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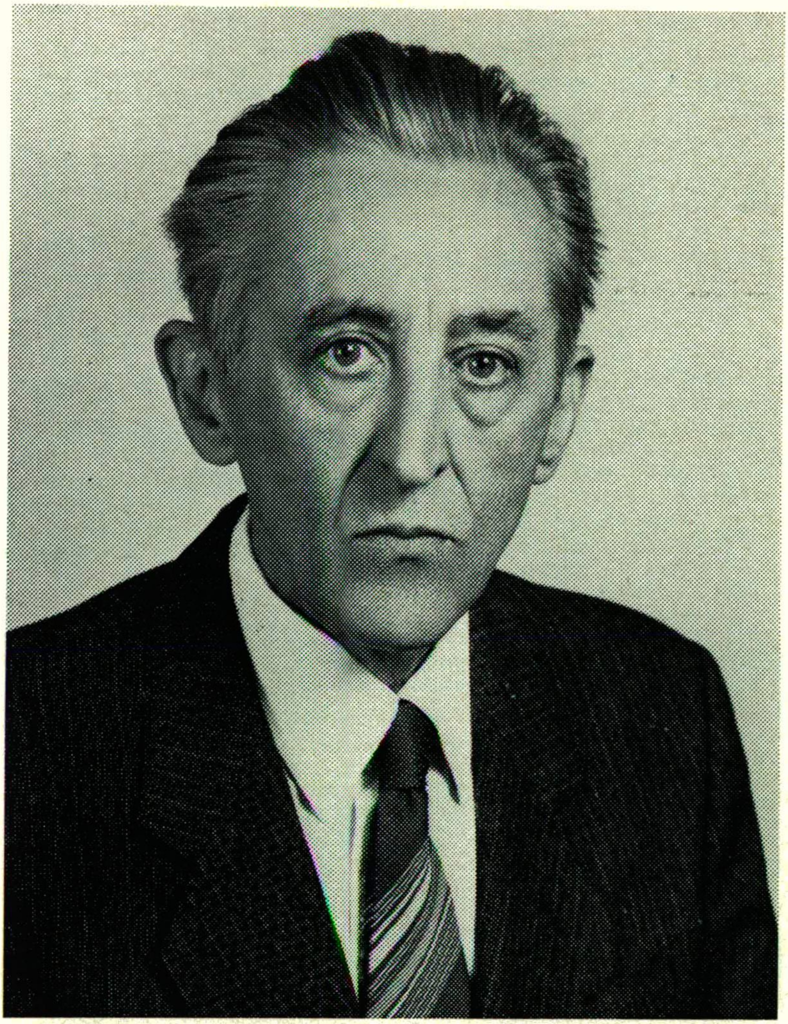

TANDORI KÁROLY

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On the convergence of eigenfunction expansions in H^s-norm

Š. A. ALIMOV and I. JOÓ

Dedicated to Professor K. Tandori on the occasion of his 60th birthday

1. Let $S_k \subset \mathbf{R}^n$ ($n \geqq 3$; $k=1, \dots, l$) be submanifolds of dimension $\dim S_k = m_k \leqq n-3$ having smooth projection to R^{m_k}, i.e. there exist coordinates $(\xi, y) = (\xi_1, \dots, \xi_{m_k}; y_1, \dots, y_{n-m_k}) \in \mathbf{R}^n$ and functions $\varphi_j^k \in C^1(R^{m_k} \to R^{n-m_k})$ such that

$$S_k = \{(\xi, y) \in R^n\colon\ y_j = \varphi_j^k(\xi),\ |\nabla \varphi_j^k(\xi)| \leqq C_j^k\}$$

and

$$S = \bigcup_{k=1}^{l} S_k.$$

Let $q \in C^\infty(\mathbf{R}^n \setminus S)$ be such a real valued function for which

$$|q(x)| \leqq c/\mathrm{dist}\,(x, S)$$

is fulfilled. Consider the Schrödinger operator $L_0 = -\Delta + q(x)\cdot$ with domain $\mathcal{D}(L_0) = C_0^\infty(\mathbf{R}^n)$. Such operators occur as the Hamiltonian of many body problem (cf. [7, XI]). E.g. in the case of two particles we have $H = -\Delta + q\,\cdot$, $\mathcal{D}(H) = C_0^\infty(\mathbf{R}^n)$, $n=6$, $m=3$, $q(x, y) = \dfrac{c_1}{|x|} + \dfrac{c_2}{|y|} + \dfrac{c_3}{|x-y|}$; $x \in \mathbf{R}^3$, $y \in \mathbf{R}^3$. In the case of homogeneous and izotropic spaces the manifolds S_k are subspaces in $\mathbf{R}^n$.

It is easy to see that the assumptions $m_k \leqq n-3$ implies $q(x) \in L_2^{\mathrm{loc}}(\mathbf{R}^n)$. Indeed; it is enough to prove this for $S_k = S$, $\dim S = m \leqq n-3$,

$$S = \{(\xi, y) \in R^n\colon\ y_j = \varphi_j(\xi);\ |\nabla \varphi_j(\xi)| \leqq C_j;\quad j = 1, 2, \dots, n-m\}.$$

Using the coordinata-transformation $(\xi, y) \to (\xi, z)$; $z_j = y_j - \varphi_j(\xi)$ we have for the Jacobian $D(\xi, z)/D(\xi, y) = 1$ and for any $0 \leqq \eta \in C_0^\infty(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} |q(x)|^2 \eta(x)\, dx = \int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} |q(\xi, z+\varphi(\xi))|^2 \eta(\xi, z+\varphi(\xi))\, dx,$$

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where

$$\varphi = (\varphi_1, \ldots, \varphi_{n-m}) \in C^1(\mathbf{R}^m \to \mathbf{R}^{n-m}).$$

On the other hand, for any $x=(\xi, y)\in \mathbf{R}^n$ and $u=(\tilde{\xi}, \varphi(\tilde{\xi}))\in S$, $|y-\varphi(\xi)| \leqq$ $\leqq |y-\varphi(\tilde{\xi})|+|\varphi(\tilde{\xi})-\varphi(\xi)| \leqq |y-\varphi(\tilde{\xi})|+|\nabla\varphi(\xi^*)|\cdot|\xi-\tilde{\xi}| \leqq c(|y-\varphi(\tilde{\xi})|+|\xi-\tilde{\xi}|)$. Hence $|y-\varphi(\xi)|^2 \leqq 2c^2(|y-\varphi(\tilde{\xi})|^2+|\tilde{\xi}-\xi|^2) \leqq 2c^2|x-u|^2$, i.e. $|y-\varphi(\xi)| \leqq c\,\mathrm{dist}\,(x, S)$, consequently

$$|q(\xi, z+\varphi(\xi))| \leqq c/\mathrm{dist}\{(\xi, z+\varphi(\xi)), S\} \leqq$$

$$\leqq \frac{c}{|z+\varphi(\xi)-\varphi(\xi)|} = \frac{c}{|z|}.$$

We have

$$\int_{\mathbf{R}^n} |q(x)|^2\eta(x)\,dx = \int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} |q(\xi, z+\varphi(\xi))|^2\eta(\xi, z+\varphi(\xi))\,dz \leqq$$

$$\leqq c\int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} \frac{1}{|z|^2}\eta(\xi, z+\varphi(\xi))\,dz < \infty.$$

It follows from the Lemma 1 of the present work that the operator L_0 is bounded below, i.e. for any $f\in C_0^\infty(\mathbf{R}^n)$

$$(L_0 f, f) = (-\Delta f, f)+(qf, f) = (\nabla f, \nabla f)+(qf, f) \geqq -c(f, f), \quad (c>0),$$

and hence by K. O. Friedrichs' theorem [6] we obtain: L_0 has selfadjoint extension L further $L \geqq -cI (c>0)$. Denote $L=\int_0^\infty \lambda dE_\lambda$ the spectral expansion of L and for any $f\in L_2(\mathbf{R}^n)$ consider the expansion $E_\lambda f$.

In [8] is proved: for any $f\in H^s(\mathbf{R}^n)$ $(0\leqq s\leqq 1)$ $\|E_\lambda f-f\|_{H^s}\to 0$ as $\lambda\to\infty$. H^s denotes the space of functions from $L_2(\mathbf{R}^n)$ with the norm

$$\|f\|_{H^s} \overset{\text{def}}{=} \|(I-\Delta)^{s/2}f\|_{L_2} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L_2} \quad ([9], 2.3.3).$$

The aim of the present note is to prove the following

Theorem 1. *For any* $f\in H^s(\mathbf{R}^n)$ $(0\leqq s\leqq 2)$ *we have*

$$\|E_\lambda f-f\|_{H^s} \to 0 \quad \text{as} \quad \lambda\to\infty. \tag{1}$$

A theory of general orthogonal series was developped by K. Tandori in the last twenty years (Cf. e.g. [2—4]). At the same time IL'IN [5] found a new theory of spectral expansions which is a special case of the general orthogonal expansions, further proved in [5] the Theorem 1 in the specail case when $q\equiv 0$ and s is integer.

From Lemma 2 below it follows, among others, by the well known Kato—Rellich theorem [7, X.2] that the operator L_0 considered is essentially selfadjoint, further $\mathscr{D}(\bar{L}_0)=\mathscr{D}(L)=H^2$.

2. For the proof of the Theorem 1 we need some lemmas. Define

$$\varrho(x) = [\mathrm{dist}\,(x, S)]^{-1}.$$

Lemma 1. *We have for any* $f \in H^1$

$$\int_{\mathbf{R}^n} \varrho(x)\,|f(x)|^2\,dx \leqq c\|f\|_{L_2}\cdot\|f\|_{H^1}. \tag{2}$$

Here and below c denotes a constant, which is independent from f and not necessarily the same in each occurences.

Proof. Using polar coordinates and the identity

$$h(r) = -2\int_r^\infty h(t)h'(t)\,dt$$

we obtain

$$\int_{\mathbf{R}^{n-m}} \frac{|g|(y)^2}{|y|}\,dy = \int_\theta \int_0^\infty r^{n-m-2}|g(r,\theta)|^2\,dr\,d\theta = -2\int_\theta\int_0^\infty \left[\int_r^\infty g(t,\theta)\frac{\partial g(t)\theta}{\partial t}\,dt\right]\times$$

$$\times r^{n-m-2}\,dr\,d\theta = \frac{2}{n-m-1}\int_\theta\int_0^\infty g(r,\theta)\frac{\partial g(r,\theta)}{\partial r}\,r^{n-m-1}\,dr\,d\theta$$

whence for $g(y)=g(\xi,y)=f(\xi,y+\varphi(\xi))$ we obtain

$$\int_{\mathbf{R}^n} \varrho(x)\,|f(x)|^2\,dx \leqq c\int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} \frac{|f(\xi, y+\varphi(\xi))|^2}{|y|}\,dy \leqq$$

$$\leqq c\int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} |f(\xi, y+\varphi(\xi))|\cdot|\nabla_y f(\xi, y+\varphi(\xi))|\,dy =$$

$$= c\int_{\mathbf{R}^n} |f(x)|\,|\nabla_y f(x)|\,dx \leqq c\|f\|_{L_2}\cdot\|f\|_{H^1}.$$

Lemma 1 is proved.

Lemma 2. *We have for any* $f \in H^1$

$$\int_{\mathbf{R}^n} \varrho^2(x)\,|f(x)|^2\,dx \leqq c\|f\|^2_{H^1}. \tag{3}$$

Proof. Using polar coordinates and the notation

$$I = I(\xi,\theta) = \int_0^\infty r^{n-m-3}|f(\xi, r, \theta)|^2\,dr \quad (\xi \in \mathbf{R})$$

we obtain

$$I \leqq 2 \int_0^\infty r^{n-m-3} \int_r^\infty \left| f(\xi, t, \theta) \frac{\partial f(\xi, t, \theta)}{\partial t} \right| dt\, dr =$$

$$= 2 \int_0^\infty \left(\left| f(\xi, t, \theta) \frac{\partial f(\xi, t, \theta)}{\partial t} \right| \int_0^t r^{n-m-3}\, dr \right) dt =$$

$$= c \int_0^\infty \left| f(\xi, t, \theta) \frac{\partial f(\xi, t, \theta)}{\partial t} \right| t^{n-m-2}\, dt \leqq$$

$$\leqq c \left(\int_0^\infty t^{n-m-3} |f(\xi, t, \theta)|^2\, dt \right)^{1/2} \left(\int_0^\infty \left| \frac{\partial f(\xi, t, \theta)}{\partial t} \right|^2 t^{n-m-1}\, dt \right)^{1/2},$$

hence

$$I \leqq c \int_0^\infty \left| \frac{\partial f(\xi, t, \theta)}{\partial t} \right|^2 t^{n-m-1}\, dt \leqq c \|\nabla_y f\|^2_{L_2(\mathbf{R}^{n-m})},$$

i.e.

$$\int_{\mathbf{R}^m} \int_\theta I\, d\theta\, d\xi \leqq c \|\nabla f\|^2_{L_2(\mathbf{R}^n)} \leqq c \|f\|_{H^1}.$$

Lemma 2 is proved.

Corollary. *The operator* L_0 *with* $D(L_0) = C_0^\infty(\mathbf{R}^n)$ *is essentially selfadjoint and* $D(\bar{L}_0) = D(L) = H^2$.

Proof. It follows from Lemma 2 and (8) (below)

$$\|qf\|_{L_2} \leqq c \|f\|_{H^1} \leqq \varepsilon \|f\|_{H^2} + c(\varepsilon) \|f\|_{L_2} = \varepsilon \|(I - \nabla) f\|_{L_2} + c(\varepsilon) \|f\|_{L_2}.$$

Because $I - \Delta$ is essentially selfadjoint and $D(\overline{I - \Delta}) = H^2$, the Corollary follows by Kato—Rellich theorem ([7], X.2.).

