & 0 \end{pmatrix}.$$

Заметим, что $\Gamma_p=VT_p$, где T_p — блочно-теплицева матрица, V — симметрическая ортогональная матрица:

$$T_p = \begin{pmatrix} \gamma_p & 0 & \cdots & 0 \\ \gamma_{p-1} & \gamma_p & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_1 & \gamma_2 & \cdots & \gamma_p \end{pmatrix} = \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{p-1} & a_{p-2} & \cdots & a_0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & I & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Очевидно, что $I-\Gamma_p^*\Gamma_p=I-T_p^*T_p$, $I-\Gamma_p\Gamma_p^*=V(I-T_pT_p^*)V$. М.-ф. $\mathscr{P}_-(z)$ и $\mathscr{Q}_-(z)$ — многочлены относительно z^{-1} с матричными коэффициентами, степени не выше $p-1$ и p, соответственно. Поэтому $z^p\mathscr{P}_-(z)$ и $z^p\mathscr{Q}_-(z)$ — многочлены относительно z степени не выше p.

Итак, задача Шура имеет решение тогда и только тогда, когда $I-T_p^*T_p\geqq 0$. Эту задачу назовём вполне неопределённой, если $I-T_p^*T_p>0$. В этом случае множество решений описывается по формуле (2), т. е. $\mathfrak{A}=\mathfrak{A}(A)$, где

$$A(z) = \begin{pmatrix} z^p\mathscr{P}_-(z) & z^p\mathscr{Q}_-(z) \\ \mathscr{Q}_+(z) & \mathscr{Q}_+(z) \end{pmatrix}.$$

М.-ф. $A(z)$ здесь является многочленом с матричными коэффициентами степени не выше p. Ниже будет показано, что она является j-внутренней. Многочлены $\mathscr{P}_+(z)$ и $\mathscr{Q}_+(z)$ получаются по блочно-теплицевой матрице T_p следующим образом. Рассматриваются матрицы X и Y порядков $mp \times mp$ и $np \times np$ соответственно, являющиеся решением системы

$$\begin{cases} X+T_pY = 0, \\ T_p^*X+Y = I_n. \end{cases}$$

Пусть X_k и Y_k $(1 \leqq k \leqq p)$ их блоки порядков $m \times m$ и $n \times n$. Тогда $p_k = Y_k Y_1^{-1/2}$, $q_k = X_{p-k+1} Y_1^{-1/2}$, так что

$$\mathscr{P}_+(z) = \sum_1^p Y_k Y_1^{-1/2} z^{k-1}, \quad \mathscr{Q}_+(z) = \sum_1^p X_{p-k+1} Y_1^{-1/2} z^k.$$

Подобным же образом определяются многочлены $z^p \mathscr{P}_-(z)$ и $z^p \mathscr{Q}_-(z)$.

3. Задача Шура является частным случаем следующей обобщённой задачи Неванлинны—Пика. Пусть заданы две внутрение м.-ф. $b_1(z)$ и $b_2(z)$ соответственно порядков $m \times m$ и $n \times n$. Требуется описать множество $\mathfrak{A}$ всех м.-ф. $f(z)$ класса $\mathbf{B}^{m \times n}$ таких, что $b_1^{-1}(z) f(z) b_2^{-1}(z)$ имеют граничные значения с заданной «главной частью» $\gamma(\zeta)$ разложения ряд Фурье:

$$\tilde{f}(\zeta) = b_1^{-1}(\zeta) f(\zeta) b_2^{-1}(\zeta) = \sum_1^\infty \gamma_k \zeta^{-k} + \dots.$$

Для $\tilde{f}(\zeta)$ имеем задачу $N(m; n)$, у которой заданная главная часть $\gamma(\zeta)$ удовлетворяет условию: $b_1(\zeta)\gamma(\zeta)b_2(\zeta)$ имеет нулевую «главную часть», т. е. разлагается в ряд Фурье по неотрицательным степеням ζ. В задаче Шура имеем $b_1(z) = z^p I_m$, $b_2(z) = I_n$. Случай, когда $b_1(z) = b(z) I_m$, $b_2(z) = I_n$, где $b(z)$ — произвольная скалярная внутренняя функция, был ранее рассмотрен в [4b]. Если, в частности, $b(z)$ — произведение Бляшке с простыми нулями z_k $(|z_k| < 1,\ 1 \leqq k < N \leqq +\infty)$, то имеем задачу, к которой сводится задача Неванлинны—Пика для м.ф. $f(z)$ класса $\mathbf{B}^{m \times n}$ с узлами интерполяции z_k: описать множество $\mathfrak{A}$ м.-ф. $f(z) (\in \mathbf{B}^{m \times n})$ с заданными значениями $f_k = f(z_k)$ $(1 \leqq k < N \leqq +\infty)$ (при $N = +\infty$ предполагается, что $\sum\limits_1^\infty (1-|z_k|) < \infty$). Как известно, впервые нескалярный вариант классической задачи Неванлинны—Пика; и даже не для м.-ф., а для оператор-функций, изучался методами теории расширения изометрических операторов в работе B. Sz.-Nagy и A. Korányi [13].

Если $b_2(z) = I_n$, а $b_1(z)$ — конечное или бесконечное матричное произведение Бляшке—Потапова [14], то имеем задачу, к которой сводится касательная задача Неванлинны—Пика, исследованная в работах [16]. Если же не только $b_1(z)$, но и $b_2(z)$ — произведения Бляшке—Потапова, то приходим к более общей задаче, нежели касательная (би-касательной задаче), в которой в интерполяционных данных одновременно фигурируют величины, связанные с $f(z)$ и $f^*(z)$.

В случае, когда обобщённая задача Неванлинны—Пика является вполне неопределённой, множество $\mathfrak{A}$ её решений описывается формулой (2), т. е. $\mathfrak{A}=\mathfrak{A}(A)$, где

$$A(\zeta)=\begin{pmatrix} b_1(\zeta) & 0 \\ 0 & b_2^{-1}(\zeta)\end{pmatrix}\begin{pmatrix}\mathcal{P}_-(\zeta) & \mathcal{Q}_-(\zeta) \\ \mathcal{Q}_+(\zeta) & \mathcal{P}_+(\zeta)\end{pmatrix}.$$

М.-ф. $\mathcal{P}_\pm(\zeta)$ и $\mathcal{Q}_\pm(\zeta)$ определяются по $\gamma(\zeta)$ по указанным ранее формулам. Покажем, что рассматриваемая в этой задаче м.-ф. $A(\zeta)$ является граничным значением j-внутренней м.-ф. Действительно, $A(\zeta)$ принимает j-унитарные значения и

$$s_{11}(\zeta)\overset{\text{def}}{=}a_{11}(\zeta)-a_{12}(\zeta)a_{22}^{-1}(\zeta)a_{21}(\zeta)=b_1(\zeta)[\mathcal{P}_+^*(\zeta)]^{-1}\in\mathbf{B}^{m\times m},$$

$$s_{12}(\zeta)\overset{\text{def}}{=}a_{12}(\zeta)a_{22}^{-1}(\zeta)=f_0(\zeta)\in\mathbf{B}^{m\times n},$$

$$s_{21}(\zeta)\overset{\text{def}}{=}-a_{22}^{-1}(\zeta)a_{21}(\zeta)=-\mathcal{P}_+^{-1}(\zeta)\mathcal{Q}_+(\zeta)\in\mathbf{B}^{n\times m},$$

$$s_{22}(\zeta)\overset{\text{def}}{=}a_{22}^{-1}(\zeta)=\mathcal{P}_+^{-1}(\zeta)b_2(\zeta)\in\mathbf{B}^{n\times n}.$$

По основной лемме из [11] получаем, что $A(z)=[a_{ik}(z)]_1^2$ j-внутренняя м.-ф.

4. Континуальные аналоги

1. Будем теперь рассматривать вместо единичной окружности вещественную прямую, а вместо единичного круга — верхнюю полуплоскость.

Через $\mathbf{B}_+^{m\times n}$ обозначим класс голоморфных при $\operatorname{Im} z>0$ м.-ф. $\mathcal{E}(z)$ порядка $m\times n$ с $\|\mathcal{E}(z)\|\leq 1$ при $\operatorname{Im} z>0$. Для $\mathcal{E}(z)$ из $\mathbf{B}_+^{m\times n}$ существуют п. в. граничные значения $\mathcal{E}(x)=\lim\limits_{y\downarrow 0}\mathcal{E}(x+iy)$, $-\infty<x<+\infty$, причём $\operatorname*{ess\,sup}\limits_{-\infty<x<+\infty}\|\mathcal{E}(x)\|=$ $=\sup\limits_{\operatorname{Im} z>0}\|\mathcal{E}(z)\|\,(=\|\mathcal{E}\|_\infty)$.

Замена переменной

$$z\mapsto(z-i)/(z+i), \tag{24}$$

отображающая замкнутую верхнюю полуплоскость на замкнутый единичный круг, биективно переводит $\mathbf{B}_+^{m\times n}$ на $\mathbf{B}^{m\times n}$, а класс $K_+^{m\times n}$ измеримых сжимающих на вещественной прямой м.-ф. $f(z)$ порядка $m\times n$ — на класс $K^{m\times n}$. При этом функционал $i(f;z)$, определённый по формуле (1), переходит в функционал

$$I(f;\,z)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\det[I_n-f^*(x)f(x)]\,\frac{\operatorname{Im} z}{|x-z|^2}\,dx\quad(\operatorname{Im} z>0,\ f\in K_+^{m\times n}), \tag{1_+}$$

так что

$$I(f)\overset{\text{def}}{=}I(f;\,i)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\ln\det[I_n-f^*(x)f(x)]\,\frac{dx}{1+x^2}. \tag{$1'_+$}$$

Для м.-ф. $A(x)=[a_{ik}(x)]_1^2$, принимающей п. в. на вещественной прямой j-унитарные значения, будем рассматривать

$$(6_+) \qquad \chi(x) = -a_{22}^{-1}(x)a_{21}(x)$$

и предполагать, что для м.-ф. $\chi(x)$ выполняются условия

$$(!_+) \qquad 1)\quad \chi \in \mathbf{B}_+^{n\times m}, \qquad 2)\quad I(\chi) < \infty.$$

Путём замены переменной (24) из теоремы 2 получается

Теорема 2_+. *Пусть м.-ф. $A(x)=[a_{ik}(x)]_1^2$ принимает j-унитарные значения п. в. на вещественной прямой и м.-ф. $\chi(x)$, определённая по формуле (6_+), удовлетворяет условию $(!_+)$. Тогда справедливо утверждение, отличающееся от утверждения теоремы 2 лишь тем, что в нём следует писать вместо переменных ζ с $|\zeta|=1$ и z с $|z|<1$ переменные x с $\operatorname{Im} x=0$ и z с $\operatorname{Im} z>0$, вместо функционала $i(f; z)$ — функционал $I(f; z)$ и вместо класса $\mathbf{B}^{m\times n}$ — класс $\mathbf{B}_+^{m\times n}$.*

Аналогичным образом из теоремы 3 получается её континуальный аналог — теорема 3_+.

2. Теоремы 2_+ и 3_+ применимы, в частности, к задаче $N_+(m; n)$, являющейся континуальным аналогом задачи $N(m; n)$. При её формулировке в полной общности возникают дополнительные трудности, связанные с тем, что преобразование Фурье от ограниченной м.-ф. является обобщённой функцией медленного роста и приходится, таким образом, использовать определённые факты из теории обобщённых функций.

Во избежание этих трудностей ограничимся рассмотрением следующей задачи $N_+^0(m; n)$, являющейся по существу частным случаем задачи $N_+(m; n)$, однако важной тем, что она имеет прямое отношение к теории рассеяния канонического дифференциального уравнения вида

$$(25) \qquad J\frac{dY}{dr} = \lambda Y + V(r)Y \quad (0 \leqq r < +\infty), \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

с суммируемым потенциалом $V=V^*(\in L_1^{2n\times 2n}(0;\ +\infty))$ [6] и её решение опирается на теорию классических интегральных операторов.

Пусть задана суммируемая м.-ф. $\Gamma_0(\in L_1^{m\times n}(0;\ +\infty))$ порядка $m\times n$. Требуется описать множество $\mathfrak{A}$ всех м.-ф. $S(x)$ из $K_+^{m\times n}$ таких, что

$$S(x) = \int_0^{+\infty} e^{-ixt}\Gamma_0(t)\,dt + \Phi(x),$$

где $\Phi(x)$ — граничное значение некоторой ограниченной при $\operatorname{Im} z>0$ м.-ф. $\Phi(z)$.

Для формулировки условия разрешимости этой задачи $N_+^0(m; n)$ и описания множества $\mathfrak{A}$ рассматривается вполне непрерывный ганкелев оператор $\mathbf{\Gamma}_0$ в пространстве $L_2^{n\times 1}(0; \infty)$, определяемый по формуле

$$(\mathbf{\Gamma}_0\xi)(t) = \int_0^{+\infty} \Gamma_0(t+s)\xi(s)\,ds, \quad \xi \in L_2^{n\times 1}(0; \infty).$$

Задача $N_+^0(m; n)$ имеет решение тогда и только тогда, когда $I - \mathbf{\Gamma}_0^*\mathbf{\Gamma}_0 \geqq 0$. Она является вполне неопределённой, если $\|\mathbf{\Gamma}_0\| < 1$. Именно к этому случаю при $m=n$ приводит задача рассеяния для канонической системы (25) с суммируемым потенциалом [6]. В этом и только этом случае существует решение задачи $S(x)$ с $\|S\|_\infty < 1$.

Замечание 4. Для задачи $N_+^0(m; n)$ в множестве решений существуют такие $S(x)$, которые представимы в виде

$$S(x) = S_0 + \int_{-\infty}^{+\infty} e^{itx}\Gamma(t)\,dt,$$

где S_0 — постоянная матрица, а $\Gamma \in L_1^{m\times n}(-\infty; +\infty)$. Более того, $S(x)$ допускает указанное представление тогда и только тогда, когда соответствующая ей в формуле (2_+) параметрическая м.-ф. $\mathscr{E}(z)$ допускает представление

$$\mathscr{E}(z) = \mathscr{E}_0 + \int_0^{+\infty} e^{izt}\hat{\mathscr{E}}(t)\,dt, \quad \hat{\mathscr{E}} \in L_1^{m\times n}(0; \infty).$$

В частности, такими являются решения задачи $N_+^0(m; n)$, дающие минимум функционалов $I(S; z)$ $(\operatorname{Im} z > 0)$, ибо для них соответвующие м.-ф. являются постоянными.

При выполнении условия $\|\mathbf{\Gamma}_0\| < 1$ описание множества $\mathfrak{A}$ получается по формуле (2_+), отличающейся от (2) лишь тем, что ζ с $|\zeta| = 1$ заменяется на x с $\operatorname{Im} x = 0$, а $\mathbf{B}^{m\times n}$ — на $\mathbf{B}_+^{m\times n}$. Для м.-ф. $A(x) = [a_{ik}(x)]_1^2$ получаются формулы (21_+), отличающиеся от (21) тем, что ζ заменяется на x. Рассматриваемые в (21_+) м.-ф. $\mathscr{P}_+(x)$ и $\mathscr{Q}_+(x)$ определяются следующим образом:

$$\mathscr{P}_+(x) = I_n - \mathscr{F}_+\{(\mathbf{I} - \mathbf{\Gamma}_0^*\mathbf{\Gamma}_0)^{-1}\mathbf{\Gamma}_0^*\Gamma_0\}(x),$$
$$\mathscr{Q}_+(x) = \mathscr{F}_+\{(\mathbf{I} - \mathbf{\Gamma}_0^*\mathbf{\Gamma}_0)^{-1}\Gamma_0^*\}(x).$$

Здесь использовано обозначение

$$\mathscr{F}_+\{G\}(x) = \int_0^{+\infty} e^{ixt}G(t)\,dt, \quad G \in L_1^{k\times n}(0; \infty).$$

М.-ф. $\mathscr{P}_-(x)$ и $\mathscr{Q}_-(x)$, рассматриваемые в (21_+), получаются по таким же формулам, что и $\mathscr{P}_+(x)$ и $\mathscr{Q}_+(x)$: следует в них заменить $\mathbf{\Gamma}_0$ на $\mathbf{\Gamma}_0^*$, $\mathbf{\Gamma}_0^*$ на $\mathbf{\Gamma}_0$, Γ_0 на

Γ_0^*, Γ_0^* на Γ и x на $-x$. Для получаемой в итоге м.-ф. $A(x)=[a_{ik}(x)]_1^2$ выполняются условия теорем 2_+ и 3_+; здесь $a_{22}^{-1}(x)=\mathscr{P}_+^{-1}(x)$ — внешняя м.-ф. класса $\mathbf{B}_+^{n\times n}$.

3. Точно так же, как ранее была решена задача Шура $S(m; n)$, сводящаяся к задаче $N(m; n)$, рассматривается её континуальный вариант — задача $S_+^0(m; n)$: описать все м.-ф. $F(z)$ из $\mathbf{B}_+^{m\times n}$ такие, что

$$F(z) = \int_0^T e^{izt}C(t)\,dt + e^{izT}\Phi(z),$$

где $C(t)(\in L_1^{m\times n}(0; T))$ — заданная м.-ф., а $\Phi(z)$ — ограниченная голоморфная при $\operatorname{Im} z>0$ м.-ф. Она сводится к задаче $N_+^0(m; n)$, поставленной для $S(x)=$ $=e^{-ixT}F(x)$ по заданной м.-ф. $\Gamma_0(t)$,

$$\Gamma_0(t) = \begin{cases} 0 & \text{при} \quad t>T, \\ C(T-t) & \text{при} \quad 0<t<T. \end{cases}$$

Так как

$$\eta(t) = \mathbf{\Gamma}_0\xi = \begin{cases} 0 \quad \text{при} \quad t>T \\ \displaystyle\int_0^{T-t} C(T-t-s)\xi(s)\,ds \quad \text{при} \quad 0<t<T, \end{cases}$$

то условие $I-\mathbf{\Gamma}_0^*\mathbf{\Gamma}_0\geqq 0$ существования решения задачи можно переписать в виде $I-\mathbf{S}_T^*\mathbf{S}_T\geqq 0$, где $\mathbf{S}_T$ — теплицев оператор, действующий из $L_2^{n\times 1}(0; T)$ в $L_2^{m\times 1}(0; T)$, определяемый по формуле

$$(\mathbf{S}_T\xi)(t) = \int_0^t C(t-s)\xi(s)\,ds \quad (0\leqq t\leqq T).$$

Условие полной неопределённости задачи $S_+^0(m; n)$ записывается в виде $I-$ $-\mathbf{S}_T^*\mathbf{S}_T>0$. Формулы для $\mathscr{P}_\pm(x)$ и $\mathscr{Q}_\pm(x)$ можно переписать, заменяя ганкелев оператор $\mathbf{\Gamma}_0$ теплицевым оператором $\mathbf{S}_T$. Описание всех решений $F(z)$ задачи $S_+^0(m; n)$ получается по формуле (2_+), в которой м.-ф. $A(z)=[a_{ik}(z)]_1^2$ — j-внутреняя при $\operatorname{Im} z>0$,

$$a_{11}(z) = e^{izT}\mathscr{P}_-(z), \quad a_{12}(z) = e^{izT}\mathscr{Q}_-(z),$$

$$a_{21}(z) = \mathscr{Q}_+(z), \qquad a_{22}(z) = \mathscr{P}_+(z).$$

Все блоки $a_{ik}(z)$ в рассматриваемой задаче оказываются м.-ф. из $H_2^{r\times n}\ominus e^{izT}H_2^{r\times n}$ $(r=m, n)$ и поэтому это целые м.-ф. Таким образом $A(z)$ целая j-внутренняя м.-ф. и к ней применима теорема 4_+ — континуальный аналог теоремы 4.

Для задачи $S_+^0(m; n)$ справедливо замечание такое же, какое было сделано ранее для задачи $N_+^0(m; n)$.

Приведенные нами результаты об описании множества решений задач $N_+^0(m; n)$ и $S_+^0(m; n)$ заимствованы из рукописи М. Г. Крейна и Ф. Э. Мелик-Адамяна (где предполагалось $m=n$), краткое извлечение из которой опубликовано в работе [6]. Следует отметить, что работа [6] была первым матричным и притом континуальным аналогом работы [4a].

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The modulus of variation of a function and the Banach indicatrix

V. O. ASATIANI and Z. A. CHANTURIA

Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

It is well known that the notion of variation of a function was introduced by C. JORDAN in 1881 in the paper [12], devoted to the convergence of Fourier series. In 1924 N. WIENER [22] generalized this notion and introduced the notion of P-variation. Finally, L. YOUNG [23] introduced the notion of Φ-variation of a function.

Definition 1 (see [23]). Let Φ be a strictly increasing continuous function on $[0, \infty)$ and $\Phi(0)=0$. f will be said to have bounded Φ-variation on $[a, b]$, or $f\in V_\Phi$, if

$$v_\Phi(f) = \sup_{\Pi} \sum_{k=1}^{n} \Phi(|f(x_k)-f(x_{k-1})|) < \infty,$$

where $\Pi=\{a\leqq x_0<x_1<\ldots<x_n\leqq b\}$ is an arbitrary partition.

If $\Phi(u)=u$, then V_Φ coincides with the Jordan class V and when $\Phi(u)=u^p$, $p>1$, it coincides with the Wiener class V_p. In 1973 Z. A. CHANTURIA [5] introduced the notion of the modulus of variation of a function.

Definition 2. Let f be bounded on $[a, b]$. The modulus of variation of the function f is the function $\upsilon(n,f)$ defined by $\upsilon(0,f)=0$ and for $n\geqq 1$,

$$\upsilon(n,f) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(x_{2k+1})-f(x_{2k})|,$$

where Π_n is an arbitrary system of disjoint intervals (x_{2k}, x_{2k+1}), $k=0, 1, \ldots n-1$, of the interval $[a, b]$.

The modulus of variation $\upsilon(n, f)$ is non-decreasing and convex upwards ([5], [19]). Such a function will be called a modulus of variation. If a modulus of variation $\upsilon(n)$ is given then the class of functions f, given on $[a, b]$, for which

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$v(n, f)=O(v(n))$ when $n\to\infty$, will be denoted by $V[v]$. It is known that if Φ is convex and $\Phi(u)\not\sim u$ *) on $[0, \delta]$ then $V_\Phi\subset V[n\Phi^{-1}(1/n)]$ is a strict inclusion ([5], [8]).

In 1925 S. Banach [3] introduced the function $N(y, f)$ for continuous functions f: for every $y\in(-\infty, +\infty)$, $N(y, f)$ is equal to the number (finite or infinite) of solutions of equations $f(x)=y$. Following I. P. Natanson [16] (p. 112) $N(y, f)$ will be called the Banach indicatrix. Banach [3] proved that a continuous function f belongs to V if and only if $N(y, f)$ is summable on $[m(f), M(f)]$, where $m(f)=\inf_{x\in[a,b]} f(x)$ and $M(f)=\sup_{x\in[a,b]} f(x)$.

S. M. Lozinski [14] generalized the notion of the Banach indicatrix for bounded functions which have only discontinuities of the first kind. Denote this class by $W(a, b)$. S. M. Lozinski [13] showed that the Banach theorem is valid without assuming the continuity of f.

One can obtain the class $W(a, b)$ from $C(a, b)$ by a monotone transformation of the argument, as it follows from the following theorem of O. D. Tsereteli [20] (p. 42) and [21] (p. 131): Let $f\in W(a, b)$. Then there exist functions χ and F satisfying the following conditions: χ increases on $[a, b]$, F is continuous on $[\chi(a), \chi(b)]$ and $f(x)=F(\chi(x))$.

The definition of Lozinski is equivalent to the following

Definition 3. Let $f\in W(a, b)$. The Banach indicatrix of f is defined by $N(y, f):=N(y, F)$, where F is determined by the relation $f(x)=F(\chi(x))$.

Since the variation of a function does not vary for monotone transformations of the argument, thus by virtue of Tsereteli's theorem, Lozinski's result is a consequence of Banach's theorem.

T. Zerekidze [24] proved the analogue of Banach's theorem for the classes V_p: If $f\in W(a, b)$ and $p>1$ then from the condition

$$\int_{-\infty}^{+\infty} [N(y, f)]^{1/p}\, dy < \infty$$

it follows that $f\in V_p$. The converse does not hold.

The purpose of the present paper is to study the relationship between the degree of summability of the Banach indicatrix and the modulus of variation of the function in question. The results obtained are then applied to some problems of the theory of Fourier series.

Let Ω be an increasing concave function on $[0, +\infty)$, $\Omega(0)=0$, $\lim_{x\to\infty}\Omega(x)=\infty$, $\lim_{x\to\infty}\frac{\Omega(x)}{x}=0$. The following theorem holds.

*) $\Phi(u)\sim\Psi(u)$ on $[a, b]$ if there exist positive constants A and B such that $A\Phi(u)<\Psi(u)<B\Phi(u)$, when $u\in[a, b]$.

Theorem 1. *If $f \in W(a, b)$ and*

$$(1) \qquad \int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy < \infty,$$

the modulus of variation $\upsilon(n, f)$ of f satisfies the following relation

$$(2) \qquad \sum_{n=1}^{\infty} [2\Omega(n) - \Omega(n-1) - \Omega(n+1)]\upsilon(n, f) < \infty.$$

The proof is based on the following lemma.

Lemma 1. *If $f \in W(a, b)$, then*

$$\upsilon(n, f) \leqq 3 \int_{m(f)}^{M(f)} N_n(y, f)\, dy,$$

where

$$N_n(y, f) = \begin{cases} N(y, f) & \textit{when} \quad N(y, f) \leqq n, \\ n & \textit{when} \quad N(y, f) > n. \end{cases}$$

Proof. By virtue of Tsereteli's theorem it suffices to prove the lemma for $f \in C(a, b)$. By the definition of the modulus of variation of a function, for any $\varepsilon > 0$ one can find $2n$ points $\{x_k^{(\varepsilon)}\}_{k=0}^{2n-1}$ such that

$$a \leqq x_0^{(\varepsilon)} < x_1^{(\varepsilon)} \leqq \ldots \leqq x_{2n-2}^{(\varepsilon)} < x_{2n-1}^{(\varepsilon)} \leqq b$$

and

$$\upsilon(n, f) \leqq \sum_{k=0}^{n-1} |f(x_{2k+1}^{(\varepsilon)}) - f(x_{2k}^{(\varepsilon)})| + \varepsilon.$$

Introduce the function

$$g_n(x) = \begin{cases} f(x_k^{(\varepsilon)}) & \text{when} \quad x = x_k^{(\varepsilon)}, \quad k = 0, 1, \ldots, 2n-1, \\ f(x_0^{(\varepsilon)}) & \text{when} \quad x = a, \\ f(x_{2n-1}^{(\varepsilon)}) & \text{when} \quad x = b, \quad \text{and} \\ \text{linear for all other} & x \in [a, b]. \end{cases}$$

Let

$$m_k = \min\{f(x_k^{(\varepsilon)}),\ f(x_{k+1}^{(\varepsilon)})\}, \quad M_k = \max\{f(x_k^{(\varepsilon)}),\ f(x_{k+1}^{(\varepsilon)})\}.$$

Then on any segment $[x_k^{(\varepsilon)}, x_{k+1}^{(\varepsilon)}]$ the equation $g_n(x) = y$, $y \in [m_k, M_k]$, has a unique solution, whereas the equation $f(x) = y$ has at least one solution, i.e., for any y, $N(y, g_n) \leqq N(y, f)$. On the other hand, $N(y, g_n) \leqq 2n+1$. Therefore

$$(3) \qquad N(y, g_n) \leqq \min\{N(y, f), 2n+1\} \leqq 3\min\{N(y, f), n\} = 3N_n(y, f).$$

Let us estimate the variation of the function g_n. We have

$$\upsilon(g_n) \geqq \sum_{k=1}^{2n-1} |f(x_{k+1}^{(\varepsilon)}) - f(x_k^{(\varepsilon)})| \geqq \upsilon(n, f) - \varepsilon,$$

whence by virtue of Banach's theorem and relation (3),

$$\upsilon(n, f) \leqq \upsilon(g_n)+\varepsilon = \int_{m(g_n)}^{M(g_n)} N(y, g_n)\, dy+\varepsilon \leqq 3 \int_{m(f)}^{M(f)} N_n(y, f)\, dy+\varepsilon.$$

Since ε is arbitrary, the lemma is proved.

Proof of Theorem 1. Introduce the notations

$$\sigma(n) = [\Omega(n)-\Omega(n-1)]n, \quad e_n = \{y;\ N(y, f) = n\},$$

$$E_n = \bigcup_{k=1}^{n} e_k = \{y;\ 1 \leqq N(y, f) \leqq n\}, \quad E_n' = \bigcup_{k=n+1}^{\infty} e_k = \{y;\ N(y, f) > n\}.$$

It is easy to see that by the properties of Ω we have

1) $$\sigma(n) \leqq \Omega(n), \quad n = 1, 2, \ldots,$$

2) $$\frac{\sigma(n)}{n} \geqq \frac{\sigma(n+1)}{n+1}, \quad n = 1, 2, \ldots.$$

Using these relations and Abel's transformation, we get

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy \geqq \int_{m(f)}^{M(f)} \sigma(N(y, f))dy =$$

(4) $$= \sum_{n=1}^{\infty} \int_{e_n} \frac{\sigma(N(y, f))}{N(y, f)} N(y, f)\, dy = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} \int_{e_n} N(y, f)\, dy \geqq$$

$$\geqq \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k}-\frac{\sigma(k+1)}{k+1}\right) \int_{E_k} N(y, f)\, dy+\frac{\sigma(n)}{n} \int_{E_n} N(y, f)\, dy.$$

In virtue of Lemma 1,

$$\upsilon(n, f) \leqq 3 \int_{m(f)}^{M(f)} N_n(y, f)\, dy = 3 \int_{E_n} N(y, f)\, dy+3n\mu E_n'.$$

From here and (4) it follows that

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy \geqq$$

$$\geqq \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k}-\frac{\sigma(k+1)}{k+1}\right)\left(\frac{1}{3}\upsilon(k, f)-k\mu E_k'\right)+\frac{\sigma(n)}{n} \int_{E_n} N(y, f)\, dy =$$

$$= \frac{1}{3} \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k}-\frac{\sigma(k+1)}{k+1}\right) \upsilon(k, f)-\sum_{k=1}^{n} \frac{\sigma(k)}{k} (k\mu E_k'-(k-1)\mu E_{k-1}')+\frac{\sigma(n)}{n} n\mu E_n+$$

$$+\frac{\sigma(n)}{n} \int_{E_n} N(y, f)\, dy \geqq$$

$$\geqq \frac{1}{3} \sum_{k=1}^{n-1} \left(\frac{\sigma(k)}{k}-\frac{\sigma(k+1)}{k+1}\right) \upsilon(k, f)-\sum_{k=1}^{n} \frac{\sigma(k)}{k} (k\mu E_k'-(k-1) E_{k-1}')+\frac{\sigma(n)}{3n} \upsilon(n, f).$$

But since

$$k\mu E'_k-(k-1)\mu E'_{k-1} = \mu E'_{k-1}-k\mu e_k$$

thus

$$\int_{m(f)}^{M(f)} \Omega\big(N(y,f)\big)\,dy \geqq \frac{1}{3}\sum_{k=1}^{n}[2\Omega(k)-\Omega(k+1)-\Omega(k-1)]\upsilon(k,f)+$$

$$+\sum_{k=1}^{n}\sigma(k)\mu e_k-\sum_{k=1}^{n}\frac{\sigma(k)}{k}\mu E'_{k-1}.$$

From the latter relation it follows that

$$\sum_{k=1}^{n}[2\Omega(k)-\Omega(k+1)-\Omega(k-1)]\upsilon(k,f) \leqq 3\int_{m(f)}^{M(f)} \Omega\big(N(y,f)\big)dy+3\sum_{k=1}^{n}\frac{\sigma(k)}{k}\mu E'_{k-1} \leqq$$

$$\leqq 3\int_{m(f)}^{M(f)} \Omega\big(N(y,f)\big)dy+3\sum_{j=1}^{\infty}\mu e_k\left(\sum_{k=1}^{\infty}\frac{\sigma(j)}{j}\right)=$$

$$=3\int_{m(f)}^{M(f)} \Omega\big(N(y,f)\big)dy+3\sum_{k=1}^{\infty}\mu e_k\Omega(k) \leqq 6\int_{m(f)}^{M(f)} \Omega\big(N(y,f)\big)\,dy<\infty.$$

Theorem 1 is proved.

We give some corollaries of Theorem 1.

Corollary 1. *If* $f\in W(a,b)$ *and for* $\alpha>0$,

$$\int_{m(f)}^{M(f)} \ln^{\alpha}\big(N(y,f)\,dy<\infty, \tag{5}$$

then

$$\sum_{n=1}^{\infty}\frac{\ln^{\alpha-1}(n+1)}{n^2}\upsilon(n,f)<\infty.$$

Proof. Like before, we may assume that $f\in C(a,b)$. Then for $y\in[m(f),M(f)]$, $N(y,f)\geqq 1$, therefore (5) is equivalent to

$$\int_{m(f)}^{M(f)} \ln^{\alpha}\big(1+N(y,f)\big)\,dy<\infty.$$

Take now $\Omega(x)=\ln^{\alpha}(1+x)$. Then

$$2\Omega(n)-\Omega(n+1)-\Omega(n-1)>c\,\frac{\ln^{\alpha-1}(n+1)}{n^2}$$

whence by virtue of Theorem 1 we obtain the statement of Corollary 1.

Corollary 2. *If* $f\in W(a,b)$ *and for* $p>1$,

$$\int_{m(f)}^{M(f)} N^{1/p}(y,f)\,dy<\infty,$$

then

$$\sum_{n=1}^{\infty} \frac{v(n, f)}{n^{2-1/p}} < \infty.$$

In fact, for the proof it is sufficient to take $\Omega(x)=x^{1/p}$.

Theorem 1 cannot be converted since Theorem 2 holds.

Theorem 2. *Let Ω satisfy the above conditions. Then there exists a function $f_0 \in C(a, b)$ for which* (2) *is valid, but* (1) *is not fulfilled.*

Proof. Let us show first that there exist an increasing sequence of integers $\{\mu_k\}_{k=0}^{\infty}$ and a sequence of positive numbers $\{b_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} [\Omega(\mu_k) - \Omega(\mu_{k-1})] b_k < \infty \tag{6}$$

and

$$\sum_{k=1}^{\infty} \Omega(\mu_k - \mu_{k-1}) b_k = \infty. \tag{7}$$

Let

$$\mu_{2k+1} = 2\left[\frac{\Omega^{-1}(2k+1)}{2}\right] - 1, \quad \mu_{2k} = 2\left[\frac{\Omega^{-1}(2k)}{2}\right], \quad \alpha_k = \frac{\Omega^{-1}(k)}{2} - \left[\frac{\Omega^{-1}(k)}{2}\right].$$

Since the function Ω^{-1} is convex and can be represented as

$$\Omega^{-1}(x) = \int_0^x P(t)\, dt,$$

where $P(t)\uparrow$ on $[0, +\infty)$, and since

$$\lim_{x\to\infty} \frac{\Omega(x)}{x} = 0,$$

we have

$$\lim_{t\to\infty} P(t) = \infty.$$

Taking into account the above facts we have

$$\mu_{k+1} - \mu_k = \Omega^{-1}(k+1) - \Omega^{-1}(k) - 2\alpha_{k+1} - 2\alpha_k - 1 \geqq \int_k^{k+1} P(t)\, dt - 5,$$

i.e., $\mu_{k+1} - \mu_k \to \infty$ when $k \to \infty$; thus $\{\mu_k\}_{k=0}^{\infty}$ increases, beginning with some number k_0. It will be assumed without loss of generality that $k_0 = 0$.

Since Ω is convex upwards, $\Omega(x-y) \geqq \Omega(x) - \Omega(y)$ for $x>y>0$, hence

$$\begin{aligned} \Omega(\mu_{k+1}) - \Omega(\mu_k) &\leqq \Omega(\Omega^{-1}(k+1) - 2\alpha_{k+1}) - \Omega(\Omega^{-1}(k) - 2\alpha_k - 1) \leqq \\ &\leqq \Omega(\Omega^{-1}(k+1)) - \Omega(\Omega^{-1}(k) - 3) \leqq k+1 - \Omega(\Omega^{-1}(k)) + \Omega(3) = 1 + \Omega(3). \end{aligned} \tag{8}$$

Denote $\lambda_k=\Omega(\mu_k-\mu_{k-1})$. It is obvious that $\lambda_k\to\infty$, $k\to\infty$.

We shall divide the set of natural numbers $\mathbf{N}$ into subsets $\mathbf{N}=\bigcap_{k=1}^{\infty}\mathbf{N}_k$ in the following way: $\mathbf{N}_1=\{1\}$. If the sets $\mathbf{N}_1,\dots,\mathbf{N}_k$ are already constructed, then $\mathbf{N}_{k+1}$ is constructed as follows: Let $\tilde{\mathbf{N}}_k=\mathbf{N}\setminus\bigcup_{i=1}^{k}\mathbf{N}_i$ and $\Lambda_k=\min_{i\in\tilde{\mathbf{N}}_k}\lambda_i$; then

$$\mathbf{N}'_{k+1}=\{n;\ \Lambda_k\leqq\lambda_n<2\Lambda_k\}.$$

If $|\mathbf{N}'_{k+1}|\geqq|\mathbf{N}_k|$ then we put $\mathbf{N}_{k+1}=\mathbf{N}'_{k+1}$ and if $|\mathbf{N}'_{k+1}|<|\mathbf{N}_k|$, then to the set $\mathbf{N}'_{k+1}$ we add $|\mathbf{N}_k|-|\mathbf{N}'_{k+1}|$ natural numbers successively, beginning with the maximal term of the set $|\mathbf{N}'_{k+1}|$. The obtained set will be $\mathbf{N}_{k+1}$.

In virtue of the construction,

$$\text{(9)}\qquad 1)\quad |\mathbf{N}_{k+1}|\geqq|\mathbf{N}_k|\quad\text{and}\quad 2)\quad \Lambda_{k+1}\geqq 2\Lambda_k.$$

Suppose

$$b_m=\frac{1}{|\mathbf{N}_k|\Lambda_k},\quad\text{when}\quad m\in N_k.$$

It is clear that by virtue of (9), $b_m\geqq b_{m+1}$. Then we have

$$\sum_{m=1}^{\infty}b_m=\sum_{k=1}^{\infty}\sum_{m\in\mathbf{N}_k}\frac{1}{|\mathbf{N}_k|\Lambda_k}=\sum_{k=1}^{\infty}\frac{1}{\Lambda_k}\leqq\Lambda_1\sum_{k=1}^{\infty}\frac{1}{2^k}<\infty,$$

whence, applying (8), we get

$$\sum_{k=1}^{\infty}[\Omega(\mu_k)-\Omega(\mu_{k-1})]\,b_k<\infty.$$

On the other hand,

$$\sum_{m=1}^{\infty}\lambda_m b_m=\sum_{k=1}^{\infty}\sum_{m\in\mathbf{N}_k}\lambda_m b_m\geqq\sum_{k=1}^{\infty}\Lambda_k\frac{1}{\Lambda_k}=\infty.$$

Thus, the required sequences are constructed.

Let $\sum_{j=k}^{\infty}b_j=y_k$ and $m_k=\mu_k-\mu_{k-1}$. Note that m_k is an odd number. Let us construct the function f_0 in the following way: divide the segment $[1/2^k,1/2^{k-1}]$ into m_k parts by means of points

$$\{x_i^{(k)}\}_{i=1}^{m_k+1},\quad 1/2^k=x_1^{(k)}<x_2^{(k)}<\dots<x_{m_k}^{(k)}<x_{m_k+1}^{(k)}=1/2^{k-1}$$

and let

$$f_0(x)=\begin{cases}\dfrac{1}{2}\{(y_k+y_{k+1})+(-1)^i(y_k-y_{k+1})\} & \text{when}\quad x=x_i^{(k)},\ i=1,\dots,m_k+1,\\ \text{linear when}\quad x\in[x_i^{(k)},x_{i+1}^{(k)}], & \\ 0\quad\text{when}\quad x=0. & \end{cases}$$

Since $y_k \to 0$ when $k \to \infty$, thus f_0 is continuous on $[0, 1]$. Further, $N(y, f_0) = m_k$ when $y \in (y_{k+1}; y_k)$, hence, using (7), we get

$$\int_{m(f)}^{M(f)} \Omega(N(y, f_0))\, dy = \sum_{k=1}^{\infty} \int_{y_{k+1}}^{y_k} \Omega(N(y, f_0))\, dy = \sum_{k=1}^{\infty} b_k \Omega(m_k) =$$

$$= \sum_{k=1}^{\infty} b_k \Omega(\mu_k - \mu_{k-1}) = \infty.$$

Next we show that

$$\sum_{n=1}^{\infty} [2\Omega(n) - \Omega(n+1) - \Omega(n-1)]\, \upsilon(n, f_0) < \infty. \tag{10}$$

Consider two auxiliary functions

$$f_1(x) = \begin{cases} y_{k+1} & \text{when} \quad x \in [1/2^k, 1/2^{k-1}), \ k = 1, 2, \ldots, \\ 0 & \text{when} \quad x = 0, \end{cases}$$

and $f_2 = f_0 - f_1$. Then, it is obvious that

$$\upsilon(n, f_0) \leqq \upsilon(n, f_1) + \upsilon(n, f_2). \tag{11}$$

In virtue of the monotonicity of the function f_1, for all n,

$$\upsilon(n, f_1) = y_2 = \sum_{i=2}^{\infty} b_i. \tag{12}$$

Let us estimate now the modulus of variation of the function f_2. For a natural n we choose the number k such that

$$\mu_{k-1} = \sum_{i=1}^{k-1} m_i < n \leqq \sum_{i=1}^{k} m_i = \mu_k.$$

Then $\upsilon(n, f_2) - \upsilon(n-1, f_2) = b_k$ whence, according to (7),

$$\begin{aligned} &\sum_{k=1}^{\infty} [\Omega(n) - \Omega(n-1)][\upsilon(n, f_2) - \upsilon(n-1, f_2)] = \\ &= \sum_{k=1}^{\infty} \sum_{n=\mu_{k-1}+1}^{\mu_k} [\Omega(n) - \Omega(n-1)]\, b_k = \sum_{k=1}^{\infty} [\Omega(\mu_k) - \Omega(\mu_{k-1})]\, b_k = B < \infty. \end{aligned} \tag{13}$$

Using the relations (11), (12), and (13) we have

$$\sum_{k=1}^{n} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)]\, \upsilon(k, f_0) \leqq \sum_{k=1}^{n} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)]\, \upsilon(k, f_1) +$$

$$+ \sum_{k=1}^{n} [\Omega(k) - \Omega(k-1)][\upsilon(k, f_2) - \upsilon(k-1, f_2)] + [\Omega(n) - \Omega(n+1)]\, \upsilon(n, f_2) \leqq$$
$$\leqq y_2[\Omega(1) + \Omega(n) - \Omega(n+1)] + B \leqq \Omega(1) y_2 + B,$$

whence the validity of relation (10) follows. Theorem 2 is proved.

The following theorem states the dependence between the degree of summability of the Banach indicatrix and the classes V_Φ.

Theorem 3. *Let* Φ *be a continuous increasing convex function on* $[0, \infty)$, $\Phi(0)=0$, $\lim\limits_{u\to 0}\dfrac{\Phi(u)}{u}=0$, *and let*

$$\Omega(x)=\begin{cases}\displaystyle\int_{1/x}^{1}\frac{1}{t\Phi^{-1}(t)}\,dt & \text{when } x\in[1,\infty),\\ 0 & \text{when } x\in[0,1).\end{cases}\tag{14}$$

If $f\in W(a,b)$ *and* (1) *is fulfilled, then* $f\in V_\Phi$.

For the proof of this theorem two lemmas are needed.

Lemma 2 (see [11], p. 111 or [19], p. 160). *Let* $0\leqq a_n\downarrow$, $0\leqq b_n\downarrow$, *and let the relations* $\sum_{i=1}^{k} a_i \leqq \sum_{i=1}^{k} b_i$ *be true for* $k=1, 2, \ldots, m$. *Then for convex functions* Φ *the inequality*

$$\sum_{i=1}^{m}\Phi(a_i)\leqq\sum_{i=1}^{m}\Phi(b_i)$$

holds.

Lemma 3. *Let* $0<a_n\downarrow$ *and let* Φ *be a convex increasing function on* $[0,\infty)$ *and* $\Phi(u)>0$ *for* $u>0$. *Then*

$$\sum_{n=1}^{\infty} a_n\frac{1}{n\Phi^{-1}(1/n)}<\infty \quad\Rightarrow\quad \sum_{n=1}^{\infty}\Phi(a_n)<\infty.\tag{15}$$

Proof. Since Φ is convex, therefore $\Phi(u)/u$ increases, and hence $u/\Phi^{-1}(u)$ also increases, i.e., the sequence $\left\{\dfrac{1}{n\Phi^{-1}(1/n)}\right\}$ decreases. Starting from this, by virtue of Cauchy's theorem on numerical series, the convergence of the first series under (15) is equivalent to that of the series

$$\sum_{n=1}^{\infty} a_{2^n}\frac{1}{\Phi^{-1}(1/2^n)}.\tag{16}$$

From the convergence of series (16) it follows that there exists a natural number n_0 such that $a_{2^n}<\Phi^{-1}(1/2^n)$ for $n>n_0$. Since $u/\Phi(u)\downarrow$, from the latter inequality we obtain

$$\frac{a_{2^n}}{\Phi(a_{2^n})}\geqq\frac{\Phi^{-1}(1/2^n)}{\Phi(\Phi^{-1}(1/2^n))}=2^n\Phi^{-1}(1/2^n)\quad\text{when}\quad n>n_0,$$

or

$$2^n\Phi(a_{2^n})\leqq a_{2^n}\frac{1}{\Phi^{-1}(1/2^n)},\quad n>n_0.$$

From this relation and from the convergence of series (16) we obtain that

$$\sum_{n=1}^{\infty} 2^n \Phi(a_{2^n}) \quad \text{and} \quad \sum_{n=1}^{\infty} \Phi(a_n)$$

converge. The lemma is proved.

Proof of Theorem 3. Let us show first that Ω satisfies the conditions of Theorem 1. In fact, we have

1) $$\lim_{x\to\infty} \frac{\Omega(x)}{x} = \lim_{x\to\infty} \frac{1}{x} \int_{1/x}^{1} \frac{1}{t\Phi^{-1}(t)}\,dt = \lim_{x\to\infty} \frac{1}{x} \int_{1}^{x} \frac{1}{t\Phi^{-1}(1/t)}\,dt =$$

$$= \lim_{x\to\infty} \frac{1}{x\Phi^{-1}(1/x)} = 0,$$

2) $$\lim_{x\to\infty} \Omega(x) = \lim \int_{1/x}^{1} \frac{dt}{t\Phi^{-1}(t)} \geqq \lim_{x\to\infty} \int_{1/x}^{2/x} \frac{dt}{t\Phi^{-1}(t)} \geqq \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1}{(2/x)\Phi^{-1}(2/x)} = \infty,$$

3) the function Ω is convex upwards, since $\Omega'(x) = \dfrac{1}{x\Phi^{-1}(1/x)}$ is a decreasing function.

Since all the conditions of Theorem 1 are fulfilled, thus (2) is also satisfied. We will show that (2) implies the relation

$$\sum_{n=1}^{\infty} [\Omega(n) - \Omega(n-1)][\upsilon(n, f) - \upsilon(n-1, f)] < \infty. \tag{17}$$

To this end it is sufficient to prove that

$$\lim_{n\to\infty} [\Omega(n) - \Omega(n-1)]\upsilon(n, f) = 0. \tag{18}$$

By virtue of the convergence of series (2), for any $\varepsilon > 0$ one can find an n such that for any $m > n$ the relation

$$\sum_{k=n}^{m} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)]\upsilon(k, f) < \varepsilon$$

holds, whence, by virtue of the monotonicity of $\upsilon(n, f)$ and the fact that $\Omega(m+1) - \Omega(m) \to 0$, $m \to \infty$, we get

$$\varepsilon > \upsilon(n, f) \sum_{k=n}^{m} [2\Omega(k) - \Omega(k+1) - \Omega(k-1)] =$$

$$= \upsilon(n, f)[\Omega(n) - \Omega(n-1) + \Omega(m) - \Omega(m+1)] \geqq \frac{1}{2} \upsilon(n, f)[\Omega(n) - \Omega(n-1)].$$

Thus (18) is proved and it proves also (17).

But then, since

$$\Omega(n)-\Omega(n-1) = \int_{n-1}^{n} \frac{dt}{t\Phi^{-1}(1/t)} \geqq \frac{1}{n\Phi^{-1}(1/n)},$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{n\Phi^{-1}(1/n)} [v(n, f)-v(n-1, f)] < \infty,$$

and this, by virtue of Lemma 3, gives

$$\sum_{n=1}^{\infty} \Phi\big(v(n, f)-v(n-1, f)\big) < \infty. \tag{19}$$

We may now show that $f\in V_\Phi$. Let us take an arbitrary partition $\Pi = \{a \leqq x_0 < x_1 < \ldots < x_m \leqq b\}$; without loss of generality it may be assumed that

$$|f(x_k)-f(x_{k-1})| \geqq |f(x_{k+1})-f(x_k)|.$$

For every $n=1, 2, \ldots, m$ we have

$$\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| \leqq v(n, f) = \sum_{k=1}^{n} \big(v(k, f)-v(k-1, f)\big).$$

Therefore, if we take $a_k = |f(x_k)-f(x_{k-1})|$ and $b_k = v(k, f)-v(k-1, f)$, and apply Lemma 2 and relation (19), we have

$$\sum_{k=1}^{m} \Phi\big(|f(x_k)-f(x_{k-1})|\big) \leqq \sum_{k=1}^{m} \Phi\big(v(k, f)-v(k-1, f)\big) \leqq$$

$$\leqq \sum_{k=1}^{\infty} \Phi\big(v(k, f)-v(k-1, f)\big) < \infty.$$

Thus, as it was required, we proved that $f\in V_\Phi$.

Corollary 3. [24] *Let $f\in W(a, b)$, and for $p>1$,*

$$\int_{m(f)}^{M(f)} [N(y, f)]^{1/p}\, dy < \infty.$$

Then $f\in V_p$.

Corollary 4. *Let $f\in W(a, b)$, and for $\alpha>1$,*

$$\int_{m(f)}^{M(f)} \ln^\alpha\big(N(y, f)+1\big) < \infty.$$

Then $f\in V_\Phi$, where $\Phi(x)=\exp(-x^{1/(1-\alpha)})$ in $(0, \delta)$; $\delta>0$.

We shall show that Theorem 3 cannot be converted.

Theorem 4. *Let the function* Φ *satisfy the conditions of Theorem* 3, *and let* Ω *be defined by* (14). *Then there exists a function* $f_0 \in V_\Phi$ *which does not satisfy relation* (1).

Proof. In virtue of Theorem 2 there exists a function f_0 which satisfies relation (2) and does not satisfy relation (1). But the previous theorem shows that from (2) it follows $f_0 \in V_\Phi$.

The results obtained will be applied to some problems of the theory of Fourier series.

1. By the well-known Jordan theorem, if a 2π-periodic continuous function f has bounded variation, then its Fourier series $\sigma(f)$ converges uniformly ([12]). This theorem was generalized by Wiener [22] for the class $C \cap V_2$, by Marcinkiewicz [15] (p. 40) for the class $C \cap V_p$, by L. Young [23] for the class $C \cap V_\Phi$, where $\Phi(u) = \exp(-u^{-\alpha})$, $0 < \alpha < 1/2$. Salem [18] obtained the most general condition on Φ, providing the uniform convergence of Fourier series of the class $C \cap V_\Phi$, which reads as follows: Let Φ be a convex increasing function, and let Ψ be a function, complementary in the sense of Young*) to the function Φ; if $f \in C \cap V_\Phi$ and

$$\sum_{n=1}^{\infty} \Psi\left(\frac{1}{n}\right) < \infty, \tag{20}$$

then $\sigma(f)$ converges uniformly.

K. I. Oskolkov [17] proved that (20) is equivalent to the condition

$$\int_0^1 \ln \frac{1}{\Phi(u)}\, du < \infty.$$

A. M. Garsia and S. Sawyer [9] proved that if $f \in C(0, 2\pi)$ and

$$\int_{m(f)}^{M(f)} \ln N(y, f)\, dy < \infty, \tag{21}$$

then $\sigma(f)$ converges uniformly.

From Corollary 1 it follows that if (21) is satisfied then

$$\sum_{k=1}^{\infty} \frac{v(k, f)}{k^2} < \infty. \tag{22}$$

But if $f \in C(0, 2\pi)$ and (22) holds true, then as it was proved in [6], the Fourier series of the function f converges uniformly, i.e., the theorem of Garsia—Sawyer is the result of Corollary 2 from [6].

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*) $\Psi(u) = \max_{v \geqq 0} \{uv - \Phi(v)\}$.

2. V. O. Asatiani [1] obtained the analogue of condition (22) for $(C, -\alpha)$-summability $(0<\alpha<1)$ of Fourier series. He proved that if $f \in C(0, 2\pi) \cap V[\upsilon]$, and for $0<\alpha<1$,

$$\sum_{k=1}^{\infty} \frac{\upsilon(k)}{k^{2-\alpha}} < \infty,$$

then $\sigma(f)$ is uniformly $(C, -\alpha)$-summable to f. From this result and Corollary 2 we have

Theorem 5. *Let* $f \in C(0, 2\pi)$ *and assume that for* $0<\alpha<1$,

$$\int_{m(f)}^{M(f)} N^{\alpha}(y, f)\, dy < \infty.$$

Then $\sigma(f)$ *is uniformly* $(C, -\alpha)$*-summable to* f.

3. Wiener's criterion on the continuity of functions of bounded variation is well known: Let

(23) $$\min\{f(x-0),\ f(x+0)\} \leqq f(x) \leqq \max\{f(x-0),\ f(x+0)\}$$

for any x, and let a_k and b_k be the Fourier coefficients of the function f, $\varrho_k = \sqrt{a_k^2+b_k^2}$. If $f \in V[0, 2\pi]$ then for f to be continuous, each of the following conditions is necessary and sufficient:

(24) $$\sum_{k=1}^{n} k^2 \varrho_k^2 = o(n),$$

(25) $$\sum_{k=1}^{n} k \varrho_k = o(n).$$

S. M. Lozinski [14] showed that instead of conditions (24) or (25) one may take

(26) $$\sum_{k=1}^{n} \varrho_k = o(\ln n),$$

(27) $$\sum_{k=n}^{\infty} \varrho_k^2 = o\left(\frac{1}{n}\right).$$

B. I. Golubov [10] applied these results to the classes V_p when $1<p<2$, and showed that for the classes V_p with $p \geqq 2$ a similar theorem does not hold. Z. A. Chanturia [7] (see also [8]) proved a theorem containing all of the previous results: If f satisfies condition (23), and its modulus of variation satisfies the condition

(28) $$\sum_{n=1}^{\infty} \frac{\upsilon^2(n, f)}{n^2} < \infty,$$

then for the function f to be continuous, each of conditions (24)—(27) is necessary and sufficient.

From the theorems of Wiener and Banach—Lozinski it follows that if $f \in W(0, 2\pi)$ satisfies condition (23) and its Banach indicatrix is summable then each of conditions (24)—(27) is necessary and sufficient for the continuity of the function f. We shall now prove a theorem which is much stronger and is in certain sense best possible.

Theorem 6. *If* $f \in W(0, 2\pi)$ *satisfies condition* (23) *and its Banach indicatrix satisfies the condition*

$$\int_{m(f)}^{M(f)} N^{1/2}(y, f)\, dy < \infty, \tag{29}$$

then each of conditions (24)—(27) *is necessary and sufficient for the continuity of* f.

Proof. It suffices to prove that (29) implies (28). By virtue of Corollary 2, (29) yields

$$\sum_{n=1}^{\infty} \frac{\upsilon(n, f)}{n^{3/2}} < \infty. \tag{30}$$

Since the general term of the last series decreases monotonically, we have

$$\frac{\upsilon(n, f)}{n^{3/2}} = o\left(\frac{1}{n}\right),$$

or $\upsilon(n, f) \leqq c n^{1/2}$. Therefore

$$\frac{\upsilon^2(n, f)}{n^2} \leqq c \frac{\upsilon(n, f)}{n^{3/2}}.$$

The latter inequality and the convergence of series (30) imply the convergence of (28), which was to be proved.

We shall now show that Theorem 6 is, in a certain sense, best possible, namely, if we take an integral class wider than (29) then Theorem 6 does not hold; more exactly, the following statement is true.

Theorem 7. *Let* Ω *be a convex upwards increasing function. If*

$$\varlimsup_{u \to \infty} \frac{\Omega(u)}{\sqrt{u}} = 0, \tag{31}$$

then there exists a function $f_0 \in C(0, 2\pi)$ *for which*

$$\int_{m(f)}^{M(f)} \Omega(N(y, f))\, dy < \infty,$$

but (24) *and* (27) *do not hold.*

Proof. By virtue of (31) we may choose an increasing sequence of even natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $n_{k+1}/n_k \geqq q > 1$ and

$$\sum_{k=1}^{\infty} \frac{\Omega(n_k)}{\sqrt{n_k}} < \infty. \tag{32}$$

Let $c_k = \sum_{i=k}^{\infty} \frac{1}{\sqrt{n_i}}$; then

$$c_k^2(n_k - n_{k-1}) = \left(\sum_{i=k}^{\infty} \frac{1}{\sqrt{n_i}}\right)^2 (n_k - n_{k-1}) \geqq \frac{1}{n_k}(n_k - n_{k-1}) = 1 - \frac{n_{k-1}}{n_k} \geqq 1 - \frac{1}{q} > 0,$$

i.e.,

$$\sum_{k=1}^{\infty} c_k^2(n_k - n_{k-1}) = \infty. \tag{33}$$

Take now $n_0 = 0$ and choose the sequence $\{B_n\}_{n=1}^{\infty}$ with $B_n = c_k$, when $n_{k-1}/2 < n \leqq n_k/2$. It is clear that $B_n \downarrow 0$, and in virtue of (33),

$$\sum_{n=1}^{\infty} B_n^2 = \infty. \tag{34}$$

Following the scheme of [8] we construct the function f_0 as follows:

$$f_0(x) = \begin{cases} B_k & \text{when } x \in I_{2k+1}, \quad k = 1, 2, \dots, \\ 0 & \text{when } x \in I_{2k}, \quad x \in \left[\frac{\pi^2}{2}, 2\pi\right] \cup \left[0, \frac{3}{2}\right], \\ \text{linear for all other} & x \text{ from } [0, 2\pi], \end{cases}$$

where I_k is a specially chosen sequence of segments such that I_k lies to the right of I_{k-1}.

The fact that if (34) is fulfilled then f_0 does not satisfy conditions (24) and (27), but

$$\int_{m(f)}^{M(f)} \Omega(N(y, f_0))\, dy < \infty,$$

is proved in [8]. In fact, using (32) we have

$$\int_{m(f)}^{M(f)} \Omega(N(y, f_0))\, dy = \sum_{k=1}^{\infty} \int_{c_{k+}}^{c_k} \Omega(N(y, f_0))\, dy =$$

$$= \sum_{k=1}^{\infty} (c_k - c_{k+1})\, \Omega(n_k) = \sum_{k=1}^{\infty} \frac{\Omega(n_k)}{\sqrt{n_k}} < \infty.$$

Theorem 7 is proved.

It should be noted, finally, that some of the results of the present paper were published without proof in [2].

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Quasisimilarity and properties of the commutant of C_{11} contractions

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

An operator T acting on the complex Hilbert space $\mathfrak{H}$ is said to have property (Q) if $T|\ker X$ and $(T^*|\ker X^*)^*$ are quasisimilar for every X in the commutant $\{T\}'$ of T. This property was introduced by UCHIYAMA [11] in connection with a conjecture of SZ.-NAGY and FOIAŞ [8].

We say that T has property (P) if $\ker X^*=\{0\}$ for every operator X in $\{T\}'$ such that $\ker X=\{0\}$.

In this note we prove that a weak C_{11} contraction has property (Q) whenever it has property (P). None of the assumptions of this result can be omitted. Indeed, there are weak C_{11} contractions (even unitary operators) that do not have property (P) and we will show that there are C_{11} contractions having property (P) but not property (Q). Since (P) is a quasisimilarity invariant in C_{11} (cf. [4]) and, as we shall see, for unitary operators (P) and (Q) are equivalent, we obtain in particular that the property of being a weak contraction and property (Q) are not quasisimilarity invariants in C_{11}.

These examples show that the results of [2] concerning weak C_0 contractions and [1] concerning C_0 contractions with property (Q) cannot be extended to the class of C_{11} contractions.

It is easy to see that our Theorem 2.7 extends (via [4]) the result of WU [12] concerning completely nonunitary C_{11} contractions with finite defect indices.

We note that every C_{11} contraction with property (P) is the direct sum of a singular unitary operator and an operator on a separable space. (Cf. [4, Corollary 5].)

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1. The residual part of a contraction

Let T be a contraction acting on the Hilbert space $\mathfrak{H}$ and let U_+ acting on $\mathfrak{K}_+$ be the minimal isometric dilation of T, that is U_+ is an isometry, $T^*=U_+^*|\mathfrak{H}$ and $\mathfrak{K}_+=\bigvee_{n\geq 0} U_+^n\mathfrak{H}$. Let $\mathfrak{K}_+=\mathfrak{M}\oplus\mathfrak{R}$ be the Wold decomposition of $\mathfrak{K}_+$ with respect to U_+, with $\mathfrak{R}=\bigcap_{n\geq 0} U_+^n\mathfrak{K}_+$.

Definition 1.1. The unitary operator $R_T=U_+|\mathfrak{R}$ is called the *residual part* of T. (Cf. [9, ch. II. 2].)

It is obvious that $R_{V\oplus T}=V\oplus R_T$ whenever V is a unitary operator.

Sz.-Nagy and Foiaş proved the following (cf. [10, Theorem 1.3]):

Proposition 1.2. *If the contractions T and T' are similar, then R_T and $R_{T'}$ are unitary equivalent.*

Let us recall that a contraction T acting on $\mathfrak{H}$ is said to be of class C_{11} if $\lim_{n\to\infty}\|T^nh\|=0$ or $\lim_{n\to\infty}\|T^{*n}h\|=0$ implies $h=0$. The following result is proved in [9, Proposition II. 3.5].

Proposition 1.3. *Any C_{11} contraction T is quasisimilar to R_T.*

It follows by [9, Proposition II. 3.4] that in the class C_{11} R_T is a quasisimilarity invariant and even a quasiaffine invariant. Therefore R_T is the unique unitary operator (up to unitary equivalence), quasisimilar to the operator T of class C_{11}.

We do not know whether R_T is in general a quasisimilarity invariant. It is easy to see that R_T is not a quasiaffine invariant; indeed, if S denotes the unilateral shift on H^2, we have $S\prec S^*$ [7] and $R_S\not\cong R_{S^*}$.

The following result follows from [9, Chapter VII, §1].

Lemma 1.4. *If T is a completely nonunitary contraction on $\mathfrak{H}$ and $\mathfrak{H}'$ is an invariant subspace for T, then $R_T\cong R_{T'}\oplus R_{T''}$, where $T'=T|\mathfrak{H}'$ and $T''=(T^*|\mathfrak{H}\ominus\mathfrak{H}')^*$.*

The following two results will help us extend this lemma to arbitrary contractions. The first of them is proved in [5, Lemma 2], while the proof of the second one is essentially the same as that in [5, Lemma 1].

Lemma 1.5. *Any absolutely continuous unitary operator is similar to a completely nonunitary contraction.*

Lemma 1.6. *Let U be a singular unitary operator and let T be a completely nonunitary contraction. Every invariant subspace $\mathfrak{M}$ of $U\oplus T$ has the form $\mathfrak{N}\oplus\mathfrak{P}$, where $\mathfrak{N}$ is invariant for U and $\mathfrak{P}$ is invariant for T.*

Theorem 1.7. *Let* T *be any contraction acting on* $\mathfrak{H}$, $\mathfrak{H}'$ *an invariant subspace for* T. *Then we have* $R_T \cong R_{T'} \oplus R_{T''}$, *where* $T' = T|\mathfrak{H}'$ *and* $T'' = (T^*|\mathfrak{H} \ominus \mathfrak{H}')^*$.

Proof. Let T_1 be another contraction acting on $\mathfrak{H}_1$, and $X: \mathfrak{H} \to \mathfrak{H}_1$ an invertible operator such that $T_1 X = XT$; set $\mathfrak{H}_1' = X\mathfrak{H}'$. Then T' and T'' are similar to $T_1' = T_1|\mathfrak{H}_1'$ and $T_1'' = (T_1^*|\mathfrak{H}_1 \ominus \mathfrak{H}_1')^*$, respectively. This shows by Proposition 1.2 that in proving the theorem we may replace T by a similar operator. It follows then from Lemma 1.5 that we may assume $T = U \oplus T_1$, where U is a singular unitary operator and T_1 is completely nonunitary. (Cf. also [9, Theorem I. 3.2].) Now Lemma 1.6 shows that we can further reduce the proof to the cases where T is a singular unitary or completely nonunitary. If T is completely nonunitary the proposition follows by Lemma 1.4. In turn, if T is a singular unitary operator, then every invariant subspace of T reduces T (cf. [6, Proposition 1.11]) and so the statement becomes obvious. The proof is complete.

2. C_{11} contractions with property (P)

The following result was proved in [4].

Proposition 2.1. *A contraction* T *of class* C_{11} *has property* (P) *if and only if* R_T *has property* (P).

Now, unitary operators having property (P) are easily characterized in terms of properties of their commutant.

Lemma 2.2. *A unitary operator* T *has property* (P) *if and only if the commutant* $\{T\}'$ *is a finite von Neumann algebra.*

Proof. Assume first that $\{T\}'$ is not finite. Then there exists a nonunitary isometry U in $\{T\}'$; in particular U is one-to-one but $\ker U^* \neq \{0\}$ so that T does not have property (P).

Conversely, if T does not have property (P), there exists X in $\{T\}'$ such that $\ker X = \{0\}$ and $\ker X^* \neq \{0\}$. If $X = UP$ is the polar decomposition of X, we have $U \in \{T\}'$ (cf. the proof of [9, Proposition II. 3.4]), $\ker U = \ker X = \{0\}$ and $\ker U^* = \ker X^* \neq \{0\}$ so that $\{T\}'$ is not finite. The lemma is proved.

It follows from the results of [3] that unitary operators having property (P) also have the following "cancellation" property: if $T \oplus U$ is unitarily equivalent to $T \oplus V$ for some unitary operators T, U and V, and $T \oplus U$ has property (P), then U and V are unitarily equivalent.

Proposition 2.3. *Let* T *be a* C_{11} *contraction having property* (P). *For every* X *in* $\{T\}'$ *the operators* $R_{T|\ker X}$ *and* $R_{(T^*|\ker X^*)^*}$ *are unitarily equivalent.*

Proof. By Theorem 1.7 we have $R_T \cong R_{T|\ker X} \oplus R_{(T^*|(\ker X)^\perp)^*} \cong R_{T|(\operatorname{ran} X)^-} \oplus \oplus R_{(T^*|\ker X^*)^*}$. The operators $(T^*|(\ker X)^\perp)^*$ and $T|(\operatorname{ran} X)^-$ are of class C_{11} (cf. [5, Lemma 5]) and they are quasisimilar (cf., e.g., [12, Corollary 3.4]), so that $R_{(T^*|(\ker X)^\perp)^*}$ and $R_{T|(\operatorname{ran} X)^-}$ are unitarily equivalent. The proposition now follows from the cancellation property described above.

An obvious consequence of Proposition 2.3 is the following.

Corollary 2.4. *Let the C_{11} contraction T be such that $T|\ker X$ and $T^*|\ker X^*$ are of class C_{11} for every X in $\{T\}'$. Then T has property (Q) if and only if it has property (P).*

The hypothesis of the preceding Corollary can be weakened; to do this we need some definitions from [5]. For a C_{11} contraction $\operatorname{lat}_1 T$ denotes the set of those invariant subspaces $\mathfrak{M}$ for T such that $T|\mathfrak{M}$ is of class C_{11}. For every invariant subspace $\mathfrak{M}$ for T there exists a largest subspace in $\operatorname{lat}_1 T$ contained in $\mathfrak{M}$, this subspace (the C_{11}-part of $\mathfrak{M}$) is denoted by $\mathfrak{M}^{(1)}$. For a subspace $\mathfrak{M}$ in $\operatorname{lat} T^*$ we set $\mathfrak{M}^{\perp_1} = (\mathfrak{M}^\perp)^{(1)}$.

Let us say that the C_{11} contraction T has property (R) if $\ker X \in \operatorname{lat}_1 T$ for every X in $\{T\}'$.

Proposition 2.5. *Let T be a C_{11} contraction having property (P). Then T has property (R) if and only if T^* has property (R).*

Proof. By [5, Lemma 5] a subspace $\mathfrak{M}$ is in $\operatorname{lat}_1 T^*$ if and only if it has the form $(\ker X)^\perp$ for some X in $\{T\}'$. It follows that T has property (R) if and only if $\mathfrak{M}^\perp \in \operatorname{lat}_1 T$ for every $\mathfrak{M}$ in $\operatorname{lat}_1 T^*$.

Let us assume that T has property (R) and $\mathfrak{M} \in \operatorname{lat}_1 T$; it follows from [5, Proposition 2] that $(\mathfrak{M}^{\perp_1})^{\perp_1} = \mathfrak{M}$. Now, $\mathfrak{M}^{\perp_1} \in \operatorname{lat}_1 T^*$ and T has property (R) so that $(\mathfrak{M}^{\perp_1})^\perp \in \operatorname{lat}_1 T$. Consequently $(\mathfrak{M}^{\perp_1})^\perp = (\mathfrak{M}^{\perp_1})^{\perp_1} = \mathfrak{M}$ and therefore $\mathfrak{M}^{\perp_1} = \mathfrak{M}^\perp$; that is $\mathfrak{M}^\perp \in \operatorname{lat}_1 T^*$. We proved that T^* has property (R).

By [4, Corollary 4] T has property (P) if and only if T^* has property (P). Thus the proof is completed by the same argument applied to T^* instead of T.

Now we can reformulate Corollary 2.4 as follows.

Theorem 2.6. *Let T be a C_{11} contraction having property (P). Then T has property (Q) if and only if $T|\ker X$ is of class C_{11} for every X in $\{T\}'$.*

Proof. The sufficiency obviously follows from Corollary 2.4 and Proposition 2.5. Conversely, if T has property (Q) and $X \in \{T\}'$, then $T|\ker X$ is of class $C_{1\cdot}$ and $(T^*|\ker X^*)^*$ is of class $C_{\cdot 1}$; it follows that both operators are of class C_{11} since they are quasisimilar. The theorem is proved.

Let us recall that a contraction T is said to be weak if $I-T^*T$ is a trace class operator and $\lambda I-T$ is invertible for some λ with $|\lambda|<1$.

Theorem 2.7. *A weak C_{11} contraction has property (P) if and only if it has property (Q).*

Proof. It is enough to prove that a weak C_{11} contraction T having property (P) also has property (Q). By virtue of Theorem 2.6 it suffices to show that, if T is a weak C_{11} contraction then $T|\ker X$ is of class C_{11} for every X in $\{T\}'$.

It is clear that $I-(T|\ker X)^*(T|\ker X)=P_{\ker X}(I-T^*T)|\ker X$ is a trace class operator. By [9, Theorem VIII. 2.1] T is invertible. Since X commutes with T^{-1}, we have that $T^{-1}(\ker X)\subset\ker X$, and so $T|\ker X$ is also invertible. Therefore $T|\ker X$ is a weak contraction of class $C_{1\cdot}$, and so by [9, Theorem VIII. 2.1] it is of class C_{11}. The theorem follows.

Corollary 2.8. *A unitary operator has property (P) if and only if it has property (Q).*

3. Examples

It is known [9, Ch. VI. 4.2] that there exist C_{11} contractions whose spectrum coincides with the closed unit disk. The following result shows that there are C_{11} contractions having property (P) whose spectrum covers the unit disk.

Proposition 3.1. *Let U be an absolutely continuous unitary operator. There exists a C_{11} contraction T such that $\sigma(T)=\{\lambda\colon |\lambda|\leqq 1\}$ and R_T is unitarily equivalent to U.*

Proof. It suffices to prove the proposition in the case U is the operator of multiplication by e^{it} on $L^2(\sigma)$, where $\sigma\subset[0,2\pi]$ has positive Lebesgue measure. Choose pairwise disjoint subsets σ_n of σ of positive measure such that $\bigcup_{n\geqq 0}\sigma_n=\sigma$ and choose a sequence $\{\varepsilon_n\}_{n\geqq 0}$ of positive numbers less than 1. For each n there exists an outer function ϑ_n (uniquely determined up to a constant factor of modulus one) such that $|\vartheta_n(e^{it})|=1$ if $t\notin\sigma_n$ and $|\vartheta_n(e^{it})|=\varepsilon_n$ if $t\in\sigma_n$. It is clear by [4, Corollary 1] that the functional model T corresponding with the characteristic function $\theta(\lambda)=\operatorname{diag}(\vartheta_0(\lambda),\vartheta_1(\lambda),\ldots)$ satisfies the condition $R_T\cong U$.

If the numbers ε_n satisfy the relation $\lim_{n\to\infty}|\sigma_n|\log\varepsilon_n=-\infty$ (where $|\sigma_n|$ denotes the Lebesgue measure of σ_n) we have $\lim_{n\to\infty}\vartheta_n(\lambda)=0$ for every λ, $|\lambda|<1$, and by [9, Theorem VI. 4.1] this implies that $\sigma(T)\supset\{\lambda\colon|\lambda|<1\}$. The proposition follows.

It is obvious that the operator T constructed in the preceding proof is not a weak contraction; in particular, a C_{11} contraction with a cyclic vector is not necessarily a weak contraction. Let us also note that if T is of class C_{11} then T and T^* cannot have eigenvalues of absolute value less than 1. Thus, if T is a C_{11} contraction and $\lambda\in\sigma(T)$, $|\lambda|<1$, then $\lambda I-T$ is one-to-one and has non-closed, dense range.

In the sequel we will identify a vector f of the Hilbert space $\mathfrak{H}$ with the operator $\mathbf{C}\to\mathfrak{H}$ defined by $\mathbf{C}\ni\lambda\mapsto\lambda f\in\mathfrak{H}$; the adjoint f^* is then defined by $f^*(g)=(g,f)$ for $g\in\mathfrak{H}$.

Lemma 3.2. *Let S be an injective contraction acting on $\mathfrak{H}$ such that $S\mathfrak{H}\neq\mathfrak{H}$. There exists a vector $f\in\mathfrak{H}$ such that the operator $(S,f)\colon\mathfrak{H}\oplus\mathbf{C}\to\mathfrak{H}$ defined by $(S,f)(h\oplus\lambda)=Sh+\lambda f$ is an injective contraction.*

Proof. It is clear that (S,f) is injective if and only if $f\notin S\mathfrak{H}$. Let us set $f=u-SS^*u$, where $u\notin S\mathfrak{H}$ and $\|u\|^2\leqq 1/2$. Then clearly $f\notin S\mathfrak{H}$ and

$$\|u\|^2+\|f\|^2\leqq\|u\|^2+\|u\|^2\leqq 1. \tag{3.1}$$

We only have to prove that (S,f) is a contraction. Indeed, let $h\oplus\lambda\in\mathfrak{H}\oplus\mathbf{C}$; we have (using the notation $D=(I-S^*S)^{1/2}$)

$$\|Sh+\lambda f\|^2\leqq\|Sh\|^2+2|\lambda|\,|(Sh,f)|+|\lambda|^2\|f\|^2=$$

$$=\|Sh\|^2+2|\lambda|\,\big|((I-SS^*)Sh,u)\big|+|\lambda|^2\|f\|^2=\|Sh\|^2+2|\lambda|\,|(SDDh,u)|+|\lambda|^2\|f\|^2\leqq$$

$$\leqq\|Sh\|^2+2|\lambda|\,\|u\|\,\|Dh\|+|\lambda|^2\|f\|^2.$$

Using the inequality $2ab\leqq a^2+b^2$ in the middle term we get

$$\|Sh+\lambda f\|^2\leqq\|Sh\|^2+\|Dh\|^2+|\lambda|^2\|u\|^2+|\lambda|^2\|f\|^2=$$

$$=\|h\|^2+|\lambda|^2(\|u\|^2+\|f\|^2)\leqq\|h\|^2+|\lambda|^2$$

by (3.1). The lemma follows.

Theorem 3.3. *There exist C_{11} contractions having property (P) but not property (Q).*

Proof. Let T' and T'' be two noninvertible C_{11} contractions acting on $\mathfrak{H}'$ and $\mathfrak{H}''$, respectively. By Lemma 3.2 we can choose vectors $f\in\mathfrak{H}'$ and $g\in\mathfrak{H}''$ such that (T',f) and (T''^*,g) are injective contractions. It is then easy to see that the operator T defined on $\mathfrak{H}'\oplus\mathbf{C}\oplus\mathfrak{H}''$ by the matrix

$$\begin{bmatrix} T' & f & 0 \\ 0 & 0 & g^* \\ 0 & 0 & T'' \end{bmatrix}$$

is a C_{11} contraction. Let us note that the invariant subspace $\mathfrak{H}' \oplus \mathbf{C}$ for T is not in $\mathrm{lat}_1 T$, while its orthocomplement $\mathfrak{H}''$ obviously belongs to $\mathrm{lat}_1 T^*$; by the proof of Proposition 2.5 and by Theorem 2.6 we infer that T does not have property (Q).

By Theorem 1.7 we have $R_T \cong R_{T'} \oplus R_0 \oplus R_{T''} \cong R_{T'} \oplus R_{T''}$, so that T has property (P) whenever T' and T'' have property (P) (cf. Proposition 2.1 and [4, Lemma 5]). The theorem follows by Proposition 3.1.

Remark 3.4. Proposition 3.1 shows in fact that the operator T in the preceding proof can be chosen so that R_T is unitarily equivalent to a given absolutely continuous unitary operator with property (P). In particular R_T could be chosen so that all its invariant subspaces are reducing (a reductive operator). This shows that the property "$\mathrm{lat}_1 T = \mathrm{lat}\, T$", generalizing reductivity, is not a quasisimilarity invariant in the class of C_{11} contractions or even in the class of C_{11} contractions having property (P).

Remark 3.5. Let us choose $T' = T''$ in the proof of Theorem 3.3; in this case we can produce an operator X in $\{T\}'$ for which $T|\ker X$ and $(T^*|\ker X^*)^*$ are not quasisimilar. Such an operator is defined by the matrix

$$\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where I denotes the identity operator on $\mathfrak{H}' = \mathfrak{H}''$.

Remark 3.6. Finally we note that we have got by Theorems 3.3, 2.6 and by the proof of Proposition 2.5 that the C_{11}-orthogonal complement $\mathfrak{L}^{\perp_1}$ of a subspace $\mathfrak{L} \in \mathrm{lat}_1 T$, where T is a C_{11} contraction with property (P), does not generally coincide with the orthogonal complement $\mathfrak{L}^{\perp}$ of $\mathfrak{L}$.

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A proof of the spectral theorem for J-positive operators

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Dedicated to Béla Sz.-Nagy on the occasion of his 70th birthday

A Krein space is a Hilbert space with the usual (positive definite) inner product (f, g) and a non-degenerate (in general, indefinite) J-inner product $[f, g]=(Jf, g)$, where J is a symmetry: $J^*=J$ and $J^2=I$. A Krein space with one (or both) of the eigenspaces of J having finite dimension is called a Pontrjagin space.

Let A be a bounded or unbounded linear operator in the Krein space $\mathfrak{H}$. If JA is selfadjoint (in the Hilbert space sense), then A is said to be J-selfadjoint. If JA is positive, that is, $[Af, f]=(JAf, f)\geqq 0$ for every f in the domain of A, then we say A is J-positive. Further, if there is a non-zero polynomial p such that $p(A)$ is J-positive we say A is J-positizable.

In 1963, M. G. KREĬN and H. LANGER [1] proved a spectral theorem for J-selfadjoint operators with real spectrum in a Pontrjagin space. The proof made use, among other things, of the J-positizability of these operators. LANGER [2] generalized the theorem to J-positizable J-selfadjoint operators with real spectrum in a Krein space (see also [3]—[5] for statement of the result). Proofs for the bounded J-positive case have also been given by M. G. KREĬN and JU. L. ŠMUL'JAN [6], T. ANDO [7], and for further generalizations by B. N. HARVEY [8] and P. JONAS [9]—[10].

In our opinion, the spectral theory based on these results has not gained the popularity it deserves. The situation can perhaps be improved by reducing the machinery required in the proofs. ANDO [7] has already made the decisive step in this direction.

Our proof below was inspired by a paper of C. S. WONG [11] and is hoped to be a further step in eliminating unnecessary tools. Restricted to the bounded J-positive case, it uses only the basic facts of Hilbert space spectral theory as treated by B. Sz.-NAGY [12] and the elements of Krein space theory [13]. In particular, neither an auxiliary space nor complex variables are needed.

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Theorem (Kreĭn, Langer). *Let A be a bounded J-positive operator on the Krein space $\mathfrak{H}$. Then to every real number $\lambda\neq 0$ there is one and only one J-selfadjoint projection E_λ on $\mathfrak{H}$ such that the function $\lambda\mapsto E_\lambda$ has the following properties:*

1. *If $\lambda\leqq\mu$, then $E_\lambda E_\mu=E_\mu E_\lambda=E_\lambda$.*
2. *If $\lambda<\mu<0$, then $[E_\lambda f, f]\geqq[E_\mu f, f]$; if $0<\lambda<\mu$, then $[E_\lambda f, f]\leqq[E_\mu f, f]$ for every $f\in\mathfrak{H}$.*
3. *If $\lambda<-\|A\|$, then $E_\lambda=O$; if $\lambda>\|A\|$, then $E_\lambda=I$.*
4. *If $\lambda\neq 0$, then the strong limit $E_{\lambda+0}$ exists and $E_{\lambda+0}=E_\lambda$.*
5. *If T is a bounded linear operator on $\mathfrak{H}$ such that $TA=AT$, then $TE_\lambda=E_\lambda T$ for every λ.*
6. *The spectrum $\sigma(A|E_\lambda\mathfrak{H})$ is contained in the interval $(-\infty,\lambda]$, while $\sigma(A|(I-E_\lambda)\mathfrak{H})$ is contained in $[\lambda,\infty)$.*

Moreover,

$$\int_{-\|A\|-0}^{\|A\|} \nu\, dE_\nu$$

is a strongly convergent improper integral with singular point 0, and

$$S:=A-\int_{-\|A\|-0}^{\|A\|} \nu\, dE_\nu$$

is a bounded J-positive operator such that $S^2=O$, $SE_\lambda=E_\lambda S=O$ if $\lambda<0$, whereas $S(I-E_\lambda)=(I-E_\lambda)S=O$ if $\lambda>0$.

Proof. The positive operator $B:=JA$ satisfies

$$A=JB. \tag{1}$$

The operator

$$C:=B^{1/2}JB^{1/2} \tag{2}$$

is selfadjoint and

$$CB^{1/2}=B^{1/2}A. \tag{3}$$

Since $\|C\|\leqq\|A\|$, the spectral decomposition of C can be written in the form

$$C=\int_{-\|A\|-0}^{\|A\|} \nu\, dF_\nu, \tag{4}$$

where $\{F_\lambda\}_{\lambda=-\infty}^{\infty}$ is the right-continuous spectral family of C. We set

$$C_\lambda:=C|F_\lambda\mathfrak{H} \quad \text{for} \quad \lambda<0, \tag{5}$$

$$C_\lambda:=C|(I-F_\lambda)\mathfrak{H} \quad \text{for} \quad \lambda>0 \tag{6}$$

and

$$E_\lambda:=JB^{1/2}C_\lambda^{-1}F_\lambda B^{1/2} \quad \text{for} \quad \lambda<0, \tag{7}$$

$$I-E_\lambda:=JB^{1/2}C_\lambda^{-1}(I-F_\lambda)B^{1/2} \quad \text{for} \quad \lambda>0. \tag{8}$$

Clearly, E_λ is a bounded operator on $\mathfrak{H}$ for every real number $\lambda \neq 0$. Further, if $\lambda < 0$ then $C_\lambda^{-1} F_\lambda$ is selfadjoint and therefore $E_\lambda^* = JE_\lambda J$, which is equivalent to E_λ being J-selfadjoint. If $\lambda > 0$, the J-selfadjointness of $I - E_\lambda$ and hence the same property of E_λ follow similarly. Thus

$$(9) \qquad E_\lambda^* = JE_\lambda J \quad \text{for every} \quad \lambda \neq 0,$$

a relation needed later on.

Let $\lambda \leqq \mu < 0$. Then from (7), (2), and the relation

$$(10) \qquad CC_\lambda^{-1} F_\lambda = F_\lambda \quad (\lambda < 0)$$

(see (5)) we obtain $E_\lambda E_\mu = E_\lambda$. Similarly, if $0 < \lambda \leqq \mu$ then (8), (2), and the relation

$$(11) \qquad CC_\lambda^{-1}(I - F_\lambda) = I - F_\lambda \quad (\lambda > 0)$$

(see (6)) yield $(I - E_\lambda)(I - E_\mu) = I - E_\mu$, that is, $E_\lambda E_\mu = E_\lambda$. Finally, in the case $\lambda < 0 < \mu$ from (7), (8), (2) and (11) we get $E_\lambda(I - E_\mu) = O$ and therefore $E_\lambda E_\mu = E_\lambda$ again. The relation $E_\mu E_\lambda = E_\lambda$ follows by taking adjoints and applying (9). Thus Property 1 is valid. Choosing $\lambda = \mu$ we see that E_λ is a projection.

Let us prove Property 2. If $\lambda < \mu < 0$, then by (7), (5), and (4)

$$[E_\mu f, f] - [E_\lambda f, f] = (C_\mu^{-1} F_\mu B^{1/2} f, B^{1/2} f) - (C_\lambda^{-1} F_\lambda B^{1/2} f, B^{1/2} f) =$$
$$= \int_\lambda^\mu \frac{1}{\nu} d(F_\nu B^{1/2} f, B^{1/2} f) \leqq 0$$

for every $f \in \mathfrak{H}$. On the other hand, if $0 < \lambda < \mu$ then by (8), (6), and (4)

$$[E_\mu f, f] - [E_\lambda f, f] = [(I - E_\lambda) f, f] - [(I - E_\mu) f, f] =$$
$$= \left(C_\lambda^{-1}(I - F_\lambda) B^{1/2} f, B^{1/2} f\right) - \left(C_\mu^{-1}(I - F_\mu) B^{1/2} f, B^{1/2} f\right) =$$
$$= \int_\lambda^\mu \frac{1}{\nu} d(F_\nu B^{1/2} f, B^{1/2} f) \geqq 0,$$

as required.

Property 3 is a simple consequence of (7), (8) and (4).

To prove Property 4, first let $\lambda < \mu < 0$. Then

$$\|E_\mu f - E_\lambda f\|^2 = \|JB^{1/2}(C_\mu^{-1} F_\mu - C_\lambda^{-1} F_\lambda) B^{1/2} f\|^2 \leqq$$
$$\leqq \|B\| \int_\lambda^\mu \frac{1}{\nu^2} d(F_\nu B^{1/2} f, B^{1/2} f) \leqq \|B\| \frac{1}{\mu^2} \|(F_\mu - F_\lambda) B^{1/2} f\|^2,$$

and the last member tends to 0 as $\mu \to \lambda + 0$. Therefore $E_{\lambda+0} = E_\lambda$ if $\lambda < 0$. A similar reasoning applies in the case $\lambda > 0$.

Next assume that T is a bounded linear operator which commutes with A; i.e., $TA = AT$. To prove $TE_\lambda = E_\lambda T$ consider the case $\lambda < 0$ first.

By (7), (5), and (4)

$$E_\lambda = JB^{1/2} \int_{-\|A\|-0}^{\lambda} \frac{1}{\nu} dF_\nu \cdot B^{1/2} \quad (\lambda < 0).$$

Choose a sequence of polynomials $\{p_n\}_1^\infty$ which is bounded on $[-\|A\|, \|A\|]$ and satisfies the relation

$$\lim_{n\to\infty} p_n(\nu) = \begin{cases} 1/\nu & \text{if } -\|A\| \leqq \nu \leqq \lambda, \\ 0 & \text{if } \lambda < \nu \leqq \|A\|. \end{cases}$$

Then $JB^{1/2}p_n(C)B^{1/2} \to E_\lambda$ strongly. Hence it is sufficient to prove that T commutes with $JB^{1/2}C^m B^{1/2}$ for $m=0, 1, 2, \ldots$. But

$$JB^{1/2} C^m B^{1/2} = A^{m+1} \quad (m = 0, 1, 2, \ldots),$$

as one can verify by induction with the help of (1) and (3). This completes the proof of Property 5 for $\lambda<0$.

If $\lambda>0$, we start from the relation

$$I - E_\lambda = JB^{1/2} \int_{\lambda}^{\|A\|} \frac{1}{\nu} dF_\nu \cdot B^{1/2} \quad (\lambda > 0)$$

obtainable from (8), (6), and (4), and conclude as above that T commutes with $I-E_\lambda$.

Just as in Hilbert space, from the consequence $AE_\lambda = E_\lambda A$ of Property 5 it follows that the subspaces $E_\lambda \mathfrak{H}$ and $(I-E_\lambda)\mathfrak{H}$ are invariant under A.

As to Property 6, we first note that the relations (1), (7)—(8), (2) and (10)—(11) imply

$$AE_\lambda = JB^{1/2} F_\lambda B^{1/2} \quad (\lambda \neq 0), \tag{12}$$

$$A(I-E_\lambda) = JB^{1/2}(I-F_\lambda) B^{1/2} \quad (\lambda \neq 0). \tag{13}$$

Since $\sigma(T_1 T_2) \subset \{0\} \cup \sigma(T_2 T_1)$ for any pair T_1, T_2 of bounded linear operators (see [14], Problem 61), from (12) and (2) we obtain $\sigma(AE_\lambda) \subset \{0\} \cup \sigma(CF_\lambda)$. Therefore $\sigma(A|E_\lambda \mathfrak{H}) \subset \{0\} \cup \sigma(C|F_\lambda \mathfrak{H})$ and, in view of (4),

$$\sigma(A|E_\lambda \mathfrak{H}) \subset \{0\} \cup (-\infty, \lambda].$$

We have to prove that if $\lambda<0$ then 0 does not belong to $\sigma(A|E_\lambda \mathfrak{H})$.

Let $\lambda<0$ and assume that $AE_\lambda f_n \to 0$ $(n\to\infty)$ for some sequence $\{f_n\}_1^\infty \subset \mathfrak{H}$. Then also $B^{1/2}AE_\lambda f_n \to 0$ or, by (12) and (2), $CF_\lambda B^{1/2} f_n \to 0$. Since, according to (4) and the assumption $\lambda<0$, the value 0 is regular for $C|F_\lambda \mathfrak{H}$, it follows that $F_\lambda B^{1/2} f_n \to 0$. Applying the operator $JB^{1/2}C_\lambda^{-1}$ and using (7) we obtain $E_\lambda f_n \to 0$ $(n \to \infty)$.

Thus 0 belongs to neither the continuous nor the point spectrum of $A|E_\lambda \mathfrak{H}$. But $A|E_\lambda \mathfrak{H}$, being a selfadjoint operator on the "negative Hilbert space" $E_\lambda \mathfrak{H}$

(cf. Properties 2—3 as well as [13], Theorems II. 3.10 and V. 3.5), has no residual spectrum. This proves one half of Property 6. The proof of the other half is similar.

Assume that $\lambda \mapsto E'_\lambda$ (λ real, $\lambda \neq 0$, E'_λ a J-selfadjoint projection on $\mathfrak{H}$) is also a function with the properties 1—6. Let $\lambda < \mu < 0$. Obviously

$$E'_\lambda = E'_\lambda E_\mu + E'_\lambda (I - E_\mu).$$

By Property 5, E'_λ and E_μ commute with A and with each other. In particular, $E'_\lambda(I-E_\mu)$ is a J-selfadjoint projection which commutes with A.

Similarly to the case of a selfadjoint projection, if E is a J-selfadjoint projection and $AE=EA$ then

$$\sigma(A) = \sigma(A|E\mathfrak{H}) \cup \sigma(A|(I-E)\mathfrak{H}) \supset \sigma(A|E\mathfrak{H}).$$

Indeed, the A-invariant subspaces $E\mathfrak{H}$ and $(I-E)\mathfrak{H}$ are orthogonal with respect to the J-inner product; therefore [13], Theorem V. 3.5, implies that they are orthogonal also in a Hilbert space with norm equivalent to the original one.

Applying this fact to the Krein spaces (cf. [13], Theorem V. 3.4) $E'_\lambda\mathfrak{H}$, $(I-E_\mu)\mathfrak{H}$, and using Property 6 we obtain

$$\sigma(A|E'_\lambda(I-E_\mu)\mathfrak{H}) \subset \sigma(A|E'_\lambda\mathfrak{H}) \cap \sigma(A|(I-E_\mu)\mathfrak{H}) \subset (-\infty, \lambda] \cap [\mu, \infty) = \emptyset.$$

But the spectrum of a selfadjoint operator on the "negative Hilbert space" $E'_\lambda(I-E_\mu)\mathfrak{H} \subset E'_\lambda\mathfrak{H}$ can be empty only if the space is zero. Thus $E'_\lambda(I-E_\mu)=O$,

$$E'_\lambda = E'_\lambda E_\mu. \tag{14}$$

(14) remains valid if $0<\lambda<\mu$, the only difference in the proof being that $E'_\lambda(I-E_\mu)\mathfrak{H} \subset (I-E_\mu)\mathfrak{H}$ now are ordinary Hilbert spaces. Letting $\mu \to \lambda+0$, from (14) and Property 4 we conclude that $E'_\lambda = E'_\lambda E_\lambda$. Similarly, $E_\lambda = E_\lambda E'_\lambda$. Therefore, in view of Property 5, $E'_\lambda = E_\lambda$.

The existence of the strong integral

$$\int_{-\|A\|-0}^{-\varepsilon} \nu \, dE_\nu,$$

where $\varepsilon > 0$, follows by reading the next relations from the right to the left (see (7), (5), and (4)):

$$\int_{-\|A\|-0}^{-\varepsilon} \nu \, dE_\nu = JB^{1/2} C_{-\varepsilon}^{-1} \int_{-\|A\|-0}^{-\varepsilon} \nu \, dF_\nu \cdot B^{1/2} = JB^{1/2} C_{-\varepsilon}^{-1} C_{-\varepsilon} F_{-\varepsilon} B^{1/2} = JB^{1/2} F_{-\varepsilon} B^{1/2}.$$

Similarly, from (8), (6), (4), and (1)

$$\int_{\varepsilon}^{\|A\|} \nu \, dE_\nu = -\int_{\varepsilon}^{\|A\|} \nu \, d(I-E_\nu) = -JB^{1/2} C_\varepsilon^{-1} \int_{\varepsilon}^{\|A\|} \nu \, d(I-F_\nu) \cdot B^{1/2} =$$

$$= JB^{1/2} C_\varepsilon^{-1} \int_{\varepsilon}^{\|A\|} \nu \, dF_\nu \cdot B^{1/2} = JB^{1/2} C_\varepsilon^{-1} C_\varepsilon (I-F_\varepsilon) B^{1/2} =$$

$$= JB^{1/2}(I-F_\varepsilon) B^{1/2} = A - JB^{1/2} F_\varepsilon B^{1/2}.$$

Thus

$$(15) \qquad \int_{-\|A\|-0}^{\|A\|} \nu \, dE_\nu = A - JB^{1/2}(F_0 - F_{-0})B^{1/2}$$

as a strong improper integral.

The operator $S := JB^{1/2}(F_0 - F_{-0})B^{1/2}$ appearing in (15) is obviously bounded and J-positive. Further $S^2 = O$, since according to (2) and (4)

$$B^{1/2}JB^{1/2}(F_0 - F_{-0}) = C(F_0 - F_{-0}) = O.$$

By the same reason, $SE_\lambda = E_\lambda S = O$ if $\lambda < 0$, and $S(I - E_\lambda) = (I - E_\lambda)S = O$ if $\lambda > 0$.

The proof is complete.

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The random martingale central limit theorem and weak law of large numbers with o-rates

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday on 29 July 1983, in friendship and great respect

1. Introduction

Although the central limit theorem (CLT) for randomly indexed sums of random variables (r. vs.) has been quite a popular field of research in the past 30 years or so in the case of independent r.vs., the situation is quite different in the more difficult case of "dependent" r.vs. This convergence theorem has been equipped with large-O rates in a variety of papers (for the independent case see, e.g., [24], [26], [16], [23]; and for the dependent case [25], [11]) as well as with little-o rates, however much less so; see [4], [23] in the independent case or [6], [18], [8], [22], [10], [19], [20] in the (classical) non-random case.

On the other hand, the random weak law of large numbers (WLLN) seems hardly — with the exception of MOGYORÓDI [17] and CSÖRGÖ and RÉVÉSZ [13] — to have been considered before, even when the r.vs. are independent. For historical comments concerning random limit theorems without rates see [1], [12], [15], and with rates [11].

The purpose of this paper is to consider a comprehensive theorem on o-rates of convergence for normalized randomly indexed sums of not necessarily independent r. vs. which will include both the CLT and WLLN. The type of convergence to be considered will essentially be weak convergence. A particular type of "weak dependency" will be assumed, just as in [11], namely the situation of martingale difference sequences (MDS).

More concretely, this means the following: Let $(X_i)_{i\in \mathbf{N}}$ be a sequence of real valued r. vs. defined on a probability space $(\Omega, \mathscr{A}, P)$; and let $(\mathscr{F}_i)_{i\in \mathbf{P}}$ $(\mathbf{P}:=\mathbf{N}\cup\{0\})$ be an increasing sequence of sub-σ-algebras of $\mathscr{A}$ such that X_i is $\mathscr{F}_i$-measurable

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for each $i \in \mathbf{N}$. Then $(X_i, \mathscr{F}_i)_{i \in \mathbf{P}}$, $X_0 := 0$, is called a MDS if

$$E[X_i | \mathscr{F}_{i-1}] = 0 \quad \text{a.s.} \quad (i \in \mathbf{N}). \tag{1.1}$$

Let us further recall the concept of a randomly indexed sum of r. vs. Let N_λ, $\lambda \in \mathbf{R}^+$, be an $\mathbf{N}$-valued r.v. defined on $(\Omega, \mathscr{A}, P)$ that is independent of the r.v. X_i, $i \in \mathbf{N}$ for each $\lambda \in \mathbf{R}^+$, and let $N_\lambda \to \infty$ in probability for $\lambda \to \infty$. The normalized random sums to be considered in this paper are of the form

$$T_{N_\lambda} := \varphi(N_\lambda) S_{N_\lambda} \tag{1.2}$$

where $S_{N_\lambda} := \sum_{i=1}^{N_\lambda}$, and where $\varphi : \mathbf{N} \to \mathbf{R}^+$ is a positive, normalizing function. The weak convergence concerns the o-rate with which $E[f(T_{N_\lambda})] - E[f(Z)]$ tends to zero for $\lambda \to \infty$. Here the limiting r.v. Z is assumed to be φ-decomposable. This means that for each $n \in \mathbf{N}$ there exist independent r.vs. Z_i, $Z_i = Z_{i,n}$, $1 \leq i \leq n$, such that the distribution P_Z of Z can be represented as

$$P_Z = P_{\varphi(n) \sum_{i=1}^{n} Z_i}. \tag{1.3}$$

With these preparations the general theorem of this paper may be stated roughly as follows: If $(X_i | \mathscr{F}_i)_{i \in \mathbf{P}}$ is a MDS, Z a φ-decomposable r.v. with zero mean such that the r-th absolute moments of the r.vs. X_i, $i \in \mathbf{N}$, and the decomposition components Z_i, $i \in \mathbf{N}$, are finite for some $r \in \mathbf{N}$, and both sequences $(X_i)_{i \in \mathbf{N}}$, $(Z_i)_{i \in \mathbf{N}}$ satisfy a generalized, random Lindeberg condition of order r (see (2.6)) and are related by

$$E\left[\left(\sum_{i=1}^{N_\lambda} E[|X_i|^r] + E[|Z_i|^r]\right)^{-1} (\varphi(N_\lambda))^{j-r} \sum_{i=1}^{N_\lambda} |E[X_i^j | \mathscr{F}_{i-1}] - E[Z_i^j]|\right] \to 0$$

for $\lambda \to \infty$ and each $1 \leq j \leq r$, then

$$|E[f(T_{N_\lambda})] - E[f(Z)]| = o_f\left\{E\left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} E[|X_i|^r] + E[|Z_i|^r]\right]\right\}$$

for $\lambda \to \infty$ for all $f \in C_B^r(\mathbf{R})$ (see definition (2.1)) provided an additional boundedness condition (see (3.4)) is assumed.

By specializing the limiting r.v. Z and the normalizing function φ the random sum CLT as well as the random WLLN, both equipped with o-rates, will be deduced as particular cases of this general theorem.

The results of this paper generalize those known in the area in several respects. It contains those of Butzer and Hahn [7], [8] for the case of independent r.vs. and classical (non-random) sums since a sequence of independent r.vs. with zero means builds a MDS. It also includes a result of A. K. Basu [3] on the CLT for "dependent" r.vs. as well as of Z. Rychlik and D. Szynal [23] on the random CLT with o-rates for independent r.vs. The fact that the moments of X_i and Z_i coincide up to the order r, a condition needed in [23] and which would correspond to condition (3.7) of this paper, is now replaced by the weaker hypothesis (3.5).

Concerning the proofs, they are based upon a modification of the Lindeberg—Trotter operator approach tailored to the situation of not necessarily independent r.vs. as well as of randomly indexed r.vs. X_i which are independent of the index variable N_λ, $\lambda\in\mathbf{R}^+$, as already applied in BUTZER—SCHULZ [11]. This time the proofs are more difficult than for the large-O theorems of [11] not so much because of their length but since they use further basic concepts of probability such as the random Lindeberg condition. So in this sense the equipment of convergence assertions with little-o rates is a more typical generalization than that with large-O rates.

Section 2 is concerned with questions of notation as well as with the definitions of generalized Lindeberg and Liapounov conditions of given order and connection between these and the Feller condition in the case of random sums. Section 3 is devoted to the general theorem of the paper stated above, and Sections 4 and 5 to the random CLT and WLLN, respectively.

2. Notations; Generalized random Lindeberg and related conditions

In the following, $C_B=C_B(\mathbf{R})$ will denote the class of all real valued, bounded, uniformly continuous functions defined on the reals $\mathbf{R}$, endowed with norm $\|f\|_{C_B}:=\sup_{x\in\mathbf{R}}|f(x)|$. For $r\in\mathbf{P}=\{0,1,2,\dots\}$ we set

$$C_B^0:=C_B,\quad C_B^r:=\{f\in C_B;\ f',f'',\dots,f^{(r)}\in C_B\},\tag{2.1}$$

the semi-norm on C_B^r given by $|g|_{C_B^r}:=\|g^{(r)}\|_{C_B}$. Lipschitz classes of index $r\in\mathbf{N}$ and order α, $0<\alpha\leqq r$, will also be needed. These are defined for $f\in C_B$ by

$$\mathrm{Lip}\,(\alpha;\ r;\ C_B):=\{f\in C_B;\ \omega_r(t;\ f;\ C_B)\leqq L_f t^\alpha,\quad t>0\},$$

where L_f is the Lipschitz constant, and

$$\omega_r(t;\ f;\ C_B):=\sup_{|h|\leqq t}\left\|\sum_{k=1}^{r}(-1)^{r-k}\binom{r}{k}f(\cdot+kh)\right\|_{C_B}$$

denotes the r-th modulus of continuity.

The concept of φ-decomposability, defined in (1.3), can be extended to randomly indexed r.vs. since the range of the index r.v. N_λ is a subset of $\mathbf{N}$. In fact, for any decomposable r.v. Z one has by (1.3)

$$P_Z=P_{\varphi(N_\lambda)\sum_{i=1}^{N_\lambda}Z_i}\quad(\lambda\in\mathbf{R}^+).\tag{2.2}$$

If the decomposition r.vs. Z_i; $i\in\mathbf{N}$, are independent of N_λ for each $\lambda\in\mathbf{R}^+$, which will be assumed in the sequel, the usual rules for conditional expectations yield

$$P_Z = \sum_{n=1}^{\infty} p_n P_{\varphi(n)\sum_{i=1}^n Z_i} \tag{2.3}$$

where $p_n=p_n(\lambda):=P\{\omega;\ N_\lambda(\omega)=n\}$. This implies that for the expectation of Z,

$$E(Z) = \sum_{n=1}^{\infty} p_n E\left[\varphi(n)\sum_{i=1}^{n} Z_i\right].$$

Another relation that will often be used in the following is

$$E[f(Z)] = E\left[f\left(\varphi(N_\lambda)\sum_{i=1}^{N_\lambda} Z_i\right)\right] = \sum_{n=1}^{\infty} p_n E\left[f\left(\varphi(n)\sum_{i=1}^{n} Z_i\right)\right] \quad (f\in C_B^r), \tag{2.4}$$

valid in view of (2.2) and (2.3); and analogously for the r.vs. T_{N_λ} (recall (1.2)), namely

$$E[f(T_{N_\lambda})] = \sum_{n=1}^{\infty} p_n E[f(T_n)]. \tag{2.5}$$

The following generalization of the well-known Lindeberg-condition will play an important role in the proofs of this paper.

Definition 1. Let $(X_i)_{i\in\mathbf{N}}$ be a sequence of real valued r.vs. having finite moments of order s, $0<s<\infty$. Then $(X_i)_{i\in\mathbf{N}}$ is said to satisfy the generalized random Lindeberg condition of order s, if for every $\delta>0$,

$$L_{N_\lambda}^s(\delta) := E\left[\frac{\sum_{i=1}^{N_\lambda}\int_{|x|\ge\delta/\varphi(N_\lambda)} |x|^s\, dF_{X_i}(x)}{\sum_{i=1}^{N_\lambda} E[|X_i|^s]}\right] \to 0 \quad \text{for} \quad \lambda\to\infty. \tag{2.6}$$

In case $r=2$ and $\varphi(N_\lambda):=s_{N_\lambda}^{-1}$, where $s_{N_\lambda}:=\left(\sum_{i=1}^{N_\lambda} E[X_i^2]\right)^{1/2}$, one obtains the usual random Lindeberg condition (cf. RYCHLIK [21]).

If the parameter λ is a positive integer n and if, for every n, the r.v. N_λ takes the value n with probability one, and if $\varphi(n):=s_n$, then (2.6) reduces to the Lindeberg condition of order s, introduced in [6], a definition which has in the meantime been taken over and used effectively by PRAKASA RAO [18], RYCHLIK and SZYNAL [22], [23] and BASU [3]. The reader should recall that there are various (different) generalizations of the Lindeberg and Liapounov conditions (see, e.g., BROWN [5], BASU [2], [3]).

The following lemma relates Lindeberg conditions of different orders. It will be shown that under an additional assumption a Lindeberg condition of higher order implies one of lower order.

Lemma 1. *If the generalized random Lindeberg condition of order $r+\varepsilon$, $r\in\mathbf{N}$, $0<\varepsilon\leqq 1$, is satisfied, then that of order r holds provided there exist constants $M, \lambda_0\in\mathbf{R}^+$ such that*

$$\left|\frac{\sum_{i=1}^{N_\lambda} E[|X_i|^{r+\varepsilon}]}{(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda} E[|X_i|^r]}\right| \leqq M \quad a.s. \quad (\lambda \geqq \lambda_0). \tag{2.7}$$

Proof. Because $|x|\geqq\delta/\varphi(N_\lambda)$ implies $|x|^{r+\varepsilon}\geqq|x|^r(\delta/\varphi(N_\lambda))^\varepsilon$, one has for arbitrary $\varepsilon>0$ according to (2.7)

$$L_{N_\lambda}^{r+\varepsilon}(\delta) \geqq E\left[\frac{\delta^\varepsilon \sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)} |x|^r\, dF_{X_i}(x)}{(\varphi(N_\lambda))^\varepsilon \sum_{i=1}^{N_\lambda} E[|X_i|^{r+\varepsilon}]}\right] \geqq$$

$$\geqq E\left[\frac{\delta^\varepsilon \sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)} |x|^r\, dF_{X_i}(x)}{M\sum_{i=1}^{N_\lambda} E[|X_i|^r]}\right] = \frac{\delta^\varepsilon}{M} L_{N_\lambda}^r(\delta).$$

The Liapounov condition of order r, introduced in [8], can also be extended to the situation of random sums just in the same manner as the Lindeberg condition.

Definition 2. Let $(X_i)_{i\in\mathbf{N}}$ be a sequence of real valued r.vs. for which the r-th order moment $(0<r<\infty)$ is finite. Then $(X_i)_{i\in\mathbf{N}}$ is said to satisfy the generalized Liapounov condition of order r, if there exists an $\varepsilon>0$ such that

$$\lim_{\lambda\to\infty} E\left[\frac{\sum_{i=1}^{N_\lambda} E[|X_i|^{r+\varepsilon}]}{(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda} E[|X_i|^r]}\right] = 0.$$

Just as in the classical case (cf. [8]) the following lemma holds.

Lemma 2. *If a sequence $(X_i)_{i\in\mathbf{N}}$ satisfies the generalized random Liapounov condition of order r, then it also satisfies the random Lindeberg condition of order r.*

Proof. Since $|x|\geqq\delta/\varphi(N_\lambda)$ implies $|x|^{r+\varepsilon}\geqq|x|^r(\delta/\varphi(N_\lambda))^\varepsilon$ for each $\varepsilon>0$, one has

$$L_{N_\lambda}^r(\delta) \leqq E\left[\frac{\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)} |x|^{r+\varepsilon}\, dF_{X_i}(x)}{\delta^\varepsilon(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda} E[|X_i|^r]}\right] \leqq E\left[\frac{\sum_{i=1}^{N_\lambda} E[|X_i|^{r+\varepsilon}]}{\delta^\varepsilon(\varphi(N_\lambda))^{-\varepsilon}\sum_{i=1}^{N_\lambda} E[|X_i|^r]}\right].$$

Since $\delta>0$ is arbitrary, the assertion follows.

It was Z. RYCHLIK [21] who extended the Feller condition to the situation of random sums. It states

Definition 3. A sequence of real valued r.vs. $(X_i)_{i\in \mathbf{N}}$ with $0<E[X_i^2]<\infty$ is said to satisfy a random Feller condition, if

$$\lim_{\lambda\to\infty} E\left[\max_{1\leq i\leq N_\lambda} \frac{E[X_i^2]}{s_{N_\lambda}^2}\right] = 0. \tag{2.8}$$

The well-known connection between the Lindeberg and Feller conditions remains also valid in the random case.

Lemma 3. *If a sequence* $(X_i)_{i\in \mathbf{N}}$ *of r.vs. with* $0<E[X_i^2]<\infty$ *satisfies condition* (2.6) *for* $s=2$ *and* $\varphi(N_\lambda)=s_{N_\lambda}^{-1}$, *then* (2.8) *is satisfied.*

Proof. For arbitrary $\delta>0$ and $1\leq i\leq N_\lambda$ one has

$$E[X_i^2] = \int_{\mathbf{R}} x^2\, dF_{X_i}(x) \leq \delta^2 s_{N_\lambda}^2 + \sum_{i=1}^{N_\lambda} \int_{|x|\geq \delta s_{N_\lambda}} x^2\, dF_{X_i}(x) \quad \text{a.s.}$$

This implies that

$$\max_{1\leq i\leq N_\lambda} \frac{E[X_i^2]}{s_{N_\lambda}^2} \leq s_{N_\lambda}^{-2} \sum_{i=1}^{N_\lambda} \int_{|x|\geq \delta s_{N_\lambda}} x^2\, dF_{X_i}(x) \quad \text{a.s.}$$

Taking expectations of both sides yields the assertion.

3. General convergence theorem for MDS with o-rates

The following main approximation theorem for MDS for random sums with "little-o" rates will be established by the Lindeberg—Trotter operator-theoretic approach as tailored to the situation for MDS in [14], this time however modified to the instance of o-rates. For this purpose, additional assumptions are necessary, namely a generalized random Lindeberg condition of order r which is needed not only for the r.vs. X_i, $i\in\mathbf{N}$, but also for the decomposition components Z_i, $i\in\mathbf{N}$, as well as a type of boundedness condition upon the higher order moments of X_i and Z_i (cf. (3.4)) in association with the φ-function.

Theorem 1. *Let* $(X_i, \mathscr{F}_i)_{i\in \mathbf{P}}$ *be a MDS,* Z *a* φ-*decomposable r.v. with* $E[Z]=0$ *such that*

$$\zeta_{r,i} := E[|X_i|^r] < \infty, \quad \xi_{r,i} := E[|Z_i|^r] < \infty \quad (i\in\mathbf{N}) \tag{3.1/2}$$

for some $r\in\mathbf{N}$. *Set*

$$M(n) := \sum_{i=1}^{n} (\zeta_{r,i}+\xi_{r,i}) \quad (n\in\mathbf{N}). \tag{3.3}$$

Further assume that

$$(3.4)\qquad (\varphi(N_\lambda))^r M(N_\lambda) = O\{E[(\varphi(N_\lambda))^r M(N_\lambda)]\}\quad a.s.\quad (\lambda\to\infty).$$

a) *If the sequences of r.vs.* $(X_i)_{i\in N}$ *as well as of decomposition components* $(Z_i)_{i\in N}$ *satisfy the generalized random Lindeberg condition* (2.6) *of order* r *and further the condition*

$$(3.5)\qquad E\left[(M(N_\lambda))^{-1}(\varphi(N_\lambda))^{j-r}\sum_{i=1}^{N_\lambda}|E[X_i^j|\mathscr{F}_{i-1}]-E[Z_i^j]|\right] = o(1)$$

for $\lambda\to\infty$ *for each* $1\leqq j\leqq r$, *one has for each* $f\in C_B^r$,

$$(3.6)\qquad |E[f(T_{N_\lambda})]-E[f(Z)]| = o_f\{E[(\varphi(N_\lambda))^r M(N_\lambda)]\}\quad (\lambda\to\infty).$$

If instead of (3.5) *the stronger condition*

$$(3.7)\qquad E[X_i^j|\mathscr{F}_{i-1}] = E[Z_i^j]\quad a.s.\quad (i\in\mathrm{N},\quad 1\leqq j\leqq r)$$

is satisfied, then the estimate (3.6) *again holds.*

b) *If the r.vs.* X_i *as well as* Z_i, $i\in\mathrm{N}$, *are identically distributed such that assumption* (3.7) *holds, and if the normalizing function* φ *satisfies the conditions*

$$(3.8)\qquad \varphi(N_\lambda) = o(1)\quad a.s.\quad (\lambda\to\infty),$$

$$(3.9)\qquad \varphi(N_\lambda) = o(E[\varphi(N_\lambda)])\quad a.s.\quad (\lambda\to\infty),$$

then $f\in C_B^r$ *implies*

$$|E[f(T_{N_\lambda})]-E[f(Z)]| = o_f\{E[(\varphi(N_\lambda))^r N_\lambda(\zeta_{r,1}+\xi_{r,1})]\}\quad (\lambda\to\infty).$$

Proof. a) Setting $R_{n,i}:=\sum_{k=1}^{i-1}X_k+\sum_{k=i+1}^{n}Z_k$, $1\leqq k\leqq n$, $n\in\mathrm{N}$, an application of Taylor's formula up to the order r to both $f(\varphi(n)R_{n,i}+\varphi(n)X_i)$ and $f(\varphi(n)R_{n,i}+\varphi(n)Z_i)$ for $f\in C_B^r$ yields

$$f(T_n)-f\left(\varphi(n)\sum_{i=1}^n Z_i\right) = \sum_{i=1}^n\sum_{j=1}^r\frac{(\varphi(n))^j}{j!}\{f^{(j)}(\varphi(n)R_{n,i})X_i^j-f^{(j)}(\varphi(n)R_{n,i})Z_i^j\}+$$

$$+\sum_{i=1}^n\frac{1}{(r-1)!}\int_0^1(1-t)^{r-1}\{f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)(\varphi(n)X_i)^r-$$

$$-f^{(r)}(\varphi(n)R_{n,i})(\varphi(n)X_i)^r+f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)Z_i)(\varphi(n)Z_i)^r-$$

$$-f^{(r)}(\varphi(n)R_{n,i})(\varphi(n)Z_i)^r\}\,dt.$$

If one divides both sides of this equation by $(\varphi(n))^r M(n)$, and then takes the expectations of both sides, one deduces

(3.10)
$$\left|(\varphi(n))^{-r}(M(n))^{-1}\left\{E\left[f(T_n)-f\left(\varphi(n)\sum_{i=1}^{n} Z_i\right)\right]\right\}\right| \leqq$$

$$\leqq (M(n))^{-1}\left\{\left|E\left[\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{(\varphi(n))^{j-r}}{j!}\{f^{(j)}(\varphi(n)R_{n,i})X_i^j-f^{(j)}(\varphi(n)R_{n,i})Z_i^j\}\right]\right|+\right.$$

$$+E\left[\sum_{i=1}^{n}\frac{1}{(r-1)!}\int_0^1(1-t)^{r-1}\{|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)X_i^r-f^{(r)}(\varphi(n)R_{n,i})X_i^r|+\right.$$

$$\left.\left.+|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)Z_i)Z_i^r-f^{(r)}(\varphi(n)R_{n,i})Z_i^r|\}\,dt\right]\right\}.$$

Since $f\in C_B^r$, $f^{(r)}$ is uniformly continuous on $\mathbf{R}$, i.e., to each $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that

(3.11)
$$|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)-f^{(r)}(\varphi(n)R_{n,i})|<\varepsilon \quad (i\in\mathbf{N})$$

if $|t\varphi(n)X_i|<\delta$, thus if $|X_i|<\delta/\varphi(n)$ since $0\leqq t\leqq 1$. Likewise one has an estimate corresponding to (3.11) when X_i is replaced by Z_i.

However, the $\zeta_{r,i}$ and $\xi_{r,i}$ are finite by hypothesis. So

(3.12)
$$E[|\{f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)-f^{(r)}(\varphi(n)R_{n,i})\}X_i^r|]=$$

$$=E[|f^{(r)}(\varphi(n)R_{n,i}+t\varphi(n)X_i)-f^{(r)}(\varphi(n)R_{n,i})|\,|X_i|^r\{\mathbf{1}_{|X_i|<\delta/\varphi(n)}+\mathbf{1}_{|X_i|\geqq\delta/\varphi(n)}\}]\leqq$$

$$\leqq \varepsilon\zeta_{r,i}+2|f|_{C_B^r}\int_{|x|\geqq\delta/\varphi(n)}|x|^r\,dF_{X_i}(x).$$

Here $\mathbf{1}_A$ denotes the indicator function of the set $A\subset\Omega$. Analogously one obtains an estimate corresponding to (3.12) when X_i is replaced by Z_i and $\zeta_{r,i}$ by $\xi_{r,i}$.

By applying the same arguments concerning conditional expectations as were used in the proof of the associated "large-O" theorem ([11, Theorem 1a)]), one has for $1\leqq j\leqq r$

(3.13)
$$\left|\sum_{i=1}^{n}E[f^{(j)}(\varphi(n)R_{n,i})(X_i^j-Z_i^j)]\right|=\left|\sum_{i=1}^{n}E[f^{(j)}(\varphi(n)R_{n,i})(E[X_i^j|\mathscr{F}_{i-1}]-E[Z_i^j])]\right|\leqq$$

$$\leqq E\left[\sum_{i=1}^{n}\{|f^j|_{C_B}|E[X_i^j|\mathscr{F}_{i-1}]-E[Z_i^j]|\}\right].$$

Let us now form the inequality (3.10), this time the sum $M(n)$ weighted with the probabilities p_n of (2.3). On account of (2.4), (2.5) and the inequalities (3.13), (3.12) and its counterpart for Z_i, this yields

$$\left|E\left[(\varphi(N_\lambda))^{-r}(M(N_\lambda))^{-1}\left\{f(T_{N_\lambda})-f\left(\varphi(N_\lambda)\sum_{i=1}^{N_\lambda}Z_i\right)\right\}\right]\right| \leqq \tag{3.14}$$

$$\leqq E\left[(M(N_\lambda))^{-1}\sum_{i=1}^{N_\lambda}\sum_{j=1}^{r}\left\{\frac{(\varphi(N_\lambda))^{j-r}}{j!}|f^{(j)}|_{C_B}|E[X_i^j|\mathscr{F}_{i-1}]-E[Z_i^j]|\right\}\right]+\varepsilon+$$

$$+2|f|_{C_B^r}\left\{(M(N_\lambda))^{-1}\left(E\left[\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{X_i}(x)\right]+E\left[\sum_{i=1}^{N_\lambda}\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{Z_i}(x)\right]\right)\right\}.$$

In view of the Lindeberg conditions for the r.vs. X_i and Z_i as well as (3.5) the right side of the foregoing inequality can be made arbitrarily small for $\lambda\to\infty$.

Now on account of condition (3.4) there exist $c_1, \lambda_0\in\mathbf{R}^+$ such that

$$c_1|(\varphi(N_\lambda))^r(M(N_\lambda))|^{-1}\geqq|E[(\varphi(N_\lambda))^r M(N_\lambda)]|^{-1} \quad \text{a.s.}$$

for each $\lambda>\lambda_0$. Since the left side of (3.14) vanishes for $\lambda\to\infty$, this implies that

$$\left|E\left[f(T_{N_\lambda})-f\left(\varphi(N_\lambda)\sum_{i=1}^{N_\lambda}Z_i\right)\right]\right| = o_f\{E[(\varphi(N_\lambda))^r M(N_\lambda)]\} \quad (\lambda\to\infty).$$

Because of (2.2), this gives the desired estimate (3.6).

It is obvious that (3.7) is sufficient for (3.5) to hold.

b) The proof of part b) follows from a) provided one can show that assumption (3.9) implies the random Lindeberg conditions for the X_i and Z_i for $i\in\mathbf{N}$. Since the X_i are now identically distributed, the Lindeberg condition for X_i reduces to

$$\lim_{\lambda\to\infty} E\left[\int_{|x|\geqq\delta/\varphi(N_\lambda)}|x|^r\,dF_{X_1}(x)\right]=0 \quad (\delta>0). \tag{3.15}$$

Because of condition (3.9), (3.15) is satisfied if

$$\lim_{\lambda\to\infty}\int_{|x|\geqq\delta/E[\varphi(N_\lambda)]}|x|^r d\,F_{X_1}(x)=0 \quad (\delta>0). \tag{3.16}$$

But in view of assumption (3.8) one has $E[\varphi(N_\lambda)]=o(1)$ for $\lambda\to\infty$. Therefore the range of integration in (3.16) approaches the empty set for $\lambda\to\infty$, and so the Lindeberg condition for X_i follows from the absolute continuity property of the Lebesgue integral. Since one can show in the same way that the assumptions of part a) are satisfied for the decomposition components Z_i, the proof of the theorem is complete.

Remark 1. Concerning the possible fulfilment of assumption (3.4), the left side of (3.4) is constant a.s. and so trivially true for usual sums, thus for $\lambda=n\in\mathbf{N}$ when the r.vs. N_n take on the value n with probability 1. A sufficient condition for the validity of (3.7) and so also for (3.5) in the case of identically distributed r.vs. $(X_i)_{i\in N}$ is the requirement $E[X_1^j]=E[Z_1^j]$, $1\leqq j\leqq r$, since then $E[X_1^j|\mathscr{F}_0]=E[X_1^j]$, where $\mathscr{F}_0:=\{\Phi, \Omega\}$.

Remark 2. It is not possible to deduce o-estimates for the *strong* convergence of the r.vs. T_{N_λ} towards the r.v. Z comparable to that in [11] with the help of the modification of a lemma of V. M. ZOLOTAREV [27] given in [11]. For the application of this lemma includes an estimate of the metric $\sup_{t\in R}|F_{T_{N_\lambda}}(t)-F_Z(t)|$ from above by $\sup_{f\in D}|E[f(T_{N_\lambda})]-E[f(Z)]|$, where $D:=\{f\in C_B^r;\ |f^{(r)}|\leqq 1\}$. But since the uniform continuity of $f^{(r)}$ is used in the proof of Theorem 1, in order to deduce a reasonable estimate the latter supremum would have to be taken over a class of functions the r-th derivatives of which are equicontinuous. In this respect one should also recall [9] concerned with connections between the rates of weak and strong convergence in the particular case of the CLT.

4. The random CLT for MDS with o-rates

We now wish to apply our general Theorem 1 to a concrete limiting r.v. Z, namely to the Gaussian distributed r.v. X^* with mean zero and variance 1. However, the resulting random CLT is not a direct application of Theorem 1 since here the random Feller condition is only needed for the r.vs. X_i, $i\in\mathbf{N}$. Together with the random Lindeberg condition for the sequence $(X_i)_{i\in\mathbf{N}}$ it implies just the random Lindeberg condition for the r.vs. Z_i. Furthermore, it is not necessary to assume in part b) of the theorem condition (4.3) which corresponds to the requirement (3.4). The special form of the normalizing function $\varphi(n)$ now makes it possible to deduce (4.3) from (4.6).

Theorem 2. *Let $(X_i, \mathscr{F}_i)_{i\in\mathbf{P}}$ be a MDS such that (3.1) holds for some $r\in\mathbf{N}, r\geqq 2$, let X^* be a Gaussian distributed r.v. with mean zero and variance* 1, *and let $(a_i)_{i\in\mathbf{N}}$ be any sequence of positive reals with $A_{N_\lambda}:=\left(\sum_{i=1}^{N_\lambda} a_i^2\right)^{1/2}$.*

a) *Assume that the sequence $(X_i)_{i\in\mathbf{N}}$ satisfies the random Lindeberg condition* (2.6) *of order r with $\varphi(N_\lambda):=A_{N_\lambda}^{-1}$, as well as a random Feller-type condition, namely*

$$\lim_{\lambda\to\infty} E\left[\max_{1\leqq i\leqq N_\lambda}\frac{a_i}{A_{N_\lambda}}\right]=0. \tag{4.1}$$

If additionally

$$E\left[\frac{A_{N_\lambda}^{r-j}\sum_{i=1}^{N_\lambda}|E[X_i^j\,|\mathscr{F}_{i-1}]-a_i^j E[X^{*j}]|}{\sum_{i=1}^{N_\lambda}(\zeta_{r,i}+a_i^r E[|X^*|^r])}\right]=o(1)\quad(1\leqq j\leqq r) \tag{4.2}$$

for $\lambda\to\infty$, as well as

$$A_{N_\lambda}^{-r}\sum_{i=1}^{N_\lambda}\zeta_{r,i}=O\left(E\left[A_{N_\lambda}^{-r}\sum_{i=1}^{N_\lambda}\zeta_{r,i}\right]\right)\quad a.s.\quad(\lambda\to\infty), \tag{4.3}$$

then one has for each $f \in C_B^r$,

(4.4)
$$|E[f(A_{N_\lambda}^{-r} S_{N_\lambda})] - E[f(X^*)]| = o_f\left\{E\left[A_{N_\lambda}^{-r} \sum_{i=1}^{N_\lambda} (\zeta_{r,i} + a_i^r E[|X^*|^r])\right]\right\} \quad (\lambda \to \infty).$$

If instead of (4.2) *the stronger condition*

(4.5)
$$E[X_i^j | \mathscr{F}_{i-1}] = a_i^j E[X^{*j}] \quad (i \in \mathbf{N}, \quad 1 \leqq j \leqq r)$$

is satisfied, then the estimate (4.4) *again holds.*

b) *If the r.vs. are identically distributed,* $a_i = 1$, $i \in \mathbf{N}$, *and if condition* (4.5) *as well as*

(4.6)
$$N_\lambda^{-1/2} = O\{E[N_\lambda^{-1/2}]\} \quad a.s. \quad (\lambda \to \infty)$$

hold, then $f \in C_B^r$ *implies for* $\lambda \to \infty$,

(4.7)
$$|E[f(S_{N_\lambda}/\sqrt{N_\lambda})] - E[f(X^*)]| = o_f\{E[N_\lambda^{(2-r)/2}(\zeta_{r,1} + E[|X^*|^r])]\}.$$

Proof. a) The r.v. X^* is φ-decomposable for each $n \in \mathbf{N}$ into n independent, normally distributed r.vs. Z_i, $1 \leqq i \leqq n$, namely $Z_i = a_i X^*$. Moreover, one can ensure as in [11, Theorem 1] that the Z_i, $i \in \mathbf{N}$, N_λ, $\lambda \in \mathbf{R}^+$, as well as the sub-σ-algebras $\mathscr{F}_i$, $i \in \mathbf{N}$, are all independent. So X^* can be decomposed in the form (2.3). Since $E[Z_i^j] = a_i^j E[X^{*j}]$ for $i \in \mathbf{N}$, assumptions (3.4) and (3.5) are satisfied on account of (4.3) and (4.2). Furthermore, the random Lindeberg condition for the X_i and the Feller-type condition (4.1) yield the random Lindeberg condition for the Z_i (cf. [23]). So Theorem 1 may be applied since the moments (3.1/2) exist here, too.

b) Setting $Z_i := X^*$, $i \in \mathbf{N}$, and $\varphi(N_\lambda) := N_\lambda^{-1/2}$ in Theorem 1b), then assumptions (3.7) and (3.9) reduce exactly to conditions (4.5) and (4.6), whereas condition (3.8) is satisfied because $N_\lambda \to \infty$ for $\lambda \to \infty$. It just remains to show that condition (4.6) suffices for the requirement

(4.8)
$$N_\lambda^{(2-r)/2} = O(E[N_\lambda^{(2-r)/2}]) \quad \text{a.s.} \quad (\lambda \to \infty),$$

namely for (3.4) with $\varphi(N_\lambda) = N_\lambda^{-1/2}$. In case $r=2$ there is nothing to prove, and (4.6) coincides with (4.8) for $r=3$. For $r \geqq 4$ one has

$$E[N_\lambda^{-1/2}] \leqq (E[N_\lambda^{-(r-2)/2}])^{1/(r-2)}$$

by Hölders inequality. This yields that (4.6) follows from (4.8). So assertion (4.7) is a consequence of Theorem 1b).

5. The random WLLN for MDS with o-rates

The final application of Theorem 1 will be the WLLN with o-error bounds for random sums in a version adapted to the applicability of this theorem. Thus instead of being concerned with the usual *stochastic* convergence of the r.vs. T_{N_λ} towards zero, namely of

$$\lim_{\lambda\to\infty} P(\{|T_{N_\lambda}| \geqq \varepsilon\}) = 0 \quad (\varepsilon > 0), \tag{5.1}$$

we plan to estimate the o-rate with which T_{N_λ} converges *weakly* to the degenerate limiting r.v. X_0; namely of

$$\lim_{\lambda\to\infty} |E[f(T_{N_\lambda})] - E[f(X_0)]| = 0 \quad (f \in C_B^r), \tag{5.2}$$

any $r \in \mathbf{N}$. As a matter of fact, the convergence definitions (5.1) and (5.2) are equivalent. Indeed, (5.1) implies (5.2), by standard arguments, and the converse holds since the limiting r.v. X_0 is a constant a.s.

Since $E[f(X_0)] = \int_{\mathbf{R}} f(x)\,dP_{X_0}(x) = f(0)$ for all $f \in C_B^r$, the following formulation of the WLLN with o-rates is feasible.

Theorem 3. *Let* $(X_i, \mathscr{F}_i)_{i\in \mathbf{P}}$ *be a MDS, and let* $r \in \mathbf{N}$.

a) *If the sequence* $(X_i)_{i\in\mathbf{N}}$ *satisfies* (3.1) *as well as the random Lindeberg condition* (2.6) *of order* r, *and if*

$$E\left[\frac{\varphi(N_\lambda)^{j-r} \sum_{i=1}^{N_\lambda} |E[X_i^j|\mathscr{F}_{i-1}]|}{\sum_{i=1}^{N_\lambda} \zeta_{r,i}}\right] = o(1) \quad (\lambda \to \infty,\ 1 \leqq j \leqq r) \tag{5.3}$$

for $\lambda \to \infty$; *as well as*

$$(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} \zeta_{r,i} = O\left\{E\left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} \zeta_{r,i}\right]\right\} \quad (\lambda \to \infty), \tag{5.4}$$

then one has for each $f \in C_B^r$;

$$|E[f(T_{N_\lambda})] - E[f(X_0)]| = o_f\left\{E\left[(\varphi(N_\lambda))^r \sum_{i=1}^{N_\lambda} \zeta_{r,i}\right]\right\} \quad (\lambda \to \infty).$$

b) *If the sequence* $(X_i)_{i\in\mathbf{N}}$ *satisfies the random Lindeberg condition of order* 1 *with* $\varphi(N_\lambda) := N_\lambda^{-1}$, *as well as* (5.4) *for* $r=1$, *and if*

$$\sum_{i=1}^{N_\lambda} E[|X_i|] = O(N_\lambda) \quad a.s. \quad (\lambda \to \infty), \tag{5.5}$$

then

$$\lim_{\lambda\to\infty} E[f(S_{N_\lambda}/N_\lambda)] = f(0). \tag{5.6}$$

c) *If the r.vs.* X_i, $i \in \mathbf{N}$, *are identically distributed,* $\zeta_1 < \infty$ *and*

$$N_\lambda^{-1} = O\{E[N_\lambda^{-1}]\} \quad (\lambda \to \infty), \tag{5.7}$$

then the random WLLN in the form (5.6) *again holds.*

Proof. a) If one chooses the decomposition components Z_i such that $P_{Z_i}=P_{X_0}$ for all $i\in\mathrm{N}$, then $P_{X_0}=\sum_{n=1}^{\infty} p_n P_{\varphi(n)\Sigma_{i=1}^n Z_i}$, and part a) follows from Theorem 1 a) since here the sum $M(n)$, defined in (3.3), reduces to $\sum_{i=1}^{n}\zeta_{r,i}$, and therefore conditions (5.3) and (5.4) are special cases of assumptions (3.5) and (3.4).

b) Part b) follows from a) with $r=1$ and $\varphi(N_\lambda)=N_\lambda^{-1}$, because condition (5.5) implies assumption (3.1), whereas (5.3) is fulfilled because of the definition (1.1) of a MDS.

c) Setting $\varphi(N_\lambda):=N_\lambda^{-1}$, part c) turns out to be a special case of Theorem 1 b) if one considers that condition (3.7) is fulfilled for $r=1$ because of (1.1) and that (3.9) reduces to assumption (5.7).

It is an open question whether the convergence assertions (5.1) and (5.2) are still equivalent to another under suitable conditions if they are equipped with rates. This is generally not the case in the corresponding situation for the CLT, see again Butzer—Hahn [9].

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Analytic generators for one-parameter cosine families

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Dedicated to B. Szőkefalvi-Nagy on the occasion of his seventieth birthday

One parameter cosine families of linear operators have been recently used in several papers on operator algebras ([6], [7], [13]). Some technical results of these papers suggested us to develop here a general theory of the analitic generator of one-parameter cosine families similarly to that presented in [3] for one-parameter groups. It is proved, that a one-parameter cosine family of 0 exponential type is uniquely determined by its analytic generator and explicit formulas are given.

We remark that the theory developed here can be used to give intrinsic characterizations for the analytic generators of one-parameter groups of automorphisms of operator algebras; this is due to the fact that while the analytic generator of such a goup frequently has "bad" spectral properties [4], the analytic generator of its "cosine part" has always a "thin" spectrum.

1. Analytic extensions of cosine families

Let us first specify the frame in which cosine families are to be considered.

We call a *dual pair of Banach spaces* any pair $(\mathbf{X}, \mathscr{F})$ of complex Banach spaces, together with a bilinear functional

$$\mathbf{X}\times\mathscr{F}\ni(x,\varphi)\to\langle x,\varphi\rangle\in\mathbf{C},$$

such that

(i) $\|x\| = \sup\limits_{\|\varphi\|\leq 1}|\langle x,\varphi\rangle|$ for any $x\in\mathbf{X}$;

(ii) $\|\varphi\| = \sup\limits_{\|x\|\leq 1}|\langle x,\varphi\rangle|$ for any $\varphi\in\mathscr{F}$;

(iii) the convex hull of any relatively $\sigma(\mathbf{X},\mathscr{F})$-compact subset of $\mathbf{X}$ is relatively $\sigma(\mathbf{X},\mathscr{F})$-compact;

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(iv) the convex hull of any relatively $\sigma(\mathscr{F}, \mathbf{X})$-compact subset of $\mathscr{F}$ is relatively $\sigma(\mathscr{F}, \mathbf{X})$-compact.

If $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces, then $(\mathscr{F}, \mathbf{X})$ endowed with the same bilinear pairing, is also a dual pair of Banach spaces. We note that if $\mathbf{X}$ is a complex Banach space and $\mathbf{X}^*$ its dual, then the pairs $(\mathbf{X}, \mathbf{X}^*)$ and $(\mathbf{X}^*, \mathbf{X})$, endowed with the natural pairing between $\mathbf{X}$ and $\mathbf{X}^*$, are dual pairs of Banach spaces. We recall that if $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces, then the uniform boundedness principle holds in $\mathbf{X}$ with respect to $\sigma(\mathbf{X}, \mathscr{F})$ ([8], Th. 2.8.6); in particular, the analyticity of $\mathbf{X}$-valued mappings of complex variable does not depend on the topology considered on $\mathbf{X}$ ([8], Th. 3.10.1). On the other hand quite general $\mathbf{X}$-valued mappings, defined on a locally compact space endowed with a Radon measure, are $\sigma(\mathbf{X}, \mathscr{F})$-integrable ([2], Prop. 1.2; [3], Prop. 1.4.).

If $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces and T is a $\sigma(\mathbf{X}, \mathscr{F})$-densely defined linear operator in $\mathbf{X}$, then one can define the adjoint $T^{\mathscr{F}}$ of T in $\mathscr{F}$ by

$$(\varphi, \psi) \in \operatorname{graph}(T^{\mathscr{F}}) \Leftrightarrow \langle x, \psi\rangle = \langle T(x), \varphi\rangle \quad \text{for all} \quad x \in \mathscr{D}_T.$$

$T^{\mathscr{F}}$ is always $\sigma(\mathscr{F}, \mathbf{X})$-closed. If moreover T is $\sigma(\mathbf{X}, \mathscr{F})$-closed, then $T^{\mathscr{F}}$ will be $\sigma(\mathscr{F}, \mathbf{X})$-densely defined and $(T^{\mathscr{F}})^{\mathbf{X}} = T$ holds ([11], IV. 7.1). Denote by $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ the Banach algebra of all $\sigma(\mathscr{F}, \mathbf{X})$-continuous linear operators on $\mathbf{X}$. For $T \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$ we have

$$T^{\mathscr{F}}(\varphi) = \varphi \circ T, \quad \varphi \in \mathscr{F} \quad \text{and} \quad T^{\mathscr{F}} \in \mathscr{B}_{\mathbf{X}}(\mathscr{F}).$$

If $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces and T is a $\sigma(\mathbf{X}, \mathscr{F})$-closed linear operator in $\mathbf{X}$; then the *resolvent set of* T is

$$\varrho(T) = \{\lambda \in \mathbf{C};\ \lambda - T \quad \text{is injective and} \quad (\lambda - T)^{-1} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})\},$$

and *the spectrum* of T is $\sigma(T) = \mathbf{C} \setminus \varrho(T)$. The standard power series argument shows that $\varrho(T)$ is open in $\mathbf{C}$, thus $\sigma(T)$ is closed. If T is also $\sigma(\mathbf{X}, \mathscr{F})$-densely defined, then $\sigma(T) = \sigma(T^{\mathscr{F}})$. We note that if $\mathscr{F} = \mathbf{X}^*$ or $\mathbf{X} = \mathscr{F}^*$, then, by the closed graph theorem, the Banach—Smulian theorem on the weak continuity of linear functionals, and the Alaoglu theorem, we have

$$\varrho(T) = \{\lambda \in \mathbf{C};\ \lambda - T \text{ is bijective}\}.$$

Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces; a *one-parameter cosine family* C in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ is a mapping $C : \mathbf{R} \to \mathscr{B}_{\mathscr{F}}(\mathbf{X})$ such that

$$C_0 = I_{\mathbf{X}}, \text{ where } I_{\mathbf{X}} \text{ is the identity map of } \mathbf{X};$$

$$C_{s+t} + C_{s-t} = 2C_s C_t \text{ for all } s, t \in \mathbf{R}.$$

It follows directly from this definition that

$$C_t = C_{-t}, \quad t \in \mathbf{R} \quad \text{and} \quad C_t C_s = C_s C_t, \quad s, t \in \mathbf{R}.$$

C is called *$\sigma(\mathbf{X}, \mathcal{F})$-continuous* if for each $x\in\mathbf{X}$ the mapping $\mathbf{R}\ni t\to C_t(x)\in\mathbf{X}$ is $\sigma(\mathbf{X}, \mathcal{F})$-continuous. In this case one can define the *dual cosine family* $C^{\mathcal{F}}$ in $\mathcal{B}_{\mathbf{X}}(\mathcal{F})$ by $C_t^{\mathcal{F}}=(C_t)^{\mathcal{F}}$, $t\in\mathbf{R}$, and $C^{\mathcal{F}}$ is $\sigma(\mathcal{F}, \mathbf{X})$-continuous. We note that a quite complete infinitesimal generator theory for strongly continuous one-parameter cosine families is done in [12].

Let C be a $\sigma(\mathbf{X}, \mathcal{F})$-continuous one-parameter cosine-family in $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$. For $z\in\mathbf{C}$, denote by $D_z=\{\zeta\in\mathbf{C};\ \operatorname{Im}\zeta\cdot\operatorname{Im}z\leqq 0,\ |\operatorname{Im}\zeta|\leqq|\operatorname{Im}z|\}$. Suppose that for some $x\in\mathbf{X}$, the mapping $\mathbf{R}\ni t\to C_t(x)\in\mathbf{X}$ has a $\sigma(\mathbf{X}, \mathcal{F})$-continuous extension on D_z which is analytic on its interior; such an extension will be called *$\sigma(\mathbf{X}, \mathcal{F})$-regular*. By the symmetry principle ([1], Ch. V, 1.6) it follows that this extension is uniquely determined. Thus we can define a linear operator C_z in $\mathbf{X}$ by

$$(x, y)\in\operatorname{graph}C_z\Leftrightarrow\mathbf{R}\ni t\to C_t(x)\in\mathbf{X}\ \text{ has a }\ \sigma(\mathbf{X}, \mathcal{F})\text{-regular extension}$$
$$\text{on } D_z \text{ whose value at } z \text{ is } y.$$

C_z is called the *analytic extension* of C at z.

Lemma 1.1. *Let $(\mathbf{X}, \mathcal{F})$ be a dual pair of Banach spaces and C a $\sigma(\mathbf{X}, \mathcal{F})$-continuous one-parameter cosine family in $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$. Then*

$$C_z=C_{-z},\quad z\in\mathbf{C},$$
$$C_{s+z}+C_{s-z}=2C_sC_z\subset 2C_zC_s,\quad s\in\mathbf{R},\quad z\in\mathbf{C}.$$

Proof. Let $z\in\mathbf{C}$. For each $x\in\mathcal{D}_{C_z}$ the mapping $D_{-z}\ni\zeta\to C_{-\zeta}(x)\in\mathbf{X}$ is $\sigma(\mathbf{X}, \mathcal{F})$-regular and extends $\mathbf{R}\ni t\to C_{-t}(x)=C_t(x)\in\mathbf{X}$, hence $x\in\mathcal{D}_{D_{-z}}$, $C_{-z}(x)=$ $=C_z(x)$. Thus $C_z\subset C_{-z}$ and changing z with $-z$, one gets also the converse inclusion.

Let further $s\in\mathbf{R}$ and $z\in\mathbf{C}$. For each $x\in\mathcal{D}_{C_z}=\mathcal{D}_{C_{s+z}}=\mathcal{D}_{C_{s-z}}$ the mappings

$$D_z\ni\zeta\to C_{s+\zeta}(x)+C_{s-\zeta}(x)\in\mathbf{X},\quad D_z\ni\zeta\to 2C_sC_\zeta(x)\in\mathbf{X}$$

are $\sigma(\mathbf{X}, \mathcal{F})$-regular extensions of

$$\mathbf{R}\ni t\to C_{s+t}(x)+C_{s-t}(x)=2C_sC_t(x)=2C_tC_s(x)\in\mathbf{X};$$

thus

$$C_s(x)\in\mathcal{D}_{C_Z},\quad C_{s+z}(x)+C_{s-z}(x)=2C_sC_z(x)=2C_zC_s(x).$$

Therefore $C_{s+z}+C_{s-z}=2C_sC_z\subset 2C_zC_s$.

According to Lemma 1.1 and to the symmetry principle, for each $z\in\mathbf{C}$, it holds

$$(x, y)\in\operatorname{graph}C_z\Leftrightarrow\mathbf{R}\ni t\to C_t(x)\in\mathbf{X}\ \text{ has a }\ \sigma(\mathbf{X}, \mathcal{F})\text{-regular extension on the strip}$$
$$\{\zeta\in\mathbf{C};\ |\operatorname{Im}\zeta|\leqq|\operatorname{Im}z|\}\ \text{whose value in } z \text{ is } y.$$

In particular, if $z\in\mathbf{C}$, $\operatorname{Im}z\neq 0$ and $x\in\mathcal{D}_{C_z}$, then by [8], Th. 3.10.1, we have that $\mathbf{R}\ni t\to C_t(x)\in\mathbf{X}$ is norm-continuous.

Lemma 1.2. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Then we have:*

(i) *for each* $z \in \mathbf{C}$ *and* $x \in \mathscr{D}_{C_z}$

$$\|C_z(x)\| \leqq \sup_{\substack{|\mathrm{Re}\,\zeta| \leqq 1 \\ \mathrm{Im}\,\zeta = \mathrm{Im}\,z}} \|C_\zeta(x)\| \cdot \sum_{\substack{k \in \mathbf{Z} \\ |k| \leqq |\mathrm{Re}\,z|}} \|C_k\|;$$

(ii) *for each* $\varepsilon > 0$ *and* $x \in \mathscr{D}_{C_{-\varepsilon i}} = \mathscr{D}_{C_{\varepsilon i}}$

$$\overline{\lim_{\delta \to +\infty}} \frac{1}{\delta} \ln \sup_{\substack{|\mathrm{Re}\,z| \leqq \delta \\ |\mathrm{Im}\,z| \leqq \varepsilon}} \|C_z(x)\| \leqq \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| = \overline{\lim_{\mathbf{Z} \ni k \to +\infty}} \frac{1}{k} \ln \|C_k\| \leqq \ln(1 + 2\|C_1\|).$$

Proof. (i) Let $\alpha \in \mathbf{R}$ and $x \in \mathscr{D}_{C_{\alpha i}}$, and denote for convenience

$$c = \sup_{|s| \leqq 1} \|C_{s+\alpha i}(x)\|.$$

We prove by induction, that for $n \geqq 1$,

$$\|C_{t+\alpha i}(x)\| \leqq c \sum_{\substack{k \in \mathbf{Z} \\ |k| \leqq n-1}} \|C_k\| \quad \text{for} \quad |t| \leqq n.$$

Indeed, the above statement holds obviously for $n = 1$. Assuming that it holds for some $n \geqq 1$ and that $n < |t| \leqq n+1$, we successively get by Lemma 1.1

$$C_{t+\alpha i}(x) = C_{|t| + \mathrm{sign}(t)\alpha i}(x) = 2C_n C_{|t| - n + \mathrm{sign}(t)\alpha i}(x) - C_{n - (|t| - n) - \mathrm{sign}(t)\alpha i}(x) =$$
$$= 2C_n C_{\mathrm{sign}(t)(|t| - n) + \alpha i}(x) - C_{-\mathrm{sign}(t)(n - (|t| - n)) + \alpha i}(x),$$
$$\|C_{t+\alpha i}(x)\| \leqq 2\|C_n\| \cdot c + c \cdot \sum_{|k| \leqq n-1} \|C_k\| = c \cdot \sum_{|k| \leqq n} \|C_k\|.$$

(ii) Again by induction, it is easy to verify that

$$\|C_k\| \leqq (1 + 2\|C_1\|)^k \quad \text{for} \quad k \geqq 1.$$

Now one can easily complete the proof.

We note that, if $f: \mathbf{R} \to \mathbf{C}$ is a Lebesgue-measurable function with

$$\int_{-\infty}^{+\infty} |f(t)| e^{\omega |t|}\, dt < +\infty$$

where $\omega > \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\|$, then there exists

$$C_f = \int_{-\infty}^{+\infty} f(t) C_t\, dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

uniquely defined by

$$\left\langle \left(\int_{-\infty}^{+\infty} f(t) C_t\, dt \right)(x), \varphi \right\rangle = \int_{-\infty}^{+\infty} f(t) \langle C_t(x), \varphi \rangle\, dt, \quad x \in \mathbf{X}, \ \varphi \in \mathscr{F}$$

([2], Prop. 1.2; [3], Prop. 1.4). For each $s\in\mathbf{R}$, we have:

$$C_f C_s = C_s C_f = \int_{+\infty}^{-\infty} f(t) C_s C_t td = \int_{-\infty}^{+\infty} f(t)\frac{1}{2}(C_{s+t}+C_{s-t})\,dt =$$

$$= \int_{-\infty}^{+\infty} \frac{f(t-s)+f(-t+s)}{2} C_t\,dt.$$

Thus, if f is additionally even, then

$$C_f C_s = C_s C_f = \int_{-\infty}^{+\infty} f(t-s) C_t\,dt, \quad s\in\mathbf{R}.$$

Lemma 1.3. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Let us denote*

$$f_\delta(t) = \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2}, \quad \delta > 0, \ t\in\mathbf{R}.$$

Then

$$C_{f_\delta}(\mathbf{X}) \subset \bigcap_{z\in\mathbf{C}} \mathscr{D}_{C_z}, \quad \delta > 0,$$

$$C_{f_\delta} C_z \subset C_z C_{f_\delta} = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\,dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X}), \quad \delta > 0, \ z\in\mathbf{C},$$

$$\sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta\to+\infty} C_{f_\delta}(x) = x, \quad x\in\mathbf{X}.$$

Proof. Let $\delta>0$. Since

$$\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} |e^{\omega|t|}\,dt < +\infty, \quad z\in\mathbf{C}, \ \omega > 0,$$

the integral

$$\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\,dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X}), \quad z\in\mathbf{C}$$

exists. It is easy to see that the mapping

$$\mathbf{C}\ni z \to \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\,dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

is analytical and extends

$$\mathbf{R}\ni s \to C_f C_s = C_s C_f = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-s)^2} C_t\,dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X}).$$

It follows that

$$C_{f_\delta}(\mathbf{X}) \subset \bigcap_{z\in\mathbf{C}} \mathscr{D}_{C_z},$$

$$C_{f_\delta} C_z \subset C_z C_{f_\delta} = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\, dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X}), \quad z\in\mathbf{C}.$$

Now let $x\in\mathbf{X}$ be arbitrary; for each $\varphi\in\mathscr{F}$ and $\delta, \varepsilon>0$ the following holds:

$$|\langle C_{f_\delta}(x)-x, \varphi\rangle| = \left|\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} \langle C_t(x)-x, \varphi\rangle\, dt\right| \leqq$$

$$\leqq \int_{|t|<\varepsilon} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} |\langle C_t(x)-x, \varphi\rangle|\, dt + \int_{|t|\geqq\varepsilon} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} |\langle C_t(x)-x, \varphi\rangle|\, dt \leqq$$

$$\leqq \sup_{|t|<\varepsilon} |\langle C_t(x)-x, \varphi\rangle| + \int_{|t|\geqq\varepsilon} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} (\|C_t\|+1)\, dt\, \|x\|\, \|\varphi\|.$$

Hence

$$\overline{\lim_{\delta\to+\infty}}\, |\langle C_{f_\delta}(x)-x, \varphi\rangle| \leqq \inf_{\varepsilon\to 0} \sup_{|t|<\varepsilon} |\langle C_t(x)-x, \varphi\rangle| = 0.$$

We can now give

Proposition 1.4. *Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces, C a $\sigma(\mathbf{X}, \mathscr{F})$-continuous one-parameter cosine family in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ and $z\in\mathbf{C}$. Then C_z is $\sigma(\mathbf{X}; \mathscr{F})$-densely defined and $\sigma(\mathbf{X}, \mathscr{F})$-preclosed. Moreover, we have*

$$\overline{C_z}^{\sigma(\mathbf{X}, \mathscr{F})} = \overline{C_z | \bigcap_{\zeta\in\mathbf{C}} \mathscr{D}_{C_\zeta}}^{\sigma(\mathbf{X}, \mathscr{F})} = (C_z^{\mathscr{F}})^{\mathbf{X}}.$$

Proof. By Lemma 1.3 it is clear that C_z and $C_z^{\mathscr{F}}$ are $\sigma(\mathbf{X}, \mathscr{F})$, resp. $\sigma(\mathscr{F}, \mathbf{X})$-densely defined. For each $x\in\mathscr{D}_{C_z}$ and $\varphi\in\mathscr{D}_{C_z^{\mathscr{F}}}$ the functions $\zeta\to\langle C_\zeta(x), \varphi\rangle$ and $\zeta\to\langle x, C_\zeta^{\mathscr{F}}(\varphi)\rangle$ defined on the strip $\{\zeta\in\mathbf{C};\ |\mathrm{Im}\,\zeta|\leqq|\mathrm{Im}\,z|\}$, are regular extensions of the function

$$\mathbf{R}\ni t \to \langle C_t(x), \varphi\rangle = \langle x, C_t^{\mathscr{F}}(\varphi)\rangle,$$

hence $\langle C_z(x), \varphi\rangle = \langle x, C_z^{\mathscr{F}}(\varphi)\rangle$. It follows $C_z \subset (C_z^{\mathscr{F}})^{\mathbf{X}}$. In particular, C_z is $\sigma(\mathbf{X}, \mathscr{F})$-preclosed.

To end the proof, we have only to prove that the domain of $(C_z^{\mathscr{F}})^{\mathbf{X}}$ is contained in the domain of $\overline{C_z | \bigcap_{\zeta\in\mathbf{C}} \mathscr{D}_{D_\zeta}}^{\sigma(\mathbf{X}, \mathscr{F})}$. Let x be in the domain of $(C_z^{\mathscr{F}})^{\mathbf{X}}$. By Lemma 1.3, for each $\delta>0$ we have $C_{f_\delta}^{\mathscr{F}} C_z^{\mathscr{F}} \subset C_z^{\mathscr{F}} C_{f_\delta}^{\mathscr{F}}$, thus

$$C_{f_\delta}(C_z^{\mathscr{F}})^{\mathbf{X}} = (C_z^{\mathscr{F}} D_{f_\delta}^{\mathscr{F}})^{\mathbf{X}} \subset (C_{f_\delta}^{\mathscr{F}} C_z^{\mathscr{F}})^{\mathbf{X}} = (C_z^{\mathscr{F}})^{\mathbf{X}} C_{f_\delta}.$$

Again by Lemma 1.3, it follows that

$$C_{f_\delta}(x) \in \bigcap_{\zeta \in \mathbf{C}} \mathscr{D}_{C_\zeta}, \quad \delta > 0,$$

$$\sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} C_{f_\delta}(x) = x,$$

$$\sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} C_z C_{f_\delta}(x) = \sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} (C_z^{\mathscr{F}})^{\mathbf{X}} C_{f_\delta}(x) =$$

$$= \sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} C_{f_\delta}(C_z^{\mathscr{F}})^{\mathbf{X}}(x) = (C_z^{\mathscr{F}})^{\mathbf{X}}(x).$$

Consequently x is in the domain of $\overline{C_z | \bigcap_{\zeta \in \mathbf{C}} \mathscr{D}_{C_\zeta}}^{\sigma(\mathbf{X}, \mathscr{F})}$.

In the sequel we shall denote $\overline{C}_z^{\sigma(\mathbf{X}, \mathscr{F})}$ and $\overline{C_z^{\mathscr{F}}}^{\sigma(\mathscr{F}, \mathbf{X})}$ simply by $\overline{C}_z$, respectively $\overline{C_z^{\mathscr{F}}}$. We call $\overline{C_i} = \overline{C_{-i}}$ *the analytic generator* of the cosine family C.

2. Spectral properties of the analytic generator of cosine families

Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces and C a $\sigma(\mathbf{X}, \mathscr{F})$-continuous one-parameter cosine family in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. We recall that by Lemma 1.2 (ii)

$$\lim_{t \to +\infty} \frac{1}{t} \ln \|C_t\| = \overline{\lim_{k \to +\infty}} \frac{1}{k} \ln \|C_k\| < +\infty.$$

On the other hand, if $\mathbf{X} \neq \{0\}$, then

$$\overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| \geqq 0.$$

Indeed, we have for each $t \in \mathbf{R}$

$$1 = \|C_0\| = \|2C_t C_t - C_{2t}\| \leqq 2\|C_t\|^2 + \|C_{2t}\| \leqq 3 \max\{\|C_t\|^2, \|C_{2t}\|\}$$

so that

$$0 = \lim_{t \to +\infty} \frac{1}{2t} \ln 1 \leqq \overline{\lim_{t \to +\infty}} \max\left\{\frac{1}{t} \ln \|C_t\|, \frac{1}{2t} \ln \|C_{2t}\|\right\} = \lim_{t \to +\infty} \frac{1}{t} \ln \|C_t\|.$$

We say that C is of 0 *exponential type* if

$$\overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| \leqq 0$$

that is, if $\mathbf{X} \neq \{0\}$,

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \ln \|C_t\| = 0.$$

Let $\mu \in \mathbf{C} \setminus (-\infty, 0]$. We denote

$$\arg \mu = \theta \text{ where } \mu = |\mu| e^{i\theta},\ |\theta| < \pi; \quad \ln \mu = \ln |\mu| + i\theta.$$

Then $\mathbf{C} \ni z \to \mu^z = e^{z \ln \mu} \in \mathbf{C}$ is an entire function. The next lemma is the main technical result of this paragraph.

Lemma 2.1. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces,* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$, $\mu \in \mathbf{C} \setminus (-\infty, 0]$ *and* $\lambda = (\mu^2+1)/2\mu$. *Then the function* $g_\lambda \colon \mathbf{R} \to \mathbf{C}$, *defined by*

$$g_\lambda(t) = \begin{cases} \dfrac{\mu}{\mu^2-1} \cdot \dfrac{\mu^{it} - \mu^{-it}}{\sin i\pi t} = \dfrac{\mu}{i(\mu^2-1)} \cdot \dfrac{\mu^{it} - \mu^{it}}{\operatorname{sh} \pi t} & \text{if } \mu \neq 1, \\ \dfrac{it}{\sin i\pi t} = \dfrac{t}{\operatorname{sh} \pi t} & \text{if } \mu = 1 \end{cases}$$

depends only on λ, *the integral*

$$C_{g_\lambda} = \int_{-\infty}^{+\infty} g_\lambda(t) C_t\, dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

exists and $C_{g_\lambda}(\lambda + C_{-i}) \subset (\lambda + C_{-i}) C_{g_\lambda}; I_{\mathbf{X}}$.

Proof. Since the roots of the equation $\lambda = (w^2+1)/2w$ are μ and μ^{-1} and

$$\frac{\mu}{\mu^2-1}(\mu^{it} - \mu^{-it}) = \frac{\mu^{-1}}{\mu^{-2}-1}(\mu^{-it} - \mu^{it}), \quad t \in \mathbf{R},$$

g_λ depends only on λ.

Choosing some ω with $0 < \omega < \pi - |\arg \mu|$, we have

$$\int_{-\infty}^{+\infty} |g_\lambda(t)| e^{\omega |t|}\, dt < +\infty \quad \text{and} \quad \omega > \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| = 0.$$

By our remarks after Lemma 1.2 it follows that $C_{g_\lambda} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$ is well defined and $C_{g_\lambda} C_s = C_s C_{g_\lambda}$. Let $x \in C_{-i}$ be arbitrary. Since the mapping

$$\{\zeta \in \mathbf{C};\ |\operatorname{Im} \zeta| \leqq 1\} \ni \zeta \to C_{g_\lambda} C_\zeta(x) \in \mathbf{X}$$

is $\sigma(\mathbf{X}, \mathscr{F})$-regular and extends

$$\mathbf{R} \ni s \to C_{g_\lambda} C_s(x) = C_s C_{g_\lambda}(x) \in \mathbf{X}$$

we have $C_{g_\lambda}(x) \in \mathscr{D}_{C_{-i}}$ and $C_{-i} C_{g_\lambda}(x) = C_{g_\lambda} C_{-i}(x)$. Consequently $(\lambda + C_{-i}) C_{g_\lambda}(x) = C_{g_\lambda}(\lambda + C_{-i})(x)$.

Finally, we show that $C_{g_\lambda} C_{-i}(x) = x - \lambda C_{g_\lambda}(x)$, that is $C_{g_\lambda}(\lambda + C_{-i})(x) = x$.

Let us first assume $\mu\neq 1$ and fix some. $0<\varepsilon<1$. Since by Lemma 1.2. (ii)

$$\overline{\lim_{\delta\to+\infty}}\frac{1}{\delta}\ln\sup_{\substack{|\mathrm{Re}\,\zeta|\leq 1\\|\mathrm{Im}\,\zeta|\leq 1}}\|C_\zeta(x)\|\leqq 0,$$

using Lemma 1.1 and the Cauchy integral theorem, we get

$$C_{g_\lambda}C_{-i}(x)=\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{it}-\mu^{-it}}{\sin i\pi t}C_{t-i}(x)\,dt+$$

$$+\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{it}-\mu^{it}}{\sin i\pi i}C_{t+i}(x)\,dt=$$

$$=\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{i(t+i\varepsilon)}-\mu^{-i(t+i\varepsilon)}}{\sin i\pi(t+i\varepsilon)}C_{t-(1-\varepsilon)i}(x)\,dt+$$

$$+\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{i(t-i\varepsilon)}-\mu^{-i(t-i\varepsilon)}}{\sin i\pi(t-i\varepsilon)}C_{t+(1-\varepsilon)i}(x)\,dt.$$

Defining the curves Γ_- and Γ_+ by

$$\Gamma_-(t)=t-(1-\varepsilon)i,\quad \Gamma_-(t)=1+(1-\varepsilon)i,\quad t\in\mathbf{R},$$

we obtain

$$C_{g_\lambda}C_{-i}(x)=\int_{\Gamma_-}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{iz-1}-\mu^{-iz+1}}{\sin(i\pi z-\pi)}C_z(x)\,dz+$$

$$+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{iz+1}-\mu^{-iz-1}}{\sin(i\pi z+\pi)}C_z(x)\,dz=$$

$$=\int_{\Gamma_-}\frac{\mu}{2(\mu^2-1)}\frac{\mu^{-iz+1}-\mu^{iz-1}}{\sin i\pi z}C_z(x)\,dz+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{-iz-1}-\mu^{iz+1}}{\sin i\pi z}C_z(x)\,dz.$$

Further, the residue theorem gives

$$\int_{\Gamma_-}\frac{\mu}{2(\mu^2-1)}\frac{\mu^{-iz+1}-\mu^{iz-1}}{\sin i\pi z}C_z(x)\,dz=x+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\frac{\mu^{-iz+1}-\mu^{iz-1}}{\sin i\pi z}Cz(x)\,dz$$

so that, using again the Cauchy integral theorem, we conclude

$$C_{g_\lambda}C_{-i}(x)=x+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{-iz+1}-\mu^{iz-1}+\mu^{-iz-1}-\mu^{iz+1}}{\sin i\pi z}C_z(x)\,dz=$$

$$=x+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{(\mu+\mu^{-1})(\mu^{-iz}-\mu^{iz})}{\sin i\pi z}C_z(x)\,dz=$$

$$=x-\lambda\int_{\Gamma_+}\frac{\mu}{\mu^2-1}\cdot\frac{\mu^{iz}-\mu^{-iz}}{\sin i\pi z}C_z(x)\,dz=x-\lambda C_{g_\lambda}(x).$$

For $\mu=1$ the proof is completely similar.

Theorem 2.2. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Then* $\sigma(\overline{C}_i)\subset[1; +\infty)$. *Moreover, for each* $\lambda\in\mathbf{C}\setminus(-\infty; -1]$ *the roots of the equation* $\lambda=(\mu^2+1)/2\mu$ *belong to* $\mathbf{C}\setminus(-\infty, 0)$ *and*

$$(\lambda+\overline{C_{-i}})^{-1}=\begin{cases}\displaystyle\int_{-\infty}^{+\infty}\frac{\mu}{i(\mu^2-1)}\cdot\frac{\mu^{it}-\mu^{-it}}{\operatorname{sh}\pi t}C_t\,dt & \text{if } \lambda\neq 1\\ \displaystyle\int_{-\infty}^{+\infty}\frac{t}{\operatorname{sh}\pi t}C_t\,dt & \text{if } \lambda=1.\end{cases}$$

Proof. Let $\lambda\in\mathbf{C}\setminus(-\infty, -1]$ be arbitrary. If one of the roots of the equation $\lambda=(\mu^2+1)/2\mu$, say μ_1, belonged to $(-\infty; 0]$, then, taking in account that the other root is μ_1^{-1}, we would have

$$\lambda=\frac{\mu_1+\mu_1^{-1}}{2}=-\frac{|\mu_1|+|\mu_1|^{-1}}{2}\leqq-1.$$

Now let $\mu\in\mathbf{C}\setminus(-\infty, 0]$ be arbitrary, with $\lambda=(\mu^2+1)/2\mu$, define $g_\lambda\colon\mathbf{R}\to\mathbf{C}$ as in Lemma 2.1. Then by this lemma $C_{g_\lambda}(\lambda+C_{-i})\subset(\lambda+C_{-i})C_{g_\lambda}=I_{\mathbf{X}}$ holds, and this implies that $C_{g_\lambda}(\lambda+\overline{C_{-i}})\subset(\lambda+\overline{C_{-i}})C_{g_\lambda}$. On the other hand, since $(\lambda+\overline{C_{-i}})C_{g_\lambda}|\mathscr{D}_{C_{-i}}=C_{g_\lambda}(\lambda+C_{-i})\subset I_{\mathbf{X}}$, and $\mathscr{D}_{C_{-i}}$ is $\sigma(\mathbf{X}; \mathscr{F})$-dense in $\mathbf{X}$, one gets easily that $(\lambda+\overline{C_{-i}})C_{g_\lambda}=I_{\mathbf{X}}$. Consequently $\lambda+\overline{C_{-i}}$ is invertible and $(\lambda+\overline{C_{-i}})^{-1}=C_{g_\lambda}\in\mathscr{B}_{\mathscr{F}}(\mathbf{X})$.

A first consequence of Theorem 2.2 is the following unicity result:

Corollary 2.3. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *and* D *two* $\sigma(\mathbf{X}; \mathscr{F})$-*continuous one-parameter cosine families of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *If* $\overline{C_{-i}}\subset\overline{D_{-i}}$, *then* $C=D$.

Proof. By Theorem 2.2 we have for each $s\in\mathbf{R}-\{0\}$

$$\left(\frac{e^{2s}+1}{2e^s}+\overline{C_{-i}}\right)^{-1}=\int_{-\infty}^{+\infty}\frac{e^s}{i(e^{2s}-1)}\cdot\frac{e^{its}-e^{-its}}{\operatorname{sh}\pi t}C_t\,dt,$$

that is,

$$\frac{e^{2s}-1}{2e^s}\left(\frac{e^{2s}+1}{2e^s}+\overline{C_{-i}}\right)^{-1}=\int_{-\infty}^{+\infty}\frac{\sin ts}{\operatorname{sh}\pi t}C_t\,dt,\quad s\in\mathbf{R}.$$

Similarly,

$$\frac{e^{2s}-1}{2e^s}\left(\frac{e^{2s}+1}{2e^s}+\overline{D_{-i}}\right)^{-1}=\int_{-\infty}^{+\infty}\frac{\sin ts}{\operatorname{sh}\pi t}D_t\,dt,\quad s\in\mathbf{R}.$$

Since $\overline{C_{-i}}\subset\overline{D_{-i}}$ implies $\left(\frac{e^{2s}+1}{2e^s}+\overline{C_{-i}}\right)^{-1}=\left(\frac{e^{2s}+1}{2e^s}+\overline{D_{-i}}\right)^{-1}$ for every $s\in\mathbf{R}$, the above considerations yield

$$\int_{-\infty}^{+\infty}\frac{\sin ts}{\operatorname{sh}\pi t}C_t\,dt=\int_{-\infty}^{+\infty}\frac{\sin ts}{\operatorname{sh}\pi t}D_t\,dt,\quad s\in\mathbf{R}.$$

Using the inequality $|\sin\alpha-\sin\beta|\leqq|\alpha-\beta|$, $\alpha,\beta\in\mathbf{R}$, and the Lebesgue dominated convergence theorem, it is easy to see that one can differentiate with respect to s under the sign of integration and we get

$$\int_{-\infty}^{+\infty}\cos ts\frac{t}{\operatorname{sh}\pi t}C_t\,dt=\int_{-\infty}^{+\infty}\cos ts\frac{t}{\operatorname{sh}\pi t}D_t\,dt.$$

In other words, for each $x\in\mathbf{X}$ and $\varphi\in\mathscr{F}$, the integrable continuous even functions

$$\mathbf{R}\ni t\to\frac{t}{\operatorname{sh}\pi t}\langle C_t(x),\varphi\rangle\quad\text{and}\quad\mathbf{R}\ni t\to\frac{t}{\operatorname{sh}\pi t}\langle D_t(x),\varphi\rangle$$

have equal Fourier cosine transforms, so they coincide. Consequently $C_t=D_t$; $t\in\mathbf{R}$.

By Corollary 2.3, $\overline{C_{-i}}$ determines C uniquely. A second consequence of Theorem 2.2 is an invariance result:

Corollary 2.4. *Let $(\mathbf{X},\mathscr{F})$ be a dual pair of Banach spaces, C a $\sigma(\mathbf{X},\mathscr{F})$-continuous one-parameter cosine family of 0 exponential type in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ and $\mathbf{Y}$ a $\sigma(\mathbf{X},\mathscr{F})$-closed linear subspace of $\mathbf{X}$. If there exists some $\lambda_0\in\mathbf{C}\setminus(-\infty,-1]$ with $(\lambda_0+\overline{C_{-i}})^{-1}\mathbf{Y}\subset\mathbf{Y}$ then $C_t\mathbf{Y}\subset\mathbf{Y}$, $t\in\mathbf{R}$.*

Proof. If $\lambda\in\mathbf{C}$ and $|\lambda-\lambda_0|<\|(\lambda_0+\overline{C_{-i}})^{-1}\|^{-1}$, then $(\lambda+\overline{C_{-i}})^{-1}$ exists and

$$(\lambda+\overline{C_{-i}})^{-1}=\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k((\lambda_0+\overline{C_{-i}})^{-k-1}\in\mathscr{B}_{\mathscr{F}}(\mathbf{X}),$$

where the series converges in the operator norm. Thus, for such λ, we have $(\lambda+\overline{C_{-i}})^{-1}\mathbf{Y}\subset\mathbf{Y}$. Let $y\in\mathbf{Y}$ be arbitrary and $\varphi\in\mathscr{F}$ such that $\langle z,\varphi\rangle=0$; $z\in\mathbf{Y}$. By the first part of the proof, the analytic function

$$\mathbf{C}\setminus(-\infty,-1]\ni\lambda\to\langle(\lambda+\overline{C_{-i}})^{-1}(y),\varphi\rangle$$

vanishes on some neighbourhood of λ_0, hence it vanishes identically. Using

Theorem 2.2 similarly as in the proof of Corollary 2.3, we get successively

$$\int_{-\infty}^{+\infty} \frac{\sin ts}{\operatorname{sh} t} \langle C_t(y), \varphi\rangle dt = 0, \quad s\in\mathbf{R},$$

$$\int_{-\infty}^{+\infty} \cos ts \frac{t}{\operatorname{sh} \pi t} \langle C_t(y), \varphi\rangle dt = 0, \quad s\in\mathbf{R},$$

$$\langle C_t(y), \varphi\rangle = 0, \quad t\in\mathbf{R}.$$

By the Hahn—Banach theorem we conclude that $C_t\mathbf{Y}\subset\mathbf{Y}$, $t\in\mathbf{R}$.

Corollary 2.5. *Let* $(\mathbf{X}, \mathcal{F})$ *be a dual pair of Banach spaces,* C *a* $\sigma(\mathbf{X}, \mathcal{F})$*-continuous one-parameter cosine family of* 0 *exponential type in* $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$ *and* $\mathbf{Y}$ *and* $\mathbf{Z}$ *two* $\sigma(\mathbf{X}, \mathcal{F})$*-closed linear subspaces of* $\mathbf{X}$. *If* $(\overline{C_{-i}})^{-1}\mathbf{Y}\subset\mathbf{Z}$, $(\overline{C_{-i}})^{-1}\mathbf{Y}\subset\mathbf{Z}$, *then* $\mathbf{Y}=\mathbf{Z}$.

Proof. Let $x\in\mathcal{D}_{C_{-2i}}$. By Lemma 1.1 $2C_sC_{-i}(x)=C_{s+i}(x)+C_{s-i}(x)$, $s\in\mathbf{R}$, so that the mapping

$$\{\zeta\in\mathbf{C};\ |\operatorname{Im}\zeta|\leqq 1\}\ni\zeta\to C_{\zeta+i}(x)+C_{\zeta-i}(x)\in\mathbf{X},$$

which is $\sigma(\mathbf{X}, \mathcal{F})$-regular, extends

$$\mathbf{R}\ni s\to 2C_sC_{-i}(x)\in\mathbf{X}.$$

Thus $C_{-i}(x)\in\mathcal{D}_{C_{-i}}$, $2C_{-i}C_{-i}(x)=x+C_{-2i}(x)$, that is $I_{\mathbf{X}}+C_{-2i}\subset 2(C_{-i})^2\subset 2(\overline{C_{-i}})^2$. But $(\overline{C_{-i}})^2$ is $\sigma(\mathbf{X}, \mathcal{F})$-closed, hence $I_{\mathbf{X}}+\overline{C_{-2i}}\subset 2(\overline{C_{-i}})^2$ and $(I_{\mathbf{X}}+\overline{C_{-2i}})^{-1}=$ $=2^{-1}(\overline{C_{-i}})^{-2}$. From the last equality, we get

$$(I_{\mathbf{X}}+\overline{C_{-2i}})^{-1}\mathbf{Y} = 2^{-1}(\overline{C_{-i}})^{-2}\mathbf{Y} \subset 2^{-1}(\overline{C_{-i}})^{-1}\mathbf{Z} \subset \mathbf{Y};$$

$$(I_{\mathbf{X}}+\overline{C_{-2i}})^{-1}\mathbf{Z} = 2^{-1}(\overline{C_{-i}})^{-2}\mathbf{Z} \subset 2^{-1}(\overline{C_{-i}})^{-1}\mathbf{Y} \subset \mathbf{Z}.$$

Since $\overline{C_{-2i}}$ is the analytic generator of the cosine family

$$\mathbf{R}\ni t\to C_{2t}\in\mathcal{B}_{\mathcal{F}}(\mathbf{X}),$$

by Corollary 2.4 it follows that $C_t\mathbf{Y}\subset\mathbf{Y}$, $C_t\mathbf{Z}\subset\mathbf{Z}$, $t\in\mathbf{R}$. In particular, by Lemma 1.3 we have

$$\overline{\mathcal{D}_{C_{-i}}\cap\mathbf{Y}}^{\sigma(\mathbf{X},\mathcal{F})} = \mathbf{Y}, \quad \overline{\mathcal{D}_{C_{-i}}\cap\mathbf{Z}}^{\sigma(\mathbf{X},\mathcal{F})} = \mathbf{Z}.$$

Using now the invariance of $\mathbf{Y}$ under the action of C and the Hahn—Banach theorem, we deduce successively that $C_{-i}(y)\in\mathbf{Y}$ and $y=(\overline{C_{-i}})^{-1}C_{-i}(y)\in(\overline{C_{-i}})^{-1}\mathbf{Y}\subset\mathbf{Z}$ holds for each $y\in\mathcal{D}_{C_{-i}}\cap\mathbf{Y}$. Thus

$$\mathbf{Y} = \overline{\mathcal{D}_{C_{-i}}\cap\mathbf{Y}}^{\sigma(\mathbf{X},\mathcal{F})} \subset \mathbf{Z}.$$

One obtains similarly also the inclusion $\mathbf{Z}\subset\mathbf{Y}$.

3. Connections with one-parameter groups of operators

Let $(\mathbf{X}, \mathcal{F})$ be a dual pair of Banach spaces. We recall that the *analytic extension* U_z of a $\sigma(\mathbf{X}, \mathcal{F})$*-continuous one-parameter group*

$$U: \mathbf{R} \ni t \to U_t \in \mathcal{B}_{\mathcal{F}}(\mathbf{X})$$

at $z \in \mathbf{C}$ is defined by

$$(x, y) \in \text{graph } U_z \Leftrightarrow \mathbf{R} \ni t \to U_t x \in \mathbf{X} \text{ has a } \sigma(\mathbf{X}, \mathcal{F})\text{-regular extension on the strip } D_z \text{ whose value at } z \text{ is } y$$

and U_z is a $\sigma(\mathbf{X}, \mathcal{F})$-closed and $\sigma(\mathbf{X}, \mathcal{F})$-densely defined linear operator in $\mathbf{X}$ ([3], Section 2). U_{-i} is called the *analytic generator* of U and provided that U is *of* 0 *exponential type,* that is,

$$\lim_{|t| \to \infty} \frac{1}{|t|} \|\ln U_t\| \leqq 0,$$

it uniquely determines U ([3], Section 4).

Proposition 3.1. *Let* $(\mathbf{X}, \mathcal{F})$ *be a dual pair of Banach spaces and* U *a* $\sigma(\mathbf{X}, \mathcal{F})$*-continuous one-parameter group in* $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$. *Then the formula*

$$C_t = \frac{1}{2} (U_t + U_{-t})$$

defines a $\sigma(\mathbf{X}, \mathcal{F})$*-continuous one-parameter cosine family in* $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$ *and*

$$\overline{C_z}^{\sigma(\mathbf{X}, \mathcal{F})} = \frac{1}{2} \overline{(U_z + U_{-z})}^{\sigma(\mathbf{X}, \mathcal{F})}, \quad z \in \mathbf{C}.$$

Proof. It is easy to verify that C is a $\sigma(\mathbf{X}, \mathcal{F})$ one-parameter cosine family in $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$.

From the definition of the analytic extensions of U, respectively C, it follows immediately that $(1/2)(U_z + U_{-z}) \subset C_z$. Thus, it remains to prove only the inclusion

$$C_z \subset \frac{1}{2} \overline{(U_z + U_{-z})}^{\sigma(\mathbf{X}, \mathcal{F})}.$$

Let $x \in \mathcal{D}_{C_z}$ be arbitrary and $f_\delta = \sqrt{\dfrac{\delta}{\pi}}\, e^{-\delta t^2}$, $\delta > 0$, $t \in \mathbf{R}$; then, by Lemma 1.3, we have

$$C_{f_\delta} C_z(x) = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t(x)\, dt =$$

$$= \frac{1}{2} \left(\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} U_t(x)\, dt + \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t+z)^2} U_t(x)\, dt \right).$$

Since

$$\mathbf{C} \ni \zeta \to \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}} e^{-\delta(t-\zeta)^2} U_t(x)\, dt \in \mathbf{X}$$

is an entire extension of

$$\mathbf{R} \ni s \to U_s U_{f_\delta}(x) \in \mathbf{X}$$

where

$$U_{f_\delta} = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}} U_t\, dt \in \mathcal{B}_{\mathcal{F}}(\mathbf{X}),$$

it follows that

$$C_{f_\delta} C_z(x) = \frac{1}{2}(U_z + U_{-z})_{f_\delta}(x), \quad \delta > 0.$$

Finally, since

$$\sigma(\mathbf{X}, \mathcal{F}) - \lim_{\delta \to +\infty} U_{f_\delta}(x) = x, \quad \sigma(\mathbf{X}, \mathcal{F}) - \lim_{\delta \to +\infty} C_{f_\delta} C_z(x) = C_z(x),$$

we conclude that x belongs to the domain of $(1/2)\overline{(U_z+U_{-z})}^{\sigma(\mathcal{F},\mathbf{X})}$ and that $(1/2)\overline{(U_z+U_{-z})}^{\sigma(\mathbf{X},\mathcal{F})}(x) = C_z(x)$.

In particular, if U is of 0 exponential type, then by Corollary 2.3, $\overline{U_{-i}+U_i}$ uniquely determines the "cosine part" $t \to U_t + U_{-t}$ of V. The "cosine part" of U has the advantage that the spectrum of its analytic generator is always included in $[1, +\infty)$, while the spectrum of U_{-i} is quite frequently $= \mathbf{C}$ (see [4]); this motivates the interest of cosine families in handling one-parameter groups of operators.

Concerning applications, we restrict ourselves to a proof of the following result (see [9] and [5], Th. 4.1):

Theorem 3.2. *Let* $\mathbf{H}$ *be a complex Hilbert space and* C *a weakly continuous one-parameter cosine family of* 0 *exponential type of self-adjoint linear operators on* $\mathbf{H}$. *Then there exists an injective, positive, self-adjoint operator* B *in* $\mathbf{H}$ *such that*

$$C_t = \frac{1}{2}(B^{it} + B^{-it}) = \cos(t \ln B), \quad t \in \mathbf{R}.$$

Proof. By Theorem 2.2, $\sigma(\overline{C_{-i}}) \subset [1, +\infty)$ and

$$(\overline{C_{-i}})^{-1} = \int_{-\infty}^{+\infty} \frac{1}{e^{\frac{\pi}{2}t} + e^{-\frac{\pi}{2}t}} C_t\, dt;$$

thus $\overline{C_{-i}}$ is self-adjoint and $\overline{C_{-i}} \geqq I_{\mathbf{H}}$. It follows that $B = \overline{C_{-i}} + ((\overline{C_{-i}})^2 - I_{\mathbf{H}})^{1/2}$ is an injective, positive, self-adjoint linear operator in $\mathbf{H}$ and

$$B^{-1} = \overline{\overline{C_{-i}} - ((\overline{C_{-i}})^2 - I_{\mathbf{H}})^{1/2}} \in \mathcal{B}(\mathbf{H})$$

(see, for example, [10], Section 128).

Now, the formula $U_t=B^{it}$, $t\in\mathbf{R}$ defines a strongly continuous one-parameter group of unitaries on $\mathbf{H}$ and $U_{-i}=B$ ([3], Th. 6.1). By Proposition 3.1 the cosine families C and $\mathbf{R}\ni t\to(1/2)(U_t+U_{-t})$ have equal analytic generators, so by Corollary 2.3

$$C_t=\frac{1}{2}(U_t+U_{-t}),\quad t\in\mathbf{R}.$$

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Three-element groupoids with minimal clones

B. CSÁKÁNY

To Professor Béla Szőkefalvi-Nagy on his seventieth birthday

A set of finitary operations on a set M is called a *clone* on M if it is closed under composition and contains all projections. The clones on a finite set M form an atomic lattice whose atoms are called *minimal clones*. The set of all term functions (polynomials in the terminology of [5]) of any algebra $\langle M; F\rangle$ is a clone on M. In this paper we give a complete list of those essentially distinct three-element algebras with one essentially binary operation whose clones of term functions are minimal.

The lattice of all clones on a finite set M is also coatomic, and the coatoms are called maximal clones. The knowledge of all maximal clones on M provides a method for deciding whether an algebra $\langle M; F\rangle$ is primal. The maximal clones on a two-element set, on a three-element set, and on any finite set have been determined in [9], [6], and [11], respectively. By the Galois connection between operations and relations on a finite set (see [4], [1]), the knowledge of all minimal clones on M enables us to decide whether a set of (finitary) relations on M generates all relations on M (in the sense of [1]). The minimal clones on a two-element set are determined in [9]; however, for sets consisting of more than two elements the problem of listing the minimal clones ([10], Problem 12) is open.

Our result may be considered as a first step towards the solution of this problem. Indeed, the complete description of the maximal clones on a three-element set suggests how the maximal clones on a finite set can behave in general, and the same may be expected for minimal clones. On the other hand, it is known ([10], p. 115) that any minimal clone on a three-element set is generated by an essentially at most ternary operation. The unary case is trivial, and here we settle the binary case.

Throughout this paper, $\mathbf{n}$ denotes the set $\{0, 1, \ldots, n-1\}$. For the sake of simplicity, we consider operations on the base set $\mathbf{3}$ only and, for brevity, we call them *functions*. The symbol $[f]$ stands for the clone generated by the function f (i.e.,

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consisting of all term functions of $\langle 3; f\rangle$). Instead of $g\in[f]$ we write also $f\to g$; we say in this case that f *produces* g. Projections will often be referred to as *trivial functions*.

We start with two basic observations (see [10], Ch. 4.4):

(α) *A clone* C *is minimal iff it contains a non-trivial function, and* $f\to g$ *for any non-trivial* $f, g\in C$.

(β) *An essentially at least binary function* f *generating a minimal clone is idempotent* (i.e., $f(x, \dots, x)=x$ holds identically).

By (β), we have to consider idempotent functions only. Such a functions has a Cayley table of form

$$(*)\qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & n_5 & n_4 \\ 1 & n_3 & 1 & n_2 \\ 2 & n_1 & n_0 & 2 \end{array}$$

where $n_i\in\mathbf{3}$ $(i=0, \dots, 5)$. The function defined by $(*)$ will be denoted by the integer $\sum_{i=0}^{5} 3^i n_i$. Thus, the functions we study will be numbered by 0, 1, ..., 728. E.g., 44 is the first binary projection (i.e., the function $f(x, y)=x$), and 424 is the second one. We shall denote our functions multiplicatively, with a subscript indicating the number of the considered function; e.g., we shall write $((xy)x)_{728}$ instead of $(x\,728\,y)728\,x$. For $f, g\in\mathbf{729}$, $f\cong g$ or $f\cong_a g$ means that $\langle\mathbf{3}; f\rangle$ and $\langle\mathbf{3}; g\rangle$ are

0 (3)	17	48	94 (6)	130 (6)
1	21	49	95	132
3	22	50	96	135 (6)
4	23	52 (6)	97	136
5	24	57 (6)	104 (3)	138
6	25	58	105	139
7	26 (6)	59	106	140 (6)
8	30	67	108	141
10 (6)	31	68 (6)	109 (6)	142
11 (6)	32	76 (6)	110	144
12	33 (6)	84 (6)	111 (6)	150 (4)
13	34	85	113	156 (6)
14	35	86	126	178 (2)
15	39 (6)	87	127	624 (1)
16	44 (2)	88	129	

Table 1

isomorphic or anti-isomorphic, respectively. The functions f and g are said to be *essentially distinct* if neither $f \cong g$ nor $f \cong_a g$ holds. In other words, the permutations of **3** and the dualization of functions generate a 12 element permutation group A on **729**, and f is essentially distinct from g iff they belong to distinct orbits of A.

For our aim, it is sufficient to study one representative from each orbit as the property of generating minimal clone is preserved under isomorphism and anti-isomorphism. We represent each orbit by its least element. In Table 1 we list the full system of representatives; the number in parentheses is the number of functions in the represented orbit if it does not equal 12.

Now we are ready to formulate the result announced above.

Theorem. *Every three-element groupoid with essentially binary operation having a minimal clone of term functions is isomorphic or anti-isomorphic to exactly one of the following twelve groupoids:*

$$\langle \mathbf{3}; f\rangle \text{ with } f \in M_2 = \{0, 8, 10, 11, 16, 17, 26, 33, 35, 68, 178, 624\}.$$

Proof. A three-element groupoid with the properties in the Theorem is, by (β), idempotent, and hence is isomorphic or anti-isomorphic to exactly one $\langle \mathbf{3}; f\rangle$ where f is an entry of Table 1. Therefore it is sufficient to prove that the functions listed in the Theorem generate minimal clones on **3** while the remaining functions in Table 1 do not. The second job is mainly of computational character, and we shall do it at first. We apply the following simple fact:

(γ) *For any function f, if there exist a clone C and a non-trivial function $g \in C$ such that $f \notin C$ and $f \to g$, then $[f]$ is not minimal.*

Put $f=3$ and let C_{ij} be the clone of all functions preserving the set $\{i, j\} \subseteq \mathbf{3}$. Then $3 \notin C_{02}$ and $3 \to ((xy)y)_3 = 0 \in C_{02}$. Thus, by ($\gamma$), [3] is not minimal. The same consideration (with C_{02} and $(xy)y$) is applicable also for the functions 4 ($((xy)y)_4 = 1$) 5 (2), 12 (9), 13 (10), 14 (11), 21 (18), 22 (19), 23 (20), 57 (33), 58 (34), 59 (35), 67 (43), 76 (52), 84 (9), 85 (10), 86 (11), 87 (15), 88 (16), 104 (182), 105 (186), 106 (187), 108 (36), 109 (37), 110 (38), 126 (207), 127 (208), 129 (210), 132 (213), 135 (9), 136 (10), 138 (42), 139 (43), 141 (69), 142 (70), and 156 (213). Similarly, by the help of $x(xy)$ we can settle the functions 30 ($(x(xy))_{30} = 37 \in C_{02}$), 31 (8), 39 (37), 48 (47), 50 (53), 95 (17), 96 (17), 97 (16), 111 (37), and 140 (26), and by $(xy)x$ the function 150 (178). Further, C_{12} and $x(xy)$ take care of 15 (17), 24 (26), 34 (43), while C_{12} with $x(yx)$ and $(xy)y$ settles 32 (40) and 113 (41), respectively. Finally, taking C_{01} and $(xy)y$, we can cast off also 144 (90).

A binary function satisfying the identity $(xy)(uv) = (xu)(yv)$ is called *medial*. If f is medial and $f \to g$, then the function g is also medial (cf. Prop. III. 3.2 in [2]). Thus, if a non-medial g produces a non-trivial medial f then $[g]$ is not minimal by (α). This is the case for $g=49$ and $f=41=(x(yx))_{49}$ as $((12)(02))_{49} = 1 \neq 2 =$

$=((10)(22))_{49}$ while one can check the mediality of 41 immediately. Therefore, [49] is not minimal.

In order to show that $[f]$ is not minimal for $f=1, 6, 25$, we apply (α) as follows. Observe that $1\to 0$; namely, $((xy)x)_1=0$; on the other hand, the binary term functions of $\langle \mathbf{3}; 0\rangle$ are 44, 424, and $(xy)_0=0$, i.e., 0 does not produce 1. Further, $(x(xy))_6=8$, but not $8\to 6$, as the binary term functions of $\langle \mathbf{3}; 8\rangle$ are 44, 424, 8, and 180. Similarly, $(x(yx))_{25}=17$, but the binary term functions of $\langle \mathbf{3}; 17\rangle$ are only 44, 424, 17, 181.

Clearly, $[f]$ is not minimal if there exists a non-trivial g such that $f\to g$ and $[g]$ is not minimal. Hence it follows that $[f]$ is not minimal for $f=7, 94$, and 130. Indeed, $((xy)x)_7=6$, $((xy)y)_{90}=91\cong_a 13$, $((xy)y)_{130}=211\cong_a 49$; and we have already shown that [6], [13], and [49] are not minimal.

[44] is the clone of all projections. Thus it remains to show that [52] is not minimal. Put $q(x, y, z)=((xy)(zx))_{52}$. Then $52\to q$ and q is not trivial, as $q(1, 0, 0)=1$ and $q(1, 2, 0)=2$. However, $q\to 52$ is not valid, since $q(x, x, y)=$ $=q(x, y, x)=q(x, y, y)=x$ and hence $[q]$ contains no essentially binary function. Now, by (α), [52] is not minimal.

Next we prove that the functions in M_2 generate minimal clones. Their Cayley tables can be seen here:

$$0:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&0\\ 0&0&2\end{array}\qquad 8:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&0\\ 2&2&2\end{array}\qquad 10:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 0&1&2\end{array}\qquad 11:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 0&2&2\end{array}$$

$$16:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 2&1&2\end{array}\qquad 17:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 2&2&2\end{array}\qquad 26:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&2\\ 2&2&2\end{array}\qquad 33:\ \begin{array}{|ccc}\hline 0&0&0\\ 1&1&0\\ 2&0&2\end{array}$$

$$35:\ \begin{array}{|ccc}\hline 0&0&0\\ 1&1&0\\ 2&2&2\end{array}\qquad 68:\ \begin{array}{|ccc}\hline 0&0&0\\ 2&1&1\\ 1&2&2\end{array}\qquad 178:\ \begin{array}{|ccc}\hline 0&0&2\\ 0&1&1\\ 2&1&2\end{array}\qquad 624:\ \begin{array}{|ccc}\hline 0&2&1\\ 2&1&0\\ 1&0&2\end{array}$$

The functions 0 and 10 are semilattice operations, hence they generate minimal clones (cf. [10], 4.4.4). It was proved by Płonka that 624 ($=2x+2y \bmod 3$) generates a minimal clone (see [8]). Demetrovics, Hannák and Marčenkov proved that [178] is also minimal ([3]; for a proof, see [7]).

Now we can prove that for each $f\in M_2$ any non-trivial binary $g\in[f]$ produces f. We do this by establishing the following property of any function in M_2:

A. *It produces no essentially binary function except itself and its dual.*

Indeed, a trivial computation shows that $x(xy)$, $x(yx)$, $(xy)x$, $(xy)y$, $(xy)(yx)\in$ $\in\{x, y, xy, yx\}$ whenever multiplication means anyone of the functions in M_2.

Hence $g\in\{x, y, xy, yx\}$ follows by induction on the depth of the shortest f-term representing g. Thus $g\to f$ provided g is not a projection.

It remains to prove that, for each $f\in M_2$, a non-trivial $g\in[f]$ *of arbitrary arity* does produce f. In view of the preceding paragraph it is enough to show only that g produces a non-trivial binary function. For the cases $f=8, 11, 16, 17, 26$ (and also for the known cases $f=0, 10, 178$) this can be done by the following argument. The restriction of f to $\mathbf{2}=\{0, 1\}$ is the minimum function $\wedge$; hence any term function g of $\langle \mathbf{3}; f\rangle$ has the form $x_1\wedge\ldots\wedge x_k$ when restricted to $\mathbf{2}$. Identifying all variables but one of g we obtain a binary function which is not a projection because its restriction to $\mathbf{2}$ is the minimum function again.

Our final task is to prove that 33, 35, and 68 generate minimal clones, too. First we check that each of them enjoys also the property

B. *It turns into a projection when restricted to a suitable two-element set.*

The two-element set in B is $\{0, 1\}$ for 33 and 35, and it is $\{1, 2\}$ for 68.

A function is called a *semi-projection* if it is not a projection and it turns into the same projection when any two of its variables are identified.

Lemma. *An idempotent function with properties* A *and* B *generates a minimal clone provided it produces no ternary semi-projection.*

Indeed, suppose that f has properties A and B, but $[f]$ is not minimal. Then there exists a non-trivial function $g\in[f]$ which does not produce f. The idempotence of f and A imply that g is at least ternary. We show that g produces an essentially ternary function. A well-known theorem of Świerczkowski ([12]; see also [5], p. 206) asserts that an at least three-element algebra with independent base set has only trivial operations. Hence it follows that any non-trivial function on $\mathbf{3}$ produces an at most ternary non-trivial function (cf. also [9], 4.4.7). In particular, g produces such a function h, which is, again by the idempotence of f and property A, essentially ternary. Let us identify two variables of h; then, by A, we always obtain a projection. Assume that two different identifications of variables furnish different projections; then the same is valid for the restriction of h to the two-element subset of $\mathbf{3}$ in property B. But this is impossible, as for a function composed from projections any identification of two variables gives the same projection. Hence h turns into the same projection under identification of any two of its variables, and, as it is not a projection, it has to be a semi-projection. We proved that f produces a semi-projection, which was needed.

In virtue of the lemma, it is enough to prove that none of 33, 35, and 68 does produce a ternary semi-projection. In these proofs, the actually considered function will be denoted as multiplication (no subscript will be used); we write pqr instead of $(pq)r$; finally, we write $f(x_1, \ldots, x_k) = x_i\ldots$ to indicate that x_i is the first-from-left entry in the term f. In this case $f(x_1, \ldots, x_k)$ can be uniquely written in the form $x_i \cdot f_1(x_1, \ldots, x_k) \ldots f_n(x_1, \ldots, x_k)$. We shall use the Cayley tables of the studied functions without further reference.

Case of 33. We need the following identities of $\langle \mathbf{3}; 33\rangle$:

(1) $$xx = x(xy) = x(yx) = x, \quad (xy)x = (xy)(yx) = (xy)y = xy.$$

Suppose that $f(x, y, z)$ is a ternary 33-term of minimal length among those which are semi-projections: let $f(x, x, y) = f(x, y, y) = f(x, y, x) = x$ and $f(a, b, c) \neq a$ for suitable $a, b, c \in \mathbf{3}$. First suppose $a=0$. Let $f(x, y, z) = f_1(x, y, z) \cdot f_2(x, y, z)$; then by (1), $f_1(x, x, y) = f_1(x, y, y) = f_1(x, y, x) = x$, and, by the minimality of f, identically $f_1(x, y, z) = x$, i.e. $f(x, y, z) = x \cdot f_2(x, y, z)$, whence $f(a, b, c) = 0 \cdot f_2(a, b, c) = 0 = a$, a contradiction. Therefore, $a \neq 0$.

Let, e.g., $a=1$, $b=2$, $c=0$. Now f is a 33-term with

(2) $$f(x, y, y) = y, \quad f(1, 2, 0) \neq 1.$$

We shall be ready if we prove that for any 33-term g satisfying the requirements in (2) there exists a shorter 33-term also satisfying (2). Observe that $g(x, y, z) = x\ldots$, otherwise (1) implies $g(x, y, y) = y$ or $g(x, y, y) = yx$. Thus $g(x, y, z) = x \cdot d_1(x, y, z) \ldots d_n(x, y, z)$, and $g(x, y, y) = x \cdot d_1(x, y, y) \ldots d_n(x, y, y)$. Hence, by (1), $d_i(x, y, y) \neq y$ for every i. On the other hand, $g(1, 2, 0) = 1 \cdot d_1(1, 2, 0) \ldots d_n(1, 2, 0) \neq 1$, showing that $d_j(1, 2, 0) = 2$ for at least one j. Now, $d_j(x, y, z) = y\ldots = y \cdot h_1(x, y, z) \ldots h_m(x, y, z)$. Using (1), we infer the existence of a k with $h_k(x, y, y) = x$. As $2 \cdot h_1(1, 2, 0) \ldots h_m(1, 2, 0) = d_j(1, 2, 0) = 2$, we have $h_i(1, 2, 0) \neq 1$ for every i. In particular, $h_k(1, 2, 0) \neq 1$, i.e. h_k is the 33-term we required. For $\langle a, b, c\rangle = \langle 1, 0, 2\rangle$, the same argument works. As (12) is an automorphism of $\langle \mathbf{3}; 33\rangle$, we do not have to deal with the case $a=2$ separately.

Case of 35. The two-variable identities of $\langle \mathbf{3}; 35\rangle$ are

(3) $$xx = x(xy) = x, \quad x(yx) = (xy)x = (xy)y = (xy)(yx) = xy.$$

As in the preceding case, we obtain that if f is a ternary 35-term which is a semi-projection of minimal length, and $f(x, y, y) = x, f(a, b, c) \neq a$, then $f(x, y, z) = x \cdot f_2(x, y, z)$, and $a=1$. Furthermore, we have $f_2 = f_{21} \cdot f_{22}$ and $f_{21}(a, b, c) = 2$. Let, e.g., $\langle a, b, c\rangle = \langle 1, 2, 0\rangle$. Then $f_{21}(x, y, z) = y \cdot g_1(x, y, z) \ldots g_n(x, y, z)$. From $f_{21}(2, 0, 0) = 0 \cdot g_1(2, 0, 0) \ldots g_n(2, 0, 0) = 0$ it follows $f_{21}(x, y, y) = y$ or $f_{21}(x, y, y) = yx$. Thus, $f_2(x, y, y)$ equals $y \cdot f_{22}(x, y, y)$ or $y \cdot x \cdot f_{22}(x, y, y)$, hence $f_2(x, y, y) = y$ or $f_2(x, y, y) = yx$ by (3). In both cases, $f(x, y, y) = xy$, a contradiction. If $\langle a, b, c\rangle = \langle 1, 0, 2\rangle$, then $f_{21}(x, y, z) = z\ldots$ follows, and we can proceed similarly.

Case of 68. Again we need the two-variable identities of $\langle \mathbf{3}; 68\rangle$:

(4) $$xx = x(xy) = (xy)y = (xy)(yx) = x, \quad x(yx) = (xy)x = xy.$$

Let f be a ternary 68-term which is a semi-projection; let $f(x, y, y) = x$ and $f(a, b, c) \neq a$. From $f(1, 0, 0) = 1$ it follows $f(x, y, z) = x \cdot f_1(x, y, z) \ldots f_n(x, y, z)$. This implies $a \neq 0$; so first suppose, e.g., $\langle a, b, c\rangle = \langle 1, 0, 2\rangle$. Now $(1 \neq) f(1, 0, 2) = = 1 \ldots = 2$. At the same time, $f(1, 0, 2) = 1 \cdot f_1(1, 0, 2) \ldots f_n(1, 0, 2)$. Hence there exists an odd number of f_i's such that $f_i(1, 0, 2) = 0$. The last equality means $f_i(x, y, z) = y \ldots$, therefore there is an odd number of f_i's whose first letter is y.

On the other hand, $f(x, y, z) = x \cdot f_1(x, y, x) \ldots f_n(x, y, x)$. The identities (4) show that there exists an even number of f_j's with $f_j(x, y, x) = y$ or $f_j(x, y, x) = yx$. Observe that $f_j(x, y, z) = y \ldots$ implies $f_j(x, y, x) = y$ or $f_j(x, y, x) = yx$, and $f_j(x, y, z) = x \ldots$ or $f_j(x, y, z) = z \ldots$ implies $f(x, y, x) = x$ or $f(x, y, x) = xy$ by (4). Hence we have an even number of f_j's with first letter y. This contradiction refutes $\langle a, b, c\rangle = \langle 1, 0, 2\rangle$. Assuming $\langle a, b, c\rangle = \langle 1, 2, 0\rangle$, we obtain a similar contradiction for the number of f_i's with first letter z. As (12) is an automorphism of $\langle \mathbf{3}; 68\rangle$, we have also settled the case $a = 2$. Theorem is proved.

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A combinatorial proof of a theorem of P. Lévy on the local time

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Dedicated to Professor B. Szőkefalvi-Nagy on the occasion of his 70th birthday

1. Introduction

Let $\{W(t);\ t\geqq 0\}$ be a Wiener process and introduce the notations

$$M(t) = \sup_{0\leqq s\leqq t} W(s), \quad Y(t) = M(t) - W(t)$$

and for any Borel set A let

$$H(A, t) = \lambda\{s:\ s\leqq t, \quad W(s)\in A\}$$

be the occupation time of W where λ is the Lebesgue measure. It is well-known that $H(A, t)$ (for any fixed t) is a random measure absolutely continuous with respect to λ with probability 1. The Radon—Nikodym derivative of H is called the local time of W and it will be denoted by η i.e. $\eta(x, t)$ is defined by

$$H(A, t) = \int_A \eta(x, t)\, dx.$$

Finally let $\eta(0, t) = \eta(t)$.

A celebrated result of P. Lévy reads as follows (see for example KNIGHT [7], Theorem 5.3.7).

Theorem A. *We have*

$$\{Y(t), M(t);\ t\geqq 0\} \overset{\mathcal{D}}{=} \{|W(t)|, \eta(t);\ t\geqq 0\}$$

i.e. the finite dimensional distributions of the vector valued process $\{(Y(t), M(t));\ t\geqq 0\}$ *are equal to the corresponding distributions of* $\{(|W(t)|, \eta(t));\ t\geqq 0\}$.

A natural question arises: what is the analogue of Theorem A in the case of random walk. In order to formulate our problem precisely introduce the following notations.

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Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with $\mathbf{P}(X_1=1)=\mathbf{P}(X_1=-1)=1/2$ and let

$$S(0)=0, \quad S(n) = X_1+X_2+\ldots+X_n \quad (n = 1, 2, \ldots);$$
$$m(n) = \max_{1\leq k\leq n} S(k), \quad y(n) = m(n)-S(n),$$
$$\xi(0, n) = \xi(n) = \mathcal{N}\{k: k \leqq n, S(k) = 0\}$$

where $\mathcal{N}\{\cdot\}$ is the cardinality of the set in brackets. Now our question is: does Theorem A remain true if we replace $W(t)$, $Y(t)$, $M(t)$; $\eta(t)$ by $S(n)$; $y(n)$, $m(n)$, $\xi(n)$ respectively (and n runs over the integers). The answer of this question is negative. This fact can be seen from the following well-known

Theorem B.

$$\mathbf{P}\{\xi(2n) = k\} = \frac{1}{2^{2n-k}}\binom{2n-k}{n} \quad (k = 0, 1, 2, \ldots, n)$$

and

$$\mathbf{P}\{m(n) = k\} = \frac{1}{2^n}\binom{n}{\left[\frac{n-k}{2}\right]} \quad (k = 0, 1, 2, \ldots, n).$$

In spite of this disappointing fact we prove that Theorem A is "nearly true" for random walks. In fact we have

Theorem 1. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with $\mathbf{P}(X_1=1)=\mathbf{P}(X_1=-1)=1/2$ defined on a probability space $\{\Omega, S, \mathbf{P}\}$. Then one can define a sequence $Z_1, Z_2, \ldots$ of i.i.d.r.v.'s on the same probability space $\{\Omega, S, \mathbf{P}\}$ such that $\mathbf{P}(Z_1=1)=\mathbf{P}(Z_1=-1)=1/2$ and*

$$n^{-1/4-\varepsilon} d\big((\hat{y}(n), \hat{m}(n)), (|S(n)|, \xi(n))\big) \to 0 \quad a.s.$$

for any $\varepsilon>0$ where

$$S(0) = T(0) = 0, \;\; S(n) = X_1+X_2+\ldots+X_n, \;\; T(n) = Z_1+Z_2+\ldots+Z_n \;\; (n = 1, 2, \ldots),$$
$$\xi(n) = \mathcal{N}\{k:\ k \leqq n, \quad S(k) = 0\}, \quad \hat{m}(n) = \max_{1\leq k\leq n} T(k), \quad \hat{y}(n) = \hat{m}(n)-T(n)$$

and d is the Euclidean distance of the vectors i.e.

$$d((a_1, a_2), (b_1, b_2)) = ((b_2-a_2)^2+(b_1-a_1)^2)^{1/2}.$$

The proof of this Theorem is very elementary and will be presented in Section 2. In Section 3 we show that Theorem A can be obtained as a simple consequence of Theorem 1.

In Section 4 we show that replacing the number of roots of the random walk $S(n)$ of Theorem 1 by the number of crossing points of that walk we can obtain a much better rate than that of Theorem 1. In Section 5 as an application of Theorem A (or that of Theorem 1) we prove a Strassen-type law of iterated logarithm for local time.

2. Proof of Theorem 1

Using the notations of Theorem 1 we also introduce the following notations:

$$\begin{aligned}\varrho_1 &= \min\{i:\ i>0,\ S_i=0\},\\ \varrho_2 &= \min\{i:\ i>\varrho_1,\ S_i=0\},\\ &\ \vdots\\ \varrho_{l+1} &= \min\{i:\ i>\varrho_l,\ S_i=0\},\ \ldots,\end{aligned}$$

$$Z_j=\begin{cases}-X_1X_{j+1} & \text{if } 1\leqq j\leqq \varrho_1-1,\\ -X_{\varrho_1+1}X_{j+2} & \text{if } \varrho_1\leqq j\leqq \varrho_2-2,\\ \quad\vdots & \\ -X_{\varrho_l+1}X_{j+l+1} & \text{if } \varrho_l-(l-1)\leqq j\leqq \varrho_{l+1}-(l+1),\\ \quad\vdots & \end{cases}$$

The following lemma is immediately clear by the above definitions.

Lemma 1.

1°. $Z_1, Z_2, \ldots$ *is a sequence of i.i.d.r.v.'s with* $\mathbf{P}(Z_1=+1)=\mathbf{P}(Z_1=-1)=1/2$.

2°.
$$T(k)-T(\varrho_l-l)=\sum_{j=\varrho_l-(l-1)}^{k} Z_j=$$
$$=-X_{\varrho_l+1}\sum_{j=\varrho_l-(l-1)}^{k} X_{j+l+1}\begin{cases}\leqq 0 & \textit{if}\ \ \varrho_l-(l-1)\leqq k\leqq \varrho_{l+1}-l-3,\\ =0 & \textit{if}\ \ k=\varrho_{l+1}-l-2,\\ =1 & \textit{if}\ \ k=\varrho_{l+1}-(l+1).\end{cases}$$

3°. $$\xi(\varrho_{l+1})=l+1=T(\varrho_{l+1}-(l+1))=\hat{m}(\varrho_{l+1}-(l+1))=\hat{m}(\varrho_{l+1}-\xi(\varrho_{l+1}))$$
$$(l=0,1,2,\ldots).$$

4°. *For any* $\varrho_{l+1}\leqq n<\varrho_{l+2}$ *we have*

$$\varrho_{l+1}-\xi(\varrho_{l+1})\leqq n-\xi(n)<\varrho_{l+2}-\xi(\varrho_{l+1})=\varrho_{l+2}-\xi(\varrho_{l+2})+1,$$

hence

$$\xi(n)=\xi(\varrho_{l+1})=\hat{m}(\varrho_{l+1}-\xi(\varrho_{l+1}))\leqq \hat{m}(n-\xi(n))$$

and

$$\xi(n)=\xi(\varrho_{l+2})-1=\hat{m}(\varrho_{l+2}-\xi(\varrho_{l+2}))-1\geqq \hat{m}(n-\xi(n))-1$$

i.e.

$$|\xi(n)-\hat{m}(n-\xi(n))|\leqq 1\quad (n=1,2,\ldots).$$

5°. $$\hat{y}(k)=\begin{cases}|S(k+1)|-1 & \textit{if}\ \ 1\leqq k\leqq \varrho_1-2,\\ 0=|S(k)|-1 & \textit{if}\ \ k=\varrho_1-1\end{cases}$$

and

$$\hat{y}(k)=\begin{cases}|S(k+l)|-1 & \textit{if}\ \ \varrho_{l-1}-(l-1)\leqq k\leqq \varrho_l-(l+1),\\ 0=|S(k+l-1)|-1 & \textit{if}\ \ k=\varrho_l-l.\end{cases}$$

The following strong laws are known:

Lemma 2.

$$\limsup_{n\to\infty} \frac{\xi(n)}{(2n\log\log n)^{1/2}} = 1 \quad a.s. \tag{2.1}$$

(cf. KESTEN [6]).

$$\limsup_{n\to\infty} \frac{\hat{m}(n)-\hat{m}(n-a_n)}{\left(a_n \log \frac{n}{a_n}\right)^{1/2}} \leqq 2^{1/2} \quad a.s. \tag{2.2}$$

(cf. CSÖRGŐ—RÉVÉSZ [4], Theorem 3.1.1) *where* $a_n=((2+\varepsilon)n \log\log n)^{1/2}$, $\varepsilon\geqq 0$.

$$\limsup_{n\to\infty} \max_{0\leqq l\leqq b_n} \frac{|S(n+l)|-|S(n)|}{\left(2b_n \log \frac{n}{b_n}\right)^{1/2}} = 1 \quad a.s. \tag{2.3}$$

(cf. CSÖRGŐ—RÉVÉSZ [4], Theorem 3.1.1 and Remark 3.1.1) *where* $b_n=n^{1/2}\log\log n$. *Consequently*

$$\limsup_{n\to\infty} \frac{\hat{m}(n)-\hat{m}(n-\xi(n))}{(2n\log\log n)^{1/4}(\log n)^{1/2}} \leqq 2^{1/2} \quad a.s. \tag{2.4}$$

Further we have

$$\liminf_{n\to\infty} 4(n^{-2}\log\log n)\varrho_n = 1 \quad a.s. \tag{2.5}$$

(cf. MIJNHEER [9], p. 53 and RÉNYI [10] p. 236).

Remark. Theorem 3.1.1 of Csörgő—Révész [4] states that

$$\limsup_{n\to\infty} \sup_{0\leqq s\leqq a_n} \frac{T(n+s)-T(n)}{\left(a_n \log \frac{n}{a_n}\right)^{1/2}} = 2^{1/2} \quad \text{a.s.}$$

what easily implies (2.2). Applying Theorem A, CSÁKI—CSÖRGŐ—FÖLDES—RÉVÉSZ [1] also proved that

$$\limsup_{n\to\infty} \frac{\hat{m}(n)-\hat{m}(n-a_n)}{\left(a_n \log \frac{n}{a_n}\right)^{1/2}} = 1 \quad \text{a.s.}$$

(2.4) of Lemma 2 and 4° of Lemma 1 together imply

Lemma 3. $$\limsup_{n\to\infty} \frac{|\xi(n)-\hat{m}(n)|}{(2n\log\log n)^{1/4}(\log n)^{1/2}} \leqq \sqrt{2} \quad a.s.$$

For any positive integer k let $l=l(k)$ be defined by $\varrho_{l-1}-(l-1)\leqq k<\varrho_l-l$. Then by (2.5) of Lemma 2 we have

Lemma 4. $\lim_{n\to\infty} \frac{l(n)}{n^{1/2}\operatorname{loglog} n} = 0 \quad a.s.$

5° of Lemma 1, (2.3) of Lemma 2 and Lemma 4 together imply

Lemma 5. $\limsup_{n\to\infty} \frac{\big|\hat{y}(n)-|S(n)|\big|}{n^{1/4}((\log n)(\operatorname{loglog} n))^{1/2}} \leqq \sqrt{2} \quad a.s.$

Lemmas 3 and 5 together prove Theorem 1.

3. Proof of Theorem A

The proof of Theorem A is based on Theorem 1 and the following invariance principle

Theorem C (Révész [11]). *Let $\{W(t);\ t\geqq 0\}$ be a Wiener process defined on a probability space $\{\Omega, S, \mathbf{P}\}$. Then on the same probability space Ω one can define a sequence $X_1, X_2, \ldots$ of i.i.d.r.v.'s with $\mathbf{P}(X_1=+1)=\mathbf{P}(X_1=-1)=1/2$ such that*

$$\lim_{n\to\infty} n^{-1/4-\varepsilon}|\xi(n)-\eta(n)| = 0 \quad a.s.$$

and

$$\lim_{n\to\infty} n^{-1/4-\varepsilon}|S(n)-W(n)| = 0 \quad a.s.$$

for any $\varepsilon>0$.

In order to prove Theorem A it is enough to prove

Lemma 6.

$$A = \{Y(t_1), Y(t_2), \ldots, Y(t_n), M(t_1), M(t_2), \ldots, M(t_n)\} \stackrel{\mathcal{D}}{=}$$
$$\stackrel{\mathcal{D}}{=} \{|W(t_1)|, |W(t_2)|, \ldots, |W(t_n)|, \eta(t_1), \eta(t_2), \ldots, \eta(t_n)\} = B$$

provided that $0<t_1<t_2<\ldots<t_n\leqq 1$.

Applying the well-known formula $\left\{\frac{W(ct)}{\sqrt{c}};\ t\geqq 0\right\} \stackrel{\mathcal{D}}{=} \{W(t);\ t\geqq 0\}$ (for any $c>0$) one gets

Lemma 7. *For any $T>0$ we have*

$$A \stackrel{\mathcal{D}}{=} \left\{\frac{Y(t_1T)}{T^{1/2}}, \frac{Y(t_2T)}{T^{1/2}}, \ldots, \frac{Y(t_nT)}{T^{1/2}}, \frac{M(t_1T)}{T^{1/2}}, \frac{M(t_2T)}{T^{1/2}}, \ldots, \frac{M(t_nT)}{T^{1/2}}\right\}$$

and

$$B \overset{\mathcal{D}}{=} \left\{ \frac{|W(t_2 T)|}{\sqrt{T}}, \frac{|W(t_2 T)|}{\sqrt{T}}, \ldots, \frac{|W(t_n T)|}{\sqrt{T}}, \frac{\eta(t_1 T)}{\sqrt{T}}, \frac{\eta(t_2 T)}{\sqrt{T}}, \ldots, \frac{\eta(t_n T)}{\sqrt{T}} \right\}.$$

By Theorem C we have

Lemma 8. *One can define a random walk* $S(1), S(2), \ldots$ *on the probability space of* W *such that*

$$\frac{|W(t_i T)|}{\sqrt{T}} = \frac{|S(t_i T)|}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \ldots, n)$$

and

$$\frac{\eta(t_i T)}{\sqrt{T}} = \frac{\xi(t_i T)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \ldots, n).$$

Applying Theorem 1 we have

Lemma 9. *Given the random walk of Lemma* 8 *one can define another random walk* $T(1), T(2), \ldots$ *such that*

$$\frac{|S(t_i T)|}{\sqrt{T}} = \frac{\hat{y}(t_i T)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \ldots, n)$$

and

$$\frac{\xi(t_i T)}{\sqrt{T}} = \frac{\hat{m}(t_i T)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \ldots, n).$$

Applying again Theorem C we get

Lemma 10. *Given the random walk* $T(1), T(2), \ldots$ *of Lemma* 9 *one can define a Wiener process* $\{\overline{W}(t);\ t > 0\}$ *such that*

$$\frac{\hat{y}(t_i T)}{\sqrt{T}} = \frac{\overline{Y}(t_i T)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \ldots, n)$$

and

$$\frac{\hat{m}(t_i T)}{\sqrt{T}} = \frac{\overline{M}(t_i T)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \ldots, n)$$

where

$$\overline{M}(t) = \sup_{0 \leq s \leq t} \overline{W}(s) \quad \text{and} \quad \overline{Y}(t) = \overline{M}(t) - \overline{W}(t).$$

Lemmas 8 and 10 together imply

Lemma 11.

$$\left(\frac{|S(t_1 T)|}{\sqrt{T}}, \frac{|S(t_2 T)|}{\sqrt{T}}, \ldots, \frac{|S(t_n T)|}{\sqrt{T}}, \frac{\xi(t_1 T)}{\sqrt{T}}, \frac{\xi(t_2 T)}{\sqrt{T}}, \ldots, \frac{\xi(t_n T)}{\sqrt{T}} \right) =$$

$$= b(T) \overset{\mathcal{D}}{\Rightarrow} B \quad \text{as} \quad T \to \infty$$

and

$$\left(\frac{\hat{y}(t_1T)}{\sqrt{T}}, \frac{\hat{y}(t_2T)}{\sqrt{T}}, \ldots, \frac{\hat{y}(t_nT)}{\sqrt{T}}, \frac{\hat{m}(t_1T)}{\sqrt{T}}, \frac{\hat{m}(t_2T)}{\sqrt{T}}, \ldots, \frac{\hat{m}(t_nT)}{\sqrt{T}}\right) =$$

$$= a(T) \overset{\mathcal{D}}{\Rightarrow} A \quad as \quad T \to \infty.$$

By Lemma 9 the limit distributions of $a(T)$ and $b(T)$ cannot be different. Hence we have Lemma 6, and hence Theorem A.

4. Roots and crossings

Theorem 1 says that the vector $(|S(n)|, \xi(n))$ can be approximated by the vector $(\hat{y}(n), \hat{m}(n))$ in order $n^{1/4+\varepsilon}$ while Theorem C says that the vector $(\eta(n), W(n))$ can be approximated by the vector $(\xi(n), S(n))$ in the same order $n^{1/4+\varepsilon}$. It is natural to ask whether this order is the best possible or not. Unfortunately we do not know the answer of this question. However we can show that considering the number of crossings $\theta(n)$ instead of the number of roots $\xi(n)$ better rates can be achieved in Theorems 1 and C.

Let

(4.1) $$\theta(n) = \mathcal{N}\{k:\ k \leqq n,\ S(k-1)S(k+1) < 0\}$$

be the number of crossings. Then we have

Theorem 2. *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d.r.v.'s with* $\mathbf{P}(X_1=1)=$ $=\mathbf{P}(X_1=-1)=1/2$ *defined on a probability space* $\{\Omega, S, \mathbf{P}\}$. *Then one can define a sequence* $Z_1, Z_2, \ldots$ *of i.i.d.r.v.'s on the same probability space* $\{\Omega, S, \mathbf{P}\}$ *such that* $\mathbf{P}(Z_1=1)=\mathbf{P}(Z_1=-1)=1/2$ *and*

(4.2) $$|\hat{m}(n)-2\theta(n)| \leqq 1,$$

(4.3) $$\big|\hat{y}(n)-|S(n)|\big| \leqq 2$$

for $n=1, 2, \ldots$; *where*

$$S(0) = T(0) = 0, \quad S(n) = X_1+X_2+\ldots+X_n, \quad T(n) = Z_1+Z_2+\ldots+Z_n$$
$$(n = 1, 2, \ldots),$$
$$\hat{m}(n) = \max_{0\leqq k\leqq n} T(k), \quad \hat{y}(n) = \hat{m}(n)-T(n).$$

Proof. Let

$$\begin{aligned}
\tau_1 &= \min\{i:\ i > 0,\ \ S(i-1)S(i+1) < 0\},\\
\tau_2 &= \min\{i:\ i > \tau_1,\ S(i-1)S(i+1) < 0\},\\
&\ \vdots\\
\tau_{l+1} &= \min\{i:\ i > \tau_l,\ S(i-1)S(i+1) < 0\}, \ldots
\end{aligned}$$

and

$$Z_j = \begin{cases} -X_1X_{j+1} & \text{if } 1 \leqq j \leqq \tau_1, \\ X_1X_{j+1} & \text{if } \tau_1+1 \leqq j \leqq \tau_2, \\ \vdots & \\ (-1)^{l+1}X_1X_{j+1} & \text{if } \tau_l+1 \leqq j \leqq \tau_{l+1} \\ \vdots & \end{cases}.$$

This transformation was given in CSÁKI and VINCZE [3]. The following lemma is clearly true.

Lemma 12.

1°. *$Z_1, Z_2, \ldots$ is a sequence of i.i.d.r.v.'s with* $\mathbf{P}(Z_1=+1)=\mathbf{P}(Z_1=-1)=1/2$.

2° $$T(k)-T(\tau_l) = \sum_{j=\tau_l+1}^{k} Z_j = (-1)^{l+1}X_1 \sum_{j=\tau_l+1}^{k} X_{j+1} =$$

$$= (-1)^{l+1}X_1\big(S(k+1)-S(\tau_l+1)\big) \begin{cases} \leqq 1 & \text{if } \tau_l+1 \leqq k \leqq \tau_{l+1}-2, \\ = 1 & \text{if } k = \tau_{l+1}-1, \\ = 2 & \text{if } k = \tau_{l+1}. \end{cases}$$

3°. $2\theta(\tau_l)=2l=T(\tau_l)=\hat{m}(\tau_l)$, $l=1, 2, \ldots$.

4°. *For any* $\tau_l \leqq n < \tau_{l+1}$ *we have* $\theta(n)=l$, $2l \leqq \hat{m}(n) \leqq 2l+1$, *consequently* $0 \leqq \hat{m}(n)-2\theta(n) \leqq 1$.

5°. $$T(k) = \begin{cases} 2l+1-|S(k+1)| & \text{if } \tau_l+1 \leqq k \leqq \tau_{l+1}-1, \\ 2l+2-|S(k)| & \text{if } k = \tau_{l+1}, \end{cases}$$

therefore

$$\hat{y}(k) = \hat{m}(k)-T(k) \leqq |S(k+1)| \leqq |S(k)|+1$$

and

$$\hat{y}(k) = \hat{m}(k)-T(k) \geqq |S(k+1)|-1 \geqq |S(k)|-2.$$

This proves Theorem 2.

Corollary. *On a rich enough probability space* $\{\Omega, S, \mathbf{P}\}$ *one can define a Wiener process* $\{W(t); t \geqq 0\}$ *and a sequence* $X_1, X_2, \ldots$ *of i.i.d.r.v.'s with* $\mathbf{P}(X_1=+1)=\mathbf{P}(X_1=-1)=1/2$ *such that*

$$\big||S(n)|-|W(n)|\big| = O(\log n) \quad a.s. \tag{4.4}$$

and

$$|2\theta(n)-\eta(n)| = O(\log n) \quad a.s. \tag{4.5}$$

where $S(n)=X_1+X_2+\ldots+X_n$, $\theta(n)$ *is defined by* (4.1) *and* $\eta(\cdot)$ *is the local time at zero of* $W(\cdot)$.

Proof. Let us start with the random walk $X_1, X_2, \ldots$. Then construct a random walk $Z_1, Z_2, \ldots$ according to Theorem 2. Then by the theorem of Komlós, Major and Tusnády [8] one can construct a Wiener process $W_1(t)$ such that

$$\sup_{k \leqq n} |T(k)-W_1(k)| = O(\log n) \quad \text{a.s.} \tag{4.6}$$

where $T(k)=Z_1+\ldots+Z_k$. Put $M_1(t)=\max_{0\leqq s\leqq t} W_1(s)$. Then according to Lévy's theorem (Theorem A), $|W(t)|=M_1(t)-W_1(t)$ is the absolute value of a Wiener process whose local time $\eta(t)=M_1(t)$. Now

$$|\eta(n)-2\theta(n)| \leqq |M_1(n)-\hat{m}(n)|+|\hat{m}(n)-2\theta(n)|,$$

where $\hat{m}(n)=\max_{1\leqq k\leqq n} T(k)$. (4.5) follows from (4.2) and (4.6). Furthermore

$$\big||S(n)|-|W(n)|\big| \leqq \big||S(n)|-\hat{y}(n)\big|+|\hat{m}(n)-M_1(n)|+|T(n)-W_1(n)|,$$

where $\hat{y}(n)=\hat{m}(n)-T(n)$. (4.4) follows from (4.3) and (4.6).

5. A Strassen-type law of iterated logarithm

Let $\{W(t);\ t\geqq 0\}$ be a Wiener process and let

$$w_T(x) = w(x) = b_T^{-1}W(xT) \quad (0 \leqq x \leqq 1)$$

where

$$b_T = (2T\log\log T)^{1/2} \quad (T > e).$$

Further let $\mathscr{S}\subset C(0,1)$ be the set of absolutely continuous functions (with respect to the Lebesgue measure) for which

$$f(0) = 0 \text{ and } \int_0^1 (f'(x))^2\,dx \leqq 1.$$

The celebrated Strassen's (functional) law of iterated logarithm says:

Theorem D. [13] *The net* $\{W_T(x);\ 0\leqq x\leqq 1\}$ *is relatively compact in* $C(0,1)$ *with probability* 1 *and the set of its limit points is* $\mathscr{S}$.

It is an interesting question to characterize the limit points of $\eta(x, T)$ as $T\to\infty$. Donsker and Varadhan [5] solved this problem. Here we intend to present a result characterizing the limit points of the net

$$y_T(x) = y(x) = b_T^{-1}\eta(0, xT) \quad (0 \leqq x \leqq 1).$$

Since $y_T(x)\ (0\leqq x\leqq 1)$ for any fixed T is a non-decreasing function, its limit points must be also non-decreasing. Introduce the following

Definition. Let $\mathcal{M}\subset\mathcal{S}$ be the set of non-decreasing elements of $\mathcal{S}$.

Then we formulate our

Theorem 3. *The net* $\{y_T(x);\ 0\leqq x\leqq 1\}$ *is relatively compact in* $C(0, 1)$ *with probability* 1 *and the set of its limit points is* $\mathcal{M}$.

Proof. This result is a trivial consequence of Theorems A and D.

It looks more interesting to characterize jointly the limit points of the vectors $\{w_T(x), y_T(x);\ 0\leqq x\leqq 1\}$ $(T\to\infty)$. Intuitively it is clear enough that $y_T(x)$ must be constant in an interval where $w_T(x)\neq 0$. Hence in order to characterize the set of possible limit points it is natural to introduce the following

Definition. Let $\mathcal{N}$ be the set of those two-dimensional vector valued functions $h(x)=(f(x), g(x))$ $(0\leqq x\leqq 1)$ for which

(i) f and g are absolutely continuous in (0, 1) with respect to the Lebesgue-measure,

(ii) $f(0)=g(0)=0$,

(iii) g is non-decreasing,

(iv) $f(x)g'(x)\equiv 0$ $(0<x<1)$,

(v) $\int\limits_0^1 (f'(x)+g'(x))^2 dx\leqq 1$.

Now we have

Theorem 4. *The net* $\{w_T(x), y_T(x);\ 0\leqq x\leqq 1\}$ $(T\to\infty)$ *is relatively compact in* $C(0, 1)\times C(0, 1)$ *with probability* 1 *and the set of its limit points is* $\mathcal{N}$.

This Theorem is again a simple consequence of Theorems A and D. Theorem 4 clearly implies the following interesting

Consequence. *The net* $\{w_T(1), y_T(1)\}=\{b_T^{-1}W(T), b_T^{-1}\eta(0, T)\}$ *is relatively compact in the plane* R^2 *with probability* 1 *and the set of its limit points is the triangle*

$$T = \{(x, y):\ -1\leqq x\leqq 1,\quad 0\leqq y\leqq |x-1|\}$$

which, in turn, also implies

$$\limsup_{T\to\infty} b_T^{-1}\big(\eta(0, T)+|W(T)|\big) = 1 \quad a.s.$$

Remark. Theorem 1 shows that our Theorems 3 and 4 as well as the above Consequence remain true if we investigate the properties of the random walk $S(1), S(2), \dots$ of the introduction instead of a Wiener process. The invariance principles of Csáki—Révész [2] and Révész [12] shows that these results can be extended for more general random walks.

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Semigroups of continuous functions

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To Professor Béla Szőkefalvi-Nagy, to our Master, to my Friend

0. Introduction. Let X be a topological space and $C(X)$ denote the set of all continuous, real-valued functions defined on X. $C(X)$ is a ring under pointwise addition and multiplication of functions. A classical theorem [2] states that the isomorphy of the rings $C(X)$ and $C(Y)$ implies the homeomorphy of X and Y provided X and Y are compact Hausdorff spaces. Somewhat surprisingly, A. N. MILGRAM [7] has shown that the same is true if one replaces the isomorphy of the rings $C(X)$ and $C(Y)$ by the isomorphy of the multiplicative semigroups of $C(X)$ and $C(Y)$.

Another generalization was furnished by E. HEWITT [5]; he replaced the condition for X and Y to be compact by that of being *realcompact* (but kept the ring isomorphy of $C(X)$ and $C(Y)$). As to the concept of a realcompact space, let us recall the following definitions.

In a topological space X, denote

(1) $$Z(f)=\{x\in X:\ f(x)=0\}$$

for $f\in C(X)$,

(2) $$Z(X)=\{Z(f):\ f\in C(X)\}.$$

A subset $\mathfrak{z}\subset Z(X)$ is said to be a *z-filter* iff

(3.a) $$\emptyset\neq\mathfrak{z}\neq Z(X),$$

(3.b) $$Z_1\in\mathfrak{z},\quad Z_2\in Z(X),\quad Z_1\subset Z_2\quad\text{implies}\quad Z_2\in\mathfrak{z}.$$

(3.c) $$Z_1,\ Z_2\in\mathfrak{z}\quad\text{implies}\quad Z_1\cap Z_2\in\mathfrak{z}.$$

A z-filter $\mathfrak{z}$ is said to be *fixed* iff $\bigcap\mathfrak{z}\neq\emptyset$, *maximal* iff $\mathfrak{z}'=\mathfrak{z}$ for every z-filter $\mathfrak{z}'\supset\mathfrak{z}$, and *real* iff $Z_n\in\mathfrak{z}$ $(n\in\mathrm{N})$ implies $\bigcap_1^\infty Z_n\in\mathfrak{z}$. Now X is said to be *realcompact* iff it is a Tychonoff space such that every real maximal z-filter is fixed.

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It is a natural question whether these two generalizations can be unified. In fact, the paper [8] contains the following statement:

Theorem A. *If X and Y are realcompact spaces such that the (multiplicative) semigroups $C(X)$ and $C(Y)$ are isomorphic then X and Y are homeomorphic.*

However, the proof in [8] of this statement is rather long, goes through arguments concerning the *lattices* $C(X)$ and $C(Y)$, and seems to contain some gaps. Therefore it is desirable to have a short proof operating directly with the semigroup structure of $C(X)$ and $C(Y)$. This is desirable also because, as it was shown in [4], Theorem A implies

Theorem B. *If X and Y are arbitrary topological spaces, then the isomorphy of the semigroups $C(X)$ and $C(Y)$ implies the isomorphy of the rings $C(X)$ and $C(Y)$.*

The proof of Theorem B is based on Theorem C below. In order to formulate it, we have to recall one more definition. Let X be a Tychonoff space, and denote by υX the set of all real maximal z-filters in X, equipped with the topology for which the sets

$$B(Z) = \{\mathfrak{z} \in \upsilon X\colon\ Z \in \mathfrak{z}\} \quad (Z \in Z(X)) \tag{4}$$

constitute a closed base; υX is realcompact and is called the *Hewitt realcompactification* of X (see the monograph [3] for more details).

Theorem C. *If X and Y are Tychonoff spaces such that the semigroups $C(X)$ and $C(Y)$ are isomorphic then υX and υY are homeomorphic.*

Theorem C contains Theorem A because υX is homeomorphic to X if X is realcompact.

One of the purposes of the present paper is to present a method furnishing a simple proof of Theorem C. However, our method furnishes essentially more. Firstly, we can consider, instead of real-valued functions, functions with values in suitable topological semigroups. Secondly (which is more important), the condition of semigroup isomorphy can be replaced by an essentially weaker condition.

1. d-mappings and d-ideals. Let S be a semigroup. For $f, g \in S$, we introduce the notation $g \rhd f$ iff f is a right divisor of g, i.e., iff there is $h \in S$ such that $g = hf$. The relation $\rhd$ is transitive; it is reflexive (i.e. a preordering) if S contains a left unity element.

If S_1 and S_2 are semigroups with the respective relations $\rhd_1$ and $\rhd_2$, we say that a mapping $\varphi\colon S_1 \to S_2$ is a *d-mapping* iff $f, g \in S_1, g \rhd_1 f$ implies $\varphi(g) \rhd_2 \varphi(f)$. A bijective mapping $\varphi\colon S_1 \to S_2$ such that both φ and φ^{-1} are d-mappings will be called a *d-isomorphism*; S_1 and S_2 are said to be *d-isomorphic* iff there exists a d-isomorphism from S_1 onto S_2. If S_1 and S_2 are semigroup isomorphic then they are clearly d-isomorphic but the converse is false; e.g., two groups of the same cardinality are always d-isomorphic (because $g \rhd f$ holds for any two elements f, g of a group S).

A subset D of a semigroup S will be a called a *d-ideal* iff

(1.1) $$\emptyset \neq D \neq S,$$

(1.2) $$f \in D, \quad g \in S, \quad g \rhd f \quad \text{implies} \quad g \in D,$$

(1.3) $f, g \in D$ implies the existence of $h \in D$ such that $f \rhd h$, $g \rhd h$.

This is a special case of the general Definition 1.2 in [6]. A d-ideal is (by (1.2)) a left semigroup ideal.

Lemma 1. *If the semigroup S contains a right unity element e, and $e \rhd f$, then f cannot belong to any d-ideal D.*

Proof. Clearly $g \rhd e$ for every $g \in S$, hence $f \in D$ would imply $D = S$. □

A d-ideal D is said to be *maximal* iff $D' = D$ holds for every d-ideal $D' \supset D$. By the Kuratowski—Zorn lemma, in a semigroup with right unity element, every d-ideal is contained in a maximal d-ideal. For a d-isomorphism $\varphi: S_1 \to S_2$ and $D \subset S_1$, $\varphi(D)$ is a (maximal) d-ideal in S_2 iff D is a (maximal) d-ideal in S_1.

2. Quasi-real semigroups. Let $\mathbf{R}$ denote the real line, $\mathbf{R}^+$ the subset $(0, +\infty)$, and $\mathbf{R}_0^+$ the subset $[0, +\infty)$. Both $\mathbf{R}^+$ and $\mathbf{R}_0^+$ are semigroups (the first one even a group) under the multiplication of real numbers, and also topological spaces as subspaces of $\mathbf{R}$ equipped with the usual topology.

A set S will be called a *quasi-real semigroup* iff

(2.1) S is a semigroup;

(2.2) S contains $\mathbf{R}_0^+$ as a subsemigroup;

(2.3) $0 \in \mathbf{R}_0^+$ is a zero element in S (i.e., $0 \cdot a = a \cdot 0 = 0$ for $a \in \mathrm{S}$);

(2.4) $1 \in \mathbf{R}_0^+$ is a unity element in S (i.e., $1 \cdot a = a \cdot 1 = a$ for $a \in \mathrm{S}$);

(2.5) For $a \in \mathrm{S}$, $a \neq 0$, there is $b \in \mathrm{S}$ such that $a \cdot b = b \cdot a = 1$ (such a b is clearly unique and will be denoted by $1/a$);

(2.6) S is a topological space;

(2.7) $\mathbf{R}_0^+$ is a subspace of S;

(2.8) The mappings $(a, b) \mapsto a \cdot b$ and $a \mapsto 1/a$ are continuous from $\mathrm{S} \times \mathrm{S}$ into S and $\mathrm{S} - \{0\}$ into S, respectively;

(2.9) There is a continuous mapping $a \mapsto |a|$ from S into $\mathbf{R}_0^+$ such that $|a \cdot b| = |a| \cdot |b|$, $|a| = a$ for $a \in \mathbf{R}_0^+$;

(2.10) The sets $V_\varepsilon = \{x \in \mathrm{S} : |x| < \varepsilon\}$ $(\varepsilon > 0)$ constitute a neighbourhood base of 0 in S.

By (2.5) and (2.9), $|a| = 0$ iff $a = 0$.

As examples of quasi-real semigroups, we can mention the semigroups $\mathbf{R}_0^+$; $\mathbf{R}$, $\mathbf{C}$ (=the complex numbers) with the usual multiplication, topology, and absolute value, further many subsemigroups of $\mathbf{C}$, e.g., those composed of the numbers with arguments $2\pi r$ where $r\in\mathbf{Q}$, or $r=m/n$ where $n\in\mathbf{N}$ is fixed and $m\in\mathbf{Z}$. These examples are commutative; a non-commutative one is furnished by the real quaternions with the usual multiplication, absolute value and the topology inherited from $\mathbf{R}^4$.

We obtain further examples from

Theorem 1. *Let* $\mathbf{G}$ *be a topological group that contains* $\mathbf{R}^+$ *as a (topological) subgroup; suppose there is a continuous homomorphism* $\alpha\colon \mathbf{G}\to\mathbf{R}^+$ *such that* $\alpha(a)=a$ *for* $a\in\mathbf{R}^+$. *Let* $\mathbf{S}=\mathbf{G}\cup\{\omega\}$ *where* $\omega\notin\mathbf{G}$, *and define*

$$a\cdot\omega=\omega\cdot a=\omega\quad(a\in\mathbf{G}),\quad \omega\cdot\omega=\omega,\quad \alpha(\omega)=0.$$

Equip $\mathbf{S}$ *with a topology in the manner that* $\mathbf{G}$ *be a subspace of* $\mathbf{S}$ *and the sets* $U_\varepsilon\cup\{\omega\}$, *where*

$$U_\varepsilon=\{x\in\mathbf{G}\colon \alpha(x)<\varepsilon\}\quad(\varepsilon>0),$$

constitute a neighbourhood base of ω. *After having identified* ω *with the real number* 0, $\mathbf{S}$ *will be a quasi-real semigroup (with* $|x|=\alpha(x)$*).*

Conversely, every quasi-real semigroup can be obtained from a topological group $\mathbf{G}$ *with the help of this construction.*

Proof. $\mathbf{S}$ fulfils (2.1)—(2.5) with the identification of ω and 0. The continuity of α implies that every U_ε is open in $\mathbf{G}$; therefore there is a topology on $\mathbf{S}$ such that $\mathbf{G}$ is a subspace of $\mathbf{S}$ and the sets $U_\varepsilon\cup\{\omega\}$ constitute a neighbourhood base of ω (see e.g. [1], (6.1.2)). Such a topology is unique because $\mathbf{G}$ is necessarily open in $\mathbf{S}$; indeed, if ω belonged to every neighbourhood (in $\mathbf{S}$) of a point $a\in\mathbf{G}$, then the filter base $\{U_\varepsilon\colon \varepsilon>0\}$ would converge to a in $\mathbf{G}$, which is in contradiction with the fact that $\left\{x\in\mathbf{G}\colon \alpha(x)>\frac{\alpha(a)}{2}\right\}$ is a neighbourhood of a. For this topology (and $|x|=\alpha(x)$), (2.6)—(2.10) are evidently true.

Conversely, if $\mathbf{S}$ is a quasi-real semigroup, define $\mathbf{G}=\mathbf{S}-\{0\}$. By (2.1)—(2.5), $\mathbf{G}$ is a group containing $\mathbf{R}^+$ as a subgroup; by (2.6)—(2.8), it is a topological group, and $\mathbf{R}^+$ is a topological subgroup of $\mathbf{G}$. By (2.9), $\alpha(x)=|x|$ defines a continuous homomorphism $\alpha\colon\mathbf{G}\to\mathbf{R}^+$, and, by (2.10), all requirements are fulfilled for $\omega=0$. □

E.g., let $\mathbf{G}$ be the set of all non-singular, real, quadratic matrices of order m (for a given $m\in\mathbf{N}$) with matrix multiplication and the topology inherited from $\mathbf{R}^{m^2}$. The diagonal matrices with all elements in the diagonal equal to the same $c>0$ constitute a topological subgroup isomorphic to $\mathbf{R}^+$; after having identified this matrix with c, define $\alpha(M)=|\det M|^{1/m}$ in order to obtain a group $\mathbf{G}$ satisfying the hypotheses of Theorem 1.

Many examples can be obtained from

Theorem 2. *Let* $\mathbf{T}$ *be an arbitrary topological group with unity element* e. *Then the direct product* $\mathbf{G}=\mathbf{T}\times\mathbf{R}^+$ *satisfies the hypotheses of Theorem* 1 *provided the elements* (e, y) *are identified with* $y>0$ *and* $\alpha(x, y)=y$. □

Observe that Theorem 1 furnishes examples that are not contained in Theorem 2. E.g., let $\mathbf{G}$ be the multiplicative group of all non-singular, real, quadratic matrices of order 2 with the topology inherited from $\mathbf{R}^4$. Identify the matrix

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad (x>0)$$

with the number x, and define $\alpha(M)=|\det M|$. If $\mathbf{G}$ were of the form $\mathbf{T}\times\mathbf{R}^+$ then $\mathbf{T}$ would be isomorphic to the subgroup of $\mathbf{G}$ consisting of the elements M such that $\alpha(M)=1$. However, this is impossible because, e.g.,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

3. d-ideals of $S(X)$. Let X be a topological space, $\mathbf{S}$ a quasi-real semigroup, and denote by $S(X)$ the set of all continuous functions from X into $\mathbf{S}$. $S(X)$ is a semigroup under pointwise multiplication of functions. Our purpose is to show that the d-ideals of the semigroup $S(X)$ are connected to the z-filters in X in the same manner as the ideals of the ring $C(X)$ are (see [3]).

For $f\in S(X)$, define

$$Z(f) = \{x\in X\colon f(x) = 0\}, \tag{3.1}$$

$$|f|(x) = |f(x)| \quad (x\in X). \tag{3.2}$$

Lemma 2. *$f\in S(X)$ implies $|f|\in C(X)$. Conversely, $g\in C(X)$, $g\geqq 0$ implies $g\in S(X)$.* □

Lemma 3. *For $f\in S(X)$, we have $Z(f)=Z(|f|)$; consequently*

$$\{Z(f)\colon f\in S(X)\} = Z(X). \quad \square$$

Lemma 4. *$Z(fg)=Z(f)\cup Z(g)$ for $f, g\in S(X)$.* □

Lemma 5. *If D is a d-ideal in $S(X)$, then*

$$Z(D) = \{Z(f)\colon f\in D\} \tag{3.3}$$

is a z-filter in X.

Proof. By Lemma 3, $Z(D)\subset Z(X)$. $D\neq\emptyset$ implies $Z(D)\neq\emptyset$. On the other hand, since the constant function 1 is a unity element in $S(X)$, and $f\in S(X), Z(f)=\emptyset$ implies $1=\frac{1}{f}\cdot f$ for $\frac{1}{f}\in S(X)$, where, of course,

$$\frac{1}{f}(x)=\frac{1}{f(x)}\quad (x\in X), \tag{3.4}$$

$f\in D$ is impossible by Lemma 1. Therefore $\emptyset\notin Z(D)$.

If $Z_1\in Z(D)$, $Z_2\in Z(X)$, $Z_1\subset Z_2$, say $Z_1=Z(f), f\in D$, $Z_2=Z(g), g\in S(X)$ (cf. Lemma 3), then, by Lemma 4, $gf\in D$ implies $Z_2=Z_2\cup Z_1=Z(gf)\in Z(D)$.

Now let $Z_1, Z_2\in Z(D)$, say $Z_1=Z(f)$, $Z_2=Z(g)$, f, $g\in D$. By (1.3), there is $h\in D$ such that $f\rhd h$, $g\rhd h$. By Lemma 4, $Z(f)\supset Z(h)$, $Z(g)\supset Z(h)$, hence $Z_1\cap Z_2\supset Z(h)\in Z(D)$. Thus $Z_1\cap Z_2\in Z(D)$ because $Z(X)$ is a lattice ([3], 1.10) so that $Z_1\cap Z_2\in Z(X)$. □

Lemma 6. *If* $\mathfrak{z}$ *is a z-filter in* X, *then*

$$Z^{-1}(\mathfrak{z})=\{f\in S(X)\colon Z(f)\in\mathfrak{z}\} \tag{3.5}$$

is a d-ideal in $S(X)$.

Proof. $\emptyset\notin\mathfrak{z}$ implies $1\notin Z^{-1}(\mathfrak{z})$, and $\mathfrak{z}\neq\emptyset$ implies $Z^{-1}(\mathfrak{z})\neq\emptyset$ by Lemma 3. If $f\in Z^{-1}(\mathfrak{z})$, $g\in S(X)$, $g\rhd f$, then $Z(g)\supset Z(f)$ by Lemma 4 so that $Z(g)\in\mathfrak{z}$, $g\in Z^{-1}(\mathfrak{z})$.

Now let $f, g\in Z^{-1}(\mathfrak{z})$. Define

$$h(x)=(|f(x)|+|g(x)|)^{1/2}\quad (x\in X).$$

Then $h\in S(X)$ by Lemma 2, and $Z(h)=Z(f)\cap Z(g)$ implies $h\in Z^{-1}(\mathfrak{z})$. We show $f\rhd h$.

For this purpose, define

$$k(x)=\begin{cases}0 & \text{if}\quad x\in Z(f),\\ f(x)\cdot\dfrac{1}{h(x)} & \text{if}\quad x\in X-Z(f).\end{cases}$$

Then $k\in S(X)$. In fact, k is obviously continuous at the points of $X-Z(f)$. The equality

$$|k(x)|=\frac{|f(x)|^{1/2}}{(|f(x)|+|g(x)|)^{1/2}}\cdot|f(x)|^{1/2}$$

shows by (2.10) that the same holds at the points of $Z(f)$. Finally $f=kh$ is obvious.

We prove $g\rhd h$ similarly. □

Lemma 7. *If* D *is a d-ideal in* $S(X)$, $\mathfrak{z}$ *a z-filter in* X, *then*

$$Z^{-1}(Z(D)) \supset D, \quad Z(Z^{-1}(\mathfrak{z})) = \mathfrak{z}. \ \square \tag{3.6}$$

Lemma 8. *If* D *is a maximal d-ideal, then* $Z(D)$ *is a maximal z-filter, and* $D = Z^{-1}(Z(D))$.

Proof. For a z-filter $\mathfrak{z}' \supset Z(D)$, we have by (3.6) $Z^{-1}(\mathfrak{z}') \supset Z^{-1}(Z(D)) \supset D$, hence $Z^{-1}(\mathfrak{z}') = Z^{-1}(Z(D)) = D$, and $\mathfrak{z}' = Z(Z^{-1}(\mathfrak{z}')) = Z(D)$. $\square$

Lemma 9. *If* $\mathfrak{z}$ *is a maximal z-filter, then* $Z^{-1}(\mathfrak{z})$ *is a maximal d-ideal.*

Proof. For a d-ideal $D' \supset Z^{-1}(\mathfrak{z})$, we have by (3.6) that $Z(D') \supset Z(Z^{-1}(\mathfrak{z})) = \mathfrak{z}$, hence $Z(D') = \mathfrak{z}$, and $D' \supset Z^{-1}(\mathfrak{z}) = Z^{-1}(X(D')) \supset D'$ so that $D' = Z^{-1}(\mathfrak{z})$. $\square$

Lemma 10. *The formulas*

$$\mathfrak{z} = Z(D), \quad D = Z^{-1}(\mathfrak{z}) \tag{3.7}$$

establish a bijection from the set of all maximal d-ideals D *in* $S(X)$ *onto the set of all maximal z-filters* $\mathfrak{z}$ *in* X. $\square$

4. Construction of υX. Let X be a Tychonoff space. Our purpose is to show that υX or, more precisely, a space homeomorphic to υX can be constructed as soon as we know the relation $\rhd$ in $S(X)$ (not necessarily the semigroup structure of $S(X)$).

In fact, the knowledge of this relation permits us to determine all d-ideals, hence all maximal d-ideals in $S(X)$; thus we have, by Lemma 10, a set from which a bijection goes onto the set of all maximal z-filters in X. In order to know υX as a set, we have to select those maximal d-ideals D for which $Z(D)$ is a *real* z-filter.

Lemma 11. *If* $f, g \in S(X)$, *then* $Z(f) \subset Z(g)$ *holds iff* g *belongs to every maximal d-ideal containing* f.

Proof. If D is a maximal d-ideal, $f \in D$, and $Z(f) \subset Z(g)$, then $Z(f) \in Z(D)$, hence $Z(g) \in Z(D)$ by Lemma 5, and $g \in D$ by Lemma 8.

Conversely, if $x \in Z(f) - Z(g)$, then $\mathfrak{z} = \{Z \in Z(X) : x \in Z\}$ is a maximal z-filter ([3], 3.18) such that $Z(f) \in \mathfrak{z}$; $Z(g) \notin \mathfrak{z}$, hence $Z^{-1}(\mathfrak{z})$ is a maximal d-ideal (by Lemma 9) such that $f \in Z^{-1}(\mathfrak{z})$, $g \notin Z^{-1}(\mathfrak{z})$. $\square$

Lemma 12. *For a maximal d-ideal* D, $Z(D)$ *is a real maximal z-filter iff* $f_n \in D$ $(n \in \mathbf{N})$ *implies the existence of* $g \in D$ *such that* $Z(g) \subset Z(f_n)$ *for* $n \in \mathbf{N}$.

Proof. If $Z(D)$ is a real z-filter, and $f_n \in D$ for $n \in \mathbf{N}$, then

$$Z_0 = \bigcap_{1}^{\infty} Z(f_n) \in Z(D),$$

hence $Z_0=Z(g)$ for some $g\in D$. Conversely, suppose $f_n\in D$, $g\in D$, $Z(g)\subset Z(f_n)$ for every $n\in\mathbf{N}$. Then Z_0 defined as above belongs to $Z(X)$ ([3], 1.14), and $Z(g)\subset Z_0$ implies $Z_0\in Z(D)$ by Lemma 5. $\square$

By Lemmas 11 and 12, the knowledge of $\rhd$ permits to determine those maximal d-ideals D for which $Z(D)\in\upsilon X$. For $f\in S(X)$, $Z=Z(f)$, the set $B(Z)$ defined by (4) is composed of all $Z(D)\in\upsilon X$ for which $f\in D$ (Lemma 8). Hence we obtain a space homeomorphic to υX by defining the points to be those maximal d-ideals D that fulfil the condition formulated in Lemma 12, and by choosing for a closed base the system of the sets $B(f)$ consisting of those points D for which $f\in D$ $(f\in S(X))$.

5. Main results. We get as an immediate consequence of the argument above:

Theorem 3. *Let X and Y be Tychonoff spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ quasi-real semigroups. Define $S_1(X)$ and $S_2(Y)$ to be the semigroups of all continuous functions $f: X\to\mathbf{S}_1$ and $g: Y\to\mathbf{S}_2$, respectively. If $S_1(X)$ and $S_2(Y)$ are d-isomorphic, then X and Y are homeomorphic. In particular, X and Y are homeomorphic provided they are realcompact.* $\square$

We obtain Theorem C as a corollary because $\mathbf{R}$ is a quasi-real semigroup and semigroup isomorphy implies d-isomorphy. One can, of course, prove this theorem directly, without making use of the definitions and results in Section 2; the statements concerning $\mathbf{S}$ quoted in Section 3 are obvious in the case $\mathbf{S}=\mathbf{R}$.

Moreover, the argument applied in [4] leads to the following sharper form of Theorem B:

Theorem 4. *For arbitrary topological spaces X and Y, if the multiplicative semigroups $C(X)$ and $C(Y)$ are d-isomorphic, then the rings $C(X)$ and $C(Y)$ are isomorphic.* $\square$

6. The case $\mathbf{S}=\mathbf{R}$. If $\mathbf{S}=\mathbf{R}$ then $S(X)=C(X)$. If we agree in calling d-ideals of a ring A the d-ideals of the multiplicative semigroup of A, Lemmas 8 and 9 imply, according to [3], 2.5:

Theorem 5. *The maximal d-ideals of the ring $C(X)$ coincide with the maximal ideals.* $\square$

It is a natural question whether there is some connection between d-ideals and ideals of $C(X)$ in general.

Lemma 13. *Every d-ideal D of a ring A is a left ideal in A.*

Proof. It suffices to prove that $f,g\in D$ implies $f-g\in D$. Now there is $h\in D$ such that $f=f_1h$, $g=g_1h$ for some $f_1,g_1\in A$, hence $f-g=(f_1-g_1)h\in D$. $\square$

In particular, every d-ideal of the (commutative) ring $C(X)$ is an ideal. The converse is not true in general. In fact, let $X=\mathbf{R}$,

(6.1) $$f_0(x)=\max(x,0), \quad g_0(x)=\min(x,0) \quad (x\in X),$$

and let I be the ideal generated by $\{f_0, g_0\}$, i.e.,

(6.2) $$I=\{ff_0+gg_0\colon f,\ g\in C(X)\}.$$

Suppose $h\in I, f_0\rhd h, g_0\rhd h$. Then

(6.3) $$f_0=f_1h, \quad g_0=g_1h, \quad f_1,\ g_1\in C(X),$$

hence $Z(h)\subset Z(f_0)\cap Z(g_0)=\{0\}$. Consequently

(6.4) $$(-\infty,0)\subset Z(f_1), \quad (0,+\infty)\subset Z(g_1).$$

Select $f, g\in C(X)$ such that $h=ff_0+gg_0$; then (by (6.3))

(6.5) $$h=(ff_1+gg_1)h$$

so that

(6.6) $$f(x)f_1(x)+g(x)g_1(x)=1$$

for $x\neq 0$ and, by continuity, for $x=0$, too. The first member of the left-hand side of (6.6) vanishes for $x<0$, the second one for $x>0$ (see (6.4)), hence both vanish for $x=0$: a contradiction.

The ideal I in the preceding example was generated by a subset of cardinality 2. For 1 instead of 2, we have the following obvious

Lemma 14. *Every proper left ideal generated by an element of a ring A with unity element is a d-ideal.* □

For another result in the same direction, let us recall that an ideal I of $C(X)$ is said to be a *z-ideal* iff $I=Z^{-1}(Z(I))$ (with a notation analogous to (3.3) and (3.5)).

Lemma 15. *Every proper z-ideal of the ring $C(X)$ is a d-ideal.*

Proof. By [3], 2.3, $Z(I)$ is a z-filter for every proper ideal I of $C(X)$, hence Lemma 6 furnishes the statement. □

On the other hand, a d-ideal of $C(X)$ need not be a z-ideal. Again for $X=\mathbf{R}$, the ideal I generated by $\{h_0\}$, where $h_0(x)=x$ for $x\in X$, is a d-ideal by Lemma 14, but fails to be a z-ideal ([3], 2.4).

We can summarize our results as follows:

Theorem 6. *We have the following implications in $C(X)$:*

$$\textit{proper z-ideal} \Rightarrow \textit{d-ideal} \Rightarrow \textit{proper ideal},$$

and none of them can be reversed in general. □

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On how long interval is the empirical characteristic function uniformly consistent?

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In honour of Professor Béla Szőkefalvi-Nagy on his seventieth birthday

Introduction, results, and discussion

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed d-dimensional random vectors, $d \geqq 1$, defined on a probability space $(\Omega, \mathcal{A}, P)$, with common distribution function $F(x)$, $x \in \mathbf{R}^d$, and characteristic function

$$C(t) = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF(x), \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathbf{R}^d$. The n^{th} empirical characteristic function of the sequence is

$$C_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{i\langle t, X_j\rangle} = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF_n(x), \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d,$$

where $F_n(x)$, $x \in \mathbf{R}^d$, denotes the empirical distribution function of $X_1, \ldots, X_n$. By any advanced form of the strong law of large numbers, $\lim_{n\to\infty} C_n(t) = C(t)$ almost surely at each fixed $t \in \mathbf{R}^d$, but more than this is still trivial. Indeed, the d-variate Glivenko—Cantelli theorem for F_n and the d-variate continuity theorem of Lévy readily imply that

$$\lim_{n\to\infty} \sup_{|t| \leqq T} |C_n(t) - C(t)| = 0 \tag{1}$$

almost surely for any fixed positive $T < \infty$; that is, in statistical terminology, C_n is a strongly uniformly consistent estimator for C on any fixed bounded subset of $\mathbf{R}^d$.

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On the other hand, $C_n(t)$ is a d-variate almost periodic function for each n, at each $\omega\in\Omega$ where it is defined, and hence if

$$\Delta_n = \sup_{t\in\mathbf{R}^d} |C_n(t)-C(t)|$$

converges to zero at *only one single* $\omega\in\Omega$, then by Satz XXVI of BOCHNER [1] the limiting function $C(t)$ must be almost periodic. But then, by a simple extension of the corresponding univariate result (Corollary 1 of Theorem 3.2.3 of LUKACS [5]; here we use the Eindeutigkeitssatz (Satz XXXVII) of BOCHNER [1] instead of the corresponding univariate uniqueness theorem of Bohr), $C(t)$ must belong to a purely discrete F, i.e., it is of the form

$$C(t) = \sum_k q_k e^{i\langle t,\lambda_k\rangle}, \quad q_k \geqq 0, \quad \sum_k q_k = 1,$$

with a finite or infinite sequence of vectors λ_k. That Δ_n *does* converge to zero almost surely in such a case was pointed out by FEUERVERGER and MUREIKA [4] for univariate characteristic functions, i.e., for discrete real random variables, and later by CSÖRGŐ [3] for $d\geqq 1$.

So if we wish to say more than (1) in the general case, then we are lead to considering the quantities

$$\Delta_n(T_n) = \sup_{|t|\leqq T_n} |C_n(t)-C(t)|$$

for some sequence $\{T_n\}$ of positive numbers converging to infinity. This has been first done by FEUERVERGER and MUREIKA [4] in the univariate case, who showed that if $d=1$ and the singular part of C vanishes at infinity, then $\lim_{n\to\infty} \Delta_n(T_n)=0$ almost surely whenever $T_n=o((n/\log n)^{1/2})$. This result was improved by CSÖRGŐ [2, 3] ($d=1$ and $d\geqq 1$, respectively) who showed that $\lim_{n\to\infty} \Delta_n(T_n)=0$ almost surely for any characteristic function whenever $T_n=o((n/\log\log n)^{1/(2d)})$. The latter result is in fact an easy consequence of Kiefer's well-known d-variate extension of the Chung—Smirnov univariate law of the iterated logarithm for F_n. This familiar rate has made us think for a longer time that it was perhaps best possible, although its dependence on the dimension appeared strange. It is in fact very far from being best possible, and the final solution presented below is rather surprising.

Theorem 1. *For any d-variate characteristic function C, if $\lim_{n\to\infty}(\log T_n)/n=0$ then $\lim_{n\to\infty}\Delta_n(T_n)=0$ almost surely.*

Theorem 2. *If $\lim_{|t_k|\to\infty}|C(t_1,\ldots,t_k,\ldots,t_d)|=0$ for some k, $1\leqq k\leqq d$, and if $\overline{\lim}_{n\to\infty}(\log T_n)/n>0$, then there exists a positive ε such that*

$$\overline{\lim_{n\to\infty}}\, P\{\Delta_n(T_n)\geqq\varepsilon\}>0.$$

We see that the rate $T_n = \exp(o(n))$ is not only best possible in general for almost sure convergence, but if we take any faster sequence T_n then even stochastic convergence cannot be retained for any characteristic function vanishing at infinity along at least one path.

The proof of Theorem 2 implies that if $\log T_{n_k} \geqq \gamma n_k$, $k=1, 2, \ldots,$ for a subsequence $\{n_k\}$ of the natural numbers and some $\gamma > 0$, then for any subsequence $\{m_k\}$ of $\{n_k\}$ the sequence

$$\sup_{|t| \leqq T_{m_k}} |C_{m_k}(t) - C(t)|$$

does not converge to zero in probability. Since the topology of stochastic convergence is metrisable, and since for every $T > 0$

$$P\{\sup_{|t| \leqq T} |C_{m_k}(t) - C(t)| > \varepsilon_k(T)\} > 0$$

with some $\varepsilon_k(T) > 0$ (the opposite could only occur in the case when $C(t) = \exp(i\langle t, \lambda\rangle)$ with some vector λ, i.e., when the distribution is degenerate at λ, but this case is excluded under the hypothesis of Theorem 2), the following somewhat sharper form of Theorem 2 is also true: *If* $\lim_{|t_k| \to \infty} |C(t_1, \ldots, t_k, \ldots, t_d)| = 0$ *for some* k, $1 \leqq k \leqq d$, *and if* $\log T_{n_k} \geqq \gamma n_k$, $k = 1, 2, \ldots,$ *for a subsequence* $\{n_k\}$ *of positive integers and some* $\gamma > 0$, *then there is a positive* ε *such that*

$$P\{\sup_{|t| \leqq T_{n_k}} |C_{n_k}(t) - C(t)| \geqq \varepsilon\} \geqq \varepsilon$$

is satisfied for all k.

The proof in the positive direction is quite straightforward. Essentially it imitates that of the easier half of the continuity theorem in conjunction with the exponential inequality of Bernštein. Exactly the same approach was taken in [2, 3] for handling the much harder problem of weak convergence, or strong approximation of the process $n^{1/2}(C_n(\cdot) - C(\cdot))$. It was not realised then that this approach is also suitable for the easier problem of uniform consistency on long intervals. On the other hand, the proof of Theorem 2 shows that the behaviour of $\Delta_n(T_n)$ is intimately connected with an old number-theoretic problem. Indeed, our starting point will be Dirichlet's classic result in diophantine approximation.

Having Theorems 1 and 2 above, further questions can be posed which may be irrelevant from the statistical point of view but are interesting as purely probabilistic problems. Set $L_k = \overline{\lim}_{|t_k| \to \infty} |C(t_1, \ldots, t_k, \ldots, t_d)|$, $k = 1, \ldots, d$. Can T_n be faster than $\exp(o(n))$ if $L = \min(L_1, \ldots, L_d) > 0$ but the distribution is not purely discrete? In the positive direction we do not have anything more than Theorem 1. In the negative direction the hardest subcase seems to be the one when $L = 1$. Otherwise a slight modification of the proof of Theorem 2 below also gives the following result: *If* $0 < L < 1$ *and* $\overline{\lim}_{n \to \infty} (\log T_n)/n > \log(2\pi/\arccos L)$, *then* $\overline{\lim}_{n \to \infty} P\{\Delta_n(T_n) \geqq \varepsilon\} > 0$ *with some positive* ε.

Proofs

Theorem 1. Let $\varepsilon>0$ be artbitrarily small, $\varepsilon\leqq 2$, and choose $K=K(\varepsilon, F)$ so large that

$$\int_{|x|\geqq K} dF(x) < \frac{\varepsilon}{8}.$$

Writing $D_n(t)=B_n(t)-B(t)$, we have

$$\Delta_n(T_n) \leqq \sup_{|t|\leqq T_n} |D_n(t)| + \sup_{|t|\leqq T_n} |B_n(t)-C_n(t)| + \sup_{|t|\leqq T_n} |B(t)-C(t)|$$

with the truncated integrals

$$B(t) = \int_{|x|\leqq K} e^{i\langle t,x\rangle}\, dF(x),$$

$$B_n(t) = \int_{|x|\leqq K} e^{i\langle t,x\rangle}\, dF_n(x) = \frac{1}{n}\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j| \leqq K),$$

where $\chi(A)$ denotes the indicator of the event A. The second term is

$$\frac{1}{n}\sup_{|t|\leqq T_n}\left|\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j| > K)\right| \leqq \frac{1}{n}\sum_{j=1}^{n}\chi(|X_j| > K),$$

and these bounds converge almost surely to $\int_{|x|>K} dF(x)$ which is also a bound for the third term.

Let us cover the cube $[-T_n, T_n]^d$ by $N_n=([(8Kd^{3/2}T_n)/\varepsilon]+1)^d$ disjoint small cubes $\Delta_1, \ldots, \Delta_{N_n}$, the edges of each of which are of length $\varepsilon/(4Kd^{3/2})$, and let $t_1, \ldots, t_{N_n}$ be the centres of these cubes. Then

$$\sup_{|t|\leqq T_n} |D_n(t)| \leqq \max_{1\leqq k\leqq N_n} |D_n(t_k)| + \max_{1\leqq k\leqq N_n}\sup_{t\in\Delta_k} |D_n(t)-D_n(t_k)| \leqq \max_{1\leqq k\leqq N_n} |D_n(t_k)| + \frac{\varepsilon}{4},$$

for

$$|D_n(s)-D_n(t)| \leqq |B_n(s)-B_n(t)| + |B(s)-B(t)| \leqq$$

$$\leqq \frac{1}{n}\sum_{j=1}^{n} |\langle s-t, X_j\rangle|\chi(|X_j| \leqq K) + \int_{|x|\leqq K} |\langle s-t, x\rangle|\, dF(x) \leqq 2\,dK|s-t|, \quad s, t\in\mathbf{R}^d.$$

(In fact, the finer almost sure upper bound

$$4\int_{|x|\leqq K}\left|\sin\frac{\langle s-t, x\rangle}{2}\right| dF(x)$$

can be given here, but this is irrelevant in the present context, yielding the same result). Summing up,

$$\Delta_n(T_n) \leqq \max_{1\leqq k\leqq N_n} |D_n(t_k)| + \frac{\varepsilon}{2} \tag{2}$$

almost surely for large enough n, the threshold depending on ω. Now

$$p_n = P\left\{\max_{1\le k\le N_n} |D_n(t_k)| > \frac{\varepsilon}{2}\right\} \leqq N_n \sup_{t\in \mathbf{R}^d} P\left\{|D_n(t)| > \frac{\varepsilon}{2}\right\} \leqq$$

$$\leqq MT_n^d \sup_{t\in \mathbf{R}^d} \left(P\left\{\frac{1}{n}\left|\sum_{j=1}^{n} R_j(t)\right| > \frac{\varepsilon}{4}\right\} + P\left\{\frac{1}{n}\left|\sum_{j=1}^{n} I_j(t)\right| > \frac{\varepsilon}{4}\right\}\right)$$

with some constant $M=M(\varepsilon, F, d)$, where the random variables

$$R_j(t) = (\cos\langle t, X_j\rangle)\chi(|X_j| \leqq K) - \int_{|x|\leqq K} \cos\langle t, x\rangle\, dF(x), \quad j = 1, \ldots, n,$$

are independent, $|R_j(t)|\leqq 2$, $ER_j(t)=0$, and

$$v^2(t) = ER_j^2(t) = \int_{|x|\leqq K} \cos^2\langle t, x\rangle\, dF(x) - \Big(\int_{|x|\leqq K} \cos\langle t, x\rangle\, dF(x)\Big)^2 \leqq 1.$$

The random functions $I_j(t)$, $j=1, \ldots, n$, are defined with the cosine function replaced by the sine; and hence these are also independent and identically distributed with $|I_j(t)|\leqq 2$, $EI_j(t)=0$ and $EI_j^2(t)\leqq 1$. Therefore the Bernštein inequality ([6], Chapter X, §1, Lemma 1) gives

$$P\left\{\frac{1}{n}\left|\sum_{j=1}^{n} R_j(t)\right| > \frac{\varepsilon}{4}\right\} \leqq \begin{cases} 2e^{-\frac{\varepsilon n}{32}}, & \text{if } \varepsilon \geqq 2v^2(t), \\ 2e^{-\frac{\varepsilon^2 n}{64 v^2(t)}}, & \text{if } \varepsilon \leqq 2v^2(t). \end{cases}$$

Since $v^2(t)\leqq 1$ and $\varepsilon\leqq 2$, the probability in question is not greater than $2\exp(-\varepsilon^2 n/64)$, and the same holds for the other one with the I_j's. Thus

$$p_n \leqq 4MT_n^d e^{-\frac{\varepsilon^2 n}{64}}.$$

Let $\delta<\varepsilon^2/(64d)$. Then for large enough n; $T_n\leqq\exp(\delta n)$, and hence $\sum_{n=1}^{\infty} p_n < \infty$. The Borel—Cantelli lemma and (2) give the desired result.

Theorem 2. Since

$$\sup_{|(t_1, \ldots, t_d)|\leqq T_n} |C_n(t_1, \ldots, t_d) - C(t_1, \ldots, t_d)| \geqq$$

$$\geqq \sup_{-T_n\leqq t_k\leqq T_n} |C_n(0, \ldots, 0, t_k, 0, \ldots, 0) - C(0, \ldots, 0, t_k, 0, \ldots, 0)|,$$

where $C_n(0, \ldots, 0, t_k, 0, \ldots, 0)$ is the empirical characteristic function of the k^{th} components of $X_1, \ldots, X_n$ and $C(0, \ldots, 0, t_k, 0, \ldots, 0)$ is the common characteristic function of these components, it is clearly enough to prove the theorem in the univariate case. We assume therefore that $d=1$, i.e., that $X=\{X_1, X_2, \ldots\}$ are independent real random variables with common characteristic function $C(t)$, $-\infty<t<\infty$, with $\lim_{|t|\to\infty} |C(t)|=0$.