Lemma 3. *For any* $f \in H^2$

$$\|Lf\|_{L_2} \leqq c \|f\|_{H^2}. \tag{4}$$

Proof. Using Lemma 1 we have for any $f \in H^2$

$$\|Lf\|_{L_2} = \|-\Delta f + qf\|_{L_2} \leqq \|\Delta f\|_{L_2} + \|qf\|_{L_2} \leqq \|f\|_{H^2} + c\|f\|_{H^1} \leqq c \|f\|_{H^2}.$$

Lemma 3 is proved.

Lemma 4. *There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that for any* $f \in H^2$

$$\|Lf\|^2_{L_2} \geqq c_1 \|f\|^2_{H^2} - c_2 \|f\|^2_{L_2}. \tag{5}$$

Proof. Using the identity

$$\|Lf\|^2_{L_2} = \|\Delta f\|^2_{L_2} - 2(qf, \Delta f) + \|qf\|^2_{L_2}$$

the ingequality $a\cdot b \leqq \varepsilon a^2+(1/4\varepsilon)\cdot b^2$ $(a, b, \varepsilon>0)$, the Cauchy—Bunyakovski inequality and Lemma 2, we have

$$|(qf, \Delta f)| \leqq \|qf\|_{L_2}\cdot\|\Delta f\|_{L_2} \leqq \varepsilon\|\Delta f\|_{L_2}^2+\frac{1}{4\varepsilon}\|qf\|_{L_2}^2,$$

hence

$$\|Lf\|_{L_2}^2 \geqq \|\Delta f\|_{L_2}^2-2|(qf, \Delta f)|+\|qf\|_{L_2}^2 \geqq$$

$$\geqq \|\Delta f\|_{L_2}^2-\varepsilon\|\Delta f\|_{L_2}^2-c(\varepsilon)\|qf\|_{L_2}^2 \geqq (1-\varepsilon)\|\Delta f\|_{L_2}^2-c(\varepsilon)\|qf\|_{L_2}^2.$$

Using

$$\|qf\|_{L_2}^2 \leqq c\|f\|_{H^1}^2 \leqq \varepsilon_1\|f\|_{H^2}^2+c(\varepsilon_1)\|f\|_{L_2}^2$$

we obtain

$$\|Lf\|_{L_2}^2 \geqq (1-\varepsilon)\|\Delta f\|_{L_2}^2-c(\varepsilon)\varepsilon_1\|f\|_{H^2}^2-c(\varepsilon)c(\varepsilon_1)\|f\|_{L_2}^2.$$

On the other hand

$$\|f-\Delta f\|_{L_2}^2 = \|f\|_{H^2}^2,$$

whence

$$\|\Delta f\|_{L_2} = \|\Delta f-f+f\|_{L_2} \geqq \|\Delta f-f\|_{L_2}-\|f\|_{L_2} \geqq \|f\|_{H^2}-\|f\|_{L_2}.$$

At last we obtain

$$\|Lf\|_{L_2}^2 \geqq \big(1-\varepsilon-c(\varepsilon)\varepsilon_1\big)\|f\|_{H^2}^2-c(\varepsilon, \varepsilon_1)\|f\|_{L_2}^2,$$

and from this (5) follows if $\varepsilon=1/2$ and ε_1 is small enough. Lemma 4 is proved.

Define $L_\mu=L+\mu I$.

Lemma 5. *There exists* $\mu_0>0$ *such that for every* $f\in C_0^\infty(\mathbf{R}^n)$

$$\|L_\mu f\|_{L_2} \geqq c_\mu\|f\|_{H^2}. \tag{6}$$

The constant C_μ *does not depends on* f.

Proof. Obviously

$$\|L_\mu f\|_{L_2}^2 = (Lf+\mu f, Lf+\mu f) = (Lf, Lf)+2\mu(Lf, f, f)+\mu^2(f, f) \geqq$$

$$\geqq (Lf, Lf)-2\mu|(qf, f)|+2\mu\|\nabla f\|_{L_2}^2+\mu\|f\|_{L_2}^2.$$

Using Lemma 1, we obtain

$$|(qf, f)| \leqq \big\|\sqrt{|q|}f\big\|_{L_2}^2 \leqq c\|f\|_{L_2}\cdot\|f\|_{H^1} \leqq \varepsilon\|f\|_{H^1}^2+\frac{1}{4\varepsilon}\|f\|_{L_2}^2. \tag{7}$$

Taking into account the definition of the H^s-norm and using the inequality

$$1+|\xi|^2 \leqq \varepsilon_1(1+|\xi|^2)^2+\frac{1}{4\varepsilon_1},$$

(which is a special case of $a \leqq \varepsilon a^2 + 1/4\varepsilon$) we obtain

$$\text{(8)} \qquad \|f\|_{H^1}^2 \leqq \varepsilon_1 \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^2 \, d\xi + \frac{1}{4\varepsilon_1} \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 \, d\xi =$$

$$= \varepsilon_1 \|f\|_{H^2}^2 + \frac{1}{4\varepsilon_1} \|f\|_{L_2}^2.$$

From (7) and (8) it follows

$$|(qf, f)| \leqq \varepsilon\varepsilon_1 \|f\|_{H^2}^2 + \frac{\varepsilon}{4\varepsilon_1} \|f\|_{L_2}^2 + \frac{1}{4\varepsilon} \|f\|_{L_2}^2.$$

Let $\varepsilon = \varepsilon_1 = \alpha/\sqrt{\mu}$, where $\alpha > 0$ will be choosen below. Summarising our estimates, we obtain

$$\|L_\mu f\|_{L_2}^2 \geqq c_1 \|f\|_{H^2}^2 - c_2 \|f\|_{L_2}^2 - 2\mu \left\{ \frac{\alpha^2}{\mu} \|f\|_{H^2}^2 + \frac{1}{4} \|f\|_{L_2}^2 + \right.$$

$$\left. + \frac{\sqrt{\mu}}{4\alpha} \|f\|_{L_2}^2 \right\} + \mu^2 \|f\|_{L_2}^2 \geqq (c_1 - 2\alpha^2) \|f\|_{H^2}^2 -$$

$$-\left(\mu^2 - \frac{\mu}{2} - \frac{\mu\sqrt{\mu}}{2\alpha}\right) \|f\|_{L_2}^2 \geqq c \|f\|_{H^2}^2,$$

if α is small enough and $\mu_0 = \mu_0(\alpha)$ is large enough. Lemma 5 is proved.

The following Lemma generalizes that of 2.4 in [11].

Lemma 6. *Let A and B be strongly positive selfadjoint operators in the Hilbert space H. Suppose*

$$\text{(9)} \qquad D(B) \subset D(A)$$

and

$$\text{(10)} \qquad \|Af\|_H \leqq c \|Bf\|_H \quad (f \in D(B))$$

are fulfilled. Then for any θ, $0 \leqq \theta \leqq 1$

$$\text{(11)} \qquad \|A^\theta f\|_H \leqq c_\theta \|B^\theta f\|_H \quad (f \in D(B)).$$

Proof. Define

$$\|f\|_{D(A)^\theta} \overset{\text{def}}{=} \|A^\theta f\|_H, \quad \theta \in \mathbf{C}.$$

According to the strong positivity of A, this norm is equivalent with that of defined in Triebel [9]. We obtain from (10) (taking into account the definition of the Petree functional)

$$K(t, f, H, D(A)) \leqq cK(t, f, H, D(B)) \quad (f \in D(B)).$$

Hence, using 1.3.2 of [9], we obtain (11). Lemma 6 is proved.

Lemma 7. *For* $\mu \geqq \mu_0$, $0 \leqq s \leqq 2$ *we have*

$$\|L_\mu^{s/2} f\|_{L_2} \leqq c_s \|f\|_{H^s} (f \in H^s), \quad D(L_\mu^{s/2}) = H^s(\mathbf{R}^n). \tag{12}$$

Proof. (12) is trivial for $s=0$ and it is proved for $s=2$ in Lemma 3. Now apply Lemma 6 for $A=L_\mu$, $B=I-\Delta$, $D(B)=H^2(\mathbf{R}^n)$. We obtain:

$$\|L_\mu^\theta f\|_{L_2} \leqq c\|(I-\Delta)^\theta f\|_{L_2} = c\|f\|_{H^{2\theta}}.$$

According to TRIEBEL [9], 1.18.10 and using the Corollary after Lemma 2, we obtain: $D(L_\mu^\theta)=H^{2\theta}(\mathbf{R}^n)$. Lemma 7 is proved.

Lemma 8. *For* $\mu \geqq \mu_0$, $0 \leqq s \leqq 2$

$$\|L_\mu^{-s/2} f\|_{H^s} \leqq c_s \|f\|_{L_2} \quad (f \in L_2(\mathbf{R}^n)). \tag{13}$$

Proof. First we prove that for every $g \in H^s$

$$\|g\|_{H^s} \leqq c_s \|L_\mu^{s/2} g\|_{L_2} \quad (0 \leqq s \leqq 2). \tag{14}$$

This estimate is trivial for $s=0$ and for $s=2$ it is proved in the Lemma 5. Use Lemma 6 for $B=L_\mu$, $A=I-\Delta$, $D(A)=H^2$, then (14) follows.

From Lemma 7 we obtain $R(L_\mu^{-s/2})=D(L_\mu^{s/2})=H^s(\mathbf{R}^n)$, whence for every $f \in L_2(\mathbf{R}^n)$ we have $L_\mu^{-s/2} f \in H^s$. Now applying (14) for the function $g \overset{\text{def}}{=} L_\mu^{-s/2} f$ $(f \in L_2(\mathbf{R}^n))$ we obtain (13). Lemma 8 is proved.

Proof of Theorem 1. Using (12) and (13) we obtain for any $f \in H^s$:

$$\|f-E_\lambda f\|_{H^s} = \|L_\mu^{-s/2} L_\mu^{s/2}(I-E_\lambda) f\|_{H^s} \leqq$$

$$\leqq c\|L_\mu^{s/2}(I-E_\lambda) f\|_{L_2} = c\|(I-E_\lambda) L_\mu^{s/2} f\|_{L_2} \to 0 \quad (\lambda \to \infty).$$

Theorem 1 is proved.

Now consider the Schrödinger operator

$$L_0 = -\Delta + q(x) \cdot, \quad x \in \mathbf{R}^3, \quad D(L_0) = C_0^\infty(\mathbf{R}^3),$$

and suppose $|q(x)| \leqq c/|x|$. Then the method of the proof of Theorem 1 gives

Theorem 2. *The operator* L_0 *is essentially selfadjoint,* $D(\bar{L}_0)=D(L)=H^2(\mathbf{R}^3)$, *where* $L \overset{\text{def}}{=} \bar{L}_0$, *further for* $0<\tau<1/2$ *we have*

$$\|E_\lambda f - f\|_{C^\tau} \to 0 \quad (\lambda \to \infty, f \in H^{3/2+\tau}(\mathbf{R}^3)). \tag{15}$$

Here E_λ is the spectral family of L and C^τ denotes the Hölder class of functions (TRIEBEL [9], 2.7.1(2)), *i.e.*

$$\|f\|_{C^\tau} \stackrel{\text{def}}{=} \sup_{x\in\mathbf{R}^3} |f(x)| + \sup_{x,y\in\mathbf{R}^3} \frac{|f(x)-f(y)|}{|x-y|^\tau}.$$

Proof of Theorem 2. Using the imbedding $H^{3/2+\tau}\subset C^\tau$ (cf. [9], 2.8.1(16), $p=2$, $n=3$) (15) follows by the method of the proof of Theorem 1.

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Parameter estimation and Kalman filtering in noisy background

MÁTYÁS ARATÓ

Dedicated to Professor Károly Tandori on his 60th birthday

The parameter estimation (identification) problem will be discussed in the presence of additive coloured noise for multidimensional stochastic processes with continuous time. We shall investigate the case when the signal and noise both are Ornstein—Uhlenbeck processes. To obtain the main tool in such investigations, the Radon—Nikodym derivative, we use the method of Kalman filtering and the important remark that the Riccati equation can be solved explicitely. Some asymptotic results for the white noise case will be discussed (the reader may compare with BALAKRISHNAN [6]—[8]).

1. Introduction. Let us have an observed process $\xi(t)$ in the form

$$\xi(t) = \theta(t, \alpha)+\varepsilon(t, \beta),\ 0 \leqq t \leqq T, \tag{1}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ denote the unknown parameters, generally vector valued, which one wishes to estimate. The $\boldsymbol{\theta}(t, \boldsymbol{\alpha})$ process means the signal, while $\boldsymbol{\varepsilon}(t, \boldsymbol{\beta})$ models the error and both they are completely specified once $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are given. When the time, t, is continuous the basic tool in statistical theory is the Radon—Nikodym derivative of the probability measures induced by the processes $\xi(t)$ and $\boldsymbol{\theta}(t)$, with respect to a standard measure, e.g. the Wiener measure in the Gaussian case. It turns out that the derivatives even in the most simple cases have complicated form. For the stationary Gaussian process case the reader can find different methods in HAJEK [10], IBRAGIMOV and ROZANOV [13], PISARENKO [19]—[21], PISARENKO and ROZANOV [22], while for diffusional type Gaussian processes we may mention ARATÓ [2]—[4], BALAKRISHNAN [6]—[8], KUTOJANC [15], LIPSTSER and SHIRYAEV [16]. In these works mostly the scalar case was studied.