Let

$$S_n(t) = S_n(t; X) = \sum_{j=1}^{n} e^{itX_j}.$$

Then $C_n(t)=n^{-1}S_n(t)$, and the theorem will easily follow from the following proposition of independent interest, in which there is no assumption whatsoever on the common characteristic function, or distribution, of the independent variables $X_1; X_2, \ldots$.

Proposition. *If* $\mathcal{N}=\{n_k\}_{k=1}^{\infty}$ *denotes an arbitrary nondecreasing sequence of natural numbers and if*

$$p_\alpha(\mathcal{N}) = \sup_{M>0} \inf_{K>0} \varlimsup_{k\to\infty} P\left\{ \sup_{K\le t\le \alpha^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \ge M \right\}$$

then $p_\alpha(\mathcal{N})>0$ *for every* $\alpha>1$.

Indeed, taking for granted the validity of this Proposition, Theorem 2 can be proved as follows. By assumption there is a $\gamma>0$ such that $T_{n_k}\ge e^{\gamma n_k}$ for some subsequence $\{n_k\}$ of the positive integers. On applying the Proposition with $\alpha=e^\gamma>1$, we obtain an $M>0$ and a $\delta>0$ such that

$$P\left\{ \sup_{K\le t\le e^{\gamma n_k}} \frac{|S_{n_k}(t)|}{n_k} \ge M \right\} \ge \delta$$

for every $K>0$. Choosing K so large that $|C(t)|<M/2$ be satisfied for $t\ge K$ and then putting $\varepsilon=M/2$, we obtain

$$\varlimsup_{k\to\infty} P\left\{ \sup_{|t|\le T_{n_k}} |C_{n_k}(t)-C(t)| > \varepsilon \right\} \ge \varlimsup_{k\to\infty} P\left\{ \sup_{K\le t\le e^{\gamma n_k}} |C_{n_k}(t)| \ge M \right\} \ge \delta,$$

which is the desired result.

In order to prove the Proposition, define

$$\beta(\mathcal{N}) = \inf\{\alpha:\ p_\alpha(\mathcal{N}) > 0\}.$$

What we have to show is that $\beta(\mathcal{N})=1$. First we establish the following properties of $\beta(\mathcal{N})$:

(i) $\beta(\mathcal{N})\le 6$ for every $\mathcal{N}$,

(ii) if $\mathcal{M}=\{m_k\}_{k=1}^{\infty}$ is another sequence of positive integers with

$$n_k-m_k = O(1), \quad k\to\infty, \tag{3}$$

then $\beta(\mathcal{N})=\beta(\mathcal{M})$.

(iii) if $2\mathcal{N}=\{2n_k\}_{k=1}^{\infty}$ then $(\beta(2\mathcal{N}))^2=\beta(\mathcal{N})$.

The proof of (i) is based on Dirichlet's theorem (see e.g. §2 of [7]) stating that if $y_1, \ldots, y_n$ are arbitrary real numbers, $K>0$ and $\alpha>1$, then there is an integer $t\in[K, K\alpha^n]$ such that with appropriate integers $v_1, \ldots, v_n$ the inequalities

$$|ty_j-v_j| < \frac{1}{\alpha}, \quad j = 1, \ldots, n,$$

are satisfied simultaneously. Applying this with $\alpha=5$ and $y_j=x_j/2\pi$, $j=1, \ldots, n$, we get that for arbitrary real numbers $x_1, \ldots, x_n$ and $K>0$ there is an integer t, $K\leqq t\leqq K5^n$, such that

$$\left|\sum_{j=1}^{n} e^{itx_j}\right| \geqq \mathrm{Re}\left\{\sum_{j=1}^{n} e^{itx_j}\right\} = \sum_{j=1}^{n}\mathrm{Re}\, e^{i(tx_j-2\pi v_j)} \geqq \sum_{j=1}^{n}\mathrm{Re}\, e^{i\frac{2\pi}{5}} = n\cos\frac{2\pi}{5}.$$

Since for every fixed K, we have $K5^n<6^n$ for all sufficiently large n, it follows that

$$\inf_{K>0}\varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq 6^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \geqq \cos\frac{2\pi}{5}\right\} = 1.$$

This means that $p_6(\mathcal{N})=1$, and hence (i) is proved.

Now suppose (3) and let $\alpha>\beta(\mathcal{N})$. If we choose α_1 in between, $\beta(\mathcal{N})<\alpha_1<\alpha$, then there exist an $M>0$ and a $\delta>0$ such that

$$\inf_{K>0}\varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq \alpha_1^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \geqq M\right\} = \delta.$$

But (3) implies that for all large enough k, $\alpha^{m_k}\geqq\alpha_1^{n_k}$ and

$$\frac{|S_{m_k}(t)|}{m_k} \geqq \frac{|S_{n_k}(t)|-|n_k-m_k|}{n_k+|n_k-m_k|} \geqq \frac{1}{2}\frac{|S_{n_k}(t)|}{n_k} - O\left(\frac{1}{n_k}\right), \quad t\in\mathbf{R},$$

and so

$$\inf_{K>0}\varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq \alpha^{m_k}} \frac{|S_{m_k}(t)|}{m_k} \geqq \frac{M}{3}\right\} \geqq$$

$$\geqq \inf_{K>0}\varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq \alpha_1^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \geqq M\right\} = \delta > 0.$$

This means that $p_\alpha(\mathcal{M})>0$. Since this is true for all $\alpha>\beta(\mathcal{N})$, we can conclude that $\beta(\mathcal{M})\leqq\beta(\mathcal{N})$. Reversing the role of $\mathcal{N}$ and $\mathcal{M}$, we obtain the opposite inequality, and hence (ii) is also proved.

Turning now to the proof of (iii), we introduce the following subsequences of the original X sequence:

$$X^{(1)} = \{X_1, X_4, X_7, X_{10}, \ldots\}, \quad X^{(2)} = \{X_2, X_5, X_8, X_{11}, \ldots\},$$

$$Y^{(1)} = \{X_2, X_3, X_5, X_6, X_8, X_9, X_{11}, X_{12}, \ldots\},$$

$$Y^{(2)} = \{X_1, X_3, X_4, X_6, X_7, X_9, X_{10}, X_{12}, \ldots\},$$

and

$$Y^{(3)} = \{X_1, X_2, X_4, X_5, X_7, X_8, X_{10}, X_{11}, \dots\}.$$

Let $\alpha < \beta(\mathcal{N})$. For each k;

$$S_{2n_k}(t;\ Y^{(3)}) = S_{n_k}(t;\ X^{(1)}) + S_{n_k}(t;\ X^{(2)}),$$

whence

$$p_{\sqrt{\alpha}}(2\mathcal{N}) = \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{ \sup_{K \leqq t \leqq (\sqrt{\alpha})^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(3)})|}{2n_k} \geqq M \right\} \leqq$$

$$\leqq \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} \left(P\left\{ \sup_{K \leqq t \leqq \alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(1)})|}{2n_k} \geqq \frac{M}{2} \right\} + \right.$$

$$\left. + P\left\{ \sup_{K \leqq t \leqq \alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(2)})|}{2n_k} \geqq \frac{M}{2} \right\} \right) \leqq$$

$$\leqq \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{ \sup_{K \leqq t \leqq \alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(1)})|}{n_k} \geqq M \right\} +$$

$$+ \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{ \sup_{K \leqq t \leqq \alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(2)})|}{n_k} \geqq M \right\} = 0+0 = 0,$$

where, at the last step, we used $\alpha < \beta(\mathcal{N})$. Thus $\alpha < \beta(\mathcal{N})$ implies $\sqrt{\alpha} \leqq \beta(2\mathcal{N})$. Therefore $\beta(\mathcal{N}) \leqq (\beta(2\mathcal{N}))^2$.

Now let $\alpha < \beta(2\mathcal{N})$. Clearly,

$$S_{n_k}(t;\ X^{(1)}) = \frac{1}{2}\{S_{2n_k}(t;\ Y^{(2)}) + S_{2n_k}(t;\ Y^{(3)}) - S_{2n_k}(t;\ Y^{(1)})\}.$$

Hence, similarly as above,

$$p_{\alpha^2}(\mathcal{N}) = \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{ \sup_{K \leqq t \leqq (\alpha^2)^{n_k}} \frac{|S_{n_k}(t;\ X^{(1)})|}{n_k} \geqq M \right\} \leqq$$

$$\leqq \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} \left(P\left\{ \sup_{K \leqq t \leqq \alpha^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(2)})|}{2n_k} \geqq \frac{M}{3} \right\} + \right.$$

$$\left. + P\left\{ \sup_{K \leqq t \leqq \alpha^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(3)})|}{2n_k} \geqq \frac{M}{3} \right\} + P\left\{ \sup_{K \leqq t \leqq \alpha^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(1)})|}{2n_k} \geqq \frac{M}{3} \right\} \right) =$$

$$= 0+0+0 = 0,$$

i.e., $\alpha < \beta(2\mathcal{N})$ implies $\alpha^2 \leqq \beta(\mathcal{N})$. Therefore the opposite inequality $(\beta(2\mathcal{N}))^2 \leqq \leqq \beta(\mathcal{N})$ also follows, and hence we have (iii).

Having now the three properties of $\beta(\mathcal{N})$, the proof of our Proposition is easy. For a positive integer m, set

$$\left[\frac{1}{2^m}\mathcal{N}\right] = \left\{\left[\frac{n_k}{2^m}\right]\right\}_{k=1}^{\infty}.$$

Since for fixed m,

$$2^m\left[\frac{n_k}{2^m}\right] - n_k = O(1), \quad k \to \infty,$$

we obtain by property (ii) that

$$\beta\left(2^m\left[\frac{1}{2^m}\mathcal{N}\right]\right) = \beta(\mathcal{N}),$$

and, by an m-fold application of property (iii), that

$$\left(\beta\left(2^m\left[\frac{1}{2^m}\mathcal{N}\right]\right)\right)^{2^m} = \beta\left(\left[\frac{1}{2^m}\mathcal{N}\right]\right).$$

Thus, by property (i),

$$\beta(\mathcal{N}) = \left(\beta\left(\left[\frac{1}{2^m}\mathcal{N}\right]\right)\right)^{\frac{1}{2^m}} \leqq 6^{\frac{1}{2^m}},$$

and since this holds for any integer $m \geqq 1$, the equality $\beta(\mathcal{N}) = 1$ follows.

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Série de Poincaré et systèmes de paramètres pour les invariants des formes binaires

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Pour le 70-ième anniversaire du Professeur Béla Sz.-Nagy

1. Introduction. Soient G un groupe, V un espace vectoriel complexe de dimension finie, $\mathbf{C}[V]$ l'algèbre des fonctions complexes polynomiales sur V, ϱ une représentation linéaire de G dans V, $\mathbf{C}[V]^G$ la sous-algèbre de $\mathbf{C}[V]$ formée des éléments $\varrho(G)$-invariants. Soient $\mathbf{C}[V]^G_n$ l'ensemble des éléments de $\mathbf{C}[V]^G$ qui sont homogènes de degré n, et $d_n = \dim \mathbf{C}[V]^G_n$. La série de Poincaré de l'algèbre graduée $\mathbf{C}[V]^G$ est $F(z) = \sum_{n \geqq 0} d_n z^n$.

Supposons désormais que G soit un groupe algébrique réductif et que la représentation ϱ soit rationnelle. Soit $(p_1, \ldots, p_r)$ un système de paramètres homogènes dans $\mathbf{C}[V]^G$ (il en existe). Alors $\mathbf{C}[V]^G$, considéré comme module sur $\mathbf{C}[p_1, \ldots, p_r]$, admet une base $(q_1 = 1, q_2, q_3, \ldots, q_s)$ formée d'éléments homogènes (cf. par exemple [7]). Si l'on pose $\deg p_i = d_i$, $\deg q_j = e_j$, on a donc

$$F(z) = \frac{z^{e_1} + z^{e_2} + \ldots + z^{e_s}}{(1-z^{d_1})(1-z^{d_2})\ldots(1-z^{d_r})}.$$

Réciproquement, supposons que $F(z)$ se mette sous la forme

$$F(z) = \frac{z^{e'_1} + z^{e'_2} + \ldots + z^{e'_t}}{(1-z^{d'_1})(1-z^{d'_2})\ldots(1-z^{d'_r})}$$

où $e'_1, \ldots, e'_t, d'_1, \ldots, d'_r$ sont des entiers >0 (r est nécessairement l'ordre du pôle 1). Existe-t-il un système de paramètres homogènes de degrés $d'_1, \ldots, d'_r$? Un contre exemple a été obtenu par R. STANLEY [7], 3.8, dans lequel G est un groupe fini. Nous allons construire un contre-exemple dans lequel $G = SL(2, \mathbf{C})$.

Avec les notations ci-dessus, la borne inférieure de s pour tous les systèmes de paramètres homogènes de $\mathbf{C}[V]^G$ a été appelée dans [2] la *complexité* de $\mathbf{C}[V]^G$;

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ce nombre fournit une manière de mesurer la «distance» de $\mathbf{C}[V]^G$ à une algèbre de polynômes. Dans le contre-exemple annoncé, nous calculerons cette complexité.

Désormais, on prend $G=SL(2, \mathbf{C})$. Soit V_d l'espace des formes binaires de degré d à coefficients complexes, dans lequel G opère canoniquement par une représentation irréductible ϱ_d (on a $\dim V_d=d+1$). Soit c_d la complexité de $\mathbf{C}[V_d]^G$. T. A. Springer a obtenu [6] l'évaluation $c_d \geqq \alpha d^{-9/2} 2^d$ où α est un nombre >0 indépendant de d. La démonstration de Springer utilise des calculs assez délicats concernant la série de Poincaré; mais, concernant les degrés des p_j, Springer n'utilise que l'inégalité évidente $\deg p_j \geqq 2$; en conservant la méthode de Springer, mais en évaluant ces degrés de manière plus détaillée, nous prouverons que, si A est un nombre $<1/2$, on a $c_d \geqq \exp(Ad \log d)$ pour d assez grand. La démonstration utilise un théorème sur la répartition des nombres premiers.

Première partie

2. Soient $V=V_5 \oplus V_1$, $\varrho=\varrho_5 \oplus \varrho_1$. La décomposition $V=V_5 \oplus V_1$ définit une bigraduation $(\mathbf{C}[V]^G_{m,n})_{m\geqq 0, n\geqq 0}$ de $\mathbf{C}[V]^G$. Soit $a_{mn}=\dim \mathbf{C}[V]^G_{m,n}$. La série de Poincaré de l'algèbre bigraduée $\mathbf{C}[V]^G$ est $\Phi(z, z')=\sum_{m,n\geqq 0} a_{mn} z^m z'^n$. Comme $\mathbf{C}[V]^G$ s'identifie à l'algèbre des covariants d'une forme binaire de degré 5, on trouve la valeur de $\Phi(z, z')$ dans [8], p. 224.

Comme dans l'introduction, graduons maintenant $\mathbf{C}[V]^G$ par le degré total. Sa série de Poincaré est $F(z)=\Phi(z, z)$. Utilisant [8], on trouve

$$F(z)=\frac{1-z^2+z^6+5z^8-3z^{10}+3z^{12}-5z^{14}-z^{16}+z^{20}-z^{22}}{(1-z^2)(1-z^4)^2(1-z^6)^2(1-z^8)}.$$

Le numérateur est divisible par $1-z^2$, d'où

$$F(z)=\frac{1+z^6+6z^8+3z^{10}+6z^{12}+z^{14}+z^{20}}{(1-z^4)^2(1-z^6)^2(1-z^8)}. \tag{1}$$

On vérifie facilement que l'écriture (1) est la forme irréductible de $F(z)$. Comme tous les coefficients du numérateurs sont $\geqq 0$, cette forme de $F(z)$ est du type considéré dans l'introduction. Comme elle est irréductible, c'est l'unique écriture minimale de $F(z)$ au sens de [2].

3. Lemme. *Soit* $\varphi=ax^5+5bx^4y+10cx^3y^2+10dx^2y^3+5exy^4+fy^5$ *une forme binaire de degré* 5. *Soit* ψ *le transvectant* $(\varphi, \varphi)_4$. *On a*

$$\frac{1}{2}\psi=(ae-4bd+3c^2)x^2+(af-3be+2cd)xy+(bf-4ce+3d^2)y^2. \tag{2}$$

Les conditions suivantes sont équivalentes:

(i) φ *a une racine d'ordre* $\geqq 4$ *en* x/y;

(ii) $\psi=0$.

La formule (2) est très facile. Pour prouver (i)⇒(ii), on peut, par action de $SL(2, \mathbf{C})$, se ramener au cas où $\varphi=ax^5+5bx^4y$; alors $\psi=0$ d'après (2). Supposons maintenant $\psi=0$, et prouvons (i), qui est d'ailleurs un casparticulier de P. GORDAN, *Vorlesungen über Invariantenentheorie*, 1885, p. 204.

Par action de $SL(2, \mathbf{C})$, on se ramène au cas où $f=0$. La condition $\psi=0$ se traduit alors par $ae-4bd+3c^2=-3be+2cd=-4ce+3d^2=0$. Si $e=0$, on trouve $d=0$, $c=0$, donc (i) est vérifié. Supposons $e\neq 0$. Par action de $SL(2, \mathbf{C})$, on peut, sans perdre les conditions précédentes, supposer que $d=0$. Alors, on trouve $c=0$, $b=0$, $a=0$, donc (i) est encore vérifié.

4. Soit $(\varphi, \varphi')\in V_5\oplus V_1=V$, avec

$$\varphi=ax^5+5bx^4y+10cx^3y^2+10dx^2y^3+5exy^4+fy^5,\ \varphi'=a'x+b'y.$$

L'algèbre $\mathbf{C}[V]^G$ s'identifie à l'algèbre des covariants de φ. On a donc en [4], p. 131, une table de générateurs de $\mathbf{C}[V]^G$. Posons

$$\psi_1=\frac{1}{2}(\varphi, \varphi)_2\in V_{6;\,2,0},\quad \psi_2=\frac{1}{2}(\varphi, \varphi)_4\in V_{2;\,2,0},\quad \psi_3=(\varphi, \psi_1)_1\in V_{9;\,3,0}.$$

(La notation $\omega\in V_{i;j,k}$ signifiera que ω est une forme homogène de degré i en x et y, dont les coefficients sont homogènes de degré j en $a, b, \ldots, f$, et de degré k en a', b'.) Alors $(\psi_2, \psi_2)_2$ est un scalaire qui dépend de φ, disons $p_1(\varphi)$, où p_1 est une fonction polynomiale homogène de degré 4 de $a, b, \ldots, f$. On a $p_1\in\mathbf{C}[V_5]^G_4$. Nous considérerons p_1 comme une fonction polynomiale bihomogène sur V, de bidegré (4, 0): $p_1\in\mathbf{C}[V]^G_{4,0}$.

Définissons de même $p_2\in\mathbf{C}[V]^G_{8,0}$ et $p_3\in\mathbf{C}[V]^G_{12,0}$ par

$$p_2(\varphi)=(\psi_2^3, \psi_1)_6,\quad p_3(\varphi)=(\psi_2^5, \varphi^2)_{10}.$$

Les fonctions p_1, p_2, p_3 s'identifient à 3 invariants fondamentaux de φ de degrés 4, 8, 12. Définissons encore $p_4\in\mathbf{C}[V]^G_{1,5}$, $p_5\in\mathbf{C}[V]^G_{2,2}$, $p_6\in\mathbf{C}[V]^G_{2,6}$, $p_7\in\mathbf{C}[V]^G_{3,9}$ par

$$p_4(\varphi, \varphi')=(\varphi, \varphi'^5)_5,\ p_5(\varphi, \varphi')=(\psi_2, \varphi'^2)_2,\ p_6=(\psi_1, \varphi'^6)_6,\ p_7=(\psi_3, \varphi'^9)_9.$$

5. Théorème. (i) *Les éléments* $p_1, p_5, p_4, p_2+p_6, p_3+p_7$ *de* $\mathbf{C}[V]^G$ *sont homogènes pour la graduation totale, de degrés* 4, 4, 6, 8, 12.

(ii) *L'ensemble* $\{p_1, p_5, p_4, p_2+p_6, p_3+p_7\}$ *est un système de paramètres pour* $\mathbf{C}[V]^G$.

(iii) *La complexité de* $\mathbf{C}[V]^G$ *est* 38.

(iv) *Il n'existe pas de système de paramètres homogènes de* $\mathbf{C}[V]^G$ *correspondant à l'écriture* (1) *de la série de Poincaré.*

(i) est évident.

(ii) Supposons que $p_1, p_5, p_4, p_2+p_6, p_3+p_7$ s'annulent pour (φ, φ'). On va prouver que (φ, φ') est instable. Comme le degré de transcendance de $\mathbf{C}[V]^G$ sur $\mathbf{C}$ est 5, cela établira (ii).

Supposons d'abord $\psi_2=0$, donc $p_2=0$. D'après le lemme 3, on se ramène au cas où $\varphi=ax^5+5bx^4y$. Alors

$$\psi_1 = (ax^3+3bx^2y)\cdot 0-(bx^3)^2 = -b^2x^6, \quad p_6 = -b^2b'^6, \quad p_4 = ab'^5-5ba'b'^4.$$

Les conditions $p_4(\varphi, \varphi')=(p_2+p_6)(\varphi, \varphi')=0$ donnent $ab'^5-5ba'b'^4=b^2b'^6=0$. Si $b'=0$, (φ, φ') est instable d'après le critère de Hilbert—Mumford. Supposons $b'\neq 0$. Alors, $b=0$, puis $a=0$, donc $\varphi=0$ et il est clair que (φ, φ') est instable.

Supposons désormais $\psi_2\neq 0$. Puisque $0=p_1(\varphi, \varphi')=(\psi_2, \psi_2)_2$, ψ_2 admet une racine double en x/y, et, par action de $SL\,(2, \mathbf{C})$, on peut supposer que cette racine est 0. D'après le lemme 3, on a alors

$$ae-4bd+3c^2 \neq 0, \tag{3}$$

$$af-3be+2cd = 0, \tag{4}$$

$$bf-4ce+3d^2 = 0. \tag{5}$$

Par ailleurs, $0=p_5(\varphi, \varphi')=(ae-4bd+3c^2)b'^2$ donc, compte tenu de (3),

$$b' = 0. \tag{6}$$

Supposons d'abord $a'=0$. Alors p_6 et p_7, qui sont de degrés >0 en (a', b'), s'annulent pour (φ, φ'). Donc $0=p_1(\varphi)=p_2(\varphi)=p_3(\varphi)$ de sorte que φ est instable dans V_5. Alors $(\varphi, \varphi')=(\varphi, 0)$ est instable dans V.

Supposons désormais $a'\neq 0$. On a $0=p_4(\varphi, \varphi')=(\varphi, \varphi'^5)_5=-fa'^5$ donc

$$f = 0. \tag{7}$$

Comme ψ_2^5 est proportionnel à x^{10}, la condition (7) entraîne que $(\psi_2^5, \varphi^2)_{10}=0$, d'où $p_3(\varphi, \varphi')=0$. Alors, $p_7(\varphi, \varphi')=0$, c'est-à-dire $(\psi_3, \varphi'^9)_9=0$. Comme $\varphi'^9=a'^9x^9$, le seul terme de ψ_3 qui intervient dans le calcul de $(\psi_3, \varphi'^9)_9$ est le terme en y^9. Il nous suffit donc de considérer les termes en xy^4 et y^5 dans φ, en xy^5 et y^6 dans ψ_1:

$$\varphi = \ldots+5exy^4+fy^5 = \ldots+5exy^4,$$

$$\psi_1 = (\ldots+dy^3)(\ldots+3exy^2)-(\ldots+3dxy^2+ey^3)^2 = \ldots-3edxy^5-e^2y^6,$$

$$\psi_3 = (\ldots+ey^4)(\ldots-e^2y^5)-(\ldots+4exy^3)\left(\ldots-\frac{3}{6}edy^4\right).$$

Le terme en y^9 de ψ_3 est donc $-e^3y^9$, d'où $0=(\psi_3, \varphi'^9)_9=e^3a'^9$. Comme $a'\neq 0$, on en déduit que

(8) $$e = 0.$$

Les conditions (5), (7), (8) donnent $d=0$. Comme $b'=0$, (φ, φ') est instable dans V d'après le critère de Hilbert—Mumford.

(iii) et (iv). Soit $(q_1, q_2, q_3, q_4, q_5)$ un système de paramètres homogènes de $\mathbf{C}[V]^G$. Soit $d_i=\deg(q_i)$. On peut supposer que $d_1\leqq d_2\leqq d_3\leqq d_4\leqq d_5$. Montrons que

(9) $$d_1d_2d_3d_4d_5 \geqq 2^{10}\cdot 3^2.$$

On a

$$F(z) = \frac{A(z)}{(1-z^{d_1})(1-z^{d_2})(1-z^{d_3})(1-z^{d_4})(1-z^{d_5})}$$

où les coefficients du polynôme $A(z)$ sont $\geqq 0$. Nous allons imiter le raisonnement de [2], §2. D'après la forme irréductible (1), $F(z)$ admet -1 comme pôle d'ordre 5, $\sqrt{-1}$ comme pôle d'ordre 3, $(1+\sqrt{-3})/2$ comme pôle d'ordre 2. Donc les d_i sont tous pairs, trois d'entre eux sont divisibles par 4, deux d'entre eux sont divisibles par 6. Le développement en série de $F(z)$ commence par $1+2z^4+3z^6$. Comme les coefficients de $A(z)$ sont $\geqq 0$, on voit que les d_i sont tous $\geqq 4$, et que, si l'on note α (resp. β) le nombre de d_i égaux à 4 (resp. 6), on a $\alpha\leqq 2$, $\beta\leqq 3$.

Supposons $\alpha=0$. Les d_i sont $\geqq 6$. Trois d'entre eux sont divisibles par 4, donc trois d'entre eux sont $\geqq 8$. Donc $\prod d_i\geqq 8^3\cdot 6^2=2^{11}\cdot 3^2$. Supposons $\alpha=1$. Alors $d_1=4$. Les entiers d_2, d_3, d_4, d_5 sont $\geqq 6$ et deux d'entre eux sont $\geqq 8$, donc $\prod d_i\geqq 4\cdot 8^2\cdot 6^2=2^{10}\cdot 3^2$. Supposons $\alpha=2$. Alors $d_1=d_2=4$. Distinguons plusieurs cas. Supposons $\beta=0$. Alors $d_3\geqq 8$. Comme deux des d_i sont divisibles par 6, deux des d_i sont $\geqq 12$, donc $\prod d_i\geqq 4^2\cdot 8\cdot 12^2=2^{11}\cdot 3^2$. Supposons $\beta=1$. Alors $d_3=6$, $d_4\geqq 8$. Comme deux des d_i sont divisibles par 6, on a $d_5\geqq 12$, donc $\prod d_i\geqq 4^2\cdot 6\cdot 8\cdot 12=2^{10}\cdot 3^2$. Supposons $\beta=3$. Alors $d_3=d_4=d_5=6$, ce qui est impossible puisque trois des d_i sont divisibles par 4.

Reste le cas $\beta=2$. Alors $d_3=d_4=6$, $d_5\geqq 8$. Comme trois des d_i sont divisibles par 4, on a $d_5\in\{8, 12, 16, \ldots\}$. Si $d_5\geqq 16$, on a $\prod d_i\geqq 4^2\cdot 6^2\cdot 16=2^{10}\cdot 3^2$. On va enfin montrer que les cas $d_5=8$, $d_5=12$ sont impossibles. Supposons $d_5=8$. On aurait donc un système de paramètres homogènes de degrés 4, 4, 6, 6, 8. Les conditions $q_1(\varphi, 0)=q_2(\varphi, 0)=\ldots=q_5(\varphi, 0)=0$ doivent entraîner l'instabilité de φ. Or $\mathbf{C}[V_5]^G$ ne contient que deux éléments homogènes algébriquement indépendants de degré $\leqq 8$, et l'annulation pour φ de deux tels invariants ne peut entraîner l'instabilité de φ puisque $\mathbf{C}[V_5]^G$ a pour degré de transcendance 3. Supposons $d_5=12$. On aurait donc un système de paramètres homogènes de degrés 4, 4, 6, 6, 12.

Considérons leurs restrictions à V_5. Comme $\mathbf{C}[V_5]_6^G=0$ et que $\dim \mathbf{C}[V_5]_4^G=1$, on obtient une contradiction comme dans le cas précédent.

On a donc prouvé (9). Comme le produit $4^2\cdot 6^2\cdot 8=2^9\cdot 3^2$ correspondant à l'écriture (1) de $F(z)$ est $<2^{10}\cdot 3^2$, on en déduit (iv). D'autre part, l'écriture de $F(z)$ correspondant au système de paramètres trouvé en (ii) donne $\prod d_i= =4^2\cdot 6\cdot 8\cdot 12=2^{10}\cdot 3^2$, et fournit donc la valeur minimale de $\prod d_i$ pour tous les systèmes de paramètres homogènes; dans cette écriture, le numérateur $A(z)$ se déduit du numérateur de (1) en multipliant par $1+z^6$; la somme des coefficients de $A(z)$ est alors

$$2(1+1+6+3+6+1+1) = 38$$

ce qui prouve (iii).

6. Il reste à savoir si l'on peut obtenir un contre-exemple analogue à 5 (iv) quand on considère une représentation *irréductible* de $SL(2, \mathbf{C})$. Cela est intéressant puisque les séries de Poincaré ont alors été calculées explicitement jusqu'à la dimension 17.

7. Le système de paramètres construit en 5 (ii) n'est pas bihomogène. En fait, il résulte de [1] qu'il n'existe aucun système de paramètres bihomogènes de $\mathbf{C}[V]^G$.

Deuxième partie

8. Lemme. *Soient n et p des entiers tels que $2\leqq p\leqq n$. On considère une forme binaire de degré n en x et y du type suivant:*

$$f(x, y) = a_0x^{n-u}y^u+a_1x^{n-u-p}y^{u+p}+a_2x^{n-u-2p}y^{u+2p}+\ldots+a_sx^{n-u-sp}y^{u+sp}$$

où $a_0, a_1, \ldots, a_s\in\mathbf{C}$. On suppose f instable. Alors $f(x, y)$ admet 0 ou ∞ comme racine en x/y de multiplicité $>n/2$.

Posons $g(X, Y)=a_0X^s+a_1X^{s-1}Y+a_2X^{s-2}Y^2+\ldots+a_sY^s$. On a $f(x, y)= =x^{n-u-sp}y^ug(x^p, y^p)$. On considère les racines en X/Y de $g(X, Y)$, à l'exclusion de 0 et ∞; soient $\omega_1, \ldots, \omega_r$ ces racines, deux à deux distinctes; la somme de leurs multiplicités est $\leqq s$. Alors les racines en x/y de $f(x, y)$, à l'exclusion de 0 et ∞, sont

$$\left(\omega_j^{1/p}, \omega_j^{1/p}\exp\left(2i\pi/p\right), \ldots, \omega_j^{1/p}\exp\left(2i\pi(p-1)/p\right)\quad (j = 1, 2, \ldots, r).\right.$$

Comme $\omega_j\neq 0, \infty$ pour tout j, chacune de ces racines est de multiplicité $\leqq s$. Or $u+sp\leqq n$, donc $s\leqq n/p\leqq n/2$. Comme f est instable, f admet une racine en x/y d'ordre $>n/2$. D'après ce qui précède, cette racine est 0 ou ∞.

9. Lemme. *Soient n, b des entiers $\geqq 1$. Soit p un nombre premier tel que $n/(2b-1)>p\geqq n/(2b+1)$. Soit $(P_1, P_2, \ldots, P_{n-2})$ un système de paramètres homo-*

gènes pour les formes binaires de degré n. Soit $\delta_j = \deg P_j$. Alors p divise δ_j pour au moins b indices j.

Toute forme binaire de degré n s'écrit $\alpha_0 x^n + \alpha_1 x^{n-1} y + \ldots + \alpha_n y^n$. Chaque P_j est un polynôme en $\alpha_0, \alpha_1, \ldots, \alpha_n$. Nous supposerons que $\delta_1, \ldots, \delta_r$ sont divisibles par p et que $\delta_{r+1}, \ldots, \delta_{n-2}$ sont non divisibles par p. Il s'agit de prouver que $r \geqq b$. Distinguons 4 cas suivant que $n/(2b-1) > p > n/2b$, $p = n/2b$, $n/2b > p > > n/(2b+1)$, $p = n/(2b+1)$,

Dans le 1 er cas, on a

$$(10) \quad 0 < p < 2p < \ldots < (b-1)p < n/2 < bp < (b+1)p < \ldots < (2b-1)p < n.$$

Considérons une forme binaire du type suivant:

$$f(x, y) = \alpha_0 x^n + \alpha_p x^{n-p} y^p + \alpha_{2p} x^{n-2p} y^{2p} + \ldots + \alpha_{(2b-1)p} x^{n-(2b-1)p} y^{(2b-1)p}.$$

Supposons $P_{r+1}(f) \neq 0$. Alors P_{r+1} contient, avec un coefficient non nul, un monôme de la forme

$$\alpha_0^{\mu_0} \alpha_p^{\mu_p} \alpha_{2p}^{\mu_{2p}} \alpha_{(2b-1)p}^{\mu_{(2b-1)p}}.$$

D'après [3], p. 32, on a $2(p\mu_p + 2p\mu_{2p} + \ldots + (2b-1)p\mu_{(2b-1)p}) = n\delta_{r+1}$. Comme p est premier et ne divise pas n, p divise δ_{r+1}, ce qui est absurde. Donc, $P_{r+1}(f) = 0$. De même, $P_{r+2}(f) = \ldots = P_{n-2}(f) = 0$.

Dans $P_1, \ldots, P_r$, remplaçons α_k par 0 toutes les fois que p ne divise pas k. On obtient des polynômes homogènes $Q_1, \ldots, Q_r$ en $\alpha_0, \alpha_p, \ldots, \alpha_{(2b-1)p}$. Les conditions

$$(11) \quad Q_1(\alpha_0, \ldots, \alpha_{(2b-1)p}) = Q_2(\alpha_0, \ldots, \alpha_{(2b-1)p}) = \ldots = Q_r(\alpha_0, \ldots, \alpha_{(2b-1)p}) = 0$$

entraînent que $P_1(f) = \ldots = P_r(f) = 0$. Par ailleurs, $P_{r+1}(f) = \ldots = P_{n-2}(f) = 0$ comme on l'a vu. Donc f est instable. D'après (10) et le lemme 8, on a $\alpha_0 = \alpha_p = \ldots \ldots = \alpha_{(b-1)p} = 0$, ou $\alpha_{bp} = \alpha_{(b+1)p} = \ldots = \alpha_{(2b-1)p} = 0$. Ainsi, les équations (11) définissent dans $\mathbf{C}^{2b}$ un cône algébrique de codimension $\geqq b$. Donc $r \geqq b$.

Dans le 2ème cas, on a

$$(12) \quad \begin{aligned} &1 < p+1 < 2p+1 < \ldots < (b-1)p+1 < n/2 < bp+1 < \\ &< (b+1)p+1 < \ldots < (2b-1)p+1 < n. \end{aligned}$$

Considérons une forme binaire du type suivant :

$$f(x, y) = \alpha_1 x^{n-1} y + \alpha_{p+1} x^{n-p-1} y^{p+1} + \ldots + \alpha_{(2b-1)p+1} x^{n-(2b-1)p-1} y^{(2b-1)p+1}.$$

Si $P_{r+1}(f) \neq 0$, on voit comme dans le 1 er cas qu'il existe des entiers μ_j tels que

$$2(\mu_1 + (p+1)\mu_{p+1} + (2p+1)\mu_{2p+1} + \ldots + ((2b-1)p+1)\mu_{(2b-1)p+1}) = n\delta_{r+1} = 2bp\delta_{r+1},$$

$$\mu_1 + \mu_{p+1} + \mu_{2p+1} + \ldots + \mu_{(2b-1)p+1} = \delta_{r+1}$$

d'où $p\mu_{p+1}+2p\mu_{2p+1}+\ldots+(2b-1)p\mu_{(2b-1)p+1}=(bp-1)\delta_{r+1}$. Donc p divise δ_{r+1}, ce qui est absurde. Donc $P_{r+1}(f)=P_{r+2}(f)=\ldots=P_{n-2}(f)$.

Dans $P_1, \ldots, P_r$, remplaçons α_k par 0 toutes les fois que $k\notin\{1, p+1, 2p+1, \ldots$ $\ldots, (2b-1)p+1\}$. On obtient des polynômes $Q_1, \ldots, Q_r$ dont l'annulation entraîne l'instabilité de f. D'après (12) et le lemme 8, on a $r\geqq b$.

Dans le 3ème cas, on a

$$0<p<2p<\ldots<bp<n/2<(b+1)p<(b+2)p<\ldots<2bp<n.$$

On considère $\alpha_0x^n+\alpha_px^{n-p}y^p+\ldots+\alpha_{2bp}x^{n-2bp}y^{2bp}$ et l'on raisonne comme dans le 1 er cas.

Dans le 4ème cas, et si $p>2$, on a

$$1<p+1<2p+1<\ldots<bp+1<n/2<(b+1)p+1<\ldots<2bp+1<n.$$

Raisonnant comme dans le 2ème cas, on a cette fois

$$2(\mu_1+(p+1)\mu_{p+1}+\ldots+(2bp+1)\mu_{2bp+1})=(2b+1)p\delta_{r+1},$$

$$\mu_1+\mu_{p+1}+\ldots+\mu_{2bp+1}=\delta_{r+1}$$

d'où $2(p\mu_{p+1}+2p\mu_{2p+1}+\ldots+2bp\mu_{2bp+1})=((2b+1)p-2)\delta_{r+1}$. Comme $p>2$, p divise δ_{r+1} et l'on termine comme plus haut. Supposons $p=2$, donc $n=2(2b+1)$. On revient à la méthode du 1 er cas, en écrivant

$$0<2<4<\ldots<2b<n/2<2b+2<\ldots<4b+2=n,$$

$$f(x, y)=\alpha_0x^n+\alpha_2x^{n-2}y^2+\alpha_4x^{n-4}y^4+\ldots+\alpha_ny^n.$$

Si $P_{r+1}(f)\neq 0$, on a $2(2\mu_2+4\mu_4+\ldots+n\mu_n)=2(2b+1)\delta_{r+1}$ donc 2 divise δ_{r+1}, ce qui est absurde. On trouve même, dans ce cas, que p divise δ_j pour au moins $b+1$ indices j.

10. Théorème. *Soit A un nombre $<1/2$. Alors $c_n\geqq\exp(An\log n)$ pour n assez grand.*

Soit $p_1>p_2>\ldots$ la suite décroissante des nombres premiers $<n$. Soit $a=a_n$ le plus petit entier $\geqq 0$ tel que $n/(2a+1)<2$. Définissons des entiers $s_1, s_2, \ldots, s_a$ par

$$\begin{gathered}
n>p_1>p_2>\ldots>p_{s_1}\geqq n/3\\
n/3>p_{s_1+1}>p_{s_1+2}>\ldots>p_{s_2}\geqq n/5\\
n/5>p_{s_2+1}>p_{s_2+2}>\ldots>p_{s_3}\geqq n/7\\
\vdots\\
n/(2a-1)>p_{s_{a-1}+1}>p_{s_{a-1}+2}>\ldots>p_{s_a}\geqq n/(2a+1)
\end{gathered}$$

(certaines lignes de ce tableau peuvent ne contenir aucun p_i).

Pour $p=p_{s_{b-1}+1}, p_{s_{b-1}+2}, \ldots, p_{s_b}$, p^b divise $\delta_1\delta_2\ldots\delta_{n-2}$ (lemme 9). Par suite,

$$\delta_1\delta_2 \ldots \delta_{n-2} \geqq p_1 p_2 \ldots p_{s_1} p_{s_1+1}^2 p_{s_1+2}^2 \ldots p_{s_2}^2 \ldots p_{s_{a-1}+1}^a p_{s_{a-1}+2}^a \ldots p_{s_a}^a$$

ou

$$\delta_1\delta_2 \ldots \delta_{n-2} \geqq (p_1 p_2 \ldots p_{s_a})(p_{s_1+1} p_{s_1+2} \ldots p_{s_a}) \ldots (p_{s_{a-1}+1} p_{s_{a-1}+2} \ldots p_{s_a}).$$

Soit $\mathscr{P}$ l'ensemble des nombres premiers. Pour x réel tendant vers $+\infty$, on a $\log \prod_{p\in\mathscr{P}, p<x} p \sim x$ ([5], th. 413 et 434). Choisissons des nombres A', A'' tels que $2A<A''<A'<1$. Soit $d=d_n$ le plus petit entier tel que $2d-1 \geqq n/\log n$. On a, pour n assez grand,

$$\prod_{\substack{p\in\mathscr{P}\\ p<n}} p \geqq \exp(A'n), \quad \prod_{\substack{p\in\mathscr{P}\\ p<n/3}} p \geqq \exp\left(A'\frac{n}{3}\right), \ldots, \prod_{\substack{p\in\mathscr{P}\\ p<n/(2d-1)}} p \geqq \exp\left(A'\frac{n}{2d-1}\right)$$

donc

$$\delta_1\delta_2 \ldots \delta_{n-2} \geqq \exp\left[A'n\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2d-1}\right)\right].$$

Quand $n\to\infty$, on a $d\sim n/2\log n$, donc

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2d-1} \sim \frac{1}{2}\log\frac{n}{\log n} \sim \frac{1}{2}\log n.$$

Par suite, pour n assez grand, on a

$$\delta_1\delta_2 \ldots \delta_{n-2} \geqq \exp\left(\frac{1}{2}A''n\log n\right). \tag{13}$$

Si le système de paramètres $(P_1, \ldots, P_{n-2})$ est choisi convenablement, on a

$$c_n/(\delta_1\delta_2 \ldots \delta_{n-2}) \geqq Bn^{-9/2} \tag{14}$$

où B est une constante >0 ([6], 3.4.12). Le théorème résulte de (13) et (14).

11. Remarque. Il est probable qu'en fait la croissance de c_n est encore plus rapide.

12. Remarque. En considérant dans le lemme 9 des puissances de nombres premiers, on peut améliorer légèrement la conclusion de ce lemme. Mais cela ne permet pas d'améliorer le théorème 10.

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On contractive ϱ-dilations

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Dedicated to Professor B. Sz.-Nagy on the occasion of his seventieth birthday

Let T be a (bounded linear) operator on a Hilbert space $\mathfrak{H}$ and ϱ a positive number. We say that W is a ϱ-dilation of T if W is an operator on a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ and

(1) $$T^n h = \varrho P W^n h \quad (h\in\mathfrak{H},\ n = 1, 2, \ldots)$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. $\mathscr{C}_\varrho$ denotes the class of those operators which have *unitary* ϱ-dilations.

The study of unitary ϱ-dilations and $\mathscr{C}_\varrho$ classes was initiated by B. Sz.-Nagy and C. Foiaş [4] and continued by a number of authors. (See [3] also for further references and [2], [5], [6] for some recent results.)

Studying operators of $\mathscr{C}_\varrho$ classes, sometimes (non-unitary) contractive ϱ-dilations can be succesfully used [1]. So the dilation space and the ϱ-dilation themselves remain "near enough" to the initial space and operator, respectively. In this note we show that, for any $T\in\mathscr{C}_\varrho$, there exists a contractive ϱ-dilation with certain additional properties. Moreover, any other contractive (especially unitary) ϱ-dilation of T is a 1-dilation of a contractive ϱ-dilation of T with such properties.

Theorem. *Let $T\in\mathscr{C}_\varrho$ and let W be any contraction satisfying* (1). *Introduce the notations*

$$\mathfrak{K}_+ = \bigvee_{n=0}^{\infty} W^n\mathfrak{H}, \quad \mathfrak{L} = \mathfrak{H}\vee(W|\mathfrak{K}_+)^*\mathfrak{H}$$

and define the contraction C on $\mathfrak{L}$ by $C = Q(W|\mathfrak{L})$, where Q denotes the orthogonal projection of $\mathfrak{K}_+$ onto $\mathfrak{L}$. Then W is a 1-dilation of C; C is a ϱ-dilation of T; and

(2) $$C^2h = CTh \quad (h\in\mathfrak{H}),$$ (2*) $$C^{*2}h = C^*T^*h \quad (h\in\mathfrak{H}),$$

(3) $$\mathfrak{L} = \mathfrak{H}\vee C\mathfrak{H},$$ (3*) $$\mathfrak{L} = \mathfrak{H}\vee C^*\mathfrak{H}.$$

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Proof. We introduce the notation

$$V = W|\mathfrak{K}_+ \quad (V\colon \mathfrak{K}_+ \to \mathfrak{K}). \tag{4}$$

For $h, g \in \mathfrak{H}$ and $n=0, 1, 2, \dots$ (4) and (1) imply that

$$(V^*(V^* - T^*)h, W^n g) = (h, W^{n+2}g) - (T^*h, W^{n+1}g) = 0.$$

This fact, (4) and the definition of $\mathfrak{K}_+$ show that $\mathfrak{K}_+ \perp (V^{*2} - V^*T^*)h \in \mathfrak{K}_+$ and consequently

$$V^{*2}h = V^*T^*h \quad (h \in \mathfrak{H}). \tag{5}$$

So (4) and the definition of $\mathfrak{L}$ show that $\mathfrak{L}$ is an invariant subspace of V^*.

Now we are going to prove by induction that W is a 1-dilation of C, i.e.

$$C^n h = QW^n h \quad (h \in \mathfrak{L},\ n = 1, 2, \dots). \tag{6}$$

For $n=1$, (6) is clear from the definition of C. If (6) is true for some positive integer n, then for $h, g \in \mathfrak{L}$ we have

$$(QW^{n+1}h, g) = (VW^n h, g) = (W^n h, V^*g) = (QW^n h, V^*g) =$$
$$= (C^n h, V^*g) = (WC^n h, g) = (QWC^n h, g) = (C^{n+1}h, g)$$

and this proves (6).

For $h \in \mathfrak{H}$ and $n=1, 2, \dots$ we have $PC^n h = PQW^n h = PW^n h = (1/\varrho)T^n h$, thus C is a ϱ-dilation of T.

If $h, g \in \mathfrak{L}$, then by (6) and (4)

$$(C^*h, g) = (h, Cg) = (h, Wg) = (h, Vg) = (V^*h, g).$$

Since $\mathfrak{L}$ is invariant for V^*, we have

$$C^* = V^*|\mathfrak{L}. \tag{7}$$

This fact, (4) and the definition of $\mathfrak{L}$ show that (3*) is true. Moreover, (5) and (7) imply (2*).

For $h, g \in \mathfrak{H}$ and $n=0$ or 1 we have

$$(C^2h - CTh, C^{*n}g) = (C^{n+2}h - C^{n+1}Th, g) = (1/\varrho)(T^{n+2}h - T^{n+2}h, g) = 0,$$

and so by (3*), $C^2h - CTh \perp \mathfrak{L}$ $(h \in \mathfrak{H})$, consequently (2) is true.

In order to prove (3), suppose that $g \in \mathfrak{L}$, $g \perp \mathfrak{H}$ and $g \perp C\mathfrak{H}$. In this case, by (2), $g \perp C^n\mathfrak{H}$ for $n=0, 1, \dots$. Now for every $h \in \mathfrak{H}$ we have

$$(g, W^n h) = (g, QW^n h) = (g, C^n h) = 0,$$

consequently, by the definition of $\mathfrak{K}_+$, $\mathfrak{K}_+ \perp g \in \mathfrak{K}_+$. This implies $g=0$. So the proof is complete.

The following two remarks show that the dilation space $\mathfrak{L}$ is "not too large".

Remark 1. $\mathfrak{L}=\mathfrak{H}$ if and only if $\varrho=1$ or $T^2=0$.

Proof. If $\mathfrak{L}=\mathfrak{H}$, then $Cg=(1/\varrho)Tg$ $(g\in\mathfrak{H})$ and so we have for every $h\in\mathfrak{H}$

$$(1/\varrho)T^2h = PC^2h = PC(1/\varrho)Th = P(1/\varrho^2)T^2h = (1/\varrho^2)T^2h.$$

This implies that $\varrho=1$ or $T^2=0$.

In order to prove the converse implication, suppose first that $\varrho=1$. In this case for $f, g\in\mathfrak{H}$ we have

$$(Ch-Th, g) = 0, \quad (Ch-Th, C^*g) = (C^2h-CTh, g) = 0.$$

Thus, by (3*) and (3), $\mathfrak{L}\perp Ch-Th\in\mathfrak{L}$, consequently $C|\mathfrak{H}=T$; and so by (3), $\mathfrak{L}=\mathfrak{H}$.

Suppose now that $T^2=0$. In this case for $h, g\in\mathfrak{H}$ we have

$$((C-(1/\varrho)T)h, C^*g) = (C^2h, g)-(1/\varrho)(CTh, g) = (1/\varrho)(T^2h, g)-(1/\varrho^2)(T^2h, g) = 0.$$

This means that $(C-(1/\varrho)T)h\perp C^*\mathfrak{H}$. Since $(C-(1/\varrho)T)h\perp\mathfrak{H}$ is also true, so by (3*), $(C-(1/\varrho)T)h\perp\mathfrak{L}$, consequently $Ch=(1/\varrho)Th$ and now (3) implies $\mathfrak{L}=\mathfrak{H}$.

Remark 2. For every $h\in\mathfrak{H}$, $Th=0$ implies $Ch=0$ and $T^*h=0$ implies $C^*h=0$.

Proof. If $Th=0$ then for every $g\in\mathfrak{H}$

$$0 = (Th, g) = \varrho(PCh, g) = \varrho(Ch, g)$$

and using (2)

$$0 = (Th, C^*g) = (CTh, g) = (C^2h, g) = (Ch, C^*g).$$

These mean that $Ch\perp\mathfrak{H}$ and $Ch\perp C^*\mathfrak{H}$, so by (3*), $Ch\perp\mathfrak{L}$ and consequently $Ch=0$.

The second implication can be proved in the same way, by using T^* in place of T and C^* in place of C.

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A note on unitary dilation theory and state spaces

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Dedicated to the 70th anniversary of Professor B. Sz.-Nagy and to the 30th anniversary of his unitary dilation theorem

1. Introduction and preliminaires

The major breakthrough in dilation theory for contractions on a Hilbert space was the existence of a minimal unitary dilation obtained in 1953 by B. SZ.-NAGY [15]. Lately, the emphasis of dilation theory for contractions has been mainly on minimal isometric dilations (functional models, characteristic functions, intertwining lifting theorems, etc.). However, the natural abstract framework for certain problems in theoretical engineering (Markov realizations of wide sense stationary Gaussian random processes [6, 9—14]) are intimately related to unitary dilations of contractions. In this way very interesting and new problems arise, in which the emphasis lies entirely on unitary dilations. Here we present a solution to one of these problems.

We follow the notation and terminology in [17]. In particular by a dilation we mean a strong (or power) dilation in [8]. Throughout U is a unitary operator on $\mathfrak{K}$ and $\mathfrak{H}$ is a subspace of $\mathfrak{K}$ such that

$$\mathfrak{K} = \bigvee_{-\infty}^{\infty} U^n \mathfrak{H}. \tag{1.1}$$

For a subspace $\mathfrak{X}$ of $\mathfrak{K}$ we denote by $T_{\mathfrak{X}}$ the compression of U to $\mathfrak{X}$, that is $T_{\mathfrak{X}}=P_{\mathfrak{X}}U|\mathfrak{X}$. A *state space* $\mathfrak{X}$ (for $\mathfrak{H}$) is a subspace of $\mathfrak{K}$ such that $\mathfrak{H}\subseteq\mathfrak{X}$ and U is the minimal unitary dilation for $T_{\mathfrak{X}}$. An operator $T_{\mathfrak{X}}$ is a *state space operator* if $\mathfrak{X}$ is a state space for $\mathfrak{H}$. A state space $\mathfrak{X}$ is *minimal* if $\mathfrak{X}$ contains no strictly proper state space. Our problem is to obtain a classification of all minimal state spaces for $\mathfrak{H}$. This problem is equivalent to certain problems which naturally occur in engineering and Markov processes [6, 9—14]. There dilation theory is mentioned but not exploited. Here we shall fully exploit dilation theory to obtain all minimal

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state spaces. It is shown that the minimal state space problem is deeply related with infinite-dimensional Jordan model theory [1, 18—21] and the notion of property (P) in [3]. In this way new results will be given in Sections 3 and 4 and many known results will be derived in a simple manner in Section 2.

To complete this section some further notation is established. If $\mathfrak{M}$ is a subspace then $\mathfrak{M}^\perp$ is its orthogonal complement. For a subspace $\mathfrak{X}$ in $\mathfrak{K}$ we define $\mathfrak{X}_+$ and $\mathfrak{X}_-$ by

(1.2) $$\mathfrak{X}_+ := \bigvee_{n\geqq 0} U^n\mathfrak{X} \quad \text{and} \quad \mathfrak{X}_- := \bigvee_{n\leqq 0} U^n\mathfrak{X}.$$

Let T be an operator in $\mathfrak{X}$; $\mathfrak{y}\subseteqq\mathfrak{X}$ is *cyclic* for T if $\mathfrak{X}=\bigvee_{n\geqq 0} T^n\mathfrak{y}$. A subspace $\mathfrak{W}$ is *semi-invariant* [4] for T if $\mathfrak{W}=\mathfrak{N}\ominus\mathfrak{M}$ where $\mathfrak{M}\subseteqq\mathfrak{N}$, and $\mathfrak{M},\mathfrak{N}$ are both invariant subspaces for T. Obviously $\mathfrak{W}=\mathfrak{M}^\perp\ominus\mathfrak{N}^\perp$. Therefore $\mathfrak{W}$ is semi-invariant for T if and only if $\mathfrak{W}$ is semi-invariant for T^*. Finally, the following lemma is needed. Its proof follows from Proposition 1.3 in [4] and the geometry of dilation theory [17]. A proof is given in [6].

Lemma 1.1. *Let U be a unitary operator on $\mathfrak{K}$ and $\mathfrak{X}$ a subspace of $\mathfrak{K}$. The following statements are equivalent:*

(a) *U is a dilation of $T_\mathfrak{X}$.*

(b) *$\mathfrak{X}$ is semi-invariant for U.*

(c) $P_{\mathfrak{X}_-}\mathfrak{X}_+=P_\mathfrak{X}\mathfrak{X}_+=\mathfrak{X}$.

(d) $P_{\mathfrak{X}_+}\mathfrak{X}_-=P_\mathfrak{X}\mathfrak{X}_-=\mathfrak{X}$.

2. Basic geometric results

In this section we develop a basic geometric structure for state spaces. The results in this section are not new. In a different form they are more or less contained in [9—14] and elsewhere. The proofs are presented for two reasons: first, for completeness; secondly and more importantly, to demonstrate the power of our approach. That is to demonstrate how minimal unitary dilation theory can be used to obtain simple proofs of these results. The following identities will be useful. If $\mathfrak{M},\mathfrak{N}$ are subspaces then

(2.1) $$\mathfrak{M} = (\overline{P_\mathfrak{M}\mathfrak{N}})\oplus(\mathfrak{M}\cap\mathfrak{N}^\perp).$$

If U is the minimal unitary dilation for $T_\mathfrak{X}$ then

(2.2) $$\mathfrak{X} = \mathfrak{X}_+\ominus(\mathfrak{X}_-)^\perp = \mathfrak{X}_-\ominus(\mathfrak{X}_+)^\perp.$$

Equation (2.2) follows from (2.1) and Lemma 1.1. It is also a consequence of the geometry of minimal unitary dilation theory, [17], Ch. II.

Let $\mathfrak{X}$ be a state space for $\mathfrak{H}$. Following [9—14], $\mathfrak{X}$ is *observable* [constructible] if $\mathfrak{X}=\overline{P_{\mathfrak{X}}\mathfrak{H}_+}$ [$\mathfrak{X}=\overline{P_{\mathfrak{X}}\mathfrak{H}_-}$], respectively. From the dilation property $\mathfrak{X}$ is observable [constructible] if and only if $\mathfrak{H}$ is cyclic for $T_{\mathfrak{X}}$ [$T_{\mathfrak{X}}^*$], respectively. The *observable,* respectively *constructible part,* of $\mathfrak{X}$ is the subspace defined by

$$\mathfrak{X}_o := \bigvee_{n\geqq 0} T_{\mathfrak{X}}^n\mathfrak{H} = \overline{P_{\mathfrak{X}}\mathfrak{H}_+} = \overline{P_{\mathfrak{X}_-}\mathfrak{H}_+}, \tag{2.3a}$$

$$\mathfrak{X}_c := \bigvee_{n\geqq 0} T_{\mathfrak{X}}^{*n}\mathfrak{H} = \overline{P_{\mathfrak{X}}\mathfrak{H}_-} = \overline{P_{\mathfrak{X}_+}\mathfrak{H}_-}. \tag{2.3b}$$

Since $\mathfrak{X}_o$ is an invariant subspace for $T_{\mathfrak{X}}$ and $\mathfrak{H}\subseteqq\mathfrak{X}_o$ it follows that U is the minimal unitary dilation for $T_{\mathfrak{X}_o}$. Obviously $\mathfrak{X}_o$ is an observable state space. In a similar manner it follows that $\mathfrak{X}_c$ is a constructible state space. The *observable and constructible part* of $\mathfrak{X}$ is

$$\mathfrak{X}_{oc} := (\mathfrak{X}_o)_c := \bigvee_{n\geqq 0} T_{\mathfrak{X}_o}^{*n}\mathfrak{H} = \overline{P_{\mathfrak{X}_o}\mathfrak{H}_-}. \tag{2.4}$$

By construction $\mathfrak{X}_{oc}$ is a constructible state space. Using the observability of $\mathfrak{X}_o$ and the fact that $T_{\mathfrak{X}_o}$ is a dilation of $T_{\mathfrak{X}_{oc}}$, it is easy to verify that $\mathfrak{X}_{oc}$ is observable. Decomposing: $\mathfrak{X}=\mathfrak{X}_o\oplus\mathfrak{X}_{\bar{o}}$ and $\mathfrak{X}_o=\mathfrak{X}_{o\bar{c}}\oplus\mathfrak{X}_{oc}$ where $\mathfrak{X}_{\bar{o}}, \mathfrak{X}_{o\bar{c}}$ are the appropriate orthogonal subspaces yields:

$$\mathfrak{X} = \mathfrak{X}_{o\bar{c}}\oplus\mathfrak{X}_{oc}\oplus\mathfrak{X}_{\bar{o}}. \tag{2.5}$$

Equation (2.5) implies that all state spaces contain a constructible and observable state space $\mathfrak{X}_{oc}$. In particular, a minimal state space is constructible and observable. If not, one could use (2.5) to obtain a smaller state space. Note $\mathfrak{X}=\mathfrak{X}_{oc}$ if and only if $\mathfrak{X}$ is observable and constructible. This proves half of

Proposition 2.1. [14] *Let $\mathfrak{X}$ be a state space. Then $\mathfrak{X}$ is constructible and observable if and only if $\mathfrak{X}$ is minimal.*

Proof (only if). Assume $\mathfrak{X}$ is constructible and observable. Let $\mathfrak{W}$ be a state space contained in $\mathfrak{X}$. Lemma 1.1 implies $T_{\mathfrak{X}}$ is a dilation for $T_{\mathfrak{W}}$. Since $\mathfrak{H}$, and thus $\mathfrak{W}$, is cyclic for $T_{\mathfrak{X}}$ we have $T_{\mathfrak{W}}P_{\mathfrak{W}}=P_{\mathfrak{W}}T_{\mathfrak{X}}$. (This fact is well known [16], p. 1.) This identity implies $\mathfrak{W}$ is an invariant subspace for $T_{\mathfrak{X}}^*$. Hence $T_{\mathfrak{W}}^*=T_{\mathfrak{X}}^*|\mathfrak{W}$. The constructibility of $\mathfrak{X}$ yields

$$\mathfrak{X} = \bigvee_{n\geqq 0} T_{\mathfrak{X}}^{*n}\mathfrak{H} \subseteqq \bigvee_{n\geqq 0} T_{\mathfrak{X}}^{*n}\ \mathfrak{W} \subseteqq \mathfrak{W}.$$

Using $\mathfrak{W}\subseteqq\mathfrak{X}$ gives $\mathfrak{X}=\mathfrak{W}$ and completes the proof.

As noted earlier Proposition 2.1 is not new [14]. Ruckebusch's proof depends upon splitting subspaces and some results in [9]. Here this result was derived directly from dilation theory.

Let $\mathfrak{X}$ be a state space. Equation (2.5) demonstrates that $\mathfrak{X}_{oc}$ is a minimal state space. In a similar manner $\mathfrak{X}_{co}:=(\mathfrak{X}_c)_o$ is a minimal state space. From any state space $\mathfrak{X}$ we can obtain possibly two different minimal state spaces, $\mathfrak{X}_{oc}$ and $\mathfrak{X}_{co}$. Obviously $\mathfrak{K}$ is a state space. The minimal state spaces $\mathfrak{P}$ and $\mathfrak{F}$ are defined by $\mathfrak{P}:=\mathfrak{K}_{co}$ and $\mathfrak{F}:=\mathfrak{K}_{oc}$. A simple calculation shows that $\mathfrak{P}=\overline{P_{\mathfrak{H}_-}\mathfrak{H}_+}$ and $\mathfrak{F}=\overline{P_{\mathfrak{H}_+}\mathfrak{H}_-}$. Notice that $\mathfrak{P}$ $[\mathfrak{F}]$ is the minimal state space for $\mathfrak{H}$ contained in the past $\mathfrak{H}_-$ [future $\mathfrak{H}_+$] of $\mathfrak{H}$ respectively. Here as in [9—14] the spaces $\mathfrak{P}$ and $\mathfrak{F}$ play an important role in our theory.

Proposition 2.2. *If $\mathfrak{X}$ is an observable [a constructible] state space then $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$ $[\mathfrak{X}_-\subseteqq\mathfrak{F}_-]$, respectively. In particular, if $\mathfrak{X}$ is minimal then $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$ and $\mathfrak{X}_-\subseteqq\mathfrak{F}_-$.*

Proof. Assume $\mathfrak{X}$ is observable. Equations (2.1) and (2.2) give

$$\mathfrak{P}\oplus(\mathfrak{P}_+)^\perp = \mathfrak{P}_- = \mathfrak{H}_- = \mathfrak{P}\oplus[\mathfrak{H}_-\cap(\mathfrak{H}_+)^\perp].$$

So $(\mathfrak{P}_+)^\perp=\mathfrak{H}_-\cap(\mathfrak{H}_+)^\perp$. This, $\mathfrak{H}_-\subseteqq\mathfrak{X}_-$ and the observability of $\mathfrak{X}$ (i.e., $\mathfrak{X}=\overline{\mathfrak{P}_{\mathfrak{X}}\mathfrak{H}_+}$ by (2.3a)) implies that $\mathfrak{X}$ is orthogonal to $(\mathfrak{P}_+)^\perp$. Hence $\mathfrak{X}\subseteqq\mathfrak{P}_+$ and $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$. A similar argument proves the other part.

The following will be useful.

Proposition 2.3. *Let $\mathfrak{X}$ be a state space and $\mathfrak{X}_c$ $[\mathfrak{X}_o]$ its constructible [observable] part, respectively. Then $\mathfrak{X}_{c+}=\mathfrak{X}_+$ and $\mathfrak{X}_{o-}=\mathfrak{X}_-$.*

Proof. Decomposing $\mathfrak{X}=\mathfrak{X}_c\oplus\mathfrak{X}_{\bar{c}}$ with (2.2) gives:

$$\mathfrak{X}_+ = \mathfrak{X}_c\oplus\mathfrak{X}_{\bar{c}}\oplus(\mathfrak{X}_-)^\perp \quad\text{and}\quad \mathfrak{X}_{c+} = \mathfrak{X}_c\oplus(\mathfrak{X}_{c-})^\perp. \tag{2.6}$$

Using $\mathfrak{X}_{c+}\subseteqq\mathfrak{X}_+$ implies:

$$(\mathfrak{X}_{c-})^\perp \subseteqq \mathfrak{X}_{\bar{c}}\oplus(\mathfrak{X}_-)^\perp. \tag{2.7}$$

To prove $\mathfrak{X}_{c+}=\mathfrak{X}_+$ it is sufficient to show that we have equality in (2.7). Assume x is in $\mathfrak{X}_{\bar{c}}\oplus(\mathfrak{X}_-)^\perp$ and x is orthogonal to $(\mathfrak{X}_{c-})^\perp$. Clearly x is in $\mathfrak{X}_{c-}$. Using $\mathfrak{X}_{c-}\subseteqq\mathfrak{X}_-$ places x in $\mathfrak{X}_-$. Hence x is in $\mathfrak{X}_{\bar{c}}$. This and (2.3b) verifies that x is orthogonal to $\mathfrak{H}_-$. Since x is in $\mathfrak{X}_{\bar{c}}$, (2.6) shows that x is orthogonal to $\mathfrak{X}_{c+}$. Combining:

$$x\perp(\mathfrak{H}_-\vee\mathfrak{X}_{c+}) \supseteqq (\mathfrak{H}_-\vee\mathfrak{H}_+) = \mathfrak{K}.$$

Therefore $x=0$ and there is equality in (2.7). The other part follows by duality.

One can easily derive Propositions 2.2 and 2.3 directly from the Lifting Theorem (Theorem 2.3 p. 66 in [17]). Let us show this for Proposition 2.3.

Alternate proof of Proposition 2.3. Since $\mathfrak{X}_c$ is an invariant subspace for $T_{\mathfrak{X}}^*$ we have $T_{\mathfrak{X}}^*Q^*=Q^*T_{\mathfrak{X}_c}^*$. Here Q^* is the operator mapping $\mathfrak{X}_c$ into

$\mathfrak{X}$ defined by $Q^*:=P_{\mathfrak{X}}|\mathfrak{X}_c$. Obviously $QT_{\mathfrak{X}}=T_{\mathfrak{X}_c}Q$ and $Q=P_{\mathfrak{X}_c}|\mathfrak{X}$. Notice that U is the minimal unitary dilation for both $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}_c}$. By the Lifting Theorem there exists a contraction R on $\mathfrak{K}$ such that

$$Q = P_{\mathfrak{X}_c}R|\mathfrak{X}, \quad R\mathfrak{X}_+ \subseteqq \mathfrak{X}_{c+} \quad \text{and} \quad UR = RU.$$

Note $Qh=h$ for all h in $\mathfrak{H}$. Thus $Rh=h$ for all h in $\mathfrak{H}$. Since U commutes with R we have $R=I$; the identity. Hence $\mathfrak{X}_+\subseteqq\mathfrak{X}_{c+}$ and $\mathfrak{X}_+=\mathfrak{X}_{c+}$.

The following will be useful. Similar results are given in [14] and elsewhere.

Proposition 2.4. *If $\mathfrak{X}$ is a constructible [an observable] state space such that $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$ [$\mathfrak{X}_-\subseteqq\mathfrak{F}_-$] then $T_{\mathfrak{X}}$ [$T_{\mathfrak{F}}$] is a quasi-affine transform of $T_{\mathfrak{P}}$ [$T_{\mathfrak{X}}$], respectively. In particular, if $\mathfrak{X}$ is a minimal state space then $T_{\mathfrak{X}}$ [$T_{\mathfrak{F}}$] is a quasi-affine transform of $T_{\mathfrak{P}}$ [$T_{\mathfrak{X}}$], respectively.*

Proof. Assume $\mathfrak{X}$ is constructible and $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$. Applying Lemma 1.1 gives (for all $h\in\mathfrak{H}$ and $n\geqq 0$),

(2.8) $$P_{\mathfrak{X}}T_{\mathfrak{P}}^{*n}h = P_{\mathfrak{X}}P_{\mathfrak{P}_+}U^{*n}h = P_{\mathfrak{X}}U^{*n}h = T_{\mathfrak{X}}^{*n}h \quad \text{and} \quad P_{\mathfrak{X}}T_{\mathfrak{P}}^{*}T_{\mathfrak{P}}^{*n}h = T_{\mathfrak{X}}^{*}P_{\mathfrak{X}}P_{\mathfrak{P}}^{*n}h.$$

Let Q be the operator mapping $\mathfrak{P}$ into $\mathfrak{X}$ defined by $Q:=P_{\mathfrak{X}}|\mathfrak{P}$. Equation (2.8) and the constructibility of $\mathfrak{P}$ implies $QT_{\mathfrak{P}}^*=T_{\mathfrak{X}}^*Q$. This, the constructibility of $\mathfrak{X}$ and $Qh=h$ for all h in $\mathfrak{H}$ shows that Q has dense range in $\mathfrak{X}$. Obviously $T_{\mathfrak{P}}Q^*=Q^*T_{\mathfrak{X}}$ where $Q^*=P_{\mathfrak{P}}|\mathfrak{X}$. This, the observability of $\mathfrak{P}$ and $Q^*h=h$ for all h in $\mathfrak{H}$ shows that Q^* has dense range in $\mathfrak{P}$. Thus Q is a quasi-affinity. This completes the first part. The second part follows by a similar argument.

3. Consequences of general dilation theory

In this section we list several results which are trivial consequences of the previous section and unitary dilation theory [17]. Some of these results have not been noted before and would be difficult to obtain without dilation theory. Others (namely (1), and part of (2)) were previously obtained without consulting dilation theory.

(1) Let $\mathfrak{X}$ be a state space for $\mathfrak{H}$. Then $T_{\mathfrak{X}}^n\to 0$ [$T_{\mathfrak{X}}^{*n}\to 0$] in the strong operator topology as $n\to\infty$ if and only if $\bigcap_{n\geqq 0} U^{*n}\mathfrak{X}_-=\{0\}$ [$\bigcap_{n\geqq 0} U^n\mathfrak{X}_+=\{0\}$], respectively.

(2) Assume $\mathfrak{X}$ is a state space and $T_{\mathfrak{X}}$ is a completely nonunitary contraction. Then $T_{\mathfrak{X}}$ is a $C_{\cdot 0}, C_{0\cdot}, C_{\cdot 1}, C_{1\cdot}$ contraction if and only if its characteristic function is inner, $*$-inner, outer, $*$-outer, respectively.

Parts (1) and (2) are trivial consequences of [17], ch. II and ch. VI. Parts (3) and (4) follow from Propositions 2.1, 2.2, 2.4 with [17] ch. II.

(3) ([14]) The set of all minimal state spaces are of the same dimension. In particular, $\mathfrak{H}$ admits a finite-dimensional state space if and only if $\mathfrak{P}$ or $\mathfrak{F}$ is finite dimensional.

(4) Assume that $T_{\mathfrak{P}}$ $[T_{\mathfrak{F}}]$ is a $C_{\cdot 0}$ $[C_{0\cdot}]$ contraction and $\mathfrak{X}$ is a minimal state space; respectively. Then $T_{\mathfrak{X}}$ is a $C_{\cdot 0}$ $[C_{0\cdot}]$ contraction respectively. In particular if $T_{\mathfrak{P}}$ is a $C_{\cdot 0}$ and $T_{\mathfrak{F}}$ is a $C_{0\cdot}$ contraction then $T_{\mathfrak{X}}$ is a C_{00} contraction.

From [17] ch. VI sec. 5 we have

(5) Assume $\mathfrak{H}$ is finite dimensional and there exists a C_{00} contraction for a state space operator. Then all minimal state space operators are C_0 contractions with finite defect indices.

Our final remark follows from Section 2 and the Jordan model theory in [1, 18—21] (see also Lemma 4.1 below).

(6) Assume that there exists a C_0 contraction for a state space operator, then all minimal state space operators are C_0 contractions, quasi-similar, and have the same Jordan model.

4. Minimality and property (P)

The results in the rest of this paper are believed to be new. In this section we obtain a classification of all minimal state spaces when $T_{\mathfrak{P}}$ is a C_0 contraction with property (P) [3]. An example is given to demonstrate that the C_0 assumption is natural to the problem. Our approach depends heavily on the infinite Jordan model theory in [1, 18—21] and C_0 contractions with property (P) [3]. Throughout we follow the notation and terminology established there. If m is an inner function then $\mathfrak{H}(m):=H^2\ominus mH^2$ and $S(m)$ is the operator on $\mathfrak{H}(m)$ defined by $S(m)f=$ $=P_{\mathfrak{H}(m)}e^{it}f$ where $f\in\mathfrak{H}(m)$. A contraction T on $\mathfrak{X}$ has *property* (P), if A on $\mathfrak{X}$ is any injection such that $AT=TA$ then A is a quasi-affinity [3]. We begin with some results in [1, 3, 18—21], which we shall need.

Lemma 4.1. *Let T on $\mathfrak{X}$ be a C_0 contraction and $\hat{T}$ on $\hat{\mathfrak{X}}$ be any contraction.*

I) *The following statements are equivalent:*

a) *T is a quasi-affine transform of $\hat{T}$.*

b) *$\hat{T}$ is a quasi-affine transform of T.*

c) *T is a quasi-similar to $\hat{T}$.*

d) *$\hat{T}$ is a C_0 contraction and $\hat{T}$ has the same Jordan model as T.*

II) *If any of* a), b), c) *or* d) *is valid and T has property (P) then $\hat{T}$ is a C_0 contraction with property (P).*

III) *T has property (P) if and only if T^* has property (P). If $\mathfrak{W}$ is semi-invariant for T and T has property (P), then $T_{\mathfrak{W}}$ $(:=P_{\mathfrak{W}}T|\mathfrak{W})$ is a C_0 contraction with property (P).*

IV) *T has property (P) if and only if given any semi-invariant subspace $\mathfrak{W}$ for T such that $T_{\mathfrak{W}}$ and T have the same Jordan model then $\mathfrak{W}=\mathfrak{X}$.*

Proof. Part I follows from [1, 18—21]. Parts II and III are in [3]. Now for part IV. Assume T has property (P) and $T_{\mathfrak{W}}$ has the same Jordan model as T. Let $\mathfrak{W}=\mathfrak{M}\ominus\mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces for T. Since $\mathfrak{N}$ is invariant for $T_{\mathfrak{M}}$ we have $T^*_{\mathfrak{W}}=T^*_{\mathfrak{M}}|\mathfrak{W}$. Let

$$\bigoplus_1^\infty S(m_i),\quad \bigoplus_1^\infty S(\tilde{m}_i),\quad \bigoplus_1^\infty S(\omega_i),\quad \bigoplus_1^\infty S(\tilde{\omega}_i)$$

be the Jordan models for $T_{\mathfrak{M}}, T^*_{\mathfrak{M}}, T_{\mathfrak{W}}, T^*_{\mathfrak{W}}$ respectively. Clearly $T^*_{\mathfrak{M}}X=XT^*_{\mathfrak{W}}$ where X is the identity operator injecting $\mathfrak{W}$ into $\mathfrak{M}$. Proposition 2 in [18] or [20] implies $\tilde{\omega}_i$ divides $\tilde{m}_i$ for all i. Furthermore, $TY=YT_{\mathfrak{M}}$ where Y is the identity operator injecting $\mathfrak{M}$ into $\mathfrak{X}$. Consulting Proposition 2 of [18] or [20] again implies m_i divides ω_i for all i. Combining, $\omega_i=m_i$ for all i. Therefore $T, T_{\mathfrak{M}}$ and $T_{\mathfrak{W}}$ all admit the same Jordan model. By Lemma 4.1.I there exists a quasi-affinity A mapping $\mathfrak{X}$ into $\mathfrak{M}$ such that $AT=T_{\mathfrak{M}}A=TA$. Since T has property (P): $\mathfrak{X}=\overline{A\mathfrak{X}}=\mathfrak{M}$ and $\mathfrak{W}=\mathfrak{X}\ominus\mathfrak{N}$ is invariant for T^*. There exists a quasi-affinity B mapping $\mathfrak{X}$ into $\mathfrak{W}$ such that $BT^*=T^*_{\mathfrak{W}}B=T^*B$. By Lemma 4.1.III, T^* also has property (P); consequently $\mathfrak{X}=\overline{B\mathfrak{X}}=\mathfrak{W}$. This completes half the proof of part IV.

The other half follows by contradiction. Assume that T does not have property (P). Then there exists an injection A on $\mathfrak{X}$ such that $TA=AT$ and $\overline{A\mathfrak{X}}\neq\mathfrak{X}$. Notice that $\overline{A\mathfrak{X}}$ is invariant for T. Lemma 4.1.I implies that $T|\overline{A\mathfrak{X}}$ and T have the same Jordan model. Since $\overline{A\mathfrak{X}}\neq\mathfrak{X}$ the proof is complete.

We begin with

Theorem 4.1. *Let $\mathfrak{H}$ admit a state space $\mathfrak{W}$ such that $T_{\mathfrak{W}}$ is a C_0 contraction with property (P). Then a state space $\mathfrak{X}$ is minimal if and only if $T_{\mathfrak{X}}$ is a C_0 contraction and has the same Jordan model as $T_{\mathfrak{F}}$ or $T_{\mathfrak{P}}$. In this case, all minimal state spaces $T_{\mathfrak{X}}$ are C_0 contractions with the same Jordan model as $T_{\mathfrak{F}}$ or $T_{\mathfrak{P}}$.*

Proof. First it is shown that $T_{\mathfrak{P}}$ is a C_0 contraction with property (P). Equation (2.5) shows that $\mathfrak{W}$ contains a minimal state space $\mathfrak{W}_{oc}$ which is semi-invariant for $T_{\mathfrak{W}}$. Lemma 4.1.III implies $T_{\mathfrak{W}_{oc}}$ is a C_0 contraction with property (P). By Proposition 2.4, $T_{\mathfrak{W}_{oc}}$ is a quasi-affine transform of $T_{\mathfrak{P}}$. By Lemma 4.1.II $T_{\mathfrak{P}}$ is a C_0 contraction with property (P).

Now assume $\mathfrak{X}$ is a minimal state space. Proposition 2.4 implies $T_{\mathfrak{X}}$ is a quasi-affine transform of $T_{\mathfrak{P}}$. By Lemma 4.1.I and the preceeding paragraph, $T_{\mathfrak{X}}$ is a C_0 contraction and has the same Jordan model as $T_{\mathfrak{P}}$.

Assume $T_{\mathfrak{X}}$ is a C_0 contraction with the same Jordan model as $T_{\mathfrak{P}}$. Lemma 4.1.I implies $T_{\mathfrak{X}}$ is a C_0 contraction with property (P). Equation (2.5) shows that $\mathfrak{X}$ contains a minimal state space $\mathfrak{X}_{oc}$ semi-invariant for $T_{\mathfrak{X}}$. Proposition 2.4 assigns the same Jordan model to both $T_{\mathfrak{X}_{oc}}$ and $T_{\mathfrak{P}}$. Hence $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}_{oc}}$ have the same Jordan model. Lemma 4.1.IV gives $\mathfrak{X}=\mathfrak{X}_{oc}$ and completes the proof.

Our classification of all minimal state spaces is given in

Theorem 4.2. *Let $\mathfrak{H}$ admit a state space such that T is a C_0 contraction with property (P). Then there is a one to one correspondence between the set of all minimal state spaces for $\mathfrak{H}$ and the set of all invariant subspaces $\mathfrak{G}$ $[\mathfrak{J}]$ for U $[U^*]$ such that*

$$\mathfrak{F}_+ \subseteqq \mathfrak{G} \subseteqq \mathfrak{P}_+ \quad [\mathfrak{P}_- \subseteqq \mathfrak{J} \subseteqq \mathfrak{F}_-], \tag{4.1}$$

respectively. In this case, the set of all minimal state spaces for $\mathfrak{H}$ are $\{\overline{P_{\mathfrak{G}}\mathfrak{H}_-}\}$ $[\{\overline{P_{\mathfrak{J}}\mathfrak{H}_+}\}]$ where $\mathfrak{G}$ $[\mathfrak{J}]$ is an invariant subspace for U $[U^*]$ satisfying (4.1), *respectively.*

Proof. Assume $\mathfrak{X}$ is a minimal state space. Obviously $\mathfrak{X}_+$ is an invariant subspace for U. Proposition 2.2 implies that $\mathfrak{G}=\mathfrak{X}_+$ satisfies (4.1). By constructibility $\mathfrak{X}=\overline{P_{\mathfrak{X}_+}\mathfrak{H}_-}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-}$.

Now assume $\mathfrak{G}$ is invariant for U and satisfies (4.1). Let $\mathfrak{X}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-}$. Notice that $\mathfrak{G}$ is a state space and $\mathfrak{X}$ is its constructible part. Proposition 2.3 gives $\mathfrak{X}_+=\mathfrak{G}_+=\mathfrak{G}$. Proposition 2.4 implies $T_{\mathfrak{X}}$ is a quasi-affine transform of $T_{\mathfrak{P}}$. $T_{\mathfrak{P}}$ is a C_0 contraction with property (P) (see the proof of Theorem 4.1). Thus $T_{\mathfrak{X}}$ is a C_0 contraction, and by Lemma 4.1.I and Theorem 4.1, $\mathfrak{X}$ is a minimal state space. Hence the correspondence $\mathfrak{X}\leftrightarrow\mathfrak{G}$ $(\mathfrak{X}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-})$ is bijective. This completes the proof of the first part. The second part follows in a similar manner.

Lemma 4.2. *Let T on $\mathfrak{X}$ be a C_0 contraction. If any one of the following statements holds then T has property (P):*

(a) *T is a weak contraction;*
(b) *T has finite multiplicity;*
(c) *T has finite defect indices.*

Lemma 4.2 follows from [2, 3, 21]. Recall that (c) implies (b) and (b) implies (a). Lemma 4.2, Theorem 4.2 and (5) in Section 3 gives

Corollary 4.2. *The conclusion of Theorem 4.2 is valid if any one of the following statements is true for any state space operator $T_{\mathfrak{X}}$.*

(i) *$T_{\mathfrak{X}}$ is a weak C_0 contraction;*
(ii) *$T_{\mathfrak{X}}$ is a C_0 contraction with finite multiplicity;*
(iii) *$T_{\mathfrak{X}}$ is a C_0 contraction with finite defect indices;*
(iv) *$T_{\mathfrak{X}}$ is a C_{00} contraction and $\mathfrak{H}$ is finite dimensional;*
(v) *$\mathfrak{H}$ is finite dimensional and $T_{\mathfrak{P}}$ or $T_{\mathfrak{F}}$ is a C_0 contraction.*

Obviously (v)$\Leftrightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i).

The following shows that one cannot remove the C_0 assumption in Theorem 4.2.

Example 4.1. Here we will construct a system U, $\mathfrak{H}, \mathfrak{K}$ and $\mathfrak{G}$ such that $\mathfrak{G}$ is an invariant subspace for U satisfying (4.1) and $\mathfrak{X}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-}$ is a not a minimal state space. To this end, let $z:=e^{it}$ and U be the bilateral shift on $L^2=\mathfrak{K}$. (Here $L^2=L^2(0, 2\pi)$ and $Uf=zf$ for f in L^2.) It is easy to see that $T_{\mathfrak{K}}=U$ has the property (P) (cf. [8], Problem 115). Let $\mathfrak{H}$ equal the one dimensional space spanned by e^z. The space $\mathfrak{G}$ will be defined later. Finally

$$(4.2) \qquad U_+ := U|H^2 \quad \text{and} \quad K^2 := L^2 \ominus (zH^2).$$

We begin the proof of our counter example. Since e^z is outer:

$$(4.3) \qquad \mathfrak{H}_+ = \mathfrak{F}_+ = H^2 \quad \text{and} \quad \mathfrak{K} = L^2 = \bigvee_{-\infty}^{\infty} U^n \mathfrak{H}.$$

Consulting [5] implies e^z is cyclic for U_+^*. A simple calculation gives:

$$(4.4) \qquad \mathfrak{F} := \overline{P_{\mathfrak{H}_+}\mathfrak{H}_-} = \bigvee_{n \geqq 0} U_+^{*n} e^z = H^2.$$

Therefore $\mathfrak{F}=\mathfrak{F}_+=H^2$. Recall $\mathfrak{P}=\overline{P_{\mathfrak{H}_-}\mathfrak{H}_+}$. Note e^z has an inverse in L^∞. Hence $\mathfrak{H}_-=\overline{e^z K^2}=e^z K^2$.

Obviously $\mathfrak{P} \subseteqq \mathfrak{H}_-$. We claim

$$(4.5) \qquad \mathfrak{P} = \mathfrak{H}_- = e^z K^2.$$

Let $e^z h^*$ be any element in $\mathfrak{H}_-$ that is orthogonal to $\mathfrak{P}$. (Here h is in H^2 and h^* is its complex conjugate.) Using $\mathfrak{P}=\overline{P_{\mathfrak{H}_-}H^2}$ shows that e^z is orthogonal to hH^2. Hence e^z is orthogonal to $h_i H^2$ where h_i is the inner part of h. Equivalently e^z is in $H^2 \ominus h_i H^2$. Since e^z is cyclic for U_+^* and $H^2 \ominus h_i H^2$ is invariant for U_+^* we have $h=0$. Therefore $e^z h^*=0$ and (4.5) holds.

Equation (4.5) gives $\mathfrak{P}_+=L^2$. Let ψ be any nonconstant inner function. Let $\mathfrak{G}=\psi^* H^2$. Then $H^2=\mathfrak{F}_+ \subseteqq \mathfrak{G} \subseteqq \mathfrak{P}_+=L^2$. Consulting [5] implies ψe^z is also cyclic for U_+^*. A calculation gives:

$$\mathfrak{X} = \overline{P_{\mathfrak{G}}\mathfrak{H}_-} = \overline{P_{\psi^* H^2} e^z K^2} = \psi^* (\overline{P_{H^2} \psi e^z K^2}) = \psi^* \Big[\bigvee_{n \geqq 0} U_+^{*n}(\psi e^z) \Big] = \psi^* H^2.$$

Hence $\mathfrak{X}=\mathfrak{G}=\psi^* H^2$. Obviously $\mathfrak{X}$ is not a minimal state space. It strictly contains the minimal state space $\mathfrak{F}=H^2$. The example is now complete.

Remark 4.1. In Example 4.1 the space $\mathfrak{H}$ only admits two minimal state spaces $\mathfrak{P}$ and $\mathfrak{F}$.

Classifications of all minimal state spaces are given in [9, 10, 13, 14] and elsewhere. It was shown in [11] that the proofs given there were not correct. In fact, Example 4.1 can be used to demonstrate that these results are not valid for certain infinite dimensional vector spaces. Recently [11] corrected some of the proofs in [9, 10, 13, 14] and showed that the classification of all minimal state spaces in [9, 10, 13, 14] (and elsewhere) were indeed valid for certain $\mathfrak{H}$ and U. It turns out that the results in [11] are equivalent to Corollary 4.2 part (v). However the methods in [11] do not extend to the general case Theorem 4.2. Here we have shown that property (P) plays an important role in obtaining all state spaces. Property (P) also plays an important role in deterministic systems theory [6] and other problems in operator theory [3, 21].

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On a representation of deterministic frontier-to-root tree transformations

FERENC GÉCSEG

To Professor B. Sz.-Nagy on his 70th birthday

In [8] M. STEINBY introduced the concept of the product of tree automata (the product of universal algebras, if we disregard the initial vectors and final states), and gave an algorithm to decide for every finite system of algebras whether or not it is isomorphically complete with respect to the product. So far, no similar result has been proved for homomorphic completeness. Moreover, by the knowledge of the author, there are no investigations concerning a system K of algebras which is complete for a system L of tree transformations in the following sense: every transformation from L can be induced by a tree transducer built (in an obvious way) on a product of algebras from K.

In this paper we introduce special types of products which are the tree automata theoretic generalizations of α_i-products of finite automata introduced in [3]. Moreover, we shall study a weaker form of the last-mentioned completeness (to be called m-completeness) with respect to the product and the α_i-products for the class of all deterministic tree transformations.

1. Notions and notations

By an *operator domain* we mean a set Σ together with a mapping $r\colon \Sigma \to N_0$ which assigns to every $\sigma \in \Sigma$ an *arity*, or *rank* $r(\sigma)$, where N_0 is the set of all nonnegative integers. For any $m \geqq 0$, $\Sigma_m = \{\sigma \in \Sigma \mid r(\sigma) = m\}$ is the set of the *m-ary operators* (or *operational symbols*). If Σ is finite then it is called a *ranked alphabet*. In the sequel we shall generally omit r in the definition of an operator domain Σ. Moreover, we shall suppose that if an operator belongs to more than one operator domain then it has the same rank in all of them.

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A finite subset $R \subseteq N_0$ is a *rank type*. It is said that the *rank type* of a ranked alphabet Σ is R if $r(\Sigma)=R$; that is R consists of all $m \in N_0$ for which $\Sigma_m \neq \emptyset$.

The set of *Σ-trees over* Z (or *Σ-polynomial symbols* with variables from Z) will be denoted by $F_\Sigma(Z)$. Moreover, for every $m \geqq 0$, $F_\Sigma^m(Z)$ is the set consisting of all trees $p \in F_\Sigma(Z)$ with $h(p) \leqq m$, where $h(p)$ is the *height* of p.

In the sequel we shall use the terms "node of a tree" and "subtree at a given node of a tree" in an informal and obvious way. Moreover, relabeling of nodes of a tree will mean that every label of a tree which is an operator is replaced by an arbitrary operator of the same rank.

The symbol X will stand for the countable set $\{x_1, x_2, \ldots\}$ of variables, and for every $n \geqq 0$, $X_n = \{x_1, \ldots, x_n\}$.

Let R be a rank type. Take an operator domain Σ of rank type R and a tree $p \in F_\Sigma(X_n)$ for some $n \geqq 0$. Consider another operator domain Ω of rank type R (not necessarily different from Σ) and a tree $q \in F_\Omega(X_n)$. We say that q is *similar* to p if the following conditions are satisfied:

(i) there exist relabelings of the nodes of p and q such that the resulting trees coincide,

(ii) if at two nodes d_1 and d_2 of p the subtrees coincide then q also has the same subtree at d_1 and d_2.

The class of all trees similar to p will be denoted by $[p]$.

Take a class S of trees. We say that S is a *shape* of rank type R if there exist a ranked alphabet Σ of rank type R, a non-negative integer $n \geqq 0$ and a tree $p \in F_\Sigma(X_n)$ such that $S=[p]$. The *height* $h(S)$ of S is $h(p)$. A shape S is *trivial* if $S=\{x_i\}$ for some $x_i \in X$. Otherwise S is called *nontrivial*. If we want to emphasize that all the frontier variables occurring in trees from S belong to X_n then we write $S(n)$ for S.

Let Σ be an operator domain. A Σ-algebra $\mathcal{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an m-ary operation $\sigma^{\mathcal{A}}: A^m \to A$, where m is the arity of σ. The operation $\sigma^{\mathcal{A}}$ is called the *realization* of σ in $\mathcal{A}$. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitly, but we write $\mathcal{A}=(A, \Sigma)$. The Σ-algebra $\mathcal{A}$ is *finite* if A is finite and Σ is a ranked alphabet. Moreover, $\mathcal{A}$ has *rank type* R if Σ is of rank type R. Finally, if p is a Σ-tree then the realization of p in $\mathcal{A}$ will be denoted by $p^{\mathcal{A}}$. If there is no danger of confusion then we omit $\mathcal{A}$ in $\sigma^{\mathcal{A}}$ and $p^{\mathcal{A}}$.

A *frontier-to-root ΣX_n-recognizer* or an *$F\Sigma X_n$-recognizer,* for short, is a system $\mathbf{A}=(\mathcal{A}, \mathbf{a}, X_n, A')$ where

(1) $\mathcal{A}=(A, \Sigma)$ is a finite Σ-algebra,

(2) $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)}) \in A^n$ is the *initial vector*,

(3) $A' \subseteq A$ is the set of final states.

If Σ and X_n are not specified then we speak about an *F-recognizer*. Moreover, let us note that in [7] we use a mapping $\alpha: X_n \to A$ instead of an initial vector.

Next we recall the concept of a tree transducer. To this we need one more set of variables $Y=\{y_1, y_2, ...\}$, and let $Y_n=\{y_1, ..., y_n\}$ for every $n \geqq 0$. Moreover, $\Xi=\{\xi_1, \xi_2, ...\}$ is the set of auxiliary variables, and $\Xi_n=\{\xi_1, ..., \xi_n\}$ for arbitrary $n \geqq 0$.

A *frontier-to-root tree transducer (F-transducer)* is a system $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, P, A')$, where

(1) Σ and Ω are ranked alphabets,

(2) X_n and Y_m are the *frontier alphabets*,

(3) A is a ranked alphabet consisting of unary operators, the *state set* of $\mathfrak{A}$. (It is assumed that A is disjoint with all other sets in the definition of $\mathfrak{A}$, except A'.)

(4) $A' \subseteq A$ is the set of *final states*,

(5) P is a finite set of *productions* of the following two types:

(i) $x \to aq$ $(x \in X_n,\ a \in A,\ q \in F_\Omega(Y_m))$,

(ii) $\sigma(a_1, ..., a_l) \to aq(\xi_1, ..., \xi_l)$ $(\sigma \in \Sigma_l,\ l \geqq 0,\ a_1, ..., a_l,\ a \in A,\ q(\xi_1, ..., \xi_l) \in F_\Omega(Y_m \cup \Xi_l))$.

The *transformation* induced by $\mathfrak{A}$ will be denoted by $\tau_{\mathfrak{A}}$. Moreover, deterministic totally defined F-transducers will be called DTF-*transducers,* too. One can easily show, that for every deterministic F-transducer $\mathfrak{A}$ there is a DTF-transducer $\mathfrak{B}$ with $\tau_{\mathfrak{A}}=\tau_{\mathfrak{B}}$. Accordingly, in this paper we deal transformations induced DTF-transducers.

To a DTF-transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, P, A')$ we can correspond an $F\Sigma X_n$-recognizer $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_n, A')$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, ..., a^{(n)})$, where

(1) $a^{(i)}=a$ if $x_i \to aq \in P$ for some q $(i=1, ..., n)$, and

(2) for arbitrary $l \geqq 0$, $\sigma \in \Sigma_l$ and $a_1, ..., a_l \in A$, $\sigma^{\mathscr{A}}(a_1, ..., a_l)=a$ if $\sigma(a_1, ..., a_l) \to aq \in P$, for some q.

This uniquely determined recognizer will be denoted by rec$(\mathfrak{A})$.

Now take an $F\Sigma X_n$-recognizer $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_n, A')$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, ..., a^{(n)})$. Define an F-transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, P, A')$ by

$$P = \{x_i \to a^{(i)}q^{(i)} \mid q^{(i)} \in F_\Omega(Y_m),\ i = 1, ..., n\} \cup$$

$$\cup \{\sigma(a_1, ..., a_l) \to \sigma^{\mathscr{A}}(a_1, ..., a_l)\, q^{(\sigma, a_1, ..., a_l)} \mid \sigma \in \Sigma_l,$$

$$l \geqq 0,\ a_1, ..., a_l \in A,\ q^{(\sigma, a_1, ..., a_l)} \in F_\Omega(Y_m \cup \Xi_l)\},$$

where the ranked alphabet Ω, the integer m and the trees in the right sides of the productions in P are fixed arbitrarily. Obviously, $\mathfrak{A}$ is a DTF-transducer. Denote by tr$(\mathbf{A})$ the class of all DTF-transducers obtained in the above way. It is easy to see that for arbitrary DTF-transducer $\mathfrak{A}$ the inclusion $\mathfrak{A} \in$ tr(rec$(\mathfrak{A})$) holds. Therefore, we have

Statement 1. *For every DTF-transducer* $\mathfrak{A}$ *there exists an F-recognizer* **A** *such that* $\mathfrak{A} \in \operatorname{tr}(\mathbf{A})$.

Before recalling the definition of products of algebras, we note that in the sequel if $\mathbf{a}$ is an n-dimensional vector then $\operatorname{pr}_i(\mathbf{a})$ $(1 \leqq i \leqq n)$ will denote its i^{th} component. Moreover, we suppose that every finite index set $I = \{i_1, \ldots, i_k\}$ is given together with a (fixed) ordering of its elements. Furthermore, for arbitrary system $\{a_{i_j} | i_j \in I\}$, $(a_{i_j} | i_j \in I)$ is the vector $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ if $i_1 < i_2 < \ldots < i_k$ is the ordering of I.

From now on we shall deal with a fixed rank type R. To exclude trivial cases, it will be assumed that for an $m > 0$, $m \in R$.

Let $\Sigma, \Sigma^1, \ldots, \Sigma^k$ be ranked alphabets of rank type R, and consider the Σ^i-algebras $\mathcal{A}_i = (A_i, \Sigma^i)$ $(i = 1, \ldots, k)$. Furthermore, let

$$\psi = \{\psi_m \colon (A_1 \times \ldots \times A_k)^m \times \Sigma_m \to \Sigma^1_m \times \ldots \times \Sigma^k_m | m \in R\}$$

be a family of mappings. Then by the *product* of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with respect to ψ we mean the Σ-algebra

$$\psi(\mathcal{A}_1, \ldots, \mathcal{A}_k, \Sigma) = \mathcal{A} = (A, \Sigma)$$

with $A = A_1 \times \ldots \times A_k$ and for arbitrary $m \in R$, $\sigma \in \Sigma_m$ and $\mathbf{a}_1, \ldots, \mathbf{a}_m \in A$,

$$\sigma^{\mathcal{A}}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \big(\sigma_1^{\mathcal{A}_1}(\operatorname{pr}_1(\mathbf{a}_1), \ldots, \operatorname{pr}_1(\mathbf{a}_m)), \ldots, \sigma_k^{\mathcal{A}_k}(\operatorname{pr}_k(\mathbf{a}_1), \ldots, \operatorname{pr}_k(\mathbf{a}_m))\big),$$

where $(\sigma_1, \ldots, \sigma_k) = \psi_m(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma)$.

(Sometimes we shall consider ψ_m to be an $(mk+1)$-ary function in an obvious sense.)

Consider the above product $\psi(\mathcal{A}_1, \ldots, \mathcal{A}_k, \Sigma) = \mathcal{A}$, and define the mappings $\psi^i \colon A^n \times F_\Sigma(X_n) \to F_{\Sigma^i}(X_n)$ $(i = 1, \ldots, k;\ n \geqq 0)$ in the following way: for arbitrary $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in A^n$ and $p \in F_\Sigma(X_n)$

(1) if $p = x_j$ $(1 \leqq j \leqq n)$ then $\psi^i(\mathbf{a}, p) = x_j$,

(2) if $p = \sigma(p_1, \ldots, p_m)$ $(\sigma \in \Sigma_m)$ then $\psi^i(\mathbf{a}, p) = \sigma^i(\psi^i(\mathbf{a}, p_1), \ldots, \psi^i(\mathbf{a}, p_m))$, where $(\sigma_1, \ldots, \sigma_k) = \psi_m(p_1^{\mathcal{A}}(\mathbf{a}), \ldots, p_m^{\mathcal{A}}(\mathbf{a}), \sigma)$.

One can easily see that the equation

$$p^{\mathcal{A}}(\mathbf{a}) = \big(\psi^1(\mathbf{a}, p)^{\mathcal{A}_1}(\operatorname{pr}_1(\mathbf{a}_1), \ldots, \operatorname{pr}_1(\mathbf{a}_n)), \ldots, \psi^k(\mathbf{a}, p)^{\mathcal{A}_k}(\operatorname{pr}_k(\mathbf{a}_1), \ldots, \operatorname{pr}_k(\mathbf{a}_n))\big)$$

holds. Moreover, for arbitrary i $(1 \leqq i \leqq k)$, $\mathbf{a} \in A^n$ and $p \in F_\Sigma(X_n)$, $\psi^i(\mathbf{a}, p) \in [p]$.

We now define special types of products. First of all let us write ψ_m in the form $\psi_m = (\psi_m^{(1)}, \ldots, \psi_m^{(k)})$, where for arbitrary $\mathbf{a}_1, \ldots, \mathbf{a}_m \in A$ and $\sigma \in \Sigma_m$,

$$\psi_m(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma) = \big(\psi_m^{(1)}(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma), \ldots, \psi_m^{(k)}(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma)\big).$$

We say that $\mathcal{A}$ is an α_i-product $(i = 0, 1, \ldots)$ if for any j $(1 \leqq j \leqq k)$ and $m \in R$, $\psi_m^{(j)}$ is independent of its u^{th} components if $(v-1)k + j + i \leqq u \leqq vk$ $(v = 1, \ldots, m)$. (Here

$\psi_m^{(j)}$ is considered an $(mk+1)$-ary function.) In the case of an α_i-product in $\psi_m^{(j)}$ we shall indicate only those variables on which $\psi_m^{(j)}$ may depend. For instance, we write $\psi_m^{(1)}(\sigma)$ for $\psi_m^{(1)}(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma)$ if $i=0$.

By the above definition, $\mathscr{A}$ is an α_i-product if for arbitrary j $(1 \leqq j \leqq k)$ and $m \in R$, $\psi_m^{(j)}$ is independent of component algebras $\mathscr{A}_u$ with $i+j \leqq u \leqq k$. If $i=0$ then we speak about a *loop-free product,* too. Moreover, if for every $m \in R$, ψ_m may depend on its last variable only then $\mathscr{A}$ is a *quasi-direct product*. If in addition, $\mathscr{A}_1 = \ldots = \mathscr{A}_k = \mathscr{B}$ then we speak about a *quasi-direct power* of $\mathscr{B}$.

One can see easily that the formation of the product, α_0-product and quasi-direct product is associative. (This is not true for the α_i-product with $i>0$.)

Let $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, P, A')$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, P', B')$ be two DTF-transducers and $m \geqq 0$ an integer. We write $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ if $\tau_{\mathfrak{A}}(p)=\tau_{\mathfrak{B}}(p)$ for every $p \in F_\Sigma^m(X_u)$.

Take a class K of algebras of rank type R. We say that K is *metrically complete* (*m-complete,* for short) with respect to the product (α_i-product) if for arbitrary DTF-transducer $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, P, A')$ and integer $m \geqq 0$ there exist a product (α_i-product) $\mathscr{B}=(B, \Sigma)$ of algebras from K, a vector $\mathbf{b} \in B^u$ and a subset $B' \subseteqq B$ such that $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ for some $\mathfrak{B} \in \operatorname{tr}(\mathbf{B})$, where $\mathbf{B}=(B, \mathbf{b}, X_u, B')$. (The name metrical completeness comes from the fact that such systems are the tree automata theoretic generalizations of metrically complete systems of finite automata introduced in [1].)

Let $\mathscr{A}=(A, \Sigma)$ be an algebra, $n \geqq 0$ an integer and $\mathbf{a} \in A^n$ a vector. For arbitrary $m \geqq 0$, set $A_{\mathbf{a}}^{(m)}=\{p^{\mathscr{A}}(\mathbf{a}) | p \in F_\Sigma^m(X_n)\}$. The system $(\mathscr{A}, \mathbf{a})$ is called *m-free* if $|A_{\mathbf{a}}^{(m)}|=|F_\Sigma^m(X_n)|$, i.e., $p \neq q$ implies $p(\mathbf{a}) \neq q(\mathbf{a})$ whenever $p, q \in F_\Sigma^m(X_n)$.

Now let $\mathscr{A}=(A, \Sigma)$, $\mathscr{B}=(B, \Sigma)$ be algebras, $n, m \geqq 0$ integers and $\mathbf{a} \in A^n$, $\mathbf{b} \in B^n$ vectors. We say that $(\mathscr{B}, \mathbf{b})$ is an *m-homomorphic image* of $(\mathscr{A}, \mathbf{a})$ if there is a mapping φ of $A_{\mathbf{a}}^{(m)}$ onto $B_{\mathbf{b}}^{(m)}$ such that

(1) $\varphi(\operatorname{pr}_i(\mathbf{a}))=\operatorname{pr}_i(\mathbf{b})$ for all $i=1, \ldots, n$,

(2) $\varphi(\sigma^{\mathscr{A}}(a_1, \ldots, a_l))=\sigma^{\mathscr{B}}(\varphi(a_1), \ldots, \varphi(a_l))$ for arbitrary $l \in R$, $\sigma \in \Sigma_l$ and $a_1, \ldots, a_l \in A_{\mathbf{a}}^{(m-1)}$.

If in addition φ is one-to-one then we speak about an *m-isomorphic image,* or we say that $(\mathscr{A}, \mathbf{a})$ and $(\mathscr{B}, \mathbf{b})$ are *m-isomorphic,* in notation $(\mathscr{A}, \mathbf{a}) \overset{m}{\cong} (\mathscr{B}, \mathbf{b})$.

We obviously have the following statements.

Statement 2. *Let $\mathscr{A}=(A, \Sigma)$ and $\mathscr{B}=(B, \Sigma)$ be algebras. Take two integers $m, n \geqq 0$ and two vectors $\mathbf{a} \in A^n, \mathbf{b} \in B^n$. If $(\mathscr{A}, \mathbf{a})$ is m-free then*

(i) *$(\mathscr{B}, \mathbf{b})$ is an m-homomorphic image of $(\mathscr{A}, \mathbf{a})$, and*

(ii) *for arbitrary $\mathbf{B}=(\mathscr{B}, \mathbf{b}, X_n, B')$ and $\mathfrak{B} \in \operatorname{tr}(\mathbf{B})$ there exist $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_n, A')$ and $\mathfrak{A} \in \operatorname{tr}(\mathbf{A})$ such that $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$.*

Statement 3. *Let $\mathscr{A}=(A, \Sigma)$ and $\mathscr{B}=(B, \Sigma)$ be algebras. Take two integers $m, n \geqq 0$ and two vectors $\mathbf{a} \in A^n, \mathbf{b} \in B^n$. If $(\mathscr{A}, \mathbf{a})$ and $(\mathscr{B}, \mathbf{b})$ are m-free then they are m-isomorphic. Conversely, if $(\mathscr{A}, \mathbf{a})$ is m-free and m-isomorphic to $(\mathscr{B}, \mathbf{b})$ then $(\mathscr{B}, \mathbf{b})$ is also m-free.*

Let $(\mathscr{A}, \mathbf{a})$ $\left(\mathscr{A}=(A, \Sigma), \mathbf{a} \in A^n\right)$ be a system, $\mathscr{B}=(B, \Sigma)$ an algebra and $m \geqq 0$ integer. We say that $(\mathscr{A}, \mathbf{a})$ can be represented *m-isomorphically* by $\mathscr{B}$ if there exists a $\mathbf{b} \in B^n$ such that $(\mathscr{A}, \mathbf{a}) \overset{m}{\cong} (\mathscr{B}, \mathbf{b})$.

Finally, we say that the α_i-product and the α_j-product $(i, j \geqq 0)$ are *metrically equivalent* (*m-equivalent*) if a system of algebras is m-complete with respect to the α_i-product if and only if it is m-complete with respect to the α_j-product. The m-equivalence between an α_i-product and the product is defined similarly. (Let us note that in [4] the term "metrical equivalence" is used in a stronger sense.)

For notions not defined here we refer the reader to [5] and [6] or [7].

2. Metrically complete systems of algebras

In this section we shall give necessary and sufficient conditions for a system of algebras to be m-complete with respect to the α_i-products $(i=0, 1, \ldots)$ and the product. It will turn out that all the α_i-products are m-equivalent to each other and they are m-equivalent to the product.

First we prove

Theorem 1. *A system K of algebras of rank type R is m-complete with respect to the α_i-product (product) if and only if for arbitrary $m, n \geqq 0$ and ranked alphabet Σ of rank type R every m-free system $(\mathscr{A}, \mathbf{a})$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a} \in A^n$ can be represented m-isomorphically by an α_i-product (product) of algebras from K.*

Proof. The sufficiency is obvious by Statements 1 and 2.

To prove the necessity take an m-free system $(\mathscr{A}, \mathbf{a})$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)})$, where $\mathscr{A}$ is of rank type R. Moreover, let Ω be a ranked alphabet such that for every $l \in R$, $|\Omega_l| \geqq |F_\Sigma^{m+1}(X_n)|$. Consider the DTF-transducer $\mathfrak{A}= =(\Sigma, X_n, A, \Omega, X_n, P, A)$, where P consists of the productions

$$x_i \rightarrow a^{(i)} x_i \quad (i=1, \ldots, n), \tag{1}$$

$$\sigma(a_1, \ldots, a_l) \rightarrow \sigma^{\mathscr{A}}(a_1, \ldots, a_l)\, \omega(\xi_1, \ldots, \xi_l) \quad (\sigma \in \Sigma_l, \omega \in \Omega_l, a_1, \ldots, a_l \in A,\ l \geqq 0) \tag{2}$$

such that $n+|\{\omega | \sigma(a_1, \ldots, a_l) \rightarrow \sigma^{\mathscr{A}}(a_1, \ldots, a_l)\omega(\xi_1, \ldots, \xi_l) \in P, a_1, \ldots, a_l \in \{p^{\mathscr{A}}(\mathbf{a}) | p \in \in F_\Sigma^m(X_n)\}\}|=|F_\Sigma^{m+1}(X_n)|$. (Since $(\mathscr{A}, \mathbf{a})$ is m-free, by our assumptions about the cardinality of Ω, P can be chosen thus.)

Now let $\mathscr{B}=(B, \Sigma)$ be an α_i-product (product) of algebras from K, $\mathbf{b}=(b^{(1)}, \ldots, b^{(n)})\in B^n$ a vector and suppose that for some $\mathbf{B}=(\mathscr{B}, \mathbf{b}, X_n, B')$ and DTF-transducer $\mathfrak{B}=(\Sigma, X_n, B, \Omega, X_n, B')\in \operatorname{tr}(\mathbf{B})$ the relation $\tau_{\mathfrak{B}} \overset{m+1}{=} \tau_{\mathfrak{A}}$ holds. We shall show that $(\mathscr{A}, \mathbf{a}) \overset{m}{\cong} (\mathscr{B}, \mathbf{b})$. To this, by Statement 3, it is enough to prove that $(\mathscr{B}, \mathbf{b})$ is m-free.

Suppose that for two trees $p_1, p_2 \in F_\Sigma^m(X_n)$ we have $p_1 \neq p_2$ and $p_1^{\mathscr{B}}(\mathbf{b})=p_2^{\mathscr{B}}(\mathbf{b})$. For an $l\in R$ with $l>0$ take a $\sigma\in\Sigma_l$ and arbitrary $r_2, \ldots, r_l \in F_\Sigma^m(X_n)$. Set $t_1=\sigma(p_1, r_2, \ldots, r_l)$ and $t_2=\sigma(p_2, r_2, \ldots, r_l)$. Then the trees q_1 and q_2 obtained by $t_1 \Rightarrow_{\mathfrak{B}}^* t_1^{\mathscr{B}}(\mathbf{b})q_1$ and $t_2 \Rightarrow_{\mathfrak{B}}^* t_2^{\mathscr{B}}(\mathbf{b})q_2$ have the same label at their roots. Moreover, by $\tau_{\mathfrak{A}} \overset{m+1}{=} \tau_{\mathfrak{B}}$, the derivations $t_1 \Rightarrow_{\mathfrak{A}}^* t_1^{\mathscr{A}}(\mathbf{a})q_1$ and $t_2 \Rightarrow_{\mathfrak{A}}^* t_2^{\mathscr{A}}(\mathbf{a})q_2$ hold, too. Thus, by the choice of P, q_1 and q_2 should have distinct labels at their roots, which is a contradiction. This ends the proof of Theorem 1.

Next we give necessary conditions for a system of algebras to be m-complete with respect to the product.

Theorem 2. *Let K be a system of algebras of rank type R which is m-complete with respect to the product. Then for arbitrary integers $m, n \geqq 0$ and nontrivial shape $S(n)$ with rank type R and height less than or equal to m, there is an algebra $\mathscr{A}=(A, \Sigma)\in K$, a vector $\mathbf{a}\in A^n$, a tree $\sigma(p_1, \ldots, p_l)\in S\cap F_\Sigma(X_n)$ $(\sigma\in\Sigma_l)$ and an operator $\sigma'\in\Sigma_l$ such that $\sigma(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a})) \neq \sigma'(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a}))$.*

Proof. Assume that there exist integers $m, n \geqq 0$ and a nontrivial $S(n)$ with $h(S(n))=k$ $(0\leqq k\leqq m)$ such that for arbitrary $\mathscr{A}=(A, \Sigma)\in K$, $\mathbf{a}\in A^n$ and $\sigma(p_1, \ldots, p_l)$, $\sigma'(p_1, \ldots, p_l)\in S(n)\cap F_\Sigma(X_n)$ $(\sigma, \sigma'\in\Sigma_l)$ the equation $\sigma(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a}))=\sigma'(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a}))$ holds. Consider a k-free system $(\mathscr{B}=(B, \Omega), \mathbf{b})$, where the ranked alphabet Ω has rank type R, $|\Omega_l|\geqq 2$ and $\mathbf{b}=(b^{(1)}, \ldots, b^{(n)})$. We show that $(\mathscr{B}, \mathbf{b})$ cannot be represented k-isomorphically by any product of algebras from K. Indeed, let

$$\mathscr{C} = (C, \Omega) = \psi(\mathscr{A}_1, \ldots, \mathscr{A}_r, \Omega) \quad (\mathscr{A}_i\in K, \quad i = 1, \ldots, r)$$

be an arbitrary product and $\mathbf{c}=(\mathbf{c}_1, \ldots, \mathbf{c}_n)\in C^n$ a vector. Take two trees $q=\omega_1(q_1, \ldots, q_l)$ and $q'=\omega_2(q_1, \ldots, q_l)$ such that $\omega_1, \omega_2\in\Omega_l$, $\omega_1\neq\omega_2$ and $q, q'\in S(n)$. Then we have

$$q(\mathbf{c}) = \big(\omega_1^1(q_1^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n)), \ldots, q_l^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n))), \ldots$$
$$\ldots, \omega_1^r(q_1^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)), \ldots, q_l^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)))\big)$$

and

$$q'(\mathbf{c}) = \big(\omega_2^1(q_1^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n)), \ldots, q_l^1(\mathrm{pr}_1(\mathbf{c}_1), \ldots, \mathrm{pr}_1(\mathbf{c}_n))), \ldots$$
$$\ldots, \omega_2^r(q_1^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)), \ldots, q_l^r(\mathrm{pr}_r(\mathbf{c}_1), \ldots, \mathrm{pr}_r(\mathbf{c}_n)))\big),$$

where $q_i^j=\psi^j(\mathbf{c}_1, \ldots, \mathbf{c}_n, q_i)$ $(i=1, \ldots, l;\ \ j=1, \ldots, r)$, $(\omega_1^1, \ldots, \omega_1^r)=\psi_l(q_1(\mathbf{c}), \ldots$ $\ldots, q_l(\mathbf{c}), \omega_1)$ and $(\omega_2^1, \ldots, \omega_2^r)=\psi_l(q_1(\mathbf{c}), \ldots, q_l(\mathbf{c}), \omega_2)$. By our remark following the definition of $\psi^i(\mathbf{a}, p)$, the inclusions $\omega_i^j(q_1^j, \ldots, q_l^j)\in S(n)$ hold for all $i\,(=1, 2)$ and $j\,(=1, \ldots, r)$. Therefore, $q(\mathbf{c})=q'(\mathbf{c})$, i.e., $(\mathscr{C}, \mathbf{c})$ is not k-free. Since $(\mathscr{C}, \mathbf{c})$ was chosen arbitrarily, by Theorem 1 and Statement 3, this contradicts the assumption that K is m-complete with respect to the product, ending the proof of Theorem 2.

We shall show that if a system of algebras satisfies the conclusions of Theorem 2 then it is m-complete with respect to the loop-free product. To this two lemmas are needed.

In the next lemma Σ will be a fixed ranked alphabet of rank type R such that for every $l\in R$, Σ_l is a two-element set: $\Sigma_l=\{\sigma_l, \sigma_l'\}$.

Lemma 3. *Let K be a system of algebras with rank type R satisfying the conclusions of Theorem 2. Then for, arbitrary $m, n\geqq 0$, every m-free system $(\mathscr{A}, \mathbf{a})$ $(\mathscr{A}=(A, \Sigma), \mathbf{a}\in A^n)$ can be represented m-isomorphically by an α_0-product of algebras from K.*

Proof. We proceed by induction on m.

Let $m=0$. It follows from our assumptions that in K there is an algebra with at least two elements. Moreover, if $0\in R$ then this algebra can be chosen in such a way that it has at least two distinct 0-ary operations. One can easily show that a quasi-direct power of this algebra 0-isomorphically represents $(\mathscr{A}, \mathbf{a})$.

Now suppose that Lemma 3 has been proved for every $k\leqq m$. Let $(\mathscr{B}, \mathbf{b})$ be an m-free system, where $\mathscr{B}=(B, \Sigma)$ and $\mathbf{b}=(b_1, \ldots, b_n)\in B^n$. Take the index set $I=\{(p, q)|p, q\in F_\Sigma(X_n),\ \ p\neq q,\ \ h(p)=m+1,\ \ h(q)\leqq m+1\}$. Consider a pair $(p, q)\in I$, and let $p=\delta_l(p_1, \ldots, p_l)$ where δ_l is σ_l or σ_l'. Then by our assumptions, there is a $\mathscr{C}^{(p)}=(C^{(p)}, \Sigma^{(p)})$ in K, an n-dimensional vector $\mathbf{c}=(c_1, \ldots, c_n)$ with components from $C^{(p)}$, a $p'=\omega(p_1', \ldots, p_l')$ $(\omega\in\Sigma_l^{(p)})$ and an $\omega'\in\Sigma_l^{(p)}$ such that $p'\in[p]$, and $\omega(p_1'(\mathbf{c}), \ldots, p_l'(\mathbf{c}))\neq\omega'(p_1'(\mathbf{c}), \ldots, p_l'(\mathbf{c}))$. Define an α_0-product $\mathscr{A}^{(p,q)}=$ $=(A^{(p,q)}, \Sigma)=\psi^{(p,q)}(\mathscr{B}, \mathscr{C}^{(p)}, \Sigma)$ in the following way: take an arbitrary node d of p different from its root. Let $t=\delta_r(t_1, \ldots, t_r)$ be the subtree of p at d, and $t'=\omega_r(t_1', \ldots, t_r')$ the subtree of p' at d. Then $\psi_r^{(p,q)(2)}(t_1^{\mathscr{B}}(\mathbf{b}), \ldots, t_r^{\mathscr{B}}(\mathbf{b}), \delta_r)=\omega_r$. In all other cases, except $\psi_l^{(p,q)(2)}(p_1^{\mathscr{B}}(\mathbf{b}), \ldots, p_l^{\mathscr{B}}(\mathbf{b}), \delta_l)$, $\psi_s^{(p,q)(2)}$ $(s\in R)$ is given arbitrarily in accordance with the definition of the α_0-product. Moreover, $\psi_s^{(p,q)(1)}$ is the identity mapping on Σ_s for every $s\in R$. Finally, let

$$\psi_l^{(p,q)(2)}(p_1^{\mathscr{B}}(\mathbf{b}), \ldots, p_l^{\mathscr{B}}(\mathbf{b}), \delta_l)=\begin{cases}\omega & \text{if}\quad q(\mathbf{a}^{(p,q)})=(b, c)\\ & \quad\text{and}\quad c\neq\omega(p_1'(\mathbf{c}), \ldots, p_l'(\mathbf{c})),\\ \omega' & \text{otherwise,}\end{cases}$$

where $\mathbf{a}^{(p,q)}=((b_1, c_1), \ldots, (b_n, c_n))$. ($q(\mathbf{a}^{(p,q)})$ is defined since $p\neq q$ and $(\mathscr{B}, \mathbf{b})$ is m-free.)

By the m-freeness of $(\mathscr{B}, \mathbf{b})$ and the choice of $(\mathscr{C}^{(p)}, \mathbf{c})$, $\mathscr{A}^{(p,q)}$ has the following properties:

(i) if t and t' are distinct trees from $F_\Sigma^m(X_n)$ then $t(\mathbf{a}^{(p,q)})\neq t'(\mathbf{a}^{(p,q)})$ since they differ at least in their first components, and

(ii) $p(\mathbf{a}^{(p,q)})\neq q(\mathbf{a}^{(p,q)})$ since they differ at least in their second components. Afterwards form the direct product $\mathscr{D}=(D, \Sigma)=\prod(\mathscr{A}^{(p,q)}|(p,q)\in I)$ and the vector $\mathbf{d}\in D^n$ with $\mathrm{pr}_j(\mathbf{d})=(\mathrm{pr}_j(\mathbf{a}^{(p,q)})|(p,q)\in I)$ $(j=1, \ldots, n)$. Obviously, the system $(\mathscr{D}, \mathbf{d})$ is $(m+1)$-free. Since the quasi-direct power is a special α_0-product and the formation of α_0-products is associative this, by Statement 3, ends the proof of Lemma 3.

Lemma 4. *Let Σ be a ranked alphabet of rank type R such that for every $l\in R$, $|\Sigma_l|\geqq 2$. Moreover fix an $l\in R$ and take the ranked alphabet Σ^l with $\Sigma_l^l=\Sigma_l\cup\{\bar\sigma\}$ and $\Sigma_k^l=\Sigma_k$ if $k\neq l$. If for certain $m, n\geqq 0$ and class K of algebras an m-free system $(\mathscr{A}, \mathbf{a})$ with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}\in A^n$ can be represented m-isomorphically by an α_0-product of algebras from K then every m-free system $(\mathscr{B}, \mathbf{b})$ with $\mathscr{B}=(B, \Sigma^l)$ and $\mathbf{b}\in B^n$ can be represented m-isomorphically by an α_0-product of algebras from K.*

Proof. Let $(\mathscr{A}, \mathbf{a})$ be an m-free system with $\mathscr{A}=(A, \Sigma)$ and $\mathbf{a}=(a^{(1)}, \ldots, a^{(n)})\in A^n$ which can be represented m-isomorphically by an α_0-product of algebras from K. Take two different fixed elements $\sigma_1, \sigma_2\in\Sigma_l$. Define two (one-factor) α_0-products $\mathscr{A}_1=(A, \Sigma^l)=\psi(\mathscr{A}, \Sigma^l)$ and $\mathscr{A}_2=(A, \Sigma^l)=\bar\psi(\mathscr{A}, \Sigma^l)$ in the following way:

(i) $$\psi_k^{(1)}(\sigma)=\bar\psi_k^{(1)}(\sigma)=\sigma \quad (\sigma\in\Sigma_k,\ k\neq l),$$

(ii) $$\psi_l^{(1)}(\sigma)=\begin{cases}\sigma & \text{if } \sigma\neq\bar\sigma,\\ \sigma_1 & \text{if } \sigma=\bar\sigma\end{cases}$$

and

(iii) $$\bar\psi_l^{(1)}(\sigma)=\begin{cases}\sigma & \text{if } \sigma\neq\bar\sigma,\\ \sigma_2 & \text{if } \sigma=\bar\sigma.\end{cases}$$

One can easily see that in $\mathscr{A}_1$ the operator $\bar\sigma$ is realized as σ_1 in $\mathscr{A}$, and in $\mathscr{A}_2$ the operator $\bar\sigma$ has the same effect as σ_2 in $\mathscr{A}$. Moreover, all other operators have the same realizations in $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$.

For every $p\in F_{\Sigma^l}(X_n)$ let $p_1=\psi^1(a^{(1)}, \ldots, a^{(n)}, p)$ and $p_2=\bar\psi^1(a^{(1)}, \ldots, a^{(n)}, p)$, that is p_i $(i=1,2)$ is obtained by replacing every occurrence of the label $\bar\sigma$ in p by σ_i. Obviously $p^{\mathscr{A}_1}(\mathbf{a})=p_1^{\mathscr{A}}(\mathbf{a})$ and $p^{\mathscr{A}_2}(\mathbf{a})=p_2^{\mathscr{A}}(\mathbf{a})$.

We show that the system $(\mathscr{B}, \mathbf{b})$, where $\mathscr{B}$ is the direct product $\mathscr{A}_1\times\mathscr{A}_2$ and $\mathbf{b}=((a^{(1)}, a^{(1)}), \ldots, (a^{(n)}, a^{(n)}))$, is m-free. Since the direct product is a special α_0-product and the formation of the α_0-product is associative this, by Statement 3, will complete the proof of Lemma 4.

Take two different trees $p, q \in F_{\Sigma'}^m(X_n)$, and let us distinguish the following three cases.

(1) None of the nodes of p and q is labelled by $\bar{\sigma}$. Then $p^{\mathscr{B}}(\mathbf{b})=(p^{\mathscr{A}}(\mathbf{a}), p^{\mathscr{A}}(\mathbf{a}))$ and $q^{\mathscr{B}}(\mathbf{b})=(q^{\mathscr{A}}(\mathbf{a}), q^{\mathscr{A}}(\mathbf{a}))$ differ in both of their components.

(2) One of p and q, say p, has a node labelled by $\bar{\sigma}$ and none of the nodes of q is labelled by $\bar{\sigma}$. If $p_1=q(=q_1=q_2)$ then $p_2\neq q_2$ since $p_1\neq p_2$. Thus, $p(\mathbf{b})$ and $q(\mathbf{b})$ differ at least in one of their components.

(3) Both p and q have nodes labelled by $\bar{\sigma}$. If $p_1=q_1$ then $p_2\neq q_2$ since $p\neq q$. Again $p(\mathbf{b})$ and $q(\mathbf{b})$ differ at least in one of their components.

Now we are ready to state and prove

Theorem 5. *A system of algebras is m-complete with respect to the product if and only if it is m-complete with respect to the* α_0*-product.*

Proof. Obviously, if a system of algebras is m-complete with respect to the α_0-product then it is m-complete with respect to the product.

Conversely, let K be a system of algebras of rank type R which is m-complete with respect to the product. Then, by Lemma 3, for arbitrary $m, n\geqq 0$ and Σ of rank type R with $|\Sigma_l|=2$ $(l\in R)$ every m-free system $(\mathscr{A}, \mathbf{a})$ $(\mathscr{A}=(A, \Sigma),\ \mathbf{a}\in A^n)$ can be represented m-isomorphically by an α_0-product of algebras from K. From this, by a repeated application of Lemma 4, we get that the previous statement is valid for arbitrary ranked alphabet Σ of rank type R if $|\Sigma_l|\geqq 2$ $(l\in R)$. Moreover, if we omit an operation in an algebra belonging to an m-free system then the resulting system is m-free, too. Therefore, by Theorem 1, K is m-complete with respect to the α_0-product, which ends the proof of Theorem 5.

From the above theorem we directly get

Corollary 6. *For arbitrary* $i, j\geqq 0$ *the* α_i*-product is metrically equivalent to the* α_j*-product.*

Since there exists a one-element system of algebras which is isomorphically complete with respect to the product ([5], [8]) and for arbitrary $m, n\geqq 0$ and ranked alphabet Σ there is an m-free system $(\mathscr{A}, \mathbf{a})$ $(\mathscr{A}=(A, \Sigma),\ \mathbf{a}\in A^n)$ such that $\mathscr{A}$ is finite, we have

Corollary 7. *There exists a one-element system of algebras which is m-complete with respect to the* α_0*-product.*

Finally, we give an m-complete system consisting of two algebras which is not isomorphically complete.

Let R be a rank type with $0\in R$ and Σ the ranked alphabet of rank type R fixed for Lemma 3. Consider the Σ-algebras $\mathscr{A}=(\{a_1, a_2\}, \Sigma)$ and $\mathscr{B}=(\{b_1, b_2\}, \Sigma)$

where

$$\sigma_0^{\mathscr{A}} = a_1, \sigma_0'^{\mathscr{A}} = a_2,$$

$$\sigma_l^{\mathscr{A}}(c_1, ..., c_l) = \sigma_l'^{\mathscr{A}}(c_1, ..., c_l) = a_1 \quad (l > 0;\ c_1, ..., c_l \in A)$$

and

$$\sigma_0^{\mathscr{B}} = \sigma_0'^{\mathscr{B}} = b_1,$$

$$\sigma_l^{\mathscr{B}}(c_1, ..., c_l) = b_1 \quad (l > 0;\ c_1, ..., c_l \in B),$$

$$\sigma_l'^{\mathscr{B}}(c_1, ..., c_l) = \begin{cases} b_2 & \text{if } c_1 = ... = c_l = b_1, \\ b_1 & \text{otherwise} \end{cases} \quad (l > 0;\ c_1, ..., c_l \in B).$$

The system $K = \{\mathscr{A}, \mathscr{B}\}$ obviously satisfies the conclusions of Theorem 2 (by $\mathscr{A}$ for the only nontrivial shape of height 0 and by $\mathscr{B}$ if the given shape is higher than 0). Therefore, K is m-complete with respect to the α_0-product. Moreover K is not isomorphically complete since for arbitrary $l \in R$ with $l > 0$, none of the equations $\sigma_l^{\mathscr{A}}(a_2, ..., a_2) = a_2$, $\sigma_l'^{\mathscr{A}}(a_2, ..., a_2) = a_2$, $\sigma_l^{\mathscr{B}}(b_2, ...; b_2) = b_2$ and $\sigma_l'^{\mathscr{B}}(b_2, ..., b_2) = b_2$ holds.

It follows from Theorem 1 in [2] that if a finite system of automata is m-complete with respect to the α_0-product then it always contains an automaton forming a simple system which is m-complete with respect to the α_0-product. One can easily see that neither $\{\mathscr{A}\}$ nor $\{\mathscr{B}\}$ is m-complete with respect to the α_0-product, showing that the existence even of a 0-ary operator (in addition to unary operators) alters the conditions of m-completeness.

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Analytic operator valued functions with prescribed local data

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Dedicated to B. Sz.-Nagy on the occasion of his seventieth birthday

This paper contains the operator generalization of the classical theorems of Mittag—Leffler and Weierstrass concerning construction of an analytic function with given local data which does not have additional singularities. The obtained results generalize earlier results of the authors on the finite dimensional case.

1. Introduction and main results

Let Ω be a domain in the complex plane $\mathbf{C}$. Consider the class Φ of all operator valued functions of the form $A(\lambda)=I+K(\lambda)$, $\lambda\in\Omega$, where $K(\lambda)$ is an analytic (in Ω) operator valued function whose values are compact operators acting in a Banach space B, with the additional property that at least one value of $A(\lambda)$ is an invertible operator. In particular, for every $A(\lambda)\in\Phi$ the spectrum $\sigma(A)=\{\lambda\in\Omega\,|\,A(\lambda)$ is not invertible$\}$ consists of isolated points in Ω. For any of these points $\lambda_0\in\sigma(A)$, in its deleted neighborhood the function $A(\lambda)^{-1}$ admits the form

$$A(\lambda)^{-1}=\sum_{j=-s}^{\infty}(\lambda-\lambda_0)^j M_j, \tag{1.1}$$

where $s>1$ is an integer and the operators $M_{-s}, M_{-s+1}, \dots, M_{-1}$ are finite dimensional (see [7]). Denote by $\mathrm{SP}\,A^{-1}(\lambda_0)$ the singular part $\sum_{j=-s}^{-1}(\lambda-\lambda_0)^i M_j$ of the Laurent series (1.1).

In this paper we shall solve the following problem: construct a function $A(\lambda)\in\Phi$ given its spectrum and the singular parts at each point of spectrum. The solution of this problem is given by the next theorem which is the main result.

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Theorem 1.1. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence (finite or infinite) of different points in a domain $\Omega \subset \mathbf{C}$ with limit points (if any) on the boundary Γ of Ω. For each λ_i, $i=1,2,\ldots$, let be given a rational operator function of the form*

$$M_i(\lambda) = \sum_{j=1}^{k_i} (\lambda-\lambda_i)^{-j} M_{ij}, \quad i = 1, 2, \ldots,$$

where M_{ij} are finite dimensional operators acting in B. Then there exists an analytic operator function $A(\lambda) \in \Phi$ such that $\sigma(A) = \{\lambda_1, \lambda_2, \ldots\}$ and $\mathrm{SP}\, A^{-1}(\lambda_i) = M_i(\lambda)$, $i=1,2,\ldots$. Moreover, $A(\lambda)$ can be chosen so that $A(\lambda) - I \in \Sigma$ for every $\lambda \in \Omega$, where Σ is the algebra of all operators acting in B which are limits (in norm) of finite dimensional operators.

Theorem 1.1 is a generalization of Theorem 4.4 from [5], which in turn may be regarded as a generalization of the classical Mittag—Leffler theorem. The proof of Theorem 1.1 is given in the next section. It uses a theorem on triviality of cocycles (see [1, 4]). Note that using this theorem it is not difficult to construct a meromorphic function with given singular parts of Laurent series, as in Theorem 1.1. However, it requires additional work to ensure that this meromorphic function is the inverse of an analytic function, and this is the bulk of the proof of Theorem 1.1.

In the course of the proof of Theorem 1.1 we obtain also the following far-reaching generalization of Weierstrass' theorem (which states the existence of a scalar analytic function with prescribed zeros and prescribed multiplicities).

Theorem 1.2. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence (finite or infinite) of different points in a domain $\Omega \subset \mathbf{C}$ with limit points (if any) on the boundary Γ of Ω. For every λ_j, let be given an operator polynomial of the form $P_j(\lambda) = I + \sum_{i=0}^{k_j} \lambda^i P_{ij}$, where P_{ij} are finite dimensional operators, such that $\sigma(P_j) = \{\lambda_j\}$. Then there exists an analytic (in Ω) operator valued function $A(\lambda)$ such that $A(\lambda) - I \in \Sigma$ for all $\lambda \in \Omega$, $\sigma(A) = \{\lambda_1, \lambda_2, \ldots\}$, and for every $j=1,2,\ldots$ the quotient $A(\lambda)P_j(\lambda)^{-1}$ is analytic and invertible at λ_j.*

As before, Σ stands for the algebra of norm limits of finite dimensional operators.

The above mentioned Weierstrass' theorem is obtained by taking $B = \mathbf{C}$ and $P_j(\lambda) = (\lambda - \lambda_j)^{k_j}$ in Theorem 1.2. Theorem 1.2 will be deduced from Theorem 1.1. The finite dimensional version of Theorem 1.2 was proved in [5].

Observe that in Theorem 1.2 one could replace the condition that P_{ij} are finite dimensional by the compactness of P_{ij}. Indeed, a polynomial $T(\lambda) = I + \sum_{i=0}^{k} \lambda^i T_i$

with compact T_i and $\sigma(T)=\{\lambda_0\}$ can be factored as follows:

$$T(\lambda) = T_0(\lambda)\left(I+\sum_{i=0}^{l} \lambda^i P_i\right)$$

with finite dimensional P_i and everywhere invertible $T_0(\lambda)$ (see Theorems 3.1 and 3.2 below). This allows us to reduce the problem to the case considered in Theorem 1.2.

2. Auxiliary results

In this section we shall prove Lemma 2.1 which will be used in the proof of Theorem 1.1, and is also of independent interest. As in Theorem 1.1, Ω stands for a domain in $\mathbf{C}$ with boundary Γ, and Σ denotes the algebra of all norm limits of finite dimensional operators acting in the Banach space B.

Lemma 2.1. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence (finite or infinite) of points in Ω with limit points (if any) in Γ. Let $Y_{j0}, Y_{j1}, \ldots, Y_{j,k_{j-1}}$, $j=1,2,\ldots$, be given operators from Σ. Then there exists an analytic operator valued function $Y(\lambda)$ $(\lambda\in\Omega)$ with $Y(\lambda)-I\in\Sigma$ for all $\lambda\in\Omega$ and such that*

$$Y(\lambda_j) = I+Y_{j0}, \quad j = 1, 2, \ldots \tag{2.1}$$

and

$$Y^{(k)}(\lambda_j) = Y_{jk}, \quad k = 1, \ldots, k_j-1, \quad j = 1, 2, \ldots. \tag{2.2}$$

If, in addition, $I+Y_{j0}$ is invertible for all $j=1,2,\ldots$, then the analytic operator function $Y(\lambda)$ can be chosen with the additional property that $Y(\lambda)$ is invertible for all $\lambda\in\Omega$.