In this paper we assume that $\boldsymbol{\theta}(t, \boldsymbol{\alpha})$ and $\boldsymbol{\varepsilon}(t, \boldsymbol{\beta})$ both are first order multidimensional, autoregressive processes, the so-called Ornstein—Uhlenbeck processes, i.e.,

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they satisfy the stochastic differential equations

$$(2) \qquad d\boldsymbol{\theta}(t) = -\boldsymbol{\alpha}\boldsymbol{\theta}(t)\,dt + \mathbf{c}_1^{1/2}\,d\mathbf{w}_1(t), \quad t \geqq 0,$$

$$d\boldsymbol{\varepsilon}(t) = -\boldsymbol{\beta}\boldsymbol{\varepsilon}(t)\,dt + \mathbf{c}_2^{1/2}\,d\mathbf{w}_2(t), \quad t \geqq 0,$$

where $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$ are standard, independent Wiener processes and $-\boldsymbol{\alpha}$, $-\boldsymbol{\beta}$ have eigenvalues with negative real parts, $\mathbf{c}_1=\mathbf{c}_1^{1/2}(\mathbf{c}_1^{1/2})^*$ and $\mathbf{c}_2=\mathbf{c}_2^{1/2}(\mathbf{c}_2^{1/2})^*$ are positive semidefinite symmetrical matrices. The $\boldsymbol{\varepsilon}(t)$ process is called *coloured noise.*

In the mathematical literature it was customary to take $\varepsilon(t)$ as a Wiener process. In the earlier engineering literature the white noise process was introduced for $\varepsilon(t)$ in a formal way as a stationary stochastic process with constant spectral density. For more rigorous treatment of the white noise case Balakrishnan introduced the Radon—Nikodym derivative of weak distributions, see [6]—[7].

In the present paper we are motivated by statistical considerations more than control theoretical, to develope some practically useful and computationally efficient closed-form expressions for estimators of the drift parameter α in (2). The expressions are developed for some limiting values in terms of the spectral characteristics of the processes for the purpose to obtain approximations in the white noise case too. First of all we remark that the Riccati equations of Kalman filtering can be solved explicitely and so one can get the Radon—Nikodym derivatives. With limiting it is possible to obtain those formulae derived earlier in the one dimensional case by Balakrishnan. The result that the filtering equations can be solved explicitely in the case of stochastic equations with constant coefficients may have other practical and theoretical consequences.

2. Explicite Kalman filtering. We assume that in (2) the processes $\boldsymbol{\theta}(t, \boldsymbol{\alpha})$ and $\boldsymbol{\varepsilon}(t, \boldsymbol{\beta})$ are p-dimensional and the Wiener processes $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$ are independent, p-dimensional and standard, i.e.,

$$\mathbf{E}\mathbf{w}_i(t) = 0, \quad \mathbf{E}\big(\mathbf{w}_i(t)\mathbf{w}_i(t)^*\big) = \mathbf{I}_p \cdot t, \quad \mathbf{w}_i(0) = 0, \quad i = 1, 2,$$

where asterisk means the transposed and $\mathbf{I}_p$ is the p-dimensional unit matrix. $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ are independent of $\boldsymbol{\theta}(0)$, $\boldsymbol{\varepsilon}(0)$ and further $\boldsymbol{\theta}(t)$ and $\boldsymbol{\varepsilon}(t)$ are stationary, which means that for the covariance functions $\big($assuming $\mathbf{E}\boldsymbol{\theta}(t)=\mathbf{E}\boldsymbol{\varepsilon}(t)=0\big)$

$$B_\theta(t) = \mathbf{E}\boldsymbol{\theta}(t+s)\boldsymbol{\theta}^*(s), \quad B_\varepsilon(t) = \mathbf{E}\boldsymbol{\varepsilon}(t+s)\boldsymbol{\varepsilon}^*(s),$$

we have

$$(3) \qquad \boldsymbol{\alpha}B_\theta(0) + B_\theta(0)\boldsymbol{\alpha}^* = \mathbf{c}_1, \quad B_\theta(t) = e^{-\boldsymbol{\alpha}|t|}B_\theta(0),$$

$$\boldsymbol{\beta}B_\varepsilon(0) + B_\varepsilon(0)\boldsymbol{\beta}^* = \mathbf{c}_2, \quad B_\varepsilon(t) = e^{-\boldsymbol{\beta}|t|}B_\varepsilon(0)$$

From (1) and (2) we get

$$d\boldsymbol{\theta}(t) = -\boldsymbol{\alpha}\boldsymbol{\theta}(t)\,dt + \mathbf{c}_1^{1/2}\,d\mathbf{w}_1(t), \tag{4}$$

$$\begin{aligned} d\boldsymbol{\xi}(t) &= d\boldsymbol{\theta}(t) + d\boldsymbol{\varepsilon}(t) = \\ &= -(\boldsymbol{\alpha}-\boldsymbol{\beta})\boldsymbol{\theta}(t)\,dt - \boldsymbol{\beta}\boldsymbol{\xi}(t)\,dt + \mathbf{c}_1^{1/2}\,d\mathbf{w}_1(t) + \mathbf{c}_2^{1/2}\,d\mathbf{w}_2(t), \end{aligned} \tag{5}$$

where $\boldsymbol{\xi}(t)$ is the observable, while $\boldsymbol{\theta}(t)$ is the unobservable component of the vector process $(\boldsymbol{\xi}^*(t), \boldsymbol{\theta}^*(t))$. The Kalman filtering equations for $\mathbf{m}(t) = E(\boldsymbol{\theta}(t)|\mathscr{F}_t^{\xi})$ and $\gamma(t) = \mathbf{E}(\boldsymbol{\theta}(t)-\mathbf{m}(t))(\boldsymbol{\theta}(t)-\mathbf{m}(t))^*$ are given by (see LIPTSER and SHIRYAEV [16] Th. 12.7)

$$\begin{aligned} d\mathbf{m}(t) = -\boldsymbol{\alpha}\mathbf{m}(t)\,dt + [\mathbf{c}_1 + \gamma(t)(\boldsymbol{\beta}-\boldsymbol{\alpha})^*][\mathbf{c}_1+\mathbf{c}_2]^{-1}\big[d\boldsymbol{\xi}(t) - \\ -\big((\boldsymbol{\beta}-\boldsymbol{\alpha})\mathbf{m}(t) - \boldsymbol{\beta}\boldsymbol{\xi}(t)\big)\,dt\big], \end{aligned} \tag{6}$$

$$\begin{aligned} \dot\gamma(t) = \frac{d\gamma(t)}{dt} &= -\boldsymbol{\alpha}\gamma(t) - \gamma(t)\boldsymbol{\alpha}^* - [\mathbf{c}_1 + \gamma(t)(\boldsymbol{\beta}-\boldsymbol{\alpha})^*][\mathbf{c}_1+\mathbf{c}_2]^{-1}\times \\ &\times[\mathbf{c}_1+\gamma(t)(\boldsymbol{\beta}-\boldsymbol{\alpha})^*]^* + \mathbf{c}_1 = -[\boldsymbol{\alpha} + \mathbf{c}_1(\mathbf{c}_1+\mathbf{c}_2)^{-1}(\boldsymbol{\beta}-\boldsymbol{\alpha})]\gamma(t) - \\ &-\gamma(t)[\boldsymbol{\alpha}+\mathbf{c}_1(\mathbf{c}_1+\mathbf{c}_2)^{-1}(\boldsymbol{\beta}-\boldsymbol{\alpha})]^* - \gamma(t)(\boldsymbol{\beta}-\boldsymbol{\alpha})^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}(\boldsymbol{\beta}-\boldsymbol{\alpha})\gamma(t) - \\ &-\mathbf{c}_1(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{c}_1 + \mathbf{c}_1 = \\ &= \mathbf{a}\gamma(t) + \gamma(t)\mathbf{a}^* - \gamma(t)\mathbf{A}^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{A}\gamma(t) + \mathbf{b}\mathbf{b}^*. \end{aligned} \tag{7}$$

The solution of the Riccati equation (7) may be given in the form (see ARATÓ [4], Lemma 2 in section 1.8)

$$\gamma(t) = e^{\tilde{\mathbf{a}}t}\Big[c_0 + \int_0^t e^{\tilde{\mathbf{a}}^*u}\mathbf{A}^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{A}e^{\tilde{\mathbf{a}}u}\,du\Big]^{-1}e^{\tilde{\mathbf{a}}^*t} + \mathbf{c} \tag{8}$$

where $\mathbf{c}$ is the positive semidefinite solution of the "algebraic Riccati equation":

$$\mathbf{a}\mathbf{c} + \mathbf{c}\mathbf{a}^* - \mathbf{c}\mathbf{A}^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{A}\mathbf{c} + \mathbf{b}\mathbf{b}^* = 0, \tag{9}$$

and the constants $\mathbf{c}_0, \tilde{\mathbf{a}}, \tilde{\mathbf{a}}^*$ are given by

$$\begin{aligned} \mathbf{c}_0^{-1} &= \gamma(0) - \mathbf{c}, \\ \tilde{\mathbf{a}} = \mathbf{a} - \mathbf{c}\mathbf{A}^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{A}, &\quad \tilde{\mathbf{a}}^* = \mathbf{a}^* - \mathbf{A}^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{A}\mathbf{c}. \end{aligned} \tag{10}$$

In the scalar case we have the special form

$$\gamma(t) = e^{-\tilde{A}t}[c_0 + B(1-e^{-\tilde{A}t})/\tilde{A}]^{-1} + c, \tag{8'}$$

where from the stationarity

$$\begin{gathered} A = 2\frac{\alpha c_2 + \beta c_1}{c_1+c_2}, \quad B = \frac{(\beta-\alpha)^2}{c_1+c_2}, \quad b = \frac{c_1c_2}{c_1+c_2}, \\ c = \frac{-A + \sqrt{A^2+4Bb}}{2B} \end{gathered} \tag{9'}$$

and

$$c_0^{-1} = \gamma(0) - c, \quad \gamma(0) = \frac{c_1 c_2}{2(\alpha c_2 + \beta c_1)}, \quad \tilde{A} = A + 2c\beta. \tag{10'}$$

From (6)—(10) by integration we can prove the following statement.

Theorem 1. *The conditional expectation of* $\boldsymbol{\theta}(t)$ *under condition* $\xi(s)$, $0 \leqq s \leqq t$, *given in* (4)—(5), *has the form*

$$\begin{aligned} \mathbf{E}(\boldsymbol{\theta}(t)|\mathscr{F}_t^\xi) = \mathbf{m}(t) &= \exp\Big\{-\int_0^t \mathbf{g}(u)\,du\Big\}\Big\{\mathbf{m}(0) + \int_0^t \exp\Big(\int_0^s \mathbf{g}(u)\,du\Big)[\mathbf{h}(s)\boldsymbol{\beta}\boldsymbol{\xi}(s)\,ds + \\ &+ \mathbf{h}(s)\,d\boldsymbol{\xi}(s)]\Big\} = \exp\Big\{-\int_0^t \mathbf{g}(u)\,du\Big\}\Big\{\mathbf{m}(0) + \exp\Big(\int_0^t \mathbf{g}(u)\,du\Big)\mathbf{h}(t)\boldsymbol{\xi}(t) + \\ &+ \mathbf{h}(0)\boldsymbol{\xi}(0) + \int_0^t \exp\Big(\int_0^s \mathbf{g}(u)\,du\Big)[\mathbf{h}(s)\boldsymbol{\beta} - \mathbf{h}'(s)]\boldsymbol{\xi}(s)\,ds - \\ &- \int_0^t \mathbf{g}(s)\exp\Big(\int_0^s \mathbf{g}(u)\,du\Big)\mathbf{h}(s)\boldsymbol{\xi}(s)\,ds\Big\}, \end{aligned} \tag{11}$$

where

$$\mathbf{g}(u) = \boldsymbol{\alpha} + \mathbf{h}(u)(\boldsymbol{\beta} - \boldsymbol{\alpha}),$$
$$\mathbf{h}(u) = [\mathbf{c}_1 + \gamma(u)(\boldsymbol{\beta} - \boldsymbol{\alpha})^*](\mathbf{c}_1 + \mathbf{c}_2)^{-1},$$

and $\gamma(t)$ *is given in* (8)—(10).

Specially in the scalar case we have

$$\begin{aligned} m(t) &= \exp(-h_1(t))\Big\{m(0) + (\beta-\alpha)^{-1}\exp(h_1(t))\Big[\frac{A}{2} + B\gamma(t) - \alpha\Big]\xi(t) + \\ &-(\beta-\alpha)^{-1}[A/2 + B\gamma(0) - \alpha]\xi(0) + (\beta-\alpha)^{-1}\int_0^t \exp(h_1(s))\Big[A(\beta-\alpha)/2 + \\ &-\beta\alpha - \frac{A^2}{4} - bB + B\gamma(s)(\beta - 2A + \alpha) - B^2\gamma^2(s)\Big]\xi(s)\,ds\Big\}, \end{aligned} \tag{11'}$$

where

$$m(0) = \frac{c_1\beta}{c_1\beta + c_2\alpha}\xi(0)$$

$$h_1(t) = \int_0^t [A/2 + B\gamma(s)]\,ds = (A/2 + Bc)t + \log\big(c_0 + B(1 - e^{-\tilde{A}t})/\tilde{A}\big)/c_0.$$