We need some preparations for the proof of Lemma 2.1. A set $M\subset\mathbf{C}$ is called *finitely connected* if M is connected and $\mathbf{C}\setminus M$ consists of a finite number of connected components. We shall use later the fact that there is a sequence of finitely connected compacts $\Omega_1'\subset\Omega_2'\subset\ldots$ such that $\Omega=\bigcup_{i=1}^{\infty}\Omega_i'$. The proof of this fact is not difficult and can be found in [4], Lemma 2.1.

The following lemma can be viewed as a local analogue of Lemma 2.1.

Lemma 2.2. *Let $\lambda_0\in\Omega$, and let Ω_0 be a finitely connected compact in Ω such that λ_0 lies in the unbounded component of $\mathbf{C}\setminus\Omega_0$. Let $X_0, X_1, \ldots, X_k$ be given operators from the algebra Σ. Then for every $\varepsilon>0$ there exists an analytic (in Ω) operator valued function $L(\lambda)$ such that $L(\lambda)\in\Sigma$ for all $\lambda\in\Omega$, and*

(i) $L^{(i)}(\lambda_0) = X_i, \quad i = 0, \ldots, k,$

(ii) $\|L(\lambda)\| \leqq \varepsilon$ *for* $\lambda\in\Omega_0$.

Proof. Put $M(\lambda)=\sum_{i=0}^{k}\frac{1}{i!}X_i(\lambda-\lambda_0)^i$, then $M^{(i)}(\lambda_0)=X_i$, $i=0, \dots, k$, and $M(\lambda)\in\Sigma$ for all $\lambda\in\Omega$. Let $\alpha=\max_{\lambda\in\Omega_0}\|M(\lambda)\|$. There exists a scalar polynomial $\varphi(\lambda)$ such that

(2.3) $$|\varphi(\lambda)|\leqq\varepsilon\alpha^{-1} \quad \text{for} \quad \lambda\in\Omega_0,$$

(2.4) $$\varphi^{(i)}(\lambda_0)=\begin{cases}1, & i=0\\ 0, & i=1, \dots, k.\end{cases}$$

Indeed, we seek for $\varphi(\lambda)$ in the form

(2.5) $$\varphi(\lambda)=1-\psi(\lambda)(\lambda-\lambda_0)^k.$$

Let $\Omega_0'\subset\mathbf{C}$ be a simply connected compact such that $\Omega_0\subset\Omega_0'$ and $\lambda_0\notin\Omega_0'$. By Runge's theorem, there exists a polynomial $\psi(\lambda)$ with $\psi(\lambda_0)=0$ and $|\psi(\lambda)-$ $-(\lambda-\lambda_0)^{-k}|<\varepsilon\alpha^{-1}\beta^{-1}$, $\lambda\in\Omega_0'$, where $\beta=\max_{\lambda\in\Omega_0'}\{|\lambda-\lambda_0|^k\}$. With this $\psi(\lambda)$ in (2.5), the conditions (2.3) and (2.4) hold true. Now put $L(\lambda)=\varphi(\lambda)M(\lambda)$ to satisfy (i) and (ii).

Proof of Lemma 2.1. We shall seek for $Y(\lambda)$ in the form of an infinite product

(2.6) $$Y(\lambda)=\prod_{j=1}^{\infty}(I+L_j(\lambda)).$$

Choose a non-decreasing sequence $\Omega_1'\subset\Omega_2'\subset\dots$ of finitely connected compacts whose union is Ω, and such that $\lambda_i\in\Omega_j'$ for $i=1, 2, \dots, j-1$, but λ_j lies in the unbounded component of $\mathbf{C}\diagdown\Omega_j'$. Let $\varphi_j(\lambda)$ be a scalar function analytic in Ω with the following properties: $\varphi_j(\lambda_j)=1$; $\varphi_j^{(k)}(\lambda_j)=0$ for $k=1, \dots, k_j-1$, $\varphi_j^{(k)}(\lambda_i)=0$ for $k=0, \dots, k_j-1$ and $i\neq j$. Put $\alpha_j=\max_{\lambda\in\Omega_j'}|\varphi_j(\lambda)|$. By Lemma 2.2 there exist analytic operator functions (even operator polynomials) $M_j(\lambda)$, $j=1, 2, \dots$, such that $M_j(\lambda)\in\Sigma$ for all $\lambda\in\Omega$ and

$$M_j^{(k)}(\lambda_j)=Y_{jk},\ k=0, \dots, k_j-1,\ \|M_j(\lambda)\|\leqq\varepsilon_j\alpha_j^{-1} \quad \text{for} \quad \lambda\in\Omega_j',$$

where ε_j is any sequence of positive numbers for which the product $\prod_{k=1}^{\infty}(1+\varepsilon_k)$ converges. Define $L_j(\lambda)=\varphi_j(\lambda)M_j(\lambda)$, $j=1, 2, \dots$; with this definition of $L_j(\lambda)$ the product (2.6) converges uniformly in every Ω_j', and consequently $Y(\lambda)$ is analytic in Ω. Moreover, $L_j^{(k)}(\lambda_j)=Y_{jk}$ for $k=0, \dots, k_j-1$, and $L_j^{(k)}(\lambda_i)=0$ for $k=0, \dots, k_j-1$ and $i\neq j$. Consequently, equalities (2.1) and (2.2) are satisfied.

Suppose now that $I+Y_{j0}$ is invertible for all $j=1, 2, \dots$. In this case we shall look for $Y(\lambda)$ in the form

(2.7) $$Y(\lambda)=\exp(X_1(\lambda))\cdot\exp(X_2(\lambda))\cdot\exp(X_3(\lambda))\cdot\dots.$$

We shall construct the operator functions $X_m(\lambda)$ by induction on m.

Choose a sequence of finitely connected compacts $\Omega_1'\subset\Omega_2'\subset\dots$ with $\bigcup_{i=1}^{\infty}\Omega_i'=\Omega$, and denote by $\Xi_{m1},\dots,\Xi_{m,p_m}$ the bounded connected components of $\mathbf{C}\diagdown\Omega_m'$. We shall assume that each Ξ_{mp}, $p=1,\dots,p_m$, contains a point μ_{mp} not belonging to Ω (otherwise consider $\Omega_m'\cup\Xi_{mp}$ in place of Ω_m'). We shall assume also that $k_i<m$ for every i such that $\lambda_i\in\Omega_m'$ (this can be arranged because the set of points $\lambda_i\in\Omega_m'$ with $k_i\geqq m$ is either empty or finite).

We construct now $X_1(\lambda)$. Put $X_{i1}=\ln(I+Y_{i0})$, where the branch of the logarithm is chosen so that $\ln 1=0$; then $X_{i1}\in\Sigma$. Let $\tilde X_1(\lambda)$ be a Σ-valued analytic function such that

$$\tilde X_1(\lambda_i)=X_{i1},\quad i=1,2,\dots$$

(such $\tilde X_1(\lambda)$ exists in view of the already proved part of Lemma 2.1). Let $\varphi_{10}(\lambda)$ be an analytic (in Ω) scalar function with only zeros at $\{\lambda_1,\lambda_2,\dots\}$, which are simple. In particular the function $-\varphi_{10}(\lambda)^{-1}$ is analytic in Ω_1', therefore there exists a rational function $\psi_1(\lambda)$ with poles (if any) in $\mu_{11},\dots,\mu_{1,p_1}$ such that

$$\max_{\lambda\in\Omega_1'}|\psi_1(\lambda)+\varphi_{10}(\lambda)^{-1}|\leqq\ln(1+\varepsilon_1)\{\max_{\lambda\in\Omega_1'}|\varphi_{10}(\lambda)|\}^{-1}\{\max_{\lambda\in\Omega_1'}\|\tilde X_1(\lambda)\|\}^{-1}.$$

Put $X_1(\lambda)=(1+\psi_1(\lambda)\varphi_{10}(\lambda))\tilde X_1(\lambda)$. Then $X_1(\lambda_i)=X_{i1}$, $i=1,2,\dots$, and $\|X_1(\lambda)\|\leqq\leqq\delta_1=\ln(1+\varepsilon_1)$ for $\lambda\in\Omega_1'$. Here $\delta_1>\delta_2>\dots$ is a sequence of positive numbers chosen in advance.

Suppose Σ-valued analytic functions $X_1(\lambda),\dots,X_n(\lambda)$ are already constructed, with the following properties for $j=1,\dots,n$:

$$X_j^{(k)}(\lambda_i)=0,\quad k=0,\dots,j-2,\quad i=1,2,\dots;\tag{2.8}$$

$$Y_{i,j-1}=[\exp(X_1(\lambda))\dots\exp(X_j(\lambda))]_{\lambda=\lambda_i}^{(j-1)},\quad k_i\geqq j;\tag{2.9}$$

$$\|X_j(\lambda)\|\leqq\delta_j\quad\text{for}\quad\lambda\in\Omega_j'.\tag{2.10}$$

(For $j=1$, replace Y_{i0} in (2.9) by $I+Y_{i0}$.) By the already proved part of Lemma 2.1, there exists a Σ-valued analytic function $\tilde X_{n+1}(\lambda)$ in Ω such that $\tilde X_{n+1}^{(k)}(\lambda_i)=0$ for $k=0,\dots,n-1$; and if $k_i\geqq n+1$, then $\tilde X_{n+1}^{(n)}(\lambda_i)=X_{i,n+1}$ $(i=1,2,\dots)$, where the operators $X_{i,n+1}\in\Sigma$ are chosen in such a way that

$$Y_{in}=[\exp(X_1(\lambda))\exp(X_2(\lambda))\dots(\exp(X_n(\lambda))(\exp(\tilde X_{n+1}(\lambda))]_{\lambda=\lambda_i}^{(n)}$$

for every λ_i with $k_i\geqq n+1$. A computation (using (2.8)) shows that one can put

$$X_{i,n+1}=[\exp(X_1(\lambda_i))\dots\exp(X_n(\lambda_i))]^{-1}\cdot$$
$$\cdot\{Y_{in}-\sum_{\substack{\alpha+\dots+\alpha_n=n\\ \alpha_i\geqq 0}}[\exp(X_1(\lambda))]_{\lambda=\lambda_i}^{(\alpha_1)}\dots[\exp(X_n(\lambda))]_{\lambda=\lambda_i}^{(\alpha_n)}\}.$$

Now put $X_{n+1}(\lambda)=\varphi_{n+1}(\lambda)\tilde X(\lambda)$ where $\varphi_{n+1}(\lambda)$ is suitably chosen (as in the construction of $X_1(\lambda)$) scalar analytic function so that (2.8)—(2.10) hold also for $j=n+1$.

The condition (2.10) ensures uniform convergence of the infinite product (2.7) in every compact set in Ω, provided δ_n are chosen to tend sufficiently fast to zero. Equalities (2.8) and (2.9) ensure that $Y(\lambda)$ defined by (2.7) satisfies (2.1) and (2.2). Finally, for $\lambda\in\Omega'_n$ we have

$$\|\exp(X_n(\lambda))^{-1}-I\| \leqq \|I-\exp(X_n(\lambda))\|\cdot\|\exp(X_n(\lambda))\| \leqq (e^{\delta_n}-1)e^{\delta_n},$$

and (assuming δ_n tend sufficiently fast to zero) the infinite product $\exp(X_n(\lambda))^{-1}\cdot$ $\cdot\exp(X_{n-1}(\lambda))^{-1}\dots\exp(X_1(\lambda))^{-1}$ converges (necessarily to $Y(\lambda)^{-1}$) uniformly in every compact set in Ω. So $Y(\lambda)$ is invertible.

Lemma 2.1 is proved completely.

Note that Lemmas 2.1 and 2.2 remain true if the algebra Σ is replaced by the algebra of all compact operators.

3. Local spectral data of an analytic operator function

In this section we shall present some facts about local spectral data of an analytic function $A(\lambda)\in\Phi$ which are relevant to the proof of Theorem 1.1. As for the case of matrix functions (see [5]), one can define the local spectral data of $A(\lambda)\in\Phi$ in several forms: one-sided (right and left eigenpairs, local divisors and singular subspaces) and two-sided (the singular part of the Laurent expansion of $A(\lambda)^{-1}$). The results concerning the relationship between the various kinds of spectral data are the same as in the finite dimensional case, with essentially the same proofs (see [5] for details). So we shall focus on the kinds of spectral data which will be used in the proof of Theorem 1.1.

Let $A(\lambda)\in\Phi$, and let $A(\lambda)^{-1}=\sum_{j=-s}^{\infty}(\lambda-\lambda_0)^jM_j$ be the Laurent series of $A(\lambda)$ in a deleted neighbourhood of $\lambda_0\in\sigma(A)$. The finite dimensional subspace

$$\operatorname{Im}\begin{bmatrix} M_{-s} & 0 & \dots 0 \\ M_{-s+1} & M_{-s} & \dots 0 \\ \vdots & \vdots & \vdots\ \vdots \\ M_{-1} & M_{-2} & \dots M_{-s}\end{bmatrix}\subset B^s$$

is called the (right) *singular subspace* of $A(\lambda)$ at λ_0. (Sometimes in this definition it is convenient to consider $A(\lambda)^{-1}=\sum_{j=-s'}^{\infty}(\lambda-\lambda_0)^j$ with $s'>s$ and $M_{-s'}=\dots$ $\dots=M_{-s+1}=0$; so the singular subspace becomes a subspace in $B^{s'}$.) An operator polynomial of the form $P(\lambda)=I+\sum_{i=0}^{k}\lambda^iP_i$, where P_i are finite dimensional operators, is called a (right) *local divisor* of $A(\lambda)$ at λ_0, if $\sigma(P)=\{\lambda_0\}$ and the operator function $A(\lambda)P(\lambda)^{-1}$ is analytic and invertible at λ_0. The local divisor of $A(\lambda)$

at λ_0 is unique up to multiplication from the left by an everywhere invertible operator polynomial $S(\lambda)$ such that $S(\lambda)-I$ is finite dimensional for all λ.

The next result provides the relationships between singular subspaces and local divisors. We need the following definition of a special left inverse (cf. [6]). Let B_f be a finite dimensional vector space, and let $Z\colon B_f\to B$, $T\colon B_f\to B_f$ be linear operators such that for some s the operator

$$Q_s(Z,T)\overset{\text{def}}{=}\begin{bmatrix} Z \\ ZT \\ \vdots \\ ZT^{s-1} \end{bmatrix}\colon B_f\to B^s$$

is left invertible. A left inverse of $Q_s(Z,T)$ is called *special* if its kernel is of the form $\{(x_1,\dots,x_s)\in B^s \mid x_i\in W_i,\ i=1,\dots,s\}$, where $W_1\supset W_2\supset\dots\supset W_s$ is a non-increasing sequence of (closed) subspaces in B. If T is invertible, a special left inverse always exists. Indeed, since $\dim B_f<\infty$, one works in the proof of existence of a special left inverse with subspaces which have a finite dimensional complement, and then the proof given in the finite dimensional case ($\dim B<\infty$) applies (see Lemma 2.1 in [6]).

Theorem 3.1. *The singular subspace R of an analytic operator function $A(\lambda)\in\Phi$ at $\lambda_0\neq 0$ determines a local divisor $P(\lambda)$ of $A(\lambda)$ at λ_0 by the formula*

$$P(\lambda)=I-ZT^{-s}(V_1\lambda^s+V_2\lambda^{s-1}+\dots+V_s\lambda),$$

where $Z\colon R\to B$ is the projector on the last coordinate in $R\subset B^s$, $T\colon R\to R$ is defined by the formula $T(x_1,\dots,x_s)=(\lambda_0x_1,\lambda_0x_2+x_1,\dots,\lambda_0x_s+x_{s-1})$, $(x_1,\dots,x_s)\in R$, and $[V_1\dots V_s]$ is a special left inverse of

$$\begin{bmatrix} Z \\ ZT^{-1} \\ \vdots \\ ZT^{-(s-1)} \end{bmatrix}.$$

Conversely, if $P(\lambda)$ is a local divisor of $A(\lambda)\in\Phi$ at λ_0, then

$$\operatorname{Ker}\begin{bmatrix} P(\lambda_0) & 0 & \dots 0 \\ P'(\lambda_0) & P(\lambda_0) & \dots 0 \\ \vdots & \vdots & \vdots \\ \frac{1}{(s-1)!}P^{(s-1)}(\lambda_0) & \frac{1}{(s-2)!}P^{(s-2)}(\lambda_0) & \dots P(\lambda_0) \end{bmatrix} \tag{3.1}$$

is the singular subspace of $A(\lambda)$ at λ_0.

The proof of Theorem 3.1 is the same as the proof of Theorem 2.4 in [5]. Note that T is invertible in view of the condition $\lambda_0 \neq 0$. Note also that T maps R into R, as easily seen from the definition of the singular subspace R. The case $\lambda_0 = 0$ can be easily reduced to the case $\lambda_0 \neq 0$ by considering $A(\lambda + a)$ in place of $A(\lambda)$, for some $a \in \mathbf{C} \setminus \{0\}$.

The number s in (3.1), which is the least positive integer such that $(\lambda - \lambda_0)^s A(\lambda)^{-1}$ is analytic at λ_0, is determined by the local divisor $P(\lambda)$ as follows:

$$s = \min\{j > 0 \mid \dim \operatorname{Ker} \mathcal{N}_j = \dim \operatorname{Ker} \mathcal{N}_{j+1}\}$$

where $\mathcal{N}_j$ is the matrix in (3.1) with j in place of s.

A function $A(\lambda) \in \Phi$ is a *right divisor* of a function $B(\lambda) \in \Phi$ if $B(\lambda) = C(\lambda) A(\lambda)$ for some $C(\lambda) \in \Phi$. The description of divisibility in terms of singular subspaces and local divisors is given by the following Theorem, the proof of which is analogous to the proof of Theorems 1.4 and 2.5 in [5].

Theorem 3.2. *The following statements are equivalent:*

(i) *$A(\lambda) \in \Phi$ is a right divisor of $B(\lambda) \in \Phi$;*

(ii) *$\sigma(A) \subset \sigma(B)$, and for any $\lambda_0 \in \sigma(A)$ the local divisor of $A(\lambda)$ is in turn a right divisor of a local divisor of $B(\lambda)$ at λ_0;*

(iii) *$\sigma(A) \subset \sigma(B)$, and the singular subspace of $A(\lambda)$ at any $\lambda_0 \in \sigma(A)$ is contained in the singular subspace of $B(\lambda)$ at λ_0.*

In particular, $A(\lambda) \in \Phi$ and $B(\lambda) \in \Phi$ are right divisors of each other if and only if $\sigma(A) = \sigma(B)$, and $A(\lambda)$ and $B(\lambda)$ have the same local divisors at each $\lambda_0 \in \sigma(A)$, or equivalently, if the singular subspaces of $A(\lambda)$ and $B(\lambda)$ at each $\lambda_0 \in \sigma(A) = \sigma(B)$ coincide.

Let us remark (this remark will not be used in the proof of Theorem 1.1) that Theorems 3.1 and 3.2 can be also stated in terms of Jordan chains of a function $A(\lambda) \in \Phi$ corresponding to λ_0. By definition, the vectors $y_0, \dots, y_{k_0-1} \in B$ form a Jordan chain of $A(\lambda)$ corresponding to λ_0 if $\sum_{i=0}^{k} \frac{1}{i!} A^{(i)}(\lambda_0) y_{k-i} = 0, k = 0, \dots, k_0 - 1$. The q Jordan chains $y_0^{(j)}, \dots, y_{k_j-1}^{(j)}, j = 1, \dots, q$, of $A(\lambda)$ corresponding to λ_0 are said to be a *canonical set* if the eigenvectors $y_0^{(1)}, \dots, y_0^{(q)}$ are linearly independent and the sum $\sum_{p=1}^{q} k_p$ is maximal possible (cf. [7]). Every Jordan chain $y_0, \dots, y_{k_0-1}$ of $A(\lambda)$ corresponding to λ_0 is a linear combination of the canonical set: namely,

$$y_m = \sum_{j=1}^{q} \alpha_j y_m^{(j)}, \quad m = 0, \dots, k_0 - 1, \quad \text{for some} \quad \alpha_j \in \mathbf{C}.$$

As in the finite dimensional case (see Theorem 2.4 in [5]) one can prove that each of the three local characteristics of an analytic function $A(\lambda) \in \Phi$ at λ_0 — singular subspace, local divisor, canonical set of Jordan chains — determines the other two. In fact, a canonical set of Jordan chains appeared implicitly in Theorem 3.1. Namely, there exists an invertible operator $S\colon R\to\mathbf{C}^r$ such that (in the notation of Theorem 3.1) the operators $ZS^{-1}\colon \mathbf{C}^r\to B$ and $STS^{-1}\colon \mathbf{C}^r\to\mathbf{C}^r$ have the following structure, in the standard orthonormal basis in $\mathbf{C}^r$:

$$ZS^{-1} = [y_0^{(1)}\dots y_{k_1-1}^{(1)}\,y_0^{(2)}\dots y_{k_2-1}^{(2)}\dots y_0^{(q)}\dots y_{k_q-1}^{(q)}],$$

where $y_0^{(j)},\dots,y_{k_j-1}^{(j)}$, $j=1,\dots,q$, is a canonical set of Jordan chains of $A(\lambda)$ corresponding to λ_0; $STS^{-1}=J_1\oplus\dots\oplus J_q$, where J_i is the Jordan block of size $k_i\times k_i$ with eigenvalue λ_0.

4. Proof of the main theorem

The following result, which will be used in the proof of Theorem 1.1, is a particular case of Theorem 2.1 in [4], see also [1], and may be regarded as a theorem on triviality of cocycles. Given a compact set $\Omega_0\subset\Omega$, we denote by $\mathrm{GL}_s(\Omega_0)$ the set of all operator valued functions $G(\lambda)$ which are analytic and invertible in some neighborhood U_G of Ω_0 (the neighbourhood depending on the function) and such that $G(\lambda)-I\in\Sigma$ for every $\lambda\in U_G$.

Proposition 4.1. *There exists a sequence of compacts* $\Omega_1\subset\Omega_2\subset\dots\subset\Omega$, $\bigcup_{i=1}^{\infty}\Omega_i=\Omega$ *with the following property: For every sequence of analytic operator functions* $G_m(\lambda)\in\mathrm{GL}_s(\Omega_m)$, $m=1,2,\dots$, *there exists a sequence* $D_m(\lambda)\in\mathrm{GL}_s(\Omega_m)$ *such that*

$$G_m(\lambda)=(D_{m+1}(\lambda))^{-1}D_m(\lambda),\quad \lambda\in\Omega_m,\quad m=1,2,\dots. \tag{4.1}$$

We are ready now to prove Theorem 1.1.

Proof of Theorem 1.1. We shall break the proof into two steps.

a) Let $R_{k_i}\subset B^{k_i}$ be the singular subspace determined by $M_i(\lambda)$:

$$R_{k_i}=\mathrm{Im}\begin{bmatrix} M_{i,k_i} & 0 & \dots & 0\\ M_{i,k_i-1} & M_{i,k_i} & \dots & 0\\ \vdots & \vdots & & \vdots\\ M_{i1} & M_{i2} & \dots & M_{i,k_i}\end{bmatrix}.$$

We shall construct first an analytic operator function $\tilde{A}(\lambda)$ with $\sigma(\tilde{A})=\{\lambda_1,\lambda_2,\dots\}$ and corresponding singular subspaces $R_{k_1},R_{k_2},\dots$, and such that $\tilde{A}(\lambda)-I\in\Sigma$ for every $\lambda\in\Omega$.

Let $\Omega_1\subset\Omega_2\subset\dots$ be the sequence of compacts as in Proposition 4.1. Observe that each Ω_m contains only a finite number of λ_i's; let S_m be the (finite) set of indices i such that $\lambda_i\in\Omega_m$, $m=1,2,\dots$. It is not difficult to see that there exists a finite dimensional subspace $B_m \subset B$ and a direct complement B'_m to B_m in B such that $M_{ij}B_m \subset B_m$ and $M_{ij}B'_m = 0$ for $j=1, \ldots, k_i$ and $i \in S_m$. Using Theorem 4.1 of [5], for every $m=1, 2, \ldots$ construct an analytic (in Ω) operator function of the form $A_m(\lambda)=I+K_m(\lambda)$; where $K_m(\lambda)B_m \subset B_m$ and $K_m(\lambda)B'_m=0$ for every $\lambda \in \Omega$, such that $\sigma(A_m(\lambda))=\{\lambda_i \mid i \in S_m\}$, and R_{k_i} is the singular subspace of $A_m(\lambda)$ corresponding to λ_i, for every $i \in S_m$. Let $G_m(\lambda)=A_{m+1}(\lambda)(A_m(\lambda))^{-1}$; Theorem 3.2 ensures that $G_m(\lambda)$ is invertible in Ω_m, $m=1, 2, \ldots$. Applying Proposition 4.1, find a sequence $D_m(\lambda) \in \mathrm{GL}_s(\Omega_m)$, $m=1, 2, \ldots$, with the property (4.1). Then $D_m(\lambda)A_m(\lambda)=D_{m+1}(\lambda)A_{m+1}(\lambda)$, $\lambda \in \Omega_m$; so in fact the function $\tilde{A}(\lambda)=D_m(\lambda)A_m(\lambda)$ $(\lambda \in \Omega_m)$ is defined and analytic in Ω. Clearly, $\tilde{A}(\lambda)-I \in \Sigma$ for every $\lambda \in \Omega$; moreover, $\sigma(\tilde{A})=\{\lambda_1, \lambda_2, \ldots\}$ with corresponding singular subspaces $R_{k_1}, R_{k_2}, \ldots$.

b) We construct $A(\lambda)$ in the form $A(\lambda)=X(\lambda)\tilde{A}(\lambda)$, where $X(\lambda)$ is an everywhere invertible analytic (in Ω) operator function such that $X(\lambda)-I \in \Sigma$ for all $\lambda \in \Omega$, and $\tilde{A}(\lambda)$ is the operator valued function constructed in the part a).

For a fixed λ_i, there exists an operator function $\tilde{A}_i(\lambda)$, analytic in a neighborhood U_i of λ_i, with the properties that $\tilde{A}_i(\lambda)-I$ is finite dimensional for $\lambda \in U_i$ and $\mathrm{SP}\, \tilde{A}_i^{-1}(\lambda_i)=\sum_{j=1}^{k_i}(\lambda-\lambda_i)^{-j}M_{ij}$ (see Theorem 4.4 in [5]). Then the singular subspaces of $\tilde{A}(\lambda)$ and $\tilde{A}_i(\lambda)$ corresponding to λ_i coincide. By Theorem 3.2, the operator function $Z_i(\lambda) \overset{\mathrm{def}}{=} \tilde{A}_i(\lambda)A(\lambda)^{-1}$ is analytic and invertible in U_i. Moreover, $Z_i(\lambda)-I \in \Sigma$; $\lambda \in U_i$. Write $Z_i(\lambda)=\sum_{j=0}^{\infty}(\lambda-\lambda_i)^j Z_{ij}$, and let $X(\lambda)$ be an everywhere invertible analytic (in Ω) operator function such that $X(\lambda)-I \in \Sigma$, $\lambda \in \Omega$, and

$$\frac{1}{j!}X^{(j)}(\lambda_i) = Z_{ij}, \quad j = 0, \ldots, k_i, \quad i = 1, 2, \ldots .$$

The existence of such $X(\lambda)$ is ensured by Lemma 2.1. Now put $A(\lambda)=X(\lambda)\tilde{A}(\lambda)$.

Let us check that the requirements of Theorem 1.1 are satisfied with this choice of $A(\lambda)$. Indeed,

$$A(\lambda)-I = X(\lambda)(\tilde{A}(\lambda)-I)+X(\lambda)-I \in \Sigma,$$

for all $\lambda \in \Omega$. For every λ_i, in a neighbourhood of λ_i we have

$$A(\lambda) = X(\lambda)\tilde{A}(\lambda) = X(\lambda)(Z_i(\lambda))^{-1} \cdot Z_i(\lambda)\tilde{A}(\lambda).$$

Now because of the choice of $X(\lambda)$ we obtain $Z_i(\lambda)(X(\lambda))^{-1}=(\lambda-\lambda_i)^{k_i+1}U_i(\lambda)+I$ for some operator function $U_i(\lambda)$ which is defined and is analytic in a neighborhood of λ_i. Hence

$$\mathrm{SP}\, A^{-1}(\lambda_i) = \sum_{j=1}^{k_i} (\lambda-\lambda_i)^{-j} M_{ij}, \quad i = 1, 2, \ldots,$$

and Theorem 1.1 is proved completely.

Finally, observe that in view of Theorem 3.1, the part a) of the proof of Theorem 1.1 provides the proof for Theorem 1.2.

In conclusion let us remark that Theorem 1.2 can be stated also in terms of singular subspaces, as well as in terms of canonical set of Jordan chains, in the same way as in the finite dimensional case (see [5]). We state it in terms of canonical set of Jordan chains: Let $\lambda_1, \lambda_2, \ldots$ be a sequence as in Theorem 1.2, and for every λ_i, let be given a set of vectors in B:

$$y_{01}^{(i)}, \ldots, y_{k_{1i}-1,1}^{(i)};\; y_{02}^{(i)}, \ldots, y_{k_{2i}-1,2}^{(i)};\; \ldots;\; y_{0q}^{(i)}, \ldots, y_{k_{qi}-1,q}^{(i)} \tag{4.2}$$

(q depends on i) with linearly independent vectors $y_{01}^{(i)}, \ldots, y_{0q}^{(i)}$. Then there exists an analytic (in Ω) operator valued function $A(\lambda)$ such that $A(\lambda)-I\in\Sigma$ for all $\lambda\in\Omega$, $\sigma(A)=\{\lambda_1, \lambda_2, \ldots\}$, and for every $i=1, 2, \ldots$, the set (4.2) is a canonical set of Jordan chains of $A(\lambda)$ corresponding to λ_i. The proof of this statement is obtained immediately from Theorem 1.2, taking into account the remark at the end of Section 3.

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The RKNG (Rellich, Kato, Sz.-Nagy, Gustafson) perturbation theorem for linear operators in Hilbert and Banach space

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th Birthday 29 July 1983

In this paper I wish to discuss, including some hitherto unpublished comments, observations, and results, a fundamental perturbation theorem for linear operators, which I shall state as follows:

$$(*)\qquad \left.\begin{array}{l} A \quad \text{has a certain property and} \\ \|Bx\| \leqq a\|x\|+b\|Ax\|, \quad b<1 \end{array}\right\} \Rightarrow A+B \quad \text{has the same property.}$$

Two of the most important instances of this theorem are for the properties:

(1) selfadjointness in a Hilbert space;

(2) contraction semigroup generator in a Banach space.

The former is a special case of the latter.

Throughout I will assume, unless specified to the contrary, that B is a regular perturbation of A, that is, that $D(B)\supset D(A)$. In general, I will not consider form versions. Nor, except for a few comments, will I consider closure (e.g., the versions with $b=1$) versions.

1. Some history

Theorem $(*)$ for the selfadjointness property (1) usually goes by the name of the Rellich—Kato (sometimes, Kato—Rellich) Theorem. It is due originally to Lord Rayleigh and E. Schrödinger in their calculus of perturbations, which involved the formal assumption that an analytic perturbation $A(\varepsilon)$ of A yields analytic transformations of the eigenvalues and eigenvectors as well. A rigorous proof was first found by F. RELLICH [1]. The theorem was employed by KATO [2] in a fundamental way in an application to quantum mechanics.

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It is generally less well known that Sz.-Nagy [3] made an original contribution to this theorem (see the discussion below for more details). I made the extension to Banach space in [4], and with the apology that four letter theorems are now all the RAGE — you must know Reed—Simon [5] to understand this pun — I have here labeled it the RKNG Theorem.

I remember reading when I was young a newspaper account of an interview with a Nobel Prize winner, and his comment that (roughly) "we are all just bricklayers in the temple of science, some of us happen to arrive at the right time to turn the corners". Certainly that is also the case even with any theorem, be it four letter or not. Thus there are many many aspects of the RKNG Theorem that I will not elaborate on at all. Most of these related facts are available from the books Reed—Simon [5] and Kato [6] or from the recent literature, see for example the recent review by Sohr [7].

It should be pointed out that the initial workers and users of these theories were not primarily interested in the RKNG Theorem in its final "clean" form, but were instead motivated by its appearance in important problems in quantum physics, boundary values problems, and elsewhere.

Rigorous proofs of convergence eventually became of interest and Rellich managed to accomplish this in a series of papers [8] including the case of unbounded operators. For the case of bounded operators his results may be described as follows. For a convergent series depending on a real perturbation parameter ε,

$$A(\varepsilon) = A_0+\varepsilon A_1+\varepsilon^2 A_2+\varepsilon^3 A_3+\ldots,$$

$A(\varepsilon)$ will be a bounded selfadjoint operator with eigenvalues $\lambda(\varepsilon)$ depending on ε. If the unperturbed operator A_0 has an isolated eigenvalue $\lambda(0)$ of finite multiplicity, so will $A(\varepsilon)$, these will be close to $\lambda(0)$ and of the same multiplicity, provided that ε is small enough. Moreover the eigenvalues $\lambda(\varepsilon)$ and the associated eigenvectors can be expanded in series in terms of the eigenvalues of A and the A_i.

In particular, then, Rellich showed that if λ_0 is an isolated eigenvalue of finite multiplicity n of a selfadjoint transformation A_0, then there exists an ε-neighborhood of the origin and real valued functions $\lambda^{(1)}(\varepsilon), \ldots, \lambda^{(n)}(\varepsilon)$ and Hilbert space elements $f^{(1)}(\varepsilon), \ldots, f^{(n)}(\varepsilon)$ such that $\lambda^{(k)}(0)=\lambda_0$, $\lambda^{(k)}(\varepsilon)$ and $f^{(k)}(\varepsilon)$ are the eigenvalues and eigenvectors, respectively, of $A(\varepsilon)$. In addition (for all values of ε); the elements $f^{(k)}(\varepsilon)$ form an orthonormal sequence and; apart from the points $\lambda^{(k)}(\varepsilon)$, no member of the spectrum of $A(\varepsilon)$ lies in the neighborhood of λ_0.

This result applies only to isolated eigenvalues of finite multiplicity and is in general not true for those of infinite multiplicity. We shall return to this point in describing below Sz.-Nagy's results.

Kato [2, 9] employed this theory to establish the selfadjointness of the basic (Schrödinger) operators of quantum mechanics and to study their spectral properties.

This work is sufficiently extensive and well documented elsewhere in the physics and mathematics literature that I shall not attempt any comprehensive account here.

It is interesting, however, that one may illustrate and demonstrate this important scientific result for the case of the Hydrogen atom operator $H=-(1/2)\Delta-1/r$ quite readily from the fundamental theorem ($*$) plus a basic inequality (Sobolev) of potential theory

$$\|u/r\| \leqq 2\|\mathrm{grad}\, u\|,$$

the norm here being that of $L^2(R^3)$. For $A=-(1/2)\Delta$ it suffices to show that the perturbation $B=-1/r$ is relatively small with respect to A (that is what the norm inequality condition in ($*$) is called). For functions u in $C_0^\infty(R^3)$ using the Sobolev inequality one has for any $\varepsilon>0$

$$\|u/r\| \leqq 2\|\mathrm{grad}\, u\| \leqq \varepsilon^{-1}\|u\|+\varepsilon\|\Delta u\|$$

and thus the selfadjointness of the full Hamiltonian H from that of the bare Hamiltonian A. The selfadjointness of the latter is easily established by Fourier transform to a multiplication operator. We also used an integration by parts and the arithmetic-geometric mean inequality (the details may be found in [10]).

Finally I would like to mention that Kato employed ($*$) (and the Sobolev inequality) in [2] not only for the nonrelativistic Schrödinger operators but also to establish the selfadjointness of the Dirac operator.

As mentioned above, although certainly known to specialists, it is not generally known that Sz.-Nagy [3, 11] made important contributions to this theory. Let us describe them here. The main result, in [3], Theorem I there, treats the behavior of an arbitrary, but isolated, part of a spectrum. Then in [3, Theorem III], Rellich's results for an isolated eigenvalue of finite multiplicity are reobtained. Also [3, Theorem II] estimates convergence conditions for the power series representation of the perturbed eigenvalues and eigenfunctions, the estimates obtained being somewhat sharper than those obtained by Rellich. Further, the approach of Rellich required the heavy function theory of Puiseux Series and the Weierstrass Vorbereitungsatz on zeros of a function of several variables.

In particular, Sz.-Nagy [3] showed the following. Let A_0 be selfadjoint and A_k, $k=1, 2, 3, \ldots$, be symmetric regular $(D(A_k)\supset D(A_0))$ perturbations satisfying

$$\|A_k f\| \leqq p^{k-1}(a\|f\|+b\|Af\|).$$

Then for $-p^{-1}<\varepsilon<p^{-1}$ the series $A(\varepsilon)\equiv A_0+\varepsilon A_1+\varepsilon^2 A_2+\ldots$ is selfadjoint for $-(p+b)^{-1}<\varepsilon<(p+b)^{-1}$. Moreover if an interval $\mu_1<\lambda<\mu_2$ contains an isolated part of A_0's spectrum, then the constantness of the spectral family $E_\lambda(\varepsilon)$ for $A(\varepsilon)$ follows from that of $E_\lambda(0)$ for A_0 for $-(p+a)^{-1}<\varepsilon<(p+b)^{-1}$ near the ends of the (μ_1, μ_2) interval, and in a neighborhood interior to the interval (μ_1, μ_2) the reduced perturbed operator $A(\varepsilon)[E_{\mu_2}(\varepsilon)-E_{\mu_1}(\varepsilon)]$ has power series expansion $\lambda_0(E_{\mu_2}(\varepsilon)-E_{\mu_1}(\varepsilon))+B_0+\varepsilon B_1+\varepsilon^2 B_2+\dots$ where the coefficients satisfy $\|B_k\| \leqq \frac{\mu_2+\mu_1}{2} \cdot$ $\cdot(\text{constant}_1)(p+\text{constant}_2)^{k-1}$.

Sz.-Nagy's approach in [3] uses the theory of the resolvent operator R_z and the associated spectral representations in the complex plane. In [11; first citation] he applies the theory to the ordinary differential equation boundary value problem $-py''=\lambda(1+\varepsilon\sigma)y$, $y(0)=y(l)=0$. In [11, second citation], some extensions to the theory of closed operators in Banach space are made, including several original ideas on the extending of the theory from bounded operators to relatively bounded ones. In [11; third citation] one finds a translation of the original paper [3].

I would not have known of these papers had not Professor Sz.-Nagy given them to me on a visit to Szeged in 1972.

My contribution [4] to the RKNG theorem came about in an indirect way. I was finishing my doctoral work in partial differential equations with L. E. Payne at the University of Maryland. Payne had already taken a position at Cornell University, and as I had some free time I was attending the lectures of S. Goldberg on operator theory. Goldberg was stuck at $b<1/2$ in a proof of the following basic defect index lemma.

Lemma [4, Lemma 1]. *Let T and B be linear operators with domains in a normed linear space X and ranges in a normed linear space Y. Suppose that T has a bounded inverse and that B is bounded with $\|B\|<b\|T^{-1}\|^{-1}$, $b<1$. Then* $\dim(Y/\overline{R(T)})=\dim(Y/\overline{R(T+B)})$.

Inasmuch as this lemma was desired by Goldberg for a proof of the basic index perturbation theorem that states that the full index of an operator is preserved under relatively small perturbations, I became interested and proved the above lemma, which also appears in the book [12] as Corollary V 1.3. Believing that the technique of proof should have wider value, upon stumbling onto Nelson's paper [13], Nelson also being stuck at $b<1/2$, I published [4] with its RKNG result for semigroup generators. Because Nelson was working in Banach space, so did I.

Theorem [4, Theorem 2]. *Let A be the infinitesimal generator of a contraction semigroup on the Banach space X, and let B be a dissipative operator with $D(B)\supset D(A)$. If there exist constants a and $b, b<1$, such that for all x in $D(A)$, $\|Bx\|\leqq a\|x\|+b\|Ax\|$ then $A+B$ is the infinitesimal generator of a contraction semigroup.*

In [14, first citation] I tried to place the doubling technique in a proper context with respect to the general Fredholm theory of linear operators in normed (not necessarily complete) spaces and in so doing gave a number of extensions of [4].

I also noted as an example of the method the following extension from $b<1/2$ to $b<1$ of a theorem of KATO [6, Theorem 3.4] on perturbation of sesquilinear forms.

Theorem [14, first citation, Theorem 4.1]. *Let* $t[u, v]$ *be a densely defined closed sectorial form with* $\mathrm{Re}\,(t[u, u]) \geqq 0$, *and let* $b[u, v]$ *satisfy*

$$|b[u, u]| \leqq a\|u\|^2 + b\,\mathrm{Re}\,(t[u, u]), \quad b < 1,$$

and $\mathrm{Re}\,(b[u, u]) \geqq -\mathrm{Re}\,(t[u, u])$ *for* u *in* $D(t) \subset D(b)$. *Then the resolvent* $R_\lambda(T_{t+b})$ *for the associated form operator exists for* $\mathrm{Re}\,\lambda < -ab^{-1}$ *and*

$$\|R_\lambda(T_t) - R_\lambda(T_{t+b})\| \leqq nb(b_n - b)^{-1}(-\mathrm{Re}\,\lambda)^{-1},$$

where n *is chosen such that* $b < b_n = (2^n - 1)/2^n$.

It should be noted that in extending the previous version from $b<1/2$ to $b<1$, I made an additional assumption that $\mathrm{Re}\,(t+b) \geqq 0$. Also for the record let me correct here two minor typographical errors: [14, p. 286, the last line] reads $\|(1-c_n)B_{n+1}\| < 2^{-(n+1)}\gamma(T) = \ldots$ and [14, p. 287, the first line] reads ... is a B_1 for $T + c_n B_{n+1}, \ldots$.

Shortly afterward, while on the faculty of the University of Minnesota, I attended a lecture by Ken-iti Sato who was visiting from Japan. Sato, who was chiefly interested in applications to probability and who, although there already for several months, I had not met, had proved the following theorem for $b<1/2$.

Theorem [14, second citation, Theorem 2.1]. *Let* A *be the infinitesimal generator of a nonnegative contraction semigroup on a Banach lattice* $\mathscr{B}$, *and let the perturbation* B *satisfy* $\|Bf\| \leqq a\|f\| + b\|Af\|$, $b<1$ *for all* f *in* $D(A) \subset D(B)$. *If* $A+B$ *is* $(\alpha+\beta)$*-weakly dispersive, then* $A+B$ *also generates a nonnegative contraction semigroup.*

During his lecture I wrote down the following "lemma": a weakly dispersive operator B is dissipative in at least one semi-inner-product, and based upon that assumption, immediately noted the extension to $b<1$ that is stated in the above theorem. Unfortunately, the just mentioned "lemma" resisted our efforts at its proof, causing the proofs in [14] to be longer than otherwise would have been necessary. Although the "Lemma" is true in a wide variety of cases and under many additional assumptions, its general validity or nonvalidity for all Banach lattices is to my knowledge not known.

A few years later a recent student of F. Browder in Chicago, B. Calvert from New Zealand, spent a postdoctoral one half year with me at the University of Colorado. Calvert had done his doctoral work on nonlinear semigroup generators in Banach lattices. Here is a nonlinear version of the RKNG theorem, which we needed for application to questions of multiplicative perturbation of generators.

Theorem [14, third citation, Lemma 1]. *Let A be m-accretive (φ) in a Banach space X, let B be single-valued with $D(B)\supset D(A)$ such that $A+B$ is accretive (ψ) such that $\psi(u, v, x+Bu, y+Bv)=\varphi(u, v, x\in Au, y\in Av)$. Suppose there exist constants a and b such that, for x_1 and x_2 in $D(A)$,*

$$\|Bx_1-Bx_2\| \leqq a\|x_1-x_2\|+b|Ax_1-Ax_2|, \quad b<1.$$

Then $A+B$ is m-accretive (ψ).

Here φ and ψ may be different duality maps and $|S|$ is the infinium of $\|s\|$ for all s in a set S. In the nonlinear case the technicalities in the statements cause some loss of interest in the results. We refer to Browder [15] for further information on the nonlinear semigroup literature.

For extensive treatments of the RKNG Theorem, selfadjoint operators, forms, and related matters, we point out, in addition to the books [5] and [6] already mentioned, two books almost totally devoted to such questions, Faris [16] and Chernoff [17].

2. The real Hilbert space case

The usual proofs of the RKNG Theorem utilize the defect index theories. In particular, for the instance of preservation of selfadjointness, it is interesting to ask if a direct proof can be worked out for real Hilbert space without employing the Von Neumann $R(A+B\pm iI)=H$ complex defect-zero criteria. A direct proof of the RKNG theorem for real spaces was worked out with my doctoral student D. K. Rao about ten years ago, and I would like to give it here since I have never seen it anywhere else.

Theorem (RKNG Theorem in real Hilbert space). *Let A be selfadjoint in a real Hilbert space and let B be a symmetric operator with $D(B)\supset D(A)$ and $\|Bx\|\leqq a\|x\|+b\|Ax\|$, $b<1$. Then $A+B$ is selfadjoint.*

Proof. It suffices to first prove the theorem for $b<1/2$. Then the doubling technique allows extension to $b<1$. Since $A+bB$ is closed and symmetric, it thus suffices to prove that for $\|Bx\|\leqq a\|x\|+\|A\|$, $b<1/2$, one has $D((A+bB)^*)\subset$ $\subset D(A+bB)$.

Let $y\in D((A+bB)^*)$. Then

$$\left|((A+bB)^*y,\ Ax)+k^2(y, x)\right| \leqq \left(\|(A+bB)^*y\|+\|y\|\right)\|x\|_1$$

where $(x, y)_1=(Ax, Ay)+k^2(x, y)$ is the inner product on $H_A\equiv D(A)$ with the A-norm. It will be advantageous to choose k such that $k>2ab(1-b)^{-1}$. The linear functional y_1^* on H_A induced from y via

$$y_1^*(x) = \left((A+bB)^*y, Ax\right)+k^2(y, x)$$

is bounded and hence there exists some y^* in $D(A)$ such that $y_1^*(x)=(y^*, x)_1 \equiv \equiv(Ay^*, Ax)+k^2(y^*, x)$. We now claim, remembering that $k>2ab(1-b)^{-1}$, that B may "inserted" so that one has the representation, for some y_B^* in $D(A)$;

$$y_1^*(x) \equiv (y^*, x)_1 = ((A+bB)y_B^*, Ax)+k^2(y_B^*, x).$$

Accepting this for the moment, for all x in $D(A^2)$ we then have

$$(y-y_B^*, ((A+bB)A+k^2)x) = 0.$$

For $k>2ab(1-b)^{-1}$ and $b<1/2$, the operator $(A+bB)A+k^2=(A^2+k^2)+bBA$ is onto because A^2+k^2 maps onto and bBA is a small perturbation. Accepting also this latter statement for the moment, we thus have y in $D(A)$ and thus $D((A+bB)^*)\subset D(A+bB)$.

There are several ways to verify the two details in the above.

For the first, from $y_1^*(x)=(Ay^*, Ax)+k^2(y^*, x)$ we may "insert" the (bBy_B^*,Ax) term as follows. The linear functional $-(bBz_0, Ax)$ induced by any vector z_0 in $D(A)$ is bounded on H_A immediately by Schwarz's inequality and the fact that B is A-small with $ab<k(1-b)/2$. Thus there exists z_1 in $D(A)$ such that $(-bBz_0,Ax)= =(z_1, x)_1$ and $\|z_1\|_1<\frac{1+b}{2}\|z_0\|_1$. Repeating this with z_1 we are led to the sequence z_n given by $(z_{n+1}, x)_1=-(bBz_n, Ax)$. Thus we may write the "telescoped sum"

$$(z_0, x)_1 = \sum_{m=0}^{n} [((A+bB)z_m, Ax)+k^2(z_m, x)]+(z_{n+1}, x)_1.$$

Let $\sum_{m=0}^{\infty} z_m=z$, and note that $\sum_{m=1}^{n}\|z_m\|<\sum_{m=1}^{n}(2^{-1}(1+b))^m\|z_0\|_1$, which guarantees absolute convergence (for $b<1$) of both $\sum_{m=0}^{\infty} z_m$ and $\sum_{m=0}^{\infty} Az_m$. By the relative boundedness $\sum_{m=0}^{\infty}\|(A+bB)z_m\|$ is similarly seen to converge and hence $\sum_{m=0}^{n}(A+bB)z_m$ converges to $(A+bB)z$ because $\sum_{m=0}^{n} z_m\to z$ and $A+bB$ is closed. Letting y^* be z_0 and y_B^* be z we thus have shown that $(y^*, x)_1=((A+bB)y_B^*, Ax)+k^2(y_B^*, x)$.

For the second detail, we note that the perturbation bBA satisfies the relative bounds

$$b\|BAz\| \leqq ab\|Ax\|+b^2\|A^2x\| \leqq (abk^{-1}2^{-1/2}+b^2)\|(A^2+k^2)x\| \leqq$$
$$\leqq ((1-b)2^{-3/2}+b^2)\|(A^2+k^2)x\|.$$

Because the coefficient of the right hand side is less than one for $b<1/2$, the surjectivity of A^2+k^2 implies that of A^2+k^2+bBA by the basic defect index lemma of [4] mentioned in the section above; more precisely and quickly, by its unbounded version [14, first citation, Theorem 2.4].

There are no doubt shorter and sharper versions of the proof given above for the RKNG Theorem for Real Hilbert Space, perhaps also a proof by complexifying and using the complex version. The latter was not evident to us at the time and in any case the proof given above holds as well for the complex case. We did not investigate the above proof in any generality (e.g., Banach space, nonlinear versions).

Although the important quantum mechanical application occurs in a complex Hilbert space setting, there are a number of suppositions that go into that choice of scalars, and the issue is not completely settled. See my remarks in [18]. Along these lines it would perhaps be interesting to have the RKNG theorem for the quaternion field, perhaps also for the Clifford algebra.

3. Regular positive perturbations

One would like to have a specialized RKNG theorem of the form: *if B is a symmetric regular positive perturbation of positive selfadjoint A, then $A+B$ is selfadjoint (or at least essentially selfadjoint).* The idea is that the regularity of B with respect to A means that B is A-bounded, and that even though the relative bound b may be greater than one, it doesn't matter. There also are a number of compelling physical examples for such a theorem. I conjectured in 1972, in some discussions with K. Jörgens who was visiting me in Boulder for the year, that such a result should hold, and spent some time on it that year.

By translation the question is equivalent to that for the more general situation of two semibounded selfadjoint operators, one regular with respect to the other, and also equivalent to the more special case of A and B both bounded below by one, $D(B)\supset D(A)$. One can write down lots of operator-theoretic conditions for the selfadjointness (better: essential selfadjointness) of $A+B$. For example, for strongly positive selfadjoint A and B, the latter a regular perturbation, $D(A^{1/2})\subset \subset D(B^{1/2})$ by forms so that

$$A+B = A^{1/2}[I+A^{-1/2}B^{1/2}\cdot B^{1/2}A^{-1/2}]A^{1/2} = A^{1/2}[I+TT^*]A^{1/2}.$$

Note that $T=A^{-1/2}B^{1/2}$ is a densely defined bounded operator and that $C=I+TT^*$ is a strongly positive essentially selfadjoint bounded densely defined operator. One has then the situation

$$\overline{A+B}\subset A^{1/2}\bar{C}A^{1/2}\subset (A+B)^*$$

where the middle term $A^{1/2}\bar{C}A^{1/2}$ may be seen to be the Friedrichs extension of $A+B$. Thus $A+B$ is selfadjoint iff C maps $D(A^{1/2})$ onto itself iff $D((A+B)^*)\cap \cap D(A^{1/2})$ is contained in $D(B)$, and $A+B$ is essentially selfadjoint iff $\overline{A^{1/2}C}=A^{1/2}\bar{C}$ iff $(A+B)^2$ is densely defined.

Unfortunately the general proposition is false, and in this final section I want to discuss why, and present some counter examples. These were obtained in May 1979 during visits with J. Weidmann in Frankfurt and with T. Kato, who provided the coup de grâce, in Paris. They are important to have, and as far as I know, are not in the published literature.

But before I conclude, I would like to mention some ways in which such questions were, and in some cases may be, avoided.

The usual dodge is to accept form sums. Then for example if A and B are positive selfadjoint and $D(A^{1/2})\cap D(B^{1/2})$ is dense, the "form sum" $A+B$ is selfadjoint. One does not even need a regular perturbation. But that is the point: one also does not in general get a "regular" sum selfadjoint on $D(A)$.

A second dodge is to invoke a positivity assumption on the product term $\langle Ax, Bx\rangle$ rather than on A and B. For example, for A selfadjoint and B a symmetric regular perturbation, the condition $\mathrm{Re}\,(Ax, Bx)\geqq 0$ implies (OKAZAWA [19]) that $A+B$ is selfadjoint. An extension of this result is that one needs (GUSTAFSON and REJTO [20]) only $\mathrm{Re}\,(Ax, Bx)\geqq -(a\|x\|^2+b\|Ax\|\;\|Bx\|)$ for $b<1$ to conclude $A+cB$ selfadjoint for all $c\geqq 0$. Further discussions of these methods may be found in [7].

A third set of variations involves closure assumptions. For example, for a regular perturbation B the selfadjointness of $A+cB$ fails only when the closure of $A+cB$ fails.

Let us conclude with some counter examples.

First, it is not sufficient that just A and the sum of $A+B$ be positive. Consider any closed symmetric positive nonselfadjoint operator T, let $A=|T|$ and $B=T-|T|$. Then B is a regular symmetric perturbation of positive selfadjoint A, with relative bound 2, $A+B$ is closed symmetric positive but not selfadjoint.

Second, a version of the result mentioned above from [20] is: for A essentially selfadjoint and B a symmetric A-bounded regular perturbation of A, if the set of values (Au, Bu) for all u in $D(A)$ is contained in some half-plane not containing $(-\infty, 0]$ then $A+B$ is essentially selfadjoint. To see that the half line $(-\infty, 0]$ must be excluded, let $Ay=-y''$ in $L^2(0, \infty)$ with $D(A)=\{y\in L^2(0, \infty)$ such that y and y' are absolutely continuous, $y\in C_0^\infty[0, \infty), y(0)=0\}$, and let $By=y''+iy'-y$ on $D(A)$. A is essentially selfadjoint and $A+B$ has no selfadjoint extensions at all. It is easy to verify that (Au, Bu) values lie in the second quadrant.

Counter examples to A and B positive essentially selfadjoint operators, one regular with respect to the other, yet $A+B$ not essentially selfadjoint, may be obtained as follows. Consider $Ay=-(py')'$, $B=(-qy')'$, with $p(x)=x^4+\sin^2(1/x)$ and $q(x)=x^4+\cos^2(1/x)$ for $0<x<1$ and continued smoothly to 1 as $x\to\infty$. A and B are essentially selfadjoint in $L^2(0, \infty)$ on the domain $C_0^\infty(0, \infty)$; but $A+B$ fails to be essentially selfadjoint on that domain. One may use other p and

q and also one can replace the oscillatory parts by such discretized oscillating functions such as $s(x)=1$ for $(2n+1)^{-1} \leqq x < (2n)^{-1}$, 0 otherwise, $c(x)=1-s(x)$.

The idea is that the wave functions must go to zero as $x \to 0$ so well for A and B to accomodate the oscillation, whereas for $A+B$, in which the oscillation has cancelled itself, the wave functions need not be so good and $A+B$ will need a larger domain of selfadjointness.

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Von Neumann's coordinatization theorem [1)]

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In Honour of Béla Szőkefalvi-Nagy on his 70th birthday

1. Notation. L denotes a complemented, modular lattice with homogeneous basis $a_1, \ldots, a_N$, $N \geqq 4$ [2, Part II, Def. 3.1]; $A^j \equiv a_1 \vee \ldots \vee a_j$; ab means $a \wedge b$; $a \dot{\vee} b$ means $a \vee b$ if $ab=0$; $L_{ji} \equiv (b \in L : b \dot{\vee} a_j = a_i \dot{\vee} a_j)$.

If $\mathscr{R}$ is a ring and $m \leqq N$, then $\mathscr{R}^N(m)$ denotes the *right* $\mathscr{R}$-module $\big((\alpha_1, \ldots, \alpha_N)$: all $\alpha_i \in \mathscr{R}$ and $\alpha_i = 0$ for $m < i \leqq N\big)$; $(\alpha_1, \ldots, \alpha_m)_N$ is an abbreviation for $(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0) \in \mathscr{R}^N(m)$; $\mathscr{R}^N \equiv \mathscr{R}^N(N)$; $L(\mathscr{R}^N(m))$ denotes the set of finitely generated submodules of $\mathscr{R}^N(m)$, ordered by inclusion.

2. Von Neumann's theorem. *In each L_{ji} $(j \neq i)$, addition and multiplication can be defined so that:*

(2.1) *The L_{ji} become regular rings with unit, isomorphic to a common regular ring $\mathscr{R}$* [2, Part II, Theorem 9.2].

(2.2) *For each j the sublattice $(b \in L : b \leqq a_j)$ is isomorphic to $L(\mathscr{R})$, the lattice of principal right ideals of $\mathscr{R}$* [2, Part II, Theorem 9.2].

(2.3) *L is isomorphic to $L(\mathscr{R}^N)$* [2, Part II, Theorem 14.1].

3. Outline of von Neumann's proof. (3.1) Choose $c_{1j} = c_{j1}$, $2 \leqq j \leqq N$, so that $c_{j1} \dot{\vee} a_j = c_{j1} \dot{\vee} a_1 = a_j \dot{\vee} a_1$; set $c_{ji} = (c_{j1} \vee c_{1i})(a_j \vee a_i)$ for $1, i, j$ all different.

(3.2) Call a family $\alpha \equiv (\alpha_{ji} \in L_{ji} : i \neq j)$ an L-number if $(\alpha_{ji} \vee c_{jk})(a_k \vee a_i) = \alpha_{ki}$ and $(\alpha_{ji} \vee c_{ik})(a_j \vee a_k) = \alpha_{jk}$. Note: For every $b \in L_{ji}$ there exists a unique L-number α with $\alpha_{ji} = b$ [2, Part II, Lemma 6.1].

(3.3) Let $\mathscr{R}$ denote the set of L-numbers with operations:

$$(\alpha+\beta)_{ji} = \big(\alpha_{jk} \vee (\beta_{ji} \vee a_k)(a_j \vee c_{ik})\big)(a_j \vee a_i),$$
$$(\alpha\beta)_{ji} = (\alpha_{jk} \vee \beta_{ki})(a_j \vee a_i).$$

(3.4) For each $\alpha \in \mathscr{R}$ and $1 \leqq j \leqq N$, define the *reach* of α into a_j by $\alpha_j^{(r)} \equiv$ $\equiv (\alpha_{ji} \vee a_i) a_j$ (does not depend on i, $i \neq j$).

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(3.5) Prove: $\alpha\gamma=\beta$ has a solution γ if and only if $\beta_j^{(r)}\leqq\alpha_j^{(r)}$ (holds for all j if for some j) [2, Part II, Lemma 9.4] or [1, (3.2) with multiplication reversed].

(3.6) Prove: For each $b\leqq a_j$: $b=e_j^{(r)}$ for some idempotent $e\in\mathscr{R}$ [2, Part II, Theorem 9.3].

(3.7) Deduce: Parts (2.1), (2.2) of the theorem hold [2, Part II, Theorem 9.2].

$(3.8)_m$ Prove: For $1\leqq m\leqq N$ there exists an isomorphism

$$\varphi_m\colon (b\in L\colon\ b\leqq A^m)\to L(\mathscr{R}^N(m)) \quad \text{with} \quad \varphi_1\subset\varphi_2\subset\ldots\subset\varphi_N.$$

Note: φ_N establishes Part (2.3) of the theorem. The outstanding difficulty in von Neumann's proof is to establish the φ_m.

4. Von Neumann's strategy to prove $(3.8)_m$. (4.1) Call b an m-element if (i) $m=1$ and $b\leqq a_1$, or (ii) $2\leqq m\leqq N$ and $b\dot{\vee} A^{m-1}\leqq A^m$.

(4.2) For each m-element b define $\varphi(b)$, a submodule of $L(\mathscr{R}^N(m))$, as follows:

(i) If $b\leqq a_1$ define $\varphi(b)\equiv(e,0,\ldots,0)\mathscr{R}$ with e idempotent and $e_1^{(r)}=b$.

(ii) If $2\leqq m\leqq N$ define $\varphi(b)\equiv(-\alpha_1,\ldots,-\alpha_{m-1},1)_N e\mathscr{R}$ with e idempotent and $e_m^{(r)}=(A^{m-1}\vee b)a_m$, with $b'\dot{\vee}e_m^{(r)}=a_m$ and $(\alpha_i)_{im}=(b\vee b'\vee A^{i-1}\vee a_{i+1}\vee\ldots\vee a_{m-1})(a_i\vee a_m)$.

Note: $\varphi(b)$ is determined uniquely by b though e, b', and the α_i may not be; also $(\alpha_i)_{im}(A^{m-1}\vee b)=(b\vee A^{i-1}\vee a_{i+1}\vee\ldots\vee a_{m-1})(a_i\vee a_m)$.

(4.3) For each $x\in L$ and decomposition $x=\bigvee_{i=1}^{N}x_i$ with x_i an i-element, (such decompositions exist for all x), assign to x the submodule $\varphi(x_1)+\ldots+\varphi(x_N)$.

$(4.4)_m$ Prove: the set $(\varphi(x_1)+\ldots+\varphi(x_m)\colon x\leqq A^m)=L(\mathscr{R}^N(m))$.

$(4.5)_m$ Prove: For decompositions $x=\bigvee_{i=1}^{m}x_i$, $y=\bigvee_{i=1}^{m}y_i$: $x\leqq y$ if and only if $\sum_{i=1}^{m}\varphi(x_i)\leqq\sum_{i=1}^{m}\varphi(y_i)$. Note: $(4.5)_m$ implies that $\varphi_m(x)\equiv\sum_{i=1}^{m}\varphi(x_i)$ has the same value for all decompositions of x; then $(4.4)_m$, $(4.5)_m$ establish $(3.8)_m$.

Von Neumann established $(4.4)_m$ without difficulty [2, Part II, Theorem 11.2]; $(4.5)_1$ follows immediately from (3.5), (3.6). But von Neumann's proof of $(4.5)_m$, $2\leqq m\leqq N$ [2, Part II, pages 168—208], is a virtuoso demonstration of mathematical technique.

5. A new proof of $(4.5)_m$, $2\leqq m\leqq N$. We use direct lattice calculations (for the case $m=2$, in particular) and reduce part of the case m (to the case $m-1$) when $3\leqq m\leqq N$.

We require the following properties of L-numbers.

$$(5.1)\qquad (\alpha-\beta)_{jk}=\big(\alpha_{ji}\vee(a_k\vee\beta_{ji})(a_j\vee c_{ik})\big)(a_j\vee a_k) \quad [1,\ (2.3)],$$

hence

$$(5.2)\qquad (\alpha-\beta)_j^{(r)} = (\alpha_{ji}\vee\beta_{ji})a_j,$$

$$(5.3)\qquad (\alpha+\beta\gamma)_{ji} = \big(\beta_{jk}\vee(\alpha_{ji}\vee a_k)(\gamma_{ki}\vee a_j)\big)(a_j\vee a_i)$$

[1, (5.2) with multiplication reversed].

6. Proof of $(4.5)_2$. We assume $x_1\leqq a_1$, $y_1\leqq a_1$, $x_2\dot{\vee} a_1\leqq a_2\vee a_1$, $y_2\dot{\vee} a_1\leqq a_2\vee a_1$ and we need to prove:

(6.1) $x_1\vee x_2\leqq y_1\vee y_2$ if and only if $\varphi(x_1)+\varphi(x_2)\leqq\varphi(y_1)+\varphi(y_2)$. Because of modularity we need consider only the case $x_1=0$, $\varphi(x_1)=0$ (use $(4.5)_1$).

Now the inequality $\varphi(x_2)\leqq\varphi(y_1)+\varphi(y_2)$ is equivalent, in turn, to each of:

$$(6.2)\qquad \big(-\alpha_1(x_2)e(x_2),\, e(x_2)\big)_N = \big(e(y_1),\, 0\big)_N\beta_1+\big(-\alpha_1(y_2)e(y_2),\ e(y_2)\big)_N\beta_2$$

for some β_1, $\beta_2\in\mathscr{R}$;

$$(6.3)\qquad e(y_2)e(x_2)=e(x_2)\quad\text{and}\quad \big(\alpha_1(y_2)e(x_2)-\alpha_1(x_2)e(x_2)\big)_1^{(r)} \leqq \big(e(y_1)\big)_1^{(r)};$$

(6.4) $(a_1\vee x_2)a_2\leqq(a_1\vee y_2)a_2$ and, (use (5.2)),

$$\big(\alpha_1(x_2)e(x_2)\big)_{13}\vee\big(\alpha_1(y_2)e(x_2)\big)_{13}\big)a_1 \leqq y_1;$$

(6.5) (i) $a_1\vee x_2\leqq a_1\vee y_2$ and
(ii) $\big(\alpha_1(x_2)e(x_2)\big)_{13}\leqq y_1\vee\big((\alpha_1(y_2)e(x_2)\big)_{13}$.

The inequality (6.5) (ii) is equivalent to each of:

$$(6.6)\qquad \big((\alpha_1(x_2))_{12}\vee(e(x_2))_{23}\big)(a_1\vee a_3) \leqq y_1\vee\big((\alpha_1(y_2))_{12}\vee(e(x_2))_{23}\big),$$

$$(6.7)\qquad \big((\alpha_1(x_2))_{12}\vee(e(x_2))_{23}\big)\big(a_1\vee a_3\vee(e(x_2))_{23}\big) \leqq y_1\vee\big(\alpha_1(y_2)\big)_{12}\vee\big(e(x_2)\big)_{23},$$

$$(6.8)\qquad \big(\alpha_1(x_2)\big)_{12}\big(a_1\vee(a_3\vee(e(x_2))_{23})a_2\big) \leqq y_1\vee\big(\alpha_1(y_2)\big)_{12},$$

$$(6.9)\qquad \big(\alpha_1(x_2)\big)_{12}\big(a_1\vee(a_1\vee(e(x_2))_{21})a_2\big) \leqq y_1\vee\big(\alpha_1(y_2)\big)_{12},$$

$$(6.10)\qquad \big(\alpha_1(x_2)\big)_{12}(a_1\vee x_2) \leqq y_1\vee\big(\alpha_1(y_2)\big)_{12}.$$

Now (6.5) (i) and (6.10) together are equivalent to:

$$(6.11)\qquad x_2\leqq y_1\vee y_2,$$

which establishes (6.1), i.e. $(4.5)_2$.

7. Proof of $(4.5)_m$ **assuming** $(4.5)_{m-1}$, $3\leqq m\leqq N$. We assume $x_1\leqq A^{m-1}$, $y_1\leqq A^{m-1}$, $x_2\dot{\vee} A^{m-1}\leqq A^m$, $y_2\dot{\vee} A^{m-1}\leqq A^m$ and we must prove

(7.1) $x_1\vee x_2\leqq y_1\vee y_2$ if and only if $\varphi_{m-1}(x_1)+\varphi(x_2)\leqq\varphi_{m-1}(y_1)+\varphi(y_2)$ where φ_{m-1} is the isomorphism on A^{m-1} determined by φ (existing since $(4.5)_{m-1}$ is assumed to hold). We may assume that $x_1=y_1$ $(=z$, say).

We recall that [2, Part II, Lemma 13.2] states: *if* $a \leqq b$ *then every* x *can be expressed as* $(x \vee a)(x \vee c)$ *for some* c *with* $a \dot{\vee} c = b$. Repeated application of this lemma shows that our z can be expressed as $z^{(1)} \wedge z^{(2)} \wedge \ldots \wedge z^{(m-1)}$ where, for each $j < (m-1)$: $z^{(j)} \vee a_j = A^{m-1}$, and $z^{(m-1)} \geqq A^{m-2}$.

It is clearly sufficient to establish (7.1) with z replaced by $z^{(j)}, j = 1, \ldots, m-1$. Thus, in (7.1), we need consider only:

case (1): $z \geqq A^{m-2}$; and *case* (2): $z \vee a_j = A^{m-1}$ for some $j < (m-1)$.

The proof of (7.1) for case (1). We use lattice calculations as in the proof of $(4.5)_2$ in § 6. With the present z, x_2, y_2,

$$\varphi_{m-1}(z) = u_1 \mathscr{R} + \ldots + u_{m-2} \mathscr{R} + u_{m-1} g \mathscr{R}$$

where u_j is the vector in $\mathscr{R}^N$ with j-th component 1 and all other components 0, and g is an idempotent with $(g)^{(r)}_{m-1} = z a_{m-1}$.

Let

$$\varphi(x_2) = (-\alpha_1, \ldots, -\alpha_{m-1}, 1)_N e \mathscr{R}, \quad \varphi(y_2) = (-\beta_1, \ldots, -\beta_{m-1}, 1)_N f \mathscr{R}.$$

Then the last inequality of (7.1) is equivalent to each of the following:

(7.2) (i) $ef = e$ and (ii) $(\beta_{m-1} - \alpha_{m-1}) e \in g \mathscr{R}$,

(7.3) (i) $(x_2 \vee A^{m-1}) a_m \leqq (y_2 \vee A^{m-1}) a_m$, i.e., $x_2 \vee A^{m-1} \leqq y_2 \vee A^{m-1}$, and (ii) $((\beta_{m-1} - \alpha_{m-1}) e)^{(r)}_{m-1} \leqq z a_{m-1}$.

Choose any $k \leqq N$ with k different from $m-1$, m. Then (7.3) (ii) is equivalent to each of the following:

$$((\beta_{m-1} e)_{m-1,k} \vee (\alpha_{m-1} e)_{m-1,k}) a_{m-1} \leqq z a_{m-1}; \tag{7.4}$$

$$(\alpha_{m-1} e)_{m-1,k} \leqq z a_{m-1} \vee (\beta_{m-1})_{m-1,m} \vee e_{mk}; \tag{7.5}$$

$$(\alpha_{m-1})_{m-1,m} (a_{m-1} \vee a_k \vee e_{mk}) \leqq z a_{m-1} \vee (\beta_{m-1})_{m-1,m} \vee e_{mk}; \tag{7.6}$$

$$(\alpha_{m-1})_{m-1,m} (a_{m-1} \vee a_m (a_k \vee e_{mk})) \leqq z a_{m-1} \vee (\beta_{m-1})_{m-1,m}. \tag{7.7}$$

The left hand side of (7.7) equals

$$(\alpha_{m-1})_{m-1,m} (a_{m-1} \vee e^{(r)}_m) = (\alpha_{m-1})_{m-1,m} (x_2 \vee A^{m-1}) = (x_2 \vee A^{m-2})(a_m \vee a_{m-1}).$$

In the presence of (7.3) (i), the right hand side of (7.7) may now be replaced by each of

$$(z a_{m-1} \vee (\beta_{m-1})_{m-1,m})(y_2 \vee A^{m-1}), \quad z a_{m-1} \vee (y_2 \vee A^{m-2})(a_m \vee a_{m-1}),$$

$$(y_2 \vee A^{m-2} \vee z a_{m-1})(a_m \vee a_{m-1}), \quad (y_2 \vee z)(a_m \vee a_{m-1}), \quad y_2 \vee z,$$

so (7.4) (ii) is equivalent to each of

$$(x_2 \vee A^{m-2})(a_m \vee a_{m-1} \vee A^{m-2}) \leqq y_2 \vee z, \quad x_2 \vee A^{m-2} \leqq z \vee y_2.$$

Thus (7.4) is equivalent to: $x_2 \leqq z \vee y_2$ and this establishes (7.1) for case (1).

The proof of (7.1) for case (2). Choose z_{m-1} so that $z_{m-1} \dot{\vee} z a_j = z(a_j \dot{\vee} a_{m-1})$. Then: $z_{m-1} \leqq z$; $z_{m-1} \dot{\vee} a_j = a_{m-1} \dot{\vee} a_j$; $z_{m-1} \dot{\vee} A^{m-2} = z \vee A^{m-2} = A^{m-1}$; and $\varphi_{m-1}(z_{m-1}) = \varphi(z_{m-1}) = (0, \ldots, -\beta, 0, \ldots, 1)_N \mathscr{R}$ with $-\beta$ in the j-th place and 1 in the $(m-1)$-th place.

Set $\bar{x}_2 = (x_2 \vee z_{m-1})(A^{m-2} \vee a_m)$, $\bar{y}_2 = (y_2 \vee z_{m-1})(A^{m-2} \vee a_m)$. Then $\bar{x}_2 A^{m-1} = 0 = \bar{y}_2 A^{m-1}$; $z \vee \bar{x}_2 = z \vee x_2$; $z \vee \bar{y}_2 = z \vee y_2$ and so the inequality $z \vee x_2 \leqq z \vee y_2$ can be expressed as: $z \vee \bar{x}_2 \leqq z \vee \bar{y}_2$.

If $\varphi(x_2) = (-\alpha_1, \ldots, -\alpha_{m-2}, -\alpha_{m-1}, 1)_N e \mathscr{R}$ and $\varphi(y_2) = (-\beta_1, \ldots, -\beta_{m-2}, -\beta_{m-1}, 1)_N f \mathscr{R}$ then (use (5.3)):

$$\varphi(\bar{x}_2) = (-\alpha_1, \ldots, -\alpha_{j-1}, -\alpha_j - \beta\alpha_{m-1}, -\alpha_{j+1}, \ldots, -\alpha_{m-2}, 0, 1)_N e \mathscr{R},$$

$$\varphi(\bar{y}_2) = (-\beta_1, \ldots, -\beta_{j-1}, -\beta_j - \beta\beta_{m-1}, -\beta_{j+1}, \ldots, -\beta_{m-2}, 0, 1)_N f \mathscr{R}$$

so the inequality

$$\varphi_{m-1}(z) + \varphi(x_2) \leqq \varphi_{m-1}(z) + \varphi(y_2)$$

can be expressed as:

$$\varphi_{m-1}(z) + \varphi(\bar{x}_2) \leqq \varphi_{m-1}(z) + \varphi(\bar{y}_2)$$

(use: $(0, \ldots, 0, -\beta, 0, \ldots, 1)_N(\beta_{m-1} - \alpha_{m-1})e$, with $-\beta$ in the j-th place and 1 in the $(m-1)$-th place, is in $\varphi_{m-1}(z)$).

Thus we need only prove (7.1) in case (2) with z, x_2, y_2 replaced by $z, \bar{x}_2, \bar{y}_2$ respectively. We may now also replace z by $\bar{z} = zA^{m-2}$. Then we observe that all of $\bar{z}, \bar{x}_2, \bar{y}_2, \leqq A^{m-2} \dot{\vee} a_m$. Hence we can apply $(4.5)_{m-1}$ with $a_1, \ldots, a_{m-2}, a_{m-1}$ replaced by $a_1, \ldots, a_{m-2}, a_m$ (replacing $(\alpha_1, \ldots, \alpha_{m-1})_N$ in $\mathscr{R}^N(m-1)$ by $(\alpha_1, \ldots, \alpha_{m-2}, 0, \alpha_{m-1})_N$ in $\mathscr{R}^N(m)$); this replacement is permitted because it preserves the order of the a_j, and the functions φ, φ_{m-2}. This establishes (7.1) for the case (2) and completes the proof of $(4.5)_m$. This completes the proof of von Neumann's theorem.

8. Supplementary remark. Call $a_1, \ldots, a_N$, $N \geqq 3$ a *Desarguesian basis* for a complemented modular lattice L if for some $c_{1j}, j > 1$:

(i) (Bjarni Jónsson) a_i is perspective to some $b_i \leqq a_1$ for $i \geqq 2$ with $b_2 = b_3 = a_1$;

(ii) $a_2 a_1 = a_3(a_2 \dot{\vee} a_1) = 0$ and $a_1 \vee \ldots \vee a_N = 1$, and

(iii) the formulae (3.3) make $\mathscr{R}$ a regular ring if, in the definition of L-number, i, j are restricted to $\{1, 2, 3\}$.

If such a Desarguesian basis for L exists, then the a_i, $i > 3$ can be altered so that $\{a_1, \ldots, a_N\}$ becomes an independent basis for L and, with some changes, the above proof of von Neumann's theorem holds; the condition (iii) can be replaced by certain Desarguesian-type lattice conditions (K. D. FRYER and I. HALPERIN, *Acta Sci. Math.*, **17** (1956), 203—249; B. JÓNSSON, *Trans. Amer. Math. Soc.*, **97** (1960), 64—94).

The proof is simplified when, in the definition of L-number, the i, j are further restricted to $j<i$; but then the use of $e_N^{(r)}$ in (4.2) above and $(e(x))_{21}$ in (6.9) above, and the use of k $(<m-1)$ in (7.5) above (when $m=N$) must be (and can be) adjusted.

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On partial asymptotic stability and instability. I (Autonomous systems)

L. HATVANI

Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

1. Introduction

Ljapunov's direct method is the most powerful tool for establishing also partial stability properties, i.e. stability properties regarding some variables only [1]—[3]. One finds, however, that in many applications it is very complicated to construct an appropriate Ljapunov function. For example, the derivative of the total mechanical energy of a holonomic mechanical system under the action of dissipative forces is negative definite with respect to velocities only, thus it cannot be used in the basic theorems to establish asymptotic stability or instability with respect to the generalized coordinates. The method of BARBASHIN and KRASOVSKIĬ [4], [5] and LASALLE's invariance principle [6] enable us to get asymptotic stability or instability by Ljapunov functions with semidefinite derivative. These methods have been extended to the study of partial stability [1], [7], [8]. However, in comparison with the stability investigations concerning all variables, a new difficulty appears: the extensions require the boundedness of all the uncontrolled coordinates along every solution. As it was shown in [9] by an example, this condition cannot be omitted. Our purpose is to replace this condition by such ones which can be checked directly, i.e. without a priori knowledge of the solutions.

We first study what we can state after having dropped the condition of boundedness of the uncontrolled coordinates. This allows us to locate the limit set of the vector function whose components are the controlled coordinates of a solution. Then we can find additional conditions on the *Ljapunov function* which assure the zero solution of an autonomous system to be partially asymptotically stable or

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unstable. Starting from the localization result mentioned above, in the continuation [10] of the present paper we give some additional conditions on the *right-hand side* of the system which imply the same properties. We apply our results to study stability properties of mechanical equilibrium in the presence of dissipative forces. As special cases we study motions of a material point along certain surfaces in a constant field of gravity.

2. Notations, definitions. Preliminaries

Consider the system of differential equations

$$\dot{x} = X(x, t), \tag{2.1}$$

where $t \in R_+ = [0, \infty)$, and $x = (x^1, \dots, x^k)$ belongs to the space R^k with a norm $|x|$. Denote by $B_k(\varrho)$ the open ball in R^k with center at the origin and radius $\varrho > 0$, $\bar{B}_k(\varrho)$ its closure in R^k. Let a partition $x = (y, z)$ $(y \in R^m,\ z \in R^n;\ 1 \leqq m \leqq k,\ n = k - m)$ be given. Assume that the function X is defined on the set Γ_y:

$$\Gamma_y = G_y \times R_+ \quad (G_y = B_m(H) \times R^n; \quad 0 < H \leqq \infty),$$

it is continuous in x, is measurable in t, and satisfies the Carathéodory condition locally (i.e. for every compact set $K \subset R^k$ there is a locally integrable $h: R_+ \to R_+$ such that $|X(x, t)| \leqq h(t)$ for all $(x, t) \in K \times R_+$), so the local existence of solutions of initial value problems is assured, and every solution has a maximal extension. We denote by $x(t) = x(t; x_0, t_0)$ any solution with $x(t_0) = x_0$. If (2.1) is autonomous, i.e. X does not depend on t, we use the notation $x(t; x_0) = x(t; x_0, 0)$. We always assume that solutions are z-continuable [2], i.e. if $x(t) = (y(t), z(t))$ is a solution of (2.1) and $|y(t)| \leqq H' < H$ for $t \in [t_0, T)$, then $x(t)$ can be continued to the closed interval $[t_0, T]$.

The zero solution of (2.1) is said to be:

y-stable if for every $\varepsilon > 0$, $t_0 \in R_+$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta(\varepsilon, t_0)$ implies $|y(t; x_0, t_0)| < \varepsilon$ for $t \geqq t_0$;

asymptotically y-stable if it is y-stable, and for every $t_0 \in R_+$ there exists a $\sigma(t_0) > 0$ such that $|x_0| < \sigma(t_0)$ implies $|y(t; x_0, t_0)| \to 0$ as $t \to \infty$;

uniformly asymptotically y-stable if it is y-stable so that the number $\delta(\varepsilon, t_0)$ can be chosen independently of t_0, and there exists a $\sigma > 0$ such that $|y(t; x_0, t_0)| \to 0$ uniformly in $x_0 \in B_k(\sigma)$ as $t \to \infty$;

y-unstable if it is not y-stable.

Let $x = \varphi(t) = (\psi(t), \chi(t))$ be a solution of (2.1) defined on an interval $[t_0, \infty)$ $(t_0 \in R_+;\ \psi: [t_0, \infty) \to R^m,\ \chi: [t_0, \infty) \to R^n)$. A point $q \in R^m$ is a *y-limit point* of the solution $x=\varphi(t)$ if there exists a sequence $\{t_i\}$ such that $t_i\to\infty$ and $\psi(t_i)\to q$ as $i\to\infty$. The *partial limit set* $\Omega_y(\varphi)$ of $x=\varphi(t)$ *with respect to* y is the set of all its y-limit points. It is easy to see that if ψ is bounded then $\Omega_y(\varphi)$ is non-empty, compact, connected and is the smallest closed set approached by $\psi(t)$ as $t\to\infty$.

The *complete trajectory* $\gamma(\xi)$ of (2.1) belonging to a non-continuable solution $\xi:(\alpha,\beta)\to R^k$ is defined by $\gamma(\xi)=\{\xi(t):\alpha<t<\beta\}$. It is known [11] that if (2.1) is autonomous then the set $E=\Omega_x(\varphi)\cap G_y$ is *semiinvariant* with respect to (2.1), which means that for every $x_0\in E$ the equation (2.1) has a solution ξ such that $\xi(0)=x_0$ and $\gamma(\xi)\subset E$.

A continuous function $V:\Gamma'_y\to R$, $\Gamma'_y=G'_y\times R_+$ $(G'_y=B_m(H')\times R^n;\ 0<H'<H)$ is a *Ljapunov function* of (2.1) if $V(0,t)\equiv 0$; V is locally Lipschitzian and

$$\dot V(x,t)=\limsup_{h\to 0+}\frac{V(x+hX(x,t),\,t+h)-V(x,t)}{h}\leqq 0$$

for all $(x,t)\in\Gamma'_y$. The function $\dot V$ is called the derivative of V with respect to (2.1).

We say that a function $a\colon R_+\to R_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $a(0)=0$. A function $V:\Gamma'_y\to R$ is said to be *positive definite in* y or *positive y-definite* if there exists a function $a\in\mathcal{K}$ such that $a(|y|)\leqq V(y,z,t)$ for all $(y,z,t)\in\Gamma'_y$.

Let us given a continuous function $W:L\times R^q\times R_+\to R$, where $L\subset R^p$ is open, $p\geqq 1$, $q\geqq 0$ are integers. Following LASALLE's notation [6], for $c\in R$ we denote by $W_p^{-1}[c,\infty]$ the set of the points $u\in R^p$ for which there is a sequence $\{(u_i,v_i,t_i)\}$ such that $u_i\to u$, $|v_i|\to\infty$, $t_i\to\infty$, $W(u_i,v_i,t_i)\to c$ as $i\to\infty$. If $W:L\to R$ (i.e. $q=0$ and W does not depend on t), then $W_p^{-1}[c,\infty]$ is the inverse under W of c and is denoted by $W^{-1}(c)$ as usual.

We say that a function $x=(y,z)\colon R_+\to R^k$ is *z-bounded* if $z\colon R_+\to R^n$ is bounded for $t\geqq 0$.

Now we can cite the following extensions of the Barbashin—Krasovskiĭ theorem to partial stability, which contain the original theorems as special cases $(y=x)$.

Theorem A (A. C. OZIRANER [9]). *Suppose that every solution of the autonomous system*

$$\dot x = X(x) \tag{2.2}$$

starting from a sufficiently small neighbourhood of the origin is z-bounded.

I. *If there exists a Ljapunov function* $V:G'_y\to R$ *such that*

(i) V *is positive y-definite and* $V(0)=0$;

(ii) *the set* $\{x\colon V(x)>0\}\cap\dot V^{-1}(0)$ *contains no complete trajectory of* (2.2), *then the zero solution of* (2.2) *is uniformly asymptotically stable.*

II *If there exists a Ljapunov function* $V:G'_y\to R$ *such that* (i′) $V(0)=0$, *and every neighbourhood of the origin contains a point* x *with* $V(x)<0$;

(ii′) *the set* $\{x\colon V(x)<0\}\cap\dot{V}^{-1}(0)$ *contains no complete trajectory of* (2.2), *then the zero solution of* (2.2) *is y-unstable.*

In [12] J. P. LaSalle pointed out that the essence of the Barbashin—Krasovskiĭ method consists in the location of the limit sets and formulated it in his "invariance principle": If $V: G'_y\to R$ is a Ljapunov function of (2.2), and $\varphi\colon R_+\to R^k$ is a solution of the same equation such that $|\varphi(t)|\leqq H''<H'$ for $t\geqq 0$, then $\Omega_x(\varphi)\subset\dot{V}^{-1}(0)\cap V^{-1}(c)$ with some $c\in R$.

3. Theorems on general differential systems

First of all we have to locate the partial limit set of solutions of

$$\dot{x} = X(x) \tag{3.1}$$

with the knowledge of boundedness only of controlled coordinates. An easy but, so it appears, useful generalization of LaSalle's invariance principle is

Lemma 3.1. *Suppose that* $V: G'_y\to R$ *is a Ljapunov function of* (3.1) *and let* $x=\varphi(t)=(\psi(t),\chi(t))$ *be a solution of* (3.1) *such that* $|\psi(t)|\leqq H''<H'$ *for* $t\in R_+$. *Then either* a) $|\chi(t)|\to\infty$ *as* $t\to\infty$ *or* b) $v(t)=V(\varphi(t))\to v_0$ *as* $t\to\infty$; *the set* $\Omega_x(\varphi)$ *is not empty and is contained in* $\dot{V}^{-1}(0)\cap V^{-1}(v_0)$.

Proof. Assume that a) is not satisfied. Then, because of the boundedness of ψ, there exist $q\in R^m$, $r\in R^n$, $v_0\in R$ and a sequence $\{t_i\}$ such that $t_i\to\infty$, $\psi(t_i)\to q$, $\chi(t_i)\to r$ and $v(t_i)\to v_0$ as $i\to\infty$. Since $\Omega_x(\varphi)$ is semi-invariant with respect to (3.1), there exists a solution $x(t)=x(t;q,r)$ of (3.1) for which $\gamma(x)\subset\Omega_x(\varphi)$. Let $\bar{t}\in R_+$ be fixed. Then there exists a sequence $\{\bar{t}_i\}$ such that $\bar{t}_i\to\infty$, $(\psi(\bar{t}_i),\chi(\bar{t}_i))\to x(\bar{t};q,r)$ and $v(\bar{t}_i)\to V(x(\bar{t};q,r))$ as $i\to\infty$. Since v is decreasing, $V(x(\bar{t};q,r))=v_0$, which completes the proof.

Denote by $P_y\colon R^k\to R^m$ the orthogonal projection from R^k into R^m, i.e. $P(x)=y$.

Lemma 3.2. *Let* V,φ *satisfy the conditions of Lemma* 3.1 *and assume, in addition, that* V *is bounded from below. Then*

$$\Omega_y(\varphi)\subset P_y(\Omega_x(\varphi))\cup V_m^{-1}[v_0,\infty],$$

where v_0 *is defined by* $v(t)=V(\varphi(t))\to v_0$ *as* $t\to\infty$.

Proof. Introduce the notations $L=\Omega_y(\varphi)$, $M=P_y(\Omega_x(\varphi))$, $N=L\setminus M$. Obviously, $M\subset L$; therefore, it is sufficient to prove that $N\subset V_m^{-1}[v_0,\infty]$. If $q\in N$, then there are a sequence $\{t_i\}$ and $v_0 \in R$ such that $t_i \to \infty$, $\psi(t_i) \to q$, $|\chi(t_i)| \to \infty$, $v(t_i) \to v_0$ as $i \to \infty$, i.e. $q \in V_m^{-1}[v_0, \infty]$. Since v is nonincreasing, v_0 is independent of q, thus $N \subset V_m^{-1}[v_0, \infty]$.

The lemma is proved.

Theorem 3.1. *Let $V: G'_y \to R$ be a positive y-definite Ljapunov function of* (3.1) *such that for every $c > 0$ the set $\dot{V}^{-1}(0) \cap V^{-1}(c)$ contains no complete trajectory. Then the zero solution of* (3.1) *is y-stable and for every solution $x(t) = (y(t), z(t))$ starting from a sufficiently small neighbourhood of the origin either* a) $V(x(t)) \to 0$ *(and, consequently, $|y(t)| \to 0$) or* b) $|z(t)| \to \infty$ *as $t \to \infty$.*

Proof. Since V is positive y-definite and $\dot{V}(x) \leqq 0$, the zero solution of (3.1) is y-stable (see [11], p. 15) and, a fortiori, every solution starting from some neighbourhood $B_k(\varrho)$ $(\varrho > 0)$ of the origin is y-bounded. Let $x = \varphi(t) = (\psi(t), \chi(t))$ be such a solution. Suppose $|\chi(t)| \nrightarrow \infty$ as $t \to \infty$. We have to prove that in this case $v(t) = V(\varphi(t)) \to 0$ as $t \to \infty$. By Lemma 3.1, $v(t) \to v_0 \geqq 0$, and there is a point $p \in R^k$ such that $p \in \Omega_x(\varphi) \subset \dot{V}^{-1}(0) \cap V^{-1}(v_0)$. The set $\Omega_x(\varphi)$ is semiinvariant with respect to (3.1); consequently, $\dot{V}^{-1}(0) \cap V^{-1}(v_0)$ contains a complete trajectory of (3.1). By the assumptions this implies $v_0 = 0$.

The proof is complete.

Theorem A.I in the previous section is a corollary of Theorem 3.1. Indeed, by Theorem 3.1, the conditions of Theorem A.I imply that the zero solution of (3.1) is y-stable and $V(x(t; x_0)) \to 0$ as $t \to \infty$ for all $x_0 \in B_k(\sigma)$ with some $\sigma > 0$. Application of the classic covering theorem of Heine—Borel—Lebesgue gives that this convergence is uniform in $x_0 \in B_k(\sigma)$ (see [9]), which implies uniform asymptotical y-stability because V is positive y-definite.

The following two theorems show how to make use of the alternative given in Lemma 3.1 and Theorem 3.1 for getting sufficient conditions for partial asymptotic stability of the zero solution of an autonomous system.

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Suppose, in addition, that $V(y, z) \to 0$ uniformly in $y \in B_m(H')$ as $\dot{V}(y, z) \to 0$ and $|z| \to \infty$. Then the zero solution of* (3.1) *is uniformly asymptotically y-stable.*

Proof. In view of Theorem 3.1 and the remark following its proof, it is sufficient to prove that the function V tends to 0 along every solution starting from some neighbourhood of the origin. Denote by $\varphi(t) = (\psi(t), \chi(t))$ an arbitrary solution of (3.1) with $|\psi(t)| \leqq H'' < H'$ $(t \in R_+)$ and let $v(t) = V(\varphi(t)) \to v_0$ as $t \to \infty$. Since $\dot{v}(t) = \dot{V}(\varphi(t)) \leqq 0$ for all $t \in R_+$ and the function v is bounded from below, there exists a sequence $\{t_i\}$ such that $t_i \to \infty$ and $\dot{V}(\varphi(t_i)) \to 0$ as $i \to \infty$. By Theorem 3.1 either $v_0 = 0$ or the sequence $\{z_i = \chi(t_i)\}$ diverges to infinity in norm as

$i\to\infty$. In the latter case we have $\dot V(\psi(t_i); z_i)\to 0$, $|z_i|\to\infty$ and, simultaneously, $V(\psi(t_i), z_i)\to v_0$ as $i\to\infty$. The last assumption of the theorem implies $v_0=0$, which completes the proof.

It may be pointed out that Theorem 3.2 improves certain results which can be obtained by the application of some basic theorems on partial uniform asymptotic stability (see [2, 13]) to autonomous systems. To illustrate this fact let us recall a theorem of K. Peiffer and N. Rouche [13, Th. IV]. For (3.1) it says that if there exists a positive y-definite Ljapunov function $V: G'_y\to R$ such that $V(x)\to 0$ uniformly in $x\in G'_y$ as $\dot V(x)\to 0$, then the zero solution of (3.1) is uniformly asymptotically y-stable. This obviously follows from Theorem 3.2. In fact, Theorem 3.2 improves this corollary as it is shown by the following example.

Consider the system

$$\dot x^1 = -x^1(1+(x^3)^2), \quad \dot x^2 = -x^2, \quad \dot x^3 = x^3-(x^3)^3 \tag{3.2}$$

and let $y=(x^1, x^2)$, $z=x^3$. The function $V(x^1, x^2, x^3)=(x^1)^2+(x^2)^2+(x^2)^2(x^3)^2$ is positive (x^1, x^2)-definite, its derivative with respect to (3.2) reads $\dot V(x^1, x^2, x^3)= -2(x^1)^2(1+(x^3)^2)-2(x^2)^2-2(x^2)^2(x^3)^4$. If $\dot V\to 0$ then

$$x^1\to 0, \quad x^2\to 0, \quad (x^2)^2(x^3)^4\to 0, \tag{3.3}$$

which do not imply that $V\to 0$; therefore, the theorem of Pfeiffer and Rouche cannot be applied to this case. On the other hand, $|x^3|\to\infty$ and (3.3) together already imply $V\to 0$ and Theorem 3.2 can be applied.

C. Risito [7] proved that the statements of Theorem A (without uniformity) remain true if instead of (ii) and (ii′) one requires the following: the set $\{(y, z): y=0\}$ is invariant and the region $\dot V^{-1}(0)\setminus\{(y, z): y=0\}$ contains no complete positive semi-trajectory. Lemma 3.2 allows us to extend this result to the case when the uncontrolled coordinates are not supposed to be bounded.

Theorem 3.3. *Let $V: G'_y\to R$ be a positive y-definite Ljapunov function of* (3.1). *Suppose that for every $c>0$*

(i) *if the set $\dot V^{-1}(0)\cap V^{-1}(c)$ contains a complete trajectory then this trajectory is contained also in the set $\{(y, z): y=0\}$;*

(ii) $V_m^{-1}[c, \infty]\subset\{0\}$.

Then the zero solution of (3.1) *is asymptotically y-stable.*

Proof. As it was shown in the proof of Theorem 3.1, the zero solution starting from some neighbourhood of the origin is y-bounded. Let $x=\varphi(t)=(\psi(t), \chi(t))$ be such a solution. We have to prove that $|\psi(t)|\to 0$ as $t\to\infty$, i.e. $\Omega_y(\varphi)=\{0\}$. The function $v(t)=V(\varphi(t))$ is nonincreasing and nonnegative, hence $v(t)\to v_0\geqq 0$. If $v_0=0$ then the statement is true because V is positive y-definite. Assume that

$v_0>0$ and there exists a q such that $0\neq q\in\Omega_y(\varphi)$. By Lemma 3.2 and condition (ii) there is an $r\in R^n$ such that $p=(q,r)\in\Omega_x(\varphi)$. The set $\Omega_x(\varphi)$ is semiinvariant, hence there exists a solution $\xi:(-\infty,\infty)\to R^k$ of (3.1) for which $\xi(0)=p$ and $\gamma(\xi)\subset\subset\Omega_x(\varphi)$. In view of Lemma 3.1 $\gamma(\xi)$ is contained also in the set $\dot V^{-1}(0)\cap V^{-1}(v_0)$. On the other hand, $q\neq 0$; therefore $\gamma(\xi)$ is not contained in the set $\{x: y=0\}$, in contradiction to condition (i) of the theorem. This means that either $v_0=0$ or $\Omega_y(\varphi)=0$, which completes the proof.

Corollary 3.1. *Let condition* (i) *in Theorem* 3.3 *be satisfied. Suppose, in addition, that there is a number* $\varrho>0$ *such that* $0<|\bar y|<\varrho$ *implies*

$$\lim_{y\to\bar y,\,|z|\to\infty} V(y,z)=\infty. \tag{3.4}$$

Then the zero solution of (3.1) *is asymptotically y-stable.*

The alternative given in Lemma 3.1 can be used also for getting sufficient conditions for the instability of the zero solution of (3.1).

Theorem 3.4. *Suppose that there is a Ljapunov function* $V: G'_y\to R$ *of* (3.1) *satisfying the following properties:*

(i) *for every* $\delta>0$ *there exists* $x_0(\delta)\in B_k(\delta)$ *with* $V(x_0(\delta))<0$;

(ii) *there is an* ε_0 $(0<\varepsilon_0<H')$ *such that for every* $c<0$ *the set*

$$\dot V^{-1}(0)\cap V^{-1}(c)\cap(\bar B_m(\varepsilon_0)\times R^n) \tag{3.5}$$

contains no complete trajectory.

Then for every δ $(0<\delta<\varepsilon_0)$ *either* a) *every curve* $\gamma_y^+(x_0(\delta))$: $t\mapsto y(t;x_0(\delta))$ $(t\in R_+)$ *leaves the ball* $B_m(\varepsilon_0)$ *in finite time, or* b) $|z(t;x_0(\delta))|\to\infty$ *as* $t\to\infty$.

Proof. If the statement is not true, then for some δ_0 $(0<\delta_0<\varepsilon_0)$ there exists a solution

$$\varphi(t)=(\psi(t),\chi(t))\quad(\varphi(0)=x_0(\delta_0)) \tag{3.6}$$

such that

$$|\psi(t)|\leqq\varepsilon_0\quad(t\in R_+);\quad v(t)=V(\varphi(t))\to v_0<0\quad(t\to\infty), \tag{3.7}$$

the limit set $\Omega_x(\varphi)$ is not empty and is contained in the set (3.5) for $c=v_0$. On the other hand, $\Omega_x(\varphi)$ is semiinvariant, thus (3.5) contains a complete trajectory in contradiction to condition (ii) of the theorem.

The proof is complete.

Corollary 3.2. *Let the conditions of Theorem* 3.4 *be satisfied. Suppose, in addition, that*

$$\liminf_{|z|\to\infty} V(y,z)\geqq 0 \tag{3.8}$$

uniformly in $y\in\bar B_m(\varepsilon_0)$. *Then the zero solution of* (3.1) *is y-unstable.*

Theorem 3.5. *Let the conditions of Theorem 3.4 be satisfied. Suppose, in addition, that the Ljapunov function V is bounded from below on the set $\bar{B}_m(\varepsilon_0)\times R^n$, and*

$$\liminf_{|z|\to\infty,\, \dot{V}\to 0} V(y, z) \geqq 0 \tag{3.9}$$

uniformly in $y\in\bar{B}_m(\varepsilon_0)$. *Then the zero solution of* (3.1) *is unstable with respect to* y.

Proof. We shall prove that for all $\delta>0$ every curve $\gamma_y^+(x_0(\delta))$ defined in Theorem 3.4 leaves the ball $B_m(\varepsilon_0)$. If it is not true then for some δ_0 $(0<\delta_0<\varepsilon_0)$ the solution (3.6) possesses properties (3.7), and, by Theorem 3.4, $|\chi(t)|\to\infty$ as $t\to\infty$. Similarly to the proof of Theorem 3.2 it can be proved that (3.9) implies $v_0=0$. This contradiction completes the proof.

4. Applications to mechanical systems

Consider a holonomic mechanical system of r degrees of freedom with time-independent constraints under the action of potential and dissipative forces. The Lagrangian equations of motions are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial P}{\partial q} + Q, \tag{4.1}$$

where the following notations [11] are used: the column vectors $q, \dot{q}\in R^r$ consist of generalized coordinates and velocities, respectively (v^{T} denotes the transposed of $v\in R^r$); the potential energy $P: q\mapsto P(q)\in R$ is continuously differentiable, $P(0)=0$; $T=T(q, \dot{q})=(1/2)\dot{q}^{\mathrm{T}}A(q)\dot{q}$ is the kinetic energy, where the symmetric matrix function $A: q\mapsto A(q)\in R^{r\times r}$ is continuously differentiable; the continuous function $Q: (q, \dot{q})\mapsto Q(q, \dot{q})\in R^r$ is the resultant of non-energic and dissipative forces with complete dissipation, i.e. there exists a function $c\in\mathscr{K}$ such that $Q^{\mathrm{T}}(q, \dot{q})\dot{q}\leqq -c(|\dot{q}|)$ for all $q, \dot{q}\in R^r$. Assume that $q=\dot{q}=0$ is an equilibrium of (4.1) and the motions starting from a neighbourhood of this equilibrium depend continuously on the initial coordinates and velocities.

L. Salvadori [11] proved that if the equilibrium at $q=0$ is isolated then it is asymptotically stable if P has a minimum there, and unstable if P has no minimum there. By means of a simple example with one degree of freedom K. Peiffer [14] showed that this theorem would be false without the condition that the equilibrium at $q=0$ is isolated. Applying our results we give sufficient conditions for asymptotic stability and instability of the equilibrium at $q=0$ (possibly non-isolated) with respect to velocities or some generalized coordinates.

For $q \in R^r$ denote by $\lambda(q)$ $(\Lambda(q))$ the smallest (largest) eigenvalue of the symmetric matrix $A(q)$. We can estimate the kinetic energy as follows

$$\frac{1}{2}\lambda(q)|\dot{q}|^2 \leqq T(q, \dot{q}) \leqq \frac{1}{2}\Lambda(q)|\dot{q}|^2 \quad (q, \dot{q} \in R^r). \tag{4.2}$$

For every $q \in R^r$ the matrix $A(q)$ is positive definite (i.e. $\lambda(q) > 0$) (see [11], p. 362); therefore, (4.1) can be rewritten into the equivalent normal form

$$\dot{x} = X(x) \quad (x = \operatorname{col}(q, \dot{q})).$$

As is known (see [11], p. 358), the derivative of the total mechanical energy $H = T + P$ with respect to (4.1) is $\dot{H}(q, \dot{q}) = Q(q, \dot{q})\dot{q} \leqq 0$; consequently, H is a Ljapunov function of (4.1). The dissipation is complete, hence for arbitrary $c \in R$ we have

$$\dot{H}^{-1}(0) \cap H^{-1}(c) = \{\operatorname{col}(q, \dot{q}) \colon \ P(q) = c, \ \dot{q} = 0\};$$

therefore, the complete trajectories of (4.1) contained in this set are the equilibria $q = q_0$, $\dot{q} = 0$ for which $P(q_0) = c$.

Sometimes we shall use a partition $q = \operatorname{col}(\hat{q}, \tilde{q})$ of the vector of generalized coordinates, where $\hat{q} \in R^s$, $\tilde{q} \in R^{r-s}$, $0 \leqq s \leqq r$ (if $s = 0$ then $\tilde{q} = q$, and the conditions and statements concerning $\hat{q}$ are to be dropped).

In his first paper on partial stability, V. V. RUMJANCEV [1] proved that in the absence of any potential forces the equilibrium $q = \dot{q} = 0$ of (4.1) is asymptotically $\dot{q}$-stable provided that there are some constants λ_0, Λ_0 such that

$$0 < \lambda_0 \leqq \lambda(q) \leqq \Lambda(q) \leqq \Lambda_0.$$

The following two corollaries generalize this result to the case of the presence of potential forces.

Corollary 4.1. *Suppose that the potential energy $P(\hat{q}, \tilde{q})$ is positive $\hat{q}$-definite and the region $\{q \colon P(q) > 0\}$ contains no equilibria. If for some H'*

$$\lim_{|\tilde{q}| \to \infty} \lambda(\hat{q}, \tilde{q}) = \infty \tag{4.1}$$

uniformly in $\hat{q} \in \bar{B}_s(H')$, then the equilibrium $q = \dot{q} = 0$ of (4.1) is asymptotically $\dot{q}$-stable.

Proof. We will apply Corollary 3.1 to equation (4.1) and Ljapunov function H with $y = \dot{q}$. Condition (4.3) and inequality (4.2) imply that H is positive $\hat{q}$-definite as well, hence the equilibrium is $\hat{q}$-stable (see [11], p. 15) and we can assume that $\hat{q}$ belongs to a bounded set. Consequently, in (4.3) we can change $|\tilde{q}| \to \infty$ into $|q| \to \infty$ and, in view of (4.2), we have $T(q, \dot{q}) \to \infty$ as $|q| \to \infty$ for every $\dot{q} \neq 0$. This means, that all conditions of Corollary 3.1 are fulfilled.

Corollary 4.2. *If*

(i) $P(q) \geqq 0$ $(q \in R^r)$;

(ii) *the set* $\{q\colon P(q) > 0\}$ *contains no equilibria;*

(iii) $\lambda(q) \geqq \lambda_0$ $(0 < \lambda_0 = \text{const.})$;

(iv) $\lim\limits_{|q| \to \infty} P(q) \leqq \infty$ *exists;*

(v) *there are* $d \in \mathcal{K}$ *and* $H' > 0$ *such that*

$$Q(q, \dot{q})\dot{q} \leqq -d(T(q, \dot{q})) \quad (q \in R^r,\ |\dot{q}| \leqq H'), \tag{4.4}$$

then the equilibrium $q = \dot{q} = 0$ *of* (4.1) *is asymptotically* $\dot{q}$*-stable.*

Proof. If $P(q) \to \infty$ as $|q| \to \infty$, then the generalized coordinates are bounded along every motion, and the statement follows from Theorem A in Section 2. Suppose the limit of P is finite. By Theorem 3.1 the equilibrium $q = \dot{q} = 0$ is $\dot{q}$-stable, and for every motion $(q(t), \dot{q}(t))$ starting from some neighbourhood of $q = \dot{q} = 0$ either $h(t) = H(q(t), \dot{q}(t)) \to 0$ or $|q(t)| \to \infty$ as $t \to \infty$. In the second case $P(q(t))$ has a finite limit, thus $T(q(t), \dot{q}(t)) \to T_0$ as $t \to \infty$. If $T_0 > 0$, then by condition (v)

$$\frac{d}{dt} h(t) = Q(q(t), \dot{q}(t))\dot{q}(t) \leqq -d(T_0) < 0 \quad (t \in R_+),$$

which is impossible, because h is non-negative. Therefore, in both cases $T(q(t), \dot{q}(t)) \to 0$ $t \to \infty$. According to (iii) and (4.2) this implies $|\dot{q}(t)| \to 0$ as $t \to \infty$, which completes the proof.

Condition (iv) is rather restrictive, but if we know more of the behaviour of generalized coordinates we can weaken it, as its only role is to assure the existence of the (finite or infinite) limit of the potential energy along motions. For example, if $P(\hat{q}, \tilde{q})$ is positive $\hat{q}$-definite then we can assume that $\hat{q} \in \bar{B}_s(H')$ with some constant H'. Suppose that $P(\hat{q}, \tilde{q}) \to P_*(\hat{q})$ as $|\tilde{q}| \to \infty$. If $P_*(\hat{q})$ is constant then (iv) is satisfied. The case of changing $P_*(\hat{q})$ can be treated by the application of the further development of the Barbashin—Krasovskiĭ method in another direction [10].

As to condition (v), it is obviously fulfilled if there is a Λ_0 such that $\Lambda(q) \leqq \Lambda_0$ for all $q \in R^r$ (namely, $d(u) = c(u/\Lambda_0)$).

If we know a priori that the generalized coordinates are bounded, then conditions (iii)—(v) can be dropped, and the statement is a consequence of Oziraner's theorem (Theorem A in Section 2). But it is worth noticing that the conditions of Corollary 4.2 can be satisfied even if the generalized coordinates are not bounded. This can be shown by the system of one degree of freedom described by the equation

$$\ddot{q} + q^3 = 0 \quad (q, \dot{q} \in R)$$

found by K. Peiffer (see [11], p. 115) in order to prove that complete dissipation does not imply stability in case $P(q) \leqq 0$.

Applying Corollary 3.1 to equation (4.1) and Ljapunov function H with $y=\hat{q}$ we obtain

Corollary 4.3. *If*

(i) *P is positive $\hat{q}$-definite;*

(ii) *the set $\{q\colon \hat{q}\neq 0\}$ contains no equilibria;*

(iii) *for every $\hat{q}_0\neq 0$, $\lim\limits_{\hat{q}\to\hat{q}_0,\,|\tilde{q}|\to\infty} P(\hat{q},\tilde{q}) = \infty$,*

then the equilibrium $q=\dot{q}=0$ is asymptotically $\hat{q}$-stable.

Assuming that the equilibrium $q=0$ is isolated with respect to the region $\{q\colon P(q)<0\}$ and P has no minimum there, W. T. Koiter [15] proved that the equilibrium is unstable. The special case $q=\hat{q}$ of our following corollary shows that it is, in fact, q-unstable.

Corollary 4.4. *Suppose that for some $H', \varepsilon_0, \lambda_0$ $(0<\varepsilon_0<H', \lambda_0>0)$ the following conditions are satisfied:*

(i) *for every δ $(0<\delta<\varepsilon_0)$ there is a $q_0(\delta)\in B_r(\delta)$ with $P(q_0(\delta))<0$;*

(ii) $\lambda(\hat{q},\tilde{q})\geqq\lambda_0>0$ $(|\hat{q}|\leqq H',\ \tilde{q}\in R^{r-s})$;

(iii) *the region $\{q\colon P(q)<0,\ |\hat{q}|<\varepsilon_0\}$ contains no equilibria.*

Then for every δ $(0<\delta<\varepsilon_0)$ either a) *the curve $\gamma_{\hat{q}}^{+}(q_0(\delta),0)\colon t\mapsto\hat{q}(t;\,q_0(\delta),0)$ $(t\in R_+)$ leaves the ball $B_s(\varepsilon_0)$ in finite time, or* b) $|\tilde{q}(t;\,q_0(\delta),0)|\to\infty$ *as $t\to\infty$.*

Proof. We can apply Theorem 3.4 with $V=H, y=\hat{q}$, observing that condition (ii) precludes the possibility of $|\dot{q}(t)|\to\infty$ without $|\tilde{q}(t)|\to\infty$ as $t\to\infty$.

Now, Corollary 3.2 yields

Corollary 4.5. *Let all conditions of Corollary* 4.4 *be satisfied. Suppose, in addition, that*

(iv) $\liminf\limits_{|\tilde{q}|\to\infty} P(\hat{q},\tilde{q}) \geqq 0$ *uniformly in* $\hat{q}\in\bar{B}_s(H')$.

Then the equilibrium $q=\dot{q}=0$ of (4.1) *is $\hat{q}$-unstable.*

Examples. Finally, in order to illustrate the results of this section we study the stability properties of the mechanical system of two degrees of freedom introduced and investigated by K. Peiffer and N. Rouche [13]. Consider a point of mass equal to 1 moving in a constant field of gravity in the inertial frame of reference $Oxyz$; Oz directed vertically upward. Suppose the point is constrained to move on a surface of equation $z=(1/2)y^2(1+x^2)$ and furthermore, that it is subjected to viscous friction. The total mechanical energy is $H=T+P$ where

$$T=\frac{1}{2}\dot{x}^2+\frac{1}{2}\dot{y}^2+\frac{1}{2}y^2[\dot{y}(1+x^2)+\dot{x}xy]^2,\quad P=\frac{g}{2}y^2(1+x^2)$$

and g is the acceleration in the gravity field. Let the dissipative forces be defined by the formulas

$$Q^1 = -\alpha \frac{\partial T}{\partial \dot{x}} \quad Q^2 = -\alpha \frac{\partial T}{\partial \dot{y}}. \tag{4.5}$$

By Rumjancev's theorem ([1], see also [11], p. 15) the equilibrium $x=y=\dot{x}=\dot{y}=0$ is $(y, \dot{x}, \dot{y})$-stable, but the coordinate x may be even unbounded. Therefore, although the system is autonomous, the earlier theorems of Barbashin—Krasovskiĭ type cannot be applied to establish asymptotic stability with respect to $(y, \dot{x}, \dot{y})$. Peiffer and Rouche proved that the stability is asymptotic with respect to $\dot{x}$. Applying Corollary 4.3 with $\hat{q}=y$ we obtain that the equilibrium is asymptotically y-stable even under arbitrary nonlinear friction with total dissipation (the special form (4.5) of the dissipative forces is not needed).

Note that by the use of the Ljapunov—Malkin theorem on the critical case of the stability investigations by first approximations (see [16], p. 113) one can prove that the equilibrium is stable with respect to all variables, the stability is asymptotic with respect to $(y, \dot{x}, \dot{y})$, and for every motion starting from some neighbourhood of the equilibrium $x(t)\to x_0=\text{const.}$ as $t\to\infty$.

Our theorems allow us to investigate the general case when the point is constrained to move on a surface of equation $z=f(x, y)$ $(f(0, 0)=0)$. Corollaries 4.3 and 4.5 yield conditions on the potential energy $P=gf(x, y)$ assuring the equilibrium $x=y=0$ to be asymptotically y-stable or y-unstable. We illustrate this by two simple examples.

Let

$$f(x, y) = \begin{cases} (1/2)y^2(1+x^2)+e^{-1/|x|}\sin^2(1/x^2) & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

By Corollary 4.3, the equilibrium $x=y=\dot{x}=\dot{y}=0$ is asymptotically y-stable in spite of the fact that the region $\{(x, y)\colon P(x, y)>0\}$ contains equilibria (see condition (ii) in Theorem A in Section 2).

As Corollary 4.5 shows, in the case of $f(x, y)=y^3/(1+x^2)$ the equilibrium $x=y=\dot{x}=\dot{y}=0$ is y-unstable.

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The behavior of the Riesz representation theorem with respect to order and topology

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Dedicated with admiration to Béla Sz.-Nagy on the occasion of his 70th birthday

The celebrated representation theorem of Frederick Riesz, the revered teacher of both Béla Sz.-Nagy and myself, can be stated in the following form: every Radon measure on an open interval of the real line is the derivative in the sense of distributions of a function which is locally of bounded variation. In the present note I want to give some precisions about how this correspondence between functions locally of bounded variation and Radon measures behaves with respect to the topologies and the order structures on the two spaces involved.

1. Let I be a non-empty open interval $]a, b[$ of the real line $\mathbf{R}$, which may be finite or infinite, i.e., $-\infty \leqq a < b \leqq \infty$. We shall only consider real-valued functions and real measures in this note. We denote by $\mathscr{V}(I)$ the vector space of all functions $f\colon I \to \mathbf{R}$ whose total variation

$$q_{\alpha\beta}(f) = \sup_{\Delta} \sum_{j=1}^{l} |f(x_j) - f(x_{j-1})|$$

is finite on every compact subinterval $[\alpha, \beta]$ of I; here the least upper bound is taken with respect to all subdivisions

(1) $$\Delta\colon\ \alpha = x_0 < x_1 < \ldots < x_l = \beta$$

of $[\alpha, \beta]$. Each $q_{\alpha\beta}$ is a semi-norm. We consider $\mathscr{V}(I)$ equipped with the non-Hausdorff locally convex topology defined by the family $(q_{\alpha\beta})$ of semi-norms, where $[\alpha, \beta]$ varies in I.

A preorder compatible with the vector space structure of $\mathscr{V}(I)$ is defined if we take for the cone P of positive elements the set of all increasing functions. Then $P \cap (-P)$ consists of the constants and is, incidentally, also the closure of $\{0\}$

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in the topology defined above. Every $f \in \mathscr{V}(I)$ is the difference of two increasing functions, i.e., $\mathscr{V}(I)=P-P$, and therefore $f(x-0)$ and $f(x+0)$ exist at every $x \in I$. Any two elements $f, g \in \mathscr{V}(I)$ have a least upper bound $\sup(f, g)$ which is determined up to an additive constant. In particular the positive variation $\Phi = = \operatorname{var}^+ f = \sup(f, 0)$ of $f \in \mathscr{V}(I)$ is defined up to an additive constant; it is an increasing function such that $\Phi - f$ is increasing and such that, whenever g is an increasing function for which $g-f$ is increasing, then $g-\Phi$ is increasing; it is given explicitly by the formula

$$\Phi(\beta)-\Phi(\alpha) = \sup_{\Delta} \sum_{j=1}^{l} \max\left(f(x_i)-f(x_{i-1}), 0\right),$$

where $a<\alpha<\beta<b$, and the least upper bound is taken with respect to all subdivisions (1). Similarly, the negative variation $\Psi = \operatorname{var}^- f = \sup(-f, 0)$ is defined, and $F = \Phi + \Psi = \operatorname{var} f = \sup(f, -f)$ is the absolute variation of f.

Since each $f \in \mathscr{V}(I)$ is locally integrable, we can associate with it the Radon measure T_f which has density f with respect to Lebesgue measure, i.e., which is given by

$$\langle T_f, \varphi\rangle = \int_I \varphi(x) f(x)\, dx$$

for all functions φ belonging to the space $\mathscr{K}(I)$ of continuous functions with compact support in I.

With $f \in \mathscr{V}(I)$ we can associate a second Radon measure, the Stieltjes measure S_f, defined by the Stieltjes integral

$$\langle S_f, \varphi\rangle = \int_I \varphi(x)\, df(x)$$

for all $\varphi \in \mathscr{K}(I)$. The inequality

$$|\langle S_f, \varphi\rangle| \leqq q_{\alpha\beta}(f) \cdot \max |\varphi(x)|, \tag{2}$$

valid for all $\varphi \in \mathscr{K}(I)$ with $\operatorname{Supp} \varphi \subset [\alpha, \beta]$, shows that S_f is indeed a Radon measure. Clearly, $f \mapsto S_f$ is a linear map from $\mathscr{V}(I)$ into the space $\mathscr{M}(I) = \mathscr{K}'(I)$ of all Radon measures on I. For $a<\alpha\leqq\beta<b$ we have

$$\begin{aligned} &S_f(]\alpha, \beta[) = f(\beta-0)-f(\alpha+0), \quad S_f([\alpha, \beta]) = f(\beta+0)-f(\alpha-0), \\ &S_f(]\alpha, \beta]) = f(\beta+0)-f(\alpha+0), \quad S_f([\alpha, \beta[) = f(\beta-0)-f(\alpha-0). \end{aligned} \tag{3}$$

The integration by parts formula

$$\int_I \varphi(x)\, df(x) = -\int_I \varphi'(x) f(x)\, dx$$

holds in particular for all φ belonging to the space $\mathscr{D}(I)$ of all infinitely differentiable functions with compact support in I. Rewriting it in the form

$$\langle S_f, \varphi \rangle = -\langle T_f, \varphi' \rangle = \langle \partial T_f, \varphi \rangle,$$

we see that S_f is the derivative ∂T_f in the sense of distributions of T_f.

The representation theorem of Frederick Riesz now states that the map $f \mapsto S_f$ is surjective from $\mathscr{V}(I)$ onto $\mathscr{M}(I)$ [2]. It follows from the formulas (3) that $S_f = 0$ if and only if there exists a constant $C \in \mathbf{R}$ such that $f(x-0) = f(x+0) = C$ at each $x \in I$. It follows furthermore from (3) that if μ is a positive measure, one can find an increasing function $f \in \mathscr{V}(I)$ for which $S_f = \mu$. It is obvious that, conversely, if f is increasing, then S_f is positive.

2. If $\mu = S_f$, we do not have necessarily $|\mu| = S_{\operatorname{var} f}$. Indeed, take $I = \mathbf{R}$ and let $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. Then $\operatorname{var}^+ f$ is the Heaviside function Y, $\operatorname{var}^- f$ is the function taking the value 0 for $x \leqq 0$ and the value 1 for $x > 0$, and

$$\operatorname{var} f = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x = 0, \\ 2 & \text{if} \quad x > 0. \end{cases}$$

Thus $S_{\operatorname{var} f} = 2\delta$ but $S_f = |S_f| = 0$. The situation improves if we suppose that $f(x)$ is between $f(x-0)$ and $f(x+0)$:

Theorem 1. *If $f \in \mathscr{V}(I)$, then*

$$S_f^+ \leqq S_{\operatorname{var}^+ f}, \; S_f^- \leqq S_{\operatorname{var}^- f}, \; |S_f| \leqq S_{\operatorname{var} f}.$$

If we assume furthermore that $f(x)$ is between $f(x-0)$ and $f(x+0)$ for every $x \in I$, then the sign of equality is valid in the three inequalities.

Proof. Set $\Phi = \operatorname{var}^+ f$, $\Psi = \operatorname{var}^- f$ and $F = \operatorname{var} f$. Both Φ and $\Phi - f$ are increasing functions, hence $S_\Phi \geqq 0$ and $S_\Phi \geqq S_f$, i.e., $S_\Phi \geqq S_f^+$. One sees similarly that $S_\Psi \geqq S_f^-$. Thus $S_F = S_\Phi + S_\Psi \geqq S_f^+ + S_f^- = |S_f|$.

Assume next that f is continuous and let μ be a positive measure on I such that $\mu \geqq S_f$. There exists an increasing function g such that $S_g = \mu$. It follows from the above remarks that we can find an increasing function h such that $S_h = \mu - S_f = S_{g-f}$ and which satisfies

$$h(x-0) = g(x-0) - f(x), \quad h(x+0) = g(x+0) - f(x) \tag{4}$$

at every $x \in I$. Since g is increasing, we have

$$g(x-0) - f(x) \leqq g(x) - f(x) \leqq g(x+0) - f(x) \tag{5}$$

at each $x \in I$. Formulas (4) and (5) imply that $g-f$ is an increasing function. It follows from the definition of $\operatorname{var}^+ f$ that $g-\Phi$ is increasing. Hence $\mu = S_g \geqq S_\Phi$, and so S_Φ is indeed the measure $S_f^+ = \sup(S_f, 0)$. One can prove in exactly the same way that $S_\Psi = S_f^- = \sup(-S_f, 0)$. Thus we have also $S_F = S_\Phi + S_\Psi = S_f^+ + S_f^- = |S_f|$.

Consider now a pure jump function $f \in \mathscr{V}(I)$. Such a function is determined by two families $(l_x)_{x\in I}$, $(r_x)_{x\in I}$ of real numbers such that for every compact subset K of I we have $\sum_{x\in K} |l_x| < \infty$ and $\sum_{x\in K} |r_x| < \infty$. Writing $j_x = l_x + r_x$ and taking $a<\alpha<\beta<b$, the function f is given up to an additive constant by the formula

$$f(\beta)-f(\alpha) = r_\alpha + \sum_{\alpha<x<\beta} j_x + l_\beta .$$

One has $f(x)-f(x-0)=l_x$, $f(x+0)-f(x)=r_x$, i.e., f is the pure jump function with left jump l_x and right jump r_x at x. If we define $l_x^+ = \max(l_x, 0)$, $l_x^- = \max(-l_x, 0)$, and similarly for r_x^+, r_x^-, then $\Phi = \operatorname{var}^+ f$ is the pure jump function with jumps l_x^+, r_x^+, $\Psi = \operatorname{var}^- f$ is the pure jump function with jumps l_x^-, r_x^-, and $F = \operatorname{var} f$ is the pure jump function with jumps $|l_x|$, $|r_x|$. The corresponding Stieltjes measures are $S_f = \sum_{x\in I} j_x \delta_x$, $S_\Phi = \sum_{x\in I} (l_x^+ + r_x^+)\delta_x$, $S_\Psi = \sum_{x\in I} (l_x^- + r_x^-)\delta_x$ and $S_F = \sum_{x\in I} (|l_x| + |r_x|)\delta_x$. On the other hand, $|S_f| = \sum_{x\in I} |j_x| \delta_x$. If we assume that $f(x)$ is between $f(x-0)$ and $f(x+0)$, then $|j_x| = |l_x| + |r_x|$ and so $S_F = |S_f|$.

Every function $f \in \mathscr{V}(I)$ can be decomposed into a sum $f = f_C + f_J$ of a continuous function $f_C \in \mathscr{V}(I)$ and a pure jump function $f_J \in \mathscr{V}(I)$; and this decomposition is unique up to an additive constant. If f satisfies the condition that $f(x)$ is between $f(x+0)$ and $f(x-0)$, then so does f_J since f has the same jumps as f_J. The measure S_{f_J} is concentrated on the countable set of points where the jumps of f are non-zero; i.e., S_{f_J} is atomic. By virtue of (3) the measure S_{f_C} of every countable set is zero, i.e., S_{f_C} is diffuse. It follows from what has been said above and from [1], Chap. V, § 5, n° 10, Proposition 15 that

$$|S_f| = |S_{f_C}| + |S_{f_J}| = S_{\operatorname{var} f_C} + S_{\operatorname{var} f_J}. \tag{6}$$

Since the functions $\operatorname{var} f_C - f_C$ and $\operatorname{var} f_J - f_J$ are increasing, so is $\operatorname{var} f_C + \operatorname{var} f_J - f$ and therefore also $\operatorname{var} f_C + \operatorname{var} f_J - \operatorname{var} f$. Thus

$$S_{\operatorname{var} f_C} + S_{\operatorname{var} f_J} \geqq S_{\operatorname{var} f} = S_F. \tag{7}$$

Combining (6) and (7) we obtain $|S_f| \geqq S_F$. The opposite inequality has already been proved as the first assertion of the theorem, hence $|S_f| = S_F$.

Finally we have $S_\Phi = (1/2)(S_F + S_f) = (1/2)(|S_f| + S_f) = S_f^+$ and $S_\Psi = (1/2)(S_F - S_f) = (1/2)(|S_f| - S_f) = S_f^-$.

3. To conclude, I want to show that the map $f \mapsto S_f$ behaves in the best possible way with respect to the topologies involved. I announced this result earlier ([2], Theorem 2) with a somewhat terse proof. The "good" proof is based on the following simple, general observation:

Theorem 2. *Let X be a locally compact, paracompact topological space. The semi-norms $\mu \mapsto |\mu|(K)$, where K runs through the compact subsets of X, define on the space $\mathcal{M}(X)$ of Radon measures the strong topology $\beta(\mathcal{M}(X), \mathcal{K}(X))$.*

Proof. Let us denote by $\mathcal{T}$ the topology defined by the semi-norms $\mu \mapsto |\mu|(K)$.

(a) Let V be a neighborhood of 0 for the topology $\mathcal{T}$. We may assume that V is of the form $\{\mu \in \mathcal{M}(X) : |\mu|(K) \leqq \varepsilon\}$, where $\varepsilon > 0$ and K is a compact subset of X. Let L be a compact neighborhood of K. The set

$$B = \{\varphi \in \mathcal{K}(X) : \operatorname{Supp} \varphi \subset L, |\varphi(x)| \leqq 1/\varepsilon\}$$

is bounded in $\mathcal{K}(X)$. Consider an arbitrary μ in the polar B° of B. If φ is a positive function in B and $\psi \in \mathcal{K}(X)$ is such that $|\psi| \leqq \varphi$, then ψ also belongs to B, and it follows from [1], Chap. III, § 1, n° 5, formula (9) that

$$\langle |\mu|, \varphi \rangle = \sup_{\substack{|\psi| \leqq \varphi \\ \psi \in \mathcal{K}(X)}} \langle \mu, \psi \rangle \leqq 1.$$

By [1], Chap. III, § 1, n° 2, Lemme 1 there exists $\varphi \in B$ such that $\chi_K \leqq \varepsilon \varphi$. Thus

$$|\mu|(K) = |\mu|(\chi_K) \leqq \varepsilon \langle |\mu|, \varphi \rangle \leqq \varepsilon$$

and therefore $\mu \in V$. We have proved that V contains the strong neighborhood B° of 0.

(b) Conversely, let W be a strong neighborhood of 0. We may assume that $W = B^\circ$ where B is a bounded subset of $\mathcal{K}(X)$. Since X is paracompact, by [1], Chap. III, § 1, n° 1, Proposition 2 (ii) there exists a compact set $K \subset X$ and a number $\gamma > 0$ such that $\operatorname{Supp} \varphi \subset K$ and $|\varphi(x)| \leqq \gamma$ for $\varphi \in B, x \in X$. The set

$$V = \{\mu \in \mathcal{M}(X) : |\mu|(K)| \leqq 1/\gamma\}$$

is a $\mathcal{T}$-neighborhood of 0. If $\mu \in V$ and $\varphi \in B$, then $|\varphi| \leqq \gamma \chi_K$ and therefore

$$|\langle \mu, \varphi \rangle| \leqq |\langle |\mu|, |\varphi| \rangle| \leqq \gamma \cdot |\mu|(\chi_K) = \gamma \cdot |\mu|(K) \leqq 1.$$

Thus $\mu \in B^\circ$ and we have proved that $V \subset W$.

If (K_ν) is a family of compact subsets of X such that each compact subset of X is contained in some K_ν, then the strong topology on $\mathcal{M}(X)$ is also defined by the family of semi-norms $\mu \mapsto |\mu|(K_\nu)$.

If X is locally compact but not paracompact, then the semi-norms $\mu \mapsto |\mu|(K)$ define the quasi-strong topology ([1], Chap. III, § 1, Exerc. 8) on $\mathcal{M}(X)$.

4. We are now ready to prove the result referred to above.

Theorem 3. *The surjective linear map $f \mapsto S_f$ from $\mathcal{V}(I)$ onto $\mathcal{M}(I)$ is a strict morphism if we equip $\mathcal{M}(I)$ with the strong topology.*

Proof. (a) We first prove that the map is continuous. Let B be a bounded subset of $\mathcal{K}(I)$. There exists a compact subinterval $[\alpha, \beta]$ of I and a number $\gamma > 0$ such that $\operatorname{Supp} \varphi \subset [\alpha, \beta]$ and $|\varphi(x)| \leqq \gamma$ for $\varphi \in B$, $x \in I$. Define a neighborhood of 0 in $\mathcal{V}(I)$ by

$$V = \{f \in \mathcal{V}(I) \colon q_{\alpha\beta}(f) \leqq 1/\gamma\}.$$

If $f \in V$ and $\varphi \in B$, then by inequality (2) we have $|\langle S_f, \varphi\rangle| \leqq 1$, i.e., S_f belongs to the neighborhood B° of 0 in $\mathcal{M}(I)$.

(b) Now we prove that the map is open. Let V be a neighborhood of 0 in $\mathcal{V}(I)$ which we may assume to be of the form

$$V = \{f \in \mathcal{V}(I) \colon q_{\alpha\beta}(f) \leqq \varepsilon\},$$

where $K = [\alpha, \beta] \subset I$ and $\varepsilon > 0$. Let W be the neighborhood $\{\mu \in \mathcal{M}(I) \colon |\mu|(K) \leqq \varepsilon\}$ of 0 in $\mathcal{M}(I)$. Given $\mu \in W$, we want to find an f in V such that $\mu = S_f$; then we will have proved that the image of V contains W.

The existence of such an f is implicit in the proof of Theorem 1 of [2] but we can also proceed as follows. Let $\mu = \mu^+ - \mu^-$, then $|\mu| = \mu^+ + \mu^-$. There exist increasing functions g and h such that $\mu^+ = S_g$, $\mu^- = S_h$. Then by virtue of the formulas (3)

$$q_{\alpha\beta}(g) \leqq g(\beta+0) - g(\alpha-0) = \mu^+(K)$$

and similarly $q_{\alpha\beta}(h) \leqq \mu^-(K)$. Setting $f = g - h$ we have $S_f = S_g - S_h = \mu^+ - \mu^- = \mu$ and

$$q_{\alpha\beta}(f) \leqq q_{\alpha\beta}(g) + q_{\alpha\beta}(h) \leqq \mu^+(K) + \mu^-(K) = |\mu|(K) \leqq \varepsilon,$$

i.e., $f \in V$.

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On the representation of distributive algebraic lattices. I

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

E. T. SCHMIDT [4] proved that every distributive lattice is isomorphic with the lattice of all compact congruences of a lattice. The analogous question for distributive semilattices is a long-standing conjecture of lattice theory. In this paper we prove a theorem which can be considered as a further evidence to this conjecture. Our result is based on a theorem of P. Pudlák. Motivated by Schmidt's result, PUDLÁK [3] discovered another method suitable to attack the problem. He first proved that every distributive semilattice is the direct limit of its finite distributive subsemilattices. This reduces the conjecture to the following

Problem. Consider the category of finite distributive lattices where the morphisms are the one-to-one 0-preserving $\vee$-homomorphisms. Is there any functor R of this category to the category of finite lattices (with lattice embeddings) such that the following hold?

(α) For any distributive lattice D, there is an isomorphism $\varphi_D\colon D\cong\mathrm{Con}(R(D))$.

(β) Whenever D_1 has a one-to-one 0-preserving $\vee$-homomorphism δ to D_2, then $R(D_1)$ has a lattice embedding $R(\delta)$ to $R(D_2)$, such that

(γ) $R(\delta_{12}\delta_{23})=R(\delta_{12})R(\delta_{23})$ for all $\delta_{12}\colon D_1\to D_2$ and $\delta_{23}\colon D_2\to D_3$ satisfying the stipulations in (β), and,

(δ) if we denote by $\mathrm{Con}\,(R(\delta))$ the mapping of $\mathrm{Con}\,(R(D_1))$ to $\mathrm{Con}\,(R(D_2))$ induced by $R(\delta)$ (that is, the one, which maps $\Theta_1\in\mathrm{Con}\,(R(D_1))$ to the congruence generated by $\{(aR(\delta), bR(\delta))\in R(D_2)^2|(a,b)\in\theta_1\}$), then the following diagram is commutative

$$\begin{array}{ccc} D_1 & \xrightarrow{\quad\delta\quad} & D_2 \\ {\scriptstyle\varphi_{D_1}}\downarrow & & \downarrow{\scriptstyle\varphi_{D_2}} \\ \mathrm{Con}\,(R(D_1)) & \xrightarrow{\mathrm{Con}\,(R(\delta))} & \mathrm{Con}\,(R(D_2)) \end{array}.$$

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In case of an affirmative answer the conjecture would follow. Indeed, for any distributive semilattice D, we can choose a directed set $\{D_\gamma\}_{\gamma\in\Gamma}$ of finite distributive subsemilattices approaching to D. By (γ) and (δ), the $\mathrm{Con}\,(R(D_\gamma))$'s form the same directed set (up to commuting isomorphisms). Therefore, the direct limit of this set is D, too. On the other hand, the $R(D_\gamma)$'s, too, form a directed set, and the semilattice of all finitely generated congruences of their direct limit is the direct limit of the $\mathrm{Con}\,(R(D_\gamma))$'s.

Pudlák carried out a modification of this program, namely, he proved the analogous statement for distributive lattices and 0-preserving lattice embeddings in the place of distributive semilattices and 0-preserving $\vee$-embeddings, and obtained a new proof of Schmidt's theorem. We are interested in the question how much of Pudlák's theorem can be proved without imposing the restriction that the embeddings be lattice embeddings. It will be shown that two finite distributive semilattices with 0 have a simultaneous representation (that is, a representation satisfying (α), (β) and (δ)), provided one of them is a 0-subsemilattice of the other. In Part II of this paper we shall derive Pudlák's theorem from this result as well as Bauer's result on the representability of countable semilattices.