To calculate $\mathbf{m}(0) = \mathbf{E}(\boldsymbol{\theta}(0)|\boldsymbol{\xi}(0))$ and $\gamma(0) = \operatorname{cov}(\boldsymbol{\theta}(0), \boldsymbol{\theta}(0)|\boldsymbol{\xi}(0))$ one can use the Gaussianness and stationarity of the process $(\boldsymbol{\theta}(t), \boldsymbol{\xi}(t))$. Let $\boldsymbol{\theta} = \boldsymbol{\theta}(0)$, $\boldsymbol{\xi} = \boldsymbol{\xi}(0)$

be a Gaussian random vector, then

$$m(0) = D_{\theta\xi} D_{\xi\xi}^{-1} \xi, \tag{12}$$

$$\gamma(0) = D_{\theta\theta} - D_{\theta\xi} D_{\xi\xi}^{-1} D_{\xi\theta},$$

where

$$\tilde{B}(0) = \begin{pmatrix} \operatorname{cov}(\boldsymbol{\theta}, \boldsymbol{\theta}) & \operatorname{cov}(\boldsymbol{\theta}, \boldsymbol{\xi}) \\ \operatorname{cov}(\boldsymbol{\xi}, \boldsymbol{\theta}) & \operatorname{cov}(\boldsymbol{\xi}, \boldsymbol{\xi}) \end{pmatrix} = \begin{pmatrix} D_{\theta\theta} & D_{\theta\xi} \\ D_{\xi\theta} & D_{\xi\xi} \end{pmatrix}$$

is the joint covariance matrix. From stationarity we obtain

$$\tilde{\mathbf{A}} \tilde{B}(0) + \tilde{B}(0) \mathbf{A}^* = -\tilde{\mathbf{B}}_w, \tag{13}$$

where

$$\tilde{\mathbf{A}} = \begin{pmatrix} -\alpha & 0 \\ \beta - \alpha & -\beta \end{pmatrix}, \quad \tilde{\mathbf{B}}_w = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_1^{1/2}(\mathbf{c}_1^{1/2} + \mathbf{c}_2^{1/2})^* \\ (\mathbf{c}_1^{1/2} + \mathbf{c}_2^{1/2})(\mathbf{c}_1^{1/2})^* & \mathbf{c}_1 + \mathbf{c}_2 \end{pmatrix},$$

and the unknown $\tilde{B}(0)$ may be gotten from the linear relations

$$\boldsymbol{\alpha} D_{\theta\theta} - D_{\theta\theta} \boldsymbol{\alpha}^* = \mathbf{c}_1,$$

$$\boldsymbol{\alpha} D_{\theta\xi} - D_{\theta\theta}(\boldsymbol{\beta} - \boldsymbol{\alpha})^* + D_{\xi\theta} \boldsymbol{\beta}^* = \mathbf{c}_1^{1/2}(\mathbf{c}_1^{1/2} + \mathbf{c}_1^{1/2})^*,$$

$$(\boldsymbol{\alpha} - \boldsymbol{\beta}) D_{\theta\xi} + \boldsymbol{\beta} D_{\theta\xi} + D_{\theta\xi}(\boldsymbol{\beta} - \boldsymbol{\alpha})^* + D_{\xi\xi} \boldsymbol{\beta}^* = (\mathbf{c}_1 + \mathbf{c}_2).$$

Explicite solution of (13) is not known in the general case.

To find different approximations for the maximum likelihood method one has to dicuss special cases and find simple forms for $\mathbf{m}(t)$ and its integral. Below we shall investigate two problems: a) the realization of $\xi(s)$ is given in $-\infty < s \leqq t$, b) the white noise approximation, when

$$\boldsymbol{\beta} \mathbf{c}_2^{-1} \boldsymbol{\beta}^* \to \boldsymbol{\beta}_0, \quad \text{if} \quad \boldsymbol{\beta} \to \infty, \tag{14}$$

where the convergence of matrix $\mathbf{A}$ is understood in the norm $\|\mathbf{A}\|^2 = \operatorname{Sp} \mathbf{AA}^*$, and $\boldsymbol{\beta}$ has full rank, p.

a) Taking the observation interval $-T \leqq s \leqq t$, $(T > 0)$, we obtain from (12) that

$$\mathbf{m}_T(-T) = \mathbf{a}_1 \xi(-T), \quad \gamma_T(-T) = \mathbf{b}_1,$$

with constant matrices $\mathbf{a}_1$ and $\mathbf{b}_1$. Assuming that the matrix

$$\mathbf{g} = \left[\boldsymbol{\alpha} - \left(\mathbf{c}_1 + \mathbf{c}(\boldsymbol{\beta} - \boldsymbol{\alpha})^* (\mathbf{c}_1 + \mathbf{c}_2)^{-1} (\boldsymbol{\beta} - \boldsymbol{\alpha})\right)\right]$$

has eigenvalues with positive real parts from (8) and (11) one can obtain

$$\gamma_T(t) \to \mathbf{c}, \quad \text{if} \quad T \to \infty, \tag{15}$$

$$e^{-\int_{-T}^{t} g(u)\, du} = \exp\{-g \cdot (t+T)\} \to 0, \quad \text{if} \quad T \to \infty.$$

So we proved.

Theorem 2. *The limit function of $m_T(t)$, when $T\to\infty$, under the condition (15) has the from*

$$\tilde{\mathbf{m}}(t) = \mathbf{h}\cdot\xi(t) + \int_{-\infty}^{t} e^{\mathbf{g}(s-t)}[\mathbf{h}\cdot\boldsymbol{\beta}]\xi(s)\,ds - \int_{-\infty}^{t} \mathbf{g}e^{\mathbf{g}(s-t)}\mathbf{h}\xi(s)\,ds, \tag{16}$$

where

$$\mathbf{h} = [\mathbf{c}_1 + \mathbf{c}(\boldsymbol{\beta}-\boldsymbol{\alpha})^*][\mathbf{c}_1+\mathbf{c}_2]^{-1}.$$

Remark. In the special case when $\alpha=\beta$ we have $g=\alpha$, $h=c_1[c_1+c_2]^{-1}$ and

$$\tilde{\mathbf{m}}(t) = c_1(c_1+c_2)^{-1}\xi(t),$$

as it could be expected.

b) If $\boldsymbol{\beta}\mathbf{c}_2^{-1}\boldsymbol{\beta}^*\to\boldsymbol{\beta}_0$, when $\boldsymbol{\beta}\to\infty$, and $\boldsymbol{\beta}_0$ is positive definite we obtain from (7)—(10)

$$\mathbf{b}\mathbf{b}^* \to \mathbf{c}_1, \quad \mathbf{a}\to-\boldsymbol{\alpha}, \tag{17}$$

$$\mathbf{A}^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}\mathbf{A}\to\boldsymbol{\beta}_0,$$

further, $\tilde{\mathbf{c}}$ is the solution of the equation

$$-\boldsymbol{\alpha}\tilde{\mathbf{c}} - \tilde{\mathbf{c}}\boldsymbol{\alpha}^* - \tilde{\mathbf{c}}\boldsymbol{\beta}_0\mathbf{c} + \mathbf{c}_1 = 0 \tag{18}$$

and

$$\tilde{\mathbf{a}} \to -\boldsymbol{\alpha} - \tilde{\mathbf{c}}\boldsymbol{\beta}_0, \tag{19}$$

$$\mathbf{c}_0^{-1} \to \tilde{\mathbf{c}}_0^{-1} = \gamma(0) - \tilde{\mathbf{c}}.$$

In this case we have

Theorem 3. *In the "white noise" case approximation*

$$\gamma_\beta(t) \sim \tilde{\gamma}(t) = \exp[(-\boldsymbol{\alpha}-\tilde{\mathbf{c}}\boldsymbol{\beta}_0)t]\Big[\tilde{\mathbf{c}}_0 + \int_0^t \exp[-(\boldsymbol{\alpha}+\tilde{\mathbf{c}}\boldsymbol{\beta}_0)^* u]\boldsymbol{\beta}_0 \exp[-(\boldsymbol{\alpha}+ \tag{20}$$

$$+\tilde{\mathbf{c}}\boldsymbol{\beta}_0)u]\,du\Big]^{-1}\exp[(-\boldsymbol{\alpha}-\tilde{\mathbf{c}}\boldsymbol{\beta}_0)^* t] + \tilde{\mathbf{c}},$$

and

$$\mathbf{m}_\beta(t) \sim \tilde{\mathbf{m}}(t) = \exp\Big(-\boldsymbol{\alpha}t + \int_0^t \tilde{\gamma}(u)\boldsymbol{\beta}_0\,du\Big)\Big\{\tilde{\mathbf{m}}(0) + \int_0^t \exp\Big(-\boldsymbol{\alpha}s + \tag{21}$$

$$+\int_0^s \tilde{\gamma}(u)\boldsymbol{\beta}_0\,du\Big)\tilde{\gamma}(s)\boldsymbol{\beta}_0\xi(s)\,ds\Big\},$$

when $\boldsymbol{\beta}\to\infty$, *and* $\boldsymbol{\beta}\mathbf{c}_2^{-1}\boldsymbol{\beta}^*\to\boldsymbol{\beta}_0$ *is positive definite.*

Remark. In the same way can be treated the case when the noise power/signal power, i.e., $B_\varepsilon(0)\cdot B_\theta^{-1}(0)$ tends to 0.

3. Parameter estimation. The parameters $\mathbf{c}_1$, $\mathbf{c}_2$ can be determined in two steps, if we want to handle them separately. Using the fact that $\xi(t)$ is a diffusional type process we obtain (with probability 1)

(22) $$\Sigma(\xi(t_i)-\xi(t_{i-1}))(\xi(t_i)-\xi(t_{i-1}))^* \to (\mathbf{c}_1+\mathbf{c}_2)T,$$
$$0 = t_0 \leqq t_1 \leqq \ldots \leqq t_n = T,$$

when $\max(t_i-t_{i-1}) \to 0$. On the other side in representation (6) the process

(23) $$d\tilde{\mathbf{w}}(t) = d\xi(t)-[(\boldsymbol{\beta}-\boldsymbol{\alpha})\mathbf{m}(t)-\boldsymbol{\beta}\xi(t)]\,dt$$

is a Wiener process with parameters

$$E\tilde{\mathbf{w}}(t) = 0, \quad E\tilde{\mathbf{w}}(t)\tilde{\mathbf{w}}^*(t) = (\mathbf{c}_1+\mathbf{c}_2)t.$$

This gives (see e.g. Th. 7.17 in [16]) that (with probability 1)

(24) $$\Sigma(\mathbf{m}(t_i)-\mathbf{m}(t_{i-1}))(\mathbf{m}(t_i)-\mathbf{m}(t_{i-1}))^* \to$$
$$\to \int_0^T [\mathbf{c}_1+((\boldsymbol{\beta}-\boldsymbol{\alpha})\gamma(s))][\mathbf{c}_1+\mathbf{c}_2]^{-1}[c_1+\gamma(s)(\boldsymbol{\beta}-\boldsymbol{\alpha})^*]\,ds,$$

if $\max(t_i-t_{i-1}) \to 0$. The last relation could be used if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ were known.

To get separately $\mathbf{c}_1$ and $\mathbf{c}_2$ one can use (22), (24) and some preliminary estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. For this purpose let us use that the covariance matrix function and the spectral density function have the form

(25) $$B_\xi(t) = \mathbf{E}\xi(t+s)\xi^*(s) = e^{-\boldsymbol{\beta}|t|}B_\varepsilon(0)+e^{-\boldsymbol{\alpha}|t|}B_\theta(0),$$
$$f_\xi(\lambda) = \frac{1}{2\pi}(i\lambda\mathbf{I}_p+\boldsymbol{\alpha})^{-1}\mathbf{c}_1[(-i\lambda\mathbf{I}_p+\boldsymbol{\alpha})^*]^{-1}+$$
$$+\frac{1}{2\pi}(i\lambda\mathbf{I}_p+\boldsymbol{\beta})^{-1}\mathbf{c}_2[(-i\lambda\mathbf{I}_p+\boldsymbol{\beta})^*]^{-1},$$

with condition (3).

Let $\hat{B}(t)$ denote the empirical covariance function

$$\hat{B}(t) = \frac{1}{T-t}\int_0^{T-t} \xi(s)\xi^*(s+t)\,ds,$$

then equating at the points t_1, t_2 to the theoretical values we have

(26) $$\hat{B}(t_1) = e^{\hat{\boldsymbol{\alpha}}|t_1|}\cdot B_\theta(0)+e^{\hat{\boldsymbol{\beta}}|t_1|}\cdot B_\varepsilon(0),$$
$$\hat{B}(t_2) = e^{\hat{\boldsymbol{\alpha}}|t_2|}\cdot B_\theta(0)+e^{\hat{\boldsymbol{\beta}}|t_2|}\cdot B_\varepsilon(0)$$

which give, together with (3) a system for the estimators $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$. We note here that these estimators are not efficient.

To improve estimators $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$, the solutions of system (26), one can use the sequential estimation, proposed by Liptser and Shiryaev [16] (chapter 12) in a modified form. E.g. let us assume that $\boldsymbol{\alpha}$ is a random vector variable with

$$\mathbf{E}\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}} \quad \text{and} \quad \gamma_\alpha(0) = \operatorname{cov}(\boldsymbol{\alpha}, \boldsymbol{\alpha}).$$

Further, from (6), with the approximation only in the diffusion coefficient $\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}$ we get

$$d\mathbf{m}(t) = -\boldsymbol{\alpha}\mathbf{m}(t)\,dt+[\mathbf{c}_1+\gamma(t)(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})][\mathbf{c}_1+\mathbf{c}_2]^{-1}\,[d\boldsymbol{\xi}(t)- -((\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})\mathbf{m}(t)-\hat{\boldsymbol{\beta}}\boldsymbol{\xi}(t))\,dt] \tag{27}$$

where $\gamma(t)$ is given by (8)—(9) with parameters $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$. Assuming that $P(\boldsymbol{\alpha}<u|\mathbf{m}(0))$ is Gaussian and the fourth moment of $\boldsymbol{\alpha}$ exists from theorem 12.8 in [16] we obtain for

$$\tilde{\boldsymbol{\alpha}}(t) = \mathbf{E}(\boldsymbol{\alpha}|\mathscr{F}_t^m),\quad \tilde{\gamma}_\alpha(t) = \mathbf{E}((\boldsymbol{\alpha}(t)-\boldsymbol{\alpha})(\boldsymbol{\alpha}(t)-\boldsymbol{\alpha})^*|\mathscr{F}_t^m), \tag{28}$$

$$\tilde{\boldsymbol{\alpha}}(t) = \Big\{\mathbf{I}+\gamma_\alpha(0)\int_0^t \mathbf{m}^*(s)[\mathbf{c}_1+\gamma(s)(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})](\mathbf{c}_1+\mathbf{c}_2)^{-1}[\mathbf{c}_1+ +\gamma(s)(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})]^*\mathbf{m}(s)\,ds\Big\}^{-1}\Big\{\hat{\boldsymbol{\alpha}}+\gamma_\alpha(0)\int_0^t \mathbf{m}^*(s)[\mathbf{c}_1+\gamma(s)(\hat{\boldsymbol{\beta}}- -\hat{\boldsymbol{\alpha}})](\mathbf{c}_1+\mathbf{c}_2)^{-1}[\mathbf{c}_1+\gamma(s)(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})]^*\,d\mathbf{m}(s)\Big\},$$

(29)

$$\tilde{\gamma}_\alpha(t) = \Big\{\mathbf{I}+\gamma_\alpha(0)\int_0^t \mathbf{m}(s)[\mathbf{c}_1+\gamma(s)(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})](\mathbf{c}_1+\mathbf{c}_2)^{-1}[\mathbf{c}_1+\gamma(s)(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\alpha}})]^*\mathbf{m}(s)ds\Big\}^{-1}\gamma_\alpha(0).$$

To improve $\hat{\boldsymbol{\beta}}$ one can use (5) with $\hat{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\alpha}}(t)$ and the same sequential procedure. This approximation may be compared with those which were proposed by LJUNG [17], see also HANNAN [11, 12], in the discrete time case. The derivation here seems more simple, but no optimality is proved.