The main result of this part is the following

Theorem. *Let D_1 and D_2 be finite distributive lattices, and let $\delta: d\mapsto d^+$ be a one-to-one 0-preserving $\vee$-homomorphism of D_1 into D_2. Then there exist lattices L_1 and L_2 such that*

(α_1) $D_i\cong\mathrm{Con}\,(L_i)$, $i=1,2$ *(these isomorphisms will be denoted by φ_i),*

(β_1) *L_1 can be embedded to L_2 (by a one-to-one lattice isomorphism, to be denoted by λ),*

(δ_0) *every congruence of $L_1\lambda$ can be extended to L_2; and, therefore the mapping γ: $\mathrm{Con}\,(L_1)\to\mathrm{Con}\,(L_2)$, taking each $\Theta\in\mathrm{Con}\,(L_1)$ to its smallest extension, that is, to the congruence generated by $\{(a\lambda, b\lambda)|(a,b)\in\Theta\}$, is also a one-to-one 0-preserving $\vee$-homomorphism, furthermore*

(δ_1) *for all $d_i\in D_i$, $i=1,2$, δ maps d_1 to d_2 if and only if γ maps $d_1\varphi_1$ to $d_2\varphi_2$. In other words, γ represents δ.*

1. Proof of (α_1). We define L_1 (see E. T. Schmidt [5], pp. 82—87) as follows. Let B_1 be the Boolean lattice generated by D_1. Let M_1 consist of all triples $(x,y,z)\in B_1^3$ satisfying $x\wedge y=x\wedge z=y\wedge z$. Let L_1 be the set of all triples in M_1 also satisfying $x\in D_1$. Then L_1 is a lattice, too, under the ordering of B_1^3. It is proven in E. T. Schmidt [5] that $D_1\cong\mathrm{Con}\,(L_1)$. For further purposes we shall recall the proof here. We need a description of the operations of L_1. The meet operation is the same as in B_1^3. However, the joins in B_1^3, M_1 and L_1 are different. They will be denoted by $\vee, \vee_M, \vee_L$, respectively (or by $\vee, \vee_{M_1}, \vee_{L_1}$, where necessary). To describe them we introduce the following operators. $(x,y,z)\mapsto$

$\mapsto(x, y, z)^{\sim}$ acts on B_1^3 and maps (x, y, z) to the smallest element of M_1 above (x, y, z). $x\mapsto\bar{x}$ acts on B_1 and maps x to the smallest element of D_1 above x. Finally, $(x, y, z)\mapsto(x, y, z)^{\wedge}$ acts on M_1 and maps (x, y, z) to the smallest element of L_1 above (x, y, z). Now we have (see [5]),

$$(x, y, z)\vee_M(x', y', z') = (x\vee x', y\vee y', z\vee z')^{\sim},$$

$$(x, y, z)\vee_L(x', y'; z') = (x\vee x', y\vee y', z\vee z')^{\sim\wedge},$$

$$(x, y, z)^{\sim} = \big(x\vee(y\wedge z), y\vee(x\wedge z), z\vee(x\wedge y)\big) \quad \text{for} \quad (x, y, z)\in B_1^3,$$

$$(x, y, z)^{\wedge} = \big(\bar{x}, y\vee(\bar{x}\wedge z), z\vee(\bar{x}\wedge y)\big) \quad \text{for} \quad (x, y, z)\in M_1.$$

Now consider any congruence α of L_1. We shall prove that α is generated by a pair $((0, 0, 0), (x, 0, 0))\in L_1^2$. (Then $x\in D_1$, and hence $D_1\cong\mathrm{Con}\,(L_1)$.) To prove this claim, let $(x, y, z)\,\alpha\,(x', y', z')$. Then, forming the meets with $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively, we obtain

$$(x, 0, 0)\,\alpha\,(x', 0, 0), \quad (0, y, 0)\,\alpha\,(0, y', 0), \quad (0, 0, z)\,\alpha\,(0, 0, z').$$

Hence $(x, 0, 0)\vee_L(0, 1, 0)=(x, 1, 0)^{\sim\wedge}=(x, 1, x)^{\wedge}=(x, 1, x)$, and $(x', 0, 0)\vee_L(0, 1, 0)= =(x', 1, x')$, thus $(x, 1, x)\,\alpha\,(x', 1, x')$. Forming the meet of both sides with $(0, 0, 1)$, we get $(0, 0, x)\,\alpha\,(0, 0, x')$. Similarly, $(0, 0, y)\,\alpha\,(0, 0, y')$. Thus the congruence generated by $((x, y, z), (x', y', z'))$ contains the pairs

$$((0, 0, x), (0, 0, x')), \quad ((0, 0, y), (0, 0, y')), \quad ((0, 0, z), (0, 0, z')).$$

It is also generated by them. Indeed, under the congruence generated by these three pairs the following pairs are also related:

$$((x, 0, 0), (x', 0, 0)), \quad ((0, y, 0), (0, y', 0)), \quad ((0, 0, z), (0, 0, z')).$$

(We have to compute as above.) Hence, computing modulo α,

$$(x, y, z) = ((x, 0, 0)\vee(0, y, 0)\vee(0, 0, z))^{\sim\wedge} =$$

$$= (x, 0, 0)\vee_L(0, y, 0)\vee_L(0, 0, z) \equiv (x', 0, 0)\vee_L(0, y', 0)\vee_L(0, 0, z') = (x', y', z').$$

The elements of the form $(0, 0, t)$ constitute a Boolean sublattice, thus the congruence generated by $((x, y, z), (x', y', z'))$ is generated by an ideal of $\{(0, 0, t)\mid t\in B_1\}$. Hence α is also generated by a pair $((0, 0, 0), (0, 0, t_\alpha))$ or, equivalently, by

$$((0, 0, 0), (0, 0, t_\alpha))\vee_L((0, 1, 0), (0, 1, 0)) = ((0, 1, 0), (\bar{t}_\alpha, 1, \bar{t}_\alpha)),$$

or by

$$((0, 1, 0), (\bar{t}_\alpha, 1, \bar{t}_\alpha))\wedge((1, 0, 0), (1, 0, 0)) = ((0, 0, 0), (\bar{t}_\alpha, 0, 0)),$$

as claimed. (For more details see [5].) Now consider the lattice of Figure 1. Let this lattice be denoted by L_2. We show that $D_2\cong\mathrm{Con}\,(L_2)$. First, however, let us give a more accurate description of this lattice. For a finite distributive lattice D let $M(D)$ (respectively, $L(D)$) denote the lattice formed from D analogously as M_1 (respectively, L_1) is formed from D_1. Furthermore, whenever D is a distributive lattice, let $B(D)$ denote the Boolean extension of D.

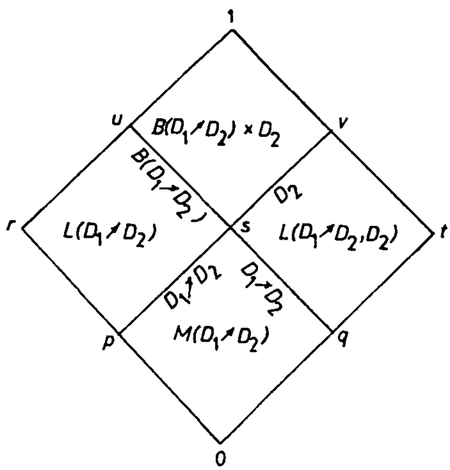

Figure 1

Finally, whenever D, D' are distributive lattices, $D \subseteq D'$, and the 0 and 1 of D are the same as those of D', then let $M(D', D)$ consist of all triples $(x, y, z) \in (D')^3$ satisfying $x \vee (y \wedge z) = y \vee (x \wedge z) = z \vee (x \wedge y)$ and let $L(D', D) = \{(x, y, z) | x \in D, (x, y, z) \in M(D', D)\}$. Now the meaning of $L(D_1 \nearrow D_2)$, $M(D_1 \nearrow D_2)$, $L(D_1 \nearrow D_2, D_2)$ and $B(D_1 \nearrow D_2) \times D_2$ of Figure 1 is clear. For the definition of $D_1 \nearrow D_2$, see [3].

As to how they are glued together note that $L(D_1 \nearrow D_2)$ contains an ideal isomorphic with $D_1 \nearrow D_2$ (the set of elements $(x, 0, 0)$, $x \in D_1 \nearrow D_2$) and $M(D_1 \nearrow D_2)$ contains such a dual ideal. The mapping which is identical on the D_i's maps the ideal of $L(D_1 \nearrow D_2)$ in question isomorphically to this dual ideal of $M(D_1 \nearrow D_2)$. Further isomorphism maps an ideal of $L(D_1 \nearrow D_2, D_2)$ to another dual ideal of $M(D_1 \nearrow D_2)$. If we identify the elements corresponding to each other under these isomorphisms we get a partial lattice (the union of $[0, u]$ and $[0, v]$ on Figure 1). It can be made into a lattice by inserting a $B(D_1 \nearrow D_2) \times D_2$ to the top of the Figure, and making analogous identifications. ($B(D_1 \nearrow D_2) \times D_2$ has an ideal isomorphic with $B(D_1 \nearrow D_2)$. This will be identified with the dual ideal of $L(D_1 \nearrow D_2)$ consisting of all those elements which are greater than or equal to all elements used in the identification between $L(D_1 \nearrow D_2)$ and $M(D_1 \nearrow D_2)$. $B(D_1 \nearrow D_2) \times D_2$ also has an ideal isomorphic with D_2, to be used for the identification with the corresponding dual ideal of $L(D_1 \nearrow D_2, D_2)$.) We show that $D_2 \cong \mathrm{Con}\,(L_2)$. Consider any two elements of L_2. The congruence generated by them is obviously a join of four congruences α_i, $i=1, 2, 3, 4$ where α_1 (respectively, $\alpha_2, \alpha_3, \alpha_4$) is generated by a pair of elements in $L(D_1 \nearrow D_2)$ (respectively, $M(D_1 \nearrow D_2)$, $L(D_1 \nearrow D_2, D_2)$, $B(D_1 \nearrow D_2) \times D_2$). If we prove that all of these congruences are generated by subintervals of $[q, t]$ containing q, then we are done. Now the same calculations that proved that $\mathrm{Con}\,(L_1) \cong D_1$ show that α_1 is generated by a subinterval of $[p, s]$ containing p; the same computations in $M(D_1 \nearrow D_2)$ and in $L(D_1 \nearrow D_2, D_2)$ yield that α_1 is generated by a subinterval of $[q, s]$ containing q as well as by a subinterval of $[q, t]$ containing q. α_2 can also be generated by elements of $L(D_1 \nearrow D_2)$ which reduces the case of α_2 to that of α_1. The case of α_3 can be reduced to that of α_2, and, finally, the case of α_4 follows from the cases of α_1 and α_3.

2. Proof of (β_1). Preparing this proof it turned out that Theorem 1 of [3], which was intended to be used in the proof of (β_1), is still not general enough. We have to prove a stronger result (Lemma 1). The proof of this result goes along the lines of [3], Theorem 1; for completeness' sake, however, we repeat part of the details.

Let $B(D_1 \nearrow D_2)$ be the Boolean lattice generated by $D_1 \nearrow D_2$. Let B_i be the Boolean lattice generated by D_i, $i=1,2$. Denote by B_1^b the Boolean lattice generated by D_1^b, where D_1^b denotes the lattice $D_1 \cup \{1\}$ with $x<1$ for all $x \in D_1$. Now we know from [3] that $D_1 \nearrow D_2$ is the lattice obtained from $D_1^b * D_2$ (the 0—1-free product) by factorizing by the congruence generated by all pairs $(d \vee d^+, d^+)$ $d \in D_1$. Now factorizing $B_1^b * B_2$ by this congruence we get a Boolean lattice generated by $D_1 \nearrow D_2$. This Boolean lattice will be denoted by $B_1^b \nearrow B_2$. Clearly $B_1^b \nearrow B_2 = B(D_1 \nearrow D_2)$. It also contains B_1, the smallest Boolean lattice generated by D_1. This follows from [3], Theorem 1, for $D_1^b \nearrow D_2$ contains D_1. (Of course B_1, like D_1, does not contain the upper bound of $B_1^b \nearrow B_2$.)

In Section 1 we defined the operator $x \mapsto \bar{x}$ mapping the Boolean algebra $B(D)$ generated by the distributive lattice D to D by associating the least upper bound $\bar{x}$ in D with the element $x \in B$. Now B_1 is embedded to $B_1^b \nearrow B_2$. Therefore, for elements of B_1, there are two possibilities to define $x \mapsto \bar{x}$, namely within B_1 as the least upper bound of an element in D_1, and within $B_1^b \nearrow B_2$ as the least upper bound of an element in $D_1 \nearrow D_2$. We are going to show (and this is the crucial point of the proof) that these two definitions coincide.

This statement includes the main theorem of [3]. Indeed, from [3], Theorem 1 it follows that the smallest Boolean lattice generated by D_1 in $B_1^b \nearrow B_2$ intersects $D_1 \nearrow D_2$ in D_1. (This is not evident, we have to use GRÄTZER [1], Corollary 10.9., or more exactly a slight generalization of this Corollary as the units of D_1 and D_2 do not coincide, however, it can be proved.) The converse is also true: [3], Theorem 1 follows from the fact that the intersection of $B(D_1)$ and $D_1 \nearrow D_2$ in $B_1^b \nearrow B_2$ is D_1.

(This is evident.) Now consider any element x of $B(D_1)$ in $D_1 \nearrow D_2$. Then $\bar{x}$ formed in $D_1 \nearrow D_2$ is x. Now applying $B(D_1) \cap (D_1 \nearrow D_2) = D_2$, that is, using [3], Theorem 1 we have that $\bar{x}$ formed in D_2 is x, too. This shows that the statement whose proof we promised is, indeed stronger than [3], Theorem 1. Now let $\bar{x}$ be the least upper bound of $x \in B_1$ in D_1 and let $\bar{\bar{x}}$ be the least upper bound of x in $D_1 \nearrow D_2$. Obviously $\bar{\bar{x}} \leqq \bar{x}$.

Lemma 1. *For all* $x \in B_1$, $\bar{x} \leqq \bar{\bar{x}}$.

Before proving Lemma 1, we have to solve the word problem of $B_1 \nearrow D_2$, where $B_1 \nearrow D_2$ denotes the lattice generated by $B_1 \cup D_2$ in $B_1^b \nearrow B_2$. A solution will be given in the following lemma.

Let Θ denote the congruence generated by the pairs $(d^+, d \vee d^+)$, $d \in D_1$, in $B_1 \nearrow D_2$. Let Q_1 denote the set of atoms of B_1. Let $\mathscr{I}(k)$ be the subset $\{j | k \not\geqq j^+\}$ of Q_1, if k is an irreducible of D_2. (There is a homomorphism of Q_1 to P_1 corresponding to the embedding $D_1 \to B_1$. For any k, $\mathscr{I}(k)$ goes to an ideal of P_1 under this homomorphism; P_i denotes the set of join-irreducibles of D_i, $i = 1, 2$; j^+ denotes $\bar{j}^+$.)

Lemma 2. *For arbitrary elements* $f, g \in B_1 \nearrow D_2$, $f \equiv g \pmod{\Theta}$ *iff, for all* k, $f(k) \equiv g(k) \pmod{\Theta(\mathscr{I}(k))}$ *where* $\Theta(\mathscr{I}(k))$ *is the congruence generated by the ideal* $\mathscr{I}(k)$.

The proof is analogous with that of [3], Theorem 2, and it will be omitted.

Now we go on to prove Lemma 1. We have to show $\bar{x} \leqq \bar{\bar{x}}$. As in [3], elements of $B_1 \nearrow D_2$ will be represented by antitone functions from P_2 to B_1. It is enough to show that for all $b \in B_1$, $f_b \leqq f$ implies $f_{\bar{b}} \leqq f$ in $B_1 \nearrow D_2$ where f_b (respectively, $f_{\bar{b}}$) is the function identically b (respectively, $\bar{b}$) and $f \in D_1 \nearrow D_2$. It suffices to show this statement for b irreducible, as the operation $b \mapsto \bar{b}$ preserves joins.

Now let j be irreducible and assume that $f_j \leqq f \pmod{\theta}$. Then, for all k, $j \leqq f(k) \pmod{\theta(\mathscr{I}(k))}$. Hence, we have either $j \leqq f(k)$ (and then also $\bar{j} \leqq f(k)$ as $f(k)$ is in D_1) or $j \equiv j \wedge f(k) \pmod{\theta(\mathscr{I}(k))}$, $j \not\leqq f(k)$. In the latter case $j \wedge f(k) = 0$, thus $j \equiv 0 \pmod{\theta(\mathscr{I}(k))}$, that is, $k \not\geqq j^+$, whence $k \not\geqq \bar{j}^+$, that is, $\bar{j} \equiv 0 \pmod{\theta(\mathscr{I}(k))}$. In either case $\bar{j} \leqq f(k) \pmod{\theta(\mathscr{I}(k))}$, whence $f_{\bar{j}} \leqq f \pmod{\Theta}$ completing the proof of Lemma 1.

Now we return to the proof of (β_1) and show that $L(D_1)$ is a sublattice of $L(D_1 \nearrow D_2)$. Consider the elements (x, y, z) of $L(D_1 \nearrow D_2)$ with $x, y, z \in B_1$ (hence $x \in D_1$, by [3], Theorem 1). These triples form a $\wedge$-subsemilattice of $L(D_1 \nearrow D_2)$. But, because of Lemma 1, the join of two such triples is the same as their join in $L(D_1)$:

$$(x, y, z) \vee_{L(D_1)} (x', y', z') = (x \vee x', y \vee y', z \vee z')^{\sim\wedge(L(D_1))}$$

$$(x, y, z) \vee_{L(D_1 \nearrow D_2)} (x', y', z') = (x \vee x', y \vee y', z \vee z')^{\sim\wedge(L(D_1 \nearrow D_2))}$$

Now the operation $\tilde{}$ does not depend upon, in which lattice the triple is considered, and Lemma 1 shows that the same is true for $\hat{}$.

3. Proof of (δ_0) and (δ_1). (δ_0) is a consequence of (δ_1), thus we need only prove (δ_1). Let $d \in D_1$. (δ_1) says that $d\delta\varphi_2 = d\varphi_1\gamma$. Now, $d\varphi_1$ is the congruence generated by $((0,0,0),(d,0,0))$ in L_1. λ takes this pair to the interval $[p,r]$. Let these elements there be denoted by $(0,0,0)_{[p,r]}$ and $(d,0,0)_{[p,r]}$ (Figure 1). Then the congruence $d\varphi_1\gamma$ is generated by this pair. With analogous notations, it is also generated by $((0,0,0)_{[q,s]},(0,0,d)_{[q,s]})$. $(L(D_1 \nearrow D_2, D_2)$ was defined such that the first component must be in D_2. Therefore, when we glue it by a $D_1 \nearrow D_2$ to $M(D_1 \nearrow D_2)$, the third (or second) component must denote the elements used in the gluing. That is $[q,s]$ is the interval $[(0,0,0),(0,0,1)]$ of $L(D_1 \nearrow D_2, D_2)$. Omitting the subscript $[q,s]$, let us meet the pair $((0,0,0),(0,0,d))$ with $(0,1,0)$ and join the result with $(1,0,0)$; so we get

$$(0,1,0) \equiv (\bar{d},1,\bar{d}) \pmod{d\varphi_1\gamma}, \quad (0,0,0) \equiv (\bar{d},0,0) \pmod{d\varphi_1\gamma},$$

and both pairs generate $d\varphi_1\gamma$, where $\bar{d}$ denotes the least upper bound of $d \in D_1$ $(\subseteq D_1 \nearrow D_2)$ in D_2 $(\nsubseteq D_1 \nearrow D_2)$. On the other hand, $d\delta = d^+$, thus $d\delta\varphi_2$ is generated by $((0,0,0),(d^+,0,0))$. We only have to prove $\bar{d} = d^+$ in $D_1 \nearrow D_2$ for all $d \in D_1$. Recall that $\bar{d}$ denotes the least upper bound of d in D_2. It suffices to show that $d_1 \leqq d_2$ $(d_i \in D_i,\ i=1,2)$ implies $d_1^+ \leqq d_2$. Besides, if we prove it for d_1 irreducible, then it is true for arbitrary d_1. This follows from the fact that $^+$ preserves joins. Now assume that $f_{d_1} \leqq f_{d_2} \pmod{\Theta}$ that is, for all $k \in P_2$

$$d_1 \leqq f_{d_2}(k) \pmod{\theta(I(k))},$$

where f_{d_1} represents the element d_1, that is, f_{d_1} takes the value d_1 identically and f_{d_2} is the characteristic function of d_2:

$$f_{d_2}(k) = \begin{cases} 1 & \text{if } k \leqq d_2, \\ 0 & \text{otherwise.} \end{cases}$$

Now $d_1 \vee f_{d_2}(k) \equiv f_{d_2}(k)$ means that for all k, the value of the function

$$d_1 \vee f_{d_2}(k) = \begin{cases} 1 & \text{if } k \leqq d_2, \\ d_1 & \text{otherwise,} \end{cases}$$

is congruent with $f_{d_2}(k)$ modulo $\Theta(\mathscr{I}(k))$. Now let us go out to $B_1 \nearrow B_2$ and form the meet with

$$\bar{f}_{d_2}(k) = \begin{cases} 0 & \text{if } k \leqq d_2, \\ 1 & \text{otherwise;} \end{cases}$$

then we obtain that for all $k \nleqq d_2$, $k \in P_2$;

$$d_1 \equiv 0 \pmod{\theta(I(k))},$$

that is, $d_1 \in \mathcal{I}(k)$, in other words $k \nleqq d_1^+$. Thus $\{k | k \in P_2, k \leqq d_1^+\} \subseteqq \{k | k \in P_2, k \leqq d_2\}$. Hence $d_1^+ \leqq d_2$, as claimed.

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Sur la structure circulaire des ensembles de points limites des sommes partielles d'une série de Taylor

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Pour le 70-ième anniversaire du Professeur Béla Sz.-Nagy

On considère une série de Taylor

$$\sum_0^\infty c_j z^j$$

sur le cercle unité $|z|=1$, et on pose $z=e^{ix}$, x réel. Les sommes partielles sont

$$S_n(x) = \sum_0^n c_j e^{ijx}.$$

On suppose que, pour $x\in F$, ensemble mesurable de mesure de Lebesgue $|F|\neq 0$, la série est sommable au sens $(C, 1)$ vers une somme $\sigma(x)$, c'est-à-dire que les sommes de Fejér

$$\sigma_n(x) = \frac{S_0(x)+\ldots+S_{n-1}(x)}{n}$$

ont une limite

$$\lim_{n\to\infty} \sigma_n(x) = \sigma(x) \quad (x\in F).$$

Un célèbre théorème de Marcinkiewicz et Zygmund dit que presque partout sur F l'ensemble des points limites des $S_n(x)$ est réunion de cercles (de rayons $\geqq 0$) centrés en $\sigma(x)$: c'est la «structure circulaire» ([1], p. 178). Le but de cet article est de préciser un peu la distribution des $S_n(x)$.

Nous considérerons que les $S_n(x)$ prennent leurs valeurs dans $\overline{\mathbf{C}}$, le plan $\mathbf{C}$ complété par un cercle à l'infini ; la convergence d'une suite complexe z_n dans $\overline{\mathbf{C}}$ équivaut à la convergence de la suite $\dfrac{z_n}{1+|z_n|}$ dans $\mathbf{C}$. A une suite complexe z_n et à un compact $K\subset\overline{\mathbf{C}}$ nous allons associer un nombre $d((z_n), K)$ qu'on pourra

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interpréter comme la densité supérieure des z_n au voisinage de K ; la définition précise fait intervenir des blocs $B_m \subset \mathbf{N}$, affectés de coefficients b_m, comme nous allons le voir un peu plus loin. Enonçons les résultats.

Théorème. *Pour presque tout* $x \in F$, *on a*

$$d((S_n(x)), K) = d((S_n(x)), K')$$

pour tout couple de compacts K *et* K' *de* $\bar{\mathbf{C}}$ *obtenus l'un à partir de l'autre par rotation autour de* $\sigma(x)$.

Corollaire 1. *Pour toute suite d'entiers* $n_j \to \infty$, *l'ensemble des points limites dans* $\bar{\mathbf{C}}$ *des* $S_{n_j}(x)$ *a presque partout la structure circulaire (c'est-à-dire invariant par les rotations de centre* 0*).*

Corollaire 2. *Pour presque tout* $x \in F$, *on a*

$$d((S_n(x) - \sigma(x)), A) \geqq \frac{1}{N} d((S_n(x)), \bar{\mathbf{C}})$$

pour tout compact «angulaire» $A = A\left(\frac{1}{N}, \varphi\right)$ *de* $\bar{\mathbf{C}}$, *défini par* $z=0$ *ou* $|\arg z - \varphi| \leqq \frac{\pi}{N}$ (N entier $\geqq 1$, φ réel).

Complément. Soit Γ un compact de $\bar{\mathbf{C}}$ ayant la structure circulaire de centre 0. Pour presque tout $x \in \Gamma$, on a

$$d((S_n(x) - \sigma(x)), A \cap \Gamma) \geqq \alpha\, d((S_n(x) - \sigma(x)), \Gamma),$$

pour tout compact «angulaire» $A = A(\alpha, \varphi)$ de $\bar{\mathbf{C}}$ défini par $z=0$ ou $|\arg z - \varphi| \leqq \pi\alpha$ ($0 \leqq \alpha \leqq 1$, φ réel).

Définissons $d((z_n), K)$. On donne une suite de parties de $\mathbf{N}$, soit (B_m), telle que $\inf B_m \to \infty$, et une suite (b_m) strictement positive (les cas les plus intéressants sont $b_m = 1$, et $b_m = \operatorname{card} B_m$). Pour chaque partie infinie de $\mathbf{N}$, soit Λ, on pose

$$d(\Lambda) = \limsup_{m \to \infty} \frac{\operatorname{card}(B_m \cap \Lambda)}{b_m}.$$

Si $b_m = \operatorname{card} B_m$, on a $0 \leqq d(\Lambda) \leqq 1$ et $d(\mathbf{N}) = 1$. Si $b_m = \operatorname{card} B_m$ et $B_m = [0, m]$, $d(\Lambda)$ est la densité supérieure de Λ au sens ordinaire, c'est-à-dire

$$d(\Lambda) = \limsup_{m \to \infty} \frac{\operatorname{card}([0, m] \cap \Lambda)}{m}.$$

Si $b_m = \operatorname{card} B_m = 1$ et $B_m = \{n_m\}$ $(\lim_{m \to \infty} n_m = \infty)$, $d(\Lambda)$ égale 1 ou 0 suivant que Λ contient une infinité ou un nombre fini de points n_m. Dans le cas général, on a $0 \leqq d(\Lambda) \leqq \infty$.

Pour un ouvert G de $\overline{\mathbf{C}}$, on pose

$$d((z_n), G) = d(\{n \in \mathbf{N} \mid z_n \in G\}).$$

On pose enfin

$$d((z_n), K) = \inf_{G \supset K} d((z_n), G)$$

la borne inférieure étant prise pour tous les ouverts G contenant K. On peut interpréter $d((z_n), K)$ comme la borne supérieure des $d(\Lambda)$ pour toutes les Λ telles que les points limites de $(z_n)_{n \in \Lambda}$ se trouvent dans K, et il est facile de voir que cette borne supérieure est atteinte. Dans le cas $b_m = \text{card } B_m$, on a

$$0 \leqq d((z_n), K) \leqq d((z_n), \overline{\mathbf{C}}) = 1.$$

Dans le cas général, on a

$$0 \leqq d((z_n), K) \leqq \infty, \quad d((z_n), K) \subset d((z_n), K') \quad \text{si} \quad K \subset K',$$

$$d((z_n), K \cup K') \leqq d((z_n), K) + d((z_n), K').$$

La dernière inégalité donne tout de suite le corollaire 2 à partir du théorème. Dans le cas $b_m = \text{card } B_m = 1$ et $B_m = \{n_m\}$ $(\lim_{m \to \infty} n_m = \infty)$, on a $d((z_n), K) = 1$ ou 0 suivant que la suite (z_{n_m}) admet un point limite dans K ou non. Le théorème dit alors que pour presque tout $x \in F$ l'ensemble des points limites de la suite $(S_{n_m}(x))$ est invariant par les rotations autour de $\sigma(x)$; c'est le corollaire 1.

Le complément se démontre comme le corollaire 2 quand $\frac{1}{\alpha}$ est entier. Pour le cas général, on a besoin d'une variante des définitions et du théorème, que voici. Si g est une fonction continue sur $\overline{\mathbf{C}}$, on pose

$$d((z_n), g) = \limsup_{m \to \infty} \frac{1}{b_m} \sum_{n \in B_m} g(z_n).$$

Si k est une somme finie de fonctions indicatrices de compacts, on pose

$$d((z_n), k) = \inf_{g \geqq k} d((z_n), g).$$

On a, pour presque tout $x \in F$,

$$d((S_n(x) - \sigma(x)), k) = d(S_n(x) - \sigma(x), k')$$

pour tout couple de fonctions k, k' obtenues l'une à partir de l'autre par rotation de centre 0: c'est la variante du théorème dont on a besoin. Nous laissons au lecteur le soin de vérifier cet énoncé (qui se démontre comme le théorème) et d'en déduire le complément.

Démontrons le théorème. La clé est une formule de Marcinkiewicz et Zygmund, dont nous indiquerons rapidement la démonstration pour la commodité du lecteur.

Lemme fondamental. *Si* $x \in F$ *et* $\alpha_n = O\left(\frac{1}{n}\right)$,

$$S_n(x+\alpha_n) - \sigma(x) = \big(S_n(x) - \sigma(x)\big)e^{in\alpha_n} + o(1).$$

Preuve. On peut supposer $x=0$ et $\sigma(0)=0$, et poser $\alpha_n = \frac{\beta}{n}$ $(\beta = \beta_n = O(1))$. La formule à prouver est

$$\sum_0^n c_j(e^{ij\alpha_n} - e^{i\beta}) = o(1)$$

soit

$$\sum_0^n c_j\left(1 - \exp\left(i\beta\left(1 - \frac{j}{n}\right)\right)\right) = o(1).$$

Or l'hypothèse $0 \in F$ et $\sigma(0)=0$ signifie

$$\sum_0^n c_j \varphi\left(\frac{j}{n}\right) = o(1)$$

pour $\varphi(x) = \inf(1-x, 0)$, donc, de façon uniforme, pour toutes les fonctions $\varphi(x)$ bornées sur $\mathbf{R}^+$, convexes, et tendant vers 0 à l'infini. Or la fonction

$$\psi(x) = \begin{cases} 1 - \exp\big(i\beta(1-x)\big) & (0 \leqq x \leqq 1) \\ 0 & (x \geqq 1) \end{cases}$$

est une combinaison linéaire de telles fonctions φ, soit

$$\psi(x) = \varphi_1(x) - \varphi_2(x) + i\varphi_3(x) - i\varphi_4(x)$$

avec $\sup \varphi_j(x) = O(1)$ si $\beta = O(1)$. La formule est donc établie.

En vue d'énoncer une proposition d'où le théorème se déduira aisément, voici encore quelques notations:

$$\Lambda_x(D) = \{n \mid S_n(x) \in D\},$$

$$E(D, d, v) = \{x \mid \forall m \geqq v \ \operatorname{card}\big(B_m \cap \Lambda_x(D)\big) \leqq db_m\}, \quad E(D, d) = \bigcup_v E(D, d, v).$$

Ainsi $x \notin E(D, d)$ signifie qu'il existe une suite $m_j \to \infty$ telle que $\operatorname{card}\big(B_{m_j} \cap \Lambda_x(D)\big) > db_{m_j}$.

Proposition. *Soit* D *un ouvert dans* $\overline{\mathbf{C}}$, $d \geqq 0$ *et* v *entier* $\geqq 1$. *Si* x $(x \in F)$ *est un point de densité de* $E(D, d, v)$ *et si* D' *est un ouvert tel que, par une rotation convenable de centre* $\sigma(x)$, $\overline{D}'$ *(adhérence de* D' *dans* $\overline{\mathbf{C}}$*) soit appliqué dans* D, *alors* $x \in E(D', d)$.

Preuve. C'est la même que dans [1]. Soit β l'angle d'une rotation de centre $\sigma(x)$ appliquant $\overline{D}'$ dans D. Comme x est point de densité de $E(D, d, v)$, il existe une suite $\alpha_n = \frac{\beta}{n} + o\left(\frac{1}{n}\right)$ telle que $x+\alpha_n \in E(D, d, \nu)$. Le lemme fondamental montre que $x \notin E(D', d)$ entraînerait $x \notin E(D, d)$, ce qui est impossible puisque $x \in E(D, d, \nu)$.

Achevons la démonstration du théorème.

Les disques «rationnels» et les angles «rationnels» forment une base dénombrable d'ouverts de $\bar{\mathbf{C}}$. Prenons désormais pour D une réunion finie de disques rationnels et d'angles rationnels, et posons

$$\mathscr{E} = \mathscr{E}((B_m), (b_m)) = \bigcup_{D,d,\nu} \left(E(D, d, \nu) - \mathscr{D}(E(D, d, \nu))\right)$$

où D parcourt les ouverts en question, d les rationnels $\geqq 0$, et ν les entiers $\geqq 1$, et où $\mathscr{D}(\ )$ désigne l'ensemble des points de densité. Comme la réunion est dénombrable, $\mathscr{E}$ est un ensemble de mesure nulle.

Montrons que si $x \in F$ et $x \notin \mathscr{E}$ on a la conclusion du théorème. En effet, si la conclusion était en défaut, on aurait

$$d\big((S_n(x)), K\big) < d < d\big((S_n(x)), K'\big)$$

avec d rationnel, K et K' compacts transformés l'un dans l'autre par une rotation de centre $\sigma(x)$. On pourrait alors choisir un $D \supset K$ et un $D' \supset K'$ vérifiant les hypothèses de la proposition, et tels que

$$d\big((S_n(x)), D\big) < d < d\big((S_n(x)), D'\big),$$

c'-est-à-dire

$$d\big(A_x(D)\big) < d < d\big(A_x(D')\big),$$

d'où

$$x \in E(D, d), \quad x \notin E(D', d)$$

ce qui est impossible d'après la proposition. Le théorème est démontré.

Littérature

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On arithmetic functions with regularity properties

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th anniversary

1. We shall say that an additive function $f(n)$ is of finite support if $f(p^\alpha)=0$ whenever p is a large prime. Let

$$P(z)=\alpha_0+\alpha_1 z+\ldots+\alpha_k z^k, \quad \alpha_k = 1, \quad \alpha_0 \neq 0$$

be an arbitrary polynomial with complex coefficients. The operators E, Δ, I are defined by the following relations:

$$Ex_n = x_{n+1}, \quad \Delta x_n = x_{n+1}-x_n, \quad Ix_n = x_n.$$

We are interested in the following problem: What is the set of additive functions $f(n)$ satisfying the relation

$$P(E)f(n) \to 0 \quad (n \to \infty). \tag{1.1}$$

This question was raised in [1]. Recently we solved it for completely additive functions. Namely, from a famous result of E. Wirsing we deduced that if a completely additive function $f(n)$ satisfies the relation

$$\frac{P(E)f(n)}{\log n} \to 0, \tag{1.2}$$

then $f(n)$ is a constant multiple of $\log n$; $f(n)=c\log n$ satisfies (1.2) with $c\neq 0$ only if $P(1)=0$. In the same paper we proved that for a completely additive function $f(n)$,

$$\frac{1}{x}\sum_{n\leqq x}|P(E)f(n)| \to 0 \quad (x \to \infty) \tag{1.3}$$

implies that $f(n)=c\log n$. The method used there cannot be applied without change to additive functions. Now we shall show how we can modify the method so as to be suitable for additive functions.

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Theorem 1.1. *If* (1.3) *holds for a complex valued additive function* $f(n)$, *then* $f(n)=c\log n+f_1(n)$ *where* $f_1(n)$ *is an additive function of finite support satisfying the recursion*

$$P(E)f_1(n)=0\quad (n=1,2,\ldots). \tag{1.4}$$

If $P(1)\neq 0$, *then* $c=0$.

Theorem 1.2. *If* $f(n)$ *is a complex valued additive function satisfying the linear recursion*

$$P(E)f(n)=0\quad (n=1,2,\ldots), \tag{1.5}$$

then

1) $f(p^\alpha)=0$ *for every prime power* p^α *satisfying* $p>k+1$,

2) $f(p^{\gamma+1})=f(p^\gamma)$ *if* $p^{\gamma+1}-p^\gamma>k+1$,

3) $f(n)$ *is periodic with* B *where* $B=\prod_{p\le k+1}p^{\gamma_p}$ *and* γ_p *is the smallest integer satisfying* $p^{\gamma_p+1}-p^{\gamma_p}>k+1$.

A modification of Theorem 1.2 was proved earlier by L. Lovász, A. Sárközy and M. Simonovits [2]. We shall deduce it immediately from Theorem 1.1.

Proof of Theorem 1.1. If the relation

$$\frac{1}{x}\sum_{n\le a}|k(E)f(n)|\to 0\quad (x\to\infty) \tag{1.6}$$

holds for a polynomial $k(z)$ then it holds for any other polynomial $K(z)$ that is a multiple of $k(z)$. Let $P(z)=\prod_{i=1}^{k}(z-\theta_i)$, and for a fixed integer $m>1$, let $Q_m(z^m)=\prod_{i=1}^{k}(z^m-\theta_i^m)$. Since $P(z)$ divides $Q_m(z^m)$, therefore

$$\frac{1}{x}\sum_{n\le x}|Q_m(E^m)f(n)|\to 0,$$

and so

$$\frac{1}{x}\sum_{mn\le x}|Q_m(E^m)f(nm)|\to 0. \tag{1.7}$$

Let $Q_m(z)=\beta_0+\beta_1z+\ldots+\beta_kz^k$ $(\beta_k=1)$; $\Delta(m,n)=\sum_{j=0}^{k}\beta_j\{f(m(n+j))-f(n+j)\}$. Then

$$\Delta(m,n)=Q_m(E^m)f(nm)-Q_m(E)f(n). \tag{1.8}$$

Applying the operator $P(E)$ and taking into account that $P(E)|P(E)Q_m(E)$, we yet that

$$\frac{1}{x}\sum_{n\le x}|P(E)\Delta(m,n)|\le\frac{1}{x}\sum_{n\le x}|P(E)Q_m(E^m)f(nm)|+\frac{1}{x}\sum_{n\le x}|P(E)Q_m(E)f(n)|,$$

whence

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}|P(E)\Delta(m,n)| = 0. \tag{1.9}$$

Let now $P>2k+1$ be a prime, and let n run over the set satisfying $P^{\nu}\|n$ with $\nu\ge 1$ fixed. Then

$$\begin{aligned}\Delta(P,n) &= \beta_0\{f(P^{\nu+1})-f(P^{\nu})\}+(\beta_1+\ldots+\beta_k)f(P) =\\ &= \beta_0\{f(P^{\nu+1})-f(P^{\nu})-f(P)\}+Q_m(1)f(P),\\ \Delta(P,n+h) &= f(P)Q_m(1)\quad (0<h\le 2k).\end{aligned}$$

Consequently

$$P(E)\Delta(m,n) = P(1)Q_m(1)f(P)+\alpha_0\beta_0\{f(P^{\nu+1})-f(P^{\nu})-f(P)\}. \tag{1.10}$$

Observing that the set of n's has a positive density, we get that

$$P(1)Q_m(1)f(P)+\alpha_0\beta_0\{f(P^{\nu+1})-f(P^{\nu})-f(P)\}. \tag{1.11}$$

Let now n run over the integers $\equiv 1 \pmod P$. Then we have $\Delta(P,n+h)= =f(P)Q_m(1)$ $(0\le h\le 2k)$, and so $P(E)\Delta(m,n)=P(1)Q_m(1)f(P)$. Repeating the above argument we get $P(1)Q_m(1)f(P)=0$. Since $P(1)\ne 0$ implies that $Q_m(1)\ne 0$, we have $f(P)=0$ provided $P(1)\ne 0$. From (1.11) we get that

$$f(P^{\nu+1})-f(P^{\nu})-f(P) = 0\quad (\nu = 1, 2, \ldots),$$

and hence $f(P^{\nu})=\nu f(P)$ $(\nu\ge 1)$.

Let P be an arbitrary prime, and let γ_0 be so large that $P^{\gamma_0}>2k+1$. Let $\varepsilon_1,\ldots,\varepsilon_{2k}$ be fixed nonnegative integers such that $P^{\gamma_0}\|n$ and $P^{\varepsilon_i}\|n+i$ $(i=1,\ldots,2k)$ hold for at least one n. Let A_{γ} denote the set of those n's for which $P^{\gamma}\|n$ and $P^{\varepsilon_i}\|n+i$ $(i=1,\ldots,2k)$. The following assertion is obvious: A_{γ} is nonempty for $\gamma\ge\gamma_0$ and it has a positive density.

Clearly $P(E)\Delta(n,P)$ is constant if n runs over the elements of A_{γ}; therefore it equals 0 on A_{γ}. Hence

$$P(E)\Delta(n_1,P)-P(E)\Delta(n_2,P) = 0$$

if $n_1\in A_{\gamma+1}$; $n_2\in A_{\gamma}$ $(\gamma\ge\gamma_0)$. Consequently

$$\alpha_0\beta_0\{f(P^{\gamma+2})-f(P^{\gamma+1})\} = \alpha_0\beta_0\{f(P^{\gamma+1})-f(P^{\gamma})\},$$

and from $\alpha_0\beta_0\ne 0$ we get that

$$\xi_{\gamma+1}=\xi_{\gamma}\quad (\gamma\ge\gamma_0),\quad \xi_{\gamma}=f(P^{\gamma+1})-f(P^{\gamma}). \tag{1.12}$$

Now we write $f(n)$ as $f_1(n)+f_2(n)$ where $f_2(n)$ is a completely additive function defined as follows: $f_2(P^{\alpha})=f(P^{\alpha})$ if $P>2k+2$. Then $f_1(P^{\alpha})=0$ if $P>2k+2$. For a smaller prime P we put $f_2(P)=\xi_{\gamma_0}$; which implies by (1.12) that $f_1(P^{j+1})= =f_1(P^{j})$ if $j\ge\gamma_0$.

Now we have shown that $f_1(n)$ is a function of finite support; and it is periodic with a period B_1. Consequently

$$P(E)(E^{B_1}-I)f_1(n) = 0.$$

Taking into account the relation

$$(E^{B_1}-I)P(E)f(n) = P(E)(E^{B_1}-I)f_1(n)+(E^{B_1}-I)P(E)f_2(n) = (E^{B_1}-I)P(E)f_2(n),$$

we have

$$\frac{1}{x}\sum_{n\leqq x}|(E^{B_1}-I)P(E)f_2(n)| \to 0. \tag{1.13}$$

From the theorem cited above we get that $f_2(n)=c\log n$. Earlier we have proved that $f(P)=f_2(P)=0$ for every large P, provided $P(1)\neq 0$. This implies that $f_2(n)\equiv 0$; furthermore, from the periodicity of $f_1(n)$ and from (1.3) we get that $P(E)f_1(n)\equiv 0$ $(n=1, 2, \dots)$.

Assume now that $P(1)=0$. Then $P(E)\,c\log n\to 0$, whence (1.3) yields that

$$\frac{1}{x}\sum_{n\leqq x}|P(E)f_1(n)| \to 0 \quad (x\to\infty).$$

Using the periodicity of $f_1(n)$ we get that $P(E)f_1(n)\equiv 0$ $(n=1, 2, \dots)$. This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since (1.5) implies (1.3), we get that $f(n)=f_1(n)+ +c\log n$, $P(E)f_1(n)=0$; moreover, by (1.5), $P(E)c\log n=P(E)(f(n)-f_1(n))\equiv 0$, which is impossible for $c\neq 0$. Therefore we have that $f(n)=f_1(n)$ is of finite support. Then there exists a K such that $f(p^\alpha)=0$ for each prime $p>K$. For an integer n let $A_K(n)$ denote the product of all prime factors of n not greater than K. Let $\delta(n)$ be the exact exponent of p in $n: p^{\delta(n)}\| n$, and set $A'_K(n)=p^{-\delta(n)}A_K(n)$.

Let n_1 be chosen so that $\delta(n_1)=\gamma\geqq 0$, $n_1\equiv p^\gamma \pmod{p^{\gamma+1}}$, and let γ be so large that $p^{\gamma+1}-p^\gamma>k+1$. Then we can find an integer n_2 satisfying the following relations: $\delta(n_2)=\gamma+1$, $A'_K(n_1)=A'_K(n_2)$, $A_K(n_1+j)=A_K(n_2+j)$ $(j=1, \dots, k)$. Taking into account the equality $f(n)=f(A_K(n))$, we get from (1.5) that

$$0 = P(E)f(n_2)-P(E)f(n_2) = \alpha_0\{f(P^{\gamma+1})-f(P^\gamma)\},$$

which by $\alpha_0\neq 0$ implies that $f(P^{\gamma+1})=f(P^\gamma)$.

Thus 1) and 2) are proved; 3) is an immediate consequence of them.

Remark. The assertion of Theorem 1.1 remains true if (1.3) is replaced by

$$\liminf \frac{1}{\log x}\sum_{n\leqq x}\frac{1}{n}|P(E)f(n)| = 0. \tag{1.3$'$}$$

2. Theorem 2.1. *Let f be a completely additive real valued function, and let P be a nonzero polynomial with rational coëfficients satisfying the relation*

$$A_P P(E) f(n) \equiv 0 \pmod 1 \tag{2.1}$$

with a suitable integer $A_P \neq 0$. Then there exists an integer B such that $f(n)=g(n)/B$, where $g(n)$ is an integer valued additive function.

First we prove the following

Lemma 2.1. *If $\Delta^k f(n) \equiv 0 \pmod 1$ $(n=1,2,\ldots)$ for a $k \geqq 1$, and $f(n)$ is completely additive, then $f(n) \equiv 0 \pmod 1$.*

Proof. Let us assume that $k=1$. Then summing the congruences $f(n+1)-f(n) \equiv 0 \pmod 1$ for $n=pu, \ldots, qu-1$, we have $f(q)-f(p) \equiv 0 \pmod 1$ for each pair p, q which by $q=np$ gives that $f(n) \equiv 0 \pmod 1$.

Now we use induction on k. Assume that our lemma is true for k, and consider the condition $\Delta^{k+1} f(n) \equiv 0 \pmod 1$. Starting from

$$\sum_{n=1}^{N-1} \Delta^{k+1} f(n) = \Delta^k f(N) - \Delta^k f(1) \equiv 0 \pmod 1,$$

we get

$$\Delta^k f(N) \equiv c \pmod 1, \quad c = \Delta^k f(1).$$

If Q is an arbitrary polynomial with integer coefficients, then

$$(E-I)^k Q(E) f(N) \equiv cQ(1) \pmod 1.$$

Let $Q(z)=Q_m(z)=\left(\frac{z^m-1}{z-1}\right)^k=(1+z+\ldots+z^{m-1})^k$. Then $(E-I)^k Q_m(E)=(E^m-I)^k$, consequently

$$(E^m-I)^k f(mN) \equiv cQ_m(1) \pmod 1;$$

furthermore;

$$(E^m-I)^k f(mN) \equiv (E-I)^k f(N) \equiv c \pmod 1,$$

whence $c(Q_m(1)-1) \equiv 0 \pmod 1$. Since $Q_m(1)=m^k$, we get $c(m^k-1) \equiv 0 \pmod 1$ $(m=2,3,\ldots)$. Therefore c is a rational number. Let $c=A/B$, where A, B are coprime integers. If $B \neq 1$, then by choosing $m=B$, we get $c(B^k-1) \equiv 0 \pmod 1$, $c \equiv 0 \pmod 1$, which is a contradiction. This completes the proof of the lemma.

Proof of Theorem 2.1. Let A be the set of all polynomials P with rational coefficients for which

$$A_P P(E) f(n) \equiv 0 \pmod 1$$

holds with a suitable integer A_P. Then A is an ideal in $R[x]$. Let $P(z)=\prod_{j=1}^{k}(z-\theta_j) \in A$. From the fundamental theorem of symmetric polynomials it follows that

$$k_m(z) = \prod_{j=1}^{k}{}' \frac{z^m-\theta_j^m}{z-\theta_j}$$

has rational coefficients; consequently

$$R_m(z^m) = \prod_{j=1}^{k} (z^m-\theta_j^m) \in A.$$

Furthermore, $R_m(E^m)f(nm)=R_m(1)f(m)+R_m(E)f(n)$. Let F be an integer such that $FR_m(E^m)f(n)\equiv 0 \pmod 1$. Then we have

$$FR_m(1)f(m)+FR_m(E)f(n) \equiv 0 \pmod 1.$$

If $R_m(1)=0$, then $R_m \in A$. If $R_m(1)\neq 0$, then applying the operator Δ we get that

$$FR_m(E)\Delta f(n) \equiv 0 \pmod 1,$$

whence $R_m(z)(z-1)\in A$.

Let P be the generator element of A, that is, a polynomial of minimum degree in A. Let $\deg P=k$. From (2.1) we get that A is not empty. If $k=0$, then our theorem is obviously true. For $k\geqq 1$ assume first that $P(1)=0$. Then

$$\delta(z) = (P(z),\ R_m(z))\in A, \tag{2.2}$$

implying $\deg\delta(z)=k$, i.e., $R_m(z)\equiv P(z)$,

$$\{\theta_1, \ldots, \theta_k\} = \{\theta_1^m, \ldots, \theta_k^m\} \quad (m = 2, 3, \ldots), \tag{2.3}$$

whence it follows that $\theta_1=\ldots=\theta_k=1$, $P(z)=(z-1)^k$. Assume now that $P(1)\neq 0$. Then

$$\delta(z) = (P(z),\ R_m(z)(z-1))\in A,$$

consequently $\deg\delta(z)=k$, and from $(z-1, P(z))=1$ we get that $P(z)=R_m(z)$ $(m=2, 3, \ldots)$, which implies (2.3), and so $\theta_1=\ldots=\theta_k=1$, which is impossible.

Thus we have proved the following assertion: If (2.1) holds with a suitable P then there exists an integer $F\neq 0$ and an integer $k>0$ such that

$$F\Delta^k f(n) \equiv 0 \pmod 1. \tag{2.4}$$

Using Lemma 2.1 with $Ff(n)$ instead of $f(n)$ we get that $Ff(n)$ is an integer for every n. This finishes the proof of the theorem.

3. Conjecture. Let $P(z)=1+\alpha_1 z+\ldots+\alpha_k z^k$ $(k\geqq 1)$ be a polynomial with at least one irrational coefficient. If a completely additive function $f(n)$ satisfies the relation $P(E)f(n)\equiv 0 \pmod 1$ $(n=1, 2, \ldots)$ then $f(n)$ is identically zero.

Theorem 3.1. *The conjecture is true for* $k=2$.

Proof. Let $\xi=f(2)$, $\eta=f(3)$. From $P(E)f(1)\equiv 0$ we get that

$$\alpha_1\xi \equiv -\alpha_2\eta \pmod 1, \tag{3.1}$$

and from $P(E)f(2)\equiv 0$ that

$$(2\alpha_2+1)\xi+\alpha_1\eta \equiv 0 \pmod 1. \tag{3.2}$$

Similarly, by considering $P(E)f(n)\equiv 0 \pmod 1$ for $n=7$ and $n=6$, and taking into account (3.1) we deduce:

$$f(7) \equiv -\alpha_1 f(8)-\alpha_2 f(9) \equiv -3\alpha_1\xi-2\alpha_2\eta \pmod 1,$$

$$f(7) \equiv \alpha_2\eta \pmod 1, \tag{3.3}$$

$$\alpha_1 f(7) \equiv -f(6)-\alpha_2 f(8) \pmod 1,$$

$$\alpha_1 f(7) \equiv -(1+3\alpha_2)\xi-\eta \pmod 1. \tag{3.4}$$

Similarly,

$$\alpha_1 f(5) \equiv -(2+\alpha_2)\xi-\alpha_2\eta \pmod 1 \quad (n=4). \tag{3.5}$$

Starting from $P(E)f(14)\equiv 0 \pmod 1$ we get

$$(\xi+4\alpha_2\xi)+f(7)+\alpha_1 f(3)+\alpha_1 f(5) \equiv 0 \pmod 1.$$

Substituting (3.3) and (3.5) into the left hand side, we get $(1+4\alpha_2)\xi+\alpha_2\eta+\alpha_1\eta-$ $-(2+\alpha_2)\xi-\alpha_2\eta\equiv 0 \pmod 1$, whence $-\xi+3\alpha_2\xi+\alpha_1\eta\equiv 0 \pmod 1$, and, by (3.2),

$$\alpha_2\xi \equiv 2\xi \pmod 1, \tag{3.6}$$

$$\alpha_1\eta \equiv -5\xi \pmod 1. \tag{3.7}$$

For $n=26$ and $n=13$ we have

$$f(2\cdot 13)+\alpha_1 f(3^3)+\alpha_2 f(2^2\cdot 7) \equiv 0 \pmod 1$$

$$f(13)+\alpha_1 f(2\cdot 7)+\alpha_2 f(3\cdot 5) \equiv 0 \pmod 1,$$

where by subtraction we get

$$\alpha_2 f(7) \equiv 3\xi-2\eta-2\alpha_1\xi \pmod 1. \tag{3.8}$$

Considering $n=5$ and taking into account (3.8) we get

$$f(5) \equiv \alpha_1\xi-\alpha_1\eta-3\xi+2\eta \pmod 1. \tag{3.9}$$

Putting now $n=25$ and $n=12$ we get that

$$f(5^2)+\alpha_1 f(2\cdot 13)+\alpha_2 f(3^3) \equiv 0 \pmod 1,$$

$$f(12)+\alpha_1 f(13)+\alpha_2 f(14) \equiv 0 \pmod 1.$$

Subtracting them and by using (3.8), (3.9) we deduce that

$$5\eta-3\xi+2\alpha_1\xi \equiv 0 \pmod 1. \tag{3.10}$$

From $n=3$ we get

$$\alpha_2 f(5) \equiv -\eta-2\alpha_1\xi \pmod 1. \tag{3.11}$$

Putting now $n=48$ we have

$$f(2^3\cdot 3)+\alpha_1 f(7^2)+\alpha_2 f(5^2\cdot 2) \equiv 0 \pmod 1,$$

and by (3.11) and (3.4) we get

$$9\xi+3\eta+4\alpha_1\xi \equiv 0 \pmod 1. \tag{3.12}$$

Since $f(2^3)+\alpha_1 f(3^2)+\alpha_2 f(2)+\alpha_2 f(5)\equiv 0 \pmod 1$, we get that

$$\alpha_2 f(5) \equiv -5\xi-2\alpha_1 \pmod 1$$

(see (3.6), (3.7)) which implies by (3.11) that

$$\eta+5\xi+2\alpha_1\xi \equiv 0 \pmod 1. \tag{3.13}$$

Now from (3.10), (3.12), and (3.13) we infer that

$$7\xi-7\eta \equiv 0 \pmod 1 \quad\text{and}\quad 4\eta-8\xi \equiv 0 \pmod 1,$$

which proves that ξ and η are rational numbers. Assume now that $\xi\neq 0$ and $\eta\neq 0$. Then (3.6) and (3.7) show that α_1 and α_2 are rational numbers, and the proof is finished. Let $\xi=0$ and $\eta\neq 0$. Then by (3.7) and (3.1) we get that α_1 and α_2 are rational numbers. In the case $\eta=0$, $\xi\neq 0$ we use (3.6) and (3.1) to derive the same result.

Finally, let us assume that $\xi=0$, $\eta=0$, and P is the smallest prime for which $f(P)\neq 0$. Since $P>3$, therefore $P+1$ is a composite number, $f(P+1)=0$, and so $\alpha_1 f(P+1)\equiv 0 \pmod 1$. Let us consider the relation

$$f(P)+\alpha_1 f(P+1)+\alpha_2 f(P+2) \equiv 0 \pmod 1. \tag{3.14}$$

If $P+2$ is a composite number then $f(P+2)=0$, and so $f(P)\equiv 0 \pmod 1$. Using that $\alpha_1 f(P)\equiv 0 \pmod 1$, $\alpha_2 f(P)\equiv 0 \pmod 1$, and that $f(P)\neq 0$, we deduce that α_1 and α_2 are integers. Assume now that $P+2$ is a prime number. If $f(P+2)=0$ then we are done as before. Let $f(P+2)\neq 0$. Then

$$f(P+2)+\alpha_1 f(P+3)+\alpha_2 f(P+4) \equiv 0 \pmod 1,$$

and $P+3$, $P+4$ are composite numbers with prime factors smaller than P, whence it follows that $f(P+3)=f(P+4)=0$ and $f(P+2)\equiv 0 \pmod 1$. Since

$$f(P+1)+\alpha_1 f(P+2)+\alpha_2 f(P+3) \equiv 0 \pmod 1,$$

we have $\alpha_1 f(P+2)\equiv 0 \pmod 1$, and so α_1 is a rational number. (3.14) implies that α_2 is also rational. The proof is complete.

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Upper estimates for the eigenfunctions of higher order of a linear differential operator

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

In several problems of the spectral theory of non-selfadjoint differential operators it occurs the need to estimate the eigenfunctions of higher order of these operators (cf. [3], [4], [5], [7], [8], [11]). These results were proved in general by the application of the mean value formulas of TITCHMARSH [2], MOISEEV [6] and JOÓ [7]. For the case of the Schrödinger operator, exact estimates were obtained in [7]. However, in case the differential operator is of order $n \geqq 3$, the mean value formula becomes rather complicated (see [6]), and it seems to be hard to obtain exact estimates by its application. In this paper, we choose another approach: using the method of variation of constants instead of the mean value formula, we trace the difficulties back to the investigation of some concrete determinants. As the result of these considerations, we obtain the formula of Theorem 1. This formula actually equals the mean value formula in case of the Schrödinger operator, but differs from it in general.

Using this formula, we extend the upper estimates of [7] to the case of an arbitrary linear differential operator. We obtain estimates not only for the eigenfunctions, but also for their derivatives. These results are formulated in Theorem 2.

Let $G \subset \mathbf{R}$ be an arbitrary open interval and consider the formal differential operator

$$Lu = u^{(n)} + p_1 u^{(n-1)} + \dots + p_n u, \tag{1}$$
$$p_1, \dots, p_n \in L^1_{\mathrm{loc}}(G) \text{ are arbitrary complex functions.}$$

Let λ be a complex number. The function $u_{-1}: G \to \mathbf{C}$, $u_{-1} \equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. As it is usual, a function $u_i: G \to \mathbf{C}$, $u_i \not\equiv 0$ $(i = 0, 1, \dots)$ is said to be an eigenfunction of order i of the operator L with the eigenvalue λ if u_i, together with its first $n-1$

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derivatives is absolute continuous on every compact subinterval of G and if for almost all $x \in G$ the equation

$$(Lu_i)(x) = \lambda u_i(x) + u_{i-1}(x)$$

holds, where u_{i-1} is an eigenfunction of order $i-1$ with the eigenvalue λ.

We shall prove the following result:

Theorem 1. *Given any pair of integers $m \geqq 0$; $n \geqq 2$ there exist entire functions f, f_{jik}, h_j $(j \geqq 0, 0 \leqq i < n, 1 \leqq k \leqq N \equiv (m+1)n)$ with $f(z) \neq 0$ for $|z| < \pi^n$ such that the following formulas are valid:*

Given any eigenfunction u_m of order $\leqq m$ of the operator (1) *with the eigenvalue $\lambda \in \mathbf{C}$, introducing for $j < m$ the functions*

$$u_j \colon G \to \mathbf{C}, \quad u_j = Lu_{j+1} - \lambda u_{j+1}, \tag{2}$$

we have that

$$\begin{aligned} f(\lambda t^n) t^{jn+i} u_{m-j}^{(i)}(x) = \sum_{k=1}^{N} f_{jik}(\lambda t^n) u_m(x+kt) + \\ + \sum_{k=1}^{N} f_{jik}(\lambda t^n) \sum_{r=0}^{m} \sum_{s=1}^{n} \int_x^{x+kt} (x+kt-\tau)^{n(r+1)-1} h_r\big(\lambda(x+kt-\tau)^n\big) p_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau \end{aligned} \tag{3}$$

for all $j \geqq 0$, $0 \leqq i < n$, and for all $x \in G$ with $x + Nt \in G$.

The functions f_{00k} are multiples of f and therefore if $j = i = 0$, this formula can be simplified by f.

Consider now the special case

$$Lu = u^{(n)} + p_2 u^{(n-2)} + \ldots + p_n u, \quad G \subset \mathbf{R} \text{ is a bounded open interval.} \tag{4}$$

It is well-known that the eigenfunctions of the operator (4) can be extended to absolute continuous functions on $\bar{G}$ (see [1]). Using Theorem 1 we shall prove the following estimates:

Theorem 2. *There exist constants*

$$\mathcal{K}_m = \mathcal{K}_m(n, |G|, \|p_2\|_1, \ldots, \|p_n\|_1), \quad m = 0, 1, \ldots,$$

($|G|$ denotes the length of G) *such that given any eigenfunction u_m of order m of the operator* (4) *with the eigenvalue $\lambda \in \mathbf{C}$, we have*

$$\|u_{m-j}^{(i)}\|_\infty \leqq \mathcal{K}_m \big(1 + |\sqrt[n]{\lambda}|\big)^{jn+i+(1/r)} \|u_m\|_r \tag{5}$$

for all $0 \leqq j \leqq m$, $0 \leqq i < n$ and $1 \leqq r \leqq \infty$.

Moreover, if $p_2, \ldots, p_n \in L^p(G)$ for some $1 < p \leqq \infty$, then there exist constants

$$\mathcal{K}_m^p = \mathcal{K}_m^p(n, |G|, \|p_2\|_p, \ldots, \|p_n\|_p), \quad m = 0, 1, \ldots,$$

such that, putting $q=(1-1/p)^{-1}$,

(6) $$\|u^{(i)}_{m-j}\|_q \leqq \mathcal{K}^{p}_{m}(1+|\sqrt[n]{\lambda}|)^{jn+i}\|u_m\|_q$$

for all $0\leqq j\leqq m$ *and* $0\leqq i<n$.

In the first section of this paper we prove Theorem 1 for the case $G=\mathbf{R}$, $p_1=p_2=\ldots=p_n\equiv 0$. In its full generality Theorem 1 is proved in Section 2. Finally, Theorem 2 will be proved in Section 3.

1. Some properties of the operator $L_0v=v^{(n)}$, $G=\mathbf{R}$. In this section v_m will denote an arbitrarily fixed eigenfunction of order $\leqq m$ of the operator L_0 with the eigenvalue $\lambda=\varrho^n$ and, for $j<m$, we introduce the functions $v_j=v^{(n)}_{j-1}-\lambda v_{j+1}$. We shall also use the notation

$$\varrho_p=\varrho e^{p\frac{2\pi i}{n}},\quad p=1,2,\ldots,n.$$

The following assertion is obvious:

Lemma 1. v_m *has the form*

(7) $$v_m(x)=\begin{cases}\sum\limits_{r=0}^{m}\sum\limits_{p=1}^{n}a_{rp}(\varrho_p x)^r e^{\varrho_p x} & \text{if } \lambda\neq 0,\\ \sum\limits_{r=0}^{m}\sum\limits_{p=1}^{n}a_{rp}x^{rn+p-1} & \text{if } \lambda=0,\end{cases}$$

with appropriate constants $a_{rp}\in\mathbf{C}$.

For any $t\in\mathbf{R}$, we define the determinant $D(\varrho t)$ of type $N\times N$ in the following way: let the $(rn+p)$-th entry of the k-th row $(1\leqq k\leqq N, 1\leqq p\leqq n, 0\leqq r\leqq m)$ be

(8) $$\frac{(k\varrho_p t)^r}{r!}e^{k\varrho_p t}.$$

One can see easily that D is an entire function with isolated roots. A more thorough investigation shows (cf. [12]) that

(9) $$D(\varrho t)\equiv C(\varrho t)^{n\frac{m(m+1)}{2}}\Big[\prod_{1\leqq p<q\leqq n}(e^{\varrho_p t}-e^{\varrho_q t})\Big]^{(m+1)^2}$$

with some constant $C\neq 0$. Let us denote the subdeterminant of $D(\varrho t)$, corresponding to the element (8) by $D_{kpr}(\varrho t)$, and define formally $D_{kpr}(\varrho t)\equiv 0$ for $r>m$.

Lemma 2. *There exist numbers* $C_{jis}\in\mathbf{C}$, *independent of the choice of* v_m, *such that for all* $j\leqq 0$, $0\leqq i<n$ *and* $x,t\in\mathbf{R}$,

(10) $$v^{(i)}_{m-j}(x)=\sum_{k=1}^{N}\Big\{\sum_{p=1}^{jn+i}C_{jis}\sum_{p=1}^{n}\varrho_p^{jn+i}\frac{D_{kps}(\varrho t)}{D(\varrho t)}\Big\}v_m(x+kt),$$

whenever $D(\varrho t)\neq 0$.

Proof. By (7) and (9), for any $1 \leqq k \leqq N$ we have

$$v_m(x+kt) = \sum_{p=1}^{n} \sum_{s=0}^{m} \frac{(k\varrho_p t)^s}{s!} e^{k\varrho_p t} w_{ps}(x), \tag{11}$$

where

$$w_{ps}(x) \equiv \sum_{r=s}^{m} a_{rp} \frac{r!}{(r-s)!} (\varrho_p x)^{r-s} e^{\varrho_p x}.$$

Hence for all $1 \leqq p \leqq n$, $0 \leqq s \leqq m$,

$$w_{ps}(x) = \sum_{k=1}^{N} \frac{D_{kps}(\varrho t)}{D(\varrho t)} v_m(x+kt). \tag{12}$$

(12) is formally true also for $s > m$ if we put $w_{ps} \equiv 0$. It follows directly from the definition of q_{ps} that for all $1 \leqq pn$, $s \geqq 0$, $i \geqq 0$, $x \in \mathbf{R}$,

$$w_{ps}^{(i)}(x) = \varrho_p^i \sum_{q=0}^{i} \binom{i}{q} w_{p,s+q}(x), \tag{13}$$

and hence

$$w_{ps}^{(n)}(x) - \lambda w_{ps}(x) = \lambda \sum_{q=1}^{n} \binom{n}{q} w_{p,s+q}(x). \tag{14}$$

In the light of (12), our assertion (10) can be written in the form

$$v_{m-j}^{(i)}(x) = \lambda^j \sum_{p=1}^{n} \varrho_p^i \sum_{s=j}^{jn+i} C_{jis} w_{ps}(x). \tag{15}$$

First we prove it for $i=0$, by induction on j. For $j=0$, (15) follows from (11) with $C_{000}=1$. Suppose the formula is true for some $j \geqq 0$; then it is true also for $j+1$. Indeed, we have by (14) and the inductive hypothesis that

$$v_{m-j-1}(x) = \lambda^j \sum_{p=1}^{n} \sum_{s=j}^{jn} C_{j0s}[w_{ps}^{(n)}(x) - \lambda w_{ps}(x)] = \lambda^j \sum_{p=1}^{n} \sum_{s=j}^{jn} C_{j0s} \lambda \sum_{q=1}^{n} \binom{n}{q} w_{p,s+q}(x) =$$

$$= \lambda^{j+1} \sum_{p=1}^{n} \sum_{r=j+1}^{(j+1)n} \left\{ \sum_{q=\max(1,r-jn)}^{\min(n,r-j)} \binom{n}{q} C_{j0,r-q} \right\} w_{pr}(x).$$

Thus (15) is true for all $j \geqq 0$, $i=0$. Hence the general case follows by (13):

$$v_{m-j}^{(i)}(x) = \lambda^j \sum_{p=1}^{n} \sum_{s=j}^{jn} C_{j0s} w_{ps}^{(i)}(x) = \lambda^j \sum_{p=1}^{n} \sum_{s=j}^{jn} C_{j0s} \varrho_p^i \sum_{q=0}^{i} \binom{i}{q} w_{p,s+q}(x) =$$

$$= \lambda^j \sum_{p=1}^{n} \varrho_p^i \sum_{r=j}^{jn+i} \left\{ \sum_{q=\max(0,r-jn)}^{\min(i,r-j)} \binom{i}{q} C_{j0,r-q} \right\} w_{pr}(x).$$

The lemma is proved.

Now we introduce some special eigenfunctions of the operator L_0 which will also be used in Section 2. Define the functions K_m: $\mathbf{C}\times\mathbf{R}\to\mathbf{C}$ in the following way:

$$K_m(\varrho, x) \equiv 0 \quad \text{if} \quad m<0;$$

$$K_0(\varrho, x) = \begin{cases} \sum_{p=1}^{n} \frac{\varrho_p}{n\lambda} e^{\varrho_p x} & \text{if} \quad \varrho \neq 0, \\ \frac{x^{n-1}}{(n-1)!} & \text{if} \quad \varrho = 0; \end{cases} \qquad (\lambda \equiv \varrho^n)$$

$$K_m(\varrho, x) \equiv \int_0^x K_0(\varrho, x-t) K_{m-1}(\varrho, t)\, dt \quad \text{if} \quad m>0.$$

Lemma 3. *For any pair of integers m and $0 \leqq i < n$,*

$$D_2^{n+i} K_m(\varrho, x) \equiv \lambda D_2^i K_m(\varrho, x) + D_2^i K_{m-1}(\varrho, x), \tag{16}$$

$$D_2^i K_m(\varrho, 0) = \begin{cases} 1 & \text{if} \quad m=0 \quad \text{and} \quad i=n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Moreover, there exist entire functions h_m^i such that $h_m^i(0)=1$ and

$$D_2^i K_m(\varrho, 2) \equiv \frac{x^{nm+n-1-i}}{(nm+n-1-i)!} h_m^i(\lambda x^n). \tag{18}$$

Consequently, for any $m \geqq 0$, $0 \leqq i < n$ and $\varrho \in \mathbf{C}$, $D_2^i K_m(\varrho, \cdot)$ is an eigenfunction of order m of the operator L_0 with the eigenvalue $\lambda = \varrho^n$.

Proof. For $m=0$, (16)—(18) can be shown by easy computation, using the identity

$$\sum_{p=1}^{n} \varrho_p^i = \begin{cases} n\lambda & \text{if} \quad i=n, \\ 0 & \text{if} \quad 0 \leqq i < n; \end{cases}$$

for $m<0$ they are obvious. Suppose they are true for some $m \geqq 0$, and we shall conclude from this their validity also for $m+1$. It suffices to show (16) and (18) for $i=0$. In fact, the cases $i>0$ of (16) and (18) hence follow by repeated derivation and (17) is a consequence of (18). Using the definition of K_{m+1} and the inductive hypothesis,

$$D_2^n K_{m+1}(\varrho, x) = \frac{d^n}{dx^n} \int_0^x K_0(\varrho, x-t) K_m(\varrho, t)\, dt =$$

$$= \sum_{j=0}^{n-1} D_2^j K_0(\varrho, 0) D_2^{n-1-j} K_m(\varrho, x) + \int_0^x D_2^n K_0(\varrho, x-t) K_m(\varrho, t)\, dt =$$

$$= K_m(\varrho, x) + \int_0^x \lambda K_0(\varrho, x-t) K_m(\varrho, t)\, dt = \lambda K_{m+1}(\varrho, x) + K_m(\varrho, x),$$

and (16) is proved. To show (18), we use the explicit forms

$$(19)\qquad h_j^0(z) \equiv \sum_{k=0}^{\infty} a_k^j z^k, \quad \lim_{k\to\infty} \sqrt[k]{|a_k^j|} = 0, \quad a_0^j = 1, \quad j = 0, 1, \dots, m.$$

We can write

$$(20)\quad K_{m+1}(\varrho, x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \sum_{k=0}^{\infty} a_k^0 (\varrho(x-t))^{nk} \frac{t^{nm+n-1}}{(nm+n-1)!} \sum_{r=0}^{\infty} a_r^m (\varrho t)^{nr}\, dt =$$

$$= x^{n(m+1)+n-1} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (\varrho x)^{(k+r)n} a_k^0 a_r^m \int_0^1 \frac{(1-\xi)^{n-1+kn}}{(n-1)!} \frac{\xi^{nm+n-1+rn}}{(nm+n-1)!}\, d\xi =$$

$$= \frac{x^{n(m+1)+n-1}}{(n(m+1)+n-1)!} \sum_{s=0}^{\infty} a_s^{m+1} (\varrho x)^{ns},$$

where

$$(21)\quad a_s^{m+1} = (n(m+1)+n-1)! \sum_{k=0}^{s} a_k^0 a_{s-k}^m \int_0^1 \frac{(1-\xi)^{n-1+nk}}{(n-1)!} \frac{\xi^{nm+n-1+(s-k)n}}{(nm+n-1)!}\, d\xi;$$

hence, in view of (19), we easily obtain

$$(22)\qquad a_0^{m+1} = 1 \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{|a_k^{m+1}|} = 0$$

(to deduce the first equality, we integrate by parts $n-1$ times). (22) shows the legality of the demonstration of (20). Finally, (20) and (22) yield (18).

Lemma 4. *Given any eigenfunction v_m of order $\leqq m$ with some eigenvalue λ, there exists a sequence $v_{k,m}$ such that $v_{k,m}$ is an eigenfunction of order $\leqq m$ with the eigenvalue $\lambda_k \neq \lambda$, $\lambda_k \to \lambda$, and for all $0 \leqq j \leqq m$, $0 \leqq i < n$ and $x \in \mathbf{R}$, we have $v_{k,m-j}^{(i)}(x) \to v_{m-j}^{(i)}(x)$ $(k \to \infty)$.*

Proof. For $\lambda \neq 0$ this is a direct consequence of Lemma 1. For $\lambda = 0$, it follows from Lemmas 1 and 3 (see (18)).

Now we prove Theorem 1 for $L = L_0$. All the following formulas will be taken for all $j \leqq 0$, $0 \leqq i < n$ and $x, t \in \mathbf{R}$. Introducing the entire functions d_{jik} by the formulas

$$(23)\qquad d_{jik}(\varrho t) \equiv \sum_{s=j}^{jn+i} C_{jis} \sum_{p=1}^{n} (\varrho_p t)^{jn+i} D_{kps}(\delta t), \quad k = 1, \dots, N$$

(see (10)), we have the identities

$$(24)\qquad D(\varrho t) t^{jn+i} v_{m-j}^{(i)}(x) = \sum_{k=1}^{N} d_{jik}(\varrho t) v_m(x+kt)$$

whenever $D(\varrho t) \neq 0$. Let μ denote the smallest multiplicity of the root 0 in the functions d_{jik}. We claim that μ is greater than or equal to the multiplicity of the root 0 in D. Indeed, in the opposite case, dividing both sides of (24) by $(\varrho t)^{\mu}$ and putting $\varrho\to 0$, $x=0$, $t=1$, we would obtain from Lemma 4 for some j, i that the identity

$$\sum_{k=0}^{N} d_{jik}^{*} v_m(k) = 0$$

holds for all eigenfunctions of order $\leqq m$ with the eigenvalue 0, i.e., for all polynomials of degree $<N$ with some coefficients d_{jik}^{*}, at least one of which differs from zero. But this is impossible because putting $v_m(x)\equiv x^r$, $r=0, 1, \ldots, N-1$, the resulting system of linear equations has the only solution $d_{ji1}^{*}=d_{ji2}^{*}=\ldots=d_{jiN}^{*}=0$.

Assume $D(\varrho t)\neq 0$. Then taking into account also (9), we can divide (24) by

$$C(\varrho t)^{n\frac{m(m+1)}{2}}\Big[\prod_{1\leqq p<q\leqq n}(\varrho_p t-\varrho_q t)\Big]^{(m+1)^2},$$

and we obtain the identities

$$f^{*}(\varrho t)\, t^{jn+i} v_{m-j}^{(i)}(x) = \sum_{k=1}^{N} f_{jik}^{*}(\varrho t)\, v_m(x+kt) \tag{25}$$

where f^{*}, f_{jik}^{*} are suitable entire functions with the properties

$$f^{*}(0)=1 \quad\text{and}\quad f^{*}(z)\neq 0 \quad\text{if}\quad |z|<\pi. \tag{26}$$

It follows from the construction of f^{*} and f_{ijk}^{*} that

$$f^{*}\big(\varrho t e^{\frac{2\pi i}{n}}\big)\equiv f^{*}(\varrho t) \quad\text{and}\quad f_{jik}^{*}\big(\varrho t e^{\frac{2\pi i}{n}}\big)\equiv f_{jik}^{*}(\varrho t);$$

therefore there exist entire functions such that

$$f^{*}(\varrho t)\equiv f(\lambda t^{n}) \quad\text{and}\quad f_{jik}^{*}(\varrho t)\equiv f_{jik}(\lambda t^{n}). \tag{27}$$

From (25)—(27) the formulas (3) of Theorem 1 follow whenever $D(\varrho t)\neq 0$. However, this last condition can be eliminated with the aid of Lemma 4. The first part of Theorem 1 is proved. To prove the second part of the theorem, it suffices to show that

$$d_{00k}(\varrho t)\equiv C_{000}\sum_{p=1}^{n} D_{kp0}(\varrho t), \quad k=1, \ldots, N$$

is a multiple of

$$(\varrho t)^{n\frac{m(m+1)}{2}}\Big[\prod_{1\leqq p<q\leqq n}(e^{\varrho_q t}-e^{\varrho_p t})\Big]^{(m+1)^2};$$

this can be shown similarly to (9). Theorem 1 for L_0 is proved.

2. Proof of Theorem 1. Using the notations of Theorem 1, introduce the functions

$$M(u_{j-r},t)\equiv(Lu_{j-r})(t)-u_{j-r}^{(n)}(t)=\sum_{s=1}^{n}p_s(t)u_{j-r}^{(n-s)}(t), \tag{28}$$

$$v_j(x)\equiv u_j(x)+\sum_{r=0}^{j}\int_a^x K_r(\varrho,x-t)M(u_{j-r},t)\,dt, \tag{29}$$

for $0\leqq r,j\leqq m$, $t,x\in G$ where a is an arbitrarily fixed point of G. First we show that

$$v_j^{(i)}(x)=u_j^{(i)}x+\sum_{r=0}^{j}\int_a^x D_2^iK_r(\varrho,x-t)M(u_{j-r},t)\,dt\quad(j\leqq m,\ 0\leqq i<n), \tag{30}$$

$$v_j=v_{j-1}^{(n)}-\lambda v_{j+1}\quad(j<m),\quad v_{-1}\equiv 0. \tag{31}$$

Indeed, using (29) and (17), we get that for any $j\leqq m$, $0\leqq i\leqq n$,

$$v_j^{(i)}(x)=u_j^{(i)}(x)+\sum_{r=0}^{j}\frac{d^i}{dx^i}\int_a^x K_r(\varrho,x-t)M(u_{j-r},t)\,dt=$$

$$=u_j^{(i)}(x)+D_2^{i-1}K_0(\varrho,0)M(u_j,x)+\sum_{r=0}^{j}\int_a^x D_2^iK_r(\varrho,x-t)M(u_{j-r},t)\,dt.$$

For $0\leqq i<n$ this implies (30) in view of (17). Now let $i=n$. Using also (29), (2), (28) and (16), we conclude that

$$v_j^{(n)}(x)-\lambda v_j(x)=$$

$$=u_j^{(n)}(x)-\lambda u_j(x)+M(u_j,x)+\sum_{r=0}^{j}\int_a^x[D_2^nK_r(\varrho,x-t)-\lambda K_r(\varrho,x-t)]M(u_{j-r},t)\,dt=$$

$$=u_{j-1}(x)+\sum_{r=0}^{j-1}\int_a^x K_r(\varrho,x-t)M(u_{j-1-r},t)\,dt,$$

whence the first part of (31) follows. $v_{-1}\equiv 0$ is obvious by (29). v_m being the restriction of an eigenfunction of order $\leqq m$ of the operator L_0 with the eigenvalue λ (by (31)); we can apply Theorem 1 for v_m. Using also (30), we obtain the identities

$$f(\lambda t^n)t^{jn+i}\left(u_{m-j}^{(i)}(x)+\sum_{r=j}^{m}\int_a^x D_2^iK_{r-j}(\varrho,x-\tau)M(u_{m-r},\tau)\,d\tau\right)=$$

$$=\sum_{k=1}^{N}f_{jik}(\lambda t^n)\left(u_m(x+kt)+\sum_{r=0}^{m}\int_a^{x+kt}K_r(\varrho,x+kt-\tau)M(u_{m-r},\tau)\,d\tau\right)$$

for all $j\geqq 0$, $0\leqq i<n$, $x\in G$ and $x+Nt\in G$. By (18) and (28) this identity would coincide with (3) if we could replace the lower bound a in all the integrals by x.

But this is allowed by the following remark: $K_r(\varrho, \cdot)$ is an eigenfunction of order r of L_0 with the eigenvalue λ (Lemma 3), and therefore we have

$$f(\lambda t^n) t^{jn+i} D_2^i K_{r-j}(\varrho, x-\tau) = \sum_{k=1}^{N} f_{jik}(\lambda t^n) K_r(\varrho, x-\tau+kt)$$

for any $j \geqq 0, 0 \leqq i < n$ and $x, t, \tau \in \mathbf{R}$. Furthermore, $D_2^i K_{r-j} \equiv 0$ for any $j > r$, $0 \leqq i < n$. Thus Theorem 1 is proved.

3. Proof of Theorem 2. Using the notations of Theorem 1, let us fix a constant C such that for all $0 \leqq j \leqq m, 0 \leqq i < n$ and $1 \leqq k \leqq N$,

(32) $$|f_{jik}(z)| \leqq C|f(z)| \quad \text{if} \quad |z| \leqq 1$$

and

(33) $$|h_j(z)| \leqq C \quad \text{if} \quad |z| \leqq N^N.$$

Assume $p_2, \ldots, p_n \in L^p(G)$ $(1 \leqq p \leqq \infty)$, and define the numbers ε, R, M_q (where $p^{-1}+q^{-1}=1$) as follows:

(34) $$\varepsilon \equiv (4N)^{-1}(b-a)^{-1/q} \quad (G \equiv (a, b)),$$

(35) $$R \equiv \min\left\{\frac{1}{|\sqrt[n]{\lambda}|}, \frac{b-a}{2N}, \min\left\{\sqrt[s-1]{\frac{\varepsilon}{C^2 N^{N+1} \|p_s\|_p}} : 2 \leqq s \leqq n\right\}\right\},$$

(36) $$M_q \equiv \max\{R^{jn+i} \|u_{m-j}^{(i)}\|_q : 0 \leqq j \leqq m, \ 0 \leqq i < n\}.$$

Using (3), (32), (33), (35), and (36), for any $0 \leqq j \leqq m, 0 \leqq i < n, a \leqq x \leqq \dfrac{a+b}{2}$ and $0 \leqq t \leqq R$ we can write

$$t^{jn+i}|u_{m-j}^{(i)}(x)| \leqq C \sum_{k=1}^{N} |u_m(x+kt)| + NC^2 \sum_{r=0}^{m} \sum_{s=2}^{n} (NR)^{n(r+1)-1} \|p_s\|_p \|u_{m-r}^{(n-s)}\|_q =$$

$$= C \sum_{k=1}^{N} |u_m(x+kt)| + N^N C^2 \sum_{r=0}^{m} \sum_{s=2}^{n} (R^{s-1}\|p_s\|_p)(R^{rn+n-s}\|u_{m-r}^{(n-s)}\|_q) \leqq$$

$$\leqq C \sum_{k=1}^{N} |u_m(x+kt)| + \varepsilon M_q,$$

i.e.,

(37) $$t^{jn+i}|u_{m-j}^{(i)}(x)| \leqq C \sum_{k=1}^{N} |u_m(x+kt)| + \varepsilon M_q.$$

First we prove (5) $(q \equiv \infty)$. Applying the operation

$$NR^{-1} \int_0^R \cdot \, dt$$

to both sides, we obtain

$$R^{j n+i}|u_{m-j}^{(i)}(x)| \leqq NCR^{-1} \sum_{k=1}^{N} \int_0^R |u_m(x+kt)|\,dt + N\varepsilon M_\infty.$$

Using the Hölder inequality, one can easily see that

$$R^{-1} \int_0^R |u_m(x+kt)|\,dt \leqq R^{-1/r}\|u_m\|_r,$$

and therefore

$$R^{j n+i}|u_{m-j}^{(i)}(x)| \leqq N^2 CR^{-1/r}\|u_m\|_r + N\varepsilon M_\infty.$$

This is true for all $a \leqq x \leqq \frac{a+b}{2}$, but one can quite similarly prove it for all $\frac{a+b}{2} \leqq x \leqq b$, too. Hence

$$R^{j n+i}\|u_{m-j}^{(i)}\|_\infty \leqq N^2 CR^{-1/r}\|u_m\|_r + N\varepsilon M_\infty,$$

and in view of (34) and (36),

$$M_\infty \leqq N^2 CR^{-1/r}\|u_m\|_r + \frac{1}{2} M_\infty,$$

$$M_\infty \leqq 2N^2 CR^{-1/r}\|u_m\|_r.$$

Hence (5) follows by (34), (35) and (36).

To prove (6), put $t=R$ in (37) and take the $L^q\left(a, \frac{a+b}{2}\right)$ norm of both sides:

$$R^{j n+i}\|u_{m-j}^{(i)}\|_{L^q\left(a, \frac{a+b}{2}\right)} \leqq NC\|u_m\|_q + (b-a)^{1/q}\varepsilon M_q.$$

A similar estimate can be obtained for $\|u_{m-j}^{(j)}\|_{L^q\left(\frac{a+b}{2}, b\right)}$, too. Therefore, in view of (36) and (34),

$$M_q \leqq 2NC\|u_m\|_q + \frac{1}{2} M_q,$$

$$M_q \leqq 4NC\|u_m\|_q.$$

Hence (6) follows by (34), (35) and (36).

Remark. For $n=2$ the functions f_{jlk}, h_j in Theorem 1 have some special properties. Using these properties, one can show with the method of the paper [7] the following stronger form of (5):

$$\|u_{m-j}^{(i)}\|_\infty \leqq \mathcal{K}_{2,m}(1+|\sqrt{\lambda}|)^{j+i}(1+|\operatorname{Re}\sqrt{\lambda}|)^{j+1/r}\|u_m\|_r.$$

Just this result was proved in [7].

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On the homotopy type of some spaces occurring in the calculus of variations

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Dedicated to Professor B. Sz.-Nagy on the occasion of his 70th birthday

1. Let $n\in\mathbf{N}$ and let $D\subset\mathbf{R}\times\mathbf{R}^n$ be an open region. Suppose $\xi_0, \xi_1\in\mathbf{R}^n$ are given such that $(0, \xi_0)$, $(1, \xi_1)\in D$. Denote by $M(D)$ the class of continuous functions $x\colon[0, 1]\to\mathbf{R}^n$ such that

$$(1)\qquad x(0)=\xi_0,\ x(1)=\xi_1,\quad\text{and}\quad \Gamma(x):=\{(t, x(t))\,|\,t\in[0, 1])\}\subset D.$$

The space of $\mathbf{R}^n$-valued continuous functions over $[0, 1]$ will be denoted by $C_n[0, 1]$. Thus $M(D)$ is a subspace of $C_n[0, 1]$. Endow $M(D)$ with the relative topology of $C_n[0, 1]$.

The global methods of the calculus of variations (see [1], [3], [5] and [6]) lead us to the following problem: how can the homotopy type of $M(D)$ be described from that of D? In this paper we establish a connection between the homotopy types of the spaces D and $M(D)$ for a rather wide class of regions D. We shall define a class of admissible regions and for this class we shall prove the following theorem.

Theorem. *Suppose $D\subset\mathbf{R}\times\mathbf{R}^n$ is an admissible region and its homotopy type is the one point union $S^{r_1}\vee S^{r_2}\vee\ldots\vee S^{r_k}$ of the spheres S^{r_i} of dimension $r_i\geq 1$ $(i=1, 2, \ldots, k)$. Then the homotopy type of $M(D)$ is the one point union $S^{r_1-1}\vee S^{r_2-1}\vee\ldots\vee S^{r_k-1}$ of the spheres S^{r_i-1} $(i=1, 2, \ldots, k)$.*

2. In this section the necessary definitions and constructions will be given.

Definition 1. The regions $D_1, D_2\subset\mathbf{R}^{n+1}$ satisfying (1) will be called t-invariantly homeomorphic, if there exists a uniformly continuous homeomorphism $\varphi\colon D_1\to D_2$ such that

$$\text{a)}\qquad \varphi(0, \xi_0)=(0, \xi_0),\ \varphi(1, \xi_1)=(1, \xi_1),$$

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b) the diagram

$$\begin{array}{ccc} \varphi: D_1 & \to & D_2 \\ {}_{\mathrm{pr}_1}\searrow & & \swarrow_{\mathrm{pr}_1} \\ & \mathbf{R} & \end{array}$$

is commutative where $\mathrm{pr}_1: \mathbf{R}^1\times\mathbf{R}^n\to\mathbf{R}^1$ is the projection of the space $\mathbf{R}^1\times\mathbf{R}^n$ onto the first factor.

Denote by $I_n\subset\mathbf{R}^n$ the n-dimensional open unit interval $\bigtimes_{i=1}^{n}]0, 1[$. Let k, i $(i\leq k)$ and r $(r\leq n)$ be positive integers and $\delta\in]0, 1/2[$ a real number. For the ordered quadruple (k, i, r, δ) define the set $Q(k, i, r, \delta)$ as the product

$$\Big(\bigtimes_{j=1}^{n-r}]0, 1[\Big)\times\Big(\bigtimes_{j=n-r+1}^{n-1}\Big[\frac{1}{2}-\delta, \frac{1}{2}+\delta\Big]\Big)\times\Big[\frac{2i-1-\delta}{2k}, \frac{2i-1+\delta}{2k}\Big].$$

Now, suppose that the positive integers n, k are given. Let $r=(r_1, r_2, ..., r_k)\in\mathbf{N}^k$ $(r_i\leq n$ for $i=1, 2, ..., k)$, $\alpha, \beta, \delta\in I_k$. Suppose that $\alpha_i<\beta_i$ for all $i=1, 2, ..., k$ and $2\delta\in I_k$. The set $D(k, r, \alpha, \beta, \delta)\subset\mathbf{R}\times\mathbf{R}^n$ will be given in the following manner:

$$D(k, r, \alpha, \beta, \delta) := \{(t, x)\in\mathbf{R}\times\mathbf{R}^n \mid t\in[0, 1], x\in I_n, \text{ and if } t\in[\alpha_i, \beta_i] \text{ then } x\notin Q(k, i, r_i, \delta_i)\}.$$

Definition 2. A region $D\subset\mathbf{R}^{n+1}$ is said to be admissible if there exist $k\in\mathbf{N}$, $r\in\mathbf{N}^k$ $(r_i\leq n, i=1, 2, ..., k)$, $\alpha, \beta, \delta\in I_k$ $(\alpha_i<\beta_i, i=1, 2, ..., k, 2\delta\in I_n)$, such that the intersection $\bigcap_{i=1}^{k}]\alpha_i, \beta_i[$ is nonempty, and D and $D(k, r, \alpha, \beta, \delta)$ are t-invariantly homeomorphic regions.

Remark. It can be easily seen that the homotopy type of the regions $D(k, r, \alpha, \beta, \delta)$ (and thus that of D) is the one point union $S^{r_1}\vee S^{r_2}\vee\ldots\vee S^{r_k}$.

Now, choose real numbers $\alpha_0, \beta_0, \alpha', \beta', t_0$ such that $0<\alpha'<\alpha_0<t_0<\beta_0<\beta'<1$. Define the function $f: [0, 1]\to[0, 1]$ in the following way:

$$f(t) := \begin{cases} t, & t\in[0, \alpha']\cup[\beta', 1], \\ \alpha' - \dfrac{t-\alpha'}{\alpha_0-\alpha'}(t_0-\alpha'), & t\in[\alpha', \alpha_0], \\ t_0, & t\in[\alpha_0, \beta_0], \\ t_0+\dfrac{t-\beta_0}{\beta'-\beta_0}(\beta'-t_0), & t\in[\beta_0, \beta']. \end{cases}$$

The restriction of f to the set $[0, \alpha_0]\cup]\beta_0, 1[$ is invertible and the inverse also can be easily given:

$$(f|_{[0,\alpha_0]\cup]\beta_0,1[})^{-1}(t) = \begin{cases} t, & t\in[0,\alpha']\cup[\beta',1], \\ \alpha'+\dfrac{t-\alpha'}{t_0-\alpha'}(\alpha_0-\alpha'), & t\in[\alpha', t_0], \\ \beta_0+\dfrac{t-t_0}{\beta'-t_0}(\beta'-\beta_0), & t\in]t_0,\beta'[. \end{cases}$$

Let $k\in\mathbf{N}$, $r\in\mathbf{N}^k$ $(r_i\leqq n,\ i=1,2,\dots,k)$, $\alpha',\beta'\in]0,1[$ $(\alpha'<\beta')$, $\delta\in\mathbf{R}^k$ $(2\delta\in I_k)$ be given. Define the subspace $M(k,r,\alpha',\beta',\delta)\subset C_n[0,1]$ as follows:

$$M(k,r,\alpha',\beta',\delta) := \Big\{x\in C_n[0,1] \,|\, x|_{[\alpha',\beta']} \ \text{const.},\ \xi(x)=:x(\alpha'),$$

$$\xi(x)\in I_n\setminus\bigcap_{i=1}^{k} Q(k,i,r,\delta),\quad x(t)=\xi_0+\frac{t}{\alpha'}(\xi(x)-\xi_0)\quad (t\in[0,\alpha']),$$

$$x(t)=\xi(x)+\frac{t-\beta'}{1-\beta'}(\xi_1-\xi(x))\quad (t\in[\beta',1])\Big\}.$$

Finally, denote by j the identity map of $[0, 1]$.

3. We start with a simple observation.

Lemma 1. *If the regions $D_1, D_2\subset\mathbf{R}^{n+1}$ satisfying the condition* (1) *are t-invariantly homeomorphic, then $M(D_1)$ and $M(D_2)$ are homeomorphic.*

Proof. Let $\varphi: D_1\to D_2$ be a t-invariant homeomorphism (in this case, obviously, the inverse $\varphi^{-1}: D_2\to D_1$ is also a t-invariant homeomorphism). Define the desired homeomorphism $\Phi: M(D_1)\to M(D_2)$ as follows:

$$(\Phi(x))(t) := \mathrm{pr}_2\varphi(t, x(t))\quad (t\in[0,1]),$$

where $\mathrm{pr}_2: \mathbf{R}^1\times\mathbf{R}^n\to\mathbf{R}^n$ is the projection of the product space $\mathbf{R}^1\times\mathbf{R}^n$ onto the second factor. From the t-invariance of the homeomorphism φ it follows immediately, that Φ is a homeomorphism. It is also clear that Φ^{-1} has a form similar to that of Φ:

$$(\Phi^{-1}(x))(t) = \mathrm{pr}_2\,\varphi^{-1}(t, x(t))\quad (t\in[0,1]).$$

From Lemma 1 it follows that it is sufficient to determine the homotopy type of the spaces $M(D(k,r,\alpha,\beta,\delta))$. We now turn to the calculation of the homotopy type of the space $M(k,r,\alpha',\beta',\delta)$. For this purpose we shall prove the following

Lemma 2. *The homotopy type of the space $M(k,r,\alpha',\beta',\delta)$ is the one point union $S^{r_1-1}\vee S^{r_2-1}\vee\dots\vee S^{r_k-1}$ of the spheres S^{r_i-1} $(i=1,2,\dots,k)$.*

Proof. It is obvious that the space $M(k, r, \alpha', \beta', \delta)$ is homeomorphic to the n-dimensional region $I_n\setminus\bigcup_{i=1}^{n} Q(k,i,r,\delta)$. The desired homeomorphism Ψ can be given by $(\xi|_{M(k,r,\alpha',\beta',\delta)})^{-1}$, where ξ is the function from the end of the 2[nd] point. Now, by the definition of the sets $Q(k,i,r,\delta)$ the region $I_n\setminus\bigcup_{i=1}^{k} Q(k,i,r,\delta)$ is homotopically equivalent to the one point union $S^{r_1-1}\vee S^{r_2-1}\vee\ldots\vee S^{r_k-1}$ of the spheres $S^{r_i-1}\,(i=1,2,\ldots,k)$.

Choose numbers $t_0\in\bigcap_{i=1}^{k}\,]\alpha_i,\beta_i[$ and $\alpha_0,\beta_0;\alpha',\beta'\in]0,1[$ such that the inequalities

$$0<\alpha'<\alpha_0<\min_{1\leq i\leq k}\{\alpha_i\}<\max_{1\leq i\leq k}\{\beta_i\}<\beta_0<\beta'<1$$

are satisfied.

Lemma 3. *The space $M(k,r,\alpha',\beta',\delta)$ is a deformation retract of the space $M(D(k,r,\alpha,\beta,\delta))$.*

Proof. A homotopy

$$F\colon [0,1]\times M(D(k,r,\alpha,\beta,\delta))\to M(D(k,r,\alpha,\beta,\delta))$$

is defined by

$$F(\tau,x):=\begin{cases} x\circ(2\tau f+(1-2\tau)j), & \tau\in[0,1/2],\\ (2-2\tau)x\circ f+(2\tau-1)\Psi(x(t_0)), & \tau\in[1/2,1],\end{cases}$$

where ψ is the function from the preceding proof.

The restriction of $x\mapsto F(\tau,x)$ to $M(k,r,\alpha',\beta',\delta)$ is the identity, because the elements of $M(k,r,\alpha',\beta',\delta)$ are constant over $[\alpha',\beta']$ and linear over the rest of $[0,1]$, consequently

$$x\circ(2\tau j+(1-2\tau)f)|_{[\alpha',\beta']}=\xi(x)=x|_{[\alpha',\beta']},$$

$$x\circ(2\tau j+(1-2\tau)f)|_{[0,\alpha']\cup[\beta',1]}=x\circ(2\tau j+(1-2\tau)j)|_{[0,\alpha']\cup[\beta',1]}=x|_{[0,\alpha']\cup[\beta',1]},$$

for $\tau\in[0,1/2]$, and

$$[(2-2\tau)x\circ f+(2\tau-1)\Psi(x(t_0)]_{[\alpha',\beta']}=(2-2\tau)x(t_0)+(2\tau-1)x(t_0)=x|_{[\alpha',\beta']},$$

$$[(2-2\tau)x\circ f+(2\tau-1)\Psi(x(t_0)]_{[0,\alpha']\cup[\beta',1]}=[(2-2\tau)x+(2\tau-1)x]|_{[0,\alpha']\cup[\beta',1]}=$$

$$=x|_{[0,\alpha']\cup[\beta',1]},$$

for $\tau\in[1/2,1]$.

The function $x\mapsto F(0,x)$ is the identity over $M(D(k,r,\alpha,\beta,\delta))$. The function $x\mapsto F(1,x)$ is a retract of $M(D(k,r,\alpha,\beta,\delta))$ onto $M(k,r,\alpha',\beta',\delta)$. The proof of Theorem follows immediately from Lemmas 1—3.

If $n=2$ and the region D is $I_3\setminus\{(t,1/2,1/2)\,|\,t\in[0,1]\}$, furthermore $\xi_0=\xi_1=(1/2,1/3)$, then the homotopy type of $M(D)$ is the one point union $\bigvee_{i=-\infty}^{\infty}S^0$ of infinitely many 0-dimensional spheres. There are as many spheres as there are different ways to wind the graphs of the functions around the omitted segment.

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Dilation theory and one-parameter semigroups of contractions

ISTVÁN KOVÁCS

To Professor Béla Szőkefalvi-Nagy on his 70th birthday

In the frame of the dilation theory, a classification of contraction operators, briefly contractions, T on a Hilbert space $\mathfrak{H}$, according to the behavior of their powers T^n and T^{*n} as $n\to\infty$, was given in [2], Chap. II in terms of the classes $C_{\alpha\beta}=C_{\alpha\cdot}\cap C_{\cdot\beta}$ $(\alpha, \beta=0, 1)$. Analogously, we may consider classes of one-parameter semigroups of contractions (T_t), i.e., representations $t\mapsto T_t$ of the additive semigroup $\mathbf{R}_+$ of the non-negative reals by contractions T_t of $\mathfrak{H}$, defined as follows:

$$\begin{aligned}
SC_{0\cdot} &= \{(T_t)\colon\ T_t x \to 0 \quad \text{for all } x\in\mathfrak{H}\},\\
SC_{\cdot 0} &= \{(T_t)\colon\ T_t^* x \to 0 \quad \text{for all } x\in\mathfrak{H}\},\\
SC_{1\cdot} &= \{(T_t)\colon\ T_t x \to 0 \quad \text{for } \ x = 0\ (x\in\mathfrak{H}) \text{ only}\},\\
SC_{\cdot 1} &= \{(T_t)\colon\ T_t^* x \to 0 \quad \text{for } \ x = 0\ (x\in\mathfrak{H}) \text{ only}\},
\end{aligned}$$

whenever $t\to\infty$. Furthermore, set $SC_{\alpha\beta}=SC_{\alpha\cdot}\cap SC_{\cdot\beta}$ $(\alpha, \beta=0, 1)$.

It might be an interesting question to know to what extent results and facts derived for the elements of $C_{\alpha\beta}$ would hold true for the elements of $SC_{\alpha\beta}$, for instance in the sense that is shown by the following observation.

One of the consequences of the paper [1] is that a contraction T which is an element of a finite-type von Neumann algebra $\mathscr{A}$ (cf. [3], Chap. V) is a unitary operator if and only if it belongs to the class $C_{1\cdot}$. According to the above considerations, a result analogous to that one might sound like this.

Theorem. *A one-parameter semigroup of contractions (T_t) the elements of which belong to a given finite-type von Neumann algebra $\mathscr{A}$ can be extended to a one-parameter group of unitary operators (U_t) of $\mathscr{A}$ if and only if $(T_t)\in SC_{1\cdot}$.*

Proof. The necessity part of the proof is evident. To show that the condition is also sufficient, consider the family of operators $F=(T_t^*T_t)_{t\in\mathbf{R}_+}$. This family

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is evidently bounded from below and is also decreasing as $t\to\infty$. In fact, if $s<t$ then as $(t-s)>0$ and $t=(t-s)+s$, for every $x\in\mathfrak{H}$ *) we have

$$0 \leqq (T_t^*T_tx|x) = (T_tx|T_tx) = (T_{t-s}T_sx|T_{t-s}T_sx) \leqq \|T_{t-s}\|^2(T_sx|T_sx) \leqq (T_s^*T_sx|x).$$

So F has a strong cluster point S in $\mathscr{A}$ which is self-adjoint and positive:

$$S = \lim\nolimits_{\text{strong}} T_t^*T_t \quad \text{as} \quad t\to\infty.$$

Moreover, S is invertible in the more general sense: $x\in\mathfrak{H}$, $Sx=0$ imply $x=0$. This follows from the fact that

$$\lim_{t\to\infty}\|T_tx\|^2 = \lim_{t\to\infty}(T_t^*T_tx|x) = (Sx|x) = 0,$$

which gives that $x=0$ by the condition of the theorem. Furthermore, for every $s\in\mathbf{R}_+$ we have $T_s^*ST_s=S$. This is immediate by the definition of S. Then for every finite normal trace φ on $\mathscr{A}$ we have $\varphi(S)=\varphi(T_s^*ST_s)=\varphi(S^{1/2}T_sT_s^*S^{1/2})$, from which we may conclude

$$(*) \qquad \varphi(S^{1/2}(I-T_sT_s^*)S^{1/2}) = 0$$

(I is the identity operator of $\mathfrak{H}$). As $S^{1/2}(I-T_sT_s^*)S^{1/2}\geqq O$ and φ is arbitrary, $(*)$ implies $S^{1/2}(I-T_sT_s^*)S^{1/2}=O$. Now, with S its square root $S^{1/2}$ is also invertible and its inverse is densely defined. Therefore, $S^{1/2}(I-T_sT_s^*)S^{1/2}=O$ implies $I-T_sT_s^*=O$, i.e., T_s^* is an isometric operator. But $\mathscr{A}$ is of finite-type, thus T_s^* and hence T_s are both unitary operators. To complete the proof, set

$$U_t = \begin{cases} T_t & \text{if} \quad t\geqq 0,\\ T_t^* & \text{if} \quad t<0.\end{cases}$$

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*) $\mathfrak{H}$ is the underlying Hilbert space of $\mathscr{A}$ and the operators under consideration.

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On measurable Hermitian indefinite functions with a finite number of negative squares

H. LANGER

Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

1. Introduction and main result

1. Let $\varkappa$ be a nonnegative integer and $0<a\leqq\infty$. We denote by $\mathfrak{P}_{\varkappa;a}$ the set of all complex functions f defined on the open interval $(-2a, 2a)$ with the following properties:

(i) $f(t)=\overline{f(-t)}$ $(-2a<t<2a)$;

(ii) the (Hermitian) kernel H_f defined by

$$H_f(t, s):=f(t-s) \quad (-a<s, t<a)$$

has $\varkappa$ negative squares;

(iii$_c$) f is continuous on $(-2a, 2a)$.

We denote by $\mathfrak{P}^m_{\varkappa;a}$ the set of all complex functions f on $(-2a, 2a)$ satisfying (i), (ii) and

(iii$_m$) f is measurable and locally bounded on $(-2a, 2a)$.

The aim of this note is to prove the following

Theorem. *The function $f\in\mathfrak{P}^m_{\varkappa;a}$ admits a unique decomposition*

$$f(t)=f_c(t)+f_s(t) \quad (-2a<t<2a) \tag{1}$$

such that $f_c\in\mathfrak{P}_{\varkappa;a}$, $f_s\in\mathfrak{P}^m_{0;a}$ and $f_s(t)=0$ a.e. on $(-2a, 2a)$.

For $\varkappa=0$ this theorem was proved by M. G. KREĬN [1], see also [2]. In this case it implies the classical result of F. RIESZ [3] stating that an arbitrary function $f\in\mathfrak{P}^m_{0;\infty}$ coincides almost everywhere with some $f_c\in\mathfrak{P}_{0;\infty}$. In connection with the paper [1] M. G. Krein asked for a generalization of his result to functions with $\varkappa$ negative squares. The Theorem above gives an affirmative answer to this question.

The proof of the Theorem will be given in section 4. Also for $\varkappa=0$ it is different from the proof of the corresponding theorem in [1]. As a main tool we use a result about the continuation of generalized functions with $\varkappa$ negative squares from a bounded interval to the whole real axis, which is perhaps of some interest in its own (see 3). In particular, it seems to be new even for $\varkappa=0$. Then it extends a classical result of M. G. Krein to positive definite generalized functions.

We mention that the decomposition of a positive definite measurable function f on R^n into the sum of a continuous positive definite function f_c and a positive definite function f_s on R^n, which vanishes almost everywhere, was proved by M. M. Crum [12], the corresponding fact for measurable positive definite functions on a locally compact group was proved by J. von Neumann and I. E. Segal, see [13].

A survey and a bibliography about positive definite functions and their generalizations can be found in [4]; continuous functions with a finite number of negative squares were considered, e.g., in [5] and [6, Parts I, IV].

2. In this section the function f is supposed to satisfy the conditions (i) and (ii). If $\varkappa=0$, then an arbitrary (even nonmeasurable) function f of this kind is bounded:

$$|f(t)| \leqq f(0) \quad (-2a < t < 2a).$$

If $\varkappa>0$ and f is continuous, then it may be unbounded at infinity (in case $a=\infty$). This holds, e.g., for $\varkappa=1$ if f has a representation of parabolic or hyperbolic type see [6, Part IV]. If f is not measurable and $\varkappa>0$, then it may be unbounded at zero. To see this we choose a (nonmeasurable) solution α of the functional equation $\alpha(t+s)=\alpha(t)\cdot\alpha(s)$, $\alpha(0)=1$, which is not locally bounded, and consider the following function f:

$$f(t) := \gamma\alpha(t)+\bar{\gamma}\overline{\alpha(t)}^{-1} \quad (-2a < t < 2a)$$

for arbitrary $0<a\leqq\infty$. The relation

$$\sum_{i,j=1}^{n} f(t_i-t_j)\xi_i\overline{\xi_j} = \gamma\sum_{i=1}^{n}\alpha(t_i)\xi_i\sum_{j=1}^{n}\alpha(t_j)^{-1}\overline{\xi_j}+\bar{\gamma}\sum_{i=1}^{n}\overline{\alpha(t_i)}\overline{\xi_i}\sum_{j=1}^{n}\overline{\alpha(t_j)}^{-1}\xi_j$$

shows that the kernel H_f has one negative and one positive square.

However, it is an open question if a measurable function f satisfying (i) and (ii) can be unbounded on some compact subinterval of $(-2a, 2a)$. Thus, we do not know whether the boundedness condition in (iii$_m$) should be imposed.

2. $\pi_\varkappa$-spaces associated with elements of $\mathfrak{P}^m_{\varkappa,2a}$

1. Let f be a complex function on $(-2a, 2a)$ satisfying the conditions (i) and (ii). We associate with f a $\pi_\varkappa$-space $\Pi_\varkappa(f)$ as follows. Consider the linear set $\mathscr{L}_0$ of all complex functions $u: s\to u(s)$ on $(-a, a)$ that are different from zero only in a finite number of points s, and equip $\mathscr{L}_0$ with the scalar product

$$[u, v] := \sum_{-a<s,t<a} f(t-s)u(s)\overline{v(t)} \quad (u, v\in\mathscr{L}_0).$$

The conditions (i) and (ii) imply that this scalar product is Hermitian and has $\varkappa$ negative squares on $\mathcal{L}_0$. Thus $\mathcal{L}_0$ can be canonically embedded into a $\pi_\varkappa$-space, which we shall denote by $\Pi_\varkappa(f)$. The element of $\Pi_\varkappa(f)$ corresponding to a function $u \in \mathcal{L}_0$ will also be denoted by u. Moreover, we introduce the functions $\varepsilon_t \in \mathcal{L}_0$, $-a<t<a$, as follows:

$$\varepsilon_t(s) := \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} \quad (-a < s < a).$$

Evidently, the elements ε_t, $-a<t<a$, generate the space $\Pi_\varkappa(f)$ and we have $[\varepsilon_s, \varepsilon_t] = f(t-s)$ $(-a<s,\, t<a)$.

Let $u_1, \ldots, u_\varkappa$ be elements of $\mathcal{L}_0$ such that

$$[u_j, u_k] = -\delta_{jk}, \quad j, k = 1, 2, \ldots, \varkappa.$$

We consider the Hilbert norm

$$\begin{aligned} \|x\|^2 &:= \sum_{j=1}^{\varkappa} |[x, u_j]|^2 + \left[x + \sum_{j=1}^{\varkappa} [x, u_j] u_j,\; x + \sum_{j=1}^{\varkappa} [x, u_j] u_j\right] = \\ &= [x, x] + 2 \sum_{j=1}^{\varkappa} |[x, u_j]|^2 \quad (x \in \Pi_\varkappa(f)) \end{aligned} \tag{2}$$

on $\Pi_\varkappa(f)$.

Lemma 1. *If the function f satisfies* (i) *and* (ii), *and is locally bounded on* $(-2a, 2a)$, *then the function $t \to \|\varepsilon_t\|$ $(-a<t<a)$ is locally bounded.*

Indeed, we have from (2)

$$\|\varepsilon_s\|^2 = [\varepsilon_s, \varepsilon_s] + 2 \sum_{j=1}^{\varkappa} |[\varepsilon_s, u_j]|^2 = f(0) + 2 \sum_t \sum_{j=1}^{\varkappa} f(t-s)\overline{u_j(t)},$$

and the statement follows from the local boundedness of f on $(-2a, 2a)$ if we observe that both summations on the right hand side are finite.

2. Now let $f \in \mathfrak{P}^m_{\varkappa;a}$. Then, besides $\Pi_\varkappa(f)$, a space $\Pi^c(f)$ can be defined as follows. Let C_a be the linear set of all continuous complex functions φ on $(-a, a)$ which vanish outside of some compact subinterval of $(-a, a)$. We define a scalar product on C_a by the formula

$$[\varphi, \psi]_C := \int_{-a}^{a} \int_{-a}^{a} f(t-s)\varphi(s)\overline{\psi(t)}\, ds\, dt \quad (\varphi, \psi \in C_a). \tag{3}$$

It will be shown in this section that the factor space of C_a modulo the isotropic subspace $\mathcal{L}^0_C$ of C_a with respect to the scalar product $[\cdot, \cdot]_C$ can be identified with some linear manifold in $\Pi_\varkappa(f)$.

To this end, for a given $\varphi\in C_a$ we define a linear functional F_φ on $\mathscr{L}_0$ by

$$F_\varphi(u) := \sum_s u(s) \int_{-a}^{a} f(t-s)\overline{\varphi(t)}\,dt \quad (u\in\mathscr{L}_0).$$

Let $(u_n)\subset\mathscr{L}_0$ be a sequence which converges to the zero element of $\Pi_\varkappa(f)$ if $n\to\infty$. Then we have

$$\Big|\sum_s u_n(s) f(t-s)\Big| = |[u_n, \varepsilon_t]| \leqq \|u_n\|\,\|\varepsilon_t\|.$$

The right hand side in this relation tends to zero if $n\to\infty$, and according to Lemma 1 this convergence holds locally uniformly with respect to $t\in(-a, a)$. This implies $F_\varphi(u_n)\to 0$ $(n\to\infty)$. Therefore F_φ is continuous and can be extended by continuity to all of $\Pi_\varkappa(f)$. Hence there exists an element $\varphi\in\Pi_\varkappa(f)$ such that

$$F_\varphi(x) = [x, \varphi] \quad (x\in\Pi_\varkappa(f)).$$

In particular,

$$(4)\qquad [\varepsilon_s, \varphi] = \int_{-a}^{a} f(t-s)\overline{\varphi(t)}\,dt,\ [u, \varphi] = \int_{-a}^{a} [u, \varepsilon_t]\overline{\varphi(t)}\,dt \quad (s\in(-a, a),\ u\in\mathscr{L}_0).$$

Next we show that the scalar product of two such elements $\varphi, \psi\in\Pi_\varkappa(f)$ coincides with (3). Indeed, if $\varphi\in C_a$, then there exists a sequence $(\varphi_n)\subset\mathscr{L}_0$ such that $\varphi_n\to\varphi$ in $\Pi_\varkappa(f)$. This implies that

$$[\varphi_n, \psi] \to [\varphi, \psi] \quad (n\to\infty), \quad [\varphi_n, \varepsilon_t] \to [\varphi, \varepsilon_t] \quad (n\to\infty),$$

and the latter convergence holds locally uniformly with respect to $t\in(-a, a)$. Thus we get

$$[\varphi, \psi] = \lim_{n\to\infty} [\varphi_n, \psi] = \lim_{n\to\infty} \int_{-a}^{a} [\varphi_n, \varepsilon_t]\overline{\psi(t)}\,dt =$$

$$= \int_{-a}^{a} [\varphi, \varepsilon_t]\overline{\psi(t)}\,dt = \int_{-a}^{a}\int_{-a}^{a} f(t-s)\varphi(s)\overline{\psi(t)}\,ds\,dt,$$

that is

$$[\varphi, \psi] = [\varphi, \psi]_C.$$

The factor space $C_a/\mathscr{L}_C^0$ can be identified with some linear manifold in $\Pi_\varkappa(f)$. In particular, the scalar product (3) has only a finite number $\varkappa'$, $0\leqq\varkappa'\leqq\varkappa$, of negative squares. The completion of $C_a/\mathscr{L}_C^0$ will be denoted by $\Pi^C(f)$. It is a $\pi_{\varkappa'}$-space for some $0\leqq\varkappa'\leqq\varkappa$, and can be identified with some (non-degenerate) subspace of $\Pi_\varkappa(f)$. Later we shall see that actually $\varkappa'=\varkappa$.

Remark 1. Instead of C_a we could have started from the space K_a of those elements of C_a which have derivatives of arbitrary order. If we again define the scalar product $[\varphi, \psi]_C$ for $\varphi, \psi\in K_a$ by the relation (3); it is easy to see that the completion of the factor space $K_a/\mathcal{L}_K^0$ coincides with $\Pi^C(f)$; here $\mathcal{L}_K^0$ denotes the isotropic subspace of K_a with respect to the scalar product $[\cdot, \cdot]_C$.

Remark 2. If the function f is continuous, that is $f\in\mathfrak{P}_{\varkappa;a}$, then the spaces $\Pi_\varkappa(f)$ and $\Pi^C(f)$ coincide. Indeed, if $s\in(-a,a)$, let $(\delta_s^{(n)})$, $n=1,2,\ldots$, be a δ_s-sequence of elements of C_a. Then it is easy to see that $\delta_s^{(n)}\to\varepsilon_s$ if $n\to\infty$, and the inclusion $\Pi_\varkappa(f)\subset\Pi^C(f)$ follows. Thus the spaces $\Pi_\varkappa(f)$ and $\Pi^C(f)$ are identical. Obviously, in this case the space $\Pi_\varkappa(f)$ is separable.

3. Generalized functions with $\varkappa$ negative squares

1. We denote by $\mathfrak{P}^d_{\varkappa;a}$ the set of all generalized functions F on $(-2a,2a)$ over the space K_{2a} with the following properties:

(i′) $(F,\varphi)=\overline{(F,\varphi^*)}$ $(\varphi\in K_{2a};\ \varphi^*(t):=\overline{\varphi(-t)})$;

(ii′) the kernel H_F on $K_a\times K_a$ defined by

$$H_F(\varphi,\psi):=(F,\varphi^\circ*\bar\psi)\quad(\varphi,\psi\in K_a,\ \varphi^\circ(t):=\varphi(-t))$$

has $\varkappa$ negative squares.

The generalized function $F\in\mathfrak{P}^d_{\varkappa;a}$ induces a scalar product

$$[\varphi,\psi]_K:=(F,\varphi^\circ*\bar\psi)\quad(\varphi,\psi\in K_a) \tag{5}$$

on K_a with $\varkappa$ negative squares. The corresponding $\pi_\varkappa$-space will be denoted by $\Pi^K_\varkappa(F)$.

Recall ([7, §4]) that a family (T_t), $0\leqq t<\infty$, of bounded linear operators in a Banach space $\mathcal{B}$ is a *generalized semigroup,* if it has the following properties:

(a) $\mathfrak{D}(T_t)=:\mathfrak{D}_t$ is a closed subspace of $\mathcal{B}$ and we have

$$\mathfrak{D}_t\subset\mathfrak{D}_{t'}\quad\text{if}\quad 0\leqq t'\leqq t,\qquad \overline{\bigcup_{t>0}\mathfrak{D}_t}=\mathcal{B};$$

(b) $T_0=I$, $T_{t+t'}=T_tT_{t'}$ $(t,t'\geqq0)$;

(c) if $t_0>0$, $x\in\mathfrak{D}_{t_0}$ and $0\leqq t,t'\leqq t_0$ then $\lim\limits_{t'\to t}T_{t'}x=T_tx$.

The *infinitesimal generator* A_0 of the generalized semigroup (T_t), $0\leqq t<\infty$, is defined as follows: $\mathfrak{D}(A_0)$ is the set of all $x\in\bigcup\limits_{t>0}\mathfrak{D}_t$ such that the limit $\lim\limits_{h\downarrow0}\frac{1}{ih}(T_hx-x)$ exists and

$$A_0x:=\lim_{h\downarrow0}\frac{1}{ih}(T_hx-x)\quad(x\in\mathfrak{D}(A_0)),$$

see [7, §4].

If $t\in R^1$, we denote by V_t the shift operator in K_a: $\varphi\in\mathfrak{D}(V_t)$ if either $\varphi=0$, or $\varphi\in K_a$ and $t+\operatorname{supp}\varphi\subset(-a,a)$, and if e.g., $t>0$,

$$(V_t\varphi)(s):=\begin{cases}\varphi(s-t) & \text{if}\quad -a+t\leqq s<a\\ 0 & \text{if}\quad -a<s\leqq-a+t\end{cases}\quad(\varphi\in\mathfrak{D}(V_t)).$$

It is easy to see that the operators V_t preserve the scalar product (5) on K_a. If $|t|$ is sufficiently small then $\vartheta(V_t)$ contains a $\varkappa$-dimensional negative subspace (with respect to (5)). Therefore, for these t the operator V_t is continuous in the norm topology of $\Pi_\varkappa^K(F)$ ([8, IX. 3]). The relation $V_{t+t'}=V_tV_{t'}$, $\operatorname{sgn} t=\operatorname{sgn} t'$, implies that all the operators V_t, $t\in R^1$, are continuous. Thus they can be extended by continuity to the closure $\overline{\vartheta(V_t)}$ in $\Pi_\varkappa^K(F)$. As a result we get a generalized semigroup of bounded isometric operators in $\Pi_\varkappa^K(F)$, which will also be denoted by (V_t), $0\leqq t<\infty$. Its infinitesimal generator is the operator

$$A_0 = i\frac{d}{dt}.$$

Evidently, $K_a\subset\vartheta(A_0)$ and $A_0K_a\subset K_a$. Moreover, the operator A_0 in $\Pi_\varkappa^K(F)$ is π-Hermitian (this either follows easily from the fact that the operators V_t are π-isometric, or can be checked directly). As it is real with respect to the involution $\varphi\to\varphi^*$ in K_a, its defect numbers are equal, and it is not hard to show (cf. [6, Part IV; §2]) that they are either $=0$ or $=1$.

Moreover, if $a=\infty$, then the operators V_t, $t\in R^1$, form a group of π-unitary operators. In this case the operator A_0 is π-self-adjoint.

2. Proposition 1. *If $F\in\mathfrak{P}^d_{\varkappa;\,a}$, there exists at least one generalized function $\tilde F\subset\mathfrak{P}^d_{\varkappa;\,\infty}$ which extends F to the whole real axis.*

Proof. We consider the operator A_0 in $\Pi_\varkappa^K(F)$. It admits at least one π-self-adjoint extension $\tilde A$ in $\Pi_\varkappa^K(F)$. Denote by $(\tilde U_t)$, $t\in R^1$, the group of π-unitary operators in $\Pi_\varkappa^K(F)$ generated by $\tilde A$, that is, $\tilde U_t:=\exp(it\tilde A)$ $(t\in R^1)$. Then, if $\varphi\in K_a$, we have $(\tilde U_t\varphi)(s)=\varphi(s-t)$ for sufficiently small $|t|$. In fact, $\tilde U_t$ is a π-unitary extension of the operator V_t.

An extension $\tilde F$ of F can now be defined as follows. Let $\varphi,\psi\in K_\infty$ be such that their supports are contained in closed intervals of length $<2a$, say,

$$\operatorname{supp}\varphi\subset(t'-a,\,t'+a),\quad \operatorname{supp}\psi\subset(t''-a,\,t''+a) \tag{6}$$

for some $t',t''\in R^1$. Then, if $\tilde V_t$ denotes the shift operator in K_∞ defined by

$$(\tilde V_t\varphi)(s):=\varphi(s-t)\quad(\varphi\in K_\infty,\ s,\,t\in R^1),$$

we have $\tilde V_{-t'}\varphi$, $\tilde V_{-t''}\psi\in K_a$. The scalar product $[\cdot,\cdot]_K$ can be extended by the relation

$$[\varphi,\psi]:=[\tilde U_{t'}\tilde V_{-t'}\varphi,\,\tilde U_{t''}\tilde V_{-t''}\psi]_K.$$

It is not hard to see that this definition is correct, that is, on K_a it gives the scalar product already defined, and it is independent of the choice of t' and t'' if only the translations $\tilde V_{-t'}$ and $\tilde V_{-t''}$ map φ and ψ, respectively, into K_a.

If $\varphi, \psi \in K_\infty$ do not satisfy conditions of the form (6), we choose a resolution of the identity (e_j), $j=1, 2, \ldots$, such that $e_j \in K_\infty$, $\operatorname{supp} e_j \subset (t_j - a, t_j + a)$ for some $t_j \in R^1$, $j=1, 2, \ldots$, and $\sum_j e_j(s) = 1$ $(s \in R^1)$. Writing $\varphi = \sum_j \varphi e_j$, $\psi = \sum_j \psi e_j$ and applying the considerations of the last paragraph to the functions $\varphi e_j, \psi e_j$, we define a scalar product on K_∞ by

$$[\varphi, \psi] := [\sum_j \tilde{U}_{t_j} \tilde{V}_{-t_j}(\varphi e_j), \sum_j \tilde{U}_{t_j} \tilde{V}_{-t_j}(\psi e_j)]_K.$$

It is not hard to show that this is a continuous bilinear functional on K_∞ which is invariant under translation. Therefore, according to [9, II, § 3.5], it is of the form $(\tilde{F}, \varphi^\circ * \bar{\psi})$ with some generalized function $\tilde{F} \in \mathfrak{P}^d_{\varkappa; \infty}$ which extends F to the real axis. The proposition is proved.

Remark. It can be shown that there is a one-to-one correspondence between all continuations $\tilde{F} \in \mathfrak{P}^d_{\varkappa; \infty}$ of $F \in \mathfrak{P}^d_{\varkappa; a}$ to the whole real axis and all generalized resolvents of the operator A_0, cf. [6, Part IV].

3. The generalized functions $F \in \mathfrak{P}^d_{\varkappa; a}$ are "conditionally positive definite" in the following sense *): There exists a polynomial p of degree $\leqq \varkappa$ such that

$$\left(p\left(i\frac{d}{dt}\right)\bar{p}\left(i\frac{d}{dt}\right)F, \varphi^\circ * \bar{\varphi}\right) \geqq 0 \quad (\varphi \in K_a), \tag{7}$$

cf. [10]. If p is chosen monic (that is, the coefficient of the term with greatest exponent is 1) and of minimal possible degree, it is unique if and only if the operator A_0 is π-self-adjoint in $\Pi^K_\varkappa(F)$, cf. [6, Part IV, § 2].

The generalized function $\tilde{F} \in \mathfrak{P}^d_{\varkappa; \infty}$ admits an (essentially unique) integral representation by means of a "spectral measure" μ of exponential growth at infinity, see [10]. This representation has the same structure as that appearing in [9, II, § 4, Theorem 3]. We decompose the integral into the sum of two integrals. One of them is taken over a bounded interval containing all singularities of the "spectral measure" μ (in [9, II, § 4, (25)] the only such singularity of μ is at zero, while in general the singularities may appear at eigenvalues of A_0 with nonpositive eigenvectors). The second integral without the "regularizing term" defines a positive definite generalized function, whereas the first integral (over the bounded interval) and those terms which are given by the nonreal spectrum correspond to a generalized function induced by a continuous function. Thus the following proposition has been proved.

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*) In [9] the generalized function F on R^1 is called "conditionally positive definite" if (7) holds with a homogeneous polynomial p.

Proposition 2. *The generalized function* $\tilde{F}\in\mathfrak{P}^d_{\varkappa;\,\infty}$ *can be decomposed as* $\tilde{F}=\tilde{F}_1+\tilde{F}_0$, *where* $\tilde{F}_1\in\mathfrak{P}_{\varkappa;\,\infty}$ *is a continuous function and* $\tilde{F}_0\in\mathfrak{P}^d_{0;\,\infty}$ *is a positive definite generalized function.*

Combining Propositions 1 and 2 we obtain:

Corollary. *The generalized function* $F\in\mathfrak{P}^d_{\varkappa;\,a}$ *can be decomposed as* $F=F_1+F_0$, *where* $F_1\in\mathfrak{P}_{\varkappa;\,a}$ *and* $F_0\in\mathfrak{P}^d_{0;\,a}$.

4. Proof of the Theorem

1. We start with the following lemma (cf. [11, IX. 2]).

Lemma 2. *Let* f_0 *be a function on* $(-2a, 2a)$ *which is locally bounded, measurable and positive definite as a generalized function.**) Then it admits a representation*

(8) $$f_0(t)=\int_{R^1} e^{i\lambda t}\,d\mu_0(\lambda)\quad\text{for almost all}\quad t\in(-2a,2a)$$

with a bounded nonnegative measure μ_0 *on* R^1.

Proof. The positive definite generalized function f_0 has an extension $\tilde{f}_0\in\mathfrak{P}^d_{0;\,\infty}$, see Proposition 1. Then $\tilde{f}_0$ is the Fourier transform of a nonnegative polynomially bounded measure μ_0 on R^1. In particular,

$$(\tilde{f}_0,\varphi)=\int_{-\infty}^{\infty}\phi(\lambda)\,d\mu_0(\lambda)\quad(\varphi\in K_a),$$

where $\phi(\lambda):=\int_{-a}^{a} e^{i\lambda t}\varphi(t)\,dt\ \ (\lambda\in R^1)$.

Let $j\in K_\infty$ be such that $j\geqq 0$, $\int_{-a}^{a} j(t)\,dt=1$, $\operatorname{supp} j\subset(-a,a)$ and for $0<\varepsilon\leqq 1$ define the functions

$$j_\varepsilon(t):=\varepsilon^{-1}\,j(t\varepsilon^{-1})\quad(t\in R^1).$$

Then

$$\int_{-\infty}^{\infty}|\hat{j}_\varepsilon(\lambda)|^2\,d\mu_0(\lambda)=(\tilde{f}_0, j_\varepsilon * j_\varepsilon)\leqq\sup_{t\in\operatorname{supp} j}|f_0(2t)|.$$

On each compact subset of R^1 the functions $\hat{j}_\varepsilon$ tend uniformly to 1 if $\varepsilon\downarrow 0$. Hence

$$\int_{-\infty}^{\infty} d\mu_0(\lambda)\leqq\sup_{t\in\operatorname{supp} j}|f_0(2t)|,$$

**) In [11, IX, 2] a bounded function on R^1 with this property is called *weakly positive definite*.

and, since j can be chosen to have arbitrarily small support, it follows that

$$\int_{-\infty}^{\infty} d\mu_0(\lambda) \leqq \varlimsup_{t\downarrow 0} |f_0(t)|.$$

The lemma is proved.

2. Now let $f\in \mathfrak{P}^m_{\varkappa;\,a}$. Then f can be considered as a generalized function, and since we have

$$[\varphi, \psi]_K = (f, \varphi^\circ * \bar{\psi}) = \int_{-a}^{a}\int_{-a}^{a} f(t-s)\varphi(s)\overline{\psi(t)}\,ds\,dt,$$

according to the results of section 2.2 this scalar product has $\varkappa'$, $0\leqq\varkappa'\leqq\varkappa$, negative squares. By the Corollary to Proposition 2 the generalized function f can be decomposed as

$$f = f_1 + f_0, \tag{9}$$

where $f_1\in\mathfrak{P}_{\varkappa';\,a}$, $f_0\in\mathfrak{P}^d_{0;\,a}$, and the equality holds in the sense of generalized functions. Evidently, $f_0=f-f_1$ can be considered as a locally bounded measurable function. Thus, by Lemma 2, it admits a representation (8) with some bounded measure μ_0. The continuous function

$$f_c(t) := f_1(t) + \int_{-\infty}^{\infty} e^{i\lambda t}\,d\mu_0(\lambda) \quad (|t|<2a) \tag{10}$$

belongs to some class $\mathfrak{P}_{\varkappa'';a}$, $0\leqq\varkappa''\leqq\varkappa$, and the relations (8), (9) and (10) imply

$$f(t) = f_c(t) + f_s(t) \quad (|t|<2a),$$

where $f_s(t)=0$ a.e. on $(-2a, 2a)$. We show that the function f_s is positive definite. To this end we first prove the following

Lemma 3. *Let g be a complex function on $(-2a, 2a)$ such that $g(t)=\overline{g(-t)}$ and $g(t)=0$ a.e. on $(-2a, 2a)$. On the linear set $\mathscr{L}_0$* (see section 2.1) *we consider the scalar product*

$$[u, v] := \sum_{-a<s,t<a} g(t-s)u(s)\overline{v(t)} \quad (u, v\in\mathscr{L}_0).$$

If there exists a $u_0\in\mathscr{L}_0$ such that $[u_0, u_0]<0$ then we can find a set $L_s\subset\mathscr{L}_0$ with the following properties:

a) $\operatorname{card} L_s > \aleph_0$,

b) $[u, u] = -1$, $[u, v]=0$ *if* $u, v\in L_s$, $u\neq v$.

Proof. Let $u_0 = \sum_{j=1}^{n}\alpha_j\varepsilon_{t_j}$, $[u_0, u_0] = -1$. We choose $\delta>0$ so that $t_j\pm\delta\in(-a, a)$

for $j=1, 2, \ldots, n$. Then

$$[V_\sigma u_0,\ V_\sigma u_0] = [u_0,\ u_0] = -1 \quad \text{for all } |\sigma| < \delta.$$

The set $\Delta_0 := \{s\colon |s|<2a,\ g(s)=0\}$ has Lebesgue measure $\lambda(\Delta_0)=2a$.

We denote by G the family of all nonempty sets Γ such that $\Gamma\subset(-\delta, \delta)$, and the relations $\sigma, \tau\in\Gamma$, $\sigma\neq\tau$ imply $\sigma-\tau+(t_j-t_k)\in\Delta_0$ for $j, k=1, 2, \ldots, n$. Then G is not empty. Indeed, for arbitrary σ, $|\sigma|<\delta$, define

$$\Delta_{jk}(\sigma) := (\sigma+(t_j-t_k)-\Delta_0)\cap(-\delta, \delta), \quad j, k = 1, 2, \ldots, n.$$

Then $\lambda(\Delta_{jk}(\sigma))=2\delta$, which implies $\lambda\left(\bigcap\limits_{j,k=1}^{n} \Delta_{jk}(\sigma)\right)=2\delta$, hence $\bigcap\limits_{j,k=1}^{n} \Delta_{jk}(\sigma)\neq\emptyset$. For an arbitrary $\tau\neq\sigma$ which belongs to this intersection we have $\{\sigma, \tau\}\in G$.

The family G is partially ordered by inclusion, and each of its totally ordered subfamilies has an upper bound. We show that the maximal elements of G are not countable. Indeed, assume that a maximal element $\Gamma_{\max}$ of G is countable: $\Gamma_{\max}=\{s_\nu\colon \nu=1, 2, \ldots\}$. Consider the set $\Delta := \bigcap\limits_{j,k,\nu} \Delta_{jk}(s_\nu)$. Then we have again $\lambda(\Delta)=2\delta$, and therefore $\Delta\setminus\Gamma_{\max}\neq\emptyset$. If $\tau\in\Delta\setminus\Gamma_{\max}$, then $\Gamma_{\max}\cup\{\tau\}\in G$, which contradicts the maximality of $\Gamma_{\max}$.

Now let $\Gamma_0\in G$, $\operatorname{card}\Gamma_0>\aleph_0$ and put $L_s := \{V_\sigma u_0\colon \sigma\in\Gamma_0\}$. Then we have

$$[V_\sigma u_0,\ V_\sigma u_0] = [u_0,\ u_0] = -1,$$

$$[V_\tau u_0, V_\sigma u_0] = \sum_{j,k=1}^{n} \alpha_j\overline{\alpha_k}g(\sigma-\tau+t_j-t_k) = 0 \quad \text{if} \quad \sigma\neq\tau.$$

The lemma is proved.

Now we show that f_s is positive definite. Assuming the contrary we find a subset L_s of $\mathscr{L}_0$ with the properties a), b) of Lemma 3 for $g=f_s$. Denote the elements of L_s by u_γ, $\gamma\in\Gamma_0$. The space $\Pi_{\varkappa''}(f_c)$ is separable. Hence there exists a countable subset Γ_1 of Γ_0 such that the elements u_γ, $\gamma\in\Gamma_1$, form a total set in the subspace of $\Pi_{\varkappa''}(f_c)$ generated by u_γ, $\gamma\in\Gamma_0$. Choose $n>\varkappa$, and mutually different elements $u^1, \ldots, u^n\in L_s$ which do not belong to $\{u_\gamma\colon \gamma\in\Gamma_1\}$. Then, if $\|\cdot\|_c$ denotes a Hilbert norm on $\Pi_{\varkappa''}(f_c)$ which corresponds to some fundamental decomposition, then to each u^j there exists a finite sum $\sum\limits_{\gamma\in\Gamma_1}\xi_\gamma^{(j)}u_\gamma$ such that for $y^j := u^j-\sum\limits_{\gamma\in\Gamma_1}\xi_\gamma^{(j)}u_\gamma$ we have

$$\|y^j\|_c^2 < \frac{1}{2n^2}, \quad j = 1, 2, \ldots, n.$$

On the other hand, denoting by $[\cdot, \cdot]_s$ the scalar product on $\mathscr{L}_0$ corresponding to f_s, we find

$$[y^j, y^k]_s = -\delta_{jk} - \sum_{\gamma\in\Gamma_1} \xi_\gamma^{(j)}\,\overline{\xi_\gamma^{(k)}}, \quad j, k = 1, 2, \ldots, n.$$

Hence, for arbitrary complex numbers $\eta_1, \dots, \eta_n$ it follows that

$$\|\eta_j y^j\|_c^2 \leqq \left(\sum_{j=1}^{n} |\eta_j| \|y^j\|_c\right)^2 \leqq \frac{1}{2n} \sum_{j=1}^{n} |\eta_j|^2,$$

$$\left[\sum_{j=1}^{n} \eta_j y^j, \sum_{j=1}^{n} \eta_j y^j\right]_s = -\sum_{j=1}^{n} |\eta_j|^2 - \sum_{\gamma \in \Gamma_1} \left|\sum_{j=1}^{n} \xi_\gamma^{(j)} \eta_j\right|^2 \leqq -\sum_{j=1}^{n} |\eta_j|^2,$$

and we get finally

$$\left[\sum_{j=1}^{n} \eta_j y^j, \sum_{j=1}^{n} \eta_j y^j\right] \leqq \sum_{j=1}^{n} |\eta_j|^2 \left(-1 + \frac{1}{2n}\right).$$

However, this is impossible, since the scalar product on the left hand side has at most $\varkappa$ negative squares on $\mathscr{L}_0$. This contradiction implies that f_s is positive definite, and the Theorem is proved.

3. The decomposition (1) can be written in a more geometric form. To this end we first observe that in (3) the right hand side can be replaced by

$$\int_{-a}^{a} \int_{-a}^{a} f_c(t-s) \varphi(s) \overline{\psi(t)}\, ds\, dt,$$

and the space $\Pi^C(f)$ can be identified with $\Pi_\varkappa(f_c)$. Therefore it is also a $\pi_\varkappa$-space and we shall write $\Pi_\varkappa^C(f)$ instead of $\Pi^C(f)$. As a nondegenerate subspace of $\Pi_\varkappa(f)$, it is the range of a π-orthogonal projector P in $\Pi_\varkappa(f)$, and we have a decomposition

$$\Pi_\varkappa(f) = \Pi_\varkappa^C(f) \oplus \Pi_0(f),$$

where $\Pi_0(f)$ is a Hilbert space with respect to the scalar product $[\cdot, \cdot]$.

Further, if $\varphi \in C_a$, then (4) yields

$$[\varepsilon_s, \varphi] = \int_{-a}^{a} f(t-s) \overline{\varphi(t)}\, dt = \int_{-a}^{a} f_c(t-s) \overline{\varphi(t)}\, dt \quad (|s| < a),$$

and if $(\delta_t^{(n)})$, $n=1, 2, \dots$, is a δ_t-sequence of elements of C_a, then we find

$$[\varepsilon_s, \delta_t^{(n)}] \to f_c(t-s) \quad (n \to \infty;\ |s|, |t| < a). \tag{11}$$

Moreover, for arbitrary $\psi \in C_a$ we have

$$[\psi, \delta_t^{(n)}] \to [\psi, \varepsilon_t] \quad (n \to \infty).$$

This relation implies $\delta_t^{(n)} \to P\varepsilon_t$ $(n \to \infty)$ in the weak topology of $\Pi_\varkappa^C(f)$ or $\Pi_\varkappa(f)$, and from (11) we get finally that

$$[\varepsilon_s, P\varepsilon_t] = f_c(t-s) \quad (|s|, |t| < a).$$

Thus the decomposition (1) can be written as

$$f(2t) = [P\varepsilon_{-t}, \varepsilon_t] + [(I-P)\varepsilon_{-t}, \varepsilon_t] \quad (|t| < a).$$

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On the strong and extra strong approximation of orthogonal series

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In honour of Professor Béla Szőkefalvi-Nagy on his seventieth birthday

1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We consider the orthogonal series

$$\sum_{n=1}^{\infty} c_n\varphi_n(x) \quad \text{with} \quad \sum_{n=1}^{\infty} c_n^2 < \infty. \tag{1}$$

By the Riesz—Fischer theorem the series (1) converges in L^2 to a square-integrable function f. Let us denote the partial sums of (1) by $s_n(x)$.

In [1] the first author proved that *if* $0<\gamma<1$ *and*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty \tag{2}$$

then

$$\frac{1}{n}\sum_{k=1}^{n}\big(s_k(x)-f(x)\big) = o_x(n^{-\gamma})$$

almost everywhere (a.e.) in (a, b).

G. Sunouchi [8] generalized this results to strong approximation, and his result was generalized by one of us ([2]) to very strong approximation as follows:

Theorem A. *Suppose that* $\alpha>0, 0<\gamma<1, 0<p<\gamma^{-1}$, *and that* (2) *is satisfied. Then*

$$C_n(f, \alpha, p, \{m_k\};\ x) := \left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{m_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{3}$$

holds a.e. for any increasing sequence $\{m_k\}$, *where* $A_n^\alpha=\binom{n+\alpha}{n}$.

This theorem with $m_k=k$ reduces to that of Sunouchi.

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Recently the first author [3] showed that in the special case $\alpha=1$ the restriction $\gamma<1$ can be omitted, i.e. *if* $\gamma>0$ *and* $0<p<\gamma^{-1}$, *then* (2) *implies that*

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_{m_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{4}$$

holds a.e. in (a, b) *for any increasing sequence* $\{m_k\}$.

In the present work, among others, we prove that the restriction $\gamma<1$ from the assumptions of Theorem A can be omitted for any $\alpha>0$ and not only for $\alpha=1$ alone.

Namely we have

Theorem 1. *If* α *and* γ *are positive numbers and* $0<p\gamma<1$ *then condition* (2) *implies that* (3) *holds a.e. in* (a, b) *for any increasing sequence* $\{m_k\}$.

We mention that Theorem 3 of [6] made a moderate step towards this result, namely it states that (3) holds for any positive γ if $\alpha>p\gamma$.

Two further generalizations of (4) were given in the papers [4] and [5], from them we can unify the following

Theorem B. *Suppose that* $\gamma>0$, $0<p\gamma<\beta$, *and that* (2) *holds. Moreover if*
(i) $\beta\leqq 2$ *or* $\beta>2$ *but at least either* $\gamma<1$ *or* $p\leqq 2$;

(ii) $p\geqq 2$ *and* $\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-2/p}<\infty$;

then

$$h_n(f, \beta, p, \{m_k\};\ x) := \left\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{m_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{5}$$

holds a.e. in (a, b) *for any increasing sequence* $\{m_k\}$.

To help the lucidity of fulfilment of the assumptions we define certain ranges of the positive parameters p and γ. Let us denote by $A(\beta)$ the range of the positive parameters p and γ determined by the condition $p\gamma<\beta$, i.e.

$$A(\beta) := \{p, \gamma\,|\,p>0,\quad \gamma>0\quad\text{and}\quad p\gamma<\beta\},$$

moreover let

$$B(\beta) := \{p, \gamma\,|\,p>2,\quad \gamma\geqq 1\quad\text{and}\quad p\gamma<\beta\}.$$

Theorem B shows that if $(p, \gamma)\in A(\beta)\setminus B(\beta)$ then (2) implies (5), but if $(p, \gamma)\in$ $\in B(\beta)$ then we can only prove (5) under an additional condition.

This phenomenon is curious, and we have had the conjecture (see [4]) that condition (2) implies (5) for any $(p, \gamma)\in A(\beta)$. Now we shall verify this conjecture, namely we prove

Theorem 2. *If* $\gamma>0$ *and* $0<p<\beta$ *then condition* (2) *implies* (5) *a.e. in* $(a; b)$ *for any increasing sequence* $\{m_k\}$.

In connection with the extra strong approximation we shall improve the following theorems given in [3] and [6].

Theorem C. *Suppose that* $\gamma>0, 0<p<\gamma^{-1}$ *and* $p\leqq 2$, *that* $\alpha>p \max(1/2, \gamma)$; *or if* $p=2$ *then* $\alpha\geqq 1$; *moreover that* (2) *holds. Then*

$$C_n(f, \alpha, p, \{\mu_k\};\ x) := \left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{\mu_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{6}$$

holds a.e. in (a, b) *for any (not necessarily monotone) sequence* $\{\mu_k\}$ *of distinct positive integers.*

Theorem D. *Suppose that* $\gamma>0, p\geqq 2$, *and that* $p\gamma<\min(\alpha, 1)$. *Then*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-(2/p)\min(\alpha,1)} < \infty \tag{7}$$

implies (6) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

The next two theorems are certain analogues of Theorems C and D with the means $h_n(f, \beta, p, \{\mu_k\}; x)$.

Theorem C'. *Suppose that* $\gamma>0, 0<p\leqq 2$ *and* $p\gamma<\min(\beta, 1)$, *moreover that* (2) *holds. Then*

$$h_n(f, \beta, p, \{\mu_k\};\ x) = o_x(n^{-\gamma}) \tag{8}$$

holds a.e. in (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

We mention that this theorem is a collected form of Theorem 1 and Proposition A of [6].

Theorem D'. *Suppose that* $\gamma>0, p\geqq 2$, *and that* $p\gamma<\min(\beta, 1)$. *Then*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-2/p} < \infty \tag{9}$$

implies (8) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

Our two new theorems including these results read as follows:

Theorem 3. *If* $\gamma>0$ *and* $0<p\gamma<\min(\alpha, 1)$ *then* (2) *implies* (6) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

Theorem 4. *If $\gamma>0$ and $0<p\gamma<\min(\beta, 1)$ then* (2) *implies* (8) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

2. In order to prove the theorems we require some lemmas.

Lemma 1 ([2], Lemma 5). *Let* $\{\lambda_n\}$ *be a monotone sequence of positive numbers such that*

$$\sum_{n=1}^{m} \lambda_{2^n}^2 \leqq K\lambda_{2^m}^2.\text{*)}$$

Then the condition

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^2 < \infty$$

implies that

$$s_{2^n}(x) - f(x) = o_x(\lambda_{2^n}^{-1})$$

holds a.e. in (a, b).

Lemma 2 ([7], Lemma 2). *If* $\sum_{n=0}^{\infty} c_n^2 < \infty$ *then for any positive* α *and* p

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} |s_k(x) - \sigma_k(x)|^p \right)^{1/p} \right\}^2 dx \leqq A(\alpha, p) \sum_{n=0}^{\infty} c_n^2,$$

where $\sigma_k(x) := (k+1)^{-1} \sum_{i=0}^{k} s_i(x)$.

Lemma 3 ([5], Lemma 3). *Let* $\varkappa>0$ *and* $\{\lambda_n\}$ *be an arbitrary sequence of positive numbers. Assuming that the condition*

$$\sum_{n=1}^{\infty} \lambda_n \left\{ \sum_{k=n}^{\infty} c_k^2 \right\}^{\varkappa} < \infty \tag{2.1}$$

implies a "certain property $T = T(\{s_n(x)\})$" *of the partial sums* $s_n(x)$ *of* (1) *for any orthonormal system, then* (2.1) *implies that the partial sums* $s_{m_k}(x)$ *of* (1) *also have the same property* T *for any increasing sequence* $\{m_k\}$, *i.e.*

if (2.1)$\Rightarrow T(\{s_n(x)\})$ *then* (2.1)$\Rightarrow T(\{s_{m_k}(x)\})$ *for any increasing sequence* $\{m_k\}$.

Lemma 4. *We have for any positive* p *and* $m\geqq 1$

$$\int_a^b \left\{ \frac{1}{2^m} \sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p \right\}^{2/p} dx \leqq K(p) \sum_{n=2^m+1}^{2^{m+1}} c_n^2, \tag{2.2}$$

*) $K, K_1, K_2, \ldots$ denote positive constants not necessarily the same at each occurrence.

where

$$\sigma_n^*(x)=\begin{cases} c_0\varphi_0(x) & \textit{if} \quad n=0, \\ \dfrac{1}{n-2^{m-1}}\sum\limits_{k=2^m}^{n}\big(s_k(x)-s_{2^m}(x)\big) & \textit{if} \quad 2^m \leqq n < 2^{m+1};\ m=0,1,\ldots. \end{cases}$$

Proof. Using Lemma 2 with $\alpha=1$ for the following partial sums and $(C, 1)$-means

$$s_n'(x) := \begin{cases} 0 & \text{if} \quad 0\leqq n\leqq 2^{m-1}, \\ s_{n+2^{m-1}}(x)-s_{2^m}(x) & \text{if} \quad 2^{m-1}<n<2^{m+1}-2^{m-1}; \end{cases} \tag{2.3}$$

and

$$\sigma_n'(x) := \frac{1}{n}\sum_{k=1}^{n} s_k'(x) = \begin{cases} 0 & \text{if} \quad 0\leqq n\leqq 2^{m-1}, \\ \sigma^*_{n+2^{m-1}}(x) & \text{if} \quad 2^{m-1}<n<2^{m+1}-2^{m-1}, \end{cases} \tag{2.4}$$

where m is an arbitrary fixed natural number, we obtain (2.2) immediately, which completes the proof.

Lemma 5. *Let* $\gamma>0$, *and* $p\geqq 2$. *Then under condition* (2) *we have that the sum*

$$\tau_1(x) := \sum_{m=1}^{\infty}\sum_{k=2^m}^{2^{m+1}-1}(k+1)^{p\gamma-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p$$

is finite a.e. in (a, b).

Proof. By $p\geqq 2$ and Lemma 4 we have that

$$\int_a^b (\tau_1(x))^{2/p}\,dx \leqq K_1\int_a^b \sum_{m=0}^{\infty} 2^{m2\gamma}\Big\{2^{-m}\sum_{k=2^m}^{2^{m+1}-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p\Big\}^{2/p} dx \leqq$$

$$\leqq K_2\sum_{m=0}^{\infty}2^{2m\gamma}\cdot\sum_{k=2^m+1}^{2^{m+1}} c_k^2 \leqq K_2\sum_{n=1}^{\infty}c_n^2 n^{2\gamma}<\infty,$$

whence by B. Levi's theorem the statement of Lemma 5 follows.

Lemma 6. *Let* $\gamma>0$ *and* $p\geqq 2$. *Then condition* (2) *implies that*

$$\tau_2(x) := \sum_{k=1}^{\infty}k^{p\gamma-1}|\sigma_k^*(x)|^p<\infty$$

a.e. in (a, b).

Proof. An elementary consideration shows that

$$\int_a^b (\tau_2(x))^{2/p}\,dx \leqq K\int_a^b\sum_{m=0}^{\infty}2^{2m\gamma}\Big\{2^{-m}\sum_{k=2^m}^{2^{m+1}-1}|\sigma_k^*(x)|^p\Big\}^{2/p} dx \leqq \tag{2.5}$$

$$\leqq K\int_a^b\sum_{m=0}^{\infty}2^{2m\gamma}\big\{\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)|^2\big\}\,dx.$$

If $2^m<k<2^{m+1}$ then

$$\sigma_k^*(x)=\frac{1}{k-2^{m-1}}\sum_{i=2^m+1}^{k}(k+1-i)\,c_i\varphi_i(x)=\sum_{i=2^m+1}^{k}\left(1-\frac{i-2^{m-1}-1}{k-2^{m-1}}\right)c_i\varphi_i(x)$$

and $\sigma_{2^m}^*(x)=0$, so using the following simple estimation

$$\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)|^2=\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)-\sigma_{2^m}^*(x)|^2\leqq$$

$$\leqq\left(\sum_{k=2^m+1}^{2^{m+1}-1}|\sigma_k^*(x)-\sigma_{k-1}^*(x)|\right)^2\leqq 2^m\sum_{k=2^m+1}^{2^{m+1}-1}|\sigma_k^*(x)-\sigma_{k-1}^*(x)|^2,$$

we obtain that

$$\text{(2.6)}\qquad \int_a^b\left\{\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)|^2\right\}dx\leqq\frac{2^m}{2^{4(m-1)}}\sum_{k=2^m+1}^{2^{m+1}-1}\sum_{i=2^m+1}^{k}(i-2^{m-1}-1)^2c_i^2\leqq$$

$$\leqq 2^6\sum_{i=2^m+1}^{2^{m+1}-1}c_i^2.$$

Hence, by (2.5), we get that

$$\int_a^b(\tau_2(x))^{2/p}\,dx\leqq K_1\sum_{m=0}^{\infty}2^{2m\gamma}\sum_{n=2^m+1}^{2^{m+1}}c_n^2\leqq K_1\sum_{n=1}^{\infty}c_n^2n^{2\gamma}<\infty,$$

and this proves Lemma 6.

Lemma 7. *Condition* (2) *with any positive* γ *implies that*

$$\text{(2.7)}\qquad \sigma_n^*(x)=o_x(n^{-\gamma})$$

holds a.e. in (a,b).

Proof. Using estimation (2.6) we immediately obtain that

$$\int_a^b\sum_{m=0}^{\infty}(2^{m\gamma}\max_{2^m<n<2^{m+1}}|\sigma_n^*(x)|)^2\,dx\leqq K_2\sum_{n=1}^{\infty}c_n^2n^{2\gamma}<\infty,$$

whence (2.7) follows, which ends the proof.

Lemma 8. *Let* $\gamma>0,\ p\geqq 2$ *and* $p\gamma<1$. *For a given sequence* $\{\mu_k\}$ *of distinct positive integers we define another sequence* $\{m_k\}$ *as follows*: $m_k=2^m$ *if* $2^m\leqq\mu_k<2^{m+1}$. *Then* (2) *implies that the sum*

$$\mu_1(x):=\sum_{k=0}^{\infty}(k+1)^{p\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p$$

is finite a.e. in (a,b).

Proof. Choosing q such that $1<q<(1-p\gamma)^{-1}$ and applying Hölder's inequality with this q and $q'=q/(q-1)$ we obtain that

$$\mu_1(x)=\sum_{m=0}^{\infty}\sum_{2^m\le\mu_k<2^{m+1}}(k+1)^{p\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma^*_{\mu_k}(x)|^p\le$$

$$\le\sum_{m=0}^{\infty}\Big\{\sum_{2^m\le\mu_k<2^{m+1}}(k+1)^{(p\gamma-1)q}\Big\}^{1/q}\Big\{\sum_{2^m\le\mu_k<2^{m+1}}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma^*_{\mu_k}(x)|^{pq'}\Big\}^{1/q'}\le$$

$$\le K\sum_{m=0}^{\infty}\Big\{\sum_{k=1}^{2^m}k^{(p\gamma-1)q}\Big\}^{1/q}\Big\{\sum_{i=2^m}^{2^{m+1}-1}|s_i(x)-s_{2^m}(x)-\sigma^*_i(x)|^{pq'}\Big\}^{1/q'}\le$$

$$\le K_1\sum_{m=0}^{\infty}2^{m(p\gamma-1/q')}\Big\{\sum_{i=2^m}^{2^{m+1}-1}|s_i(x)-s_{2^m}(x)-\sigma^*_i(x)|^{pq'}\Big\}^{1/q'}.$$

Hence, by Lemma 4 and $p\ge 2$, we get that

$$\int_a^b(\mu_1(x))^{2/p}\,dx\le K_2\sum_{m=0}^{\infty}2^{m2\gamma}\sum_{n=2^m+1}^{2^{m+1}}c_n^2<\infty,$$

which proves Lemma 8.

Lemma 9. *Let $\gamma>0$, $p\ge 2$ and $p\gamma<1$. Then, for any given sequence $\{\mu_k\}$ of distinct positive integers, the sum*

$$\mu_2(x):=\sum_{k=0}^{\infty}(k+1)^{p\gamma+1}|\sigma^*_{\mu_k}(x)|^p$$

is finite a.e. in (a,b) if (2) holds.

Proof. In a similar way as in the proof of Lemma 8 we obtain with Hölder's inequality $(1<q<(1-p\gamma)^{-1}$ and $1/q+1/q'=1)$ that

$$\int_a^b(\mu_2(x))^{2/p}\,dx\le\int_a^b\sum_{m=0}^{\infty}\Big\{\sum_{2^m\le\mu_k<2^{m+1}}(k+1)^{p\gamma-1}|\sigma^*_{\mu_k}(x)|^p\Big\}^{2/p}dx\le$$

$$\le\int_a^b\sum_{m=0}^{\infty}2^{m(p\gamma-1/q')(2/p)}\Big\{\sum_{i=2^m}^{2^{m+1}-1}|\sigma^*_i(x)|^{pq'}\Big\}^{2/pq'}dx\le$$

$$\le\sum_{m=0}^{\infty}2^{2m\gamma}\int_a^b\Big\{\frac{1}{2^m}\sum_{k=2^m}^{2^{m+1}-1}|\sigma^*_k(x)|^{pq'}\Big\}^{2/pq'}dx.$$

From this step we can continue the proof as in Lemma 6, and so we obtain the conclusion.

3. Proof of Theorem 1. Putting

$$C_n(x):=C_n(f,\alpha,p,\{k\};x),$$

and if $2^m \leqq n < 2^{m+1}$ $(m \geqq 2)$ holds, then

$$C_n(x) \leqq K\left(\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{2^m-1} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}+\right. \tag{3.1}$$

$$\left.+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}\right) =: K(C_n^{(1)}(x)+C_n^{(2)}(x)).$$

Here the first term $C_n^{(1)}(x)$, by (4); has the order $o_x(n^{-\gamma})$, namely it is known that for any $\beta>-1$, $0<K_1<\frac{A_n^\beta}{n^\beta}<K_2$.

Next we estimate $C_n^{(2)}(x)$ as follows:

$$C_n^{(2)}(x) \leqq K\left(\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^p\right\}^{1/p}+\right. \tag{3.2}$$

$$+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_{2^{m-1}}(x)-f(x)|^p\right\}^{1/p}+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p\right\}^{1/p}+$$

$$\left.+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_{2^m}(x)-f(x)|^p\right\}^{1/p}+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|\sigma_k^*(x)|^p\right\}^{1/p}\right) =: K\sum_{i=1}^{5} D_n^{(i)}(x).$$

An easy consideration shows in view of Lemma 1 and Lemma 7 that

$$D_n^{(2)}(x)+D_n^{(4)}(x)+D_n^{(5)}(x) = o_x(n^{-\gamma}). \tag{3.3}$$

To estimate $D_n^{(1)}(x)$ and $D_n^{(3)}(x)$ we use again Hölder's inequality with such a q to be chosen so that $q>1$ and $(\alpha-1)q>-1$. Then

$$D_n^{(1)}(x) \leqq \frac{1}{(A_n^\alpha)^{1/p}}\left\{\sum_{k=2^{m-1}+1}^{2^m-1}(A_{n-k}^{\alpha-1})^q\right\}^{1/pq}\left\{\sum_{k=2^{m-1}+1}^{2^m-1}|s_k-s_{2^{m-1}}-\sigma_k^*|^{pq'}\right\}^{1/pq'} \leqq$$

$$\leqq K\left\{\frac{1}{2^m}\sum_{k=2^{m-1}}^{2^m-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^{pq'}\right\}^{1/pq'} =: D_m^*(x),$$

whence by Lemma 4 we obtain that

$$\sum_{m=1}^{\infty}\int_a^b (2^{m\gamma}D_m^*(x))^2\,dx \leqq K_1\sum_{m=1}^{\infty}2^{2m\gamma}\sum_{n=2^{m-1}+1}^{2^m}c_n^2<\infty, \tag{3.4}$$

which implies that

$$D_n^{(1)}(x) = o_x(n^{-\gamma}) \tag{3.5}$$

also holds a.e. in (a, b).

Similarly

$$D_n^{(3)}(x) \leqq \frac{1}{(A_n^\alpha)^{1/p}} \left\{ \sum_{k=2^m}^{n} (A_{n-k}^{\alpha-1})^q \right\}^{1/pq} \left\{ \sum_{k=2^m}^{n} |s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^{pq'} \right\}^{1/pq'} \leqq$$

$$\leqq K\left\{ \frac{1}{2^{m+1}} \sum_{k=2^m}^{2^{m+1}-1} |s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^{pq'} \right\}^{1/pq'} = D_{m+1}^*(x),$$

and so

(3.6) $$D_n^{(3)}(x) = o_x(n^{-\gamma})$$

also holds a.e. in (a, b) by (3.4).

Collecting the estimates given under (3.1), (3.2), (3.3), (3.5) and (3.6) we obtain that

$$C_n(f, \alpha, p, \{k\}; x) = o_x(n^{-\gamma})$$

a.e. in (a, b). Hence, using Lemma 3 with $\varkappa=1$, $\lambda_n=n^{2\gamma-1}$ and $T(\{s_n(x)\}):=$ $:=C_n(f, \alpha, p, \{k\}; x)=o(n^{-\gamma})$, the statement of Theorem 1 follows obviously.

The proof is complete.

Proof of Theorem 2. Denote

$$h_n(x) := h_n(f, \beta, p, \{k\}; x).$$

By Theorem B we can assume that $p>2$, namely otherwise (5) holds. Then with $2^m \leqq n < 2^{m+1}$

(3.7) $$h_n(x) \leqq K\left\{ n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1} |s_k(x)-f(x)|^p \right\}^{1/p} \leqq$$

$$\leqq K_1 \left(\left\{ n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1} |s_k(x)-s_{2^\nu}(x)-\sigma_k^*(x)|^p \right\}^{1/p} + \right.$$

$$\left. + \left\{ n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1} |s_{2^\nu}(x)-f(x)|^p \right\}^{1/p} + \left\{ n^{-\beta} \sum_{k=1}^{2^{m+1}} k^{\beta-1} |\sigma_k^*(x)|^p \right\}^{1/p} \right) =$$

$$=: K_1 \sum_{i=1}^{3} d_n^{(i)}(x).$$

By Lemma 1 and $\beta > p\gamma$ it is easy to show that $d_n^{(2)}(x)=o_x(n^{-\gamma})$, namely

(3.8) $$d_n^{(2)}(x) = \left\{ n^{-\beta} \sum_{\nu=0}^{m} 2^{\nu\beta} o_x(2^{-\nu\gamma p}) \right\}^{1/p} = o_x(n^{-\gamma}).$$

But if we observe that Lemma 5 and Lemma 6 imply that, as $m \to \infty$,

$$2^{m(p\gamma-1)} \sum_{k=2^m}^{2^{m+1}} |s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p = o_x(1)$$

and

$$2^{m(p\gamma-1)} \sum_{k=2^m}^{2^{m+1}} |\sigma_k^*(x)|^p = o_x(1)$$

hold a.e. in (a, b), then by the use of these estimates we can easily verify that

$$d_n^{(1)}(x)+d_n^{(3)}(x) = o_x(n^{-\gamma})$$

also holds a.e. in (a, b).

Indeed, by $\beta > p\gamma$, we have that

$$d_n^{(1)}(x) = \left\{n^{-\beta} \sum_{\nu=0}^{m} 2^{\nu(\beta-1)} o_x(2^{\nu(1-p\gamma)})\right\}^{1/p} = o_x(n^{-\gamma})$$

and similarly

$$d_n^{(3)}(x) \leqq \left\{n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}} k^{\beta-1} |\sigma_k^*(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}).$$

Summing up our partial estimations we get that

$$h_n(f, \alpha, p, \{k\}; x) = o_x(n^{-\gamma}),$$

whence Lemma 3, as in the proof of Theorem 1, conveys the assertion of Theorem 2.

Proof of Theorem 3. At first we prove the special case $\alpha=1$. Then, for $0<p\leqq 2$, Theorem C gives (6), so we assume that $p>2$. Next $\{m_k\}$ denotes the sequence defined in Lemma 8. Using this notation we have

$$\left\{\frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x)-f(x)|^p\right\}^{1/p} \leqq K\left(\left\{\frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p\right\}^{1/p} +\right.$$

$$(3.9) \qquad \left. +\left\{\frac{1}{n+1} \sum_{k=0}^{n} |\sigma_{\mu_k}^*(x)|^p\right\}^{1/p} + \left\{\frac{1}{n+1} \sum_{k=0}^{n} |s_{m_k}(x)-f(x)|^p\right\}^{1/p}\right) =$$

$$=: K\big(\mu_n^{(1)}(x)+\mu_n^{(2)}(x)+\mu_n^{(3)}(x)\big).$$

Lemma 8 and Lemma 9 prove that

$$\mu_n^{(1)}(x) = o_x(n^{-\gamma}) \quad \text{and} \quad \mu_n^{(2)}(x) = o_x(n^{-\gamma})$$

a.e. in (a, b).

To prove the same estimation for $\mu_n^{(3)}(x)$ we define a new sequence $\{N_n(m)\}$. Let $N_n(m)$ denote the number of μ_k lying in the interval $[2^m, 2^{m+1})$ and $k \leqq n+1$. It is obvious that

$$N_n(m) \leqq \min(n+1, 2^m) \quad \text{and} \quad \sum_{m=0}^{\infty} N_n(m) = n+1.$$

If $2^{l-1} \leqq n < 2^l$, then we obtain with the aid of Lemma 1 and $p\gamma < 1$ that

$$(\mu_n^{(3)}(x))^p = \frac{1}{n+1} \sum_{k=0}^{n} o_x(m_k^{-\gamma p}) = \frac{1}{n+1} \sum_{m=0}^{\infty} N_n(m) o_x(2^{-mp\gamma}) =$$

$$= \frac{1}{n+1} \left\{ \sum_{m=0}^{l-1} 2^m o_x(2^{-mp\gamma}) + \sum_{m=l}^{\infty} (n+1) o_x(2^{-mp\gamma}) \right\} = o_x(2^{-lp\gamma}) = o_x(n^{-p\gamma}),$$

which proves

$$\mu_n^{(3)}(x) = o_x(n^{-\gamma}),$$

and thus by (3.9)

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x) - f(x)|^p \right\}^{1/p} = o_x(n^{-\gamma}) \tag{3.10}$$

holds a.e. in (a, b).

If $\alpha > 1$ then (6) is an immediate consequence of (3.10) because of the relation $A_{n-k}^{\alpha-1}/A_n^{\alpha} = O\left(\frac{1}{n}\right)$ $(0 \leqq k \leqq n)$.

If $0 < \alpha < 1$ we can choose q such that $p\gamma < \frac{1}{q} < \alpha$. Then with $q' = \frac{q}{q-1}$ the inequality $(\alpha - 1)q' > -1$ is fulfilled.

Now using Hölder's inequality we obtain that

$$C_n(f, \alpha, p, \{\mu_k\}; x) \leqq \left\{ \frac{1}{(A_n^{\alpha})^{q'}} \sum_{k=0}^{n} (A_{n-k}^{\alpha-1})^{q'} \right\}^{1/pq'} \left\{ \sum_{k=0}^{n} |s_{\mu_k}(x) - f(x)|^{pq} \right\}^{1/pq} \leqq$$

$$\leqq K \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x) - f(x)|^{pq} \right\}^{1/pq}.$$

Hence, by $pq\gamma < 1$, using (3.10) we get the assertion of Theorem 3.

Proof of Theorem 4. The case $\beta = 1$ is identical with the special case $\alpha = 1$ of Theorem 3. The cases $\beta > 1$ and $0 < \beta < 1$ may be proved similarly to the cases $\alpha > 1$ and $0 < \alpha < 1$ above, choosing q such that $p\gamma < \frac{1}{q} < \beta$ for $\beta < 1$. We omit the proof.

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Представление функций в выпуклых областях обобщенными рядами экспонент

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Пусть D — конечная выпуклая область. Известно (см. [1]), что любую функцию $F(z)$, аналитическую в D, можно представить рядом экспонент

$$F(z)=\sum_{n=1}^{\infty} A_n e^{\lambda_n z}, \quad z\in D.$$

Показатели λ_n зависят лишь от области D, имеют конечную верхнюю плотность, ряд в области D сходится абсолютно, а внутри — равномерно.

Здесь будет приведен класс A функций $f(z)$ экспоненциального типа, обладающих свойством: любая функция $\Phi(z)$, аналитическая в выпуклой области D $(0\in D)$, представляется в D равномерно сходящимся внутри D рядом

$$\Phi(z)=\sum_{n=1}^{\infty} B_n f(\lambda_n z), \quad z\in D.$$

1. Определение класса A. По определению функции $f(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n$ принадлежит классу A, если $a_n\neq 0$ $(n\geqq 0)$ и функции

$$\eta(t)=\sum_{n=0}^{\infty}\frac{a_n}{t^{n+1}}, \quad \eta_1(t)=\sum_{n=0}^{\infty}\frac{1}{a_n t^{n+1}}$$

регулярны вне отрезка $[0,1]$ вещественной оси.

Классу A принадлежит функция e^z, в этом случае

$$\eta(t)=\eta_1(t)=\sum_{n=0}^{\infty}\frac{1}{t^{n+1}}=\frac{1}{t-1}.$$

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Имеем

$$\varlimsup_{n\to\infty} \sqrt[n]{|a_n|} \leqq 1, \quad \varlimsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{a_n}\right|} \leqq 1,$$

откуда следует, что существует $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}=1$. Следовательно, $f(z)$ — целая функция экспоненциального типа равного единице.

Отметим, что $\eta(t)$ — функция, ассоциированная по Борелю с $f(z)$.

2. Достаточный признак принадлежности функции классу A.

Теорема 1. *Пусть* $a_n=\varphi(n)$, $\frac{1}{a_n}=\varphi_1(n)$, $n\geqq N$, *где* $\varphi(z)$ *и* $\varphi_1(z)$ — *функции, аналитические в полуплоскости* $\operatorname{Re} z\geqq N$ и

(1) $$|\varphi(z)|<e^{\varepsilon|z|},\ |\varphi_1(z)|<e^{\varepsilon|z|},\ \operatorname{Re} z \geqq N,\ |z|>r_0(\varepsilon),\ \forall\varepsilon>0.$$

Тогда $f(z)\in A$.

Доказательство. Возьмем φ_0, $0<\varphi_0<\frac{\pi}{2}$ и нецелое $p\geqq N$. Пусть q — точка пересечения луча $\arg z=\varphi_0$ с прямой $\operatorname{Re} z=p$. Обозначим Γ контур, составленный из лучей $[q, \infty e^{i\varphi_0})$, $[\bar{q}, \infty e^{-i\varphi_0})$ и отрезка $[q, \bar{q}]$. Положим

(2) $$\psi(t)=\frac{1}{2i}\int\limits_{\Gamma}\frac{\varphi(z)e^{-zt}\,dz}{\sin\pi z}.$$

На лучах, входящих в состав Γ, в силу (1),

(3) $$\left|\frac{\varphi(z)}{\sin\pi z}\right|<A(\varepsilon)e^{(-\pi\sin\varphi_0+\varepsilon)|z|},\quad \forall\varepsilon>0.$$

Отсюда следует, что интеграл (2) сходится и представляет собой аналитическую функцию в угле $B(\varphi_0)$, одна сторона которого проходит через точку πi под углом $\left(\frac{\pi}{2}-\varphi_0\right)$ к вещественной оси, а другая сторона симметрична первой стороне относительно вещественной оси.

Оценка вида (3) остается справедливой на дугах $|z|=r_m=m\pi+\frac{\pi}{2}$ $(m=1, 2, \ldots)$, $|\arg z|\leqq\varphi_0$. Поэтому, если Γ_m — замкнутый контур, составленный из части Γ, лежащей в круге $|z|\leqq r_m$, и дуги $|z|=r_m$, $|\arg z|\leqq\varphi_0$, то для t из угла $|\arg t|<\frac{\pi}{2}-\varphi_0$

$$\psi(t)=\lim_{m\to\infty}\frac{1}{2i}\int\limits_{\Gamma_m}\frac{\varphi(z)e^{-zt}\,dz}{\sin\pi z}=\sum_{n>p}(-1)^n\varphi(n)e^{-nt}.$$

Отметим теперь, что

$$\psi(t) = \frac{1}{2i} \int_{\operatorname{Re} z = p} \frac{\varphi(z) e^{-zt}\, dz}{\sin \pi z}. \tag{4}$$

Этот интеграл сходится в полосе $|\operatorname{Im} t| < \pi$. Значит, $\psi(t)$ — функция, аналитическая в этой полосе и в полуплоскости $\operatorname{Re} t > 0$. Отсюда получаем, что и функция

$$\psi_1(t) = \sum_{n=0}^{\infty} (-1)^n a_n e^{-nt}$$

— аналитическая в полосе $|\operatorname{Im} t| < \pi$ и в полуплоскости $\operatorname{Re} t > 0$. В силу этого, функция $\eta(t)$ — аналитическая вне отрезка $[0,1]$.

Также доказывается, что и $\eta_1(t)$ регулярна вне $[0,1]$. Поэтому $f(z) \in A$.

Замечание. Из представления (4) следует, что

$$|\psi(t)| < C|e^{-pt}|, \quad |\operatorname{Im} t| < \pi - \delta, \quad \delta > 0. \tag{5}$$

Отсюда вытекает, что вне угла $|\arg t| < \delta$ (при $t \to 0$)

$$|\eta(t)| < \frac{C}{|t|^{p+1}}, \quad |\eta_1(t)| < \frac{C}{|t|^{p+1}} \quad (C = C(\delta)). \tag{6}$$

3. Обращение теоремы 1.

Теорема 2. *Пусть функции* $\eta(t)$ *и* $\eta_1(t)$ *регулярны вне отрезка* $[0, 1]$ *и вне каждого угла* $|\arg t| < \delta$ *удовлетворяют условию* (6). *Тогда имеются функции* $\varphi(z)$ *и* $\varphi_1(z)$, *аналитические при* $\operatorname{Re} z \geqq N$ *и удовлетворяющие условию* (1), *такие, что* $a_n = \varphi(n)$, $\dfrac{1}{a_n} = \varphi_1(n)$, $n \geqq N$.

Доказательство. Положим

$$\psi(t) = \sum_{n > p} (-1)^{n-1} a_n e^{-nt}. \tag{7}$$

Согласно условиям теоремы, эта функция регулярна в полуплоскости $\operatorname{Re} t > 0$ и в полосе $|\operatorname{Im} t| < \pi$, причем в меньшей полосе $|\operatorname{Im} t| < \pi - \delta$ (при $t \to -\infty$) она имеет оценку (5). При $t \to +\infty$ в полосе $|\operatorname{Im} t| < \pi - \delta$

$$|\psi(t)| < C|e^{-p_1 t}|, \quad p_1 > p. \tag{8}$$

Рассмотрим интеграл

$$\varphi(z) = \frac{\sin \pi z}{\pi} \int_{-\infty + ic}^{\infty + ic} \psi(t) e^{zt}\, dt, \quad -\pi < c < \pi. \tag{9}$$

В силу (5) и (8) он сходится для z из полосы: $p<\operatorname{Re} z<p_1$ и значение не зависит от параметра c. Имеем

$$\varphi(z)=\frac{\sin \pi z}{\pi}\int\limits_{-\infty}^{0}\psi(t)e^{tz}\,dt+\frac{\sin \pi z}{\pi}\int\limits_{0}^{\infty}\psi(t)e^{tz}\,dt=\varphi_1(z)+\varphi_2(z).$$

Функция $\varphi_1(z)$ регулярна в полуплоскости $\operatorname{Re} z>p$, причем $\varphi_1(n)=0$, $n>p$. Функция $\varphi_2(z)$ регулярна в полуплоскости $\operatorname{Re} z<p_1$.

Возьмем φ_0, $0<\varphi_0<\frac{\pi}{2}$. Имеем

$$\varphi_2(z)=\frac{\sin \pi z}{\pi}\int\limits_{0}^{\varepsilon e^{i\varphi_0}}\psi(t)e^{tz}\,dt+\frac{\sin \pi z}{\pi}\int\limits_{\varepsilon e^{i\varphi_0}}^{\infty e^{i\varphi_0}}\psi(t)e^{tz}\,dt=\frac{\sin \pi z}{\pi}\varphi_3(z)+\varphi_4(z).$$

Функция $\varphi_3(z)$ — целая экспоненциального типа не выше ε. Изучим функцию $\varphi_4(z)$. Когда $\arg z=\varphi$ удовлетворяет условию: $\frac{\pi}{2}<\varphi+\varphi_0<\frac{3\pi}{2}$, получаем

$$\varphi_4(z)=\frac{\sin \pi z}{\pi}\int\limits_{\varepsilon e^{i\varphi_0}}^{\infty e^{i\varphi_0}}\Big(\sum_{n>p}(-1)^{n-1}a_n e^{-nt}\Big)e^{tz}\,dt=$$

$$=-\frac{\sin \pi z}{\pi}e^{\varepsilon_0 z}\sum_{n>p}(-1)^{n-1}\frac{a_n e^{-\varepsilon_0 n}}{z-n},\quad \varepsilon_0=\varepsilon e^{i\varphi_0}.$$

Правая часть — целая функция экспоненциального типа, в точке $z=n$ она имеет значение равное a_n. Кроме того,

$$|\varphi_4(x)|<e^{2\varepsilon x},\quad x>x_0(\varepsilon).$$

В итоге функция $\varphi(z)$ регулярна и экспоненциального типа в полуплоскости $\operatorname{Re} z>p$, причем $\varphi(n)=a_n$ $(n>p)$ и

$$|\varphi(x)|<e^{\varepsilon x},\ x>x_1(\varepsilon),\quad \forall\varepsilon>0. \tag{10}$$

Пусть $z=p_2+iy$, $p<p_2<p_1$. Из (9) получаем

$$|\varphi(z)|<M|\sin \pi z|e^{-cy},\quad -\pi<c<\pi.$$

Выбирая для $y>0$ величину c близкой к π, а для $y<0$ — близкой к $-\pi$, получим, что

$$|\varphi(z)|<B(\varepsilon)e^{\varepsilon|y|},\quad z=p_2+iy,\quad \forall\varepsilon>0.$$

Отсюда и из (10) вытекает, что

$$|\varphi(z)|<e^{\varepsilon|z|},\ \operatorname{Re} z\geqq p_2,\ |z|>r_0(\varepsilon),\quad \forall\varepsilon>0.$$

Функция $\varphi(z)$ обладает всеми необходимыми свойствами. Также доказывается наличие нужной функции $\varphi_1(z)$.

4. Два примера. В первом примере

$$f(z) = e^z + ae^{qz}, \quad 0 < q < 1.$$

Здесь

$$a_n = 1+aq^n, \ \frac{1}{a_n} = \frac{1}{1+aq^n}, \ \varphi(z) = 1+ae^{z\ln q}, \ \varphi_1(z) = \frac{1}{1+ae^{z\ln q}} \quad (\ln q < 0).$$

Видим, что функции $\varphi(z)$ и $\varphi_1(z)$ обладают нужными свойствами и потому $f(z)\in A$.

Во втором примере

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n, \quad a_n = 1+(-1)e^{-n^2} \quad (n \geqq 0).$$

Имеем

$$\eta(t) = \frac{1}{t-1} + \sum_{n=0}^{\infty} \frac{(-1)^n e^{-n^2}}{t^{n+1}}, \quad \eta_1(t) = \frac{1}{t-1} - \sum_{n=0}^{\infty} \frac{(-1)^n e^{-n^2}}{(1+(-1)^n e^{-n^2})t^{n+1}}.$$

Ряды сходятся при всех $t\neq 0$. Поэтому у $\eta(t)$ и $\eta_1(t)$ только две особенности $t=0$ и $t=1$, значит, $f(z)\in A$. Отметим, что при $t\to 0$, когда $t<0$, функция

$$\eta(t) = \frac{1}{t-1} + \frac{1}{t} \sum_{n=0}^{\infty} \left(-\frac{1}{t}\right)^n e^{-n^2}$$

стремится к ∞ быстрее любой степени $\left(\frac{1}{t}\right)$. В силу этого не существуют функции $\varphi(z)$ и $\varphi_2(z)$ с указанными выше свойствами.

5. Преобразование M. По определению преобразование M переводит функцию $F(z) = \sum_{n=0}^{\infty} A_n z^n$ в функцию

$$\Phi(z) = M(F) = \sum_{n=0}^{\infty} B_n z^n, \ B_n = a_n A_n \quad (n \geqq 0).$$

Теорема 3. *Пусть* $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ *принадлежит классу* A. *Если функция* $F(z)$ *регулярна в области* E, *звездообразной относительно начала координат, то и функция* $\Phi(z) = M(F)$ *регулярна в* E, *причем если* K — *компакт из* E, *а* C — *замкнутый контур, лежащий в* E, *охватывающий компакт* K *и звездообразный относительно начала координат, то*

$$|\Phi(z)| < N \max_{t\in C} |F(t)|, \quad z\in K, \tag{11}$$

где постоянная N *не зависит от* $F(z)$.

Доказательство. Рассмотрим функцию

$$\psi(z) = \frac{1}{2\pi i} \int_C \eta\left(\frac{t}{z}\right) F(t) \frac{dt}{z}, \tag{12}$$

где C — звездообразный относительно начал координат замкнутый контур, лежащий в E, и z лежит внутри C. Точка $\left(\frac{t}{z}\right)$ не может попасть на отрезок $[0,1]$, поэтому $\psi(z)$ — функция, аналитическая внутри C. Но C — произвольный контур из E, значит, функция $\psi(z)$ регулярна в области E.

Пусть $z \in K$, а $t \in C$. Тогда $\left|\eta\left(\frac{t}{z}\right)\right| \leqq N_1$ и из (12) следует

$$|z\psi(z)| \leqq N_1 l \max_{t\in C} |F(t)|$$

(l — длина контура C). Можно считать, что K содержит в себе некоторый круг $|z| \leqq \delta$. Тогда

$$|\psi(z)| \leqq \frac{N_1 l}{\delta} \max_{t\in C} |F(t)|, \quad z\in K, \quad |z| \geqq \delta,$$

$$|\psi(z)| \leqq \frac{N_1 l}{\delta} \max_{t\in C} |F(t)|, \quad |z| \leqq \delta$$

(второе неравенство справедливо, в силу принципа максимума модуля). Таким образом,

$$|\psi(z)| \leqq N \max_{t\in C} |F(t)|, \ z\in K, \ N = \frac{N_1 l}{\delta}.$$

Осталось доказать, что $\psi(z) = \Phi(z)$. Пусть $|z|$ достаточно мал. Тогда

$$\eta\left(\frac{t}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n z^{n+1}}{t^{n+1}}, \quad t\in C,$$

$$\psi(z) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{2\pi i} \int_\Gamma \frac{F(t)\,dt}{t^{n+1}}\right) z^n = \sum_{n=0}^{\infty} a_n A_n z^n = \Phi(z).$$

Аналогично устанавливается, что если $\Phi(z)$ регулярна в E, то фикция $F(z) = M^{-1}(\Phi)$ также регулярна в E и

$$|F(z)| < N_1 \max_{t\in C} |\Phi(t)|, \quad z\in K. \tag{13}$$

Отметим следующие свойства преобразования M:

1. $M(F_1+F_2) = M(F_1) + M(F_2)$;
2. $M(cF) = cM(F)$;

3. Пусть функции $F_n(z)$ $(n \geqq 1)$ регулярны в области E (звездообразной относительно начала координат) и $\{F_n(z)\}$ равномерно сходится внутри E к $F(z)$. Тогда внутри E равномерно $M(F_n) \to M(F)$;

4. Если функции $\Phi_n(z)$ регулярны в E и $\{\Phi_n(z)\}$ равномерно сходится внутри E к $\Phi(z)$, то внутри E равномерно $M^{-1}(\Phi_n) \to M^{-1}(\Phi)$;

5. $M(e^{\lambda z}) = f(\lambda z)$.

Свойства 3 и 4 вытекают на основании оценок (11) и (13).

6. Разложение в ряд. Как было указано в начале статьи, каждую функцию $\Phi(z)$, аналитическую в конечной выпуклой области D, можно представить в виде

$$F(z) = \sum_{n=1}^{\infty} A_n e^{\lambda_n z}, \quad z \in D \tag{14}$$

(показатели зависят только от области D, сходимость — равномерная внутри D).

Теорема 4. *Пусть D — конечная выпуклая область, $0 \in D$ и $f(z) \in A$. Тогда каждую функцию $\Phi(z)$, аналитическую в D, можно представить в виде*

$$\Phi(z) = \sum_{n=1}^{\infty} A_n f(\lambda_n z) \tag{15}$$

(сходимость внутри D — равномерная).

Доказательство. Положим $F(z) = M^{-1}(\Phi)$. Функцию $F(z)$ представим рядом (14). Тогда

$$\Phi(z) = M(F) = M\left(\sum_{n=1}^{\infty} A_n e^{\lambda_n z}\right) = \sum_{n=1}^{\infty} A_n M(e^{\lambda_n z}) = \sum_{n=1}^{\infty} A_n f(\lambda_n z), \quad z \in D.$$

7. Формулы для коэффициентов. Пусть $A(\bar{D})$ — класс функций, аналитических на замыкании $\bar{D}$. В случае, когда $F(z) \in A(\bar{D})$, имеются формулы для определения коэффициентов A_n в разложении (14). Именно, пусть $L(\lambda)$ — целая функция экспоненциального типа и вполне регулярного роста с индикатрисой роста $h(\varphi) = K(-\varphi)$ $(K(\varphi)$ — опорная функция области $\bar{D})$ и простыми нулями $\lambda_n = |\lambda_n| e^{i\varphi_n}$ $(n \geqq 1)$, причем

$$|L'(\lambda_n)| > e^{[h(\varphi_n) - \varepsilon]|\lambda_n|}, \quad n > n_0(\varepsilon), \quad \forall \varepsilon > 0.$$

Тогда в разложении (14) в качестве показателей можно взять нули λ_n этой функции $L(\lambda)$, а в качестве коэффициентов — величины

$$A_n = \frac{1}{2\pi i} \int_C \psi_n(t) F(t)\, dt \quad (n \geqq 1),$$

где $\psi_n(t)$ — функции, ассоциированные по Борелю с функциями

$$\frac{L(\lambda)}{(\lambda-\lambda_n)L'(\lambda_n)}$$

(они регулярны вне $\bar{D}$), а C — замкнутый контур, охватывающий $\bar{D}$, на котором и внутри которого $F(t)$ — аналитическая функция.

Имеем

$$A_n=\frac{1}{2\pi i}\int\limits_C \psi_n(t)M^{-1}(\Phi)\,dt=\frac{1}{2\pi i}\int\limits_C \psi_n(t)\left\{\frac{1}{2\pi i}\int\limits_{C_1}\eta_1\left(\frac{u}{t}\right)\Phi(u)\frac{du}{t}\right\}dt.$$

Здесь C_1 — замкнутый выпуклый контур, охватывающий контур C, на котором и внутри которого $\Phi(u)$ — аналитическая функция. Поменяв порядок интегрирования, получим

$$A_n=\frac{1}{2\pi i}\int\limits_{C_1}\gamma_n(u)\Phi(u)\,du \quad (n\geqq 1), \tag{16}$$

где

$$\gamma_n(u)=\frac{1}{2\pi i}\int\limits_C \psi_n(t)\eta_1\left(\frac{u}{t}\right)\frac{dt}{t}.$$

Функции $\gamma_n(u)$ регулярны вне $\bar{D}$ и $\gamma_n(\infty)=0$. Заметим, что

$$\frac{1}{2\pi i}\int\limits_{C_1}\gamma_n(u)f(\lambda_m u)\,du=\frac{1}{2\pi i}\int\limits_C \psi_n(t)e^{\lambda_m t}\,dt=\delta_{nm}.$$

Таким образом, $\{\gamma_n(u)\}$ — система, биортогональная системе $\{f(\lambda_n z)\}$. С помощью ее коэффициенты в разложении (15) определяются по формулам (16) (при условии $\Phi(z)\in A(\bar{D})$).

8. Пример функции $f(z)$, когда теорема 4 не имеет места. Положим $f(z)=e^z+e^{qz}$. Тогда

$$\eta(t)=\frac{1}{t-1}+\frac{1}{t-q}.$$

Если $q\notin[0,1]$, то $f(z)$ не принадлежит классу A (при $q\in[0,1]$ она принадлежит классу A). Покажем, что в этом случае существуют выпуклые области D ($0\in D$), в которых представление произвольной аналитической в D функции $\Phi(z)$ рядом (15) невозможнло.

Имеем

$$f(z)=2e^{\mu z}\cos\lambda z,\quad \mu=\frac{1+q}{2},\quad \lambda=\frac{1-q}{2i}.$$

Лемма 1. *Вне окрестностей* $\left|z \mp \frac{k\pi-\pi/2}{\lambda}\right| < \frac{1}{k^\alpha}$ $(k \geqq 1, \alpha > 0)$ *и* $|z|>1$

(17) $$|f(z)| > \frac{B}{|z|^\alpha} \max(|e^z|, |e^{qz}|), \quad B>0,$$

где B — постоянная.

Доказательство. Вне указанных окрестностей

$$|\cos \lambda z| > \frac{B}{|z|^\alpha} \exp\{|\mathrm{Im}(\lambda z)|\}, \quad B > 0.$$

Пусть z лежит вне этих окрестностей. В случае $\mathrm{Im}(\lambda z) \geqq 0$ получаем

$$|f(z)| > \frac{B}{|z|^\alpha} \exp\{\mathrm{Re}(\mu z) + \mathrm{Im}(\lambda z)\}.$$

Но $(\mu = i\lambda + q)$

$$\mathrm{Re}(\mu z) + \mathrm{Im}(\lambda z) = \mathrm{Re}(qz) + \mathrm{Re}(i\lambda z) + \mathrm{Im}(\lambda z) = \mathrm{Re}(qz).$$

Кроме того $(\mu = 1 - i\lambda)$

$$\mathrm{Re}(\mu z) + \mathrm{Im}(\lambda z) \geqq \mathrm{Re}(\mu z) = \mathrm{Re}\, z - \mathrm{Re}(i\lambda z) = \mathrm{Re}\, z + \mathrm{Im}(\lambda z) \geqq \mathrm{Re}\, z.$$

Отсюда и следует неравенство (17).

В случае $\mathrm{Im}(\lambda z) < 0$ имеем

$$|f(z)| > \frac{B}{|z|^\alpha} \exp\{\mathrm{Re}(\mu z) - \mathrm{Im}(\lambda z)\}.$$

В этом случае $(\mu = 1 - i\lambda)$

$$\mathrm{Re}(\mu z) - \mathrm{Im}(\lambda z) = \mathrm{Re}\, z - \mathrm{Re}(i\lambda z) - \mathrm{Im}(\lambda z) = \mathrm{Re}\, z$$

и, кроме того $(\mu = i\lambda + q)$,

$$\mathrm{Re}(\mu z) - \mathrm{Im}(\lambda z) \geqq \mathrm{Re}(\mu z) = \mathrm{Re}(qz) - \mathrm{Im}(\lambda z) \geqq \mathrm{Re}(qz).$$

Опять получаем (17).

Лемма 2. *Пусть* $\overline{\lim\limits_{n\to\infty}} \frac{n}{|\lambda_n|} < \infty$ *и ряд* $\sum\limits_{n=1}^{\infty} A_n f(\lambda_n z)$ *сходится в окрестности* $|z - z_0| < \delta$ $(z_0 \neq 0)$. *Тогда в некоторой окрестности* $|z - z_0| < \delta_0 < \delta$ *сходятся ряды*

(18) $$\sum_{n=1}^{\infty} A_n e^{\lambda_n z}, \quad \sum_{n=1}^{\infty} A_n e^{q\lambda_n z}.$$

Доказательство. Вне кружков

$$C_k^{(n)}: \left|z \mp \frac{k\pi - \pi/2}{\lambda_n}\right| < \frac{1}{|\lambda_n| k^\alpha} \quad (k \geqq 1)$$

согласно лемме 1, имеет место оценка

$$|f(\lambda_n z)| > \frac{B}{|\lambda_n z|^\alpha} \max(|e^{\lambda_n z}|, |e^{q\lambda_n z}|). \tag{19}$$

Пусть $0<r<|z_0|-\delta<|z_0|+\delta<R$. Кружки $C_k^{(n)}$, которые имеют общие точки с окрестностью K: $|z-z_0|<\delta$, таковы, что при $n>n_0$

$$r<\frac{k\pi}{|\lambda_n|}<R.$$

Отсюда $\frac{r|\lambda_n|}{\pi}<k<\frac{R|\lambda_n|}{\pi}$ и, значит, сумма диаметров $C_k^{(n)}$, имеющих общие точки с окрестностью K, не превосходит величины

$$\frac{R|\lambda_n|}{\pi}\cdot\frac{2}{|\lambda_n|\left(\frac{r|\lambda_n|}{\pi}\right)^\alpha}=\frac{a}{|\lambda_n|^\alpha}.$$

Сумма диаметров указанных кружков, когда n меняется от N до ∞, не превосходит

$$\alpha_N \equiv a\sum_{n=N}^{\infty}\frac{1}{|\lambda_n|^\alpha}\to 0, \quad N\to\infty, \quad \alpha>1.$$

Пусть $\alpha_N<\delta$. Тогда найдется окружность $|z-z_0|=\delta_1$, $0<\delta_1<\delta$, которая не пересекается с рассматриваемыми кружками и на ней, следовательно, выполняется оценка (19) при $n\geq N$. Возьмем на этой окружности точки γ_1, γ_2, γ_3 так, чтобы они образовывали треугольник E, внутри которого лежит точка z_0. Согласно условию

$$|A_n f(\lambda_n\gamma_j)|\leqq M \quad (n\geqq N;\ j=1,2,3).$$

Отсюда, на основании (19), получаем

$$|A_n e^{\lambda_n\gamma_j}|<M_1|\lambda_n|^\alpha, \quad |A_n e^{q\lambda_n\gamma_j}|<M_1|\lambda_n|^\alpha \quad (n\geqq N).$$

Учтем еще, что, в силу неравенства $a^x b^{1-x}<a+b$, $a>0$, $b>0$, $0\leqq x\leqq 1$, для точек z отрезка $[z_1, z_2]$ выполняется соотношение ($z=\beta z_1+(1-\beta)z_2$, $0\leqq\beta\leqq 1$)

$$|e^{\lambda_n z}|\leqq|e^{\lambda_n z_1}|+|e^{\lambda_n z_2}|.$$

Поэтому на границе треугольника E, а, следовательно, и внутри него

$$|A_n e^{\lambda_n z}|\leqq 3M_1|\lambda_n|^\alpha, \quad |A_n e^{q\lambda_n z}|\leqq 3M_1|\lambda_n|^\alpha. \tag{20}$$

Возьмем внутри E окружность $|z-z_0|=\varrho$ и для точек этой окружности запишем

$$|A_n e^{\lambda_n z_0}|=|A_n e^{\lambda_n z}||e^{-\lambda_n(z-z_0)}|.$$

Пусть $|z-z_0|=\varrho$, $\arg(z-z_0)=-\arg\lambda_n$. Тогда, в силу (20)

$$|A_n e^{\lambda_n z_0}| \leqq 3M_1|\lambda_n|^\alpha e^{-\varrho|\lambda_n|}, \quad n \geqq N.$$

Значит, первый из рядов (18) в точке z_0 сходится. Аналогично убедимся, что и второй ряд в точке z_0 сходится. Итак, из сходимости ряда $\sum_{n=1}^{\infty} A_n f(\lambda_n z)$ в круге $|z-z_0|<\delta$ вытекает сходимость рядов (18) в центре круга. Но тогда, в силу этого, ряды (18) сходятся в некоторой окрестности $|z-z_0|<\delta_0$.

Лемма 3. *Пусть D — область сходимости ряда* (15), *а G и G_q — области сходимости рядов* (18). *Тода $D=G\cap G_q$.*

Доказательство, в силу леммы 2, очевидно.

Пусть D — прямоугольник: $|\mathrm{Re}\, z|<\varepsilon$, $-h_1<\mathrm{Im}\, z<h_2$, $h_2>h_1>0$. Возьмем функцию $\Phi(z)=\dfrac{1}{z-\varepsilon}$. Допустим, что имеет место разложение (15). Если $z\in D$, то $z\in G$ и $qz\in G$. Когда z пробегает D, в это время точка qz будет пробегать прямоугольник D_q, получаемый из D растяжением в $|q|$ раз и поворотом вокруг начала координат на угол $\varphi_0=\arg q$. Считаем, что $|q|\leqq 1$ и $\varphi_0\neq 0$. Пусть $\varphi_0\neq\pi$. Так как область сходимости ряда Дирихле — выпуклая, то, значит, область G содержит в себе выпуклую оболочку прямоугольников D и D_q. Выпуклая оболочка содержит в себе круг $|z|<R$, радиус R которого зависит от q, h_1 и ε, причем при $\varepsilon\to 0$ этот радиус, убывая, стремится к некоторому предельному значению $R_0>0$. Видим, что область G содержит в себе круг E_0: $|z|<R_0$. Но тогда и $G_q\supset E_0$. Отсюда вытекает, что ряд (15) сходится и представляет собой аналитическую функцию $\Phi(z)=\dfrac{1}{z-\varepsilon}$ в круге E_0. Но этого не может быть, если ε мало. Еще проще доводится до противоречия и случай, когда $\varphi_0=\pi$. Итак, прямоугольник малой ширины и не симметричный относительно точки $z=0$ не может служить областью D, в которой любая аналитическая функция разлагалась бы в ряд (15).

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ОТДЕЛ ФИЗИКИ И МАТЕМАТИКИ
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Self-dual polytopes and the chromatic number of distance graphs on the sphere

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

0. Introduction

Let S^{n-1} denote the unit sphere in the n-dimensional euclidean space and let $0<\alpha<2$. Construct a graph $G(n,\alpha)$ on the points of S^{n-1} by connecting two of them iff their distance is exactly α. We shall study the chromatic number of the graph obtained this way and prove that this chromatic number is at least n. This answers a question of ERDŐS and GRAHAM [2], who conjectured that this chromatic number tends to infinity with n.

Let us modify the definition of the graph and construct another graph $B(n,\alpha)$ by connecting two points of S^{n-1} if and only if their distance is at least α. The graph $B(n,\alpha)$ obtained this way is often called Borsuk's graph because a classical theorem of BORSUK [1] implies (in fact, is equivalent to) the result that $B(n,\alpha)$ has chromatic number at least $n+1$. Since, however, $G(n,\alpha)$ is a proper subgraph of $B(n,\alpha)$, Borsuk's theorem has no immediate bearing on the chromatic number of $G(n,\alpha)$.

If α is larger than the side of a regular simplex inscribed in the unit ball, then it is easy to describe an $(n+1)$-coloration of $B(n,\alpha)$ (and so, a fortiori, of $G(n,\alpha)$). Let R be the regular simplex inscribed in S^{n-1} and use the facet of R intersected by the segment $0X$ as the "color" of $X\in S^{n-1}$. Hence if $\alpha>\sqrt{2(n+1)/n}$ then

$$\chi(G(n,\alpha)) \leqq \chi(B(n,\alpha)) = n+1.$$

It is easy to see that the colors of the vertices of R can be chosen different, and hence this is also true for $\alpha=\sqrt{2(n+1)/n}$.

In this paper we apply a lower bound on the chromatic number of a general graph, derived in [4], to an appropriate subgraph of $G(n,\alpha)$. It is interesting to

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remark that to prove this lower bound in [4], Borsuk's theorem was used. Thus in this sense we do establish a connection between the chromatic numbers of $B(n, \alpha)$ and $G(n, \alpha)$.

In section 1 we define, construct and study certain polyhedra called strongly self-dual. It seems that these polyhedra merit interest on their own right. In section 2 we state the general lower bound on the chromatic number mentioned above and apply it to prove our main result. In section 3 we discuss the question of sharpness of our results.

1. Strongly self-dual polytopes

Let P be a convex polytope in $\mathbf{R}^n$. We say that P is *strongly self-dual* if the following conditions hold.

(1) P is inscribed in the unit sphere S^{n-1} in $\mathbf{R}^n$ (so that all vertices of P lie on the sphere S^{n-1});

(2) P is circumscribed around the sphere S' with center 0 and radius r for some $0<r<1$ (so that S' touches every facet of P);

(3) There is a bijection σ between vertices and facets of P such that if v is any vertex then the facet $\sigma(v)$ is orthogonal to the vector v.

If $n=2$ then the strongly self-dual polytopes are precisely the odd regular polygons. If $n \geqq 3$ then there are strongly self-dual polytopes with a more complicated structure.

Let us start with proving some elementary properties of strongly self-dual polytopes.

Lemma 1. *If v_1, v_2 are vertices of a strongly self-dual polytope P and v_1 is a vertex of the facet $\sigma(v_2)$ then v_2 is a vertex of the facet $\sigma(v_1)$.*

Proof. Let v be any vertex of P. The inequality defining $\sigma(v)$ is $v \cdot x \geqq -r$. For $v=v_2$, the vector $x=v_1$ lies on the facet $\sigma(v_2)$, and so $v_2 \cdot v_1=-r$. But by interchanging the role of v_1 and v_2, we obtain that v_2 lies on $\sigma(v_1)$.

Call a diagonal of a strongly self-dual polytope *principal* if it connects a vertex v to a vertex of the facet $\sigma(v)$. The proof of Lemma 1 implies:

Lemma 2. *Every principal diagonal of a strongly self-dual polytope is of the same length.*

This length α will be called the *parameter* of P. Clearly $\alpha=\sqrt{2+2r}$. As $r>0$, we have $\alpha>\sqrt{2}$. This trivial inequality can be improved. We show that the least possible value of the parameter of a strongly self-dual polytope in a given space is the side length of the regular simplex inscribed in the unit ball:

Lemma 3. *Let P be a strongly self-dual polytope in $\mathbf{R}^n$ with parameter α. Then* $\alpha \geqq \sqrt{2(n+1)/n}$.

Proof. We prove more generally that *if a polytope P is inscribed in S^{n-1} and contains the origin, then it has a pair of vertices at a distance at least $\sqrt{2(n+1)/n}$ apart.* Since the principal diagonals of a strongly self-dual polytope are obviously its longest diagonals, this will imply the Lemma.

Observe further that we may assume that P is a simplex, since if a polytope contains the origin then some of its vertices span a simplex which also contains it.

So let P be a simplex inscribed in S^{n-1} and containing the origin. Let P' be its facet nearest 0, and let z be the orthogonal projection of 0 on P'. It is easy to see that P' contains z. Let $t=|z|$. We claim that $t \leqq 1/n$. In fact, let $v_0, \dots, v_n$ be the vertices of P. Then since 0 is in P, we can write

$$\sum_{i=0}^{n} \lambda_i v_i = 0 \quad \text{with} \quad \lambda_i \geqq 0, \quad \sum_{i=0}^{n} \lambda_i = 1.$$

We may assume without loss of generality that $\lambda_0 \leqq 1/(n+1)$. Consider the point

$$w_0 = \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_0} v_i = \frac{-\lambda_0}{1-\lambda_0} v_0.$$

This point is on the boundary of P. Furthermore, $|w_0| = \lambda_0/(1-\lambda_0) \leqq 1/n$. Hence the facet of P nearest to the origin is at a distance at most $1/n$, which proves that $t \leqq 1/n$.

By induction on n, we may assume that the facet P' contains two vertices whose distance is at least

$$\sqrt{\frac{2n}{n-1}}\sqrt{1-t^2} \geqq \sqrt{\frac{2n}{n-1}}\sqrt{1-\frac{1}{n^2}} = \sqrt{\frac{2(n+1)}{n}}.$$

This proves the Lemma.

We do not know which values of α can be parameters of strongly self-dual polytopes, except in the trivial case $n=2$. But the following result will be sufficient for our purposes.

Theorem 1. *For each $n \geqq 2$ and $\alpha_1 < 2$ there exists a strongly self-dual polytope in $\mathbf{R}^n$ with parameter at least α_1.*

Proof. We give a construction by induction on n. For $n=2$ the assertion is obvious.

Let $n \geqq 3$ and let P_0 be a strongly self-dual polytope in dimension $n-1$ such that the parameter α_0 of P_0 satisfies $\alpha_0 > \alpha_1$. Thus the radius r_0 of the inscribed ball of P_0 satisfies $r_0 > r_1 = \alpha_1^2/2 - 1$.

We begin with an auxiliary construction in the plane. Let C be the unit circle in $\mathbf{R}^2$ and let E be an ellipse with axes 2 and $2r_0$, concentrical with C. Thus E touches C in two points x and y. Choose any t with $r_0>t>r_1$ and let C_t denote the circle concentrical with C and with radius t. It is clear by a continuity argument that t can be chosen so that we can inscribe an odd polygon $Q=(x_0=x, \ldots, x_{2k+1}=x)$ in E so that the sides of Q are tangent to C_t. Let α be an orthogonal affine transformation mapping E on C and let $y_0=x_0, y_1, \ldots, y_{2k+1}=x_0$ be the images of $x_0, x_1, \ldots, x_{2k+1}$ under α.

Consider C as the "meridian" of S^{n-1} with x as the "north pole". Let S^{n-2} be the "equator" and suppose the P_0 is inscribed in the "equator". Let, for each vertex v of P_0; M_v be the "meridian" through v (so M_v is a one-dimensional semicircle). Let L_i denote the "parallel' through y_i $(i=1, \ldots, k)$. We denote by $u(v, i)$ the intersection point of M_v and L_i. Further, let $u(v, 0)=x$ for all v. We define the polytope

$$P = \operatorname{conv}\{u(v, i)\colon\ v\in V(P_0);\ i = 0, \ldots, k\}.$$

(Here $V(P_0)$ denotes the set of vertices of P_0.) We prove that P is a strongly self-dual polytope with parameter $\sqrt{2+2r}>\alpha_1$.

Claim 1. *The facets of P are*

$$\operatorname{conv}\{u(v, k)\colon v\in V(P_0)\}$$

and

$$F^{(j)} = \operatorname{conv}\{u(v, i)\colon\ v\in V(F),\ \ i\in\{j, j+1\}\}$$

where F is a facet of P_0 and $0\leqq j\leqq k-1$.

Proof. Consider the affine hull $A_F^{(j)}$ of the points $u(v,j)$ $(v\in V(F))$. Then $A_F^{(j)}$ and $A_F^{(j+1)}$ are parallel affine $(n-2)$-spaces $(1\leqq j\leqq k-1)$ and so they span a unique hyperplane $B_F^{(j)}$. For $j=0$, let $B_F^{(0)}$ denote the hyperplane through the affine $(n-2)$-space $A_F^{(1)}$ and x. We denote by $H_F^{(j)}$ the closed halfspace bordered by $B_F^{(j)}$ and containing the origin. Clearly $P\subset H_F^{(j)}$.

Let, further, B_0 be the affine hull of the points $u(v, k)$ $(v\in V(P_0))$ and let H_0 be the closed halfspace bordered by B_0 and containing the origin. Again, $P\subset H_0$. It is easy to see that

$$P = \bigcap_F \bigcap_{j=0}^{k-1} H_F^{(j)} \cap H_0.$$

This proves the Claim since each $B_F^{(j)}$ as well as B_0 are spanned by the vertices of P.

Claim 2. *The ball concentrical with S^{n-1} and with radius t touches every facet of P.*

Proof. This is clear for the facet B_0. Consider $B_F^{(j)}$. Let N be the 2-dimensional plane through 0 and x, and orthogonal to $B_F^{(j)}$; without loss of generality we may assume that N intersects S^{n-1} in the circle C featured in the auxiliary construction. Then since P_0 is a strongly self-dual polytope with inscribed ball radius r_0, it follows that N intersects $A_F^{(j)}$ and $A_F^{(j+1)}$ in the points x_j and x_{j+1}, respectively. Thus it intersects $B_F^{(j)}$ in the line through x_j and x_{j+1}. Since by construction, the circle C_t touches this line, it follows that the ball about 0 with radius t touches the hyperplane $B_F^{(j)}$.

Claim 3. *B_0 is orthogonal to the vector y_0. $B_{F_v}^{(k-j)}$ is orthogonal to the vector $u(v, j)$, where F_v is the facet of P_0 opposite to the vertex v.*

Proof. The first assertion is trivial. To prove the second, we use induction on j. Let w be any vertex of P_0. First we show that $u(w, k)$ is orthogonal to $B_{F_w}^{(0)}$. This follows easily on noticing that the plane D through x, 0 and $u(w, k)$ is orthogonal to $A_{F_w}^{(k)}$ by the hypothesis that P_0 is strongly self-dual, and since $A_{F_w}^{(k)} \| B_{F_w}^{(0)}$, it follows that D is also orthogonal to $B_{F_w}^{(0)}$. Since $|x-u(w,k)|=\alpha=$ $=\sqrt{2+2t}$, considering this plane D we see easily that $u(w, k)$ is orthogonal to $B_{F_w}^{(0)}$. Consequently, $u(w, k)$ is at a distance α from all vertices of the facet $B_{F_w}^{(0)}$.

We can repeat the same argument to show that $u(v, 1)$ is orthogonal to $B_{F_v}^{(k-1)}$, and then the same argument can be used to show that $u(v, k-1)$ is orthogonal to $B_{F_v}^{(1)}$, etc. This proves Claim 3 as well as Theorem 1.

2. The chromatic number of distance graphs

We now use the existence of strongly self-dual polytopes to derive lower bounds on the chromatic number of $G(n, \alpha)$, the graph obtained by connecting all pairs of points on the unit sphere S^{n-1} at distance α apart.

In [4] the following lower bound on the chromatic number of a graph was proved. Let G be a finite graph, and define its *neighborhood complex* $N(G)$ as the simplicial complex with vertex set $V(G)$, where a subset $A \subseteq V(G)$ forms a simplex if any only if the points of A have a neighbor in common.

Theorem A. *Let G be a graph and suppose that $N(G)$ is k-connected* ($k \geqq 0$). *Then* $\chi(G) \geqq k+3$.

The main result of this section is the following.

Theorem 2. *The graph formed by the principal diagonals of a strongly self-dual polytope in $\mathbf{R}^n$ has chromatic number $n+1$.*

One half of this Theorem follows immediately from Theorem A and the next Lemma.

Lemma 4. *Let P be a strongly self-dual polytope and let G_P be the graph formed by its vertices and principal diagonals. Then $N(G_P)$ is homotopy equivalent to the surface of P.*

Proof. Let $\overline{N(G_P)}$ denote the geometric realization of $N(G_P)$. Consider the natural bijection φ from the vertex set of $\overline{N(G_P)}$ onto the vertex set of P, and extend φ affinely over the simplices of $\overline{N(G_P)}$. This results in a continuous mapping $\bar{\varphi}\colon \overline{N(G_P)} \to \partial P$ since by the definition of the neighborhood complex and of G_P, each simplex of $\overline{N(G_P)}$ is mapped into a facet of P.

On the other hand, let $\psi = \varphi^{-1}$. Subdivide each facet of P into simplices without introducing new vertices, and let K denote the resulting simplicial complex. Then ∂P may be viewed as a geometric realization of K. Extend ψ affinely over the simplices in $\bar{K}$, to obtain a continuous mapping $\bar{\psi}\colon \partial P \to \overline{N(G_P)}$.

Now $\bar{\varphi} \circ \bar{\psi} = \mathrm{id}_{\partial P}$. Further, $\bar{\psi} \circ \bar{\varphi}$ is a simplicial map of $\overline{N(G_P)}$ into itself such that $(\bar{\psi} \circ \bar{\varphi})(S) \cup S$ is contained in a simplex of $\overline{N(G_P)}$, for every simplex S of $\overline{N(G_P)}$. Hence $\bar{\psi} \circ \bar{\varphi}$ is homotopic to $\mathrm{id}_{\overline{N(G_P)}}$, and the Lemma follows.

To complete the proof of Theorem 2, it suffices to remark that $G_P \subseteq G(n, \alpha) \subseteq \subseteq B(n, \alpha)$, and even $B(n, \alpha)$ is $(n+1)$-colorable as $\alpha \geqq \sqrt{2(n+1)/n}$ by Lemma 3.

Corollary 1. *If there exists a strongly self-dual polytope in $\mathbf{R}^n$ with parameter α, then $\chi(G(n, \alpha)) = n+1$.*

To treat the values α which are not parameters of strongly self-dual polytopes, we need a simple lemma.

Lemma 5. *Let $\alpha < \beta < 2$. Then $G(n-1, \beta)$ is isomorphic to a subgraph of $G(n, \alpha)$.*

Proof. Consider a hyperplane at distance $\sqrt{1-\alpha^2/\beta^2}$ from 0. This intersects the unit sphere in an $(n-2)$-sphere with radius α/β, and hence the restriction of $G(n, \alpha)$ to this hyperplane is isomorphic with $G(n-1, \beta)$.

By Theorem 1 and Lemma 5 we obtain the following.

Corollary 2. *For any $\alpha < 2$, $\chi(G(n, \alpha)) \geqq n$.*

3. Concluding remarks

To determine the chromatic number of $G(n, \alpha)$ exactly appears to be a difficult question. For small values of α, $\chi(G(n, \alpha))$ grows probably exponentially fast with n; a similar result for euclidean spaces was proved by FRANKL and WILSON [3].

The situation is simpler when α is large; in this paper we have shown that for $\alpha > \sqrt{2(n+1)/n}$,

$$n \leqq \chi(G(n, \alpha)) \leqq n+1,$$

where the upper bound is attained by infinitely many values of α. If $n=2$, then the lower bound is attained for every α which is not the length of a diagonal of a regular odd polygon. We do not know if the lower bound is ever attained for $n \geqq 3$.

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Necessary and sufficient condition for the maximal inequality of convex Young functions

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

1. Young functions

Let $\varphi(t)$ be a non-decreasing and left-continuous function defined on $[0, +\infty)$ such that $\varphi(0)=0$ and $\lim_{t\to+\infty} \varphi(t)=+\infty$. For $x\geqq 0$ define

$$\Phi(x) = \int_0^x \varphi(t)\,dt.$$

Then Φ is non-decreasing, continuous and convex. Φ is called a Young function.

The conjugate Young function is defined as follows: for $t>0$ put $\psi(t)= =\sup\{x>0: \varphi(x)<t\}$ and let $\psi(0)=0$. One can show that ψ satisfies all the properties imposed on φ. Further, we trivially have

$$\psi(\varphi(x)) \leqq x \leqq \psi(\varphi(x)+0). \tag{1}$$

The Young function

$$\Psi(x) = \int_0^x \psi(t)\,dt$$

is said to be conjugate to Φ.

The pair (Φ, Ψ) of mutually conjugate Young functions satisfies the following inequality of Young:

$$xy \leqq \Phi(x)+\Psi(y) \quad \text{for arbitrary} \quad x \geqq 0,\ y \geqq 0.$$

Equality holds if and only if $y\in[\varphi(x), \varphi(x+0)]$ or $x\in[\psi(y), \psi(y+0)]$.

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We say that Φ satisfies the moderated growth condition if one of the following three equivalent conditions is met:

$$\limsup_{x\to+\infty}\frac{\varphi(c_1x)}{\varphi(x)}<+\infty \quad \text{for some constant} \quad c_1>1, \tag{2}$$

$$\limsup_{x\to+\infty}\frac{\Phi(c_2x)}{\Phi(x)}<+\infty \quad \text{for some constant} \quad c_2>1, \tag{3}$$

$$p=\limsup_{x\to+\infty}\frac{x\varphi(x)}{\Phi(x)}<+\infty. \tag{4}$$

In this note the quantity p is referred to as the power of Φ. The power q of the conjugate Young function Ψ is defined similarly. One can easily prove that

$$\liminf_{x\to+\infty}\frac{x\varphi(x)}{\Phi(x)}=\frac{q}{q-1}. \tag{5}$$

(Here and in the sequel let $\frac{1}{0}=+\infty$, $\frac{+\infty}{+\infty}=1$, $\frac{1}{+\infty}=0$ by definition.) Further, for arbitrary constant $c>1$ we have

$$c^{\frac{q}{q-1}}\leqq\liminf_{x\to+\infty}\frac{\Phi(cx)}{\Phi(x)}\leqq\limsup_{x\to+\infty}\frac{\Phi(cx)}{\Phi(x)}\leqq c^p. \tag{6}$$

The above assertions and further information about the theory of Young functions can be found, e.g., in [4] and in [8].

We prove the following

Lemma. *Let (Φ, Ψ) be a pair of conjugate Young functions. In order that the power q of Ψ be finite it is necessary and sufficient that the condition*

$$\limsup_{x\to+\infty}\frac{1}{\varphi(x)}\int_1^x\frac{\varphi(t)}{t}\,dt=\beta<+\infty \tag{7}$$

be satisfied.

Proof. Integrating by parts yields

$$\frac{1}{\varphi(x)}\int_1^x\frac{\varphi(t)}{t}\,dt=\frac{\Phi(x)}{x\varphi(x)}-\frac{\Phi(1)}{\varphi(x)}+\frac{1}{\varphi(x)}\int_0^x\frac{\Phi(t)}{t\varphi(t)}\frac{\varphi(t)}{t}\,dt. \tag{8}$$

Combining this with (5) we obtain that for arbitrary $\varepsilon>0$

$$\frac{1}{\varphi(x)}\int_1^x\frac{\varphi(t)}{t}\,dt\leqq\left(\frac{q-1}{q}+\varepsilon\right)\left(1+\frac{1}{\varphi(x)}\int_1^x\frac{\varphi(t)}{t}\,dt\right)+O\left(\frac{1}{\varphi(x)}\right)$$

holds, hence $\beta\leqq q-1$. Thus the growth condition implies (7).

Conversely, let y denote $\psi(2x)$. Recalling (1) we can write

$$\frac{1}{\varphi(y)}\int_1^y \frac{\varphi(t)}{t}\,dt \geqq \frac{1}{2x}\int_{\psi(x)}^{\psi(2x)} \frac{\varphi(t)}{t}\,dt \geqq \frac{\varphi(\psi(x)+0)}{2x}\int_{\psi(x)}^{\psi(2x)} \frac{dt}{t} \geqq \frac{1}{2}\log\frac{\psi(2x)}{\psi(x)}.$$

From this it follows that

$$\limsup_{x\to+\infty}\frac{\psi(2x)}{\psi(x)} \leqq e^{2\beta},$$

thus (7) implies the growth condition.

2. The maximal inequality

Definition. We say that for the Young function Φ the maximal inequality is valid with some constants $a, b\geqq 0$ depending only on Φ if for arbitrary non-negative submartingale $(X_n, \mathscr{F}_n)$, $n\geqq 1$, with the maximum $X_n^* = \max\limits_{1\leqq k\leqq n} X_k$ we have

$$E(\Phi(X_n^*)) \leqq a+E(\Phi(bX_n)) \quad n = 1, 2, \ldots. \tag{9}$$

Several papers have been devoted to such type of inequalities, e.g., [1], [3], [7].

The main purpose of the present note is to characterize all the Young functions Φ for which the maximal inequality is valid.

Theorem 1. *Let (Φ, Ψ) be a pair of conjugate Young functions. In order that Φ satisfy the maximal inequality in the above sense it is necessary and sufficient that the power q of Ψ be finite.*

Proof. Although the sufficiency part of the present assertion is already known (cf. [7]), for the sake of completeness we present here a proof to it. Suppose that Ψ obeys the growth condition. Then for arbitrary $b>q$ one can find a constant $a\geqq 0$ to satisfy the inequality $x\psi(x)\leqq a+b\Psi(x)$ for all $x\geqq 0$. We prove that the maximal inequality is valid for Φ with the same constants a and b. To this end we recall the following inequality due to Doob:

$$\lambda P(X_n^* \geqq \lambda) \leqq E(X_n I(X_n^* \geqq \lambda)) \quad \text{for} \quad \lambda \geqq 0.$$

Here $I(\cdot)$ stands for the indicator of the event in the brackets. For any $c>0$ define $X_k'=\min(X_k, c)$ and set

$$X_n^{**} = \max_{1\leqq k\leqq n} X_k' = \min(X_n^*, c).$$

On the basis of the Doob inequality we have

$$\lambda P(X_n^{**} \geqq \lambda) \leqq E(X_n I(X_n^{**} \geqq \lambda)).$$

Integrating this on $[0, +\infty)$ with respect to the measure generated by $\varphi(\lambda)$ we get

$$\int_0^{+\infty} \lambda E(I(X_n^{**} \geqq \lambda))\, d\varphi(\lambda) \leqq \int_0^{+\infty} E(X_n I(X_n^{**} \geqq \lambda))\, d\varphi(\lambda).$$

Applying the Fubini theorem to both sides we obtain

$$E\Big(\int_0^{X_n^{**}} \lambda\, d\varphi(\lambda)\Big) \leqq E(X_n \varphi(X_n^{**})).$$

By partial integration

$$\int_0^x \lambda\, d\varphi(\lambda) = x\varphi(x) - \int_0^x \varphi(\lambda)\, d\lambda = x\varphi(x) - \Phi(x) = \Psi(\varphi(x)),$$

whence

$$E(\Psi(\varphi(X_n^{**}))) \leqq \frac{1}{b} E(bX_n \varphi(X_n^{**})).$$

Using the Young inequality on the right-hand side yields

$$E(\Psi(\varphi(X_n^{**}))) \leqq \frac{1}{b}\left[E(\Phi(bX_n)) + E(\Psi(\varphi(X_n^{**})))\right].$$

From this it follows that

$$(b-1)E(\Psi(\varphi(X_n^{**}))) \leqq E(\Phi(bX_n)),$$

since X_n^{**} is bounded by c. Now by the assumption

$$\Phi(x) = x\varphi(x) - \Psi(\varphi(x)) \leqq \psi(\varphi(x)+0)\varphi(x) - \Psi(\varphi(x)) \leqq a + (b-1)\Psi(\varphi(x)),$$

from which it follows that

$$E(\Phi(X_n^{**})) \leqq a + E(\Phi(bX_n)).$$

Let c tend to $+\infty$, then $X_n^{**} \to X_n^*$ and the monotone convergence theorem completes the proof of the sufficiency part of our assertion.

Necessity. Suppose that the maximal inequality is valid for Φ with some constants a, b. We can set $b \geqq 1$. Let us define a sequence $\{x_n\}$ of numbers with the following properties:

$$x_1 = 1, \quad x_n < x_{n+1} < 2x_n \quad \text{for} \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} x_n = +\infty$$

and

$$\limsup_{n\to\infty} \frac{1}{\varphi(bx_n)} \int_1^{bx_n} \frac{\varphi(t)}{t}\, dt = \limsup_{x\to+\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}\, dt. \tag{10}$$

Let Ω be the set of the positive integers and let $\mathscr{A}$ be the σ-field of all subsets of Ω. On the measurable space $(\Omega, \mathscr{A})$ we define the probability P by the formula

$$P(\{n\}) = \frac{1}{x_n} - \frac{1}{x_{n+1}}, \quad n = 1, 2, \ldots.$$

Let $\mathscr{F}_n$ be the σ-field generated by the partition

$$(\{1\}, \{2\}, \ldots, \{n-1\}, \{n, n+1, \ldots\}).$$

Clearly we have $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots$. Further, for $n=1; 2, \ldots$ define the random variable X_n by

$$X_n(\omega) = x_n I(\omega \geqq x_n), \quad \omega \in \Omega.$$

It is easy to see that $(X_n; \mathscr{F}_n)$ is a nonnegative martingale and that

$$X_n^*(\omega) = \begin{cases} x_\omega, & \text{if} \quad \omega < n \\ x_n, & \text{if} \quad \omega \geqq n. \end{cases}$$

In virtue of the maximal inequality we have

$$\sum_{k=1}^{n-1} \Phi(x_k)\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) + \frac{1}{x_n}\Phi(x_n) \leqq a + \frac{1}{x_n}\Phi(bx_n). \tag{11}$$

The sum of the left hand side of (11) can be estimated as follows:

$$\sum_{k=1}^{n-1} \Phi(x_k)\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) = \sum_{k=1}^{n-1} \Phi(x_k)\frac{1}{2}\int_{x_k/2}^{x_{k+1}/2} \frac{1}{t^2}\,dt \geqq \frac{1}{2}\int_1^{x_n/2} \frac{\Phi(t)}{t^2}\,dt.$$

Integrating by parts we obtain

$$\frac{1}{2}\int_1^{x_n/2} \frac{\Phi(t)}{t^2}\,dt = \frac{1}{2}\left[\Phi(1) - \frac{\Phi(x_n/2)}{x_n/2} + \int_1^{x_n/2} \frac{\varphi(t)}{t}\,dt\right],$$

hence (11) implies

$$\frac{1}{2}\int_1^{x_n/2} \frac{\varphi(t)}{t}\,dt \leqq a + \frac{1}{x_n}\Phi(bx_n).$$

On the other hand,

$$\frac{1}{2}\int_{x_n/2}^{bx_n} \frac{\varphi(t)}{t}\,dt \leqq \frac{1}{2}\varphi(bx_n)\log 2b,$$

consequently

$$\frac{1}{\varphi(bx_n)}\int_1^{bx_n} \frac{\varphi(t)}{t}\,dt \leqq \frac{2a}{\varphi(bx_n)} + 2b\frac{\Phi(bx_n)}{bx_n\varphi(bx_n)} + \log 2b.$$

Keeping in mind the property (10) of the sequence $\{x_n\}$ we conclude

$$\limsup_{x \to +\infty} \frac{1}{\varphi(x)}\int_1^x \frac{\varphi(t)}{t}\,dt \leqq 2b + \log 2b,$$

thus by our Lemma Ψ fulfils the growth condition.

3. Estimates for the best constants in the maximal inequality

Denote by b^* the infimum of the constants b the maximal inequality is valid with. b^* appears to measure somehow the rate of growth of the Young function Φ: the faster Φ grows, the smaller b^* is. Hence it would be of interest either to find the connection between b^* and the quantities introduced while formulating the growth condition, or to give some estimates at least. The assertion proved below may be regarded as the first step in this direction.

Theorem 2. *Let (Φ, Ψ) be a pair of conjugate Young functions with powers p and q, respectively. Then*

$$\frac{p}{p-1} \leqq b^* \leqq q.$$

Proof. The upper estimate for b^* follows immediately from the proof of the sufficiency part of Theorem 1.

For the lower estimate suppose the maximal inequality is valid for Φ with some constants $a \geqq 0$ and $b < \frac{p}{p-1}$. From this we derive a contradiction. In view of Theorem 1 the case $p=1$ may be left out of consideration.

Define $\Omega=\{1, 2, \dots, n\}$, let $\mathcal{A}$ be the σ-field of all subsets of Ω and let $P(\{\omega\})=\frac{1}{n}$; $\omega \in \Omega$. On the probability space $(\Omega, \mathcal{A}, P)$ define the nonnegative martingale $(X_k; \mathcal{F}_k)$, $k=1, \dots, n$, as follows: let $\mathcal{F}_{n+1-k}$ be the σ-field generated by the partition

$$(\{1, 2, \dots, k\}, \quad \{k+1\}, \dots, \{n\})$$

and let

$$X_{n+1-k}(\omega) = \begin{cases} c\omega^{-1/p}, & \text{if} \quad \omega > k \\ \dfrac{c}{k} \sum\limits_{i=1}^{k} i^{-1/p}, & \text{if} \quad \omega \leqq k, \end{cases}$$

where $c = \Phi^{-1}(n)/b$.

Clearly, $(X_k, \mathcal{F}_k)$ has the martingale property. One can easily see that

$$X_n^*(\omega) = \frac{c}{\omega} \sum_{i=1}^{\omega} i^{-1/p} \geqq \frac{c}{\omega} \frac{p}{p-1} (\omega^{1-1/p} - 1),$$

from which we have

$$X_n^*(\omega) > b(1+\varepsilon) X_n(\omega) \quad \text{for} \quad \omega \geqq k_0,$$

where $\varepsilon > 0$ satisfies $b(1+\varepsilon) < \frac{p}{p-1}$ and the threshold k_0 does not depend on n.

Hence

$$E(\Phi(X_n^*)) \geqq \frac{1}{n} \sum_{\omega=k_0}^{n} E(\Phi(b(1+\varepsilon)X_n(\omega))) \geqq (1+\varepsilon)\frac{1}{n} \sum_{\omega=k_0}^{n} E(\Phi(bX_n(\omega))) \geqq$$

$$\geqq (1+\varepsilon)\left[E(\Phi(bX_n)) - \frac{k_0}{n}\Phi(bc)\right] = (1+\varepsilon)\left[E(\Phi(bX_n)) - k_0\right].$$

Applying the maximal inequality to the martingale $(X_k, \mathscr{F}_k)$ we obtain

$$a + E(\Phi(bX_n)) \geqq (1+\varepsilon)[E(\Phi(bX_n)) - k_0]. \tag{12}$$

Now let n tend to infinity. Then from (6) it follows that

$$\liminf_{n\to\infty} \frac{1}{n}\Phi(bX_n(\omega)) = \liminf_{n\to\infty} \frac{\Phi(bc\omega^{-1/p})}{\Phi(bc)} \geqq \frac{1}{\omega}$$

for arbitrary fixed positive integer ω. Consequently,

$$\lim_{n\to\infty} E(\Phi(bX_n)) = +\infty$$

which contradicts (12).

4. Remarks

(i) *Convexity inequality.* We say that for the Young function Ψ the convexity inequality is valid with some constants $a, b \geqq 0$, if for arbitrary sequence $\{Z_n\}$ of nonnegative random variables and increasing sequence $\{\mathscr{F}_n\}$ of σ-fields

$$E\left[\Psi\left(\sum_{i=1}^{n} E(Z_i|\mathscr{F}_i)\right)\right] \leqq a + E\left[\Psi\left(b\sum_{i=1}^{n} Z_i\right)\right], \quad n = 1, 2, \ldots$$

holds. By the duality theorem of [6] the maximal inequality is valid for a Young function Φ if and only if the convexity inequality holds for Ψ, the conjugate to Φ. So Theorem 1 of the present note affords also a necessary and sufficient condition for a Young function Ψ to satisfy the convexity inequality, namely, that Ψ should meet the growth condition.

(ii) *An open problem.* Denote

$$\liminf_{x\to+\infty} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\,dt$$

by α. Returning to (8) we can see that $\alpha \geqq \frac{1}{p}(1+\alpha)$, thus $\alpha \geqq \frac{1}{p-1}$. If we rewrite this into the form

$$\frac{p}{p-1} \leqq \alpha + 1 \leqq \beta + 1 \leqq q,$$

the following problem arises. Is it true that

$$\alpha+1 \leqq b^* \leqq \beta+1 \tag{13}$$

holds for every Young function Φ the conjugate of which has a finite power? If Φ itself also satisfies the growth condition, another proof of the maximal inequality shows that $b^* \leqq p\beta$ (see [5]). Since $\beta+1 \leqq p\beta$ always holds, the upper bound in (13) seems to be rather sharp if not false.

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Extension of Banach's principle for multiple sequences of operators

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Dedicated to Professor Béla Sz.-Nagy on his 70th birthday

1. Introduction

Let $(X, \mathscr{F})$ be a measurable space with a positive finite measure μ. Denote by $S=S(X, \mathscr{F})$ the set of the a.e. finite real-valued functions on X measurable with respect to $\mathscr{F}$. As is well-known, X endowed with the distance notion

$$d(\varphi, \psi)=\int_X \frac{|\varphi(x)-\psi(x)|}{1+|\varphi(x)-\psi(x)|}\, d\mu(x) \quad (\varphi, \psi \in S)$$

is a complete metric space (a so-called Frèchet space), and the convergence notion induced by d is equivalent with the convergence in measure.

Let B be a Banach space and let $T\colon B \to S$ be an operator. As usual, T is said to be *subadditive* if

(i) $|T(f+g)(x)| \leqq |Tf(x)|+|Tg(x)|$ a.e. on X for every $f, g \in B$,

and *positive homogeneous* if

(ii) $|T(\alpha f)(x)|=|\alpha Tf(x)|$ a.e. on X for every $\alpha \geqq 0$ and $f \in B$.

We shall deal only with subadditive and positive homogeneous operators T on B (sometimes these operators are said to be *convex*, too) for which the following condition is also satisfied:

(iii) T is *continuous in measure*, i.e. if $f_n, f \in B$ and $\|f_n-f\| \to 0$ as $n \to \infty$, then for every $\varepsilon>0$ we have

$$\mu\{x\colon |Tf_n(x)-Tf(x)|>\varepsilon\} \to 0 \quad \text{as} \quad n \to \infty.$$

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In certain cases we shall need a further property of the operators T, namely

(iv) $|Tf(x)-Tg(x)| \leqq \varkappa\{|T(f-g)(x)|+|T(g-f)(x)|\}$ a.e. on X for every $f, g \in B$,

where $\varkappa$ is a positive constant.

It is clear that if T is a linear operator, then (iv) is satisfied with $\varkappa=1/2$. Another example is the following: If T is an operator with properties (i) and

(v) T is *positive,* i.e. $Tf(x) \geqq 0$ a.e. on X for every $f \in B$,

then T possesses property (iv). In fact, now

$$Tf(x) = T(f-g+g)(x) \leqq T(f-g)(x)+Tg(x)$$

and similarly

$$Tg(x) \leqq T(g-f)(x)+Tf(x),$$

whence (iv) follows with $\varkappa=1$.

We note that if we replace property (ii) by

$\overline{\text{(ii)}}$ $|T(\alpha f)(x)|=|\alpha Tf(x)|$ a.e. on X for every real number α and $f \in B$,

then we can replace property (iv) by

$\overline{\text{(iv)}}$ $|Tf(x)-Tg(x)| \leqq 2\varkappa|T(f-g)(x)|$ a.e. on X for every $f, g \in B$.

Now, it is not hard to check that $\overline{\text{(iv)}}$ in the special case $2\varkappa=1$ implies property (i). So, if $\overline{\text{(ii)}}$ and (v) are satisfied, then properties (i) and $\overline{\text{(iv)}}$ with $2\varkappa=1$ are equivalent to each other.

2. Banach's principle for single series

Given an ordinary sequence $\{T_n: n=1, 2, \ldots\}$ of operators, we shall put, for every $f \in B$,

$$T^*f(x) = \sup_{n \geqq 1} |T_n f(x)|.$$

It is obvious that if the sequence $\{T_n f(x)\}$ is convergent a.e. on X for every $f \in B$, then a fortiori we also have that

(1) $$T^*f(x) < \infty \text{ a.e. on } X \text{ for every } f \in B.$$

The following results are well-known (see [1] and also [2, pp. 1—4], where the operators T_n are supposed to be linear, but the proofs apply, after some simple modifications, to the more general operators indicated in Section 1).

Theorem 0. *Let the operators* T_n *possess properties* (i)—(iv). *If condition* (1) *is satisfied, then the set of those* $f \in B$ *for which the sequence* $\{T_n f(x)\}$ *is a.e. convergent is closed.*

This immediately yields

Corollary. *Let the operators* T_n *possess properties* (i)—(iv). *If condition* (1) *is satisfied and the sequence* $\{T_n f(x)\}$ *is a.e. convergent for a set of* $f \in B$ *which is dense in* B, *then* $\{T_n f(x)\}$ *is a.e. convergent for every* $f \in B$.

The next lemma plays a decisive role in the proof of Theorem 0 and sometimes is called Banach's principle in a strict sense.

Lemma 0. *Let the operators* T_n *possess properties* (i)—(iii). *If condition* (1) *is satisfied, then there exists a positive, nonincreasing function* $C(\lambda)$, *defined for* $\lambda > 0$ *and tending to zero as* $\lambda \to \infty$ *such that*

$$\mu\{x: T^* f(x) > \lambda \|f\|\} \leqq C(\lambda) \quad \textit{for every} \quad \lambda > 0 \quad \textit{and} \quad f \in B.$$

A simple consequence is the following

Corollary. *Let the operators* T_n *possess properties* (i)—(iii). *If condition* (1) *is satisfied, then* T^* *is continuous in measure, even uniformly in* f.

3. Extension to multiple sequences using the convergence notion in Pringsheim's sense

Let $\mathcal{N}^d$ be the set of all d-tuples $\mathbf{k} = (k_1, \ldots, k_d)$ with positive integers for coordinates, where $d \geqq 1$ is a fixed integer. As usual, put

$$\mathbf{k} = (k_1, \ldots, k_d) \leqq (m_1, \ldots, m_d) = \mathbf{m} \quad \text{iff} \quad k_j \leqq m_j \quad (j = 1, \ldots, d),$$

$$\mathbf{k} \pm \mathbf{m} = (k_1 \pm m_1, \ldots, k_d \pm m_d), \ \mathbf{km} = (k_1 m_1, \ldots, k_d m_d), \quad \text{and} \quad \mathbf{1} = (1, \ldots, 1).$$

We recall that a d-multiple sequence $\{t_{\mathbf{m}}: \mathbf{m} \in \mathcal{N}^d\}$ of real numbers is said to be *convergent in Pringsheim's sense* if for every $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ so that $|t_{\mathbf{k}} - t_{\mathbf{m}}| < \varepsilon$ whenever

$$\text{(2)} \qquad \min(k_1, \ldots, k_d) \geqq M \quad \text{and} \quad \min(m_1, \ldots, m_d) \geqq M.$$

We consider a d-multiple sequence $\{T_{\mathbf{k}}: \mathbf{k} \in \mathcal{N}^d\}$ of operators having properties (i)—(iii) or (i)—(iv) enumerated in Section 1. It is a simple fact that the sequence $\{T_{\mathbf{k}} f(x)\}$ is convergent a.e. on X in Pringsheim's sense for a given $f \in B$ if and only if

$$\lim_{M \to \infty} \sup_{\text{under}(2)} |T_{\mathbf{k}} f(x) - T_{\mathbf{m}} f(x)| = 0 \quad \text{a.e. on } X,$$

or equivalently, for every $\varepsilon>0$,

$$\mu\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| > \varepsilon\} \to 0 \quad \text{as} \quad M \to \infty. \tag{3}$$

On the other hand, it is clear that if $\{T_{\mathbf{k}}f(x)\}$ is convergent a.e. on X in Pringsheim's sense for every $f\in B$, then we also have

(4)
$$T_*f(x) = \inf_{M=1,2,\ldots} \sup_{\min(k_1,\ldots,k_d)\geqq M} |T_{\mathbf{k}}f(x)| < \infty \quad \text{a.e. on } X \quad \text{for every} \quad f\in B.$$

For the sake of brevity, we write

$$T_{*M}f(x) = \sup_{\min(k_1,\ldots,k_d)\geqq M} |T_{\mathbf{k}}f(x)| \quad (M=1,2,\ldots).$$

The basic fact is again that condition (4) itself already implies the continuity of the operator T_* in measure, uniformly in f. Vice versa, it will be also seen that in certain cases such a continuity property for T_* is all that is needed to establish the a.e. convergence of the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ in Pringsheim's sense for every $f\in B$.

The following theorem extends Theorem 0.

Theorem 1. *Let the operators* $T_{\mathbf{k}}$, $\mathbf{k}\in\mathcal{N}^d$, *possess properties* (i)—(iv). *If condition* (4) *is satisfied, then the set of those* $f\in B$ *for which the d-multiple sequence* $\{T_{\mathbf{k}}f(x)\}$ *is a.e. convergent in Pringsheim's sense is closed.*

This implies the next

Corollary 1. *Let the operators* $T_{\mathbf{k}}$ *possess properties* (i)—(iv). *If condition* (4) *is satisfied and the d-multiple sequence* $\{T_{\mathbf{k}}f(x)\}$ *is a.e. convergent in Pringsheim's sense for a set of* $f\in B$ *which is dense in* B, *then* $\{T_{\mathbf{k}}f(x)\}$ *is a.e. convergent in Pringsheim's sense for every* $f\in B$.

The continuity property of T_* mentioned above is expressed in the following

Lemma 1. *Let the operators* $T_{\mathbf{k}}$, $\mathbf{k}\in\mathcal{N}^d$, *possess properties* (i)—(iii). *If condition* (4) *is satisfied, then there exists a positive, nonincreasing function* $C(\lambda)$, *defined for* $\lambda>0$ *and tending to zero as* $\lambda\to\infty$ *such that*

(5)
$$\mu\{x\colon \sup_{\min(k_1,\ldots,k_d)\geqq\lambda} |T_{\mathbf{k}}f(x)| > \lambda\|f\|\} \leqq C(\lambda) \quad \textit{for every} \quad \lambda>0 \quad \textit{and} \quad f\in B.$$

This immediately yields $\mu\{x\colon T_*f(x)>\lambda\|f\|\}\leqq C(\lambda)$, which can be reformulated as follows:

Corollary 2. *Let the operators* $T_{\mathbf{k}}$ *possess properties* (i)—(iii). *If condition* (4) *is satisfied, then* T_* *is continuous in measure, even uniformly in* f.

Proof of Lemma 1. It is modelled upon the proof of Lemma 0 (see in [2, pp. 2—3]).

By (ii), we need only establish (5) for $\|f\|=1$. Let an $\varepsilon>0$ be given. Owing to (4) for every $f\in B$ there exists an M, possibly depending on ε and f, such that

$$\mu\{x:\ T_{*M}f(x)>M\}\leqq\varepsilon.$$

In other words, this means that

$$B=\bigcap_{M=1}^{\infty}\{f:\ \mu\{x:\ T_{*M}f(x)>M\}\leqq\varepsilon\}.$$

We shall show that each set on the right of the last equality is closed. To this effect, observe that for each M,

$$\{f:\ \mu\{x:\ T_{*M}f(x)>M\}\leqq\varepsilon\}=\bigcap_{N=M}^{\infty}\{f:\ \mu\{x:\ T_{*MN}f(x)>M\}\leqq\varepsilon\}, \tag{6}$$

where

$$T_{*MN}f(x)=\max_{\substack{M\leqq\min(k_1,\dots,k_d)\leqq\\ \leqq\max(k_1,\dots,k_d)\leqq N}}|T_{\mathbf{k}}f(x)|\quad (M,\ N=1,2,\dots;\ M\leqq N).$$

By (i), for every f and g in B we have

$$|T_{*MN}f(x)-T_{*MN}g(x)|\leqq T_{*MN}(f-g)(x)+T_{*MN}(g-f)(x).$$

Consequently, for every $\delta>0$,

$$\mu\{x:\ |T_{*MN}f(x)-T_{*MN}g(x)|>\delta\}\leqq$$

$$\leqq\sum_{k_1=M}^{N}\dots\sum_{k_d=M}^{N}\left[\mu\left\{x:\ |T_{\mathbf{k}}(f-g)(x)|>\frac{\delta}{2}\right\}+\mu\left\{x:\ |T_{\mathbf{k}}(g-f)(x)|>\frac{\delta}{2}\right\}\right].$$

Since each operator $T_{\mathbf{k}}$ is continuous in measure (property (iii)), hence it follows that the operators T_{*MN} are also continuous in measure. Therefore, each of the sets

$$\{f:\ \mu\{x:\ T_{*MN}f(x)>M\}\leqq\varepsilon\}$$

is closed, and thus so is the set in (6).

Now we apply the Baire category theorem and conclude that one of the sets in (6) contains a closed ball, say with some center $f_0\in B$ and radius $\varrho>0$. This means that if $f\in B$ and $\|f-f_0\|\leqq\varrho$, then

$$\mu\{x:\ T_{*M}f(x)>M\}\leqq\varepsilon.$$

In other words, if $g\in B$ and $\|g\|\leqq 1$, then

$$\mu\{x:\ T_{*M}(f_0+\varrho g)(x)>M\}\leqq\varepsilon.$$

This yields

(7)
$$\mu\left\{x\colon T_{*M}g(x) > \frac{2M}{\varrho}\right\} \leqq \mu\{x\colon T_{*M}(f_0+\varrho g)(x) > M\} + \mu\{x\colon T_{*M}f_0(x) > M\} \leqq 2\varepsilon$$
$$\text{for every} \quad g\in B,\ \|g\| \leqq 1.$$

It is not hard to verify that (7) already implies (5) to be proved. In fact, put

$$C(\lambda) = \sup_{\|g\|\leqq 1} \mu\{x\colon T_{*[\lambda]}f(x) > \lambda\},$$

where by $[\lambda]$ we denote the integral part of $\lambda>0$. Inequality (7) shows that $C(\lambda)\leqq 2\varepsilon$ if $\lambda\geqq\max(M, 2M/\varrho)$. Thus we have

(8)
$$\lim_{\lambda\to\infty} C(\lambda) = 0$$

and our assertion is proved.

Proof of Theorem 1. Denote by $\mathscr{C}$ the set of $f\in B$ for which the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ is a.e. convergent in Pringsheim's sense. We are to show that if for a given $f\in B$ it is true that for every $\varepsilon>0$ there is a $g\in\mathscr{C}$ such that $\|f-g\|<\varepsilon$, then $f\in\mathscr{C}$ as well.

By (iv),

$$|T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| \leqq |T_{\mathbf{k}}f(x)-T_{\mathbf{k}}g(x)|+|T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)|+|T_{\mathbf{m}}g(x)-T_{\mathbf{m}}f(x)| \leqq$$
$$\leqq \varkappa[|T_{\mathbf{k}}(f-g)(x)|+|T_{\mathbf{k}}(g-f)(x)|+|T_{\mathbf{m}}(g-f)(x)|+|T_{\mathbf{m}}(f-g)(x)|]+|T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)|.$$

Thus, for every $\lambda>0$ and $M\geqq 1$,

(9)
$$\mu\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| > \lambda\|f-g\|\} \leqq$$
$$\leqq \mu\left\{x\colon T_{*M}(f-g)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\}+\mu\left\{x\colon T_{*M}(g-f)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\}+$$
$$+\mu\left\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)| > \frac{\lambda}{5}\|f-g\|\right\}.$$

Let us fix a $\delta>0$ and an $\varepsilon>0$. In virtue of (5) and (8) we get

$$\mu\{x\colon T_{*M}(f-g)(x) > M\|f-g\|\} \leqq C(M) \leqq \frac{\delta}{3}$$

if M is large enough, say $M\geqq M_1$, independently of $g\in\mathscr{C}$. Taking $\lambda=5\varkappa M_1$; hence it follows

(10)
$$\mu\left\{x\colon T_{*M}(f-g)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\}+$$
$$+\mu\left\{x\colon T_{*M}(g-f)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\} \leqq \frac{2\delta}{3} \quad \text{for} \quad M\geqq M_1.$$

Now let us choose $g\in\mathscr{C}$ in such a way that $\lambda\|f-g\|\leqq\varepsilon$. Due to (3), there exists an M_2 such that

(11) $$\mu\left\{x:\ \sup_{\text{under}(2)}|T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)|>\frac{\lambda}{5}\|f-g\|\right\}\leqq\frac{\delta}{3}\quad\text{for}\quad M\geqq M_2.$$

Collecting together (9)—(11), we can infer

$$\mu\{x:\ \sup_{\text{under}(2)}|T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)|>\varepsilon\}\leqq\delta\quad\text{for}\quad M\geqq\max(M_1,M_2).$$

Since δ and ε are arbitrary, we obtain relation (3). But this is equivalent to the a.e. convergence of the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ in Pringsheim's sense.

4. Extension to multiple sequences using the notion of regular convergence

Following HARDY [3] (cf. [5], where this kind of convergence was rediscovered and called "convergence in a restricted sense") we say that a d-multiple series

$$\sum_{\mathbf{k}\in\mathscr{N}^d}b_{\mathbf{k}}=\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}b_{k_1,\dots,k_d}$$

of real numbers is *regularly convergent* if for every $\varepsilon>0$ there exists an $M=M(\varepsilon)$ so that

(12) $$\left|\sum_{\mathbf{m}\leqq\mathbf{k}\leqq\mathbf{n}}b_{\mathbf{k}}\right|=\left|\sum_{k_1=m_1}^{n_1}\cdots\sum_{k_d=m_d}^{n_d}b_{k_1,\dots,k_d}\right|<\varepsilon$$

whenever

(13) $$\max(m_1,\dots,m_d)\geqq M\quad\text{and}\quad\mathbf{n}\geqq\mathbf{m}.$$

It is a trivial fact that the regular convergence of series (12) implies the convergence of the rectangular partial sums

$$s_{\mathbf{m}}=\sum_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}}b_{\mathbf{k}}\qquad(\mathbf{m}\in\mathscr{N}^d)$$

in Pringsheim's sense.

Given a d-multiple sequence $\{t_{\mathbf{m}}:\mathbf{m}\in\mathscr{N}^d\}$ of real numbers, first we define the "total" finite differences $\Delta t_{\mathbf{m}}$ as follows

$$\Delta t_{\mathbf{m}}=\sum_{\eta_1=0}^{1}\cdots\sum_{\eta_d=0}^{1}(-1)^{d-\eta_1-\dots-\eta_d}t_{m_1-1+\eta_1,\dots,m_d-1+\eta_d}$$

with the agreement that $t_{k_1,\dots,k_d}$ is taken to equal 0 if $k_j=0$ for at least one j, $1\leqq j\leqq d$. Then we consider the d-multiple series

(14) $$\sum_{\mathbf{m}\in\mathscr{N}^d}\Delta t_{\mathbf{m}},$$

whose rectangular partial sums coincide with the t_m. Now we say that the d-multiple sequence $\{t_{\mathbf{m}}\}$ is regularly convergent if series (14) is regularly convergent. In other words, this requires that for every $\varepsilon>0$ there exists an $M=M(\varepsilon)$ so that

$$\left|\sum_{\eta_1=0}^{1} \cdots \sum_{\eta_d=0}^{1} (-1)^{\eta_1+\cdots+\eta_d} t_{\boldsymbol{\eta}\mathbf{m}+(\mathbf{1}-\boldsymbol{\eta})\mathbf{n}}\right| < \varepsilon, \quad \boldsymbol{\eta} = (\eta_1, \ldots, \eta_d),$$

whenever (13) is satisfied. For brevity, denote by $\Delta_{\mathbf{m},\mathbf{n}} t_{\mathbf{k}}$ the expression between the absolute signs.

After these preliminaries, consider again a d-multiple sequence $\{T_{\mathbf{k}}\colon \mathbf{k}\in\mathcal{N}^d\}$ of operators possessing properties (i)—(iv). The a.e. regular convergence can be characterized as follows. The d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ is regularly convergent a.e. on X for an $f\in B$ if and only if

$$\lim_{M\to\infty} \sup_{\text{under}\,(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| = 0 \quad \text{a.e. on } X,$$

or equivalently, for every $\varepsilon>0$,

$$\mu\Big\{x\colon \sup_{\text{under}\,(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \varepsilon\Big\} \to 0 \quad \text{as} \quad M \to \infty. \tag{15}$$

It is obvious that if $\{T_{\mathbf{k}}f(x)\}$ is regularly convergent a.e. on X for every $f\in B$, then a fortiori we also have that

$$T^*f(x) = \sup_{\mathbf{k}\in\mathcal{N}^d} |T_{\mathbf{k}}f(x)| < \infty \quad \text{a.e. on } X \text{ for every } f\in B. \tag{16}$$

The fundamental fact is again that condition (16) itself already implies that the operator T^* is continuous in measure, uniformly in f. Indeed, both Lemma 0 and its Corollary are plainly true for the set $\{T_{\mathbf{k}}\colon \mathbf{k}\in\mathcal{N}^d\}$ of operators under properties (i)—(iii) and condition (16).

The extension of Theorem 0 reads as follows.

Theorem 2. *Let the operators* $T_{\mathbf{k}}, \mathbf{k}\in\mathcal{N}^d$, *possess properties* (i)—(iv). *If condition* (16) *is satisfied, then the set* $\mathscr{C}$ *of those* $f\in B$ *for which the d-multiple sequence* $\{T_{\mathbf{k}}f(x)\}$ *is a.e. regularly convergent is closed.*

An immediate consequence is that if the a.e. regular convergence of $\{T_{\mathbf{k}}f(x)\}$ is established when f belongs to some special class which is dense in B, then the a.e. regular convergence of $\{T_{\mathbf{k}}f(x)\}$ for every $f\in B$ is completely equivalent to the fulfilment of inequality (16).

Proof of Theorem 2. We have to prove that if $f\in B$ is such that for every $\varepsilon>0$ there is a $g\in\mathscr{C}$ for which $\|f-g\|<\varepsilon$, then $f\in\mathscr{C}$ as well. To this end, we prove (15).

A simple estimation shows that

$$\mu\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \lambda \|f-g\|\} \leqq$$

$$\leqq \mu\left\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x) - \Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\lambda}{2} \|f-g\|\right\} +$$

$$+\mu\left\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\lambda}{2} \|f-g\|\right\}.$$

As to the first term on the right, we illuminate the situation in the particular case $d=2$:

$$|\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x) - \Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| \leqq |T_{n_1 n_2} f(x) - T_{n_1 n_2} g(x)| + |T_{m_1 n_2} f(x) - T_{m_1 n_2} g(x)| +$$

$$+ |T_{n_1 m_2} f(x) - T_{n_1 m_2} g(x)| + |T_{m_1 m_2} f(x) - T_{m_1 m_2} g(x)| \leqq$$

$$\leqq \varkappa[|T_{n_1 n_2}(f-g)(x)| + |T_{n_1 n_2}(g-f)(x)| + \dots] \leqq 4\varkappa[T^*(f-g)(x) + T^*(g-f)(x)].$$

So, it can be easily seen that

$$\mu\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \lambda \|f-g\|\} \leqq \tag{17}$$

$$\leqq \mu\left\{x:\ T^*(f-g)(x) > \frac{\lambda}{2^{d+2}\varkappa} \|f-g\|\right\} + \mu\left\{x:\ T^*(g-f)(x) > \frac{\lambda}{2^{d+2}\varkappa} \|f-g\|\right\} +$$

$$+\mu\left\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\lambda}{2} \|f-g\|\right\}.$$

Owing to Lemma 0, applied this time to $\{T_{\mathbf{k}}:\ \mathbf{k} \in \mathcal{N}^d\}$, we obtain

$$\mu\left\{x:\ T^*(f-g)(x) > \frac{\lambda}{2^{d+2}\varkappa} \|f-g\|\right\} \leqq C\left(\frac{\lambda}{2^{d+2}\varkappa}\right), \tag{18}$$

independently of $g \in \mathscr{C}$. By choosing $\lambda = 1/\varepsilon\varepsilon_1$ and taking $\|f-g\| \leqq \varepsilon^2 \varepsilon_1$, where $\varepsilon_1 > 0$ will be chosen later on, we get from (17) and (18) that

$$\mu\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \varepsilon\} \leqq \tag{19}$$

$$\leqq 2C\left(\frac{1}{2^{d+2}\varkappa\varepsilon\varepsilon_1}\right) + \mu\left\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\varepsilon}{2}\right\}.$$

By (8), the first term on the right tends to zero as $\varepsilon_1 \to 0$. Given a $\delta > 0$, we can fix $\varepsilon_1 > 0$ so that this term does not exceed $\delta/2$. Then using the fact that $g \in \mathscr{C}$, the second term on the right-hand side of (19) can be made less than $\delta/2$ by choosing M sufficiently large, say $M \geqq M_0$. To sum up, we conclude that

$$\mu\{x:\ \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \varepsilon\} \leqq \delta \quad \text{for} \quad M \geqq M_0.$$

The proof of Theorem 2 is complete.

5. Application to a problem of summability of multiple orthogonal series

Let $\Phi=\{\varphi_{\mathbf{k}}(x)\colon \mathbf{k}\in\mathcal{N}^d\}$ be an orthonormal system (in abbreviation: ONS) on X. We shall consider the d-multiple series

$$\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(x), \tag{20}$$

where $\{c_{\mathbf{k}}\colon \mathbf{k}\in\mathcal{N}^d\}$ is a d-multiple sequence of real numbers (coefficients) for which

$$\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2<\infty. \tag{21}$$

By the Riesz—Fischer theorem the sum of series (20) exists in the sense of the mean convergence in $L^2(X)$-metric. In the following we shall be interested in the pointwise summability of series (20).

Let $\mathcal{A}=\{a_{\mathbf{m},\mathbf{k}}\colon \mathbf{m},\mathbf{k}\in\mathcal{N}^d\}$ be a given "d-multiple matrix" of real numbers with the following two properties:

$$a_{\mathbf{m},\mathbf{k}}\to a_{\mathbf{k}} \quad\text{as}\quad \min(m_1,\ldots,m_d)\to\infty \quad\text{for every } \mathbf{k}\in\mathcal{N}^d \tag{22}$$

and this convergence is regular in the sense of Section 4, and

$$\sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}^2<\infty \quad\text{for every } \mathbf{m}\in\mathcal{N}^d. \tag{23}$$

The so-called $\mathcal{A}$-means of series (20) are formed as follows

$$t_{\mathbf{m}}(x)=\sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}c_{\mathbf{k}}\varphi_{\mathbf{k}}(x)\quad(\mathbf{m}\in\mathcal{N}^d),$$

which results in a series-sequence transformation. By (21) and (23), the $\mathcal{A}$-means exist in the sense of $L^2(X)$-metric. Now, series (20) is said to be *$\mathcal{A}$-summable* (regularly or in Pringsheim's sense) if $\{t_{\mathbf{m}}(x)\colon \mathbf{m}\in\mathcal{N}^d\}$ as a d-multiple sequence is (regularly or in Pringsheim's sense, respectively) convergent.

We need the *modified Lebesgue functions* $L_M^*(\mathcal{A},\Phi;x)$ of the system Φ with respect to the summation method $\mathcal{A}$ defined in the following way. We set

$$K_{\mathbf{m}}(\mathcal{A},\Phi;x,y)=\sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}\varphi_{\mathbf{k}}(x)\varphi_{\mathbf{k}}(y)\quad(\mathbf{m}\in\mathcal{N}^d).$$

Again by (23), the kernel $K_{\mathbf{m}}(\mathcal{A},\Phi;x,y)$ as a function of y exists in the sense of $L^2(X)$-metric for almost every x. Consequently, the integral

$$L_M^*(\mathcal{A},\Phi;x)=\int_X\Big(\max_{\max(m_1,\ldots,m_d)\le M}|K_{\mathbf{m}}(\mathcal{A},\Phi;x,y)|\Big)\,d\mu(y)\quad(M=1,2,\ldots)$$

exists for almost every x and even belongs to $L^2(X)$.

Now we are ready to state

Theorem 3. *Suppose that $\Phi=\{\varphi_{\mathbf{k}}(x)\colon \mathbf{k}\in\mathcal{N}^d\}$ is an ONS on X, $\{c_{\mathbf{k}}\colon \mathbf{k}\in\mathcal{N}^d\}$ is a sequence of coefficients satisfying condition* (21), *and $\mathcal{A}=\{a_{\mathbf{m},\mathbf{k}}\colon \mathbf{m},\mathbf{k}\in\mathcal{N}^d\}$*

is a matrix of real numbers satisfying conditions (22) *and* (23). *If*

$$L := \int_X \{\sup_{M=1,2,\dots} L_M^*(\mathcal{A}, \Phi; x)\}^2 \, d\mu(x) < \infty, \tag{24}$$

then series (20) *is regularly* $\mathcal{A}$*-summable a.e. on* X.

This theorem in the special case $d=1$ is due to TANDORI [6].
First we prove the following

Lemma 2. *Under the conditions of Theorem* 3, *except* (22), *we have*

$$\int_X (\sup_{\mathbf{m}\in\mathcal{N}^d} |t_{\mathbf{m}}(x)|) \, d\mu(x) \leqq \{2L^{1/2} + (\sup_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{1},\mathbf{k}}^2)^{1/2}\}\{\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2\}^{1/2}. \tag{25}$$

Proof of Lemma 2. It will be done by a modification of the well-known classical method (see, e.g. [4] and also [6]).

For every positive integer M and $x\in X$ define $\mathbf{M}(x)=(M_1(x), \dots, M_d(x))\in\mathcal{N}^d$ in a unique way such that $1\leqq M_j(x)\leqq M$ for each $j=1, \dots, d$ and

$$t_{\mathbf{M}(x)}(x) = \max_{\max(m_1, \dots, m_d)\leqq M} t_{\mathbf{m}}(x) \quad (M = 1, 2, \dots).$$

Using the representation

$$t_{\mathbf{M}(x)}(x) = \int_X (\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y))(\sum_{\mathbf{n}\in\mathcal{N}^d} a_{\mathbf{M}(x),\mathbf{n}}\varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(y)) \, d\mu(y),$$

Fubini's theorem and the Schwarz inequality imply that

$$\int_X t_{\mathbf{M}(x)}(x) \, d\mu(x) = \int_X \{(\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)) \int_X \sum_{\mathbf{n}\in\mathcal{N}^d} a_{\mathbf{M}(x),\mathbf{n}}\varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(y) \, d\mu(x)\} \, d\mu(y) \leqq$$

$$\leqq \int_X \{|\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)| \int_X (\max_{\max(m_1, \dots, m_d)\leqq M} |\sum_{\mathbf{n}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{n}}\varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(y)|) \, d\mu(x)\} \, d\mu(y) =$$

$$= \int_X |\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)| L_M^*(\mathcal{A}, \Phi; y) \, d\mu(y) \leqq \{L \sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2\}^{1/2},$$

the last inequality is by (24). Applying Beppo Levi's theorem, hence it follows that

$$\int_X \{\sup_{\mathbf{m}\in\mathcal{N}^d} t_{\mathbf{m}}(x)\} \, d\mu(x) \leqq \{L \sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2\}^{1/2}.$$

Repeating this argument for $-t_{\mathbf{m}}(x)$, which corresponds to the system $\{-\varphi_{\mathbf{k}}(x) \colon \mathbf{k}\in\mathcal{N}^d\}$, we obtain

$$\int_X \{\sup_{\mathbf{m}\in\mathcal{N}^d} (-t_{\mathbf{m}}(x))\} \, d\mu(x) \leqq \{L \sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2\}^{1/2}.$$

Now, the wanted inequality (25) follows from the elementary relation

$$\sup_{\mathbf{m}\in\mathcal{N}^d} |t_{\mathbf{m}}(x)| \leqq \sup_{\mathbf{m}\in\mathcal{N}^d} t_{\mathbf{m}}(x) + \sup_{\mathbf{k}\in\mathcal{N}^d} (-t_{\mathbf{m}}(x)) + |t_{\mathbf{1}}(x)|.$$

Proof of Theorem 3. We recall that the set $l^2(\mathcal{N}^d)$ of those d-multiple sequences $\mathfrak{c}=\{c_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ for which condition (21) is satisfied, endowed with the usual vector operations and Euclidean norm, is a Banach space. The operators

$$\mathfrak{c} \to T_{\mathbf{m}}\mathfrak{c}(x) = t_{\mathbf{m}}(x)\colon l^2(\mathcal{N}^d) \to L^2(X) \quad (\mathbf{m}\in\mathcal{N}^d)$$

are clearly linear and continuous in $L^2(X)$-metric, a fortiori in measure. The continuity in $L^2(X)$-metric is shown by the estimate

$$\int_X t^2_{\mathbf{m}}(x)\, d\mu(x) = \sum_{\mathbf{k}\in\mathcal{N}^d} a^2_{\mathbf{m},\mathbf{k}} c^2_{\mathbf{k}} \leqq \big(\max_{\mathbf{k}\in\mathcal{N}^d} a^2_{\mathbf{m},\mathbf{k}}\big) \sum_{\mathbf{k}\in\mathcal{N}^d} c^2_{\mathbf{k}}.$$

Due to Lemma 2, for every $\mathfrak{c}\in l^2(\mathcal{N}^d)$,

$$T^*\mathfrak{c}(x) = \sup_{\mathbf{m}\in\mathcal{N}^d} |t_{\mathbf{m}}(x)| < \infty \quad \text{a.e. on } X. \tag{26}$$

For every $\mathfrak{c}\in l^2(\mathcal{N}^d)$ and $M=1,2,\ldots$ define $\mathfrak{c}^{(M)}=\{c^{(M)}_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ as follows

$$c^{(M)}_{\mathbf{k}} = \begin{cases} c_{\mathbf{k}} & \text{if } \max(k_1, \ldots, k_d) \leqq M, \\ 0 & \text{otherwise.} \end{cases}$$

It is also clear that these "finite sequences" $\mathfrak{c}^{(M)}$ constitute a dense subset in $l^2(\mathcal{N}^d)$. Furthermore, (22) yields

$$T_{\mathbf{m}}\mathfrak{c}^{(M)}(x) = \sum_{k_1=1}^{M}\cdots\sum_{k_d=1}^{M} a_{\mathbf{m},\mathbf{k}} c_{\mathbf{k}} \varphi_{\mathbf{k}}(x) \to \sum_{k_1=1}^{M}\cdots\sum_{k_d=1}^{M} a_{\mathbf{k}} c_{\mathbf{k}} \varphi_{\mathbf{k}}(x) \tag{27}$$
$$\text{as} \quad \min(m_1, \ldots, m_d) \to \infty \quad \text{for every} \quad M = 1, 2, \ldots$$

and even this convergence is regular in the sense of Section 4.

On the basis of (26) and (27), Theorem 2 is applicable and results that the d-multiple sequence $T_{\mathbf{m}}\mathfrak{c}(x)=t_{\mathbf{m}}(x)$ is regularly convergent a.e. on X for every $\mathfrak{c}\in l^2(\mathcal{N}^d)$. This finishes the proof of Theorem 3.

On closing, we formulate a slight generalization of Theorem 3. To this effect, let $\Lambda=\{\lambda_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ be a d-multiple sequence of positive numbers, which is non-decreasing in the sense that $\lambda_{\mathbf{k}}\leqq\lambda_{\mathbf{m}}$ whenever $\mathbf{k}\leqq\mathbf{m}$. Denote by $\Phi/\sqrt{\Lambda}$ the system $\{\varphi_{\mathbf{k}}(x)/\sqrt{\lambda_{\mathbf{k}}}: \mathbf{k}\in\mathcal{N}^d\}$. Then

$$L^*_M\left(\mathcal{A}, \frac{\Phi}{\sqrt{\Lambda}};\ x\right) = \int_X \left(\max_{\max(m_1,\ldots,m_d)\leqq M} \left|K_{\mathbf{m}}\left(\mathcal{A}, \frac{\Phi}{\sqrt{\Lambda}};\ x, y\right)\right|\right) d\mu(y) \quad (M = 1, 2, \ldots),$$

where

$$K_{\mathbf{m}}\left(\mathcal{A}, \frac{\Phi}{\sqrt{\Lambda}};\ x, y\right) = \sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}} \frac{\varphi_{\mathbf{k}}(x)\varphi_{\mathbf{k}}(y)}{\lambda_{\mathbf{k}}} \qquad (\mathbf{m}\in N^d).$$

The following theorem can be proved analogously to as Theorem 3 is proved.

Theorem 4. *Suppose that* $\Phi=\{\varphi_{\mathbf{k}}(x)\colon \mathbf{k}\in\mathcal{N}^d\}$ *is an ONS on* X, $\Lambda=\{\lambda_{\mathbf{k}}\}$ *is a nondecreasing sequence of positive numbers,* $\{c_{\mathbf{k}}\}$ *is a sequence of coefficients satisfying the condition*

$$\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2\lambda_{\mathbf{k}}^2<\infty, \tag{28}$$

and $\mathcal{A}=\{a_{\mathbf{m},\mathbf{k}}\}$ *is a matrix of real numbers satisfying conditions* (22) *and* (23). *If*

$$\int_X\Big\{\sup_{M=1,2,\ldots} L_M^*\Big(\mathcal{A},\frac{\Phi}{\sqrt{\Lambda}};\, x\Big)\Big\}^2 d\mu(x)<\infty,$$

then series (20) *is regularly* $\mathcal{A}$*-summable a.e. on* X.

Here we have to consider the set $l_\Lambda^2(\mathcal{N}^d)$ of those d-multiple sequences $\mathfrak{c}=\{c_{\mathbf{k}}\colon \mathbf{k}\in\mathcal{N}^d\}$ for which condition (28) is satisfied. Introducing the usual vector operations and the norm

$$\|c\|_\Lambda=\Big\{\sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2\lambda_{\mathbf{k}}^2\Big\}^{1/2},$$

$l_\Lambda^2(\mathcal{N}^d)$ becomes also a Banach space.

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Non-horizontal geodesics of a Riemannian submersion

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

1. Introduction. For Riemannian manifolds M and P, a submersion $\pi: P \to M$ is a smooth mapping of P onto M which has maximal rank and preserves the length of horizontal vectors. A tangent vector to P at x is called *horizontal* if it is orthogonal to the fiber $\pi^{-1} \circ \pi(x)$ through x, *vertical* if it is tangent to the fiber. The fundamental concepts of a Riemannian submersion were introduced by B. O'Neill [2]. The horizontal geodesics of P were studied in [3].

Our aim here is to investigate the non-horizontal geodesics of P and to characterize them with their "projections" on the basic manifold M and on the fibers through their points. As an application we shall get a stability property of some fibers with respect to the geodesic flow of a class of Riemannian submersions.

We use the method of moving frame; for the notation and the basic relations of the invariants of a submersion we refer to [1].

The paper is organised as follows. Section 2 is devoted to the basic concepts of a Riemannian submersion. In Section 3 we discuss the translation of fibers along a curve of M defined by the horizontal subspaces and the relation of this translation to the Riemannian parallel translation. In Section 4 we treat the equation of geodesics as we need. In Section 5 we apply our result in a special class of Riemannian submersions where the translation of fibers is homothetic transformation. Finally in Section 6 we investigate the stability of fibers with respect to the geodesic flow in the above discussed class of submersions.

Throughout this paper the indices $i, j, k, \dots, a, b, c, \dots$ and $\alpha, \beta, \gamma, \dots$ will run from 1 to $n+k$, form 1 to n and from $n+1$ to $n+k$, respectively, where $n = \dim M$, $n+k = \dim P$. The summation convention will be adopted.

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2. Adapted frames. Let $\{L(P), p, P\}$ and $\{L(M), q, M\}$ denote the principal fiber bundle of linear frames of P and M, respectively. The bundle of adapted frames $\{L_M(P), p, P\}$ over P of the submersion $\pi: P \to M$ is defined as a subbundle of $\{L(P), p, P\}$ consisting of frames $\{x; e_1, \dots, e_{n+k}\} \in L_x(P)$ such that the vectors $e_1, \dots, e_n$ are horizontal and the vectors $e_{n+1}, \dots, e_{n+k}$ are vertical. The structure group of the adapted frame bundle is isomorphic to the group $GL(n) \times \times GL(k) \subset GL(n+k)$.

Let ω and φ denote the $\mathbf{R}^{n+k}$-valued canonical form and the $\mathrm{gl}(n+k)$-valued Riemannian commection form on $L(P)$. ω and φ satisfy the structure equation

$$d\omega = -\varphi \wedge \omega \quad \text{or} \quad d\omega^i = -\varphi^i_k \wedge \omega^k$$

where ω^i and φ^i_k are the components of the forms ω and φ with respect to the canonical bases of $\mathbf{R}^{n+k}$ and $\mathrm{gl}(n+k)$.