Now let us return to the maximum likelihood estimators. It is known that generally no finite system of sufficient statistics exists if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are unknown.

From a Girsanov's type theorem (see [16] Th. 7.19) and (6) we obtain for the Radon—Nikodym derivative (assuming $\boldsymbol{\xi}(0)=0$)

$$\frac{d\mathbf{P}_\xi}{d\mathbf{P}_{\tilde{w}}}(\boldsymbol{\xi}(t)) = \exp\Big\{-\int_0^t [(\boldsymbol{\alpha}-\boldsymbol{\beta})\mathbf{m}(s)+\boldsymbol{\beta}\boldsymbol{\xi}(s)]^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}d\boldsymbol{\xi}(s)- -\frac{1}{2}\int_0^t [(\boldsymbol{\alpha}-\boldsymbol{\beta})\mathbf{m}(s)+\boldsymbol{\beta}\boldsymbol{\xi}(s)]^*(\mathbf{c}_1+\mathbf{c}_2)^{-1}[(\boldsymbol{\alpha}-\boldsymbol{\beta})\mathbf{m}(s)+\boldsymbol{\beta}\boldsymbol{\xi}(s)]\,ds\Big\}. \tag{30}$$

To find the distribution of the exponent in (30) in special cases has a very short history see ARATÓ and BENCZÚR [5], NOVIKOV [18], KONCZ [14].

Example. In the approximate "white noise" case (see case b) in §2.) we obtain for the Radon—Nikodym derivative (30) that it equals to

$$\exp\left\{-\frac{1}{2}\int_0^t [-\boldsymbol{\beta}_0\tilde{\mathbf{m}}(s)+\boldsymbol{\beta}_0\xi(s)]^*[-\boldsymbol{\beta}_0\tilde{\mathbf{m}}(s)]+\boldsymbol{\beta}_0\xi(s)]\,ds\right\}, \tag{31}$$

where $\tilde{\mathbf{m}}(s)$ and $\tilde{\gamma}(s)$ are given by (20) and (21), respectively. Neglecting in $\tilde{\mathbf{m}}(t)$ with the term $\tilde{\mathbf{m}}(0)$, i.e., assuming that

$$\tilde{\mathbf{m}}(t) = \int_0^t \exp\left\{-\boldsymbol{\alpha}(t-s)-\int_0^t \tilde{\gamma}(u)\boldsymbol{\beta}_0\,du\right\}\tilde{\gamma}(s)\boldsymbol{\beta}_0\xi(s)\,ds = \int_0^t f(t,s)\xi(s)\,ds, \tag{32}$$

and taking $\tilde{\gamma}(t)=\tilde{\mathbf{c}}$ we obtain for the log likelihood function

$$L(\boldsymbol{\alpha},\boldsymbol{\beta}_0) = -\frac{1}{2}\int_0^t \left[-\int_0^s \bar{e}^{(\boldsymbol{\alpha}+\tilde{\mathbf{c}}\boldsymbol{\beta}_0)(s-u)}\tilde{\mathbf{c}}\boldsymbol{\beta}_0\xi(u)\,du+\xi(s)\right]^*\boldsymbol{\beta}_0^*\boldsymbol{\beta}_0\times \tag{33}$$

$$\times\left[\xi(s)-\int_0^s \bar{e}^{(\boldsymbol{\alpha}+\tilde{\mathbf{c}}\boldsymbol{\beta}_0)(s-u)}\tilde{\mathbf{c}}\boldsymbol{\beta}_0\xi(u)\,du\right]ds.$$

From (33) the system of transcendental equations for the maximum likelihood estimators of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is given in the following way

$$\frac{\partial L(\boldsymbol{\alpha},\boldsymbol{\beta}_0)}{\partial\boldsymbol{\alpha}} = 0, \quad \frac{\partial L(\boldsymbol{\alpha},\boldsymbol{\beta}_0)}{\partial\boldsymbol{\beta}} = 0.$$

Specially in the scalar case one gets

$$\tilde{\mathbf{c}} = \frac{-\alpha+\sqrt{\alpha^2+c_1\beta_0}}{\beta_0}, \quad \tilde{\gamma}(0) = \frac{c_1}{2\alpha},$$

$$\tilde{c}_0^{-1} = \frac{c_1}{2\alpha}-\tilde{c}, \quad \tilde{A}_0 = \alpha+\beta_0\tilde{c},$$

$$\tilde{\gamma}(t) = \bar{e}^{2\tilde{A}_0t}[\tilde{c}_0+\beta_0(1-\bar{e}^{2\tilde{A}_0t})/2\tilde{A}_0]^{-1}+\tilde{c},$$

and this gives with $\tilde{\gamma}(t)\approx\tilde{c}$

$$L(\alpha,\beta_0) = -\frac{1}{2}\beta_0^2\int_0^t\left[\xi(s)-\int_0^s \bar{e}^{A_0(s-u)}\xi(u)\tilde{c}\,du\right]^2 ds.$$

The maximum likelihood equations are the following

$$\beta_0 \int_0^t \left[\xi(s) - \int_0^s e^{\tilde{A}_0(s-u)} \tilde{c}\xi(u)\,du\right]^2 ds + \beta_0^2 \int_0^t \left[\xi(s) - \int_0^s e^{\tilde{A}_0(s-u)} \tilde{c}\xi(u)\,du\right] \Big\{ \int_0^s \exp(-\tilde{A}_0 s +$$

$$+ A_0 u) c \frac{1}{2} c_1 (\alpha^2 + c_1\beta_0)^{-1/2} \xi(u)\,du - \beta_0^{-2} \int_0^s e^{A_0(s-u)} \times \tag{34}$$

$$\times \left[\frac{1}{2} c_1 (\alpha^2 + c_1\beta_0)^{-1/2} \beta_0 - (-\alpha + (\alpha^2 + c_1\beta_0)^{1/2})\right] \xi(u)\,du \Big\}\, ds = 0$$

$$\int_0^t \left[\xi(s) - \int_0^s e^{\tilde{A}_0(s-u)} \tilde{c}\xi(u)\,du\right] \left[\int_0^s e^{\tilde{A}_0(s-u)} \alpha (\alpha^2 + c_1\beta_0)^{-1/2} (-\alpha + \tag{35}$$

$$+ (\alpha^2 + c_1\beta_0)^{1/2}) \xi(u)\,du\right] ds = 0.$$

If β_0 is apriori given (known) and only α must be estimated from (35) we obtain the equation

$$\left(-\alpha + \sqrt{\alpha^2 + c_1\beta_0}\right) \int_0^t \left(\int_0^s \exp\left\{ \frac{\sqrt{\alpha^2 + c_1\beta_0}}{\beta_0} (u-s) \right\} \xi(u)\,du \right)^2 ds =$$

$$= \int_0^t \left(\int_0^s \exp\left\{ \frac{\sqrt{\alpha^2 + c_1\beta_0}}{\beta_0} (u-s) \right\} \xi(u)\,du \right) \xi(s)\,ds.$$

A similar equation, derived in another way, was given by Pisarenko [19], [21].

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Factoring compact operator-valued functions

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Dedicated to the 60th anniversary of Prof. K. Tandori

1. Let $\mathfrak{H}$ be a separable Hilbert space and denote by $\mathcal{L}(\mathfrak{H})$, $\mathcal{K}(\mathfrak{H})$, $\mathcal{C}_1(\mathfrak{H})$ and $\mathcal{F}(\mathfrak{H})$ the sets of all bounded linear operators, compact operators, trace-class, and finite-rank operators acting on $\mathfrak{H}$, respectively. We set $C=\{\lambda\in\mathbf{C}\colon |\lambda|=1\}$ and, if $\mathfrak{X}$ is a Banach space and $1\leqq p\leqq\infty$, we denote by $L^p(\mathfrak{X})$ the space of (classes of) Bochner integrable functions f defined on C, with values in $\mathfrak{X}$, such that

$$\|f\|_p=\left(\frac{1}{2\pi}\int_0^{2\pi}\|f(e^{it})\|^p\,dt\right)^{1/p}<\infty\quad(p<\infty),\qquad \|f\|_\infty=\operatorname{ess\,sup}\|f(e^{it})\|<\infty.$$

Assume that $\mathfrak{H}$ is infinite dimensional. It is fairly easy to see that for any function $Z\in L^1(\mathcal{K}(\mathfrak{H}))$ one can find functions $X, Y\in L^2(\mathcal{K}(\mathfrak{H}))$ such that $Z=Y^*X$ in $L^1(\mathcal{K}(\mathfrak{H}))$, i.e., $Z(e^{it})=Y(e^{it})^*X(e^{it})$ almost everywhere on C.

In this paper we show that under certain conditions, the functions X and Y can be chosen such that the $\mathfrak{H}$-valued functions $X(e^{it})h$, $Y(e^{it})h$ ($e^{it}\in C$, $h\in\mathfrak{H}$) belong to a certain prescribed functional model space. The methods used here were developed successively in [8], [6], and [2]. Another factorization theorem for Hilbert—Schmidt valued functions is proved in [1], where the operator-theoretic implications of such factorizations are studied in some detail. We hope that these factorization theorems will prove to be relevant in the study of infinite dimensional systems.

The research in this paper was essentially completed in 1981 and partially inspired our subsequent work.

2. We will use the notation $H^p(\mathfrak{X})$ for the Hardy subspace of $L^p(\mathfrak{X})$. If $\mathfrak{X}=\mathbf{C}$ we write H^p and L^p for $H^p(\mathfrak{X})$ and $L^p(\mathfrak{X})$, respectively. If σ is a measurable subset of C, then $L^p(\sigma, \mathfrak{X})$ will denote the space of all functions $f\in L^p(\mathfrak{X})$ that vanish almost everywhere off σ.

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2.1. Definition. A subset S of the unit ball of H^2 is said to be *dominating* for the measurable subset σ of C if the closed absolutely convex hull of the set $\{|\varphi|^2\chi_\sigma:\ \varphi\in S\}$ coincides with the unit ball of $L^1(\sigma)$:

$$\overline{\mathrm{aco}}\{|\varphi|^2\chi_\sigma:\ \varphi\in S\} = \{f\in L^1(\sigma):\ \|f\|_1 \leqq 1\}. \tag{2.2}$$

Here, as usual, χ_σ denotes the characteristic function of the set σ.

In order to provide some motivation for the preceding definition let us compare it with the following one, suggested by one of the basic theorems of Brown, Shields and Zeller [4] (these authors only consider the case in which $\sigma=C$).

2.3. Definition. A subset A of the unit disc $D=\{\lambda:\ |\lambda|<1\}$ is said to be *dominating* for the measurable subset σ of C if almost every point of σ is a nontangential limit of a sequence of points in A.

To see the connection between the two definitions let us consider the functions

$$p_\mu(z) = (1-|\mu|^2)^{1/2}(1-\bar{\mu}z)^{-1}, \quad z\in C, \quad \mu\in D, \tag{2.4}$$

which belong to H^2, and indeed to H^∞, and for which $|p_\mu(z)|^2$ equals the Poisson kernel function $P_\mu(z)=(1-|\mu|^2)|1-\bar{\mu}z|^{-2}$; $\|P_\mu\|_1=1$.

2.5. Proposition. *If a subset $A\subset D$ is dominating for $\sigma<C$, then $S=\{p_\mu:\ \mu\in A\}\subset H^2$ is also dominating for σ.*

(See Lemma 1.2 a) in [2].)

2.6. Lemma. *Assume that $\mathfrak{H}$ is a separable Hilbert space, σ is a measurable subset of C, and S is a subset of the unit ball of H^2, dominating for σ. Then the closure in $L^1(\sigma, \mathscr{K}(\mathfrak{H}))$ of the set*

$$\Sigma = \{\sum_j |\varphi_j^2|\chi_\sigma C_j \quad (\textit{finite sums}):\quad \varphi_j\in S,\quad C_j\in\mathscr{F}(\mathfrak{H}),\quad \sum_j \|C_j\| \leqq 1\}$$

coincides with the unit ball of $L^1(\sigma, \mathscr{K}(\mathfrak{H}))$.