The fundamental tensors of the submersion are of the form

$$A = A_{\beta}{}^{a}{}_{c} e_a \otimes \omega^\beta \otimes \omega^c, \quad T = T_{\beta}{}^{a}{}_{\gamma} e_a \otimes \omega^\beta \otimes \omega^\gamma$$

where $\{e_1, \dots, e_{n+k}\}$ is an adapted frame and $\{\omega^1, \dots, \omega^{n+k}\}$ is its dual coframe. The Riemannian metric tensor of P can be written as

$$g = g_{ab}\omega^a \otimes \omega^b + g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta$$

on the adapted frame bundle. The metric tensor of the basic manifold is $\tilde{g} = g_{ab}\tilde{\omega}^a \otimes \tilde{\omega}^b$, where $\omega^a = \pi^* \tilde{\omega}^a$. The Riemannian connection form ψ of M defines a form on $L_M(P)$ in a natural way whose components are denoted also by ψ^a_b.

Proposition 1. *The fundamental tensors A, T of the submersion and the Riemannian connection forms of P and M are related by the equations*

$$\varphi^a_b = \psi^a_b + (1/2) A_{\gamma}{}^{a}{}_{b}\omega^\gamma, \tag{1}$$

$$\varphi^a_\beta = (1/2) A_{\beta}{}^{a}{}_{c}\omega^c + (1/2) T_{\beta}{}^{a}{}_{\gamma}\omega^\gamma, \tag{2}$$

$$\varphi^\alpha_b = -(1/2) A^{\alpha}_{\cdot bc}\omega^c - (1/2) T^{\alpha}_{\cdot b\gamma}\omega^\gamma, \tag{3}$$

where

$$g_{\alpha\beta} A^{\alpha}_{\cdot bc} = g_{ab} A_{\beta}{}^{a}{}_{c}, \quad g_{\alpha\beta} T^{\alpha}_{\cdot b\gamma} = g_{ab} T_{\beta}{}^{a}{}_{\gamma},$$

and the tensors A and T satisfy

$$A_{\gamma ab} + A_{\gamma ba} = 0, \quad T_{\beta}{}^{a}{}_{\gamma} = T_{\gamma}{}^{a}{}_{b}, \tag{4}$$

Proof. The more detailed description of the adapted frame bundle and the proof of equations (1), (2), (4) can be found in [1], pp. 155—158 (orthonormed frames are used).

To prove the equation (3) we note that the Riemannian metric tensor satisfies

$$dg_{ij} - \varphi^k_i g_{kj} - \varphi^k_j g_{ik} = 0.$$

Since $g_{a\beta}=0$, we have

$$0 = dg_{a\beta} = g_{ac}\varphi^c_\beta + g_{\beta\gamma}\varphi^\gamma_a.$$

Using (2) we get

$$g_{\beta\gamma}\varphi^\gamma_a = -g_{ac}\varphi^c_\beta = -g_{ac}(1/2)(A_\beta{}^c{}_d\omega^d + T_\beta{}^c{}_\delta\omega^\delta) = -g_{\beta\gamma}(1/2)(A^\gamma{}_{\cdot ad}\omega^d + T^\gamma{}_{\cdot a\delta}\omega^\delta).$$

This completes the proof.

3. Translation of fibers. Let be given a curve $y(t)$ of the basic manifold M defined on an open interval $t\in I\subset \mathbf{R}$. If $t_0\in I$, $x_0\in\pi^{-1}(y(t_0))$, there is a unique horizontal curve $x(x_0, t)\,(t\in I)$ of P satisfying $x(x_0, t_0)=x_0$ and $\pi\circ x(x_0, t)=y(t)$. These curves define a 1-parameter family of maps $\tau_{t_0,t}\colon \pi^{-1}(y(t_0))\to\pi^{-1}(y(t))$ along the curve $y(t)$, $t\in I$, such that

$$\tau_{t_0,t}x_0 := x(x_0, t).$$

This map is called translation of fibers along the curve $y(t)$ of M. The derivative map $\tau_{t_0,t*}$ induces a translation of vertical vectors along horizontal curves. A vertical vectorfield Z on P is constant with respect to this translation if and only if the Lie derivative $\mathscr{L}_Y Z=[Y,Z]=0$, where Y is a horizontal vectorfield defined in a neighbourhood of $\pi^{-1}(y(t))\,(t\in I)$ satisfying $Y(x(t))=\dot{x}(t)$ for the horizontal lifts $x(t)$ of $y(t)$. Here the dot denotes the derivation by t.

Proposition 2. *If $Y=Y^a e_a$ and $Z=Z^\alpha e_\alpha$ are horizontal and vertical vectorfields on P then the expression $\nabla_Y Z-\mathscr{L}_Y Z$ is a $(1, 1)$-type tensorfield satisfying*

$$\nabla_Y Z-\mathscr{L}_Y Z = (1/2)A_\gamma{}^a{}_c Y^c Z^\gamma e_a - (1/2)T^\alpha{}_{\cdot c\gamma}Y^c Z^\gamma e_\alpha.$$

Proof. We fix a point $x_0\in P$. Let $U\subset M$ be a neighbourhood of $y_0=\pi(x_0)\in M$, and $t\in I\mapsto y(t)\in M$ is a curve in U such that $y(t_0)=y_0\ (t_0\in I)$ and $\dot{y}(t_0)=\pi_* Y(x_0)$. Let $\bar{Y}$ be a horizontal vectorfield defined on a neighbourhood of x_0 such that $\bar{Y}(x(t))=\dot{x}(t)$ for horizontial lifts $x(t)$ of $y(t)$. Let be given a frame field $\{\tilde{e}_1(y), ..., \tilde{e}_n(y)\}$ on U and an adapted frame field $\{e_1(x), ..., e_{n+k}(x)\}$ on a neighbourhood of x_0 such that $\pi_* e_a(x)=\tilde{e}_a(\pi(x))$. For a smooth function f on P we denote the components of its differential with respect to the adapted coframe $\{\omega^i\}$ dual to $\{e_i\}$ by $\partial_i f$, i.e., $df=(\partial_i f)\omega^i$. We can write for $\bar{Y}=\bar{Y}^a e_a$

$$\nabla_Y Z-\mathscr{L}_Y Z = \nabla_Z \bar{Y} = \left((\partial_\gamma \bar{Y}^a)Z^\gamma + \varphi^a_c(Z)\bar{Y}^c\right)e_a + \varphi^\alpha_c(Z)\bar{Y}^c e_\alpha.$$

Since the components $\bar{Y}^a$ are constant on the fiber we have $\partial_\gamma \bar{Y}^a=0$. By Proposition 1 we get

$$\nabla_Y Z-\mathscr{L}_Y Z = \nabla_Z\bar{Y} = \varphi^a_c(Z)\bar{Y}^c e_a + \varphi^\alpha_c(Z)\bar{Y}^c e_\alpha = (1/2)A_\gamma{}^a{}_c Z^\gamma\bar{Y}^c e_a - (1/2)T^\alpha{}_{\cdot c\gamma}Z^\gamma\bar{Y}^c e_\alpha,$$

since the forms ψ^a_c are lifted from a form on $L(M)$ and therefore $\psi^a_c(Z)=0$. At the point x_0; $\bar{Y}(x_0)=Y(x_0)$, and the proof is complete.

4. Equation of geodesics. Let $x(s)$ $(s\in I)$ be an arc-length parametrized curve in P, $y(s)=\pi\circ x(s)$ is its projection curve in the basic manifold M and $z(s)=\tau_{s,s_0}x(s)$ is its development in the fiber $\pi^{-1}(y(s_0))$, $s_0\in I$. Comma denotes the derivation by s. The tangent vector $x'(s)$ can be written as $x'(s)=Y(s)+Z(s)$, where $Y(s)$ is its horizontal part and $Z(s)$ is its vertical part. The curve $x(s)$ is a geodesic if and only if the equations

$$Y^{a\prime}+\varphi^a_c(x')Y^c+\varphi^a_\gamma(x')Z^\gamma=0,\quad Z^{\alpha\prime}+\varphi^\alpha_c(x')Y^c+\varphi^\alpha_\gamma(x')Z^\gamma=0$$

are satisfied, where $Y=Y^a e_a$, $Z=Z^\alpha e_\alpha$ for an adapted frame field $\{e_i(s)\}$ along $x(s)$. By Proposition 1 these equations can be written in the form

$$Y^{a\prime}+\psi^a_c(Y)Y^c+A_\beta{}^a{}_c Z^\beta Y^c+(1/2)T_\beta{}^a{}_\gamma Z^\beta Z^\gamma=0, \tag{5}$$

$$Z^{\alpha\prime}+\varphi^\alpha_\gamma(x')Z^\gamma-(1/2)A^\alpha_{\cdot bc}Y^bY^c-(1/2)T^\alpha_{\cdot c\beta}Y^cZ^\beta=0. \tag{6}$$

We know $A^\alpha_{\cdot bc}Y^bY^c=0$ by the skew symmetry of A. The tangent vector y' of the projection curve $y(s)$ has the components Y^a with respect to the frame $\tilde e_a=\pi_*e_a$ hence the equations (5) are equivalent to

$$\tilde\nabla_s y'=-(A_\beta{}^a{}_c Z^\beta Y^c+(1/2)T_\beta{}^a{}_\gamma Z^\beta Z^\gamma)\tilde e_a. \tag{7}$$

From the equations (6) and (7) it follows immediately that the horizontal lifts of geodesics of M are geodesics of P (in this case $Z^\alpha\equiv 0$).

It is also clear that if a geodesic $x(s)$ of P is horizontal at $s_0\in I$ then it is horizontal for all $s\in I$.

We investigate here the non-horizontal geodesics of P. If $x(s)$, $s\in I$, is a non-horizontal curve then $(\tau_{s,s_0}x(s))'\neq 0$ for all $s\in I$. Let us denote by $\bar x_\sigma(s)$ the horizontal lift of $y(s)=\pi\circ x(s)$ through $x(\sigma)$ that is $\bar x_\sigma(\sigma)=x(\sigma)$ $(\sigma\in I)$. Let $Y(x)$ be a horizontal vectorfield defined in a neighbourhood of the curve $x(s)$ such that $Y(\bar x_\sigma(s))=\bar x'_\sigma(s)$ for all $\sigma, s\in I$. Let $Z(x)$ be a vertical vectorfield defined in a neighbourhood of the curve $x(s)$ such that $Z(\bar x_\sigma(s_1))=\tau_{s_2,s_1*}Z(\bar x_\sigma(s_2))$ is satisfied for all $s_1,s_2\in I$, where τ_{s_1,s_2} is the translation $\pi^{-1}(y(s_1))\to\pi^{-1}(y(s_2))$ along $y(s)$. From the definition of the vectorfields Y and Z we get $\mathcal{L}_YZ=0$. It follows from (6) that

$$0=(\partial_bZ^\alpha)Y^b+\varphi^\alpha_\gamma(Y)Z^\gamma+(\partial_\beta Z^\alpha)Z^\beta+\varphi^\alpha_\gamma(Z)Z^\gamma-(1/2)T^\alpha_{\cdot c\beta}Z^\beta Y^c.$$

Since

$$((\partial_bZ^\alpha)Y^b+\varphi^\alpha_\gamma(Y)Z^\gamma)e_\alpha=\nabla_YZ-\varphi^a_\gamma(Y)Z^\gamma e_a,$$

and by Proposition 2,

$$\nabla_YZ=\mathcal{L}_YZ+(1/2)A_\gamma{}^a{}_cY^cZ^\gamma e_a-(1/2)T^\alpha_{\cdot c\gamma}Y^cZ^\gamma e_\alpha,$$

which together with $\mathcal{L}_YZ=0$ imply

$$((\partial_\beta Z^\alpha)Z^\beta+\varphi^\alpha_\gamma(Z)Z^\gamma)e_\alpha-\varphi^a_\gamma(Y)Z^\gamma e_a+(1/2)A_\gamma{}^a{}_cY^cZ^\gamma e_a-T^\alpha_{\cdot c\gamma}Y^cZ^\gamma e_\alpha=0.$$

By Proposition 1 $\varphi^\alpha_\gamma(Y)Z^\gamma=(1/2)A_\gamma{}^a{}_c Y^c Z^\gamma$, we find

$$(8)\qquad (\partial_\beta Z^\alpha)Z^\beta+\varphi^\alpha_\gamma(Z)Z^\gamma = T^\alpha_{\cdot c\gamma}Y^c Z^\gamma.$$

Now, we fix a parameter $s_0\in I$. If we map the fibers $\pi^{-1}(y(s))$ onto $\pi^{-1}(y(s_0))$ using the translation τ_{s,s_0} along $y(s)$, we get the curve $z(s):=\tau_{s,s_0}x(s)$ on $\pi^{-1}(y(s_0))$. The vertical vectors $\{e_{n+1}(s), \dots, e_{n+k}(s)\}$ of the adapted frame field $\{e_i(s)\}$ are mapped into the vertical frame field $\overset{*}{e}_\alpha(s):=\tau_{s,s_0*}e_\alpha(s)$ along $z(s)$; the vertical vectors $Z(x(s))$ and $Z(z(s))$ have the same components with respect to the corresponding frames. Thus the left hand side of the equation (8) is $(\overset{*}{\nabla}_s z')(s_0)$; where $\overset{*}{\nabla}_s$ is the induced covariant derivation along $z(s)$ on the submanifold $\pi^{-1}(y(s_0))$ defined by the induced Riemannian connection form $\varphi^\alpha_\gamma(Z)$.

We can summarize the obtained results.

Theorem 1. *Let $x(s)$, $s\in I$, be an arc-length parametrized curve in P. It is a geodesic of P if and only if*

(i) *the projection curve $y(s)=\pi\circ x(s)$ satisfies*

$$\tilde{\nabla}_s y' = -\pi_*[A(x')x'+(1/2)T(x', x')] = -[A_\beta{}^a{}_c Z^\beta Y^c+(1/2)T_\beta{}^a{}_\gamma Z^\beta Z^\gamma]\tilde{e}_a$$

where $\tilde{\nabla}_s$ is the covariant derivative in the basic manifold M;

(ii) *for all $s_0\in I$ the development $z(s)=\tau_{s,s_0}x(s)$ of $x(s)$ in the fiber $\pi^{-1}(y(s_0))$ satisfies*

$$\overset{*}{\nabla}_s z' = T^\alpha_{\cdot c\beta}Y^c Z^\beta e_\alpha \quad at \quad s = s_0,$$

where $\overset{*}{\nabla}_x$ is the induced covariant derivative in the fiber $\pi^{-1}(y(s_0))$.

Proof. We have proved already that the conditions (i) and (ii) are necessary for a geodesic $x(s)$ of P. The sufficiency follows from the fact that the conditions (i) and (ii) give a second order differential equation for $x(s)$, it has for all initial points and tangent vectors a unique solution which has to be the same curve as the geodesic with this initial point and tangent vector.

5. Homothetic fibers. Here we give a more detailed discussion of the case which can be obtained from a Riemannian submersion whose fibers are totally geodesic submanifolds by a bundle-like homothetic deformation with a positive smooth function defined on the basic manifold (cf. [1]).

If $g_{ab}\omega^a\otimes\omega^b+g_{\alpha\beta}\omega^\alpha\otimes\omega^\beta$ is the metric tensor of the submersion $\{P, \pi, M\}$ with totally geodesic fibers (i.e., $T_\beta{}^a{}_\gamma\equiv 0$ and the translation of the fibers is isometry), the submersion with the metric tensor

$$g_{ab}\,\omega^a\otimes\omega^b+\exp(-\varrho)\,g_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta,$$

where $\varrho: M \to \mathbf{R}$ is a smooth function, is a Riemannian submersion such that the translation of the fibers is homothetic map. In this case the second fundamental tensor of the fibers is of the form

$$T_{\beta a \gamma} = \varrho_a \exp(-\varrho) g_{\beta\gamma}, \quad \text{where} \quad d\varrho = \varrho_a \omega^a. \tag{9}$$

(cf. Corollary of Theorem 2 in [1], p. 161.)

Theorem 2. *Let $\{P, \pi, M\}$ be a Riemannian submersion satisfying* (9). *The curve $x(s)$ $(s \in I)$ is an arc-length parametrized geodesic if and only if*

(i) *the projection curve $y(s) = \pi \circ x(s)$ satisfies*

$$\tilde{\nabla}_s y' = -\pi_*[A(x')x'] - (c/2)\operatorname{grad}(\exp \varrho(y(s)))$$

for a positive constant c;

(ii) *the development $z(s) = \tau_{s,s_0} x(s)$ in the fiber $\pi^{-1}(y(s_0))$ is a geodesic parametrized with the speed $\|z'\| = \sqrt{c} \cdot \exp \varrho(y(s))$, where the constant satisfies in an initial point $s_0 \in I$,*

$$\|y'(s_0)\|^2 + c \cdot \exp 2\varrho(y(s_0)) = 1.$$

Proof. The conditions (i) and (ii) of Theorem 1 give the equations

$$\tilde{\nabla}_s y' = -\pi_* A(x')x' - (1/2)(\operatorname{grad} \varrho)\langle z', z'\rangle, \tag{10a}$$

$$\overset{*}{\nabla}_s z' = \varrho' z'. \tag{10b}$$

The equation (10a) can be obtained immediately by substitution of (9). For the proof of (10b) we note that in our case the translation of fibers τ_{s,s_0} is homothetic and consequently affine map, thus for all $s_1, s_2 \in I$ the developments $z_1(s)$ and $z_2(s)$ in the fibers $\pi^{-1}(y(s_1))$ and $\pi^{-1}(y(s_2))$, respectively, satisfy $\tau_{s_1,s_2 *}(\overset{*}{\nabla}_s z_1') = \overset{*}{\nabla}_s z_2'$. It follows that the condition (ii) of Theorem 1 can be considered in a fixed fiber $\pi^{-1}(y(s_0))$ $(s_0 \in I)$ for all $s \in I$. The right hand side of the equation (10b) can be obtained by (9)

$$\overset{*}{\nabla}_s z' = T^{\alpha}_{\cdot c\beta} Y^c Z^\beta \overset{*}{e}_\alpha = z'(\varrho_c Y^c) = z'\varrho'.$$

The equation (10b) means that the curve $z(s)$ is a geodesic in $\pi^{-1}(y(s_0))$, its speed can be computed as follows:

$$\overset{*}{\nabla}_s(\exp(-\varrho)z') = -\varrho' \exp(-\varrho)z' + \exp(-\varrho)\overset{*}{\nabla}_s z' = 0,$$

that is

$$\langle z', z'\rangle = \exp(-\varrho) \cdot c \exp 2\varrho = c \cdot \exp \varrho, \quad c = \text{constant}.$$

We substitute this in equation (10a):

$$\tilde{\nabla}_s y' = -\pi_* A(x')x' - (1/2)(\operatorname{grad} \varrho) c \cdot \exp \varrho = -\pi_* A(x')x' - (c/2)\operatorname{grad}(\exp \varrho),$$

and the necessity of the conditions of Theorem 2 is proved. But they are sufficient since for all initial points and tangent vectors these two second order equations have the same unique solution.

Corollary 1. *A vertical curve in a fiber* $\pi^{-1}(y_0)$ *is a geodesic of* P *if and only if it is a geodesic in the submanifold* $\pi^{-1}(y_0)$ *and* $(\operatorname{grad} \varrho)(y_0)=0$. *In this case the fiber* $\pi^{-1}(y_0)$ *is a totally geodesic submanifold.*

Corollary 2. *The function* $\Phi(y, y'): TM\to\mathbf{R}$ *defined by* $\Phi(y, y')=$ $=(1/c)\langle y', y'\rangle+\exp\varrho(y)$ *is constant along a projection curve of a geodesic.*

Proof. By Theorem 2 we have for a geodesic $x(s)$,

$$1=\langle x', x'\rangle=\langle y', y'\rangle+c\cdot\exp\varrho=c\cdot\Phi(y, y').$$

Remark. The constant c along a geodesic $x(s)$ can be expressed by the angle θ between the geodesic $x(s)$ and the fiber:

$$c=\exp(-\varrho)(1-\langle y', y'\rangle)=\exp(-\varrho)\cos^2\theta.$$

Thus the statement that $\sqrt{c}=\exp\left(-(1/2)\varrho\right)\cos\theta$ is constant is a generalization of Clairaut's Theorem on surface of revolution.

Corollary 3. *If* $\{P, \pi, M\}$ *is a Riemannian submersion with totally geodesic fibers, the curve* $x(s)$ *is a geodesic of* P *if and only if the following conditions are satisfied:*

(i) *let* σ *denote the arc-length parameter of the projection curve* $y=\pi\circ x$, *the first vector of curvature is*

$$\tilde{\nabla}_\sigma\frac{dy}{d\sigma}=-\pi_* A\left(\frac{dx}{d\sigma}\right)\frac{dx}{d\sigma},$$

(ii) *the development* $z(s)=\tau_{s,s_0}x(s)$ *of* $x(s)$ *in the fiber* $\pi^{-1}(y(s_0))$ *is a constant speed geodesic.*

Proof. Theorem 2 implies in the case $\varrho=$constant $\|y'\|^2=1-\|z'\|^2=$constant, therefore σ is proportial to s. It follows

$$\tilde{\nabla}_\sigma\frac{dy}{d\sigma}=\|y'\|^{-2}\nabla_s y'=-\|y'\|^{-2}\pi_* A(x')x'=-\pi_* A\left(\frac{dx}{d\sigma}\right)\frac{dx}{d\sigma}.$$

Example. Let us consider the Hopf bundle $\pi\colon S^{2m+1}\to\mathbf{CP}(m)$ of the unit sphere over the complex projective space equipped with the Fubini—Study metric. It is a Riemannian submersion with totally geodesic fibers (cf. [2], p. 466).

Its tensor A can be expressed in the form $A(Z)Y=\langle Z, N\rangle JY$, where J is the almost complex tensor on $\mathbf{C}^{m+1}$, N is the tangent unit vectorfield of the fibers defined by JM for the unit normal vectorfield M of S^{2m+1}, Z and Y are arbitrary vertical and horizontal tangent vectorfields of S^{2m+1}.

We get that the curve $x(s)$ is a geodesic of S^{2m+1} if and only if
(i) the first vector of curvature of $y(s)$ is expressed by

$$\tilde{\nabla}_\sigma \frac{dy}{d\sigma} = -\zeta J \frac{dy}{d\sigma},$$

where σ is the arc-length parameter of $y(s)$ and the vertical part Z of $\dfrac{dx}{d\sigma}$ is $Z=\zeta N$. We have

$$\tilde{\nabla}_\sigma \left(J\frac{dy}{d\sigma}\right) = J\tilde{\nabla}_\sigma \frac{dy}{d\sigma} = \zeta \frac{dy}{d\sigma},$$

that is the curve $y(s)$ is a real 2-plane curve in $\mathbf{CP}(m)$ of curvature $-\zeta$ contained in the complex projective line (2-sphere) spanned by y' and Jy'.

(ii) ζ=constant, that is the fiber curve $z(\sigma)=\tau_{\sigma,\sigma_0}x(\sigma)$ is of the speed $\dfrac{dz}{d\sigma}=z'\colon \dfrac{d\sigma}{ds}=\zeta\colon \|y'\|=\dfrac{\zeta}{\sqrt{1-\zeta^2}}$, with respect to the arc-length parameter σ of the basic curve $y(\sigma)$, which is a circle of curvature $-\zeta$ in a complex projective line in $\mathbf{CP}(m)$ $(-1<\zeta<1)$.

6. Stable fibers of the geodesic flow. As we observed in Corollary 1 of Theorem 2 if $y_0\in M$ is a critical point of the function $\varrho\colon M\to\mathbf{R}$, $\pi^{-1}(y_0)$ is a total geodesic submanifold of P, or equivalently the tangent bundle $T(\pi^{-1}(y_0))$ is an invariant submanifold of TP with respect to the geodesic flow on TP.

The fiber $\pi^{-1}(y_0)$ is called stable (with respect to the geodesic flow) if for any $\varepsilon>0$ it is possible to find a $\delta=\delta(\varepsilon)>0$ such that if an arc-length parametrized geodesic $x(s)$ satisfies the initial conditions

$$d(\pi\circ x(s_0),\ y_0) < \delta, \quad \|\pi_* x'(s_0)\| < \delta$$

then the inequalities $d(\pi\circ x(s), y_0)<\varepsilon$, $\|\pi_* x'(s)\|<\varepsilon$ hold for any $s\in\mathbf{R}$, where d is the distance on M. (For simplicity we suppose that the manifold P is complete and the geodesics of P are defined for all $s\in\mathbf{R}$.)

Theorem 3. *If the function $\varrho\colon M\to\mathbf{R}$ has at the point $y_0\in M$ strict local minimum, the fiber $\pi^{-1}(y_0)$ is stable with respect to the geodesic flow.*

Proof. Since the function $\exp\varrho$ has at y_0 strict local minimum, the strict inequality $\exp\varrho(y)>\exp\varrho(y_0)$ if $y\neq y_0$, is true in a neighbourhood $V\subset M$. If $y'\neq 0$

$$\Phi(y, y') = (1/c)\langle y', y'\rangle + \exp\varrho(y) > \exp\varrho(y) \geqq \exp\varrho(y_0)$$

for $y\in V$, that is the function $\Phi(y, y')$ has a strict local minimum at $\{y_0, 0\}\in TM$. Let be given an $\varepsilon>0$ such that the ε-neighbourhood of y_0 is contained in V. We consider the values of Φ on the boundary of the neighbourhood defined by the inequalities $d(y, y_0)<\varepsilon$, $\|y\|<\varepsilon$. The function Φ reaches its minimum Φ^* on this compact set (if ε is sufficiently small) and $\Phi^*>\exp\varrho(y_0)$. We can find a neighbourhood $d(y, y_0)<\delta$, $\|y'\|<\delta$ in TM such that here $\Phi(y, y')<\Phi^*$. If the initial point and tangent vector of $y(s)=\pi\circ x(s)$ satisfy the inequalities $d(y(s_0), y_0)<\delta$; $\|y'(s_0)\|<\delta$ then $\Phi(y(s_0), y'(s_0))<\Phi^*$. But by Corollary 2 of Theorem 2 Φ is constant along $y(s)=\pi\circ x(s)$ if $x(s)$ is a geodesic of P, consequently $\Phi(y(s), y'(s))<\Phi^*$ for all $s\in\mathbf{R}$. Therefore the curve $\{y(s), y'(s)\}=\{\pi\circ x(s), \pi_* x'\}$ cannot attain the boundary of the ε-neighbourhood of $\{y_0, 0\}$, because there would be $\Phi\geqq\Phi^*$.

This completes the proof.

Corollary. *If $\{P, \pi, M\}$ is a submersion with 1-dimensional fibers, then at the strict minimum point $y_0\in M$ of the function $\varrho\colon M\to\mathbf{R}$ the fiber geodesic $\pi^{-1}(y_0)$ is stable.*

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On a Paley-type inequality

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

In this paper a new space similar to the dyadic Hardy spaces is investigated. This space is defined by a shift-invariant norm and it is proved that for $1<p<\infty$ this norm is equivalent to the L^p-norm.

1. Introduction

The spaces $L^p=L^p(0,1)$ $(1<p<\infty)$ are considered as real Banach spaces of real-valued functions with the usual norms $\| \;\|_p$. The "dyadic Hardy spaces" are denoted by $\mathbf{H}^p$. The spaces $\mathbf{H}^p$ $(1\leqq p<\infty)$ coincide with the space of all L^1 functions, quadratic variations of which belong to L^p. The quadratic variation $Q(f)$ of the function $f\in L^1$ is defined by

$$Q(f):=\left(\sum_{n=0}^{\infty}|\Delta_n(f)|^2\right)^{1/2} \tag{1}$$

where $\Delta_n(f)=E_n(f)-E_{n-1}(f)$ $(n=0,1,\ldots)$, $E_{-1}f=0$ and $E_n(f)$ denotes the 2^n-th partial sum of the Walsh—Fourier series of f. The operator E_n is equal to the conditional expectation with respect to the σ-algebra generated by the intervals $[k2^{-n}, (k+1)2^{-n})$ $(k=0,1,\ldots,2^n-1)$. The dyadic $\mathbf{H}^p$-norm of the function f is

$$\|f\|_{\mathbf{H}^p}:=\|Q(f)\|_p \quad (1\leqq p<\infty). \tag{2}$$

It was proved by R. E. A. C. Paley [1] that for $1<p<\infty$ there exist constants c_p and c_p' depending only on p such that

$$c_p'\|f\|_p\leqq\|Q(f)\|_p\leqq c_p\|f\|_p \quad (1<p<\infty), \tag{3}$$

i.e., for $1<p<\infty$ the L^p-norm and the $\mathbf{H}^p$-norm are equivalent. In the case $p=1$ the inequality (3) is not true. B. Davis [2] has proved (in a more general form) that

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the $\mathbf{H}^1$-norm of f is equivalent to the L^1-norm of the dyadic maximal function $E^*(f)$ of f: $\|Q(f)\|_1 \sim \|E^*(f)\|_1$ where $E^*(f) = \sup_n |E_n(f)|$. Furthermore, it is known that

$$\|E^*(f)\|_p \sim \|Q(f)\|_p \sim \int_0^1 \|T(f;\ x)\|_p \,dx \quad (1 \leqq p < \infty) \tag{4}$$

where

$$T(f;\ x) := \sum_{n=0}^{\infty} r_n(x) \Delta_n(f)$$

and $r = (r_n,\ n \in \mathbf{N})$ $(\mathbf{N} := \{0, 1, 2, \ldots\})$ denotes the Rademacher system. A special case of (3) is the well-known Khintchine inequality:

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^{1/2} \sim \left\|\sum_{n=0}^{\infty} a_n r_n\right\|_p \quad (1 < p < \infty).$$

The L^p-norms $(1 < p < \infty)$ are invariant with respect to the dyadic shift operators $s_n(f) := f\Psi_n$ $(n \in \mathbf{N})$, where the Ψ_n-s are the Walsh—Paley functions, i.e., $\|f\|_p = \|f\Psi_n\|_p$ $(1 < p < \infty, n \in \mathbf{N})$. The $\mathbf{H}^1$-norm has not this property. An easy computation shows that for the functions

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if} \quad 0 \leqq x < 2^{-n}, \\ 0, & \text{if} \quad 2^{-n} \leqq x < 1 \end{cases} \quad (n \in \mathbf{N})$$

we have

$$\|Q(D_{2^n})\|_1 > 3^{-1/2} n, \quad \|Q(\Psi_{2^n} D_{2^n})\|_1 = 1. \tag{5}$$

We introduce the following shift-invariant norm: for $1 \leqq p < \infty$ let

$$\|f\|_{\mathbf{H}_p^*} := \left\|\sup_n Q(f\Psi_n)\right\|_p,$$

and denote by $\mathbf{H}_p^*$ the set of L^1 functions f, for which $\|f\|_{\mathbf{H}_p^*} < \infty$. Obviously, $\mathbf{H}_p^* \subseteqq \mathbf{H}_p$. By means of (5) a function f_0 can be constructed such that $\|f_0\|_{\mathbf{H}^1} < \infty$ and $\|f_0\|_{\mathbf{H}_1^*} = \infty$. In [3] it was proved that the sublinear operator

$$Q^*(f) = \sup_n Q(f\Psi_n) \quad (f \in L^1)$$

has weak type (2, 2), i.e., there exists a constant C independent of f such that for every $y > 0$,

$$\text{mes}\,\{x \in [0, 1)\colon\ Q^*(f)(x) > y\} < C\|f\|_2^2/y^2.$$

In this paper we give the following generalization of the above result.

Theorem. 1. *For* $1 < p < \infty$ *the* $\mathbf{H}_p^*$*-norm is equivalent to the* L^p*-norm:*

$$\|Q^*(f)\|_p \sim \|f\|_p \quad (1 < p < \infty). \tag{6}$$

2. *There exists a function in* $\mathbf{H}_1$ *with infinite* $\mathbf{H}_1^*$*-norm.*

The first part of Theorem is a consequence of the following

Lemma 1. *The operators*

$$Q_N^*(f)=\sup_{m<2^N}\left(\sum_{n=1}^{N-1}|\Delta_n(f\Psi_m)|^2\right)^{1/2}\quad (N\in\mathbf{N})\tag{7}$$

are of restricted weak type (p,p) *for every* $1<p<\infty$, *i.e., for every measurable set* $H\in[0,1)$,

$$\operatorname{mes}\{x:\ Q_N^*(\chi_H)(x)>y\}<C_p\|\chi_H\|_p^p/y^p\quad (y>0),\tag{8}$$

where χ_H *is the characteristic function of the set* H *and* C_p *is a constant depending only on* p.

It is easy to see that for every $f\in L^1$ there exists a linear operator $L_f\colon L^1\to L^1$ such that

$$\text{i) } L_f(f)=Q_N^*(f),\quad \text{ii) } |L_f(g)|\leqq Q_N^*(g)\quad (g\in L^1)\tag{9}$$

hold. Indeed, for $x\in[0,1)$ let $0\leqq M(x)<2^N$ be such a number for which

$$Q_N^*(f)(x)=\left(\sum_{n=1}^{N-1}|\Delta_n(f\Psi_{M(x)})(x)|^2\right)^{1/2}.$$

Furthermore, let

$$L_f(g)(x)=\sum_{n=1}^{N-1}\varepsilon_n(x)\,\Delta_n(g\Psi_{M(x)})(x),$$

where

$$\varepsilon_m(x)=\operatorname{sign}\Delta_m(f\Psi_{M(x)})(x)\Big/\left(\sum_{n=0}^{N-1}|\Delta_n(f\Psi_{M(x)})|^2\right)^{1/2}\quad (1\leqq m\leqq N).$$

It is obvious that for the linear operator L_f (9) is satisfied, and by (9) ii) it is also of restricted weak type (p,p) for $1<p<\infty$. Applying the Stein—Weiss interpolation theorem (see, e.g., [5], p. 191) we get that the operator $L_f\colon L^p\to L^p$ $(1<p<\infty)$ and consequently on the basis of (9) i) the operators $Q_N^*\colon L^p\to L^p$ $(1<p<\infty)$ are also uniformly bounded.

Since

$$Q^*(f)\leqq\sup_m|E_0(f\Psi_m)|+\sup_m\left(\sum_{n=1}^{\infty}|\Delta_n(f\Psi_m)|^2\right)^{1/2}=\sup_m|E_0(f\Psi_m)|+\lim_{N\to\infty}Q_N^*(f),$$

we have

$$\|Q^*(f)\|_p\leqq C_p^*\|f\|_p\quad (1<p<\infty),$$

and by the Paley-inequality,

$$c_p'\|f\|_p<\|Q(f)\|_p\leqq\|Q^*(f)\|_p.$$

This proves (6).

Let us introduce another shift-invariant norm by means of the maximal function

$$E^{**}(f) := \sup_{m,n\in\mathbf{N}} |E_n(f\Psi_m)|$$

as follows: let

$$\|f\|_p^* := \|E^{**}(f)\|_p \quad (1 \leqq p < \infty).$$

Since $E^*(f) \leqq E^{**}(f) \leqq E^*(|f|)$, the Doob-inequality (see [4]), implies that $\|f\|_p^* \sim \|f\|_p$ $(1<p<\infty)$, i.e., for $1<p<\infty$ the $\mathbf{H}_p$-norm is equivalent to the $\|\ \|_p^*$-norm. We do not know whether the $\mathbf{H}_1$-norm and the $\|\ \|_1^*$-norm are equivalent or not.

2. Two lemmas

Let

$$\mathcal{I}_N := \{[k2^n, (k+1)2^n)\colon 0 \leqq n < N, \quad (k+1)2^n < 2^N, \ k, n \in \mathbf{N}^n\},$$

and for an interval $I=[k2^n, (k+1)2^n)$ we set $m(I)=k2^n$, $|I|=2^n$ and

$$E_I(f) = \sum_{n\in I}\left(\int_0^1 f\Psi_n\,dx\right)\Psi_n.$$

Then, $E_n(f)=E_{[0,2^n)}(f)$ and for all $j\in I=[k2^n, (k+1)2^n)$ we have $E_I(f)=E_n(f\Psi_j)\Psi_j$.

By means of the intervals of $\mathcal{I}_N$ the function $Q_N^*(f)$ can be written in the form

$$Q_N^*(f) = \sup_{j<2^N}\Big(\sum_{j\in I} |\Delta_I(f)|^2\Big)^{1/2},$$

where $\Delta_I(f)=E_{I_+}(f)-E_I(f)$ and I_+ denotes the interval for which $I\subset I_+$ and $|I_+|=2|I|$ hold.

To estimate $Q_N^*(f)$ we use an elementary observation with respect to series, in which the indices of the terms are the elements of $\mathcal{I}_N$. We need the following

Lemma 2. *Let $g_I\colon [0,1)\to\mathbf{R}$ $(I\in\mathcal{I}_N)$ be a sequence of functions and $B_I\subset[0,1)$ $(I\in\mathcal{I}_N)$ a sequence of increasing sets* (i.e., $I\subseteqq J$ *implies* $B_I\subseteqq B_J$). *Further let* $A_I=B_I\setminus\bigcap_{J\subset I} B_J$. *Then*

$$\sup\Big\{\Big|\sum_{I\subseteqq J\subset K}\chi_{B_J}g_J\Big|\colon I\subset K,\ I,K\in\mathcal{I}_N\Big\} \leqq G := 2\sup_{I\in\mathcal{I}_N}\chi_{A_I}\sup_{I\subset K}\Big|\sum_{I\subseteqq J\subset K} g_J\Big|. \text{ [1)]} \tag{10}$$

Proof. To prove (10), let $x\in[0,1)$ and $S_{IK}=\Big|\sum_{I\subseteqq J\subset K}\chi_{B_J}g_J\Big|$. We show that $S_{IK}(x)\leqq G(x)$.

If $S_{IK}(x)\neq 0$, then the (linearly ordered) set $\{J\in\mathcal{I}_N\colon I\subseteqq J\subset K, x\in B_J\}$ is not empty. Denote by $\bar{I}$ the minimum element (with respect to the ordering $\subseteqq$) of

1) $J\subset K$ means that $J\subseteqq K$ and $J\neq K$.

this set. If $I \subset \bar{I}$, then by the definition of $\bar{I}$ we have that for $I \subseteqq J \subset \bar{I}$, $x \notin B_J$. Let I^* be such an element of the set $\tilde{\mathscr{I}} = \{J \in \mathscr{I}_N \colon J \subset \bar{I},\ x \in B_J\}$ $(\neq \emptyset)$, for which $|I^*| = \min\{|J| \colon J \in \tilde{\mathscr{I}}\}$. From the definition of I^* it follows that for every $J \subset I^*$ we have $J \notin \tilde{\mathscr{I}}$. Thus, for such J's, $x \notin B_J$ and consequently $x \in A_{I^*}$. From these we get

$$|S_{IK}(x)| = |S_{IK}(x)| = |S_{I^*K}(x) - S_{I^*I}(x)| \leqq$$

$$\leqq \chi_{A_{I^*}}(x) \Big| \sum_{I^* \subseteqq J \subset K} g_J(x) \Big| + \chi_{A_{I^*}}(x) \Big| \sum_{I^* \subseteqq J \subset K} g_J(x) \Big| \leqq G(x),$$

and (10) is proved.

Let

$$F_I f = \sup\{|E_I(f)| \colon J \subset I,\ 2|J| = |I|\} \quad (I \in \mathscr{I}_N,\ |I| \geqq 2),$$

$$(11) \qquad F_I f = |E_I(f)| \quad (I \in \mathscr{I}_N,\ |I| = 1),$$

$$F_I^* f = \sup\{F_J f \colon J \subseteqq I\}, \quad F^* f = \sup\{F_I^* f \colon I \in \mathscr{I}_N\}.$$

The σ-algebra generated by the intervals $[k2^{-n}, (k+1)2^{-n})$ $(k = 0, 1, \ldots, 2^n - 1)$ will be denoted by $\mathscr{A}_n$ $(n \in \mathbb{N})$ and for $I \in \mathscr{I}_N$, $|I| = 2^n$, set $\mathscr{A}_I = \mathscr{A}_n$. The sequence $(E_I(f), I \in \mathscr{I}_N)$ is predictable. Indeed, since $E_I(f) = E_{I'}(f) + E_{I''}(f)$ $(I = I' \cup I''$, $I' \cap I'' = \emptyset)$, $F_I^* f$ is $\mathscr{A}_{n-1}$-measurable and $|E_I(f)| < 2F_I^* f$.

For $y > 0$ let

$$(12) \qquad \begin{gathered} B_I^y = \{x \in [0, 1) \colon (F_I^* f)(x) > y\}, \quad A_I^y = B_I^y \setminus \bigcup_{J \subset I} B_J^y, \\ C_I^y = \{x \in [0, 1) \colon (F_I^* f)(y) \leqq ey\}. \end{gathered}$$

Then the following statement is true.

Lemma 3. *For every* $y > 0$,

$$(13) \qquad \sum_{I \in \mathscr{I}_N} \operatorname{mes} A_I^y < \frac{1}{y^2} \int_{\{F^* f > y\}} |f|^2 \, dx.$$

Proof. On the basis of the definition of A_I^y and B_I^y it is obvious that $(F_I f)(x) > y$ if $x \in A_I^y$. Let

$$D_{I'}^y = \{x \in A_I^y \colon |E_{I'}(f)(x)| > y\}, \quad D_{I''}^y = A_I^y \setminus D_{I'}^y,$$

where $I' \subset I$, $I'' = I \setminus I'$ and $2|I'| = |I|$. We set

$$P_I = \chi_{D_{I'}} E_{I'} + \chi_{D_{I''}} E_{I''}.$$

Since $E_I E_J \doteq 0$ if $I \cap J = \emptyset$, and $\chi_{A_I^y} \chi_{A_J^y} = 0$, if $I \subset J$, on the basis of the $\mathscr{A}_I$-homogeneity of E_I (which means $E_I(\lambda f) = \lambda E_I f$, if λ is $\mathscr{A}_I$-measurable) we get that the P_I's are orthogonal projections, i.e., $P_I P_J = \delta_{IJ} P_I$ $(I, J \in \mathcal{I}_N)$. Thus

$$\|\chi_{\{F^* f > y\}} f\|_2^2 \geqq \Big\| \sum_{I \in \mathcal{I}_N} P_I f \Big\|_2^2 = \sum_{I \in \mathcal{I}_N} \|P_I f\|_2^2 =$$

$$= \sum_{I \in \mathcal{I}_N} \int_{D_{I'}} |E_{I'} f|^2 \, dx + \int_{D_{I''}} |E_{I''} f|^2 \, dx \geqq y^2 \sum_{I \in \mathcal{I}_N} \operatorname{mes} A_I^y,$$

and Lemma 3 is proved.

3. Proof of Lemma 1

Let

$$\varepsilon_I^y = \frac{1}{y} \chi_{\{(1/e) F_{I_+}^* f \leqq y < F_{I_+}^* f\}} = \frac{1}{y} \chi_{B_{I_+}^y} \chi_{C_{I_+}^y} \quad (y > 0). \tag{14}$$

Then ε_I^y is $\mathcal{A}_I$-measurable and

$$\Big(\int_0^{+\infty} \varepsilon_I^y \, dy \Big) \Delta_I f = \Delta_I f.$$

Using this, the quadratic variation can be estimated as follows:

$$Q_n(f) = \Big(\sum_{n \in I \in \mathcal{I}_N} |\Delta_I f|^2 \Big)^{1/2} = \Bigg(\sum_{n \in I \in \mathcal{I}_N} \Bigg| \int_0^{+\infty} \varepsilon_I^y \Delta_I f \, dy \Bigg|^2 \Bigg)^{1/2} \leqq$$

$$\leqq \int_0^{+\infty} \Big(\sum_{n \in I \in \mathcal{I}_N} |\varepsilon_I^y \Delta_I f|^2 \Big)^{1/2} dy,$$

and by Lemma 2 we have

$$Q_N^*(f) < \int_0^{+\infty} \sup_{I \in \mathcal{I}_N} R_I^y f \, dy,$$

where

$$R_I^y f = 2 \chi_{A_I^y} \Big(\sum_{I \subseteqq J \in \mathcal{I}_N} |\varepsilon_I^y \Delta_I f|^2 \Big)^{1/2},$$

and consequently

$$\chi_{\{F^* f < \lambda\}} Q_N^*(f) \leqq \int_0^{\lambda} \sup_{I \in \mathcal{I}_N} R_I^y \, dy. \tag{15}$$

Using Abel's transformation, an easy computation shows that

$$\Big| \sum_{I \subseteqq J \in \mathcal{I}_N} \varepsilon_I^y \Delta_I f \Big| \leqq 4e,$$

thus by the Paley-inequality we get

$$\begin{aligned} \|\chi_{A_I^y} R_I^y\|_p &\leqq C_p \Big\| \sum_{I \subseteqq J \in \mathcal{I}_N} \varepsilon_I^y \Delta_I (f \chi_{A_I^y}) \Big\|_p = \\ &= C_p \Big\| \chi_{A_I^y} \sum_{I \subseteqq J \subseteqq \mathcal{I}_N} \varepsilon_I^y \Delta_I f \Big\|_p < 4e C_p \|\chi_{A_I^y}\|_p. \end{aligned} \tag{16}$$

Let first $p>2$. Then by (13) and (15),

$$\|\chi_{\{F^*f\leqq\lambda\}}Q_N^*(f)\|_{2p}\leqq\int_0^\lambda\Big(\sum_{I\in\mathcal{I}_N}\|R_I^y f\|_{2p}^{2p}\Big)^{1/2p}dy\leqq$$

$$\leqq 2(4eC_{2p})^{2p}\int_0^\lambda\Big(\sum_{I\in\mathcal{I}_N}\operatorname{mes}A_I^y\Big)^{1/2p}dy\leqq C_p'\int_0^\lambda\Big(\int_{\{F^*f>y\}}|f|^2/y^2\,dx\Big)^{1/2p}dy\leqq$$

$$\leqq C_p'\Big(\int_0^\lambda y^{-1/2}\,dy\Big)\Big(\int_0^1(F^*f)^{p-2}|f|^2\,dx\Big)^{1/2p}\leqq 2C_p'\lambda^{1/2}\Big(\int_0^1|F^*f|^p\Big)^{1/2p}.$$

Using the maximal inequality $\|F^*f\|_r\leqq(r/(r-1))\|f\|_r$ $(r>1)$ we get

$$\lambda^p\operatorname{mes}\{Q_N^*(f)>\lambda,\ F^*f\leqq\lambda\}<C_p''\|f\|_p^p,$$

and on the basis of the maximal inequality (8) follows for every $f\in L^p$ $(p\geqq 2)$.

Let now $1<p<2$ and $f=\chi_H$. By a simple integral transformation (15) can be written in the form

$$\chi_{\{F^*f\leqq\lambda^p\}}Q_N^*(f)<\lambda\int_0^{\lambda^{p-1}}\sup_{I\in\mathcal{I}_N}R_I^{\lambda t}f\,dt,$$

and since $\sup_I R_I^{\lambda t}f=\chi_{\{F^*f>\lambda t\}}\sup_I R_I^{\lambda t}f$, by $F^*f\leqq 1$ we have

$$\chi_{\{F^*f\leqq\lambda^p\}}Q_N^*(f)<\int_0^{\lambda_1}\sup_{I\in\mathcal{I}_N}R_I^{\lambda t}f\,dt,\tag{17}$$

where $\lambda_1=\min(\lambda^{p-1},\lambda^{-1})\leqq 1$. The condition $t\leqq\lambda^{p-1}$ yields $\lambda^{-2}\leqq t^{-(2-p)/(p-1)}\lambda^{-p}$, thus by (13), (16), and (17) with $q=2((2-p)/(p-1)+2)$ we have

$$\|\chi_{\{F^*f<\lambda^p\}}Q_N^*(f)\|_q\leqq\lambda\int_0^{\lambda_1}\Big(\sum_{I\in\mathcal{I}_N}\|R_I^{\lambda t}f\|_q^q\Big)^{1/q}dt\leqq$$

$$\leqq\lambda C_q\int_0^{\lambda_1}\Big(\sum_{I\in\mathcal{I}_N}\operatorname{mes}A_I^{\lambda t}\Big)^{1/q}dt<(\operatorname{mes}H)^{1/q}\lambda C_q\int_0^{\lambda_1}(\lambda t)^{-2/q}\,dt\leqq$$

$$\leqq C_q\lambda^{1-p/q}(\operatorname{mes}H)^{1/q}\int_0^1 t^{-1/2}\,dt=2C_q\lambda^{1-p/q}(\operatorname{mes}H)^{1/q}.$$

From this we obtain

$$\lambda^p\operatorname{mes}\{Q_N^*(f)>\lambda;\ F^*f\leqq\lambda^p\}\leqq\bar{C}_p\operatorname{mes}H.$$

This and the maximal inequality gives (8).

4. Proof of the second part of Theorem

Let

$$f=\sum_{n=0}^{\infty} 2^{-n/2} r_{2^n} D_{2^{2^n}}.$$

Since $\|D_{2^s}\|_1=1$ $(s\in\mathbf{N})$, this series is absolute convergent a.e. and $f\in L^1$. It is obvious that

$$E^*f\leqq \sum_{n=0}^{\infty} 2^{-n/2} D_{2^{2^n}},$$

and consequently $E^*f\in L^1$, i.e., $\|f\|_{\mathrm{H}_1}<\infty$. On the basis of $Q(r_{2^n}f)\geqq 2^{-n/2}Q(D_{2^{2^n}})$ we have

$$\|Q^*(f)\|_1\geqq\|Q(r_{2^n}f)\|_1\geqq 2^{-n/2}\|Q(D_{2^{2^n}})\|_1\geqq 3^{-1/2}2^{n/2}\quad(n\in\mathbf{N}),$$

thus $\|f\|_{\mathrm{H}_1^*}=\infty$.

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Moment problem for dilatable semigroups of operators

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

0. Introduction

The Main Theorem in the dilation theory of operators on Hilbert space due to Sz.-Nagy appeared in the Appendix to the third edition of [1]. The applications presented also in [1] show its central role in operator theory. At the same time Stinespring [11] described the so-called completely positive (linear) maps between C^*-algebras as (in a general sense) dilatable operator valued (linear) functions. It is also a generalization of Neumark's theorem (see [1]) on the dilatability of positive operator measures, a source of dilation theory.

On the other hand, Sz.-Nagy proved (see [1]) a moment theorem for self-adjoint operators generalizing a result of R. V. Kadison concerning a Schwarz-inequality for operator valued functions. Although it is also a consequence of the Main Theorem we think it has a more general character. Namely, given a *-semigroup in a C^*-algebra and an operator valued function on this *-semigroup, a moment theorem for the existence of a (completely positive) linear operator-function on the whole C^*-algebra can be formulated. This generalizes also Stinespring's theorem. Moreover we treat the moment problem for operators in the general case, when we assume only that the restrictions of the operators in question to some given subset (not assumed to be a subspace) of the Hilbert space are given. It is a new aspect for the existence of a single positive (hence for a self-adjoint) operator on Hilbert space and a self-adjoint semibounded operator also. The familiar Krein and Friedrichs extension is thus generalized and joined to moment and dilation problems.

The scalar valued case gives also a new insight into the classical Hausdorff moment problem, giving a solution analogous to that of the trigonometric moment

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problem by Riesz and Fejér. In any case, our solution differs from those of Hausdorff and Riesz—Fejér.

We give a new characterization of subnormal operators, too, along the lines of our argument.

For other applications, e.g., factorization questions for operators, moment problems for contraction and subnormal operators and their generalizations, see [4, 5, 2, 3, 10].

The author is deeply indebted to Professor Béla Szőkefalvi-Nagy for his advices.

1. Main problems and results

Given a *-semigroup G, a subset X of a Hilbert space H, and a function $f\colon G\times X\to H$, it is natural to seek an operator valued function F on G assuming its values in $B(H)$, the C^*-algebra of all bounded linear operators on H, such that

$$F(g)x = f(g, x) \quad (g\in G,\ x\in X) \tag{1}$$

holds. In this case F is called an operator representation of f.

We shall treat only the case when F in (1) is *dilatable* (in a general sense), i.e., when there is a Hilbert space K with a continuous linear operator $V\colon K\to H$ and a *-representation S of G on K such that

$$F(g) = VS(g)V^* \quad (g\in G). \tag{2}$$

Here F is *strongly dilatable* if V satisfies $VV^*=I_H$. A C^*-seminorm p on a *-semigroup G is a submultiplicative function $p\colon G\to \mathbf{R}^+$ with $p(g^*g)=p(g)^2$ implying $p(g^*)=p(g)$ $(g\in G)$.

Theorem 1.1. *A given H-valued function f on $G\times X$ has a dilatable operator representation F if and only if there exist $M\geqq 0$ and a C^*-seminorm p on G such that*

$$\Big\|\sum_{h,x} c_{h,x} f(h, x)\Big\|^2 \leqq M \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}\big(f(k^*h, x), y\big) \tag{3}$$

holds for each finite sequence $\{c_{h,x}\}$ of complex numbers indexed by elements of $G\times X$, and

$$\big(f(g^*g, x), x\big) \leqq Mp(g)^2\|x\|^2 \quad (g\in G;\ x\in X). \tag{4}$$

Theorem 1.2. *Assume that G has an identity e and that the function f on $G\times X$ satisfies*

$$f(e, x) = x \quad (x\in X), \tag{5}$$

$$\vee\{f(g, x)\colon\ g\in G,\ x\in X\} = H. \tag{6}$$

This function f is of the form (1) with strongly dilatable F if and only if there is an H-valued function φ on $G\times f(G\times X)$ such that

$$\varphi(g, f(e, x)) = \varphi(e, f(g, x)) = f(g, x) \quad (g\in G;\ x\in X), \tag{7}$$

$$\|\varphi(g, f(g', x))\| \leqq p(g)\|f(g', x)\| \quad (g, g'\in G; x\in X) \tag{8}$$

for some C^*-seminorm p on G, and

$$\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h, \xi)\Big\|^2 \leqq \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}(\varphi(k^*h, \xi), \eta) \tag{9}$$

for each sequence $\{c_{h,\xi}\}$ of complex numbers indexed by elements of $G\times f(G\times X)$.

Let now G be a (multiplicative) *-semigroup in a given C^*-algebra A. In this case a $B(H)$-valued operator function F on G is *A-dilatable* if there is a *-representation S of the C^*-algebra A on some Hilbert space K with a continuous linear operator $V: K\to H$ such that (2) holds. In other words F has a completely positive (linear) extension to the whole A (hence a "moment" F is given for this completely positive map). We shall treat a more general setting by restricting the data to a subset of the Hilbert space.

Theorem 1.3. *Let G be a (multiplicative) *-semigroup in the C^*-algebra A whose linear span is norm dense in A. An H-valued function f on $G\times X$ is of the form* (1) *with an A-dilatable operator function F if and only if there is a constant $M\geqq 0$ such that*

$$\begin{aligned}
&\Big\|\sum_{h,x} c_{h,x}f(h, x)\Big\|^2 \leqq M\sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}(f(k^*h, x), y),\\
&\sum_h\sum_k c_h\bar{c}_k(f(k^*h, x), x) \leqq M\|x\|^2\Big\|\sum_h c_h h\Big\|^2 \quad (x\in X)
\end{aligned} \tag{10}$$

hold for each finite sequence $\{c_{h,x}\}$ or $\{c_h\}$ of complex numbers indexed by elements of $G\times X$ and G, respectively.

Theorem 1.4. *Assume that A has an identity e such that the *-subsemigroup G of A which spans A, contains e too. An H-valued function f on $G\times X$ with* (5)—(6) *is of the form* (1) *with a strongly A-dilatable operator function F if and only if there is an H-valued function φ on $G\times f(G\times X)$ with* (7) *and such that*

$$\begin{aligned}
&\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h, \xi)\Big\|^2 \leqq \sum_{h,\xi}\sum_{k,\eta} c_{k,\xi}\bar{c}_{k,\eta}(\varphi(k^*h, \xi), \eta),\\
&\sum_h\sum_k c_h\bar{c}_k(\varphi(k^*h, \xi), \xi) \leqq \|\xi\|^2\Big\|\sum_h c_h h\Big\|^2 \quad (\xi\in f(G\times X))
\end{aligned} \tag{11}$$

hold for any finite sequence $\{c_{h,\xi}\}$ or $\{c_h\}$ of complex numbers indexed by elements of $G\times f(G\times X)$ and G, respectively.

Proof of the necessity. (1.1) Assuming that the H-valued function f on $G\times X$ has form (1) with dilatable operator function F on G, we have, for any finite sequence $\{c_{h,x}\}$ $(h\in G, x\in X)$ of complex numbers,

$$\Big\|\sum_{h,x} c_{h,x}f(h,c)\Big\|^2 = \Big\|\sum_{h,x} c_{h,x}F(h)x\Big\|^2 = \Big\|V\sum_{k,x} c_{h,x}S(h)V^*x\Big\|^2 \leqq$$

$$\leqq \|V\|^2\sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}(VS(k^*h)V^*x, y) = \|V\|^2\sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}(f(k^*h,x),y).$$

Moreover,

$$\big(f(g^*g,x),x\big) = \big(F(g^*g)x,x\big) = \big(VS(g^*g)V^*x,x\big) =$$

$$= \|S(g)V^*x\|^2 \leqq \|S(g)\|^2\|V^*\|^2\|x\|^2 = \|V\|^2\|S(g)\|^2\|x\|^2$$

holds for each $g\in G$, $x\in X$. This yields (4) with the C^*-seminorm p on G defined (by the *-representation S of G on the Hilbert space K) as $p(g):=\|S(g)\|$ $(g\in G)$.

(1.2) Defining φ on $G\times f(G\times X)$ by

$$\varphi\big(g, f(g',x)\big) := F(g)f(g',x) \quad (g, g'\in G;\ x\in X) \tag{12}$$

we deduce (7) and (8) from (5) and (1) as follows:

$$\varphi\big(g, f(e,x)\big) = F(g)f(e,x) = F(g)x = f(g,x),$$

$$\varphi\big(e, f(g,x)\big) = F(e)f(g,x) = VS(e)V^*f(g,x) = VV^*f(g,x) = f(g,x),$$

$$\big\|\varphi\big(g,f(g',x)\big)\big\| \leqq \|F(g)\|\|f(g',x)\| \leqq \|V\|\|S(g)\|\|V^*\|\|f(g',x)\| \leqq \|S(g)\|\|f(g',x)\|.$$

To prove (9), let $\{c_{h,\xi}\}$ be any finite sequence of complex numbers indexed by elements of $G\times f(G\times X)$. We have then

$$\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h,\xi)\Big\|^2 = \Big\|\sum_{h,\xi} c_{h,\xi}F(h)\xi\Big\|^2 = \Big\|V\sum_{k,\xi} c_{h,\xi}S(h)V^*\xi\Big\|^2 \leqq$$

$$\leqq \|V\|^2\sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}\big(VS(k^*h)V^*\xi,\eta\big) = \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}\big(\varphi(k^*\eta),\xi),\eta\big).$$

(1.3) Assuming that F is A-dilatable, we know that S is a *-representation of the C^*-algebra A. Hence $\|S\|\leqq 1$. From (1.1) we see furthermore

$$\frac{1}{\|V\|^2}\Big\|\sum_{h,x} c_{h,x}f(h,x)\Big\|^2 \leqq \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}\big(f(k^*h,x),y\big),$$

$$\sum_h\sum_k c_h\bar{c}_k\big(f(k^*h,x),x\big) \leqq \Big\|\sum_h c_hS(h)V^*x\Big\| = \Big\|S\Big(\sum_h c_hh\Big)V^*x\Big\|^2 \leqq$$

$$\leqq \|S\|^2\Big\|\sum_h c_hh\Big\|^2\|V^*\|^2\|x\|^2 \leqq \|x\|^2\Big\|\sum_h c_hh\Big\|^2$$

for any $x\in H$, whence (10) follows.

(1.4) Applying φ defined above in (12) we have by (12) and by the relations $\|V\|=1, \|S\|\leqq 1$ that

$$\Big\|\sum_{h,\xi} c_{h,\xi}\varphi(h,\xi)\Big\|^2 \leqq \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{c}_{k,\eta}(\varphi(k^*h,\xi),\eta),$$

$$\sum_h\sum_k c_h\bar{c}_k(\varphi(k^*h,\xi),\xi) = \Big\|\sum_h c_h S(h)V^*\xi\Big\|^2 = \Big\|S\Big(\sum_h c_h h\Big)V^*\xi\Big\|^2 \leqq$$

$$\leqq \|S\|^2\Big\|\sum_h c_h h\Big\|^2\|V^*\|^2\|\xi\|^2 \leqq \Big\|\sum_h c_h h\Big\|^2\|\xi\|^2 \quad (\xi\in f(G\times X)).$$

Proof of the sufficiency. (1.1) Let f be an H-valued function on $G\times X$ satisfying (3) and (4) with a C^*-seminorm p on G. Further let K_0 be the linear space of all finitely supported complex valued functions on $G\times X$. Each element of K_0 has the form $\sum_{h,x} c_{h,x}\delta(h,x)$, where $\delta(h,x)$ denotes the function assuming the value 1 in $(h,x)\in G\times X$ and 0 otherwise, and $\{c_{h,x}\}$ $(h\in G, x\in X)$ is a finite sequence of complex numbers. With this notation we can define two operations on K_0, the first of which is the linear map V into H given by

$$V\Big(\sum_{h,x} c_{h,x}\delta(h,x)\Big) := \sum_{h,x} c_{h,x}f(h,x), \tag{13}$$

and the second one the translation operation on K_0 by elements of G, defined for any g in G by

$$S(g)\Big(\sum_{h,x} c_{h,x}\delta(h,x) := \sum_{h,x} c_{h,x}\delta(gh,x). \tag{14}$$

Remark also that the map $g\mapsto S(g)$ constitutes an endomorphism of G. Lastly, we define a semi-inner product $\langle\cdot,\cdot\rangle$ on K_0 by

$$\Big\langle\sum_{h,x} c_{h,x}\delta(h,x), \sum_{k,y} d_{k,y}\delta(k,y)\Big\rangle := \sum_{h,x}\sum_{k,y} c_{h,x}\bar{d}_{k,y}(f(k^*h,x),y). \tag{15}$$

Observe the nonnegativity of the right hand side in (5) in view of (3). Now we are in a position to construct a Hilbert space K with a continuous linear map $V: K\to H$ and a *-representation S of G on K such that (1) holds with F satisfying (2), too. We obtain a pre-Hilbert space by factorizing K_0 with respect to the null space $N:=\{\xi\in K_0: \langle\xi,\xi\rangle=0\}$ and by taking the induced inner product on K_0/N. The completion of K_0/N is a Hilbert space, say K. For simplicity, denote also by $\delta(h,x)$ the image of $\delta(h,x)\in K_0$ in K under the factorization. Thus K_0 is viewed as a norm dense subset of K and V is a densely defined bounded (cf. (3)) linear operator from K into H. Thus V has a unique continuous extension to K which is denoted naturally also by V. Lastly we have to show that $S(g)$ $(g\in G)$ induces a bounded linear operator also denoted by $S(g)$ on K such that $S_g^*= =S(g^*)$ for any g in G. To this end let $\xi=\sum_{h,x} c_{h,x}\delta(h,x)$ be taken from a dense subset of K and we show

$$\|S(g)\xi\| \leqq p(g)\|\xi\| \quad (g\in G) \tag{16}$$

which is enough for our purpose. Indeed, we have

$$\|S(g)\xi\|^2 = \Big\|\sum_{h,x} c_{h,x}\delta(gh, x)\Big\|^2 = \sum_{h,x}\sum_{k,y} c_{h,x}\bar{c}_{k,y}\big(f(k^*g^*gh, x), y\big) =$$
$$= \langle S(g^*g)\xi, \xi\rangle \leqq \|S(g^*g)\xi\|\|\xi\|,$$

and hence, by induction on n,

$$\|S(g)\xi\|^{2^n} \leqq \|S((g^*g)^{2^{n-1}})\xi\|^2\|\xi\|^{2^n-2} = \|\xi\|^{2^n-2}\Big\|\sum_{h,x} c_{h,x}\delta\big((g^*g)^{2^n-2}h, x\big)\Big\|^2 \leqq$$
$$\leqq \|\xi\|^{2^n-2}\Big(\sum_{h,x}|c_{h,x}|\|\delta((g^*g)^{2^n-2}h, x)\|\Big)^2 = \|\xi\|^{2^n-2}\Big(\sum_{h,x}|c_{h,x}|\big(f(h^*(g^*g)^{2^n-1}h, x), x^{1/2}\Big)^2 \leqq$$
$$\leqq \|\xi\|^{2^n-2}\Big(\sum_{h,x}|c_{h,x}|M^{1/2}p((g^*g)^{2^{n-2}}h)\|x\|\Big)^2 \leqq \qquad \text{(by (4))}$$
$$\leqq \|\xi\|^{2^n-2}Mp(g)^{2^n}p(h)^2\Big(\sum_{h,x}|c_{h,x}|\|x\|\Big)^2 \quad (n = 1, 2, \ldots).$$

This implies (16) for $n\to\infty$.

Now $S(g)^*=S(g^*)$ follows for any g in G by observing

(17) $$\langle S(g^*)\xi, \eta\rangle = \langle \xi, S(g)\eta\rangle \quad \text{for each} \quad \xi, \eta\in K_0.$$

Indeed, if $\xi=\sum_{h,x} c_{h,x}\delta(h, x)$, $\eta=\sum_{k,y} d_{k,y}\delta(k, y)$, then both sides of (17) are equal to

$$\sum_{h,x}\sum_{k,y} c_{h,x}\bar{d}_{k,y}\big(f(k^*g^*h, x), y\big).$$

To complete the proof of (1.1) we show

(18) $$VS(g)V^*x = f(g, x) \quad (g\in G;\ x\in X).$$

By (13), it suffices to see that $S(g)V^*x=\delta(g, x)$. But

$$\Big\langle S(g)V^*x, \sum_{k,y} d_{k,y}\delta(k, y)\Big\rangle = \Big(x, VS(g^*)\Big(\sum_{k,y} d_{k,y}\delta(ky)\Big)\Big) =$$
$$= \Big(x, V\Big(\sum_{k,y} d_{k,y}\delta(g^*k, y)\Big)\Big) = \Big(x, \sum_{k,y} d_{k,y}f(g^*k, y)\Big) = \Big\langle \delta(g, x), \sum_{k,y} d_{k,y}\delta(k, y)\Big\rangle$$

holds for any $\sum_{k,y} d_{k,y}\delta(k, y)\in K_0$, verifying our assertion.

(1.2) We shall adopt the argument used in the proof of (1.1) by replacing X by $f(G\times X)$ and f by φ. (9) is a translation of (3) into the new situation and (4) implies (8) with $M\equiv 1$ since

$$\big\|\varphi(g, f(g', x))\big\|^2 \leqq \big(\varphi(g^*g, f(g', x)), f(g', x)\big) \leqq$$
$$\leqq \big\|\varphi(g^*g, f(g', x))\big\|\,\|f(g', x)\| \leqq p(g)^2\|f(g', x)\|^2$$

for any $g, g'\in G$, $x\in X$. Now we define $V, S, \langle\ ,\ \rangle$ by

(13') $$V\Big(\sum_{h,\xi} c_{h,\xi}\delta(h, \xi)\Big) := \sum_{h,\xi} c_{h,}\varphi(h, \xi),$$

(14') $$S(g)\Big(\sum_{h,\xi} c_{h,\xi}\delta(h, \xi)\Big) := \sum_{h,\xi} c_{h,\xi}\delta(gh, \xi),$$

(15') $$\Big\langle\sum_{h,\xi} c_{h,\xi}\delta(h, \xi), \sum_{k,\eta} d_{k,\eta}\delta(k, \eta)\Big\rangle := \sum_{h,\xi}\sum_{k,\eta} c_{h,\xi}\bar{d}_{k,\eta}\big(\varphi(k^*h, \xi), \eta\big).$$

In consequence, we have a *-representation S of G on some suitable Hilbert space K with a continuous linear operator $V\colon K\to H$ such that

$$VS(g)V^*f(g',x)=\varphi(g,f(g',x)) \quad (g,g'\in G;\ x\in X). \tag{19}$$

By (3) and (7) this implies (18) since

$$VS(g)V^*x=VS(g)V^*f(e,x)=\varphi(g,f(e,x))=f(g,x) \quad (g\in G;\ x\in X).$$

Finally, since $S(e)=I_K$, by (7) we have

$$VV^*f(g,x)=VS(e)V^*f(g,x)=\varphi(e,f(g,x))=f(g,x) \quad (g\in G;\ x\in X)$$

proving $VV^*=I_K$ (cf. (16)).

(1.3) First of all (10) implies (4) if we take the C^*-(semi)norm p on G defined in terms of the norm $\|\cdot\|$ of A as $p(g)=\|g\|$ $(g\in G)$ since

$$(f(g^*g,x),x)\leqq M\|x\|^2\|g\|^2 \quad (g\in G,\ x\in X).$$

As a consequence of the proof of (1.1), we have a *-representation of G on a suitable Hilbert space K such that (18) holds true. This proves (1) for a dilatable operator function F satisfying (2). But we need the A-dilatability of F. The key step is at hand: we shall prove the extendibility of S from G to A. To this end we have only to show

$$\Big\|\sum_g \lambda_g S(g)\Big\|\leqq\Big\|\sum_g \lambda_g g\Big\| \tag{20}$$

for each finite sequence $\{\lambda_g\}$ of complex numbers indexed by elements of G (because G spans A). Putting $a=\sum\limits_g \lambda_g g\in A$, $\xi=\sum\limits_{h,x} c_{h,x}\delta(h,x)\in K$ we have for $S(a)=\sum\limits_g \lambda_g S(g)$

$$\|S(a)\xi\|^2=\langle S(a^*a)\xi,\xi\rangle\leqq\|S(a^*a)\xi\|\,\|\xi\|,$$

and thus by induction, for any $n=0,1,2,\ldots,$

$$\begin{aligned}
\|S(a)\xi\|^{2^n} &\leqq \|S((a^*a)^{2^{n-2}})\xi\|^2\|\xi\|^{2^n-2}=\|\xi\|^{2^n-2}\Big\|\sum_{h,x} c_{h.x}S((a^*a)^{2^{n-2}})\delta(h,x)\Big\|^2=\\
&=\|\xi\|^{2^n-2}\Big\|\sum_{h,x}\sum_s c_{h,x}\lambda_s\delta(g_sh,x)\Big\|^2\leqq\|\xi\|^{2^n-2}\Big(\sum_x\Big\|\sum_{h,s} c_{h,x}\lambda_s\delta(g_sh,x)\Big\|\Big)^2=\\
&=\|\xi\|^{2^n-2}\Big\{\sum_x\Big(\sum_{h,s}\sum_{k,t} c_{h,x}\bar c_{k,x}\lambda_s\bar\lambda_t(f(k^*g_t^*g_sh,x),x)\Big)^{1/2}\Big\}^2\leqq\\
&\leqq\|\xi\|^{2^n-2}\Big(\sum_x M^{1/2}\|x\|\Big\|\sum_{h,s} c_{h,x}\lambda_s g_s h\Big\|\Big)^2\leqq\|\xi\|^{2^n-2}M\Big(\sum_x\|x\|\Big\|\sum_h c_{h,x}h\Big\|\Big)^2\Big\|\sum_s\lambda_s g_s\Big\|^2=\\
&=\|(a^*a)^{2^{n-2}}\|^2\|\xi\|^{2^n-2}M\Big(\sum_x\|x\|\Big\|\sum_h c_{h,x}h\Big\|\Big)^2=\|a\|^{2^n}\|\xi\|^{2^n-2}M\Big(\sum_x\|x\|\Big\|\sum_h c_{h,x}h\Big\|\Big)^2.
\end{aligned}$$

By passing to $n\to\infty$ we see that $\|S_a\xi\|\leqq\|a\|\,\|\xi\|$, which proves (20). Here the notation $(a^*a)^{2^n}=\sum\limits_s \lambda_s g_s$ was used for $a=\sum\limits_g \lambda_g g$.

(1.4) Similarly as before we adopt the preceding argument for our purpose such that X is replaced by $f(G\times X)$ and f by φ. (11) is then a simple translation of (10) into the new setting. The definitions (13′), (14′) and (15′) yield a *-representation of A on a suitable Hilbert space K with a continuous linear operator $V\colon K\to H$ such that (19) holds also true. The proof of (18) and $VV^*=I_H$ is the same as in (1.2).

2. Applications

(i) Let G be the trivial semigroup $G=\{e\}$. A familiar identification $G\times X\approx X$ implies the following results.

Theorem 2. 1. *Let f be an H-valued function given on a subset X of the Hilbert space H. There exists a self-adjoint operator F on H with $mI_H\leqq F\leqq MI_H$ and extending f if and only if*

$$\Big\|\sum_x c_x(f(x)-mx)\Big\|^2\leqq (M-m)\Big(\sum_x c_x(f(x)-mx),\ \sum_x c_x x\Big) \tag{21}$$

holds for any finite sequence $\{c_x\}$ $(x\in X)$ of complex numbers.

Proof. Since a self-adjoint operator F is bounded by m and M from below and above, respectively, if and only if $0\leqq F-mI_H$ and $\|F-mI_H\|\leqq M-m$, we have the assertion by Theorem 1.1. Indeed, (3) is the same as (21) ((4) is immediately satisfied with $p(e)=1$ if $M-m$ is replaced by M) for $f-mI_x$.

Corollary 2.1 (Krein (see [1])). *Let f be a symmetric and bounded linear operator from a linear subspace X of the Hilbert space H. Then there exists a self-adjoint operator F on H extending f and with the same norm.*

Theorem 2.1.1. *Let b be an H-valued function given on a subset Y of the Hilbert space H with norm dense linear hull in H. There exists a semi-bounded self-adjoint operator B with bound 1 from below and extending b if and only if*

$$\Big\|\sum_x c_x x\Big\|^2\leqq \Big(\sum_x c_x x,\ \sum_x c_x b(x)\Big) \tag{22}$$

holds for each finite sequence $\{c_x\}$ $(x\in Y)$ of complex numbers.

Proof. The necessity of (22) is evident so we omit the proof. To prove the sufficiency of (22) let f be the inverse map of b (the existence of which is an easy consequence of (22)). Since (22) is the same as (21) with $m=0$, $M=1$ we have by Theorem 2.1 a positive operator F with norm $\leqq 1$ which extends f. Hence F has an inverse $B=F^{-1}$ too. B is the desired operator. The proof is complete.

Corollary 2.1.1 (FRIEDRICHS (see [1])). *Let b be a symmetric operator bounded from below by* 1 *defined on a dense linear subspace Y of the Hilbert space H. Then there exists a self-adjoint operator B bounded from below by* 1, *and extending b.*

(ii) If H is a one dimensional Hilbert space, Theorem 1.3 (with the usual identifications $B(H)\cong \mathbf{C}$, $X=\{1\}$) gives us a new solution to the classical Hausdorff moment theorem differing also from Riesz' solution.

Theorem 2.2. *Let G be a (multiplicative) *-semigroup of a C^*-algebra A, spanning a norm dense *-subalgebra in A. A complex valued function f given on G has a (necessarily unique) positive linear extension to A if and only if there is a constant $M>0$ for which*

$$(1/M)\Big|\sum_g c_g f(g)\Big|^2 \leqq \sum_g \sum_h c_g \bar{c}_h f(h^* g) \leqq M\Big\|\sum_g c_g g\Big\|^2 \tag{23}$$

holds for each finite sequence $\{c_g\}$ $(g\in G)$ of complex numbers.

Corollary 2.2.1. *Let f be a complex valued function on a C^*-algebra A. f is a positive linear functional on A if and only if* (23) *holds with $G=A$.*

Corollary 2.2.2. *Let Ω be a compact subset of the real line and let $\{\mu_n\}_{n=0}^{\infty}$ be a given sequence of complex numbers. There is a positive (bounded) measure μ on Ω such that*

$$\int_\Omega t^n \, d\mu = \mu_n \quad \textit{for} \quad n = 0, 1, 2, \ldots$$

if and only if

$$0 \leqq \sum_m \sum_n c_m \bar{c}_n \mu_{m+n} \leqq \mu_0 \max_{t\in\Omega} \Big|\sum_n c_n t^n\Big|^2 \tag{24}$$

holds for each finite sequence $\{c_n\}_{n\geqq 0}$ of complex numbers.

Corollary 2.2.3. *Let Ω be a compact subset of the complex plane and let $\{\mu_{m,n}\}_{m,n=0}^{\infty}$ be a given double sequence of complex numbers. There is a (necessarily unique) positive (bounded) measure μ on Ω such that*

$$\int_\Omega (\bar{\lambda})^m \lambda^n \, d\mu(\lambda) = \mu_{m,n} \quad \textit{for} \quad m, n = 0, 1, 2, \ldots$$

if and only if

$$0 \leqq \sum_j \sum_k c_j \bar{c}_k \mu_{m_j+n_k, m_k+n_j} \leqq \mu_{0,0} \max_{\lambda\in\Omega} \Big|\sum_j c_j (\bar{\lambda})^{m_j} \lambda^{n_j}\Big|^2 \tag{25}$$

holds for each finite sequence $\{c_j\}_{j\geqq 0}$ of complex numbers.

3. Operator problems

Theorem 3.1. *Let G be a (multiplicative) *-semigroup in a C^*-algebra A generating a norm dense *-subalgebra of A. An operator valued function $f: G \to B(H)$ is A-dilatable if and only if there is a constant $M \geqq 0$ such that*

$$\Big\|\sum_g f(g)x_g\Big\|^2 \leqq M \sum_g \sum_h (f(h^*g)x_g, x_h) \tag{26}$$

and

$$\sum_g \sum_h c_g \bar{c}_h (f(h^*g)x, x) \leqq M\|x\|^2 \Big\|\sum_g c_g g\Big\|^2 \quad (x \in H) \tag{27}$$

hold for every finite sequence $\{x_g\}_{g\in G}$ and $\{c_g\}_{g\in G}$ in H and $\mathbf{C}$, respectively.

Sketch of the proof (for details see [9]). The following function f on $G \times X$ given by

$$f(g, x) = f(g)x \quad (g \in G;\ x \in X),$$

where $X = H$, produces (26) and (27) along (10) with $x_g = \sum_x c_{g,x} x$. The necessity of (26) and (27) thus follows. For the proof of the sufficiency we have to change the argument used in the proof of Theorem 1.3 replacing K_0 by the linear space of H-valued functions on G with finite support, that is, δ_g is replaced by $\delta_g x_g$ with $x_g \in H$ for $g \in G$. An easy analysis of the proof of the sufficiency part of Theorem 1.3 shows our statement.

Corollary 3.1.1. *Let G be a (multiplicative) *-semigroup in a commutative C^*-algebra A generating a norm dense *-subalgebra in A. An operator valued function $f: G \to B(H)$ is A-dilatable if and only if there is a constant $M > 0$ such that*

$$\Big|\sum_g c_g (f(g)x, x)\Big|^2 \leqq \|x\|^2 \sum_g \sum_h c_g \bar{c}_h (f(h^*g)x, x) \leqq M\|x\|^4 \Big\|\sum_g c_g g\Big\|^2 \quad (x \in H) \tag{28}$$

holds for each finite sequence $\{c_g\}_{g\in G}$ of complex numbers.

Proof. Since (26) and (27) imply (28) (by setting $x_g = c_g x$ for $g \in G, x \in X$), the necessity of (28) is obvious. For the sufficiency, the function $g \mapsto (f(g)x, x)$ on G has a (unique) positive linear extension by Theorem 2.2 for any fixed x in H. The norm of this extension is $\leqq M^{1/2}\|x\|^2$. Hence we obtain a positive linear extension F of f to the whole of A. But a result of Stinespring [11, Theorem 4] ensures that F is automatically A-dilatable.

The next result solves the operator moment problem of Sz.-Nagy in a new way.

Corollary 3.1.2. *Let Ω be a compact subset of the real line and let $\{A_n\}_{n=0}^{\infty}$ be a sequence of operators on a Hilbert space H. There is a positive (bounded)* *operator measure* F *on* Ω *such that*

$$\int_{\Omega} t^n(F(dt)x, x) = (A_n x, x) \quad \textit{for} \quad n = 0, 1, 2, \dots; \quad x \in H$$

if and only if

(29) $$0 \leqq \sum_m \sum_n c_m \bar{c}_n (A_{m+n} x, x) \leqq \|A_0\| \, \|x\|^2 \max_{t \in \Omega} \Big|\sum_n c_n t^n\Big|^2 \quad (x \in H)$$

holds for any finite sequence $\{c_n\}_{n \geqq 0}$ *of complex numbers.*

Proof. (29) is a version of (28) for $G = \{t^n\}_{n=0}^{\infty}$ $(t \in \Omega)$ and $A = C(\Omega)$ with $f(t^n) = A_n$ $(n = 0, 1, 2, \dots)$. Thus Corollary 3.1.1 implies the statement.

Corollary 3.1.3. *Let* Ω *be a compact subset of the complex plane and let* $\{A_{m,n}\}_{m,n=0}^{\infty}$ *be a double sequence of operators on a Hilbert space* H. *There is a positive (bounded) operator measure* F *on* Ω *such that*

$$\int_{\Omega} (\bar{\lambda})^m \lambda^n (F(dt)x, x) = (A_{m,n} x, x) \quad (m, n = 0, 1, 2, \dots; \ x \in H)$$

if and only if

(30)
$$0 \leqq \sum_j \sum_k c_j \bar{c}_k (A_{m_j + n_k, m_k + n_j} x, x) \leqq \|A_{0,0}\| \, \|x\|^2 \max_{\lambda \in \Omega} \Big|\sum_j c_j (\bar{\lambda})^{m_j} \lambda^{n_j}\Big|^2 \quad (x \in H)$$

holds for any finite sequence $\{c_j\}_{j \geqq 0}$ *of complex numbers.*

Proof. (30) is a version of (28) for $G = \{(\bar{\lambda})^m \lambda^n\}_{m,n=0}^{\infty}$ $(\lambda \in \Omega)$, $A = C(\Omega)$ with $f((\bar{\lambda})^m \lambda^n) = A_{m,n}$ $(m, n = 0, 1, 2, \dots)$. Thus Corollary 3.1.1 implies Corollary 3.1.3.

It follows a new characterization of subnormal operators (for the definition see [1]).

Corollary 3.1.4. *Let* B *be an operator in* $B(H)$ *for a Hilbert space* H. B *is subnormal if and only if*

(31) $$0 \leqq \sum_{k,l} \sum_{m,n} c_{k,l} \bar{c}_{m,n} (B^{*(k+n)} B^{l+m} x, x) \leqq \|x\|^2 \max_{\lambda \in \Omega} \Big|\sum_{m,n} c_{m,n} (\bar{\lambda})^m \lambda^n\Big|^2 \quad (x \in H)$$

holds for any finite double sequence $\{c_{m,n}\}_{m,n \geqq 0}$ *of complex numbers, where* Ω *denotes the spectrum of* B.

Proof. B is subnormal if and only if the function

$$(\bar{\lambda})^m \lambda^n \mapsto B^{*m} B^n \quad (m, n = 0, 1, 2, \dots; \ \lambda \in \Omega)$$

is $C(\Omega)$-dilatable (for details see [9]). But this is equivalent to (31).

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DEPARTMENT OF MATH. ANALYSIS II
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On the overconvergence of complex interpolating polynomials. II
Domain of geometric convergence to zero

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To Professor B. Szőkefalvi-Nagy on his seventieth birthday

1. Introduction. We continue here with developments concerning extensions of Walsh's Theorem on the overconvergence of sequences of differences of interpolating polynomials. As the title suggests, we are interested in determining *precisely* those domains in the complex plane for which (cf. [1]) the sequence

$$\{p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)\}_{n=1}^{\infty} \tag{1.1}$$

converges geometrically to zero for all $f \in A_\varrho$, where A_ϱ is the set of functions analytic in the circle $|z| < \varrho$ and having singularity on $|z| = \varrho$ ($\varrho > 1$). Here $p_{n-1}(z, Z, f)$ is the Lagrange interpolating polynomial of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ of degree $\leqq n-1$ based on the nodes determined by the n^{th} row of the infinite triangular matrix $Z = \{z_{k,n}\}_{k=1}^{n}{}_{n=1}^{\infty}$, and

$$Q_{n-1,l}(z, f) := \sum_{k=0}^{n-1} \left(\sum_{j=0}^{l-1} a_{k+jn} \right) z^k \quad (l = 1, 2, \ldots). \tag{1.2}$$

2. Constructions. As for Z, we now assume the stronger hypothesis (than that used in [1]) that there exists a real number ϱ' with $1 \leqq \varrho' < \varrho$ for which

$$1 \leqq |z_{k,n}| \leqq \varrho' < \varrho \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots). \tag{2.1}$$

As in [1], we set

$$\omega_n(t, Z) := \prod_{k=1}^{n} (t - z_{k,n}) \quad (n = 1, 2, \ldots), \tag{2.2}$$

and

$$G_l(z, R) = G_l(z, R, Z) := \varlimsup_{n \to \infty} \left\{ \max_{|t|=R} \left| (1 - t^{-ln}) \frac{z^n - 1}{t^n - 1} - \frac{\omega_n(z, Z)}{\omega_n(t, Z)} \right| \right\}^{1/n}, \tag{2.3}$$

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for any $R > \varrho'$ and any complex number z. We also set

$$\hat{G}_l(z, \varrho) := \inf_{\varrho' < R < \varrho} G_l(z, R). \tag{2.4}$$

With these definitions, we first establish

Proposition 1. *For any complex number $z \neq 1$ and any positive integer l, there holds*

$$G_l(z, R) \leqq \frac{\max\{|z|; 1\}}{R^{l+1}} \quad (z \neq 1, R > \varrho'). \tag{2.5}$$

Proof. From the techniques of [1], we see, via the maximum principle, that

$$\max_{|t|=R}\left|(1-t^{-ln})\frac{z^n-1}{t^n-1} - \frac{\omega_n(z, Z)}{\omega_n(t, Z)}\right| =$$

$$= \max_{|t|=R}\left|\frac{(t^{(l-1)}+t^{(l-2)n}+\ldots+1)(z^n-1)\omega_n(t, Z) - t^{ln}\omega_n(z, Z)}{t^{ln}\omega_n(t, Z)}\right| =$$

$$= R^{-ln}\max_{|t|=R}\left|\frac{(t^{(l-1)n}+t^{(l-2)n}+\ldots+1)(z^n-1)\omega_n(t, Z) - t^{ln}\omega_n(z, Z)}{\prod_{k=1}^{n}\left(R - \frac{\bar{z}_{k,n}t}{R}\right)}\right| \geqq$$

$$\geqq R^{-ln}\frac{|(z^n-1)\omega_n(0, Z)|}{R^n} \geqq \frac{|z^n-1|}{R^{(l+1)n}},$$

as $|\omega_n(0, Z)| \geqq 1$ from (2.1) and (2.2). Thus from the definition of $G_l(z, R)$ in (2.3), (2.5) immediately follows. Q. E. D.

Now define

$$\Delta_l(z) = \Delta_l(z, \varrho, Z) := \sup_{f \in A_\varrho} \overline{\lim_{n\to\infty}} |p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)|^{1/n} \tag{2.6}$$

for any complex number z. Then we have

Proposition 2. *For any z with $|z| > \varrho$,*

$$\hat{G}_l(z, \varrho) \geqq \Delta_l(z) \geqq G_l(z, \varrho). \tag{2.7}$$

Proof. Let E denote the matrix of nodes of interpolation formed from the roots of unity. Then for any $f \in A_\varrho$ and $\varepsilon > 0$, we have by [1, (1.9)]

$$|p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)| \leqq |p_{n-1}(z, Z, f) - p_{n-1}(z, E, f)| + |p_{n-1}(z, E, f) -$$

$$- Q_{n-1,l}(z, f)| \leqq \frac{1}{2\pi}\left|\int_\Gamma \frac{f(t)}{t-z}\left(\frac{\omega_n(z, Z)}{\omega_n(t, Z)} - \frac{z^n-1}{t^n-1}\right)dt\right| + \left(\frac{|z|}{\varrho^{l+1}} + \varepsilon\right)^n \leqq$$

$$\leqq \frac{M_f R}{|z|-R}\max_{|t|=R}\left|\frac{z^n-1}{t^n-1} - \frac{\omega_n(z, Z)}{\omega_n(t, Z)}\right| + \left(\frac{|z|}{\varrho^{l+1}} + \varepsilon\right)^n \leqq$$

$$\leqq \frac{M_f R}{|z|-R}\left\{(G_l(z, R)+\varepsilon)^n + R^{-ln}\frac{|z^n-1|}{R^n-1}\right\} + \left(\frac{|z|}{\varrho^{l+1}} + \varepsilon\right)^n \quad (\varrho' < R < \varrho < |z|),$$

where $\Gamma=\{t: |t|=R\}$, $M_f=\max_{z\in\Gamma}|f(z)|$, provided $n\geqq n_0=n_0(\varepsilon)$. Hence by Proposition 1,

$$\varlimsup_{n\to\infty} |p_{n-1}(z, Z, f)-Q_{n-1,l}(z, f)|^{1/n} \leqq \max\left\{G_l(z, R)+\varepsilon, \frac{|z|}{R^{l+1}}, \frac{|z|}{\varrho^{l+1}}+\varepsilon\right\} \leqq$$

$$\leqq \max\left\{G_l(z, R), \frac{|z|}{\varrho^{l+1}}\right\}+\varepsilon.$$

But here $\varepsilon>0$ and R $(\varrho'<R<\varrho)$ were arbitrary. Thus again by Proposition 1

$$\varlimsup_{n\to\infty} |p_{n-1}(z, Z, f)-Q_{n-1,l}(z, f)|^{1/n} \leqq \inf_{\varrho'<R<\varrho} G_l(z, R) =: \hat{G}_l(z, \varrho).$$

As this inequality holds for all $f\in A_\varrho$, this gives from (2.6) that

$$\Delta_l(z) \leqq \hat{G}_l(z, \varrho),$$

the desired first inequality of (2.7).

Next, for any u with $|u|=\varrho$ and with $f_u(z):=(u-z)^{-1}\in A_\varrho$, a direct computation gives that

$$(2.8)\quad p_{n-1}(z, Z, f_u)-Q_{n-1,l}(z,f_u) = \frac{1}{u-z}\left\{(1-u^{-ln})\frac{z^n-1}{u^n-1}-\frac{\omega_n(z, Z)}{\omega_n(u, Z)}+u^{-ln}\right\}.$$

Now by Proposition 1, $G_l(z, \varrho)>\varrho^{-l}$ $(|z|>\varrho)$. Thus we may choose an $\varepsilon>0$ with

$$(2.9)\qquad \varrho^{-l}+\varepsilon < G_l(z, \varrho) \quad (|z| > \varrho).$$

Further let $\{n_j\}_{n=1}^{\infty}$ be an infinite sequence of positive integers with $n_1<n_1<\dots$ (dependent on z) such that

$$\max_{|t|=\varrho}\left|(1-t^{-ln_j})\frac{z^{n_j}-1}{t^{n_j}-1}-\frac{\omega_{n_j}(z, Z)}{\omega_{n_j}(t, Z)}\right| > (G_l(z, \varrho)-\varepsilon)^{n_j} \quad (j = 1, 2, \dots)$$

(cf. Definition (2.3)). Now, choose u_j with $|u_j|=\varrho$ (which is also dependent on z) so that

$$(2.10)\quad \begin{aligned}&\left|(1-u_j^{-ln_j})\frac{z^{n_j}-1}{u_j^{n_j}-1}-\frac{\omega_{n_j}(z, Z)}{\omega_{n_j}(u_j, Z)}\right| = \\ &= \max_{|t|=\varrho}\left|(1-t_j^{-ln_j})\frac{z^{n_j}-1}{t^{n_j}-1}-\frac{\omega_{n_j}(z, Z)}{\omega_{n_j}(t, Z)}\right| > (G_l(z, \varrho)-\varepsilon)^{n_j},\end{aligned}$$

for each $j=1, 2, \dots$. With $n=n_j$ and $u=u_j$, it follows from (2.8) and (2.10) that

$$|p_{n_j-1}(z, Z, f_{u_j})-Q_{n_j-1,l}(z, f_{u_j})| > \frac{1}{|z|+\varrho}\{(G_l(z, \varrho)-\varepsilon)^{n_j}-\varrho^{-ln_j}\},$$

for all $j=1, 2, \dots$. Now, following the construction of [1], there is an $\tilde{f}$ (dependent on z) in A_ϱ for which

$$(2.11)\quad |p_{n_j-1}(z, Z,\tilde{f})-Q_{n_j-1,l}(z,\tilde{f})| \geqq \frac{1}{3(|z|+\varrho)n_j}\{(G_l(z, \varrho)-\varepsilon)^{n_j}-\varrho^{-ln_j}\}$$

for all $j=1, 2, \ldots$. Thus, by (2.9)

$$\varlimsup_{n\to\infty} |p_{n-1}(z, Z, \tilde{f})-Q_{n-1,l}(z, \tilde{f})|^{1/n} \geqq G_l(z, \varrho)-\varepsilon,$$

and as $\tilde{f}$ is some element in A_ϱ, then from the definition in (2.6),

$$\Delta_l(z) \geqq G_l(z, \varrho)-\varepsilon.$$

But, as this holds for every $\varepsilon>0$ with $\varrho^{-l}+\varepsilon<G_l(z, \varrho)$, then

$$\Delta_l(z) \geqq G_l(z, \varrho),$$

the desired last inequality of (2.7). Q. E. D.

As an obvious consequence of (2.7) of Proposition 2, we have

Corollary 3. *Let z be any complex number with $|z|>\varrho$ for which $\hat{G}_l(z, \varrho)<1$. Then, the sequence* (1.1) *converges geometrically to zero for each $f\in A_\varrho$.*

As a consequence of the proof of Proposition 2, we further have

Corollary 4. *Let z be any complex number with $|z|>\varrho$ for which $G_l(z, \varrho)>1$. Then, there is a function $\tilde{f}$ (depending on z) in A_ϱ for which the sequence* (1.1) *(with f replaced by $\tilde{f}$) is unbounded.*

Proof. If $G_l(z, \varrho)=1+2\eta$ where $\eta>0$, choose $\varepsilon>0$ sufficiently small so that $G_l(z, \varrho)-\varepsilon>1+\eta>1$. Then, (2.11) directly shows that the sequence (1.1) (with f replaced by $\tilde{f}$) is unbounded. Q. E. D.

Obviously, Corollary 4 and Proposition 1 imply that the sequence (1.1) is necessarily unbounded for some $\tilde{f}$ in A_ϱ, whenever $|z|>\varrho^{l+1}$. The same conclusion was deduced in [1].

Open questions. 1. Is $\hat{G}_l(z, \varrho)=G_l(z, \varrho)$?

2. Assuming the answer is "yes" for the previous question, then $\mathfrak{G}:=$ $:=\{z\colon G_l(z, \varrho)=1\}$ divides the complex plane into sets where either one has geometric convergence to zero for all f in A_ϱ or unboundedness of the sequence (1.1) for some f in A_ϱ. What does $\mathfrak{G}$ look like?

3. In general, one would not suspect that $\mathfrak{G}$ is a circle, even though this is the case for all examples treated in the literature. Can one construct cases (i.e. matrices Z) where indeed $\mathfrak{G}$ is not a circle? This suggests considering $Z=\{z_{k,n}\}$ where $\{z_{k,n}\}_{k=1}^n$ are not uniformly distributed, as $n\to\infty$.

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On symplectic actions of compact Lie groups with isotropy subgroups of maximal rank

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

Let (M, ω) be a symplectic manifold, G a connected Lie group and $\Phi: G \times M \to M$ a symplectic action. If the action Φ has a momentum map $\mu: M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$ of G, then there is an action $\Psi: G \times \mathfrak{g}^* \to \mathfrak{g}^*$ such that μ is equivariant with respect to the actions Φ, Ψ. In this setting are derived the results of A. A. Kirillov, B. Kostant, J. M. Souriau and of others concerning Hamiltonian systems with symmetries ([1], pp. 276—311). Restriction to the case where G is compact offers a situation with peculiar features, a subject which seems to deserve special concern. A result pertaining to the above case is presented below. In fact, it is shown that if G is compact and the isotropy subgroups of Φ are of maximal rank then all the orbits of Φ are equivariantly isomorphic.

The concepts and results applied subsequently are in conformity with those laid down in the work of R. Abraham and J. E. Marsden [1], however, in the notations there are some deviations.

The following lemma presents a simple but for the subsequent results essential observation.

Lemma 1. *Let (M, ω) be a symplectic manifold, G a connected Lie group and $\Phi: G \times M \to M$ a symplectic action with a momentum map $\mu: M \to \mathfrak{g}^*$. Then the kernel of the tangent linear map $T_z\mu: T_zM \to T_{\mu(z)}\mathfrak{g}^*$ is given by*

$$\operatorname{Ker} T_z\mu = (T_z G(z))^{\perp}$$

where the orthogonal complement is taken with respect to the symplectic form ω.

Proof. Let $Z \in T_zM$ and $\varphi: I \to M$ a curve with $\varphi(0) = z, \varphi'(0) = Z$. Then the following holds for any fixed $X \in \mathfrak{g}$ according to the definition of the momentum

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map:

$$\frac{d}{d\tau}\langle\mu(\varphi(\tau)), X\rangle\Big|_{\tau=0} = Z\langle\mu(x), X\rangle = d(\langle\mu(x), X\rangle)(Z) = (i(\bar{X})\omega)(Z) = \omega(\bar{X}, Z)$$

where $\bar{X}$ is the infinitesimal generator of the action Φ corresponding to the element X of the Lie algebra $\mathfrak{g}$ of G. On the other hand, the following is obviously valid:

$$\frac{d}{d\tau}\langle\mu(\varphi(\tau)), X\rangle\Big|_{\tau=0} = \langle\iota_\xi\circ T_z\mu Z, X\rangle$$

where $\iota_\zeta\colon T_\zeta\mathfrak{g}^*\to\mathfrak{g}^*$ is the canonical isomorphism at $\zeta=\mu(z)$. Consequently, the following is obtained:

$$\langle\iota_\xi\circ T_z\mu Z, X\rangle = \omega(\bar{X}, Z) \quad \text{for} \quad X\in\mathfrak{g}.$$

Therefore $Z\in\operatorname{Ker} T_z\mu$ if and only if $Z\in(T_zG(z))^\perp$ holds, since $T_zG(z)$ is spanned by the values of $\bar{X}$ at z as X runs through $\mathfrak{g}$.

Let now (M,ω) be a symplectic manifold and $\langle\,,\rangle$ a Riemannian metric on M. Then there is a unique tensor field A of type $(1;1)$ on M such that $\omega(X,Y)=\langle AX, Y\rangle$ holds for any vector fields $X, Y\in\mathcal{T}(M)$. Moreover, since ω is non-degenerate, $A_z\colon T_zM\to T_zM$, the value of A at the point $z\in M$ is an automorphism of the tangent space. Consider now with respect to the inner product $\langle\,,\rangle_z$ the polar decomposition $A_z=S_z\circ J_z$ of A_z, then the symmetric tensor S_z and the orthogonal tensor J_z are uniquely defined since A_z is injective ([2], pp. 169—170). Thus, tensor fields S, J are obtained on M. Moreover, the tensor field J is an almost complex structure on M and $\langle X, Y\rangle=\omega(JX, Y)$ holds for arbitrary vector fields $X, Y\in\mathcal{T}(M)$ according to a basic result ([1], pp. 172—174).

The tensor field J is called the *almost complex structure defined by the symplectic form ω and by the Riemannian metric $\langle\,,\rangle$.*

The following corollary is a consequence of the preceding lemma and of the above mentioned facts.

Corollary. *Let (M,ω) be a symplectic manifold, G a connected Lie group and $\Phi\colon G\times M\to M$ a symplectic action with a momentum map $\mu\colon M\to\mathfrak{g}^*$. Moreover, let there be a Riemannian metric $\langle\,,\rangle$ on M which is left invariant by the action Φ and let J be the almost complex structure defined by ω and $\langle\,,\rangle$. Then the kernel of the tangent linear map $T_z\mu$ is given at any point $z\in M$ by*

$$\operatorname{Ker} T_z\mu = J_z(N_zG(z))$$

where $N_zG(z)$ is the orthogonal complement of the tangent space $T_zG(z)$ with respect to the inner product $\langle\,,\rangle_z$.

Proof. According to the above mentioned relation of ω, $\langle\ ,\rangle$, and J the following inclusions obviously hold:

$$J(N_zG(z)) \subset (T_zG(z))^{\perp}, \quad J^{-1}((T_zG(z)^{\perp}) \subset N_zG(z)$$

for $z\in M$. Consequently, the Corollary follows directly from the preceding lemma.

The following lemma which is again a consequence of the relation of ω, $\langle\ ,\rangle$, and J, is essential for the subsequent results.

The lemma concerns the induced action on the tangent bundle. In fact, if an action $\Phi: G\times M\to M$ is given, then by $\Phi_g(z)=\Phi(g, z)$, $z\in M$, a diffeomorphism $\Phi_g: M\to M$ is defined for any $g\in G$. Consequently the tangent linear map $T\Phi_g: TM\to TM$ is a transformation of TM for $g\in G$. Thus an action of G on TM is obtained which is called the *induced action* of G on TM.

Lemma 2. *Let (M,ω) be a symplectic manifold, G a connected Lie group, $\Phi: G\times M\to M$ a symplectic action, and $\langle\ ,\rangle$ a Riemannian metric on M which is left invariant by the action Φ. Then the almost complex structure J defined by ω and $\langle\ ,\rangle$ is equivariant for the induced action of G on TM; in other words, $T\Phi_g\circ J=J\circ T\Phi_g$ is valid for any element g of G.*

Proof. Let A be the tensor field defined by ω and $\langle\ ,\rangle$ on M and S, J those obtained by the polar decomposition of A. The invariance of ω and $\langle\ ,\rangle$ yields that the following is valid for arbitrary vector fields $X, Y\in\mathscr{T}(M)$ and $g\in G$:

$$\langle T\Phi_g^{-1}\circ A\circ T\Phi_g X, Y\rangle = \langle A\circ T\Phi_g X, T\Phi_g Y\rangle =$$
$$= \omega(T\Phi_g X, T\Phi_g Y) = \omega(X,Y) = \langle AX, Y\rangle.$$

But then $A=T\Phi_g^{-1}\circ A\circ T\Phi_g$ holds for $g\in G$. Consequently, the following is valid, too:

$$A = S\circ J = (T\Phi_g^{-1}\circ S\circ T\Phi_g)\circ(T\Phi_g^{-1}\circ J\circ T\Phi_g), \quad g\in G.$$

But, then $T\Phi_g^{-1}\circ S\circ T\Phi_g$, $T\Phi_g^{-1}\circ J\circ T\Phi_g$ yields a polar decomposition of A for $g\in G$, since the Riemannian metric $\langle\ ,\rangle$ is left invariant by the action Φ. Since A_z, $z\in M$, is injective, its polar decomposition is unique, as mentioned before. Consequently, the validity of

$$J = T\Phi_g^{-1}\circ J\circ T\Phi_g, \quad g\in G$$

is obtained which yields the assertion of the lemma.

In order to state a corollary of the preceding lemma the introduction of a concept is convenient. In fact, let $\Phi: G\times M\to M$ be a smooth action of a connected Lie group G. Then

$$R_z = \{X \mid T_z\Phi_g X = X \quad \text{for} \quad g\in G_z \quad \text{where} \quad X\in T_zM\}$$

is a subspace of the tangent space T_zM at any point z of the manifold M.

Corollary. *Let* (M, ω) *be a symplectic manifold,* G *a connected Lie group,* $\Phi: G\times M\to M$ *a symplectic action and* $\langle\, ,\rangle$ *a Riemannian metric on* M *which is left invariant by the action* Φ*. If* J *is the almost complex structure defined by* ω *and* $\langle\, ,\rangle$ *then* $J_z(R_z)=R_z$ *holds at any point* $z\in M$*.*

Proof. If $X\in R_z$ then $T_z\Phi_g(J_z(X))=J_z(T_z\Phi_z(X))=J_z(X)$ holds for $g\in G_z$, and this implies the assertion of the corollary.

In case of a smooth action of a compact connected Lie group there is a standard classification of the orbits of the action and accordingly principal, exceptional and singular orbits are distinguished. As a result of R. Palais [5] shows the above classification can be introduced in case of isometric actions of connected Lie groups so that almost all the fundamental results concerning smooth actions of compact Lie groups remain valid. Therefore, if $\Phi: G\times M\to M$ is an isometric action of a connected Lie group on a Riemannian manifold M then there are points $z\in M$ such that $G(z)$ is a principal orbit; moreover, if in this case

$$T_z M = T_z G(z)\oplus N_z G(z)$$

is the orthogonal decomposition with respect to the Riemannian metric $\langle\, ,\rangle$, then $N_zG(z)\subset R_z$ holds.

For the formulation of the next lemma the introduction of the following concept is convenient. Consider a smooth action $\Phi: G\times M\to M$ of a connected Lie group G on a differentiable manifold M and a non-zero tangent vector $X\in T_zG(z), z\in M$; it is said that X is an *isotropy fixed tangent vector* for the action Φ provided that the following is valid:

$$X = T_z\Phi_g X \quad \text{for} \quad g\in G_z.$$

Some results concerning basic properties of the above concept will be given elsewhere.

Lemma 3. *Let* (M, ω) *be a symplectic manifold,* G *a connected Lie group,* $\Phi: G\times M\to M$ *a symplectic action with a momentum map* $\mu: M\to \mathfrak{g}^*$ *and* $\langle\, ,\rangle$ *a Riemannian metric on* M *which is left invariant by the action* Φ*. If the action* Φ *has no isotropy fixed tangent vectors then*

$$\operatorname{Ker} T_z\mu = N_z G(z)$$

holds at any point $z\in M$ *such that* $G(z)$ *is a principal orbit of the action.*

Proof. Let $z\in M$ be such that $G(z)$ is a principal orbit and consider the orthogonal decomposition

$$T_z M = T_z G(z)\oplus N_z G(z)$$

with respect to the Riemannian metric $\langle\, ,\rangle$. Let $X\in R$ and

$$X = X'+X'',\quad X'\in T_z G(z),\ X''\in N_z G(z)$$

its corresponding decomposition. Then both X' and X'' are left fixed by the action $T_z\Phi_g: T_zM \to T_zM$, $g \in G_z$. Thus, the assumption that Φ has no isotropy fixed tangent vectors implies that $X'=0$ holds. Consequently, $R_z \subset N_zG(z)$ is valid. On the other hand, the assumption that $G(z)$ is a principal orbit implies that $N_zG(z) \subset R_z$ holds. Thus, $N_zG(z)=R_z$. Now, the corollaries to Lemma 1 and to Lemma 2 as well as the preceding assertion yield that

$$\operatorname{Ker} T_z\mu = J_z(N_zG(z)) = J_z(R_z) = R_z = N_zG(z)$$

holds. Thus, the assertion of the lemma is proved.

The following theorem presents the result already mentioned at the beginning. The rank of compact Lie groups occurring here is taken in the usual sense given in terms of the maximal tori or of the Cartan subalgebras.

Theorem. *Let (M, ω) be a symplectic manifold, G a compact connected Lie group and $\Phi: G \times M \to M$ a symplectic action with a momentum map. If the isotropy subgroups of Φ are of maximal rank then all the orbits of Φ are equivariantly isomorphic.*

Proof. Since the group G is compact, there is a Riemannian metric $\langle , \rangle$ on M which is left invariant by the action Φ. The assumption that the isotropy subgroups of Φ are of maximal rank implies that Φ has no isotropy fixed tangent vectors. In fact, assume that there is a $V \in T_zG(z)$ for some $z \in M$ which is an isotropy fixed tangent vector of Φ. Consider now the maps

$$\pi_z: G \to G/G_z, \quad \varepsilon_z: G/G_z \to G(z),$$

which are the canonical projection and the canonical equivariant isomorphism, and fix a reductive decomposition $\mathfrak{g}=\mathfrak{g}\oplus\mathfrak{m}_z$ where $\mathfrak{g}_z \subset \mathfrak{g}$ is the subalgebra of the Lie algebra $\mathfrak{g}$ corresponding to the isotropy subgroup G_z. Then with the usual identifications $T_eG=\mathfrak{g}$, $T_eG_z=\mathfrak{g}_z$, a restricted map

$$T_o\varepsilon_z \circ T_e\pi_z: \mathfrak{m}_z \to T_zG(z)$$

is obtained where $o=\pi_z(e)$, and this restricted map is a vector space isomorphism which is equivariant for the following actions:

$$\operatorname{Ad}(g): \mathfrak{m}_z \to \mathfrak{m}_z, \quad T_z\Phi_g: T_zG(z) \to T_zG(z), \quad g \in G_z.$$

Now, the existence of the vector $V \in T_zG(z)$ yields an element X of the Lie algebra $\mathfrak{g}$ such that

$$X \in \mathfrak{m} - \{0\} \quad \text{and} \quad [\mathfrak{g}_z, X] = 0$$

are valid. Since $G_z \subset G$ is of maximal rank, there is a Cartan subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ included in $\mathfrak{g}_z$. But then $[\mathfrak{k}, X]=0$ holds, and therefore X is in the normalizer of $\mathfrak{k}$. Since $X \notin \mathfrak{k}$ is valid, a contradiction is obtained with the definition of Cartan subalgebras. Therefore, the action Φ has no isotropy fixed tangent vectors. Let now $\mu: M \to g^*$ be a momentum map of the action Φ. Then the preceding lemma applies and yields that

$$\operatorname{Ker} T_z \mu = N_z G(z)$$

is valid for $z \in M$ provided that $G(z)$ is a principal orbit of the action Φ.

Fix now a $z \in M$ such that $G(z)$ is a principal orbit of Φ and consider the component F_z containing the point z of the following set

$$\{x \mid \Phi(g, x) = x \quad \text{for} \quad g \in G_z, \text{ where } x \in M\}.$$

Then, F_z is a totally geodesic submanifold of the Riemannian manifold M according to a fundamental result ([4], pp. 59—61) and F_z intersects every orbit of the action Φ [6]. Let $F_z' \subset F_z$ be the set of points $x \in F_z$ such that $G(x)$ is a principal orbit. Then, F_z' is an open, everywhere dense subset of F_z in consequence of the Principal Isotropy Type Theorem. Moreover, observations made in the proof of the preceding lemma imply that

$$T_x F_z = N_x G(x)$$

holds for $x \in F_z'$. Therefore, the assertion of Lemma 3 yields that

$$\operatorname{Ker} T_z \mu = T_x F_z$$

is valid for $x \in F_z'$. But then $\mu(F_z')$ is a single point and consequently $\mu(F_z)$ is a single point too. Consider now the action $\Psi: G \times g^* \to g^*$ on the dual space g^*, which is associated with the action Φ ([1], pp. 276—294). The image of μ is a single orbit of the action Ψ, owing to the facts that μ is equivariant for Φ and Ψ, that F_z intersects every orbit of Φ, and that $\mu(F_z)$ is a single point.

The restriction of $T_x\mu$ to $T_xG(x)$ is injective provided that $G(x)$ is a principal orbit of Φ, as Lemma 3 implies this. Therefore, the action Φ cannot have singular orbits, since the image of μ is a single orbit of Ψ as observed above. Thus, μ restricted to an orbit of Φ is a covering map. Let now $z \in M$ be such that $G(z)$ is a principal orbit. Then F_z is intersected the same number of times by any principal orbit of Φ. Since in any neighbourhood of an exceptional orbit there are principal ones; F_z is intersected the same number of times by an exceptional orbit of Φ as by the principal ones. Therefore, the existence of exceptional orbits and properties of the momentum map μ imply the existence of different intersecting totally geodesic submanifolds F_z. But the fact that two different ones among such submanifolds intersects entails obviously the existence of singular orbits. Consequently, the action Φ has no exceptional orbits either. Thus, the action Φ has only principal orbits; and this fact implies the assertion of the theorem.

As its following corollary shows, the preceding theorems has consequences concerning the structure of the symplectic manifold as well.

Corollary. *Let* (M, ω) *be a symplectic manifold,* G *a compact connected Lie group and* $\Phi: G\times M\to M$ *a symplectic action with a momentum map. If the isotropy subgroups of the action* Φ *are of maximal rank, then* M *is the total space of a differentiable fibre boundle, where the base manifold is the orbit space of the action* Φ *and the fibers are diffeomorphic to a finite covering of a fixed orbit of the coadjoint action* $\mathrm{Ad}^*: G\times \mathfrak{g}^*\to \mathfrak{g}^*$.

Proof. Since G is compact, there is a momentum map $\tilde{\mu}: M\to \mathfrak{g}^*$ of the action Φ such that the associated action $\tilde{\Psi}: G\times\mathfrak{g}^*\to\mathfrak{g}^*$ is equivalent to the coadjoint action of G. In fact, let $\mu: M\to\mathfrak{g}^*$ be a momentum map of Φ and $\sigma: G\to\mathfrak{g}^*$ the coadjoint cocycle associated to μ; then the associated action Ψ is given as follows:

$$\Psi(g, \xi) = \mathrm{Ad}^*(g^{-1})\xi+\sigma(g) \quad \text{where} \quad (g, \xi)\in G\times\mathfrak{g}^*.$$

Since the group G is compact, the action Ψ must have a fixed point $\zeta\in\mathfrak{g}^*$ and therefore

$$\Psi(g, \zeta) = \mathrm{Ad}^*(g^{-1})\zeta+\sigma(g) = \zeta$$

holds for every $g\in G$. Consequently, the associated action Ψ is given as follows:

$$\Psi(g, \xi) = \mathrm{Ad}^*(g^{-1})(\xi-\zeta)+\zeta \quad \text{where} \quad (g, \xi)\in G\times\mathfrak{g}^*.$$

According to the preceding theorem μ maps to a single orbit of Ψ and is a covering map on each orbit of Φ which are all of the same type. Consequently, the orbits of Φ are diffeomorphic to a finite covering of a single orbit of the coadjoint action. Since the orbits of Φ are all of the same type, the assertion of the corollary follows now by a basic theorem on the union of orbits of the same type of smooth actions of compact Lie groups ([3], pp. 6—9).

Added in proof. The author is indebted to Professor J. E. Marsden for the information that Lemmas 1, 2 and 3 are partially contained in the paper Symmetry and bifurcations of momentum mappings, *Comm. Math. Phys.*, **78** (1981), 445—478 by J. M. ARMS, J. E. MARSDEN and V. MONCRIEF. Moreover, the author is thankful to Professor J. J. Duistermaat for acquainting him with conjectures that some results of the paper such as the Corollary to the Theorem can be developed further. A detailed account of the above observations together with further related results will be presented in a forthcoming paper.

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Some weak-star ergodic theorems

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

0. Introduction. Let M be a von Neumann algebra and let G be a group of $*$-automorphisms of M. It is proved in [3] that if the family of G-invariant normal states is faithful on M (i.e., M is G-finite), then for every $t \in M$, the w^*-closed convex hull of $\{gt : g \in G\}$ contains exactly one G-invariant element. In the present paper we prove the converse of this theorem in the case where M is σ-finite and G is abelian. We present our results in the more general setting of arbitrary Banach spaces.

1. Results. Throughout this paper B denotes a Banach space and B^* its dual space. We denote by $L_{w*}(B^*)$ the space of w^*-continuous linear operators in B^*, equipped with the topology of pointwise w^*-convergence. Every element g of $L_{w*}(B^*)$ is a bounded operator in B^* such that there exists a unique bounded linear operator g_* in B for which $(g_*)^* = g$. Throughout this paper G will denote a bounded commutative semigroup $G \subset L_{w*}(B^*)$. We shall study the implications of the following condition:

(U) For every $t \in B^*$, the w^*-closed convex hull of the orbit $\{gt : g \in G\}$ contains a unique G-invariant element, which will be denoted by t^G.

(The fact that this closed convex hull contains at least one G-invariant element follows from the Kakutani—Markov fixed point theorem (cf. [2], V. 10. 6), in view of the w^*-compactness of the unit ball of B^*.)

Theorem 1. *Suppose that condition* (U) *is satisfied and either B is a separable Banach space or G is a separable topological subspace of $L_{w*}(B^*)$. Then the mapping $t \to t^G$ $(t \in B^*)$ is a bounded linear projection P acting in B^*. We have $gP = Pg = P$ and P is the limit, in $L_{w*}(B^*)$, of a sequence of elements of the convex hull of G.*

Theorem 2. *Suppose that either B or G is separable. If condition* (U) *is satisfied and B is weakly complete, then the mapping $t \to t^G$ $(t \in B^*)$ is a w^*-continuous*

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linear projection P *such that* $gP=Pg=P$. *The operator* P *belongs to the sequential closure, in* $L_{w*}(B^*)$, *of the convex hull* $\mathrm{co}\,G$ *of* G. *Moreover, for every* $v_0\in\mathrm{co}\,G$ *and every* w^**-neighborhood* N *of zero there exists* $v_1\in\mathrm{co}\,G$ *such that* $vv_1v_0t-t^G\in N$ *for every* $v\in\mathrm{co}\,G$ *and* $t\in B^*$ *such that* $\|t\|\leqq 1$.

Proposition 1. *The hypotheses of Theorem* 2 *are satisfied if:*

(a) $B=L^1(X,S,m)$, *where* (X,S,m) *is a positive localizable measure space* (*then* $B^*=L^\infty(X,S,m)$);

(b) G *is a bounded commutative semigroup of* w^**-continuous linear operators in* $L^\infty(X,S,m)$, *satisfying condition* (U);

(c) *Either* $L^1(X,S,m)$ *or* G *is separable.*

Proposition 2. *The assertions of Theorem* 2 *hold if:*

(a) B^* *is a* W^**-algebra* M;

(b) G *is a bounded commutative semigroup of* w^**-continuous linear mappings of* M *into itself, satisfying condition* (U).

(c) *Either* M *is* σ*-finite or* G *is separable.*

Corollary. *Let* M *be a von Neumann algebra and let* G *be a commutative group of* $*$*-automorphisms of* M, *satisfying condition* (U). *If* M *is* σ*-finite or* G *is separable, then* M *is* G*-finite* (for this notion, cf. [3]).

2. Proofs. For the proof of Theorem 1 we need the following two lemmas.

Lemma 1. *Let* $G=\{g_1,g_2,\ldots\}$ *be countable and let* B *be separable. Suppose that condition* (U) *is satisfied. Then for every* $t\in B^*$, *the sequence* $\left\{\frac{1}{n^n}\sum_{i_1,\ldots,i_n=1}^{n} g_1^{i_1}\ldots g_n^{i_n}t\right\}_{n=1}^{\infty}$ w^**-converges to* t^G.

Lemma 2. *Let* B_1 *be a* G_**-invariant closed subspace of* B, *i.e., let* $g_*\varphi\in B_1$ *for* $g_*\in G_*$, $\varphi\in B_1$. *Furthermore, let* $B_1^\perp=\{t\colon(\varphi,t)=0$ *for all* $\varphi\in B_1\}$ *and let the dual space* B_1^* *of* B_1 *be identified canonically with the quotient space* $B^*/B_1^\perp$. *If* G *acting on* B^* *satisfies condition* (U), *then* G *acting on* B_1^* *also satisfies condition* (U).

Proof of Lemma 1. Let $v_n=\frac{1}{n^n}\sum_{i_1,\ldots,i_n=1}^{n} g_1^{i_1}\ldots g_n^{i_n}$ and let $t\in B^*$. We have to prove that the sequence $\{v_nt\}$ w^*-converges to t^G. To prove this, we show that every subsequence $\{v_{n_k}t\}$ of $\{v_nt\}$ contains a subsequence $\{v_{n_{k_l}}t\}$ which w^*-converges to t^G. Since the sequence $\{v_nt\}$ is a bounded sequence in B^* and every closed ball in B^* is metrizable compact in the w^*-topology (cf. [2], V. 4.2, V. 5.1), this will imply that $v_nt\to t^G$ in the w^*-topology as $n\to\infty$. Let $\{v_{n_k}t\}$ be a subsequence of $\{v_nt\}$. Since $\{v_{n_k}t\}$ is a bounded sequence, it contains a w^*-convergent subsequence $\{v_{n_{k_l}}t\}$ (by the above remark). We have to prove that the limit of

$\{v_{n_{k_l}}t\}$ is t^G. Since the limit of $\{v_{n_{k_l}}t\}$ obviously belongs to the w^*-closed convex hull of $\{gt : g\in G\}$, we only have to prove that it is G-invariant. Pick a positive integer s. Let $n\geqq s$. Then g_s appears in v_n. By the commutativity of G we have:

$$\|g_s v_n t - v_n t\| =$$

$$= \left\| \frac{1}{n^n} \sum_{i_1,\dots,i_n=1}^{n} g_g^{i_1} \dots g_{s-1}^{i_{s-1}} g_s^{i_s+1} g_{s+1}^{i_{s+1}} \dots g_n^{i_n} t - \frac{1}{n^n} \sum_{i_1,\dots,i_n=1}^{n} g_1^{i_1} \dots g_n^{i_n} t \right\| =$$

$$= \left\| \frac{1}{n^n} \sum_{i_1,\dots,i_{s-1},i_{s+1},\dots,i_n=1}^{n} (g_1^{i_1} \dots g_{s-1}^{i_{s-1}} g_s^{n+1} g_{s+1}^{i_{s+1}} \dots g_n^{i_n} t - g_1^{i_1} \dots g_{s-1}^{i_{s-1}} g_{s+1}^{i_{s+1}} \dots g_n^{i_n} t) \right\| \leqq$$

$$\leqq \frac{2n^{n-1}\|G\|\cdot\|t\|}{n^n} = \frac{2}{n}\|G\|\cdot\|t\|,$$

where $\|G\| = \sup\{\|g\| : g\in G\}$. If now $n=n_{k_l}$ and $l\to\infty$, then $n_{k_l}\to\infty$, and consequently, $\|g_s v_{n_{k_l}} t - v_{n_{k_l}} t\|\to 0$ by the above. Hence $g_s v_{n_{k_l}} t \to \lim\limits_{l\to\infty} v_{n_{k_l}} t$. On the other hand, by the w^*-continuity of g_s we have $g_s v_{n_{k_l}} t \to g_s \lim\limits_{l\to\infty} v_{n_{k_l}} t$. Consequently, $g_s \lim\limits_{l\to\infty} v_{n_{k_l}} t = \lim\limits_{l\to\infty} v_{n_{k_l}} t$. Since g_s was an arbitrary element of G, we have proved that $\lim\limits_{l\to\infty} v_{n_{k_l}}$ is G-invariant, and consequently,

$$\lim_{l\to\infty} v_{n_{k_l}} t = t^G.$$

Proof of Lemma 2. Since $GB_1^\perp \subset B_1^\perp$, the semigroup G acts on $B_1^* = B^*/B_1^\perp$, and Lemma 2 makes sense. Let $f\in B_1^*$ and let f_0 be a G-invariant element of the w^*-closed convex hull of $\{gf : g\in G\}$. There exists a net v_n of elements of $\mathrm{co}G$ such that $v_n f\to f_0$ in the w^*-topology of B_1^*. The element $f\in B_1^*$ is canonically identified with a coset $t+B_1^\perp$ $(t\in B^*)$ and for every $g\in G$, the element gf is identified with $gt+B_1^\perp$. The convergence relation $vf\to f_0$ means that for every $\varphi\in B_1$, $(\varphi, v_n t)$ converges, the limit being (φ, f_0). For every $\varphi\in B_1$, $g\in G$ we have $(g_*\varphi, f_0) = (\varphi, g_*^* f_0) = (\varphi, gf_0) = (\varphi, f_0)$. Consequently, f_0 is a G_*-invariant bounded linear form on B_1.

Since closed balls are w^*-compact in B^*, there is a subnet v_1 of the net v_n for which $v_1 t$ converges in the w^*-topology of B^*. Let us denote the limit by t_0. The element $t_0\in B^*$ belongs to the w^*-closed convex hull of $\{gt : g\in G\}$ and

$$(*) \qquad (\varphi, t_0) = (\varphi, f_0) \quad \text{for} \quad \varphi\in B_1.$$

Since G acting on B^* satisfies condition (U), there is a net w_k in $\mathrm{co}G$ such that $w_k t_0 \to t^G$ in the w^*-topology of B^*. For $\varphi\in B_1$ we have: $(\varphi, t^G) = \lim\limits_k (\varphi, w_k t_0) = \lim\limits_k (w_{k*}\varphi, t_0) = \lim\limits_k (w_{k*}\varphi, f_0) = \lim\limits_k (\varphi, f_0) = (\varphi, f_0)$. (Here the next to the last equality holds because f_0 is G_*-invariant on B_1 and the equality before the next to the last equality holds because of $(*)$ and the G_*-invariance of B_1.) Consequently, $(\varphi, t^G)=(\varphi, f_0)$ for $\varphi\in B_1$, i.e., f_0 is the restriction of t^G to B_1. Since f_0 was an arbitrary element in the w^*-closed convex hull of $\{gf: g\in G\}$, the lemma is proved.

Proof of Theorem 1. Throughout this proof we assume that condition (U) is satisfied for G acting on B^*.

(1) First we assume that B is separable. This implies the separability of G. Indeed, let $\{\varphi_n\}_{n=1}^{\infty}$ be a dense sequence in the unit ball of B. Let T be the set of B-valued sequences bounded by $\|G\|=\sup\{\|g\|: g\in G\}$. If $\alpha, \beta\in T$, we define $\varrho(\alpha, \beta)$ by the equality

$$\varrho(\alpha, \beta) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|\alpha_n-\beta_n\|}{1+\|\alpha_n-\beta_n\|}.$$

Then ϱ is a metric on T. We have $\alpha^{(k)}\to\alpha$ in this metric if and only if $\alpha_n^{(k)}\to\alpha_n$ $(k\to\infty)$ for every $n=1, 2, \ldots$. Since B is separable, so is T. Let $g\in G$ and let us define an element α^g of T by the equalities $\alpha_n^g=g_*\varphi_n$ $(n=1, 2, \ldots)$. The mapping $g\to\alpha^g$ is a homeomorphism of G_* onto a subset of T if G_* is considered with the topology of pointwise strong convergence on B and T is considered with the topology induced by the metric ϱ. Since T has a countable dense subset, we may infer that so does G_* (because of the metrizability of T). Since taking adjoints of operators is a weak—weak* continuous operation, G contains a countable subset G_0 which is dense in G in the topology of $L_{w^*}(B^*)$.

Now let G_0 be a countable dense subset of G in the topology of $L_{w^*}(B^*)$. Then the G_0-invariant elements of B^* are the same as the G-invariant elements of B^* and for every $t\in B^*$, the w^*-closed convex hull of $\{gt: g\in G_0\}$ coincides with the w^*-closed convex hull of $\{gt: g\in G\}$. Consequently, if in addition, we choose G_0 to be a subsemigroup of G (for example, we replace G_0 by the subsemigroup generated by G_0), then G satisfies condition (U) if and only if G_0 does.

Now we can apply Lemma 1 to the separable Banach space B and countable semigroup G_0. We obtain that there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $\operatorname{co} G_0$ such that for every $t\in B^*$, $v_n t\to t^{G_0}=t^G$ in the w^*-topology as $n\to\infty$. Consequently, the mapping $t\to t^G$ is a bounded linear projection, to be denoted by P, acting in B^*. Since $(gt)^G=gt^G=t^G$ for $g\in G$, $t\in B^*$, we have: $gP=Pg=P$. This completes the proof of Theorem 1 in case B is separable.

(2) Suppose G is separable, i.e., there exists a countable subset G_0 of G which is dense in G in the topology of $L_{w^*}(B^*)$. We may assume that G_0 is a subsemigroup of G. The first part of the proof shows that it is sufficient to prove the theorem for G_0. However, we cannot apply Lemma 1 because B may not be separable. Consequently, we also have to appeal to Lemma 2. Let $g_1, g_2, \ldots$ be all different elements of G_0 and let $v_n=\frac{1}{n^n}\sum_{i_1,\ldots,i_n=1}^{n} g_1^{i_1}\ldots g_n^{i_n}$. We are going to prove that

for every $t \in B^*$, $v_n t \to t^G$ in the w^*-topology of B^* as $n \to \infty$. All the assertions of Theorem 1 will follow from this in the same way as in part (1) of this proof.

Let φ_0 be an arbitrary element of B. Let us denote by B_1 the Banach subspace spanned by the elements $g_{1*}\varphi_0, g_{2*}\varphi_0, \dots$. The subspace B_1 is G_{0*}-invariant. We may apply Lemma 2 and obtain that G_0, acting on $B_1^* = B^*/B_1^\perp$, also satisfies condition (U). Since B_1 is separable and G_0 is countable, Lemma 1 may be applied. We obtain that for every $f \in B_1^*$, $v_n f \to f^G$ in the w^*-topology of B_1^* as $n \to \infty$. In view of the identification $B_1^* = B^*/B_1^\perp$, this implies that for every $t \in B^*$, the sequence $\{(\varphi_0, v_n t)\}_{n=1}^\infty$ is convergent. (It may be seen directly that it converges to (φ_0, t^{G_0}); however, we choose another way of proving this, which we think is easier to follow.) Since φ_0 was an arbitrary element of B and $\|v_n t\| \leqq \|G\| \cdot \|t\|$; we obtain that for every $t \in B^*$, the sequence $\{v_n t\}_{n=1}^\infty$ w^*-converges to an element Pt of B^*. It is easy to see that Pt is G_0-invariant. Therefore, $Pt = t^{G_0}$ $(= t^G)$.

Proof of Theorem 2. The hypotheses of Theorem 1 are satisfied. Consequently, there is a sequence $\{v_n\}_{n=1}^\infty$ in $\operatorname{co} G$ such that for every $t \in B^*$, $v_n t \to t^G$ in the w^*-topology of B^* as $n \to \infty$. Now let $\varphi \in B$ be given. For every $t \in B^*$, we have $(v_{n*}\varphi - v_{m*}\varphi, t) = (\varphi, (v_n - v_m)t) \to 0$ as $n, m \to \infty$. Consequently, the sequence $\{v_{n*}\varphi\}_{n=1}^\infty$ is a weak Cauchy sequence in B. Since B is assumed to be weakly complete, there exists an element of B, to be denoted by $P_*\varphi$, for which $(v_{n*}\varphi, t) \to$ $\to (P_*\varphi, t)$ $(n \to \infty)$ for every $t \in B^*$. It is easy to see that P_* is a bounded linear operator in B. As $n \to \infty$, we have: $(\varphi, v_n t) = (v_{n*}\varphi, t) \to (P_*\varphi, t) = (\varphi, P_*^* t)$ for $\varphi \in B$, $t \in B^*$. Consequently, $v_n \to P_*^*$ in $L_{w*}(B^*)$ as $n \to \infty$. Since P_*^* is obviously w^*-continuous, we obtain the assertions of Theorem 2 (except the last assertion) if we put $P = P_*^*$.

The last assertion of Theorem 2 may be proved as follows. First we prove that for every $\varphi \in B$, the closed convex hull of $\{g_*\varphi : g_* \in G_*\}$ contains exactly one G_*-invariant element (namely, $P_*\varphi$). Here we may take either weak or strong closure, because the strong closure of a convex subset of a Banach space coincides with its weak closure (cf. [2], V. 3.13). Let $\varphi \in B$ and let $\hat{\varphi}$ be a G_*-invariant element in the closure of $(\operatorname{co} G_*)\varphi$. Then there exist $w_n \in \operatorname{co} G_*$ such that $w_n\varphi \to \hat{\varphi}$ strongly as $n \to \infty$. We have $P_* w_n \varphi \to P_*\hat{\varphi}$. Here $P_* w_n \varphi = P_*\varphi$ (because $P_* g_* = P_*$ for $g \in G$) and $P_*\hat{\varphi} = \hat{\varphi}$ (because P_* is a weak limit of elements of $\operatorname{co} G_*$ and $\hat{\varphi}$ is G_*-invariant). Therefore, $\hat{\varphi} = P_*\varphi$. On the other hand, if $g \in G$, then $g_* P_* \varphi =$ $= P_*\varphi$, i.e., $P_*\varphi$ is G_*-invariant. Therefore, $P_*\varphi$ is the unique G_*-invariant element in the closure of $(\operatorname{co} G_*)\varphi$. Since this is true for every $\varphi \in B$, the following holds according to [1]: For every $\varphi \in B$, every $\varepsilon > 0$ and every $v_{0*} \in \operatorname{co} G_*$ there exists $v_{1*} \in \operatorname{co} G_*$ such that $\|v_* v_{1*} v_{0*}\varphi - P_*\varphi\| < \varepsilon$. This inequality is equivalent to the following: $|([v_* v_{1*} v_{0*} - P_*]\varphi, t)| < \varepsilon$ for all $t \in B^*$ such that $\|t\| \leqq 1$ or

$|(\varphi, [vv_1v_0-P]t)|<\varepsilon$ for all $t\in B^*$ such that $\|t\|\leqq 1$. The last assertion of Theorem 2 follows immediately from this.

Proofs of Propositions 1,2 and the corollary of Proposition 2. In Proposition 1, $L^1(X, S, m)$ is weakly complete (cf. [2], IV. 8.6); consequently, the hypotheses of Theorem 2 are satisfied. In Proposition 2, the predual of M is weakly complete (cf. [4], Proposition 1); consequently, the hypotheses of Theorem 2 are satisfied. The corollary to Proposition 2 is simply a special case of Proposition 2.

3. Remarks and problems.

Remark 1. It follows from the author's other results (to be published) that even if G is not commutative and G and B are not separable and condition (U) is satisfied, then the mapping $t\to t^G$ $(t\in B^*)$ is a bounded linear projection contained in the closure, in $L_{w*}(B^*)$, of the convex hull of G.

Remark 2. It follows from the author's other results (to be published) that even if B is not weakly complete and condition (U) is satisfied, then a weaker version of the last assertion of Theorem 2 holds.

Problem 1. Is Theorem 2 true without the hypothesis that B is weakly complete?

Problem 2. Is Theorem 2 true without the hypothesis of separability of B or G? (In this case we can only expect P to be in the closure of co$G$, instead of the sequential closure of co$G$.)

Problem 3. Are the results of this paper true without the hypothesis that G is commutative?

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Random walk on a finite group

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his seventieth birthday July 29, 1983

1. Introduction. This paper has its origin in a study of random walks on regular polytopes. Regular polytopes in two dimensions (regular polygons) and in three dimensions (regular polyhedra or Platonic solids) have been known from ancient times. Four- and higher-dimensional polytopes were discovered by L. SCHLÄFLI [12] before 1853. For the theory of regular polytopes we refer to the books of H. S. M. COXETER [5], P. H. SCHOUTE [13] and D. M. Y. SOMMERVILLE [16].

Let $\mathfrak{P}$ be a regular polytope with σ vertices whose rectangular Cartesian coordinates are $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{\sigma-1}$. Denote by q the number of edges emanating from each vertex of $\mathfrak{P}$. We define two distance functions on the vertices of $\mathfrak{P}$. The distance $D(\mathbf{x}_r, \mathbf{x}_s)$ is the number of edges in a shortest path joining $\mathbf{x}_r$ and $\mathbf{x}_s$. The distance $\|\mathbf{x}_r - \mathbf{x}_s\|$ is the Euclidean distance between $\mathbf{x}_r$ and $\mathbf{x}_s$.

Let us suppose that in a series of random steps a traveler visits the vertices of $\mathfrak{P}$. The traveler starts at a given vertex and in each step, independently of the past journey, chooses as the destination one of the neighboring vertices with probability $1/q$. Denote by $\mathbf{v}_n$ $(n=1, 2, \ldots)$ the position of the traveler at the end of the n-th step, and by $\mathbf{v}_0$ the initial position. An important problem in the theory of probability is to determine $p(n)$, the probability that the traveler returns to the initial position at the end of the nth step. By symmetry we can choose any vertex, say $\mathbf{x}_0$, as the initial position and thus

$$p(n) = \mathbf{P}\{\mathbf{v}_n = \mathbf{x}_0 \,|\, \mathbf{v}_0 = \mathbf{x}_0\} \tag{1}$$

for all $n \geqq 0$.