Proof. The dual space of $L^1(\sigma, \mathscr{K}(\mathfrak{H}))$ can be identified with $L^\infty(\sigma, \mathscr{C}_1(\mathfrak{H}))$ *via* the bilinear form

$$\langle F, G\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(F(e^{it})G(e^{it})\right)dt, \quad F\in L^\infty(\sigma, \mathscr{C}_1(\mathfrak{H})), \quad G\in L^1(\sigma, \mathscr{K}(\mathfrak{H})). \tag{2.7}$$

The set Σ is convex and balanced, and the Hahn—Banach theorem implies that it suffices to show that

$$\sup\{|\langle F, G\rangle|:\ G\in\Sigma\} = \|F\|_\infty \quad \text{for} \quad F\in L^\infty(\sigma, \mathscr{C}_1(\mathfrak{H})). \tag{2.8}$$

Since Σ is clearly contained in the unit ball of $L^1(\sigma, \mathscr{K}(\mathfrak{H}))$, we have

$$|\langle F, G\rangle| \leqq \|F\|_\infty \|G\|_1 \leqq \|F\|_\infty \quad \text{for} \quad G\in\Sigma,$$

and so it remains to check the opposite inequality. The hypothesis that S is dominating for σ shows that the closure of Σ contains all functions of the form fK with $f\in L^1(\sigma)$, $K\in\mathscr{K}(\mathfrak{H})$, $\|f\|_1\leqq 1$, $\|K\|\leqq 1$. Let $\{K_n\}$ be a dense sequence in the unit ball of $\mathscr{K}(\mathfrak{H})$. Then we have $\|R\|=\sup_n \{|\mathrm{tr}(RK_n)|\}$ for every $R\in\mathscr{C}_1(\mathfrak{H})$. Thus, if $F\in L^\infty(\sigma, \mathscr{C}_1(\mathfrak{H}))$, we have

$$\|F(z)\|_1 = \sup_n \{|\mathrm{tr}\,(F(z)K_n)|\}, \quad z\in C.$$

(Here and in the sequel we indicate by $z\in C$ a relation that holds almost everywhere on C.) The following calculation is justified for $z\in C$:

$$\|F(z)\| = \sup_n \{|\mathrm{tr}\,(F(z)K_n)|\} \leqq \sup_n \{\|\mathrm{tr}\,(F(\cdot)K_n)\|_\infty\} =$$
$$= \sup_n \sup \left\{\left|\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,\mathrm{tr}\,(F(e^{it})K_n)\,dt\right| : f\in L^1(\sigma),\ \|f\|_1 \leqq 1\right\} =$$
$$= \sup_n \sup \{|\langle F, fK_n\rangle| : f\in L^1(\sigma),\ \|f\|_1 \leqq 1\} =$$
$$\leqq \sup \{|\langle F, fK\rangle| : f\in L^1(\sigma),\ K\in\mathscr{K}(\mathfrak{H}),\ \|f\|_1 \leqq 1,\ \|K\| \leqq 1\} =$$
$$\leqq \sup \{|\langle F, G\rangle| : G\in\Sigma^-\} = \sup \{|\langle F, G\rangle| : G\in\Sigma\}.$$

This yields $\|F\|_\infty \leqq \sup \{|\langle F, G\rangle| : G\in\Sigma\}$; hence (2.8) holds and the proof is completed.

3. We recall now some notation pertaining to functional model spaces; cf. [7]. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be separable Hilbert spaces, and assume that $\Theta\colon D\to\mathscr{L}(\mathfrak{F}, \mathfrak{F}')$ is contractive and analytic: $\|\Theta(\lambda)\|\leqq 1$ and $\Theta(\lambda)=\sum_0^\infty \lambda^k\Theta_k$ $(\Theta_k\in\mathscr{L}(\mathfrak{F}, \mathfrak{F}'))$ for $\lambda\in D$. The strong limit $\Theta(z)=\lim_{r\to 1-0}\Theta(rz)$, and hence $\Delta(z)=(I-\Theta(z)^*\Theta(z))^{1/2}$ also exist for $z\in C$, are strongly measurable functions on C, and generate an analytic Toeplitz operator $T_\Theta\in\mathscr{L}(H^2(\mathfrak{F}), H^2(\mathfrak{F}'))$ and a multiplication operator $\Delta\in\mathscr{L}(L^2(\mathfrak{F}))$ by setting, for $z\in C$,

$$(T_\Theta u)(z) = \Theta(z)u(z), \quad u\in H^2(\mathfrak{F}), \quad \text{and} \quad (\Delta v)(z) = \Delta(z)v(z), \quad v\in L^2(\mathfrak{F}).$$

Next we construct the function space

$$\mathfrak{K}_+ = H^2(\mathfrak{F}')\oplus(\Delta L^2(\mathfrak{F}))^-,$$

the bar indicating norm closure, and observe that

$$Vu = T_\Theta u\oplus\Delta u, \quad u\in H^2(\mathfrak{F}), \tag{3.1}$$

defines an isometry from the space $H^2(\mathfrak{F})$ into the space $\mathfrak{K}_+$. As a consequence,

$$\mathfrak{G} = VH^2(\mathfrak{F}) \quad \text{and} \quad \mathfrak{H}(\Theta) = \mathfrak{K}_+ \ominus \mathfrak{G}$$

are subspaces of $\mathfrak{K}_+$; $\mathfrak{H}(\Theta)$ is the "functional model space" associated with the contractive analytic function Θ.

All these spaces will be regarded as subspaces of the Hilbert function space $L^2(\mathfrak{F}' \oplus \mathfrak{F})$.

Note the following relation between the adjoints of the operator T_Θ and the operator $\Theta \in \mathscr{L}(L^2(\mathfrak{F}), L^2(\mathfrak{F}'))$ of multiplication by the function $\Theta(z)$ on $L^2(\mathfrak{F})$:

$$(T_\Theta^* v)(z) = [\Theta^* v]_+(z), \quad v \in H^2(\mathfrak{F}'), \quad z \in C,$$

where we denote by $[\ \ ]_+$ the natural orthogonal projection of any (scalar- or vector-valued) function space of type L^2 onto its subspace H^2.

Let us also note that for any fixed $z_0 \in C$ for which $\Theta(z_0)$ has sense, $V(z_0)a = \Theta(z_0)a \oplus \Delta(z_0)a$, $a \in \mathfrak{F}$, defines an isometry of $\mathfrak{F}$ into $\mathfrak{F}' \oplus \mathfrak{F}$.

It will be of importance for us to consider elements of $\mathfrak{H}(\Theta)$ of the form $P_{\mathfrak{H}(\Theta)}(u \oplus 0)$, $u \in H^2(\mathfrak{F}')$. Straightforward calculation gives

$$P_{\mathfrak{H}(\Theta)}(u \oplus 0) = (u \oplus 0) - VT_\Theta^* u,$$

and hence,

$$\|P_{\mathfrak{H}(\Theta)}(u \oplus 0)\|^2 = \|u\|^2 - \|T_\Theta^* u\|^2, \tag{3.2}$$

where the norms are in the spaces $L^2(\mathfrak{F}' \oplus \mathfrak{F})$, $L^2(\mathfrak{F}')$, and $L^2(\mathfrak{F})$, respectively.

If $u(z) = \varphi(z)a$, $z \in C$, where $\varphi \in H^2$ and $a \in \mathfrak{F}'$, let us denote $P_{\mathfrak{H}(\Theta)}(u \oplus 0)$ by $\varphi \circ a$.[1] Thus, we have

$$(\varphi \circ a)(z) = (\varphi(z)a \oplus 0) - V(z)(T_\Theta^*(\varphi a))(z), \quad z \in C. \tag{3.3}$$

From (3.2) it is clear that

$$\|\varphi \circ a\|_2 \leqq \|\varphi\|_2 \, \|a\|. \tag{3.4}$$

On the other hand, we deduce from (3.3) that

$$\|(\varphi \circ a)(z)\| \leqq |\varphi(z)| \, \|a\| + \|[\Theta^* \varphi a]_+(z)\|, \quad z \in C.$$

In the special case of the functions $\varphi_m(z) = z^m$ $(m = 0, 1, \ldots)$ we have

$$[\Theta^* \varphi_m a]_+(z) = \sum_{k=0}^{m} z^{m-k} \Theta_k^* a.$$

Since $\Theta_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \Theta(e^{it})\,dt$ is a contraction we deduce that

$$\|[\Theta^* \varphi_m a]_+(z)\| \leqq (m+1)\|a\|, \quad z \in C,$$

[1]) In the paper [2] the notion $\varphi \circ a$ was only used for the functions $\varphi = p_\mu$ $(\mu \in D)$ defined in (2.4), and $p_\mu \circ a$ was then denoted in the shorter form $\mu \circ a$.

and we conclude:

$$\|(\varphi_m \circ a)(z)\| \leqq (m+2)\|a\|. \tag{3.5}$$

Let us notice, furthermore, that for any $f \in \mathfrak{F}$ and $f' \in \mathfrak{F}'$,

$$((\varphi_m \circ a)(z), f' \oplus f) = \left(a, z^{-m} f' - \sum_{k=0}^{m} z^{k-m} \Theta_k V(z)^* (f' \oplus f)\right), \quad z \in C.$$

Hence we see that if a runs through a sequence in $\mathfrak{F}'$ converging weakly to 0, then $(\varphi_m \circ a)(z)$ also converges weakly to 0 in $\mathfrak{F}' \oplus \mathfrak{F}$, $z \in C$. By virtue of linearity of $\varphi \circ a$ with respect to φ this statement extends to the finite linear combinations of the functions φ_m, that is, to all polynomials $p(z) = \sum_0^M c_m z^m$.

Assume now that $\mathfrak{H}$ is a separable Hilbert space and denote by $L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$ the space of norm-square integrable functions Y with values $Y(z)$ compact operators in $\mathscr{L}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F})$. The adjoint Y^* of such a function, $Y^*(z) = (Y(z))^*$, $z \in C$, is in $L^2(\mathscr{K}(\mathfrak{F}' \oplus \mathfrak{F}, \mathfrak{H}))$. Since the function $(\varphi \circ a)(z)$ has its values in $\mathfrak{F}' \oplus \mathfrak{F}$ (indeed, $\varphi \circ a \in \mathfrak{H}(\Theta)$), the function

$$(Y^*(\varphi \circ a))(z), \quad z \in C,$$

makes sense, is measurable, satisfies

$$\|Y^*(\varphi \circ a)\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \|Y^*(e^{it})(\varphi \circ a)(e^{it})\|\, dt \leqq$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} \|Y^*(e^{it})\|\, \|(\varphi \circ a)(e^{it})\|\, dt \leqq \|Y^*\|_2 \|\varphi \circ a\|_2,$$

and, by virtue of (3.4),

$$\|Y^*(\varphi \circ a)\|_1 \leqq \|Y^*\|_2 \|\varphi\|_2 \|a\|. \tag{3.6}$$

3.7. Lemma. *For any given $Y \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$, $\varphi \in H^2$, and for any sequence of elements $a_n \in \mathfrak{F}'$ weakly tending to* 0, *we have*

$$\|Y^*(\varphi \circ a_n)\|_1 \to 0 \quad as \quad n \to \infty.$$

Proof. As weakly convergent sequences are bounded, we may assume that $\|a_n\| \leqq 1$. Fix $\varepsilon > 0$ and choose a polynomial p such that $\|\varphi - p\|_2 < \varepsilon$ (any sufficiently large partial sum of the power series of φ does it). As we have already proved, $(p \circ a_n)(z)$ converges weakly to 0 in $\mathfrak{F}' \oplus \mathfrak{F}$ for $z \in C$. Compactness of the values of Y implies then that $Y^*(p \circ a_n)(z)$ converges strongly to 0 in $\mathfrak{H}$, i.e.,

$$\|Y^*(p \circ a_n)(z)\| \to 0, \quad z \in C.$$

On the other hand, we deduce from (3.5) that $\|(p\circ a_n)(z)\|\leqq M_p\|a_n\|\leqq M_p$ for a finite constant M_p only dependent on p; and therefore,

$$\|Y^*(p\circ a_n)(z)\| \leqq \|Y^*(z)\| M_p.$$

As $\|Y^*(z)\|$ is square integrable by assumption, it is L^1-integrable too, so we can apply the Lebesgue dominated convergence theorem to get

$$\|Y^*(p\circ a_n)\|_1 \to 0.$$

In its turn, (3.6) yields, when applied to $\varphi-p$ in place of φ:

$$\big\|Y^*((\varphi-p)\circ a_n)\big\|_1 \leqq \|Y^*\|_2\|\varphi-p\|_2\|a_n\| \leqq \|Y^*\|_2\varepsilon.$$

So we have

$$\|Y^*(\varphi\circ a_n)\|_1 \leqq \big\|Y^*\big((\varphi-p)\circ a_n\big)\big\|_1+\|Y^*(p\circ a_n)\|_1 \leqq \|Y^*\|_2\varepsilon+o(1).$$

As $\varepsilon>0$ was chosen arbitrary this concludes the proof of Lemma 3.7.

A function $Y\in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}'\oplus\mathfrak{F}))$ will be said to be $\mathfrak{H}(\Theta)$-*oriented* if $Yh\in\mathfrak{H}(\Theta)$ for every $h\in\mathfrak{H}$. The following examples will be of particular interest.

3.8. Lemma. *Let* $\varphi\in H^2$ *and* $A\in\mathscr{F}(\mathfrak{H}, \mathfrak{F}')$. *Then there exists a* $\mathfrak{H}(\Theta)$-*oriented function* $\varphi\circ A\in L^2(\mathscr{F}(\mathfrak{H}, \mathfrak{F}'\oplus\mathfrak{F}))$ *such that*

$$(\varphi\circ A)(z)h = (\varphi\circ Ah)(z), \quad z\in C, \quad h\in\mathfrak{H},$$

and

$$\|\varphi\circ A\|_2 \leqq \|\varphi\|_2(\operatorname{rank} A)^{1/2}\|A\|.$$

Proof. Choose an orthonormal basis $\{e_j\}_1^r$ in $(\ker A)^\perp$ ($r=\operatorname{rank} A$); then we have $Ah=\sum_1^r (h, e_j)a_j$ for $a_j=A\,e_j$ and for all $h\in\mathfrak{H}$; and hence,

$$(\varphi\circ Ah)(z) = \sum_1^r (h, e_j)(\varphi\circ a_j)(z), \quad z\in C. \tag{3.9}$$

Recall that each $\varphi\circ a_j$ is a vector in the space $L^2(\mathfrak{F}'\oplus\mathfrak{F})$ (indeed, in its subspace $\mathfrak{H}(\Theta)$), and therefore is a class of equivalent measurable functions. Choose representative functions from each of these classes, say $(\varphi\circ a_j)\hat{}$ $(j=1, \ldots, r)$, which are defined and finite valued at every point of C. The sum in (3.9) formed with these representatives is linear in $h(\in\mathfrak{H})$ and therefore yields, for every fixed z on C, a linear operator of rank $\leqq r$, which we may denote by $(\varphi\circ A)\hat{}(z)$. By virtue of the inequalities

$$\Big\|\sum_1^r (h, e_j)(\varphi\circ a_j)\hat{}(z)\Big\| \leqq \sum_1^r |(h, e_j)|\,\|(\varphi\circ a_j)\hat{}(z)\| \leqq$$

$$\leqq \Big(\sum_1^r |(h, e_j)|^2\Big)^{1/2}\Big(\sum_1^r \|(\varphi\circ a_j)\hat{}(z)\|^2\Big)^{1/2} \leqq \|h\|\Big(\sum_1^r \|(\varphi\circ a_j)\hat{}(z)\|^2\Big)^{1/2},$$

and also using (3.4) we get

$$\|(\varphi\circ A)^{\wedge}\|_2^2=\frac{1}{2\pi}\int_0^{2\pi}\|(\varphi\circ A)^{\wedge}(e^{it})\|^2\,dt\leqq$$

$$\leqq\sum_1^r\frac{1}{2\pi}\int_0^{2\pi}\|(\varphi\circ a_j)^{\wedge}(e^{it})\|^2\,dt\leqq\sum_1^r\|\varphi\|_2^2\|Ae_j\|^2\leqq\|\varphi\|_2^2 r\|A\|^2.$$

Different choices of the representatives $(\varphi\circ a_j)^{\wedge}$ yield equivalent functions $(\varphi\circ A)^{\wedge}$; denoting their equivalence class by $\varphi\circ A$, relation (3.9) tells us that $(\varphi\circ A)(z)h=$ $=(\varphi\circ Ah)(z)$ a.e. on C; and all the further requirements in the lemma are clearly satisfied also. This finishes the proof of Lemma 3.8.

3.10. Lemma. *The operator valued function* $\varphi A\oplus 0$ $(\varphi\in H^2,\ A\in\mathscr{F}(\mathfrak{H},\mathfrak{F}'))$ *defined by* $(\varphi A\oplus 0)(z)h=\varphi(z)Ah\oplus 0$ $(h\in\mathfrak{H},\ z\in C)$ *also belongs to* $L^2(\mathscr{F}(\mathfrak{H},\mathfrak{F}'\oplus\mathfrak{F}))$, *and so does the difference*

$$VT_\Theta^*\varphi A=(\varphi A\oplus 0)-(\varphi\circ A);$$

moreover, we have

$$\|VT_\Theta^*\varphi A\|_2\leqq\|T_\Theta^*|\varphi A\mathfrak{H}\|_2\|\varphi\|_2(\operatorname{rank}A)^{1/2}\|A\|.$$

Proof. The notation is motivated by the fact that, using relation (3.3), we get

$$(VT_\Theta^*\varphi A)(z)h=(\varphi A\oplus 0)(z)h-(\varphi\circ A)(z)h=$$
$$=\big(\varphi(z)Ah\oplus 0\big)-(\varphi\circ Ah)(z)=V(z)(T_\Theta^*\varphi Ah)(z),\quad z\in C.$$

The inequality follows by reasons analogous to those applied in the proof of Lemma 3.8.

Remark. Since $V(z)$ is an isometry, one easily sees that the same inequality holds for $T_\Theta^*\varphi A\in L^2(\mathscr{F}(\mathfrak{H},\mathfrak{F}))$ too.

We shall assume *from now on* that $\dim\mathfrak{F}'=\infty$.

3.11. Lemma. *Assume that* $\varphi\in H^2$ *and that* $A_j\in\mathscr{F}(\mathfrak{H},\mathfrak{F}')$ $(j=1,2,\ldots)$ *are such that the norms and ranks are bounded, say*

$$\|A_j\|\leqq M,\quad \operatorname{rank}A_j\leqq N\quad (j=1,2,\ldots).$$

If, in addition, the ranges $A_j\mathfrak{H}$ $(j=1,2,\ldots)$ *are pairwise orthogonal in* $\mathfrak{F}'$, *then*

$$\lim_{j\to\infty}\|Y^*(\varphi\circ A_j)\|_1=\lim_{j\to\infty}\|(\varphi\circ A_j)^*Y\|_1=0$$

for every $Y\in L^2(\mathscr{K}(\mathfrak{H},\mathfrak{F}'\oplus\mathfrak{F}))$.

Proof. As $(Y^*(\varphi\circ A_j)(z))^*=(\varphi\circ A_j)^*Y(z)$, $z\in C$, equality of the limits is obvious, and therefore we may treat the first limit only.

Because rank $A_j \leqq N$, there exists an orthonormal sequence $\{e_{jn}\}_{n=1}^{N} \subset \mathfrak{H}$ such that

$$A_j h = \sum_{n=1}^{N} (h, e_{jn}) a_{jn} \quad \text{for} \quad a_{jn} = A_j e_{jn}, \quad h \in \mathfrak{H}, \quad \text{and} \quad j = 1, 2, \ldots.$$

It follows that

$$Y^*(\varphi \circ A_j h) = \sum_{n=1}^{N} (h, e_{jn}) Y^*(\varphi \circ a_{jn}).$$

As $|(h, e_{jn})| \leqq \|h\|$ this implies that

$$\|Y^*(\varphi \circ A_j)\|_1 \leqq \sum_{n=1}^{N} \|Y^*(\varphi \circ a_{jn})\|_1.$$

Now, for fixed n, the sequence $\{a_{jn}\}_{j=1}^{\infty}$ is orthogonal (because $a_{jn} \in A_j \mathfrak{H}$) and bounded (by M), and therefore weakly convergent to 0. Thus, by virtue of Lemma 3.7, we have $\lim_{j \to \infty} \|Y^*(\varphi \circ a_{jn})\|_1 = 0$. Summing for n and applying the above inequality we obtain that $\lim_{j \to \infty} \|Y^*(\varphi \circ A_j)\|_1 = 0$, and the proof is done.

4. We shall be keeping $\mathfrak{F}$, $\mathfrak{F}'$ and Θ fixed, with $\dim \mathfrak{F}' = \infty$. For every nonzero function $\varphi \in H^2$ the space $\varphi \mathfrak{F}'$ can be considered as a subspace of $H^2(\mathfrak{F}')$, and hence we can define the operator $A_\varphi \in \mathscr{L}(\varphi \mathfrak{F}', H^2(\mathfrak{F}))$ by

$$A_\varphi(\varphi a) = T_\Theta^*(\varphi a), \quad a \in \mathfrak{F}'.$$

Note that $\varphi \mathfrak{F}'$ is also infinite dimensional, so that the following notation makes sense:

$$\eta_\Theta(\varphi) = \inf \sigma_e\big((A_\varphi^* A_\varphi)^{1/2}\big),$$

where σ_e denotes the essential spectrum. This function is in the following relation with the function

$$\hat{\eta}_\Theta(\mu) = \inf \sigma_e\big((\Theta(\mu)\Theta(\mu)^*)^{1/2}\big), \quad \mu \in D$$

considered in [2]:

$$\eta_\Theta(p_\mu) = \hat{\eta}_\Theta(\mu), \quad \mu \in D.$$

Indeed, denoting by $\Phi(\mathfrak{F}')$ the set of finite codimensional subspaces of $\mathfrak{F}'$, and applying relation (2.13) of [2], we deduce:

$$\eta_\Theta(p_\mu) = \inf \sigma_e\big((A_{p_\mu}^* A_{p_\mu})^{1/2}\big) = \sup_{\mathfrak{F}_0 \in \Phi(\mathfrak{F}')} \inf_{\substack{a \in \mathfrak{F}_0 \\ \|a\|=1}} \|A_{p_\mu}(p_\mu a)\|_2 =$$

$$= \sup_{\mathfrak{F}_0 \in \Phi(\mathfrak{F}')} \inf_{\substack{a \in \mathfrak{F}_0 \\ \|a\|=1}} \|T_\Theta^* p_\mu a\|_2 = \sup_{\mathfrak{F}_0 \in \Phi(\mathfrak{F}')} \inf_{\substack{a \in \mathfrak{F}_0 \\ \|a\|=1}} \|\Theta(\mu)^* a\| =$$

$$= \inf \sigma_e\big((\Theta(\mu)\Theta(\mu)^*)^{1/2}\big) = \hat{\eta}_\Theta(\mu).$$

The set

$$R_\Theta = \{\varphi \in H^2 \colon 0 < \|\varphi\|_2 \leqq 1, \ \eta_\Theta(\varphi) = 0\}$$

will play an important role in the sequel. Clearly,

$$R_\Theta \supset \{p_\mu \colon \mu \in D,\ \hat{\eta}_\Theta(\mu) = 0\}.$$

4.1. Lemma. *Assume that* $\{\varphi_j\}_1^\infty \subset R_\Theta$, *and that* $\{\varepsilon_j\}_1^\infty$ *and* $\{N_j\}_1^\infty$ *are sequences of positive reals and positive integers, respectively. Then there exists a sequence* $\{\mathfrak{F}'_j\}_1^\infty$ *of pairwise orthogonal subspaces of* $\mathfrak{F}'$ *satisfying the following conditions:*

(*i*) $\dim \mathfrak{F}'_j = N_j$, (ii) $\|T_\Theta^* | \varphi_j \mathfrak{F}'_j\| \leqq \varepsilon_j \quad (j = 1, 2, \ldots)$.

Proof. Denote by E_j the spectral measure of the self-adjoint operator $(A_{\varphi_j}^* A_{\varphi_j})^{1/2}$. Since $\eta_\Theta(\varphi_j) = 0$, the space $E_j[0, \varepsilon_j](\varphi_j \mathfrak{F}')$ must be infinite dimensional. A straightforward inductive argument proves the existence of orthogonal subspaces $\mathfrak{F}'_j \subset \mathfrak{F}'$ with $\dim \mathfrak{F}'_j = N_j$ and such that $\varphi_j \mathfrak{F}'_j \subset E_j[0, \varepsilon_j](\varphi_j \mathfrak{F}')$. These subspaces also satisfy condition (ii) so the proof is complete.

We are now almost ready to prove the factorization theorem which is our main aim in this paper. The proof will be by stepwise approximation, and the basic step is as follows.

4.2. Lemma. *Suppose that the set* R_Θ *is dominating for the measurable set* $\sigma \subset C$. *If we are given* $\eta > 0$, *a function* $Z \in L^1(\mathscr{K}(\mathfrak{H}))$, $\mathfrak{H}(\Theta)$*-oriented functions* $X, Y \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$, *and a positive number* ω *such that*

$$\|\chi_\sigma(Z - Y^* X)\|_1 < \omega,$$

then there exist $\mathfrak{H}(\Theta)$*-oriented functions* $X', Y' \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$ *such that*

(i) $\|\chi_\sigma(Z - Y'^* X')\|_1 < \eta$; *and* (ii) $\|X' - X\|_2 < \omega^{1/2}$, $\|Y' - Y\|_2 < \omega^{1/2}$.

Proof. Fix $\varepsilon > 0$ and ω' such that $\|\chi_\sigma(Z - Y^* X)\|_1 < \omega' < \omega$. An easy application of Lemma 2.6 shows that we can find an integer $n > 0$, functions $\varphi_1, \varphi_2, \ldots, \ldots, \varphi_n \in R_\Theta$, and operators $C_1, C_2, \ldots, C_n \in \mathscr{F}(\mathfrak{H})$ such that

$$\Big\|\chi_\sigma\Big(Z - Y^* X - \sum_{j=1}^{n} |\varphi_j|^2 C_j\Big)\Big\|_1 < \varepsilon \quad \text{and} \quad \sum_{j=1}^{n} \|C_j\| < \omega'. \tag{4.3}$$

Choose now a new positive number δ that will also depend on n. We apply Lemma 4.1 to produce a system $\{\mathfrak{F}'_{ij} \colon 1 \leqq i < \infty,\ 1 \leqq j \leqq n\}$ of pairwise orthogonal subspaces of $\mathfrak{F}'$ such that $\dim \mathfrak{F}'_{ij} = \operatorname{rank} C_j$ and

$$\|T_\Theta^* | \varphi_j \mathfrak{F}'_{ij}\| < \delta \quad (1 \leqq i < \infty,\ 1 \leqq j \leqq n). \tag{4.4}$$

We can then choose isometries $W_{ij} \colon C_j^* \mathfrak{H} \to \mathfrak{F}'_{ij}$, write the polar decompositions $C_j = U_j A_j$ with $A_j = (C_j^* C_j)^{1/2}$, and set

$$B_{ij} = W_{ij} A_j^{1/2}, \quad D_{ij} = W_{ij} A_j^{1/2} U_j^*.$$

We clearly have $C_j = D_{ij}^* B_{ij}$ and $\|B_{ij}\| = \|D_{ij}\| = \|C_j\|^{1/2} < \omega'^{1/2}$. In addition, the spaces $\{B_{ij}\mathfrak{H}: 1\leqq i<\infty, 1\leqq j\leqq n\}$ and $\{D_{ij}\mathfrak{H}: 1\leqq i<\infty, 1\leqq j\leqq n\}$ are pairwise orthogonal.