Since $\{\mathbf{v}_n;\, n \geqq 0\}$ is a homogeneous Markov chain with state space $\{\mathbf{x}_0, \mathbf{x}_1, \ldots$ $\ldots, \mathbf{x}_{\sigma-1}\}$, the problem of finding $p(n)$ has a straightforward solution. We determine the incidence matrix of the graph of the polytope, form its nth power, and any diagonal element divided by q^n yields $p(n)$.

However, we can also find $p(n)$ in another way. Divide the σ vertices of $\mathfrak{P}$ into disjoint sections $S_0, S_1, \ldots, S_m$ such that S_0 contains only a single vertex, say $\mathbf{x}_0$, and define a sequence of random variables $\{\xi_n;\, n \geqq 0\}$ so that

$$(2) \qquad \xi_n = j \quad \text{if} \quad \mathbf{v}_n \in S_j.$$

In terms of ξ_n $(n \geqq 0)$ we can write that

$$(3) \qquad p(n) = \mathbf{P}\{\xi_n = 0 \mid \xi_0 = 0\}.$$

We would like to define $S_0, S_1, \ldots, S_m$ so that the sequence $\{\xi_n;\, n \geqq 0\}$ forms a Markov chain, and its state space $\{0, 1, \ldots, m\}$ contains fewer states than that of $\{\mathbf{v}_n;\, n \geqq 0\}$. This can be done in each case and consequently it is more advantageous to use (3) than (1) for the determination of $p(n)$.

If $\mathfrak{P}$ is any regular polytope other than the four-dimensional 24-cell, 600-cell and 120-cell and if

$$(4) \qquad S_j = \{\mathbf{x}_r\colon\, D(\mathbf{x}_r, \mathbf{x}_0) = j\}$$

for $j = 0, 1, \ldots, m$ where now $0, 1, \ldots, m$ are the possible values of $D(\mathbf{x}_r, \mathbf{x}_0)$ $(r = 0, \ldots, \sigma-1)$; then the sequence $\{\xi_n;\, n \geqq 0\}$ defined by (2) is a homogeneous Markov chain.

If $\mathfrak{P}$ is any regular polytope other than the four-dimensional 120-cell, and if

$$(5) \qquad S_j = \{\mathbf{x}_r\colon\, \|\mathbf{x}_r - \mathbf{x}_0\| = d_j\}$$

for $j = 0, 1, \ldots, m$ where now $d_0, d_1, \ldots, d_m$ are the possible values of $\|\mathbf{x}_r - \mathbf{x}_0\|$ $(r = 0, 1, \ldots, \sigma-1)$ arranged in increasing order of magnitude, then the sequence $\{\xi_n;\, n \geqq 0\}$ defined by (2) is a homogeneous Markov chain.

However, if $\mathfrak{P}$ is a four-dimensional 120-cell and if S_j is defined by (4) or (5), then $\{\xi_n;\, n \geqq 0\}$ is not a Markov chain. Since the distances $D(\mathbf{x}_r, \mathbf{x}_s)$ and $\|\mathbf{x}_r - \mathbf{x}_s\|$ remain invariant under rotations and reflections of $\mathfrak{P}$, we expect that in the general case the definition of the sections of $\mathfrak{P}$ should be based on the rotations and reflections of $\mathfrak{P}$, that is, on the symmetry group G of $\mathfrak{P}$.

If $g \in G$ and if g carries $\mathbf{x}_r$ into $\mathbf{x}_s$, then we write $\mathbf{x}_r g = \mathbf{x}_s$. Let H be the stabilizer of $\mathbf{x}_0$, that is,

$$(6) \qquad H = \{g\colon\, \mathbf{x}_0 g = \mathbf{x}_0 \quad \text{and} \quad g \in G\}.$$

For any $g \in G$ define the double coset

$$(7) \qquad C(g) = HgH = \{h_1 g h_2\colon\, h_1 \in H \quad \text{and} \quad h_2 \in H\}.$$

Any two double cosets $C(g_1)$ and $C(g_2)$ are either disjoint or identical. Denote by $C_0, C_1, \ldots, C_m$ all the disjoint double cosets of type (7). In particular, let $C_0 = H$. The double cosets $C_0, C_1, \ldots, C_m$ determine a partition of G.

Now define

$$(8) \qquad S_j = \{\mathbf{x}_r\colon\, \mathbf{x}_r = \mathbf{x}_0 g \quad \text{and} \quad g \in C_j\}$$

for $j=0, 1, \ldots, m$. We can check that if S_j is defined by (8), then for each regular polytope the sequence $\{\xi_n; n\geqq 0\}$ defined by (2) is a Markov chain and m is much smaller than σ.

Since $D(\mathbf{x}_r g, \mathbf{x}_s g)=D(\mathbf{x}_r, \mathbf{x}_s)$ and $\|\mathbf{x}_r g-\mathbf{x}_s g\|=\|\mathbf{x}_r-\mathbf{x}_s\|$ for all $g\in G$, every vertex $\mathbf{x}_r$ belonging to S_j, defined by (8), has the same distance $D(\mathbf{x}_r, \mathbf{x}_0)$ from $\mathbf{x}_0$, and the same distance $\|\mathbf{x}_r-\mathbf{x}_0\|$ from $\mathbf{x}_0$. Thus (8) reduces to (4) and (5) in the indicated particular cases.

We can generalize the random walk discussed above by assuming that the traveler in each step, independently of the past journey, chooses a vertex at random as the destination, and the transition probability $\mathbf{P}\{\mathbf{v}_n=\mathbf{x}_s \mid \mathbf{v}_{n-1}=\mathbf{x}_r\}$ depends either on the distance $D(\mathbf{x}_r, \mathbf{x}_s)$, or on the distance $\|\mathbf{x}_r-\mathbf{x}_s\|$, or more generally,

$$\mathbf{P}\{\mathbf{v}_n = \mathbf{x}_s \mid \mathbf{v}_{n-1} = \mathbf{x}_r\} = p_\nu \tag{9}$$

if $\mathbf{x}_s=\mathbf{x}_r g$ and $g\in C_\nu$ $(\nu=0, 1, \ldots, m)$. If S_j $(j=0, 1, \ldots, m)$ is defined by (8) and if ξ_n is defined by (2), then in this more general random walk too, the sequence $\{\xi_n; n\geqq 0\}$ forms a homogeneous Markov chain.

In this paper we shall consider a generalization of the random walk discussed above. Specifically, we shall be concerned with a random walk on a finite group and give a general method for the determination of the n-step transition probabilities.

2. Random walk on a group. The random walk described in the Introduction is a particular case of the general model defined in this section.

Let G be a finite group which is partitioned into nonempty disjoint subsets $C_0, C_1, \ldots, C_m$ such that C_0 contains e, the identity element of G. The number of elements in C_0 is denoted by $N(C_0)=\omega$. The index set of the partition is $I=\{0, 1, \ldots, m\}$.

Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ be a sequence of mutually independent random elements each belonging to G. A sequence of discrete random variables $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ is defined such that

$$\xi_n = j \quad \text{if} \quad \gamma_0\gamma_1 \ldots \gamma_n \in C_j. \tag{10}$$

The sequence $\{\xi_n; n\geqq 0\}$ defines a random walk on the group G, or more precisely, on the partition $\{C_0, C_1, \ldots, C_m\}$.

In what follows we assume that for $n=1, 2, \ldots$ the probability $\mathbf{P}\{\gamma_n=g\}$ does not depend on the particular g, it depends only on the class C_ν $(\nu\in I)$ which contains g. We write

$$\mathbf{P}\{\gamma_n = g\} = p_\nu/\omega \tag{11}$$

for $n\geqq 1$ and $g\in C_\nu$. However, the distribution $\mathbf{P}\{\gamma_0=g\}$, $g\in G$, may be chosen arbitrarily.

Our first aim is to find a condition which guarantees that the sequence $\{\xi_n;\ n\geqq 0\}$ is a Markov chain. It is easy to see that if the following Condition (i) is satisfied, then $\{\xi_n;\ n\geqq 0\}$ is a homogeneous Markov chain with state space $I=\{0, 1, \ldots, m\}$.

Condition (i). *For any $g_1\in C_i$ the number of ordered pairs (g_2, g_3) for which $g_2\in C_\nu$, $g_3\in C_j$ and $g_1 g_2 = g_3$ is independent of the particular choice of g_1, and is equal to $\omega a_{ij\nu}$ for $i, j, \nu\in I$.*

We define the matrices

$$\mathbf{A}_\nu = [a_{ij\nu}]_{i,j\in I} \tag{12}$$

for $\nu\in I$.

We use the notation $\mathbf{I}=[\delta_{ij}]_{i,j\in I}$ for an $(m+1)\times(m+1)$ unit matrix. Here and throughout this paper δ_{ij} denotes the Kronecker symbol, that is,

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i\neq j. \end{cases} \tag{13}$$

Let us define $\sigma_0, \sigma_1, \ldots, \sigma_m$ such that the number of elements in C_ν is $N(C_\nu)=$ $=\sigma_\nu\omega$ for $\nu\in I$. Obviously, $\sigma_0=1$, and the order of G is $N(G)=\sigma\omega$ where

$$\sigma = \sum_{\nu=0}^{m} \sigma_\nu. \tag{14}$$

It follows immediately from the definition of $a_{ij\nu}$ $(i, j, \nu\in I)$ that $a_{0j\nu}=\sigma_j\delta_{j\nu}$ for $j, \nu\in I$ and that

$$\sum_{j=0}^{m} a_{ij\nu} = \sigma_\nu \tag{15}$$

for any $i\in I$.

We shall frequently use the diagonal matrix

$$\mathbf{D} = [\delta_{ij}\sigma_j^{1/2}]_{i,j\in I} \tag{16}$$

where the square root is positive.

Since the sum of the probabilities (11) for all $g\in G$ is necessarily equal to 1, the parameters $p_0, p_1, \ldots, p_m$ should satisfy the requirement

$$\sum_{\nu=0}^{m} \sigma_\nu p_\nu = 1. \tag{17}$$

According to the above consideration if Condition (i) is satisfied, then $\{\xi_n;\ n\geqq 0\}$ is a homogeneous Markov chain with state space $I=\{0, 1, \ldots, m\}$. The transition probabilities

$$\mathbf{P}\{\xi_n = j \mid \xi_{n-1} = i\} = p_{ij} \quad (i, j\in I) \tag{18}$$

are given by

$$p_{ij} = \sum_{\nu=0}^{m} a_{ij\nu} p_\nu. \tag{19}$$

If we use the notation (12), then the transition probability matrix

$$\pi = [p_{ij}]_{i,j\in I} \tag{20}$$

can be expressed in the following way

$$\pi = \sum_{\nu=0}^{m} p_\nu \mathbf{A}_\nu. \tag{21}$$

The n-step transition probabilities

$$\mathbf{P}\{\xi_n = j \mid \xi_0 = i\} = p_{ij}^{(n)} \tag{22}$$

for $i, j \in I$ and $n \geqq 0$ can be determined as the elements of the matrix

$$\pi^n = [p_{ij}^{(n)}]_{i,j\in I}. \tag{23}$$

In particular, we have

$$p(n) = \mathbf{P}\{\xi_n = 0 \mid \xi_0 = 0\} = p_{00}^{(n)} \tag{24}$$

for $n \geqq 0$.

The main problem is to determine the n-th power of π defined by (21) and (12). Since the elements of π depend on the parameters $p_0, p_1, \ldots, p_m$, at first sight it seems we should determine π^n separately for each choice of the parameters $p_0, p_1, \ldots, p_m$. However, we shall demonstrate that if the partition $\{C_0, C_1, \ldots, C_m\}$ satisfies also Condition (ii) stated below, then we can derive a general formula for π^n which is valid for any choice of the parameters $p_0, p_1, \ldots, p_m$.

Condition (ii). *For any $\nu \in I$ there is a $\nu' \in I$ such that $g \in C_\nu$ implies that $g^{-1} \in C_{\nu'}$.*

Condition (ii) implies that if $\nu' = \nu$, then C_ν contains the inverse of each of its elements. If $\nu' \neq \nu$, then $C_{\nu'}$ consists of the inverses of the elements of C_ν. Obviously, $\sigma_{\nu'} = \sigma_\nu$ for all $\nu \in I$. The integers $0', 1', \ldots, m'$ form a permutation of $0, 1, \ldots, m$. Always, $0' = 0$. We define the corresponding permutation matrix $\mathbf{\Delta}$ by

$$\mathbf{\Delta} = [\delta_{ij'}]_{i,j\in I} \tag{25}$$

where δ_{ij} is defined by (13). We have $\mathbf{\Delta}' = \mathbf{\Delta}$ where $\mathbf{\Delta}'$ is the transpose of $\mathbf{\Delta}$, and $\mathbf{\Delta}^2 = \mathbf{I}$ where $\mathbf{I}$ is an $(m+1) \times (m+1)$ unit matrix.

Since $\sigma_{\nu'} = \sigma_\nu$ for all $\nu \in I$, we have also

$$\mathbf{D\Delta} = \mathbf{\Delta D} \tag{26}$$

where $\mathbf{D}$ is defined by (16).

Finally, we note that if C_ν denotes also the sum of the elements of G which belong to C_ν, then we can interpret $C_0, C_1, \ldots, C_m$ as elements of the group algebra (Frobenius algebra) of G. If $C_0, C_1, \ldots, C_m$ satisfy Condition (i), then the elements $C_0, C_1, \ldots, C_m$ form a basis of a subalgebra $\mathscr{A}$ of the group algebra of G. The elements of $\mathscr{A}$ are $\alpha_0 C_0 + \alpha_1 C_1 + \ldots + \alpha_m C_m$ where $\alpha_0, \alpha_1, \ldots, \alpha_m$ are complex numbers. If in addition $C_0, C_1, \ldots, C_m$ satisfy Condition (ii), then $\mathscr{A}$ reduces to a so-called Schur algebra. See D. E. LITTLEWOOD [10, pp. 242, 257], [9, pp. 22, 43], I. SCHUR [15], H. WIELANDT [21], [22], O. TAMASCHKE [18], [19], [20], F. ROESLER [11] and M. BRENDER [2].

3. Examples. Here are a few examples for partitions of finite groups satisfying Conditions (i) and (ii).

Example 1. Let H be a subgroup of a finite group G. For each $g \in G$ let us form the class

$$C(g) = \{hgh^{-1}:\ h \in H\}. \tag{27}$$

Any two classes $C(g_1)$ and $C(g_2)$ are either disjoint or identical. Denote by $C_0, C_1, \ldots, C_m$ all the disjoint classes of type (27). Then the partition $\{C_0, C_1, \ldots, C_m\}$ satisfies Conditions (i) and (ii), and $\{\xi_n;\ n \geqq 0\}$ defined by (10) is a homogeneous Markov chain. In this case $C_0 = \{e\}$, $\omega = 1$ and $a_{ij\nu}$ $(i, j, \nu \in I)$ are nonnegative integers.

If, in particular, $H = G$, then $C_0, C_1, \ldots, C_m$ are the conjugacy classes of G, and the problem of finding π^n leads in a natural way to the definition of group characters. (See F. G. FROBENIUS [7], W. BURNSIDE [3], [4] and I. SCHUR [14].)

Example 2. Let H be again a subgroup of a finite group G. For each $g \in G$ let us form the double coset

$$C(g) = HgH = \{h_1 g h_2:\ h_1 \in H \quad \text{and} \quad h_2 \in H\}. \tag{28}$$

Any two classes $C(g_1)$ and $C(g_2)$ are either disjoint or identical. Denote by $C_0, C_1, \ldots, C_m$ all the disjoint classes of type (28). Then the partition $\{C_0, C_1, \ldots, C_m\}$ satisfies Conditions (i) and (ii), and $\{\xi_n;\ n \geqq 0\}$ defined by (10) is a homogeneous Markov chain. In this case $C_0 = H$, ω is the order of H, and $a_{ij\nu}$ $(i, j, \nu \in I)$ are nonnegative integers.

If G is the symmetry group of a regular polytope $\mathfrak{P}$, and if H is the stabilizer of a given vertex of $\mathfrak{P}$, then $\{\xi_n;\ n \geqq 0\}$ defines a random walk on the vertices of $\mathfrak{P}$. (See J. S. FRAME [6] and L. TAKÁCS [17].)

Example 3. Let G be the automorphism group of a distance-transitive finite connected graph. A graph is distance-transitive if for any four vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$

satisfying $D(\mathbf{x}_1, \mathbf{x}_2)=D(\mathbf{y}_1, \mathbf{y}_2)$ there is an automorphism $g\in G$ such that $\mathbf{y}_1=\mathbf{x}_1 g$ and $\mathbf{y}_2=\mathbf{x}_2 g$. Let

$$C_j = \{g\colon D(\mathbf{x}_0 g, \mathbf{x}_0) = j\} \tag{29}$$

for $j=0, 1, \ldots, m$ where m is the diamater of the graph and $\mathbf{x}_0$ is a fixed vertex. Then the partition $C_0, C_1, \ldots, C_m$ satisfies Conditions (i) and (ii), and $\{\xi_n; n\geqq 0\}$ defined by (10) is a Markov chain. (See D. G. HIGMAN [8] and N. BIGGS [1].)

4. The matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$. If a partition $\{C_0, C_1, \ldots, C_m\}$ of a finite group G satisfies Conditions (i) and (ii), the elements of the matrices $\mathbf{A}_\nu$ $(\nu\in I)$ defined by (12) can be determined by the direct use of Condition (i). However, the elements of the matrices $\mathbf{A}_\nu$ $(\nu\in I)$ also satisfy remarkable relations, and our next aim is to prove these.

In what follows we assume that $\{C_0, C_1, \ldots, C_m\}$ is a partition of a finite group, that Conditions (i) and (ii) are satisfied, and that $\mathbf{A}_\nu$ $(\nu\in I)$ is defined by (12).

Theorem 1. *We have*

$$\sigma_i a_{ij\nu} = \sigma_j a_{ji\nu'} \tag{30}$$

and

$$a_{ij\nu} = a_{i'\nu j} \tag{31}$$

for all $i, j, \nu\in I$.

Proof. By Condition (i) the number of triplets (g_1, g_2, g_3) satisfying the requirements $g_1\in C_i$, $g_2\in C_\nu$, $g_3\in C_j$ and $g_1 g_2=g_3$ is $\omega^2\sigma_i a_{ij\nu}$. Since $g_1 g_2=g_3$ if and only if $g_3 g_2^{-1}=g_1$ or $g_1^{-1}g_3=g_2$, and since now by Condition (ii) $g_2^{-1}\in C_{\nu'}$ and $g_1^{-1}\in C_{i'}$, therefore we have

$$\sigma_i a_{ij\nu} = \sigma_j a_{ji\nu'} = \sigma_i a_{i'\nu j}. \tag{32}$$

This proves (30) and (31).

Equations (30) and (31) can conveniently be expressed in matrix notation. By (30) we have

$$\mathbf{D}^2\mathbf{A}_\nu = \mathbf{A}'_{\nu'}\mathbf{D}^2 \tag{33}$$

for $\nu\in I$ where the prime means transposition and $\mathbf{D}$ is defined by (16). By (33)

$$\mathbf{D}\mathbf{A}_\nu\mathbf{D}^{-1} = \mathbf{D}^{-1}\mathbf{A}'_{\nu'}\mathbf{D} = (\mathbf{D}\mathbf{A}_{\nu'}\mathbf{D}^{-1})'. \tag{34}$$

Accordingly, if $\nu'=\nu$, then $\mathbf{D}\mathbf{A}_\nu\mathbf{D}^{-1}$ is a real symmetric matrix.

By (31) we obtain that

$$[a_{ij\nu}]_{i,\nu\in I} = \mathbf{\Delta}[a_{i\nu j}]_{i,\nu\in I} = \mathbf{\Delta}\mathbf{A}_j \tag{35}$$

for $j\in I$ where $\mathbf{\Delta}$ is the permutation matrix defined by (25).

If we interpret $C_0, C_1, \ldots, C_m$ as elements of the group algebra (Frobenius algebra) of G, and C_ν is the sum of all those elements of G which belong to C_ν, then by Condition (i) and (30) we can write that

$$C_i C_\nu = \omega \sum_{j=0}^{m} a_{ji\nu'} C_j \tag{36}$$

for any $i \in I$ and $\nu \in I$.

If we arrange the products $C_i C_\nu$ $(i=0, 1, \ldots, m)$ in the form of a row vector, then by (36) we get

$$[C_0 C_\nu, C_1 C_\nu, \ldots, C_m C_\nu] = \omega [C_0, C_1, \ldots, C_m] \mathbf{A}_\nu, \tag{37}$$

for $\nu \in I$, and by (31) and (36)

$$[C_i C_0, C_i C_1, \ldots, C_i C_m] = \omega [C_0, C_1, \ldots, C_m] \mathbf{\Delta A}_i \mathbf{\Delta} \tag{38}$$

for $i \in I$. By (35) and (36) it follows that

$$[C_i C_\nu]_{i,\nu \in I} = \sum_{j=0}^{m} \frac{\omega}{\sigma_j} (\mathbf{D}^2 \mathbf{\Delta A}_j) C_j. \tag{39}$$

Theorem 2. *We have*

$$\mathbf{A}_i \mathbf{A}_j = \sum_{\nu=0}^{m} a_{\nu i j'} \mathbf{A}_\nu \tag{40}$$

for all $i, j \in I$.

Proof. By (36) and (37)

$$\begin{aligned}[C_0 C_j C_i, C_1 C_j C_i, \ldots, C_m C_j C_i] &= \omega \sum_{\nu=0}^{m} a_{\nu j i} [C_0 C_\nu, C_1 C_\nu, \ldots, C_m C_\nu] = \\ &= \omega^2 [C_0, C_1, \ldots, C_m] \sum_{\nu=0}^{m} a_{\nu j i'} \mathbf{A}_\nu.\end{aligned} \tag{41}$$

On the other hand by the repeated applications of (37) we get

$$\begin{aligned}[C_0 C_j C_i, C_1 C_j C_i, \ldots, C_m C_j C_i] &= [C_0 C_j, C_1 C_j, \ldots, C_m C_j] C_i = \\ &= \omega [C_0, C_1, \ldots, C_m] \mathbf{A}_{j'} C_i = \omega [C_0 C_i, C_1 C_i, \ldots, C_m C_i] \mathbf{A}_{j'} = \\ &= \omega^2 [C_0, C_1, \ldots, C_m] \mathbf{A}_{i'} \mathbf{A}_{j'}.\end{aligned} \tag{42}$$

A comparison of (41) and (42) shows that

$$\mathbf{A}_{i'} \mathbf{A}_{j'} = \sum_{\nu=0}^{m} a_{\nu j i'} \mathbf{A}_{\nu'} \tag{43}$$

for all $i \in I$ and $j \in I$. If in (43) we replace i, j, ν by $i'; j', \nu'$ respectively and take into consideration that $a_{\nu' j' i} = a_{\nu i j'}$, then we get (40).

Theorem 3. *The matrices* $\mathbf{\Delta A}_j\mathbf{\Delta}$ *and* $\mathbf{A}_\nu$ *commute, that is*

(44) $$\mathbf{\Delta A}_j\mathbf{\Delta A}_\nu = \mathbf{A}_\nu\mathbf{\Delta A}_j\mathbf{\Delta}$$

for all $j\in I$ *and* $\nu\in I$.

Proof. If $\nu\in I$ is fixed, then by (37)

(45) $$[C_iC_\nu C_k]_{i,k\in I} = [(C_iC_\nu)C_k]_{i,k\in I} = \omega\mathbf{A}'_{\nu'}[C_iC_k]_{i,k\in I}$$

and by (38)

(46) $$[C_iC_\nu C_k]_{i,k\in I} = [C_i(C_\nu C_k)]_{i,k\in I} = \omega[C_iC_k]_{i,k\in I}\mathbf{\Delta A}_\nu\mathbf{\Delta}.$$

If we put (39) into (45) and (46), and compare the coefficients of C_j in the two expressions, then we obtain that

(47) $$\mathbf{A}'_{\nu'}\mathbf{D}^2\mathbf{\Delta A}_j = \mathbf{D}^2\mathbf{\Delta A}_j\mathbf{\Delta A}_\nu\mathbf{\Delta}$$

or

(48) $$\mathbf{D}^2\mathbf{A}_\nu\mathbf{\Delta A}_j = \mathbf{D}^2\mathbf{\Delta A}_j\mathbf{\Delta A}_\nu\mathbf{\Delta}$$

which proves (44).

Theorem 4. *If*

(49) $$C_jC_\nu = C_\nu C_j$$

for all $j\in I$ *and* $\nu\in I$, *or equivalently, if*

(50) $$a_{ij\nu'} = a_{i\nu j'}$$

holds for all $i, \nu, j\in I$, *then the matrices* $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_m$ *commute in pairs.*

Proof. First, we observe that (36) implies that (49) holds if and only if (50) holds. By (33) we can express (50) in the following equivalent form

(51) $$\mathbf{A}_{\nu'} = \mathbf{\Delta A}_\nu\mathbf{\Delta}$$

for all $\nu\in I$ where $\mathbf{\Delta}$ is defined by (25). If we make use of (51), then (44) reduces to the equation

(52) $$\mathbf{A}_{j'}\mathbf{A}_\nu = \mathbf{A}_\nu\mathbf{A}_{j'}$$

which is valid for all $j\in I$ and $\nu\in I$. This proves that

(53) $$\mathbf{A}_j\mathbf{A}_\nu = \mathbf{A}_\nu\mathbf{A}_j$$

for all $j\in I$ and $\nu\in I$.

The converse of Theorem 4 is obvious. If (53) holds for all $j\in I$ and $\nu\in I$, then by (40), (50) necessarily holds and this implies (49).

We note that if $\nu'=\nu$ for all $\nu\in I$, then (50) is satisfied because by (31) $a_{ij\nu}=a_{i\nu j}$.

If we consider a Schur algebra with basis $C_0, C_1, \dots, C_m$, then by the above results we can make several conclusions. If we put $i=0$ or $\nu=0$ in (36), then we obtain that C_0/ω is the unit element of the Schur algebra. The matrix representations $\mathbf{T}_1$ and $\mathbf{T}_2$ defined by

$$\mathbf{T}_1(C_\nu) = \omega[a_{i\nu j'}]_{i,j\in I} = \omega\mathbf{\Delta}\mathbf{A}_\nu\mathbf{\Delta} \tag{54}$$

and

$$\mathbf{T}_2(C_\nu) = \omega[a_{ji\nu'}]_{i,j\in I} = \omega\mathbf{A}'_{\nu'} = \omega\mathbf{D}^2\mathbf{A}_\nu\mathbf{D}^{-2} \tag{55}$$

are the right regular matrix representation, and the left regular matrix representation respectively. Accordingly, the matrix representation defined by

$$\mathbf{T}(C_\nu) = \omega\mathbf{A}_\nu \tag{56}$$

for $\nu\in I$ is equivalent to the regular matrix representation of the Schur algebra.

The Schur algebra is commutative if and only if (50) is satisfied.

5. The determination of π^n. We suppose again that $\{C_0, C_1, \ldots, C_m\}$ is a partition of a finite group G and that this partition satisfies Conditions (i) and (ii). Our aim is to determine the nth power of the matrix

$$\pi = \sum_{\nu=0}^{m} p_\nu\mathbf{A}_\nu, \tag{57}$$

where the matrices $\mathbf{A}_\nu$ $(\nu\in I)$ are defined by (12) and $p_0, p_1, \ldots, p_m$ are arbitrary real or complex numbers. If $p_0, p_1, \ldots, p_m$ are nonnegative real numbers satisfying (17), then (57) reduces to the transition probability matrix of the Markov chain $\{\xi_n;\ n\geqq 0\}$ defined by (10).

We observe that if p_ν and $p_{\nu'}$ are complex conjugate numbers for every $\nu\in I$ then the matrix $\mathbf{D}\pi\mathbf{D}^{-1}$ is a Hermitian matrix, and consequently the eigenvalues of π are real numbers. This follows from the identity

(58)
$$p_\nu\mathbf{A}_\nu + p_{\nu'}\mathbf{A}_{\nu'} = \frac{(p_\nu+p_{\nu'})-i(p_\nu-p_{\nu'})}{2}(\mathbf{A}_\nu+\mathbf{A}_{\nu'}) + \frac{i(p_\nu-p_{\nu'})}{2}(1-i)(\mathbf{A}_\nu+i\mathbf{A}_{\nu'})$$

and from (34) which implies that

$$\mathbf{D}(\mathbf{A}_\nu+\mathbf{A}_{\nu'})\mathbf{D}^{-1} \tag{59}$$

is a real symmetric matrix for all $\nu\in I$, and

$$(1-i)\mathbf{D}(\mathbf{A}_\nu+i\mathbf{A}_{\nu'})\mathbf{D}^{-1} \tag{60}$$

is a Hermitian matrix for all $\nu\in I$.

If $p_0, p_1, \ldots, p_m$ are real numbers satisfying the requirements $p_{\nu'}=p_\nu$ for all $\nu\in I$, then $\mathbf{D}\pi\mathbf{D}^{-1}$ is a real symmetric matrix.

We shall use the following method for the determination of π^n. For all $\nu\in I$ let us define

$$\mathbf{\Gamma}_\nu = \mathbf{X}^{-1}\mathbf{D}\mathbf{A}_\nu\mathbf{D}^{-1}\mathbf{X} \tag{61}$$

where $\mathbf{D}$ is given by (16) and for the time being $\mathbf{X}$ is any nonsingular $(m+1)\times\times(m+1)$ matrix.

Theorem 5. *The matrices* $\mathbf{\Gamma}_\nu$ $(\nu\in I)$ *defined by* (61) *satisfy the following equations*

$$\mathbf{\Gamma}_i\mathbf{\Gamma}_j = \sum_{\nu=0}^{m} a_{\nu i j'}\mathbf{\Gamma}_\nu \tag{62}$$

for all $i\in I$ *and* $j\in I$. *The coefficients* $a_{\nu ij'}$ *are defined by* (12).

Proof. If we multiply (40) by $\mathbf{X}^{-1}\mathbf{D}$ from the left and by $\mathbf{D}^{-1}\mathbf{X}$ from the right, then we get (62).

Form (34) it follows immediately that

$$\mathbf{\Gamma}_{\nu'} = \mathbf{\Gamma}'_\nu \tag{63}$$

for all $\nu\in I$. If, in particular, $\nu=\nu'$, then $\mathbf{\Gamma}_\nu$ is a symmetric matrix.

Usually, if we calculate $\mathbf{\Gamma}_\nu$ for a few values of ν by (61), then we can easily determine $\mathbf{\Gamma}_\nu$ for all $\nu\in I$ by (62). If every $\mathbf{\Gamma}_\nu$ $(\nu\in I)$ is known, then by (61)

$$\mathbf{A}_\nu = \mathbf{D}^{-1}\mathbf{X}\mathbf{\Gamma}_\nu\mathbf{X}^{-1}\mathbf{D} \tag{64}$$

for $\nu\in I$, and (57) can be expressed in the following form

$$\pi = \mathbf{D}^{-1}\mathbf{X}\left(\sum_{\nu=0}^{m} p_\nu\mathbf{\Gamma}_\nu\right)\mathbf{X}^{-1}\mathbf{D}. \tag{65}$$

Thus

$$\pi^n = \mathbf{D}^{-1}\mathbf{X}\left(\sum_{\nu=0}^{m} p_\nu\mathbf{\Gamma}_\nu\right)^n\mathbf{X}^{-1}\mathbf{D} \tag{66}$$

for all $n\geqq 0$.

We shall use (66) for finding π^n for all $n\geqq 0$. In what follows we shall show that we can chose the matrix $\mathbf{X}$ in such a way that the matrices $\mathbf{\Gamma}_\nu$ $(\nu\in I)$ are all block-diagonal matrices of the same type. The determination of π^n by (66) is particularly simple if each $\mathbf{\Gamma}_\nu$ $(\nu\in I)$ is a diagonal matrix.

To find a suitable $\mathbf{X}$ let us consider the matrix

$$\mathbf{M} = \sum_{\nu=0}^{m} c_\nu\mathbf{A}_\nu \tag{67}$$

where c_ν $(\nu\in I)$ are real numbers satisfying the requirements $c_{\nu'}=c_\nu$ for all $\nu\in I$. Since (59) is a real symmetric matrix for $\nu\in I$, therefore $\mathbf{D\Delta M\Delta D}^{-1}$ is also a real symmetric matrix. Consequently, there exists a real orthogonal matrix $\mathbf{X}$ such that

$$\mathbf{D\Delta M\Delta D}^{-1} = \mathbf{XLX}' \tag{68}$$

where $\mathbf{L}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{M}$. Let us suppose that the columns of $\mathbf{X}$ are arranged in such a way that the diagonal elements of $\mathbf{L}$ form a nonincreasing sequence. If $\mathbf{M}$ has r distinct eigenvalues with multiplicities $m_1, m_2, \ldots, m_r$ respectively, then the diagonal elements of $\mathbf{L}$ form r blocks containing $m_1, m_2, \ldots, m_r$ identical numbers.

Theorem 6. *If $\mathbf{X}$ is an orthogonal matrix satisfying* (68) *and if $\mathbf{L}$ is a diagonal matrix whose diagonal elements form r blocks containing $m_1, m_2, \ldots, m_r$ identical elements, then each $\mathbf{\Gamma}_\nu$ $(\nu\in I)$, defined by* (61), *is a block-diagonal matrix containing r blocks such that the i-th block is an $m_i \times m_i$ matrix $(i=1, 2, \ldots, r)$.*

Proof. By Theorem 3 the matrices $\mathbf{D\Delta M\Delta D}^{-1}$ and $\mathbf{DA}_\nu\mathbf{D}^{-1}$ commute. Thus by (64) and (68) we have

$$\mathbf{L\Gamma}_\nu = \mathbf{\Gamma}_\nu\mathbf{L} \tag{69}$$

for all $\nu\in I$. Let us form the (i, k)-entry of both sides of (69). Since $\mathbf{L}$ is a diagonal matrix, we can conclude from (69) that the (i, k)-entry of $\mathbf{\Gamma}_\nu$ is necessarily 0 if the i-th and k-th diagonal elements of $\mathbf{L}$ are distinct. Consequently, each $\mathbf{\Gamma}_\nu$ is a block-diagonal matrix of the type specified in Theorem 6. This completes the proof of Theorem 6.

If the matrix $\mathbf{M}$ defined by (67) has only simple eigenvalues then by Theorem 6 the matrices $\mathbf{\Gamma}_\nu$ $(\nu\in I)$ defined by (61) are diagonal matrices. We can prove that if for some choice of the real numbers $c_0, c_1, \ldots, c_m$ the matrix $\mathbf{M}$ defined by (67) has only simple real eigenvalues, then $\mathbf{\Delta}=\mathbf{I}$; that is, $\nu'=\nu$ for all $\nu\in I$. In this case $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute in pairs. In any other case the matrix $\mathbf{M}$ has multiple eigenvalues and our aim is to choose c_ν $(\nu\in I)$ in such a way that the sum of the squares of the multiplicities of the eigenvalues of $\mathbf{M}$ be as small as possible, that is, in Theorem 6 the sum $m_1^2+m_2^2+\ldots+m_r^2$ be as small as possible. Usually we attain the minimum if $\mathbf{M}=\mathbf{A}_\nu$ for some $\nu=\nu'$.

From Theorem 6 we can conclude that if $\mathbf{T}$ is a matrix representation of the algebra $\mathscr{A}$ with basis $C_0, C_1, \ldots, C_m$ and if $\mathbf{T}(C_\nu)=\omega\mathbf{\Gamma}_\nu$ for $\nu\in I$, then $\mathbf{T}$ is equivalent to the regular matrix representation of $\mathscr{A}$ and $\mathbf{T}$ can be expressed as the direct sum of r matrix representations of $\mathscr{A}$. Actually, by a footnote of H. Wielandt [21] (p. 386) the algebra $\mathscr{A}$ is semi-simple and consequently it is completely reducible, that is, $\mathscr{A}$ is the direct sum of simple matrix algebras over the field of complex numbers.

Examples for the application of the method developed here will be given in another paper. Now we would like to mention only briefly the case of a random walk on a four-dimensional 120-cell. We shall use the same notation as in the Introduction. A 120-cell has $\sigma=600$ vertices and from each vertex $q=4$ edges emanate. Let us consider the random walks $\{\mathbf{v}_n; n\geqq 0\}$ and $\{\xi_n; n\geqq 0\}$ defined in the Introduction. Now $\{\mathbf{v}_n; n\geqq 0\}$ is a Markov chain and the state space contains 600 states. If we define the sections of the 120-cell by (4) then $m=15$, but $\{\xi_n; n\geqq 0\}$

is not a Markov chain. If we define the sections by (5), then $m=30$, and $\{\xi_n;\ n\geqq 0\}$ is still not a Markov chain. However, if we define the sections by (8), then $\{\xi_n;\ n\geqq 0\}$ becomes a Markov chain and $m=44$. Now the transition probability matrix π is given by (21) and $\mathbf{\Gamma}_\nu$ is defined by (61). By an appropriate choice of $\mathbf{X}$ we can achieve that each $\mathbf{\Gamma}_\nu$ becomes a block-diagonal matrix containing 15 one by one, 6 two by two, and 6 three by three matrices. If $\mathbf{\Gamma}_\nu\ (\nu\in I)$ and $\mathbf{X}$ are known numerically, then we can determine the n-step transition probabilities explicitly by (66). The numerical data are used only to determine certain integers. First, we can determine explicitly the eigenvalues of $\mathbf{\Gamma}_\nu\ (\nu\in I)$ by solving quadratic and cubic equations with integer coefficients. The coefficients of these equations are determined by the traces of the first two or three powers of the block-matrices in each $\mathbf{\Gamma}_\nu$ and all these traces are integers. The eigenvalues of π can also be obtained by solving quadratic and cubic equations whose coefficients are quadratic and cubic forms of $p_0, p_1, \ldots, p_{44}$ and depend only on the traces of the first two or three powers of the block-matrices in $\mathbf{\Gamma}_\nu\ (\nu\in I)$ and on $a_{ij\nu}\ (i, j, \nu\in I)$. The numerical values of the elements of the matrix $\mathbf{X}$ are used only to determine certain integers which are the coefficients of the n-th powers of the eigenvalues of π in the expression for $600p_{ij}^{(n)}$. Since the numerical calculations are used only to determine certain integers, no high precision is needed. The expressions for the n-step probabilities are straightforward, but lengthy because of the large number of parameters $p_0, p_1, \ldots, p_{44}$.

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Über einen Zusammenhang zwischen der Grössenordnung der Partialsummen der Fourierreihe und der Integrabilitätseigenschaft der Funktionen

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Herrn Professor Béla Szőkefalvi-Nagy zum 70. Geburtstag gewidmet

Es sei Φ die Klasse der Funktionen φ mit $\varphi(0)=1$, die im Intervall $[0,\infty)$ zweimal differentierbar, monoton wachsend und von unten konkav sind, und für die

$$\varphi(x) \geqq c_1 \log\log x \quad (x\in[0,\infty)), \quad \varphi(x)=o(\log x) \quad (x\to\infty)$$

erfüllt sind. ($c_1, c_2, \ldots$ bezeichnen positive Konstanten.) Für eine $\varphi\in\Phi$ sei $L\varphi(L)$ die Klasse der meßbaren Funktionen f mit $\int_0^{2\pi} |f(x)|\varphi(|f(x)|)dx<\infty$. Die n-te Partialsumme der Fourierreihe der Funktion $f\in L(0,2\pi)$ bezeichnen wir mit $s_n(f;x)$.

In dieser Note werden wir den folgenden Satz beweisen.

Satz. *Es sei $\varphi\in\Phi$. Gibt es eine Funktion $f\in L(0,2\pi)$ derart, daß*

$$\varlimsup_{n\to\infty} \frac{1}{\varphi(n)} |s_n(f;\ x)| > 0 \tag{1}$$

in einer Menge vom positiven Maß gilt, dann für jede positive Zahl $\varepsilon(<1)$ gibt es eine meßbare Funktion F mit

$$\int_0^{2\pi} |F(x)|\,\varphi^{1-\varepsilon}(|F(x)|)\,dx<\infty,$$

daß die Fourierreihe von F fast überall divergiert.

Betreffs des Satzes bemerken wir folgendes. Nach einem bekannten Satz (s. z. B. [6], Vol. I., S. 66.), im Falle $f\in L(0,2\pi)$ gilt $s_n(f;x)=o(\log n)$ fast überall. Weiterhin, nach einem bekannten Satz (YUNG-MING CHEN [4]), für jede positive,

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monoton wachsende Folge $\varphi(n)$ mit $\varphi(n)=o(\log\log n)$ gibt es eine Funktion $f\in L(0, 2\pi)$, für die (1) fast überall gilt.

Andererseits hat P. SJÖLIN [2] bewiesen, daß im Falle $f\in L(\log^+ L)(\log^+\log^+ L)$ die Fourierreihe von f fast überall konvergiert. (α^+ bezeichnet den positiven Teil von α.) Weiterhin, nach einem bekannten Satz (s. z. B. YUNG-MING CHEN [5]) für jede positive Zahl $\varepsilon(<1)$ gibt es eine Funktion $f\in L(\log^+\log^+ L)^{1-\varepsilon}$ derart, daß die Fourierreihe von f fast überall divergiert.

Aus dem Satz folgt die folgende Behauptung.

Gibt es eine Funktion $\varphi\in\Phi$ mit der Eigenschaft, daß aus $f\in L\varphi(L)$ die Konvergenz fast überall der Fourierreihe von f folgt, dann gilt $\lim\limits_{n\to\infty}\frac{1}{\varphi^{1+\varepsilon}(n)}s_n(f;\ x)=0$ *für jede* $\varepsilon>0$ *und für jede* $f\in L(0, 2\pi)$ *fast überall.*

Beweis des Satzes. Es sei Z_k die Menge der positiven ganzen Zahlen n, für die $2^k\leqq\varphi(n)<2^{k+1}$ $(k=0, 1, \ldots)$ gilt. Auf Grund der Definition von Φ gibt es eine nichtnegative ganze Zahl k_0, daß im Falle $k\geqq k_0$ $Z_k\neq\emptyset$ ist; die Elemente von Z_k bezeichnen wir in natürlicher Anordnung mit $n_k, n_k+1, \ldots, n_{k+1}-1$ $(k=k_0, k_0+1, \ldots)$. Weiterhin, auf Grund der Definition von Φ gibt es eine nichtnegative ganze Zahl k_1 $(\geqq k_0)$, daß $n_{k+1}/n_k\geqq q>1$ $(k=k_1, k_1+1, \ldots)$ auch besteht.

Auf Grund der Voraussetzung des Satzes gibt es eine positive Zahl m und eine meßbare Menge $E(\subseteqq(0, 2\pi))$ mit $\operatorname{mes} E>0$ derart, daß

$$\overline{\lim_{n\to\infty}}\,\frac{1}{\varphi(n)}|s_n(f;\ x)|>m\quad(x\in E)\tag{2}$$

gilt. Es sei

$$E_k=\left\{x\in(0, 2\pi):\ \max_{n_k\leqq n<n_{k+1}}\frac{1}{\varphi(n)}|s_n(f;\ x)|>\frac{m}{2}\right\}\quad(k=k_1, k_1+1, \ldots).$$

Dann ist

$$\sum_{k=k_1}^{\infty}\operatorname{mes} E_k=\infty.\tag{3}$$

Im entgegengesetzten Falle existiert nähmlich für fast jeden x eine von x abhängige positive ganze Zahl $m(x)$ mit

$$\frac{1}{\varphi(n)}|s_n(f;\ x)|\leqq\frac{m}{2}\quad(n\geqq n(x)),$$

woraus

$$\overline{\lim_{n\to\infty}}\,\frac{1}{\varphi(n)}|s_n(f;\ x)|\leqq\frac{m}{2}$$

fast überall folgt, was (2) widerspricht. Weiterhin aus (3) folgt, daß es eine nicht-negative ganze Zahl i^* $(0 \leqq i^* \leqq 2)$ derart existiert, daß

(4) $$\sum_{i=i^{**}}^{\infty} \operatorname{mes} E_{3i+i^*} = \infty$$

gilt, wobei i^{**} die kleinste positive ganze Zahl bezeichnet, für die $3i^{**}+i^* \geqq k_1$ besteht.

Wir setzen für $i = i^{**}, i^{**}+1, \dots$

$$T_i(x) = \frac{1}{\varphi(n_{3i+i^*})}\left(V_{n_{3i+i^*+1}, n_{3i+i^*+2}}(f;\ x) - V_{n_{3i+i^*-1}, n_{3i+i^*}}(f;\ x)\right),$$

wobei im Falle $n < m$

$$V_{n,m}(f;\ x) = s_n(f;\ x) + \sum_{k=n+1}^{m}\left(1 - \frac{k-n}{m-n}\right)(a_k(f)\cos kx + b_k(f)\sin kx)$$

die verallgemeinerte de la Vallée Poussinsche Mittel der Fourierreihe

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx)$$

bezeichnet. Die n-te $(C, 1)$-Mittel dieser Fourierreihe bezeichnen wir mit $\sigma_n(f; x)$. Da im Falle $n < m$ $V_{n,m}(f; x) = \dfrac{m-1}{m-n}\sigma_{m-1}(f;\ x) - \dfrac{n-1}{m-n}\sigma_{n-1}(f;\ x)$ ist, und im Falle $f \in L(0, 2\pi)$ $a_n(f), b_n(f) = o(1)$ $(n \to \infty)$, weiterhin

$$\int_0^{2\pi} |\sigma_n(f;\ x)|\, dx \leqq c_2 \int_0^{2\pi} |f(x)|\, dx \quad (n = 1, 2, \dots)$$

bestehen (s. z. B. [6], Vol. I., S. 52., bzw. S. 137.), auf Grund der Definition von n_k und $T_i(x)$ ergibt sich:

(5) $$\int_0^{2\pi} |T_i(x)|(\varphi(|T_i(x)|))^{1-\varepsilon}\, dx \leqq$$

$$\leqq \int_0^{2\pi} |V_{n_{3i+i^*+1}, n_{3i+i^*+2}}(f;\ x) - V_{n_{3i+i^*-1}, n_{3i+i^*}}(f;\ x)| \frac{\varphi^{1-\varepsilon}(c_3 n_{3i+i^*+2})}{\varphi(n_{3i+i^*})}\, dx \leqq$$

$$\leqq \frac{c_4}{(2^{3i+i^*})^{\varepsilon}} \quad (i = i^{**}, i^{**}+1, \dots).$$

Aus (4), auf Grund eines bekannten Satzes (s. z. B. [6], Vol. II., S. 165.) gibt es eine Folge $\{x_i\}_{i=i^{**}}^{\infty}$ von reellen Zahlen derart, daß für die Menge

$$F_i = \{x + x_i:\ x \in E_{3i+i^*}\} \quad (i = i^{**}, i^{**}+1, \dots)$$

(6) $$\operatorname{mes} \overline{\lim_{i \to \infty}} F_i = 2\pi$$

ist.

Es sei

$$\sum_{i=i^{**}}^{\infty} r_i(t) T_i(x-x_i), \tag{7}$$

wobei $r_i(t)=\operatorname{sign}\sin 2^i\pi t$ die i-te Rademachersche Funktion bezeichnet. Die n-te Partialsumme der trigonometrischen Reihe (7) bezeichnen wir mit $R_n(x, t)$. Aus (5) folgt

$$\sum_{i=i^{**}}^{\infty} \int_0^{2\pi} |r_i(t) T_i(x-x_i)| (\varphi(|r_i(t) T_i(x-x_i)|))^{1-\varepsilon} dx \leqq c_4 \sum_{i=i^{**}}^{\infty} \frac{1}{(2^{3i+i^*})^{\varepsilon}} < \infty,$$

woraus wegen $\varphi(x) \geqq 1$ $(x \geqq 0)$ erhalten wir, daß die Reihe (7) bei jedem t fast überall zu einer Funktion $f_t(x) \in L(0, 2\pi)$ konvergiert, und die Reihe (7) die Fourierreihe von $f_t(x)$ ist.

Wir werden eine bekannte Methode anwenden. (S. [3].) Es sei $\Phi_\varepsilon(x)=x(\varphi(x))^{1-\varepsilon}$ und $\psi_\varepsilon(x)=\Phi_\varepsilon(\sqrt{x})$ $(x \geqq 0)$. Man kann leicht zeigen, daß $\psi_\varepsilon(x)$ auch eine von unten konkave Funktion in $[0, \infty)$ ist. Durch Anwendung der Jensenschen Ungleichung und auf Grund der Voraussetzungen über φ ergibt sich

$$\int_0^{2\pi} \int_0^1 \Phi_\varepsilon(|R_{n_{3i_2+i^*+2}}(x, t) - R_{n_{3i_1+i^*-1}}(x, t)|)\, dx\, dt =$$

$$= \int_0^{2\pi} \Big(\int_0^1 \Phi_\varepsilon(|R_{n_{3i_2+i^*+2}}(x, t) - R_{n_{3i_1+i^*-1}}(x, t)|)\, dt \Big) dx =$$

$$= \int_0^{2\pi} \Big(\int_0^1 \psi_\varepsilon((R_{n_{3i_2+i^*+2}}(x, t) - R_{n_{3i_1+i^*-1}}(x, t))^2)\, dt \Big) dx \leqq$$

$$\leqq \int_0^{2\pi} \psi_\varepsilon\Big(\int_0^1 (R_{n_{3i_2+i^*+2}}(x, t) - R_{n_{3i_1+i^*-1}}(x, t))^2\, dt \Big) dx =$$

$$= \int_0^{2\pi} \psi_\varepsilon\Big(\sum_{i=i_1}^{i_2} T_i^2(x-x_i) \Big) dx \leqq \sum_{i=i_1}^{i_2} \int_0^{2\pi} \psi_\varepsilon(T_i^2(x-x_i))\, dx =$$

$$= \sum_{i=i_1}^{i_2} \int_0^{2\pi} \Phi_\varepsilon(|T_i(x)|)\, dx \leqq c_4 \sum_{i=i_1}^{i_2} \frac{1}{(2^{3i+i^*})^{\varepsilon}} \leqq c_5 \frac{1}{(2^{3\varepsilon})^{i_1}} \quad (i_1 < i_2)$$

auf Grund (5). Für $i_2 \to \infty$ erhalten wir

$$\int_0^{2\pi} \int_0^1 \Phi_\varepsilon(|f_t(x) - R_{n_{3i_1+i^*-1}}(x, t)|)\, dx\, dt \leqq c_5 \frac{1}{(2^{3\varepsilon})^{i_1}} \quad (i_1 = i^{**}, i^{**}+1, \ldots).$$

Daraus folgt

$$\int_0^{2\pi} \Phi_\varepsilon(|f_t(x) - R_{n_{3i_1+i^*-1}}(x, t)|)\, dx \to 0 \quad (i_1 \to \infty)$$

bei fast jedem t. Wegen $\Phi_\varepsilon(2x)\leqq c_6\Phi_\varepsilon(x)$ $(x\geqq 0)$ ist $R_{n_{3i_1+i^*-1}}(x, t)\in L(\varphi(L))^{1-\varepsilon}$ für jedes t, auf Grund von (5), und so gilt $f_t(x)\in L(\varphi(L))^{1-\varepsilon}$ bei fast jedem t. Es gibt also eine Zahl t_0 derart, daß die Reihe (7) im Falle $t=t_0$ die Fourierreihe einer Funktion $F(x)=f_t(x)$ ist, und $F\in L(\varphi(L))^{1-\varepsilon}$ besteht; wir können auch $r_i(t_0)\neq 0$ $(i=i^{**}, i^{**}+1, \ldots)$ annehmen.

Auf Grund der Definition von F gilt für $i=i^{**}, i^{**}+1, \ldots$

$$\max_{n_{3i+i^*}\leqq n<n_{3i+i^*+1}} |s_n(F; x)-s_{n_{3i+i^*}}(F; x)| =$$

$$= \max_{n_{3i+i^*}\leqq n<n_{3i+i^*+1}} |s_n(f; x)-s_{n_{3i+i^*}}(f; x)| \frac{1}{\varphi(n_{3i+i^*})} \geqq$$

$$\geqq \max_{n_{3i+i^*}\leqq n<n_{3i+i^*+1}} \frac{|s_n(f; x)|}{\varphi(n)} \frac{\varphi(n)}{\varphi(n_{3i+i^*})} - \frac{|s_{n_{3i+i^*}}(f; x)|}{\varphi(n_{3i+i^*})}.$$

Da $n_{3(i+1)+i^*}/n_{3i+i^*}\geqq q>1$ $(i=i^{**}, i^{**}+1, \ldots)$ gilt, auf Grund eines Satzes von R. A. Hunt [1]

$$s_{n_{3i+i^*}}(f; x) = o(\log\log n_{3i+i^*}) = o(\varphi(n_{3i+i^*}))$$

fast überall besteht. Daraus, und aus (2) erhalten wir, daß im Falle $x\in \overline{\lim\limits_{i\to\infty}} F_i$

$$\max_{n_{3i+i^*}\leqq n<n_{3i+i^*+1}} |s_n(F; x)-s_{n_{3i+i^*+1}-1}(F; x)| \geqq \frac{m}{2}-o_x(1)$$

für unendlich vieles i erfüllt ist. Daraus und aus (6) folgt, daß

$$\overline{\lim_{n,m\to\infty}} |s_n(F; x)-s_m(F; x)| > 0$$

fast überall besteht.

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Solutions to three problems concerning the overconvergence of complex interpolating polynomials

V. TOTIK

To Professor B. Szőkefalvi-Nagy on his seventieth birthday

The aim of this note is to solve the problems raised in [1] by J. SZABADOS and R. S. VARGA. We keep the notations of [1].

The answer to the first problem is positive: $\hat{G}_l(z, \varrho)=G_l(z, \varrho)$. By the definition of $\hat{G}_l(z, \varrho)$ it is sufficient to show that for fixed z, $G_l(z, \varrho)$ is a monotonically decreasing continuous function of ϱ. By Hadamard's three-circle-theorem

$$I_n(\varrho) = \frac{1}{n}\log\left|\max_{|t|=\varrho}(1-t^{-ln})\frac{z^n-1}{t^n-1}-\frac{\omega_n(z, Z)}{\omega_n(t, Z)}\right|$$

is a convex function of $\log\varrho$ on the interval (ϱ', ∞). The proof of [1, Proposition 1] and a trivial estimate yield

$$K_1\log\frac{|z|}{\varrho^{l+1}} \leqq I_n(\varrho) \leqq K_2\log\frac{|z|+\varrho'}{\varrho-\varrho'}, \tag{1}$$

hence

$$\log G_l(z, \varrho) = \limsup_{n\to\infty} I_n(\varrho)$$

is also a convex function of $\log\varrho$ and thus it is continuous in ϱ. Since by (1)

$$\lim_{\varrho\to\infty}\log G_l(z, \varrho) = -\infty,$$

the convexity of $\log G_l(z, \varrho)$ implies its decrease on (ϱ', ∞) as was stated above.

After these the results of [1] imply the formula

$$\Delta_l(z, \varrho, Z) = G_l(z, \varrho) = \max\left(\frac{|z|}{\varrho}g(z, \varrho); \frac{|z|}{\varrho^{l+1}}\right) \quad (|z|>\varrho), \tag{2}$$

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where

$$g(z,\varrho)=\frac{\varrho}{|z|}\limsup_{n\to\infty}\left\{\max_{|t|=\varrho}\left|\frac{z^n-1}{t^n-1}-\frac{\omega_n(z,Z)}{\omega_n(t,Z)}\right|\right\}^{1/n}=$$

$$=\limsup_{n\to\infty}\left\{\max_{|t|=\varrho}\left|1-\frac{\omega_n(z,Z)}{z^n-1}\frac{t^n-1}{\omega_n(t,Z)}\right|\right\}^{1/n}.$$

Now turning to the second and third problems of [1] we may assume that (geometric) overconvergence takes place at least at one point z_0, $|z_0|>\varrho$, because these problems have interest only from the point of view of the overconvergence. In this case we prove the following rather surprising (see [1]) result.

Theorem. *If* $\Delta_l(z_0,\varrho,Z)<1$ *for some* $|z_0|>\varrho$ *then*

$$\mathfrak{G}=\{z\,|\,\Delta_l(z,\varrho,Z)=1\}$$

is a circle with center at the origin, $\Delta_l(z,\varrho,Z)<1$ *inside and* $\Delta_l(z,\varrho,Z)>1$ *outside this circle.*

Proof. Let

$$g_n(z,t)=\left|1-\frac{\omega_n(z,Z)}{z^n-1}\frac{t^n-1}{\omega_n(t,Z)}\right|$$

and

$$h(\varrho)=\limsup_{n\to\infty}\left\{\max_{|t|=\varrho}\left|1-\frac{t^n-1}{\omega_n(t,Z)}\right|\right\}^{1/n}.$$

First we prove the equality

$$g(z,\varrho)=h(\varrho)\quad(|z|\geqq\varrho>\varrho') \tag{3}$$

provided either side is less than one.

Suppose $g(z,\varrho)<q<1$. Then for some n_0 and $n\geqq n_0$ we have

$$g_n(z,t)\leqq q^n\quad(|t|=\varrho,\quad n\geqq n_0). \tag{4}$$

This yields

$$1-q^n\leqq\left|\frac{\omega_n(z,Z)}{z^n-1}\frac{t^n-1}{\omega_n(t,Z)}\right|\leqq 1+q^n\quad(|t|=\varrho),$$

and so for any $|t_1|=|t|=\varrho$,

$$1-Kq^n\leqq\left|\frac{t^n-1}{\omega_n(t,Z)}\Big/\frac{t_1^n-1}{\omega_n(t_1,Z)}\right|\leqq 1+Kq^n. \tag{5}$$

For fixed t the function inside the absolute value marks is homomorphic for $|t_1|>\varrho$ without zeros and with removal singularity at $t_1=\infty$, so we obtain from the maximum modulus principle that (5) holds for all $|t_1|\geqq\varrho$. Letting $t_1\to\infty$ we get

$$1-Kq^n\leqq\left|\frac{t^n-1}{\omega_n(t,Z)}\right|\leqq 1+Kq^n\quad(|t|=\varrho).$$

Again by the maximum modulus principle this is true for every $|t| \geqq \varrho$, so specially

(6) $$1-Kq^n \leqq \left|\frac{z^n-1}{\omega_n(z, Z)}\right| \leqq 1+Kq^n.$$

(4) and (6) yield

(7) $$\left|\frac{t^n-1}{\omega_n(t, Z)}-\frac{t_1^n-1}{\omega_n(t_1, Z)}\right| \leqq Kq^n$$

for every $|t|=|t_1|=\varrho$ and hence also for every $|t|=\varrho \leqq |t_1|$. Letting here $t_1 \to \infty$ it follows

$$\left|1-\frac{t^n-1}{\omega_n(t, Z)}\right| \leqq Kq^n \quad (|t|=\varrho,\ n \geqq n_0)$$

by which $h(\varrho) \leqq q$. Since $g(z, \varrho)<q<1$ was arbitrary, we obtain that $h(\varrho) \leqq g(z, \varrho)$.

Now let us suppose conversely that $h(\varrho)<q<1$. Then for some n_0 we have

$$\left|1-\frac{t^n-1}{\omega_n(t, Z)}\right| < q^n \quad (|t|=\varrho,\ n \geqq n_0)$$

Applying the maximum modulus principle once more we obtain that

$$\frac{z^n-1}{\omega_n(z, Z)}=1+\varepsilon_n(z), \quad |\varepsilon_n(z)| \leqq q^n, \quad |z| \geqq \varrho,$$

by which

$$\frac{\omega_n(z, Z)}{z^n-1}=1+\eta_n(z), \quad |\eta_n(z)| \leqq Kq^n.$$

Multiplying this by

$$\frac{t^n-1}{\omega_n(t, Z)}=1+\varepsilon_n(t), \quad |\varepsilon_n(t)| \leqq q^n, \quad |t|=\varrho$$

it follows readily that

$$g_n(z, t) \leqq Kq^n \quad (|t|=\varrho,\ |z| \geqq \varrho, \quad n \geqq n_0),$$

and the inequality $g(z, \varrho) \leqq h(\varrho)$ can be deduced as the opposite inequality above.

Let

$$\varphi(\varrho)=\max\left(\frac{h(\varrho)}{\varrho};\ \frac{1}{\varrho^{l+1}}\right).$$

So far we have proved (see (2) and (3)) the formula

(8) $$\Delta_l(z; \varrho, Z)=|z|\varphi(\varrho) \quad (|z|>\varrho)$$

under the assumption $\min(\varphi(\varrho), \Delta_l(z)/|z|)<1/\varrho$, and by the first part of our paper here $\varphi(\varrho)$ $(\varrho>\varrho')$ is a monotonically decreasing convex function of ϱ with bounds

$$\frac{1}{\varrho^{l+1}} \leqq \varphi(\varrho) \leqq \frac{1}{\varrho-\varrho'}.$$

The Theorem follows immediately from (8).

Remarks. 1. We have proved somewhat more, namely for a $\varrho>\varrho'$, (geometric) overconvergence occurs if and only if

$$\limsup_{n\to\infty}\max_{|t|=\varrho}\left|1-\frac{t^n-1}{\omega_n(t,Z)}\right|^{1/n}<1,$$

and in this case the "overconvergence radius" is

$$\frac{1}{\varphi(\varrho)}=\min\left(\varrho^{l+1};\ \frac{\varrho}{\limsup\limits_{n\to\infty}\max\limits_{|t|=\varrho}\left|1-\dfrac{t^n-1}{\omega_n(t,Z)}\right|^{1/n}}\right).$$

2. The Theorem does not hold without the assumption "$\Delta_l(z_0,\varrho,Z)<1$ for some $|z_0|>\varrho$". Indeed, if the points $z_{k,n}\,(1\leqq k\leqq n)$ are "very near to $(-1)^n$" then the interior of the set

$$\{z\,|\,\Delta_l(z,\varrho,Z)<1\}$$

is the common part of the discs $|z-1|<\varrho-1$ and $|z+1|<\varrho-1$ and $\mathfrak{G}$ is on its boundary.

3. The formula (8) yields very easily the following result of J. Szabados and R. S. Varga (see [2, Theorem 2, 3]): If

$$\limsup_{n\to\infty}\sqrt[n]{\max_{1\leqq k\leqq n}|z_{k,n}-\exp 2\pi ik/n|}\leqq\delta<1$$

then

$$\Delta_l(z,\varrho,Z)\leqq\frac{|z|}{\varrho}\max\left(\frac{1}{\varrho^l},\delta\right).$$

Indeed, for any $\varepsilon>0$ we have for large n

$$\left|1-\frac{t^n-1}{\omega_n(t,Z)}\right|^{1/n}=\left|1-\prod_{k=1}^{n}\frac{t-\exp 2\pi ik/n}{t-z_{k,n}}\right|^{1/n}\leqq$$

$$\leqq\left|\prod_{k=1}^{n}\left(1+\frac{(\delta+\varepsilon)^n}{\varrho-\varrho'}\right)-1\right|^{1/n}\leqq(Kn)^{1/n}(\delta+\varepsilon),$$

and so $h(\varrho)\leqq\delta$.

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One-sided convergence conditions for Lagrange interpolation based on the Jacobi roots

P. VÉRTESI

To Professor B. Szőkefalvi-Nagy for his 70th birthday

1. Indroduction. We investigate the Lagrange interpolation for continuous functions on the Jacobi abscissas. By conditions of new type uniform convergence theorem will be established on the whole interval $[-1, 1]$.

2. Notations and preliminary results. Let $\alpha, \beta > -1$, say, $\alpha \geqq \beta$, and let

$$-1 \equiv x_{n+1,n}^{(\alpha,\beta)} < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{2n}^{(\alpha,\beta)} < x_{1n}^{(\alpha,\beta)} < x_{0n}^{(\alpha,\beta)} \equiv 1 \tag{2.1}$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ ($n=1, 2, \ldots$; see e.g. G. SZEGŐ [3]). Let us denote by

$$L_n^{(\alpha,\beta)}(f, x) = \sum_{k=1}^{n} f(x_{kn}^{(\alpha,\beta)})\, l_{kn}^{(\alpha,\beta)}(x), \quad n = 1, 2, \ldots, \tag{2.2}$$

the Lagrange interpolatory polynomials of degree $\leqq n-1$ based on the nodes (2.1), i.e., $l_{kn}^{(\alpha,\beta)}(x)$ is the k-th fundamental polynomial of the Lagrange interpolation ($n=1, 2, \ldots$). If $f \in C$ (f is continuous on $[-1, 1]$) then $L_n^{(\alpha,\beta)}(f, x)$ generally do not tend uniformly to $f(x)$ in $[-1, 1]$ (if $n \to \infty$). However, if we suppose

a) $$\omega(f, t) = \begin{cases} o(|\ln t|^{-1}) & \text{if} \quad -1 < \alpha \leqq -1/2 \\ o(t^{\alpha+1/2}) & \text{if} \quad -1/2 < \alpha < 1/2 \end{cases}$$

when $t \to 0$, or

b) $f \in BC$ if $-1 < \alpha < 1/2$,

then

$$\lim_{n \to \infty} \| L_n^{(\alpha,\beta)}(f, x) - f(x) \| = 0 \tag{2.3}$$

(see [3], 14.4 and P. VÉRTESI [4], respectively). Here $\omega(f, t)$ is the modulus of continuity of f in $[-1, 1]$, $BC = \{f; f \in C$ and is of bounded variation on $[-1, 1]\}$ and $\|g(x)\|_{[a,b]} = \sup_{a \leqq x \leqq b} |g(x)|$; $\|\cdot\|$ stands for $\|\cdot\|_{[-1,1]}$.

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In this note a general convergence criterion is proved, from which, among others, the above mentioned theorems can be deduced.

3. Results. 3.1. We say that $f\in C$ satisfies the one-sided Dini—Lipschitz condition (shortly $f\in sDL$) if

$$f(x+h)-f(x) \leqq \frac{\varepsilon(h)}{|\ln h|}, \quad -1 \leqq x \leqq x+h \leqq 1, \tag{3.1}$$

where $\varepsilon(h)\geqq 0$ $(h\geqq 0)$ and $\lim_{h\to 0}\varepsilon(h)=0$. This definition was introduced by G. P. NEVAI [2]. He proved that for any fixed $\alpha, \beta>-1$ and $[a, b] \subset (-1; 1)$,

$$\lim_{n\to\infty} \|L_n^{(\alpha,\beta)}(f, x)-f(x)\|_{[a,b]} = 0 \quad \text{if} \quad f\in sDL.$$

3.2. According to (3.1) and L. V. ZIZIASVILI [7] we shall define the next δ-modulus for a bounded function defined on $[-1, 1]$ as follows:

$$\delta(f, t) = \sup_{\substack{0\leqq h\leqq t \\ -1\leqq x\leqq x+h\leqq 1}} [f(x+h)-f(x)], \quad t \geqq 0. \tag{3.2}$$

It is easy to see the next properties.

1) $0\leqq\delta(f, t)\leqq\omega(f, t)$,

2) $\delta(f, t)\leqq\delta(f, T)$ if $0\leqq t\leqq T$,

3) $\lim_{t\to 0} \delta(f, t)=0$ if $f\in C$,

4) $\delta(f, t)=0$ for any $0\leqq t\leqq t_0$ $(t_0>0)$ if $f\in C$, and is monotone decreasing,

5) $\delta(f, nt)\leqq n\delta(f, t)$, $\delta(f, \lambda t)\leqq(\lambda+1)\delta(f, t)$ where n is a positive integer, λ is a positive real number,

6) $\delta(f_1+f_2, t)\leqq\delta(f_1, t)+\delta(f_2, t)$.

Moreover, by definition

$$f(x)-f(x+h)+\delta(f, t) \geqq 0, \quad 0 \leqq h \leqq t. \tag{3.3}$$

Finally, illustrating 1) let us remark that, e.g., for $g(x)=(x+1)^{1/2}-(x+1)^{1/4}$ we have $c_1t^{1/2}\leqq\delta(f, t)\leqq c_2t^{1/2}$ but $c_3t^{1/4}\leqq\omega(f, t)\leqq c_4t^{1/4}$.

3.3. By these definitions we can prove

Theorem 3.1. *Let* $-1<\gamma=\max(\alpha, \beta)<1/2$ *be fixed. If* $f\in C$ *and*

$$\delta(f, t) = \begin{cases} o(|\ln t|^{-1}) & \text{if } -1 < \gamma \leqq -1/2 \\ o(t^{\gamma+1/2}) & \text{if } -1/2 < \gamma < 1/2 \end{cases} \tag{3.4}$$

then

$$\lim_{n\to\infty} \|L_n^{(\alpha,\beta)}(f, x)-f(x)\| = 0. \tag{3.5}$$

3.4. To obtain (3.5) from 2a) we remark that from 2a) (by 3.2.1)) we get (3.4); moreover if $f\in BC$ then $f=f_1-f_2$ where f_1 and f_2 are monotone decreasing, i.e., $\delta(f_1, t)=\delta(f_2, t)\equiv 0$. Hence by Theorem 3.1 we have (3.5). These mean, Theorem 3.1 includes the statements of 2, indeed.

3.5. If $\gamma\geqq 1/2$, (3.4) generally does not involve (3.5) even if $\delta(f, t)=0$. More exactly if $\mu=\min(\alpha, \beta)$ then we have

Theorem 3.2. *If $\gamma>1/2$ or, if $\gamma\geqq 1/2$ and $\mu=1/2$, then there exists a continuous monotone decreasing function g for which* (3.5) *does not hold.*

3.6. Remarks. a) Theorems corresponding to Theorems 3.1 and 3.2 can be obtained for continuous monotone increasing functions g considering that now $(-g)$ is monotone decreasing.

b) It is worthwhile to state the next

Corollary 3.3. *If $-1<\gamma<1/2$ then for any $f\in BC$ we have* (3.5). *On the other hand, if $\gamma>1/2$ or, if $\gamma\geqq 1/2$ and $\mu=1/2$, then* (3.5) *does not hold for a certain $g\in BC$.*

c) It is easy to prove Theorem 3.1 if $-1\leqq\beta\leqq\alpha<1/2$. Further, we can prove the corresponding theorems if we consider the Lagrange polynomials based on $\{x_{kn}^{(\alpha,\beta)}\}_{k=0}^{n}$, $\{x_{kn}^{(\alpha,\beta)}\}_{k=1}^{n+1}$ or $\{x_{kn}^{(\alpha,\beta)}\}_{k=0}^{n+1}$, respectively. Omitting the details we refer to [4] and [6].

4. Proofs. 4.1. Proof of Theorem 3.1. By $f=f(x), f_i=f(x_i)$ and $l_i=l_i(x)$,

$$2[f(x)-L_n^{(\alpha,\beta)}(f, x)] = 2\sum_{k=1}^{n}[f(x)-f(x_k)]\,l_k(x) =$$
$$(4.1)\qquad = (f-f_1)\,l_1+\sum_{k=1}^{n-1}(f-f_k)(l_k+l_{k+1})+\sum_{k=1}^{n-1}(f_k-f_{k+1})\,l_{k+1}+(f-f_n)\,l_n.$$

To estimate the sums first we prove

Lemma 4.1. *Let $-1<\alpha, \beta$ and $\varepsilon, \eta>0$ be fixed. If $k\geqq M_1$, $\vartheta_k\leqq\pi-\varepsilon$; then for any $x\in[-1+\eta, 1]$ we have*

$$(4.2)\qquad |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\,|\bar{l}_{kn}^{(\alpha,\beta)}(x)|\left[\frac{1}{k}+\frac{k}{(k+j)(|k-j|+1)}\right]$$

uniformly in x and k. Here $|\bar{l}_k(x)|=\max(|l_k(x)|, |l_{k+1}(x)|)$, and M_1 depends on α and β.

To obtain (4.2) we shall use the next relations. If $x_{kn}=x_{kn}^{(\alpha,\beta)}=\cos\vartheta_{kn}^{(\alpha,\beta)}$, $0\leqq k\leqq n+1$ (with $x_0\equiv 1, x_{n+1}\equiv -1$), then

$$(4.3)\qquad \vartheta_{k+1,n}^{(\alpha,\beta)}-\vartheta_{kn}^{(\alpha,\beta)} \sim \frac{1}{n}, \quad k = 0, 1, \ldots, n;$$

$$(4.4)\qquad |P_n^{(\alpha,\beta)}(x)| \sim |\vartheta-\vartheta_j|\,\vartheta_j^{-\alpha-1/2}n^{1/2} \sim |x-x_j|\,\vartheta_j^{-\alpha-3/2}n^{1/2}$$

uniformly in $x\in[-1+\eta, 1]$; moreover, with $P_n^{(\alpha,\beta)}(x)=P_n(x)$,

$$(4.5)\qquad P_n'(x_k)=(-1)^{k-1}\sqrt{\frac{n}{\pi}}\,\frac{1+O(k^{-1})}{2\left(\sin\frac{\vartheta_k}{2}\right)^{\alpha+3/2}\left(\cos\frac{\vartheta_k}{2}\right)^{\beta+3/2}}$$

uniformly in k if $k\geqq M_0$ and $0<\vartheta_k\leqq\pi-\varepsilon$,

$$(4.6)\qquad |P_n'(x_k)|\sim k^{-\alpha-3/2}n^{\alpha+2},\quad 0\leqq\vartheta_k\leqq\pi-\varepsilon,$$

$$(4.7)\qquad P_n^{(\alpha,\beta)}(x)=(-1)^nP_n^{(\beta,\alpha)}(-x).$$

(Here $x_j=\cos\vartheta_j$ is the nearest root to x $(j=j(n))$; for the symbol "$\sim$", which may depend on α,β,ε and η, see [3], 1.1; the sources of (4.3)—(4.6) can be found in [4]; ($\varepsilon>0$ and $\eta>0$ are arbitrary fixed values).)

If $|k-j|>1$, we can write

$$(4.8)\qquad l_k(x)+l_{k+1}(x)=P_n(x)\{[P_n'(x_k)(x-x_k)]^{-1}+[P_n'(x_{k+1})(x-x_{k+1})]^{-1}\}.$$

It is easy to see that

$$\{\dots\}=\frac{P_n'(x_k)+P_n'(x_{k+1})}{P_n'(x_k)P_n'(x_{k+1})(x-x_{k+1})}+\frac{x_k-x_{k+1}}{P_n'(x_k)(x-x_k)(x-x_{k+1})}=I_1+I_2.$$

If $k\geqq M_0$, $\vartheta_k\leqq\pi-\varepsilon$, then by (4.5) and $K(\vartheta)=2\left(\sin\frac{\vartheta}{2}\right)^{\alpha+3/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+3/2}$ we obtain after a simple calculation, that

$$|P_n'(x_k)+P_n'(x_{k+1})|=\sqrt{\frac{n}{\pi}}\left|\frac{[1+O(k^{-1})]K(\vartheta_{k+1})-[1+O(k^{-1})]K(\vartheta_k)}{K(\vartheta_k)K(\vartheta_{k+1})}\right|\sim\frac{\sqrt{n}}{kK(\vartheta_k)},$$

if k is big enough. I.e., if $k\geqq M_1$ and $\vartheta_k\leqq\pi-\varepsilon$ then $I_1\sim[kP_n'(x_k)(x-x_k)]^{-1}$. On the other hand, by (4.3), $|(x_k-x_{k+1})/(x-x_{k+1})|\sim k[(k+j)(|k-j|+1)]^{-1}$, so $I_2\sim k\{[(k+j)(|k-j|+1)]|P_n'(x_k)|\,|x-x_k|\}^{-1}$. Now, using (4.8), we obtain (4.2) if $|k-j|>1$, $k\geqq M_1$, $\vartheta_k\leqq\pi-\varepsilon$. (We used that $|P_n'(x_k)(x-x_k)|\sim|P_n'(x_{k+1})(x-x_{k+1})|$.) The statement is obvious if $|k-j|\leqq 1$.

By (4.2) we get as in [4]: If $\psi=\min(2;1.5-\alpha)$, then

$$(4.9)\qquad \left|\sum_{k=1}^{n-1}(f-f_k)(l_k+l_{k+1})\right|=O(1)\sum_{i=1}^{n}\omega\left(f,\frac{\sin\vartheta}{n}i+\frac{i^2}{n^2}\right)\frac{1}{i^\psi}$$

uniformly in $x\in[-1+\eta,,1]$

(see, e.g., [4], 4.10, where $\sum|f-f_k|\,|l_kk^{-1}|$ (which, by (4.2), is analogous to $\sum|f-f_k|\,|l_k+l_{k+1}|$) is estimated).

4.2. By (3.3) the second sum of (4.1) can be written as follows ($\delta_k=\delta(f, x_k-x_{k+1})$, $l_k=f_k\equiv 0$ if $k\neq 1, 2, \ldots, n$):

$$\left|\sum_{k=1}^{n-1}(f_k-f_{k+1})l_{k+1}\right| \leqq \left|\sum_{k=1}^{n-1}(f_k-f_{k-1}-\delta_k)|l_{k+1}|\right| + \sum_{k=1}^{n-1}\delta_k|l_{k+1}| \leqq$$

$$\leqq \left|\sum_{k=1}^{n-1}(f_k-f_{k+1})|l_{k+1}|\right| + 2\sum_{k=1}^{n-1}\delta_k|l_{k+1}| \leqq \left|\sum_{k=j-m+1}^{j+m-1}(f_k-f_{k+1})|l_{k+1}|\right| +$$

$$+\left|\sum_{k=1}^{j-m-1}\left[\sum_{i=1}^{k}(f_i-f_{i+1})\right](|l_{k+1}|-|l_{k+2}|)\right| + \left||l_{j-m+1}|\sum_{i=1}^{j-m}(f_i-f_{i+1})\right| +$$

$$+\left|\sum_{k=j+m}^{n-2}\left[\sum_{i=j+m}^{k}(f_i-f_{i+1})\right](|l_{k+1}|-|l_{k+2}|)\right| + \left||l_n|\sum_{i=j+m}^{n-1}(f_i-f_{i+1})\right| +$$

$$+2\sum_{k=1}^{n-1}\delta_k|l_{k+1}| = K_1+K_2+K_3+K_4+K_5+K_6,$$

where $1\leqq m=m(n)\leqq n$ will be determined later.

4.3. To estimate K_1 we need

Lemma 4.2. *Let* $-1<\alpha, \beta$ *and* $\eta>0$ *be fixed. Then*

$$\sum_{k=j-m}^{j+m}|l_{kn}^{(\alpha,\beta)}(x)| = O(1)[\ln 2m+m^{\alpha+1/2}] \tag{4.10}$$

uniformly in $x\in[-1+\eta, 1]$ *and* m, $1\leqq m\leqq n$.

If $n=O(m)$, (4.10) is well known ([3], 14.4). So let $m=o(n)$.

a) If $j\leqq m$, we obtain

$$\sum_{k=j-m}^{j+m} \leqq \sum_{k=1}^{j/2}+\sum_{k=j/2}^{2j}+\sum_{k=2j}^{2m} = J_1+J_2+J_3.$$

(Here and later $\sum_{k=a}^{b}$ stands for $\sum_{k=[a]}^{[b]}$.)

By (4.3)—(4.6), $l_k(x)=O(1)k^{\alpha+3/2}[j^{\alpha+1/2}(k+j)(|k-j|+1)]^{-1}$, which implies $J_1=O(1)$, $J_2=O(1)\ln 2j=O(1)\ln 2m$ and $J_3=O(1)(m/j)^{\alpha+1/2}=O(1)(1+m^{\alpha+1/2})$.

b) If $m\leqq j<2m$, we have

$$\sum_{k=j-m}^{j+m} \leqq J_1+J_2+\sum_{k=2j}^{3m},$$

which can be estimated as above.

c) Finally, if $j\geqq 2m$, we have

$$\sum_{k=j-m}^{j+m}|l_k(x)| = O(1)\sum_{k=j-m}^{j+m}(|k-j|+1)^{-1} = O(\ln 2m).$$

4.4. So by Lemma 4.2 we have

$$K_1 = O\left(\omega\left(f, \frac{1}{n}\right)\right)(\ln 2m + m^{\alpha+1/2}). \tag{4.11}$$

4.5. To estimate K_2 and K_4 we prove

Lemma 4.3. *Let* $-1<\alpha, \beta$ *and* $\eta>0$ *be fixed. Then*

$$\sum_{k=1}^{n} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\left(1+\frac{r(\alpha, n)}{j^{\alpha+1/2}}\right), \tag{4.12}$$

$$\sum_{k=j-m}^{j+m} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\left(1+\frac{r(\alpha, m)}{j^{\alpha+1/2}}\right), \tag{4.13}$$

$$\sum_{k=1}^{j-m} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\frac{\ln 2m}{m}, \tag{4.14}$$

$$\sum_{k=j+m}^{n} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\left(\frac{\ln 2m}{m}+\frac{r(\alpha, n)}{j^{\alpha+1/2}}\right) \tag{4.15}$$

uniformly in $x\in[-1+\eta, 1]$ *and* m, $1\leq m\leq n$. *Here*

$$r(u, v) = \begin{cases} v^{u-1/2} & \text{if } u \neq 1/2, \\ \ln v & \text{if } u = 1/2. \end{cases}$$

A. Indeed, to prove (4.12) we write the sum as follows:

$$\sum_{k=1}^{n} \dots = \sum_{k=1}^{M_1-1} \dots + \sum_{k=M_1}^{cn} \dots + \sum_{k=cn+1}^{n} \dots = I_1+I_2+I_3 \quad (0<c<1).$$

By (4.2)—(4.6), if $1\leq k\leq n-1$, then

$$|l_k(x)+l_{k+1}(x)| = O(1)\frac{k^{\alpha+1/2}}{j^{\alpha+1/2}(k+j)(|k-j|+1)}+\frac{k^{\alpha+5/2}}{j^{\alpha+1/2}(k+j)^2(|k-j|+1)^2}, \tag{4.16}$$

i.e., $I_1=O(j^{-\alpha-5/2})$. For I_2 we can write (if, say, $2j<cn$),

$$I_2 = \sum_{k=M_1}^{j/2} + \sum_{k=j/2+1}^{2j} + \sum_{k=2j+1}^{cn} = O(1)\left[\frac{1}{j}+1+\frac{r(\alpha, n)}{j^{\alpha+1/2}}\right].$$

If, say, $\eta=2\varepsilon$, then by (4.7),

$$I_3 = O(1)\frac{n^\alpha}{j^{\alpha+1/2}}\sum_{k=1}^{n}\frac{k^{\beta+1/2}}{n^{\beta+2}} = O(1)\frac{n^{\alpha-1/2}}{j^{\alpha+1/2}}.$$

By these estimations we obtain (4.12). Similar arguments apply for the other cases including the estimation of the n-th term. Now we sketch the remaining three formulae.

B. To prove (4.13), we write

$$\sum_{k=j-m}^{j+m} = \sum_{k=j-m}^{j} + \sum_{k=j+1}^{j+m} = J_1 + J_2.$$

Here $J_1 < \sum_{k=1}^{2j}$, which can be estimated by $O(1)$ (see I_2, above). Now, by (4.12), we can suppose that $m=o(n)$. To estimate J_2, we proceed as follows.

a) First let $m \leqq 2j$. If $n=O(j)$, then $\sum_{k=j}^{j+m} \leqq \sum_{k=j}^{(1+\varrho)j}$ for arbitrary $\varrho>0$ if n is big enough. But this sum can be estimated by $O(1)$, if ϱ is small enough. On the other hand, if $j=o(n)$, then $\sum_{k=j}^{j+m} \leqq \sum_{k=j}^{3j}$ which again can be estimated by $O(1)$.

b) If $m \geqq 2j$, then we have $\sum_{k=j}^{j+m} \leqq \sum_{k=j}^{2j} + \sum_{k=2j}^{m}$, which can be estimated by $O(1)(1+r(\alpha, m)j^{-\alpha-1/2})$ (see the previous estimations for I_2 and I_3).

C. To obtain (4.14), we argue as follows. If $m \leqq j/2$, then $\sum_{k=1}^{j-m} \leqq \sum_{k=1}^{j/2} + \sum_{k=j/2}^{j-m}$, which can be estimated by $O(j^{-1}\ln 2j)=O(m^{-1}\ln 2m)$ (see (4.16) and the above considerations). On the other hand, if $j>m>j/2$, we can estimate as follows:

$$\sum_{k=1}^{j-m} < \sum_{k=1}^{j/2} = O(j^{-1}) = O(m^{-1}) \quad \text{(see (4.16))}.$$

D. Now we estimate $\sum_{k=j+m}^{n}$.

a) First let $m \leqq 2j$. Then, if, say, $3j<cn$ $(0<c<1)$, we can write

$$\sum_{k=j+m}^{n} = \sum_{k=j+m}^{3j} + \sum_{k=3j+1}^{n} = J_3 + J_4.$$

Here $J_3 = O(j^{-1}(\ln 2j - \ln 2m)) = O(m^{-1}\ln 2m)$ (see (4.16)), moreover $J_4 = O(r(\alpha, n)j^{-\alpha-1/2})$ (see the corresponding parts of I_2 and I_3).

b) If $m>2j$, then

$$\sum_{k=j+m}^{n} < \sum_{k=m}^{2m} + \sum_{k=2m}^{n} = J_5 + J_6.$$

By (4.16), $J_5 = O(m^{\alpha-1/2}j^{-\alpha-1/2})$. Further, if, say, $3m<cn$ $(0<c<1)$, we can write $J_6 = \sum_{k=2m}^{cn} + \sum_{k=cn+1}^{n} = J_7 + I_3$. Here by (4.16) $J_7 = O(r(\alpha, m)j^{-\alpha-1/2})$, moreover using the estimation for I_3, finally we get (4.15).

The remaining cases can be treated analogously. Thus we have proved Lemma 4.2.

4.6. Let us estimate K_2. By 4.2 and (4.14), using the fact that $\big||\alpha|-|\beta|\big|\leqq|\alpha+\beta|$, we get

$$(4.17)\quad K_2=\left|\sum_{k=1}^{j-m-1}(f_1-f_{k+1})(|l_{k+1}|-|l_{k+2}|)\right|\leqq\sum_{k=1}^{j-m-1}|f_{k+1}-f_1|\,|l_{k+1}+l_{k+2}|\leqq$$

$$\leqq 2\|f\|\sum_{k=1}^{j-m-1}|l_{k+1}+l_{k+2}|=\|f\|\,O\left(\frac{\ln 2m}{m}\right).$$

Similarly,

$$(4.18)\qquad K_4\leqq 2\|f\|\sum_{k=j+m}^{n-2}|l_{k+1}+l_{k+2}|=\|f\|\,O(1)\left(\frac{\ln 2m}{m}+\frac{r(\alpha,n)}{j^{\alpha+1/2}}\right).$$

4.7. To estimate K_3 and K_5 we remark that for any $\alpha>-1$, $|l_k(x)|=O(1)k^{\alpha+3/2}j^{-\alpha-1/2}(k+j)^{-1}(|k-j|+1)^{-1}$ if $x\in[-1+\eta,1]$ and $\vartheta_k\leqq\pi-\varepsilon$, which can be obtained using the above arguments. I.e., if $j-m\geqq 2$, we have

$$(4.19)\qquad K_3=O(1)\frac{(j-m)^{\alpha+3/2}}{j^{\alpha+1/2}(2j-m)m}|f_1-f_{j-m+1}|=\frac{O(1)\|f\|}{m}.$$

If $0\leqq j-m<2$, then $K_3=O(1)j^{-\alpha-3/2}m^{-1}\omega(f,n^{-2})=O(1)\|f\|m^{-1}$. For K_5 we have by (4.7)

$$(4.20)\qquad K_5=O(1)\frac{n^{\alpha-\beta-2}}{j^{\alpha+1/2}}|f_n-f_{j+m}|=\frac{O(1)\|f\|}{n^{1/2}}.$$

4.8. Now we estimate $|[f(x)-f(x_1)]l_1(x)|=K_7$ (see (4.1)). By $l_k(x)=O(1)k^{\alpha+3/2}[j^{\alpha+1/2}(k+j)(|k-j|+1)]^{-1}$,

$$(4.21)\quad K_7=O(1)\omega\left(f,\frac{j^2}{n^2}\right)\frac{1}{j^{\alpha+5/2}}=O(1)\omega\left(f,\frac{j^{3/2}}{n^{3/2}}\right)\frac{1}{j^{3/2}}=O(1)\omega\left(f,\frac{1}{n^{3/2}}\right).$$

Using similar estimations we get that

$$(4.22)\qquad K_8\overset{\text{def}}{=}|f(x)-f(x_n)|\,|l_n(x)|=O(n^{-3/2}).$$

4.9. Summarizing the estimations (4.1), (4.9), (4.11) and (4.17)—(4.22), we have that for $x\in[-1+\eta,1]$ and $-1<\alpha<1/2$,

(4.23)

$$|L_n^{(\alpha,\beta)}(f,x)-f(x)|=O(1)\left[\sum_{i=1}^{n}\omega\left(f,\frac{i}{n}\right)\frac{1}{i^{\psi}}+\omega\left(f,\frac{1}{n}\right)(\ln 2m+m^{\alpha+1/2}+n^{\alpha-1/2})+\right.$$

$$\left.+\|f\|\left(\frac{\ln 2m}{m}+\frac{n^{\alpha-1/2}}{j^{\alpha+1/2}}\right)+\delta\left(f,\frac{1}{n}\right)\sum_{k=1}^{n}|l_k(x)|+n^{-1/2}\right].$$

Here $O(1)$ does not depend on f. Now let $m(n)$ tend to infinity (with n) so that, say, $\lim_{n\to\infty}m\omega\left(\frac{1}{n}\right)=0$. Using that for $\alpha>-\frac{1}{2}$, $\sum_{k=1}^{n}|l_k(x)|=O(n^{\alpha+1/2})$ and that

$\delta\left(f, \frac{1}{n}\right)=o(1/n^{\alpha+1/2})$, we obtain the statement for $x\in[-1+\eta, 1]$ in virtue of $\sum_{i=1}^{n}\omega\left(f, \frac{i}{n}\right)i^{-\psi}=o(1)$ (see [4], 3.2). If $-1<\alpha\leqq-1/2$, we can use the relations $\sum_{k=1}^{n}|l_k(x)|=O(\ln n)$ and $\delta(f, 1/n)=o(1/\ln n)$. Now by (4.7) we obtain the theorem for the whole interval $[-1, 1]$.

4.10. Proof of Theorem 3.2. First let $\alpha=1/2+2\varrho$, $\varrho>0$ and $\beta>-1$, say. Furthermore, let $\omega(t)$ be a modulus of continuity with $\lim_{t\to+0}\omega(t)t^{-1}=\infty$, $\omega_2(t):=t\,\omega(t)$, and $C(\omega_2)=\{f; f\in C \text{ and } \omega(f', t)\leqq a(f)\omega(t)\}$. We quote P. VÉRTESI [5], Theorem 8.1: There exists a function $h\in C(\omega_2)$ for which

$$\varlimsup_{n\to\infty}\frac{|L_n^{(\alpha,\beta)}(h, 1)-h(1)|}{n^{\alpha+1/2}\omega_2\left(\frac{1}{n}\right)}\geqq 1 \quad \text{for any} \quad \alpha, \beta>-1. \tag{4.24}$$

If $\omega_2(t)=t^{1+\varrho}$, then by (4.24), $|L_n(h, 1)-h(1)|\geqq n^{1+2\varrho}n^{-1-\varrho}=n^{\varrho}$ $(n=n_1, n_2, \ldots)$ i.e., $\varlimsup_{n\to\infty}|L_n(h, 1)|=\infty$. On the other hand, $h\in\mathrm{Lip}\,1$, from where $h\in BC$, i.e., $h=h_1-h_2$, where $h_1, h_2\in C$ and are monotone decreasing. But then, say, $\varlimsup_{n\to\infty}|L_n(h_1, 1)|=\infty$.

4.11. Now let $\alpha=\beta=1/2$. To obtain Theorem 3.2, we use the next statement (see H. HAHN [1]): If for the arbitrary fixed interpolatory matrix $\{x_{kn}\}$ $(k=1,2,\ldots, n;$ $n=1, 2, \ldots)$ in $[-1, 1]$, the interpolatory polynomials $L_n(f, x)$ converge for every function f of bounded variation at any point where f is continuous, then if $t\in[-1, 1]$ and differs from the nodes x_{kn}, we have

$$\lim_{n=\infty}\sum_{x_{kn}<t}l_{kn}(x)=0 \quad \text{if} \quad t<x \tag{4.25}$$

$$\lim_{n=\infty}\sum_{x_{kn}>t}l_{kn}(x)=0 \quad \text{if} \quad t>x. \tag{4.26}$$

We shall see that, e.g., (4.25) does not hold if $\alpha=\beta=1/2$, $x=1$ and $t=0$. Indeed, if $n=4s$, then by [3], (4.1.7) and $x=\cos\vartheta$,

$$\sum_{x_k<0}l_k^{(1/2,1/2)}(1)=\sum_{x_k<0}\left[(-1)^{k-1}\frac{\sin(n+1)\vartheta}{(n+1)\sin\vartheta}\frac{\sin^2\vartheta_k}{(\cos-\cos\vartheta_k)}\right]_{\vartheta=0}=$$
$$=\sum_{x_k<0}(-1)^{k-1}(1+x_k)=(x_{2s+1}-x_{2s+2})+(x_{2s+3}-x_{2s+4})+\ldots+(x_{4s-1}-x_{4s})>1-x_s=$$
$$=1-\cos\frac{s\pi}{4s+1}>\frac{2-\sqrt{2}}{2}.$$

I.e., there exists an $f\in BC$ for which (3.5) does not hold. As in 4.10, we get the proper monotone decreasing function.

4.12. Finally, if $\alpha=1/2$ and $\beta>1/2$, then by (4.7) and the argument of 4.10 we obtain the statement.

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Asymptotically commuting finite rank unitary operators without commuting approximants

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

The following is an old unsolved problem: Given selfadjoint operators $A_n, B_n \in \mathcal{L}(\mathcal{H}_n)$, $\dim \mathcal{H}_n < \infty$ $(n=1,2,\dots)$, such that $\sup_n (\|A_n\| + \|B_n\|) < \infty$ and $\lim_{n\to\infty} \|[A_n, B_n]\| = 0$, do there exist selfadjoint operators $A_n', B_n' \in \mathcal{L}(\mathcal{H}_n)$ so that $[A_n', B_n'] = 0$ and $\lim_{n\to\infty} (\|A_n - A_n'\| + \|B_n - B_n'\|) = 0$? We present in this note an example showing that the answer to the corresponding question for unitaries instead of selfadjoints is negative.

We shall take $\mathcal{H}_n = \ell^2(\mathbf{Z}/n\mathbf{Z})$ consisting of functions $\xi\colon \mathbf{Z}/n\mathbf{Z} \to \mathbf{C}$ and consider the unitary operators

$$\begin{aligned}(U_n\xi)(k+n\mathbf{Z}) &= \xi(k-1+n\mathbf{Z}),\\ (V_n\xi)(k+n\mathbf{Z}) &= \exp(2k\pi i/n)\,\xi(k+n\mathbf{Z})\end{aligned} \qquad (k = 0, 1, \dots, n-1).$$

Proposition. *Let U_n, V_n be the unitary operators defined above. Then we have $\lim_{n\to\infty} \|[U_n, V_n]\| = 0$, but there do not exist unitary operators $U_n', V_n' \in \ell^2(\mathbf{Z}/n\mathbf{Z})$ such that $[U_n', V_n'] = 0$ and $\lim_{n\to\infty} (\|U_n - U_n'\| + \|V_n - V_n'\|) = 0$.*

Proof. We have $U_nV_n = \exp(-2\pi i/n)V_nU_n$, which implies $\|[U_n, V_n]\| \to 0$ as $n \to \infty$. Assuming the existence of the commuting approximants U_n', V_n' we will reach a contradiction.

Consider on the unit circle $\mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$ the arcs $\Gamma, \Gamma', \Gamma'', \Phi^{(1)}, \Phi^{(2)}$ given respectively by

$$\Gamma\colon \frac{\pi}{5} \leqq \arg z < \frac{4\pi}{5}, \quad \Gamma'\colon \frac{2\pi}{5} \leqq \arg z < \frac{3\pi}{5}, \quad \Gamma''\colon 0 \leqq \arg z < \pi,$$

$$\Phi^{(1)}\colon 0 \leqq \arg z < \frac{2\pi}{5}, \quad \Phi^{(2)}\colon \frac{3\pi}{5} \leqq \arg z < \pi.$$

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Let E_n be the spectral projection of V'_n corresponding to Γ and let $E'_n, E''_n, F_n^{(1)}, F_n^{(2)}$ be the spectral projections of V_n corresponding to $\Gamma', \Gamma'', \Phi^{(1)}, \Phi^{(2)}$, respectively. Note that $E''_n = E'_n + F_n^{(1)} + F_n^{(2)}$. Also, since $[V'_n, U'_n] = 0$, we have $[U'_n, E_n] = 0$ and hence

(1) $$\|[U_n, E_n]\| \to 0 \quad \text{as} \quad n \to \infty.$$

We shall use the following folklore-type fact. If N_n, N'_n are normal operators, $\|N_n - N'_n\| \to 0$, $\|N_n\| < C$ and P_n, P'_n are spectral projections of N_n, respectively N'_n, corresponding to Borel sets Ω, Ω' such that $\bar{\Omega} \cap \bar{\Omega}' = \emptyset$, then we have $\|P_n P'_n\| \to 0$. This gives, in particular,

$$\lim_{n\to\infty} \|(I - E''_n) E_n\| = \lim_{n\to\infty} \|(I - E_n) E'_n\| = 0.$$

It is also easily seen that $\lim_{n\to\infty} \|F_n^{(1)} E_n F_n^{(2)}\| = 0$. So we find selfadjoint projections $\tilde{E}_n$ such that $E'_n \leq \tilde{E}_n \leq E''_n$ and $\lim_{n\to\infty} \|\tilde{E}_n - E_n\| = 0$. One may define $\tilde{E}_n$ for instance as follows. Let $X_n = E'_n + F_n^{(1)} E_n F_n^{(1)} + F_n^{(2)} E_n F_n^{(2)}$ so that $\|X_n - E_n\| \to 0$ and hence $\|X_n^2 - X_n\| \to 0$. Define $\tilde{E}_n$ (for n big enough) as the spectral projection of X_n for the interval $[1/2, 2]$. Remark also that $\tilde{E}_n = \tilde{F}_n^{(1)} + E'_n + \tilde{F}_n^{(2)}$ where $\tilde{F}_n^{(1)} \leq F_n^{(1)}$, $\tilde{F}_n^{(2)} \leq F_n^{(2)}$ are selfadjoint projections.

Consider now the projection $E_n^+ = F_n^{(1)} + E'_n + \tilde{F}_n^{(2)}$ and assume from now on $n \geq 10$. We have

(2) $$E_n^+ \leq E''_n$$

and

$$(I - E_n^+) U_n F_n^{(1)} = (I - E_n^+) U_n \tilde{F}_n^{(1)} = 0,$$

so that

$$(I - E_n^+) U_n E_n^+ = (I - E_n^+) U_n \tilde{E}_n = (I - E_n^+)(I - \tilde{E}_n) U_n \tilde{E}_n.$$

Since, by (1), $\lim_{n\to\infty} \|(I - \tilde{E}_n) U_n \tilde{E}_n\| = 0$, we infer that

(3) $$\lim_{n\to\infty} \|(I - E_n^+) U_n E_n^+\| = 0.$$

Define the isometric operator $W_n \colon \ell^2(\mathbf{Z}/n\mathbf{Z}) \to \ell^2(\mathbf{Z}_{\geq 0})$, by

$$(W_n \xi)(k) = \begin{cases} 0 & \text{if } \ k \geq n, \\ \xi(k + n\mathbf{Z}) & \text{if } \ 0 \leq k < n. \end{cases}$$

Then for $P_n^+ = W_n E_n^+ W_n^*$ and the unilateral shift S on $\ell^2(\mathbf{Z}_{\geq 0})$, we have

$$W_n (I - E_n^+) U_n E_n^+ W_n^* = W_n (I - E_n^+) W_n^* W_n U_n W_n^* W_n E_n^+ W_n^* =$$
$$= (W_n W_n^* - P_n^+) S P_n^+ = (I - P_n^+) S P_n^+$$

since, by (2), $(W_n U_n W_n^* - S) P_n^+ = 0$ and $(I - W_n W_n^*) S P_n^+ = 0$. Thus we have rank $P_n^+ < \infty$, $\text{s-}\lim_{n\to\infty} P_n^+ = I$ and, using (3), $\lim_{n\to\infty} \|(I - P_n^+) S P_n^+\| = 0$. This contradicts the non-quasitriangularity of the unilateral shift [1] and hence concludes the proof.

Remark. The approximation problems for selfadjoint and unitary operators can be interpreted in terms of singular extensions (see [2], [3]). Consider the C^*-algebra

$$\mathcal{A} = \{(T_n)_1^\infty \mid T_n \in \mathcal{L}(\mathcal{H}_n), \sup_n \|T_n\| < \infty\}$$

and $\mathcal{I} \subset \mathcal{A}$, the ideal of sequences $(T_n)_1^\infty$ such that $\lim_{n\to\infty} \|T_n\| = 0$. Then the approximation problem for selfadjoint operators amounts to the question whether every $*$-homomorphism $C(X) \to \mathcal{A}/\mathcal{I}$ can be lifted to a $*$-homomorphisms $C(X) \to \mathcal{A}$, where $X = [0, 1] \times [0, 1]$ and the problem for unitary operators to the same question for $X = \mathbf{T}^2$, the 2-torus. In connection with this we should mention that from our strong non-splitting result in [4] for the singular extension in the C^*-algebra of the Heisenberg group one can construct a $*$-homomorphism $C_0(\mathbf{R}^2) \to \mathcal{A}/\mathcal{I}$ which does not lift (here $C_0(\mathbf{R}^2)$ denotes the continuous functions on $\mathbf{R}^2$ vanishing at infinity). Adjoining a unit to $C_0(\mathbf{R}^2)$ one gets a C^*-algebra isomorphic to $C(S^2)$, where S^2 is the two-sphere, and hence the answer to the lifting problem is negative also for $X = S^2$. Like $[0, 1] \times [0, 1]$, the spaces $\mathbf{T}^2$ and S^2 are two-dimensional, but it seems that the counterexamples for $\mathbf{T}^2$ and S^2 are not due only to the dimension of these spaces but rather to their non-zero two-dimensional cohomology and hence it seems improbable that these examples will have a direct bearing on the problem for selfadjoint operators.

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J. M. Bismut—L. Gross—K. Krickeberg, Ecole d'Eté de Probabilités de Saint-Flour X—1980, X + 313 pages;

Edité par P. L. Hennequin (Lecture Notes in Mathematics, 876, 929), Springer-Verlag, Berlin—Heidelberg—New York, 1981, 1982.

These are the two new volumes of the now traditional Saint-Flour summer school series. Both volumes contain three longer survey articles of a subject area in probability or mathematical statistics. Bickel (72 pages) describes the recent flourishment of robust estimation theory concentrating in a very welcome way on the mathematical technique and not only on motivation as most authors do on this field. El Karoui (166 pages) gives a precise and unified account on stochastic control theory, represeting many results of various authors in the last three decades in the language of the French general theory of stochastic processes. Yor (42 pages) investigates a general stochastic filtration equation which containes most such equations in the literature. Bismut (100 pages) provides a shorter preliminary description of his "mecanique aléatoire" than in his later monograph (same *Lecture Notes*, 866, to be reviewed in the next volume of these *Acta*) which has come out earlier. Gross (104 pages) covers equilibrium thermodinamics, equilibrium statistical mechanics and random fields. Finally, Krickeberg (109 pages) overviews the statistical theory of point processes.

Sándor Csörgő (Szeged)

J. Bourgain, New Classes of $\mathscr{L}^p$-Spaces (Lecture Notes in Mathematics, 889), V+143, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

For normed linear spaces E and F let

$$d(E, F) = \inf \{\|T\|\,\|T^{-1}\| \mid T\colon E \to F \text{ is an onto isomorphism}\}$$

(in case E and F are not isomorphic, take $d(E, F)=\infty$). If $1\leqq p\leqq\infty$ and $1\leqq\lambda<\infty$ then a Banach space X is called a $\mathscr{L}^p_\lambda$-space provided for any finite dimensional subspace E of X there is a finite dimensional subspace F of X satisfying $E\subseteqq F$ and $d(F, l^p(\dim F))\leqq\lambda$. Now a $\mathscr{L}^p$-space is a $\mathscr{L}^p_\lambda$ space for some $\lambda<\infty$.

This concept has turned out to be very useful in the local investigation of Banach spaces, although it has also many consequences on the global structure of the space.

In the book the author provides new constructions for $\mathscr{L}^p$-spaces which solve several open problems in the negative. The examples of $\mathscr{L}^p$-spaces $(1<p<\infty)$ and $\mathscr{L}^1$-spaces are related and are constructed using trees on the integers.

Familiarity with the theory of Banach spaces, measures and universal algebras is necessary when reading the book, which is designed especially for research workers in this topic. Several open problems are also mentioned, so that Bourgain's work well illustrates the goal of the "Lecture Notes" program: "new developments in mathematical research and teaching-quickly, informally and at a high level".

V. Totik (Szeged)

K. L. Chung, Lectures from Markov Processes to Brownian Motion (Grundlehren der mathematischen Wissenschaften, 249), VIII+239 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This book begins at the beginning with the Markov property, followed quickly by the introduction of optimal times and martingales. These three topics in the discrete parameter setting are fully discussed in an earlier book of the author: A Course in Probability Theory (Academic Press, 1974, second edition). The Course may be considered as a general background. But apart from the material on discrete parameter martingale theory cited in § 1.4, the book is self-contained.

Chapter 2 serves as an interregnum between the more concrete Feller processes and Hunt's axiomatic theory. Strong and moderate Markov properties of a Feller process are established with certain measurability properties. Chapter 3 contains the basic theory as formulated by Hunt, including hitting times, recurrent and transient Hunt processes and the characterization of the hitting (balayage) operator. Properties of the Brownian motion are discussed in Chapter 4; the treatment of Schrödinger's equation by the Feynman—Kac method is new. In the last chapter a number of notable results in classical potential theory are established by the methods developed in the earlier chapters.

Each chapter ends with a section of historical remarks and a number of regrettably omitted topics are mentioned. The book contains a lot of exercises as proper extensions of the text. Graduate students and professional mathematicians will benefit from the clear, uncluttered treatment emphasizing fundamental concepts and methods.

Lajos Horváth (Szeged)

Combinatorial Mathematics VIII. Proceedings of the Eighth Australian Conference on Combinatorial Mathematics, Held at Deakin University, Geelong, Australia, August 25—29, 1980, XIV+359 pages. Edited by Kevin L. McAveney (Lecture Notes in Mathematics, Vol. 884) Springer-Verlag, Berlin—Heidelberg—New York, 1981.

These conference proceedings contain two expository papers, nine invited papers and twenty contributed papers. A great part of papers investigate symmetric combinatorial structures (vertex-transitive graph, finite projective plane, two-distance set, latin square, design). Many of the authors belong to the Australian school of combinatorics. The titles of expository papers are: R. G. Stanton and R. C. Mullin, Some properties of H-designs; R. G. Stanton and H. C. Williams, Computation of some number-theoretic coverings. The titles of invited papers are : B. Alspach, The search for long paths and cycles in vertex-transitive graphs and digraphs; C. C. Chen and N. F. Quimpo, On strongly hamiltonian abelian group graphs; R. L. Graham, Wen-Ching Winnie Li and J. L. Paul, Monochromatic lines in partitions of $\mathbf{Z}^n$; J. S. Hwang, Complete stable marriages and systems of I—M preferences; P. Lorimer, The construction of finite projective planes; R. C. Read, A survey of graph generation techniques; J. J. Seidel, Graphs and two-distance sets; J. Sheehan, Finite Ramsey theory is hard; R. G. Stanton, Further results on covering integers of the form $1+k2^n$ by primes.

L. A. Székely (Szeged)

Combinatorics and Graph Theory. Proceedings of the Symposium Held at the Indian Statistical Institute, Calcutta, February 25—29, 1980, VII+500 pages. Edited by S. B. Rao (Lecture Notes in Mathematics, Vol. 885), Springer-Verlag, Berlin—Heidelberg—New York, 1981.

These proceedings consist of 9 invited papers and 36 contributed papers. Eight of them investigate the degree sequence of several graphs. One can find many papers concerning designs. association schemes and enumeration problems.

The list of invited addresses is: C. Berge, Diperfect graphs; P. Erdős, Some new problems and results in graph theory and other branches of combinatorial mathematics; E. V. Krishnamurty, A form invariant multivariable polynomial representation of graphs; L. Lovász and A. Schrijver, Some combinatorial applications of the new linear programming algorithm; K. Balasubramanian and K. R. Parthasarathy, In search of a complete invariant for graphs; D. K. Ray-Chaudhuri, Affine triple systems; F. C. Bussemaker, R. A. Mathon and J. J. Seidel, Tables of two-graphs; S. S. Shrikhande and N. M. Singhi, Designs, adjacency multigraphs and embeddings: a survey; G. A. Patwardhan and M. N. Vartak, On the adjungate of a symmetrical balanced incomplete block design with $\lambda=1$.

L. A. Székely (Szeged)

E. B. Dynkin, Markov Processes and Related Problems of Analysis. Selected Papers (London Mathematical Society Lecture Note Series 54), VI+312 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1982.

It is widely acknowledged that Professor Dynkin's work in the last two decades has given a new shape to the theory of Markov processes. And as the role and importance of this theory within the whole theory of stochastic processes and in various appled branches cannot really be overemphasized, this is not a small thing. According to his own preface, Dynkin's new approach to Markov processes, and especially to the Martin boundary theory and the theory of duality, has the three distinctive features that the general non-homogeneous theory precedes the homogeneous one, that all the theory is invariant with respect to time reversion, and that the regularity properties of a process are formulated not in topological terms but in terms of behaviour of certain real-valued functions along almost all paths. This collection contains nine influential papers by him. The first seven of these, from 1960, 1964, 1969, 1971, 1972, 1973 and 1975, were originally published in the *Uspehi Matematicheskih Nauk* and translated into English in the *Russian Mathematical Surveys,* the eighth is his 1978 *Annals of Probability* paper, and the last one appeared in the *Transactions of the American Mathematical Society* in 1980. The author has revised the entire text of the English translations and corrected a few slips in the originals. Workers in Markov processes will find it very useful to have these papers in one volume.

Sándor Csörgő (Szeged)

P. D. T. A. Elliott, Probabilistic Number Theory I, Mean-Value Theorems, **II,** Central Limit Theorems (Grundlehren der mathematischen Wissenschaften 239, 240), XXXIII+359 pages, XXXIV+341 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979, 1980.

This monograph gives an excellent introduction to probabilistic number theory and summarizes its fundamental results. The first volume begins with some necessary results from measure theory and the theory of probability. The greatest part of these theorems are proved in the first chapter and the proofs of the remaining theorems can be found in every monograph on probability and measure theory.

After a discussion on the Selberg sieve method and the forms of the prime number theorem the author studies certain finite probability spaces, paying particular attention to a model of Kubilius which plays a crucial role in some later chapters. Chapter 4 contains the Turán—Kubilius inequality and its dual, and their connection with the inequality of the large sieve. New proofs of the classical Erdős and Erdős—Wintner theorems on the distribution of the values of additive arithmetic functions are presented. The first volume ends with the Halász method and the study of multiplicative arithmetic functions.

In the second volume the author studies the value distribution of arithmetic functions, allowing unbounded renormalisations. The methods involve a synthesis of probability and number theory, sums of independent random variables playing an important role. In particular, he investigates to what extent one can simulate the behaviour of additive arithmetic functions by that of suitably defined independent random variables. Subsequent methods involve both Fourier analysis on the line and the application of Dirichlet series.

Many additional topics are considered, a problem of Hardy and Ramanujan, local properties of additive arithmetic functions, the rate of convergence to the normal law and the arithmetic simulation of all stable laws. A number of conjectures is formulated in Chapter 17 and a list of unsolved problems is given in Chapter 23. The historical background of various results is discussed, forming an integral part of the text. The reader gets acquainted with further results on each topic and the references cover broad parts of the literature.

These very nice books may be recommended for everybody who is interested in probabilistic number theory. A graduate course may be based on a selection of results from the first volume.

Lajos Horváth (Szeged)

T. M. Flett, Differential Analysis, VIII+359 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1980.

The book is concerned with the differential calculus of functions taking values in normed spaces.

In the first chapter such basic results on functions of one variable are treated as the mean value theorems, the increment inequality and monotonicity theorems. The second chapter is a good survey on the modern existence, uniqueness and continuation results in the theory of differential equations and inequilities even for infinite-dimensional systems. The third chapter deals with the Fréchet differential, which forms the basis of the calculus of functions of a vector variable. There are two other types of differentials for this purpose. The Gâteaux (or directional) differential is commonly discussed in the literature for this case. In the last chapter the author gives a detailed account also on the Hadamard differential which can be required for certain considerations in tangent spaces and in aspects of the theory of differential equations (for example, "differentiation along the curve").

A large part of the book is devoted to applications. Besides ordinary differential equations and inequalities, the author studies extremum problems for functions of a vector variable, Ljapunov stability, geometry of tangents, the Newton—Kantorovich method, etc. A great number of examples and exercises can be found in the book.

The chapters are concluded by very interesting long historical notes. For example, at the end of the first chapter the reader finds the adventurous history of the mean value and monotonicity theorems with the original proofs and methods. (It is interesting that Ampère, a pupil of Lagrange who later achieved fame for his researches in electricity and magnetism, attempted to show that every real-valued function has a derivative everywhere, and Chauchy's treatment of the mean value theorem stemmed from this paper.)

This nicely presented book is not only a very good monograph but also an excellent textbook in advanced calculus.

L. Hatvani (Szeged)

Functional Analysis in Markov Processes. Proceedings of the International Workshop held at Katata, Japan, August 21—26, 1981, and of the International Conference held at Kyoto, Japan, August 27—29, 1981. Edited by M. Fukushima (Lecture Notes in Mathematics, 923), V+307 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This volume comprises 15 original research papers based on lectures given at the above joint meetings. The first three longer papers (S. Kusuoka, Analytic functionals of Wiener process and absolute continuity; Y. Le Jan, Dual markovian semigroups and processes; M. Tomisaki, Dirichlet forms associated with direct product diffusion processes) are based on the main three-hour lectures and represent the main features of the meetings. The authors of the 12 shorter papers are Albeverio and Høegh—Krohn (2 papers), Fukushima, Getoor and Sharpe, Guang Lu and Minping, Gundy and Silverstein, Itô, Kanda, Kotani and S. Watanabe, Oshima, Pitman and Yor, and Stroock. Some of these papers apply the functional analytic theory of Markov processes to various branches of physics.

Sándor Csörgő (Szeged)

Azriel Levy, Basic Set Theory (Perspectives in Mathematical Logic), XIV+391 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

The value of a good textbook on a subject can hardly be overestimated. I consider Levy's book such a good elementary-advanced work on a discipline which has just entered into the maturity age. The goal is to present basic set theory but the material is fairly up to date, several results from the late seventies are also incorporated. Many routine proofs are left to the reader but this only increases the legibility of the book. Also, a lot of exercises are presented — these range from almost trivial ones to advanced problems.

Basic Set Theory consists of two independent parts. The first one is devoted to the development of pure set theory. Here the material is common with many other books; the framework is with von Neumann's classes. Constructibility and forcing is excluded but the last chapter deals with the versions of the axiom of choice.

The second part is best named as selected topics. Although only a few topics are selected — the elements of description set theory; Boolean algebras and Martin's axiom and some infinite combinatorics — these enable the reader to get a sight of current research let alone their usefulness and applicability in other mathematical disciplines.

I recommend Levy's book both to lecturers on set theory and to students who may get acquainted with this challenging field through this well-written work.

V. Totik (Szeged)

Logic Symposia. Proceedings, Hakone 1979, 1980, Edited by G. H. Müller, G. Takeuti and T. Tugué (Lecture Notes in Mathematics, 891), XI+934 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

Two symposia on the foundations of mathematics were held at Gōra, Hakone, in Japan, on March 21—24, 1979 and February 4—7, 1980 mainly with Japanese participants. This book contains 15 papers read at these symposia. M. Hanazawa writes about Aronszajn trees, S. Hayashi about set theories in toposes, K. Hirose and F. Nakayasu about Spector second order classes, Y. Kakuda about precipitousness of ideals, T. Kawai about axiom systems of nonstandard set theory, S. Meahara about transfinite induction in an initial segment of Cantor's second number class, T. Miytake about proofs in recursive arithmetic, N. Motohashi about definability theorems, K. Namba about Boolean valued combinatorics, H. Ono and A. Nakamura about the connection of the undecidability of certain extensions with finite automata, I. Shinoda about sections and envelopes of type 2 objects, G. Takeuti and S. Titani about Heyting valued universes of intuitionistic set theory, S. Tugue and H. Nomoto about the independence of an elementary analysis problem, T. Uesu about intuitionistic theories and, finally, M. Yasugi writes about the Hahn—Banach extension theorem. Let us record here the result of S. Tugue and H. Nomoto: There are sets $A\subseteq\mathbf{R}$ for which the statement "For any sequence $\{a_k\}$ of real numbers, if $\lim_{k\to\infty} e^{2\pi i a_k t}=1$ for every $t\in A$, then $\{a_k\}$ converges to 0" is independent of the axioms of *ZFC*.

V. Totik (Szeged)

Péter Major, Multiple Wiener—Itô Integrals. With applications to limit theorems (Lecture Notes in Mathematics, 894), VII+127 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

This monograph is about some recent and very deep results on the asymptotic behaviour of partial sums of discrete and generalised, "really" dependent random fields suggested by some important problems in statistical physics and in the theory of infinite particle systems. A modified version of multiple Wiener—Itô integrals, a notion originally designed for the study of nonlinear functionals over Gaussian fields, have proved to be a useful tool in the investigation of this renormalisation limit problem of random fields. Almost all results and proofs in this area are related to these integrals. This fact is what necessitated this clearly and elegantly written monograph. The section headings are: 1. On a limit problem, 2. Wick polinomials, 3. Random spectral measures, 4. Multiple Wiener—Itô integrals, 5. The proof of Itô's formula. The diagram formula and some of its consequences, 6. Subordinated fields. Construction of self-similar fields, 7. On the original Wiener—Itô integral, 8. Non-central limit theorems. The last, ninth section gives the history of the problems and poses a number of unsolved problems.

The volume is a self-contained exposition and is indispensable for anyone interested in the above problems and generally in self-similar processes.

Sándor Csörgő (Szeged)

Mathematics Tomorrow, Edited by L. A. Steen, 250 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

Three years ago the Joint Projects Committee on Mathematics and the Conference Board of Mathematical Sciences (USA) prepared a volume of essays: Mathematics Today: Twelwe Informal Essays. Mathematics Tomorrow continues the theme of Mathematics Today. It is written by individuals and it contains opinions and predictions about the direction that mathematics — research and education — should take in the future.

Mathematics Tomorrow is divided into four parts: What is Mathematics?; Teaching and Learning Mathematics; Issues of Equality and Mathematics for Tomorrow. No doubt, the most exciting section is the first one which tries to determine the relationship between "pure" and "applied" mathematics and the effect of this on mathematics teaching. Here some authors argue for radical reform, others express their concern because of the pragmatic trend in recent projects. Let us present here some valuable opinions:

"Pure mathematics can be practically useful and applied mathematics can be artistically elegant" (P. Halmos).

"Applied mathematics cannot get along without pure, as an anteater cannot get along without ants, but not necessarily the reverse" (P. Halmos).

"If the habit of understanding is lost at an elementary level, or never learned, it will not reappear when the problems become more complicated" (T. Poston).

"The new core for the mathematics major might consist of only one year calculus, one semester of linear algebra, and one semester of real analysis. ... we should even consider the extreme case that the new core might be the empty set" (W. F. Lucas).

"...if the overall enrollment decline in higher education reaches a point of serious cuts in departmental sizes, then many other groups will decide that they too can teach their own mathematics series courses. ... On the other hand many golden opportunities still exist ... The choice is up to the mathematics cummunity, but it must act quickly and in a meaningful way" (W. F. Lucas).

"The applications enthusiasts hold all the cards. They have behind them the power and influence of the natural organizations and commissions. They are reshaping the mathematics curriculum in their own image ... But I ask for a favor. Let one course, just one, remain pure ... And one day when the wind is right I'll do the Cauchy Integral Formula for the last time ... and the students will see the curve and the thing inside and the lazy integral that makes the function value appear as suddenly as my palm when I open my hand. They will see pure mathematics ... And we owe Paul Halmos a chance to see that some mathematics students know that his subject exists." (J. P. King).

V. Totik (Szeged)

N. H. McClamroch, State Models of Dynamic Systems, VIII+248 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

It often happens that results of pure mathematics do not come to applications even in such cases when they are undoubtedly applicable in general. This may be caused by the difficulties of synthetizing the physical argumentation and mathematical formalism. The first step toward this synthesis is finding an appropriate mathematical model for the dynamical system investigated. The model gives a bridge between the reality and the mathematical theory. Modeling is a very complicated phase of the investigation: it requires the knowledge not only of means of mathematics but also the proper theory of the object to be modelled. The book gives an excellent glance into this art. Its sub-title really characterizes its style: it is "a case study approach". At the beginning of every chapter there is a short abstract of the necessary concepts and results from system theory, which is followed by special cases of a very wide spectrum: temperature in a building, electrical circuits, DC motor, vertical ascent of a deep sea diver, automobile suspension system, magnetic loudspeaker, liquid level in a leaky tank, spread of an epidemic, continuous flow stirred tank chemical reactor, motion of a rocket near Earth, etc. The models are classified according to two independent points of view: linear-nonlinear and first order — higher order models are distinguished, respectively.

In many cases a detailed mathematical analysis is not possible for some reason or other. Then a computer simulation, based on the state equations may be required. To make it easier the book presents a special purpose simulation language (Continuous System Modeling Program).

We can recommend this excellent book for undergraduate students, users of mathematics, and for everybody interested in applications of mathematics.

L. Hatvani (Szeged)

Padé Approximation and its Applications. Proceedings, Amsterdam 1980, Edited by M. G. de Bruin and H. van Rossum, (Lecture Notes in Mathematics 888), VI+382 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

The book contains 29 papers delivered at the conference on "Padé and Rational Approximation, Theory and Applications" held at the Institute voor Propedeutische Wiskunde of the University of Amsterdam, October 29—31, 1980. This conference was the sixth in a series of conferences on the above subject in Europe (Canterbury 1972, Toulon 1975, Lille 1977 and 1978, Antwerp 1979), which well illustrates the interest in Padé approximation and related subjects.

The four invited lectures were: C. Brezinski: The long history of continued fractions and Padé approximants; P. R. Graves—Morris: Efficient reliable rational approximation; H. Werner: Nonlinear splines, some applications to singular problems; and L. Wuytack: The conditioning of the Padé approximation problem.

V. Totik (Szeged)

N. U. Prabhu, Stochastic Storage Processes. Queues, Insurance Risk, and Dams (Applications of Mathematics, 15), VI+140 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

The main classes of stochastic models investigated in this book are queueing, insurance risk and dams. The stochastic processes underlying these models are usually (but not always) Markovian, in particular, random walks and Lévy processes. In order to answer important questions concerning these models we have to study various aspects of these processes such as the maximum and minimum functionals and hitting times.

The Introduction contains the definitions of single-server queueing systems, inventory models, storage models, insurance risk and continuous time inventory and storage models. The book is in two parts. In part I the author presents the theory of single-server queues with the first come, first served discipline. This part is based on the close connection between random walks and queueing problems. In the first three sections of Chapter 1, the author proves some important theorems on ladder processes, renewal functions and the maximum and minimum of random walks. The results described here provide answers to most of the important questions concerning this general system, but in special cases of Poisson arrivals or exponential service time, or systems with priority queue disciplines, there still remain some questions. These latter are more appropriately formulated within the framework of continuous time storage models, which is developed in part II.

In part II the author considers a model in which the input is a Lévy process and the output is continuous and is at a unit rate except when the store is empty. In spite of its simplicity, the concepts underlying this model and techniques used in its analysis are applicable in a wide variety of situations, for example, in insurance risk and queueing systems with first come, first served discipline, or priority disciplines of the static or dynamic type.

The book is clearly written, supplied with exercises at the end of sections, but the references are old and not enough to orient in recent developments in stochastic storage processes. It is recommended for every graduate student who has a background in elementary probability theory and wishes to begin studies in this part of applied probability.

Lajos Horváth (Szeged)

Recent Results in Stochastic Programming. Proceedings, Oberwolfach 1979, Edited by P. Kall and A. Prékopa, Lecture Notes in Economics and Mathematical Systems, VI+237 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

This volume contains the papers presented at a meeting on stochastic programming, held at Oberwolfach, January 28—February 3, 1979. It is divided into two parts.

The first, theoretical part consists of the papers by Bereanu, Bol, Brosowski, Groenewegen and Wessels, Heilmann, Haneveld and Rinott about the topics: (stochastic-) parametric programs, multi-stage SLP, minimax rules for SLP, convexity problems, etc.

The second part is devoted to the applications concerning water resources, portfolio selection, asphalt mixing, network planning, etc. Here the authors are: Deák, Dupacová, Kall, Kallberg and Ziemba, Karreman, Kelle, Marti, Prékopa.

An important note from the preface of the volume: "during the last two decades knowledge, theoretical and computational, on stochastic programming, and practical experience with it, have been developed so far, that neglecting a priori the stochastic nature of parameters ... can no longer be justified."

V. Totik (Szeged)

Robert B. Reisel, Elementary Theory of Metric Spaces (A Course in Constructing Mathematical Proofs), 120 pages, Universitext, Springer-Verlag New York, Inc., 1982.

It is only the second goal of this book to teach the elementary theory of metric spaces, the first goal is to teach an understanding of proofs and their constructions on an appropriate field of mathematics.

The book is offered to junior students having a level of mathematical maturity, e.g. a complete course in calculus. The author requires a lot of self-contained work from his students: he gives all the definitions (including the set-theoretical ones), shows examples, states theorems and leaves proofs to the reader. A number of hints is given and the more difficult proofs can be found in an appendix. He says "I think that the best way to use this book is in a seminar ... it could, however, be used in a lecture course where many of the proofs would be assigned to the students. It would be suitable as the text or a supplementary text in courses in general topology, real analysis or advanced calculus." A solitary student studying this book needs a teacher who criticizes his proofs.

The material of the book worked well in the author's seminar at Loyola University of Chicago in the past fifteen years.

L. A. Székely (Szeged)

S. H. Saperstone, Semidynamical Systems in Infinite Dimensional Spaces (Applied Mathematical Sciences, 37), VII+474 pages, Spriger-Verlag, New York—Heidelberg—Berlin, 1981.

Since 1927, when G. D. Birkhoff published his classical monograph on dynamical systems the so-called topological dynamics has given a framework for the qualitative theory of solutions of differencial equations of certain types. Nowadays many attempts have been made to extend results of this theory to new types of equations.

As is known a dynamical system is formed by a group of transformations of a Hausdorff topological space into itself. The family of solutions of an autonomous system of differential equations forms such a group of transformation, but the more general systems (nonautonomous ones, functional-differential equations etc.) do not have this property. But some representations of solutions can be embedded into an appropriate function space where they generally form no longer a group but only a semigroup, in other words, a semidynamical system. Typically the space is not locally compact. The author makes it clear which properties of dynamical systems can be generalized to semidynamical ones, and what special kinds of properties the semidynamical systems have.

Titles of the chapters describe well the topics involved: I. Basic definitions and properties, II. Invariance, limit sets and stability, III. Motions in metric space, IV. Nonautonomous ordinary differential equations, V. Semidynamical systems in Banach space, VI. Functional differential equations, VII. Stochastic dynamical systems, VIII. Weak semidynamical systems and processes.

Each chapter is followed by exercises, notes and comments, and an extensive bibliography. Most of the source material is from the 1960's and 1970's and was previously available only in journals.

This book will be very useful for both mathematicians and users of mathemtics interested in the qualitative theory of differential equations and its applications.

L. Hatvani (Szeged)

Vjačeslav V. Sazonov, Normal Approximation — Some Recent Advances (Lecture Notes in Mathematics, 879), VII+105 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

Although there are many central themes in modern probabilistic research, the historically first such theme, the central limit theorem, will undoubtedly always remain one. These notes, based on a series of lectures the author gave in 1979 at the University of California, Los Angeles, and at Moscow State University, are devoted to the study of the rate of convergence in the central limit theorem for independent and identically distributed random elements in finite dimensional Euclidean spaces and in real separable Hilbert spaces. The aim of the monograph is to outline the main directions and methods in the recent progress of the field. There are basically two main methods here. Quite naturally the author has chosen to emphasize the direct method of convolutions, initiated by Bergström and developed by the author himself, rather than the method of characteristic functions.

Sándor Csörgő (Szeged)

Séminaire de Probalilités XV, 1979/80. Avec table générale des exposés de 1966/67 à 1978/79, IV+704 pages;

Séminaire de Probabilités XVI, 1980/81, V+622 pages;

Séminaire de Probabilités XVI, 1980/81. **Supplément: Géométrie Differentielle Stochastique,** III+285 pages;

Edité par J. Azéma et M. Yor (Lecture Notes in Mathematics, 850, 920, 921), Springer-Verlag, Berlin—Heidelberg—New York, 1981, 1982, 1982.

These three volumes of the traditional seminar notes, centred originally in Strasbourg and now, beginning with the volume XIV, in Paris, show the French school of probability in its best again. The main theme is of course the traditional "general" theory of stochastic processes, but there are many other topics dealt with. It would be impossible to list these here since there are altogether 109 papers in the three volumes (27 in English), but almost everybody working in stochastic processes will find at least one indispensable for him. Even so, the martingale approach to some Wiener—Hopf problems in a two-part longer paper by London, McKean, Rogers and Williams in volume XVI deserves special mention. It is a great help for the readers of this series, and especially for those who do not travel often enough to France, that volume XV contains a complete list of contents of the first fourteen volumes with editorial notes on the correction, rectification, extension, or improvements of many papers in subsequent volumes. The supplement volume on stochastic differential geometry, containing six longer papers by Schwartz, Meyer, Emery, Darling and Azencott, appears to be very important. It provides "the present state of art" of a new and vigorously developing branch of stochastics.

Sándor Csörgő (Szeged)

E. B. Dynkin—A. A. Yushkevich, Controlled Markov Processes (Grundlehren der mathematischen Wissenschaften, 235), XVII+289 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

Optimal control of a stochastic process means the optimal choice of some free parameters influencing the future evolution of the underlying random process, in order to achieve optimal dynamics (minimal costs or maximal rewards). The selection of the control parameters is made on the basis of observations of the past of the process. Evidently this model describes the situation in many engineering and economic problems. Hence stochastic control is by right considered as an applied mathematical discipline.

The original Russian version of the present book was written in the time (1975) when the authors, both of them outstanding mathematicians, worked with the Central Institute of Mathematical Economics in Moscow. They accomplished a work, very rare in the literature, which is equally exciting for mathematicians and specialists motivated by economics. The volume actually deals with the optimal control of discrete-time Markov chains, or, in other terminology, with multi-stage Markov decision problems. In this respect the title is somewhat misleading as the theory of continuous-time processes, where entirely different mathematical problems arise, is not included. Neither are considered computational aspects of the determination of the optimal strategy.

Presenting the material the authors approach step by step from simpler to more advanced problems, always keeping an eye on applications. This way the reader never looses contact with practice, and the necessity of each step towards increasing abstraction is sufficiently motivated. As prerequisites only basic probability and measure theory are required.

Summing up, the present book can be recommended both to mathematicians wishing to cultivate applied probability and to economists intending to solve their problems by mathematical methods. The level of applied mathematical literature would considerably increase if everyone cast a glance at Dynkin and Yuskevich's presentation before sitting down to write his own monograph.

D. Vermes (Szeged)

N. V. Krylov, Controlled Diffusion Processes (Applications of Mathematics, 14), XII+308 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

What can hope a mathematician moving from discrete models to continuous ones? Gain on conformity with reality? More aesthetic simplified theory? Challenging new difficulties? The step between optimal control theory of discrete-time random processes and their continuous-time analogues can offer all the three sorts of rewards. The author seems to prefer confrontation of difficulties to conformity and simplicity. For him the theory of continuous-time stochastic processes is a source of hard enough mathematical problems most of which he masters triumphally.

The general object of stochastic control theory was outlined in the previous review. In order to understand the particularities of continuous-time control problems, the origins and the practical motivations of their theory and to estimate correctly the arising difficulties, the non-specialist should first consult the excellent introduction: W. H. Fleming and R. W. Rishel, Deterministic and Stochastic Optimal Control (Springer-Verlag 1975). Together with the monograph under review, the two books constitute indeed an excellent, high-level presentation of the control theory of continuous processes.

Roughly speaking the determination of the optimal strategy for a controlled diffusion process is equivalent to the solution of a possibly degenerated non-linear parabolic PDE with non-continuous right-hand side — the so-called Bellman equation. But in what sense does this equation have solutions and in what class of functions is there a unique solution? These are the central questions of the theory and once the author answers them, he arrives at a unified theory of classical calculus of variations and of the control of continuous deterministic and stochastic processes. But the technical machinery yielding this theory is not less significant. Some techniques of estimation will perhaps find as much applications in mathematics itself as the whole theory will do in practice.

The style of the book resembles an extended research article on a vividly developing field rather than an explanation of an applications-oriented mathematical theory. In order to make the author's important results available to English reading specialists in a shortest possible time, Springer-Verlag translated the Russian original without any change. (Not even bibliographical hints to meanwhile published proofs of stated results were included.)

The chapter-headings are: 1. Introduction; 2. Auxiliary propositions; 3. General properties of a playoff function; 4. The Bellman equation; 5. The construction of ε-optimal strategies 6. Controlled processes with unbounded coefficients; Appendices, Notes, Bibliography, Index. As "auxiliary propositions" the author presents L_p estimates for stochastic integrals, existence of diffusions with measurable coefficients, Markov property and parameter dependence of solutions of stochastic equations, Itô's formula in Sobolev spaces.

D. Vermes (Szeged)

I. I. Gihman—A. V. Skorohod, Controlled Stochastic Processes, VII+237 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979. (Translated from the Russian by S. Kotz.)

Is it necessary to recommend to readers the most recent monograph of the well-known authors of 12 previous volumes covering most of the theory of stochastic processes, initiators of several important mathematical theories? The 13th of their books shows once again the brilliant talent of its authors as, seemingly without any serious reading of the existing literature on stochastic control theory, they arrive with a single blow at the proximity of the results achieved in the last decade on this rapidly developing field. Specialists, familiar with the alternative ways leading to the same results, will find interesting the authors' approach via weak approximation which has several advantages as pointed out in H. J. Kushner, Probability Methods for Approximation in Stochastic Control and for Elliptic Equations (Academic Press, 1977).

As prerequisites, rudiments of measure theory and functional analysis and a knowledge of the theory of stochastic processes are supposed, the latter at the approximate level of the three-volume treatise by the same authors. Besides specialists of stochastic control theory the book will turn out to be useful to any mathematician learning Russian. As the English translation preserves the typically Russian structures and English terms are chosen much nearer to their Russian originals than the commonly used English terminology, the text suits ideally to translation exercises (from English to Russian).

D. Vermes (Szeged)

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