We make a final choice using Lemma 3.11: We choose integers $i_1, i_2, \ldots, i_n$ one by one in this order, such that, upon setting

$$B_j = B_{i_j, j}, \quad D_j = D_{i_j, j} \quad (1 \leqq j \leqq n)$$

we have

$$\|\Phi^*(\varphi_j \circ B_j)\|_1 < \delta, \quad \|\Phi^*(\varphi_j \circ D_j)\|_1 < \delta \quad \text{for } \Phi = X \text{ and } Y \ (1 \leqq j \leqq n). \tag{4.5}$$

With all these choices made (we still have to say what conditions ε and δ must satisfy!) we define

$$X' = X + \sum_1^n (\varphi_j \circ B_j), \quad Y' = Y + \sum_1^n (\varphi_j \circ D_j).$$

From the general relation $\varphi \circ A = (\varphi A \oplus 0) - VT_\Theta^* \varphi A$, where V is the isometry in (3.1), $\varphi \in H^2$, and $A \in \mathscr{F}(\mathfrak{H}, \mathfrak{F}')$, we have

$$\|X' - X\|_2 \leqq \Big\|\sum_1^n \varphi_j B_j\Big\|_2 + \sum_1^n \|T_\Theta^* \varphi_j B_j\|_2.$$

Since the operators B_j have pairwise orthogonal ranges, we have

$$\Big\|\sum_1^n \varphi_j B_j\Big\|_2 \leqq \Big(\sum_1^n \|B_j\|^2\Big)^{1/2} = \Big(\sum_1^n \|C_j\|\Big)^{1/2}.$$

On the other hand, using inequality (4.4) we deduce from Lemma 3.10 that

$$\|T_\Theta^* \varphi_j B_j\|_2 \leqq \delta\, (\text{rank } B_j)^{1/2} \|B_j\| = \delta\, (\text{rank } C_j)^{1/2} \|C_j\|^{1/2}. \tag{4.6}$$

We conclude:

$$\|X' - X\|_2 \leqq \omega'^{1/2} + \delta \omega'^{1/2} \sum_1^n (\text{rank } C_j)^{1/2}.$$

The same inequality obviously holds for $\|Y' - Y\|_2$ too. It is now clear that δ can be chosen so that the inequalities (ii) of the statement are verified (note that δ is chosen *after* the C_j, $1\leqq j\leqq n$).

In order to verify (i) we first note that the orthogonality of the spaces $\mathfrak{F}'_{ij}$ implies that

$$(\varphi_i D_i)^* \varphi_j B_j = 0 \quad (i \neq j), \quad (\varphi_j D_j)^* \varphi_j B_j = |\varphi_j|^2 D_j^* B_j = |\varphi_j|^2 C_j.$$

We have

$$Y'^* X' = Y^* X + Y^* \Big(\sum_1^n \varphi_j \circ B_j\Big) + \Big(\sum_1^n \varphi_j \circ D_j\Big)^* X + Q,$$

where

$$Q = \sum_1^n \sum_1^n (\varphi_j \circ D_j)^* (\varphi_i \circ D_i) = \sum_1^n (\varphi_j D_j \oplus 0)^* (\varphi_j B_j \oplus 0) +$$

$$+ \sum_1^n \big((\varphi_j D_j \oplus 0)^* R + S^* (\varphi_j B_j \oplus 0)\big) + S^* R,$$

with

$$R = \sum_1^n V T_\Theta^* (\varphi_j B_j), \quad S = \sum_1^n V T_\Theta^* (\varphi_i D_i).$$

Comparing these relations and applying the obvious inegality $\|LM\|_1 \leqq \|L\|_2 \|M\|_2$ we obtain:

$$\|\chi_\sigma (Z - Y'^* X')\|_1 \leqq \Big\|\chi_\sigma (Z - Y^* X - \sum_1^n |\varphi|_j^2 C_j\Big\|_1 + \sum_1^n \big(\|Y^* (\varphi_j \circ B_j)\|_1 +$$

$$+ \|(\varphi_j \circ D_j)^* X\|_1\big) + \sum_1^n (\|\varphi_j\|_2 \|D_j\| \cdot \|R\|_2 + \|S\|_2 \cdot \|\varphi_j\|_2 \|B_j\|) + \|S\|_2 \|R\|_2.$$

Applying inequalities (4.3), (4.5), (4.6) and the fact that $\|D_j\| = \|B_j\| = \|C_j\|^{1/2} < \omega$, we infer

$$\|\chi_\sigma (Z - Y'^* X')\|_2 \leqq \varepsilon + 2n\delta + 2\delta\omega \sum_1^n (\text{rank } C_j)^{1/2} + \delta^2 \omega \Big(\sum_1^n (\text{rank } C_j)^{1/2}\Big)^2,$$

and it clearly follows that (i) is verified if ε and δ are chosen appropriately small. The lemma is proved.

We prove now the main result of this paper, which is a rather standard self-improvement of Lemma 4.2. It actually generalizes Theorem A of [2], case $\vartheta = 0$.

4.7. Theorem. *Suppose that the set R_Θ is dominating for the measurable set $\sigma \subset C$. If we are given a function $Z \in L^1(\mathscr{K}(\mathfrak{H}))$, and $\mathfrak{H}(\Theta)$-oriented functions $X, Y \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$, then for any number ω for which $\|\chi_\sigma (Z - Y^* X)\|_1 < \omega$, there exist $\mathfrak{H}(\Theta)$-oriented functions $X', Y' \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$ such that*

(i) *$Z(z) = (Y'^* X')(z)$ for almost every $z \in \sigma$, and*

(ii) *$\|X' - X\|_2 < \omega^{1/2}$, $\|Y' - Y\|_2 < \omega^{1/2}$.*

Proof. Choose ω' such that $\|\chi_\sigma (Z - Y^* X)\|_1 < \omega' < \omega$, and a positive number $\vartheta < 1$ such that $(1 - \vartheta^{1/2})^{-1} \omega'^{1/2} < \omega^{1/2}$. Set $X_0 = X$ and $Y_0 = Y$. An inductive application of Lemma 4.2 shows the existence of $\mathfrak{H}(\Theta)$-oriented functions X_n, Y_n $(n \geqq 1)$ satisfying for $n \geqq 0$ the inequalities

$$\|\chi_\sigma (Z - Y_n^* X_n)\|_1 < \vartheta^n \omega', \quad \text{and} \quad \|X_{n+1} - X_n\|_2, \ \|Y_{n+1} - Y_n\|_2 \leqq (\vartheta^n \omega')^{1/2}.$$

These inequalities show that $\{X_n\}$ and $\{Y_n\}$ are Cauchy sequences in $L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$

and, upon setting $X'=\lim_{n\to\infty} X_n$, $Y'=\lim_{n\to\infty} Y_n$, we have

$$\|X'-X\|_2 \leqq \sum_{n=0}^{\infty} \|X_{n+1}-X_n\|_2 \leqq \sum_{n=0}^{\infty} (\vartheta^n\omega')^{1/2} = (1-\vartheta^{1/2})^{-1}\omega'^{1/2} < \omega^{1/2}$$

and, analogously, $\|Y'-Y\|_2<\omega^{1/2}$. It also follows that $\|\chi_\sigma(Z-Y'^*X')\|_1=0$, and this concludes the proof of our theorem.

We remind the reader of the fact from [2] that the relations in the introductory part of Section 4 imply that the assumption of Theorem 4.7 concerning the set R_Θ is certainly satisfied if the right essential spectrum of the model operator $S(\Theta)$ is dominating for the set σ.

A natural continuation of the circle of ideas in this paper takes place in [1], where applications to invariant subspaces and reflexivity are discussed.

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*) It should be remarked that the proof of Lemma 2.4 in [2], p. 249, needs some correction because the term $(\omega^{1/2}r^{1/2}+\omega^{1/2}r^{1/2}+\omega r)\varepsilon$ in the estimate of $\|\Omega\|$ contains the number r depending on ε, and therefore may not be small for ε small. However, this situation can be easily remedied by requiring in the choice of the sequence $\{b_m\}_1^r$ in (2.23), that $\|l(\mu_m\circ b_m)^*\|_1\leqq\varepsilon/r$, instead of $\leqq\varepsilon$. (The term in question changes then to $(2\omega^{1/2}+\omega)\varepsilon$.)

Further minor corrections for the same page in [2]: in the 2nd rows from above and from below change S for S_θ, and (2.19) for (2.20), respectively.

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An extension of the Lindeberg—Trotter operator-theoretic approach to limit theorems for dependent random variables I. General convergence theorems; approximation theorems with o-rates

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Dedicated to Professor Károly Tandori on the occasion of his sixtieth birthday, in high esteem

1. Introduction. This paper is concerned with a generalization of the classical Lindeberg—Trotter operator-theoretic approach to the central limit theorem (=CLT) (see e.g. [24], [15, p. 113], [10, p. 248], [20, p. 223], [18, p. 207]), so far restricted to independent random variables (=r.v.'s), to the general case of arbitrary *dependent* r.v.'s. One of the great advantages of the classical Trotter approach is that it can also cover the weak law of large numbers (=WLLN), indeed any limit theorem dealing with convergence in distribution of r.v.'s and, above all, it can even cover limit theorems equipped with O-rates or o-rates of convergence, all in the case that the r.v.'s are independent (see e.g. [6], [11, p. 157], [19], [22], [5], [21]). A further advantage of the method is that it is elementary in the sense that it does not use Fourier analytic machinery at all.

Any attempt to generalize the Trotter approach to the situation of dependent r.v.'s leads to principal difficulties. Already in the „resctrictedly" dependent case of martingale difference sequences (MDS) and arrays (MDA) did the Trotter approach have to be modified considerably in order to cover the particular type of dependency in question (see e.g. [1], [23], [9], [2], [7], [8]). In order to comprehend these difficulties let us recall the basic principles of the Trotter approach.

If $(X_k)_{k\in\mathbf{N}}$ is a sequence of independent r.v.'s and f any function belonging to the space C_B (see Section 2 for definition), the Trotter operator $V_{X_k}\colon C_B\to C_B$ associated with X_k is defined (cf. (3.1)) for each $y\in\mathbf{R}$ as the expectation of the r.v.

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$f(X_k+y)$,

$$V_{X_k}f(y) = E[f(X_k+y)] \quad (k\in\mathbf{N}).$$

One of the basic properties of this operator, the proof of which rests upon the relation $P_{\sum_{k=1}^n X_k} = \mathop{*}\limits_{k=1}^{n} P_{X_k}$, valid for the distributions P_{X_k} of independent r.v.'s X_k, is

$$V_{\sum_{k=1}^n X_k} f = V_{X_1}V_{X_2}\dots V_{X_n}f \quad (f\in C_B;\ n\in\mathbf{N}). \tag{1.1}$$

If $(Z_k)_{k\in\mathbf{N}}$ is a further sequence of r.v.'s which are independent not only amongst themselves but also of the X_k, then (1.1) leads to the basic inequality

$$\|V_{\sum_{k=1}^n X_k} f - V_{\sum_{k=1}^n Z_k} f\|_{C_B} \leqq \sum_{n=1}^{n} \|V_{X_k}f - V_{Z_k}f\|_{C_B} \tag{1.2}$$

valid for any $f\in C_B$, where $\|g\|_{C_B} = \sup_{y\in\mathbf{R}} |g(y)|$.

Now an equivalent formulation of the CLT for independent, identically distributed r.v.'s states that

$$|E[f(n^{-1/2}S_n)] - E[f(X^*)]| = o(1) \quad (n\to\infty) \tag{1.3}$$

for any $f\in C_B$, where $S_n = \sum_{k=1}^n X_k$, and X^* is a standard normally distributed r.v. This is a particular case ($y=0$!) of

$$\|V_{n^{-1/2}S_n} f - V_{X^*}f\|_{C_B} = o(1) \quad (n\to\infty)$$

for any $f\in C_B$. In order to be able to include such limit theorems under (1.2), X^* must be $(n^{-1/2})$-decomposable in the form

$$P_{X^*} = P_{n^{-1/2}\sum_{k=1}^n Z_k}, \tag{1.4}$$

where the decomposition components Z_k are $\mathcal{N}(0,\sigma_k^2)$-distributed r.v.'s with $\sigma_k^2 = \mathrm{Var}\,[X_k]$ which may, without loss of generality (see [3, p. 164]), be chosen to be independent amongst themselves as well as of the r.v.'s X_k. In that case one has by (1.4), noting that the Trotter operator just involves the distribution of the associated r.v.,

$$V_{X^*}f = V_{n^{-1/2}\sum_{k=1}^n Z_k} f \quad (f\in C_B).$$

So establishing (1.3) just amounts to showing, noting (1.2) with all r.v.'s multiplied by the factor $n^{-1/2}$, that

$$\|V_{n^{-1/2}X_k} f - V_{n^{-1/2}Z_k} f\|_{C_B} = o\left(\frac{1}{n}\right) \quad (f\in C_B;\ n\to\infty). \tag{1.5}$$

Assertions for single differences of type (1.5) can easily be estimated by assuming that the moments of the r.v.'s X_k and Z_k coincide up to the order 2.

If one would wish to equip the CLT in the form (1.3) or other weak limit theorems with rates, it would suffice to supply (1.5) with rates better than $o(1/n)$, which is possible if the corresponding moments of higher orders are equal to another. However, the whole procedure is only applicable to *independent* r.v.'s since the basic properties used, namely (1.1) and (1.2), are only valid for such r.v.'s.

A first indication that cognate methods of proof could possibly be applicable to *dependent* r.v.'s is the paper [12] by Z. Govindarajulu who established the WLLN for triangular arrays of dependent r.v.'s. Foı this purpose he used a property corresponding to (1.1), one tailored to the situation of dependent r.v.'s; but he had to replace inequality (1.2) by estimates of a different type.

The chief aim of this paper, however, is to present an operator-theoretic approach that allows one to generalize the Trotter operator-technique, one that has stood the test, to dependent r.v.'s. The development of the present approach may in some sense be compared with that of Trotter's: similarly as did Lindeberg's proof of the CLT of 1922 (cf. [17]) serve Trotter as the basis for his operator approach, so did Govindarajulu's paper give the impulse to our definition of a „conditional Trotter operator" (cf. Def. 1 in Section 3). However, its applicability is not only confined to a proof of the CLT or WLLN. These theorems will, much more, be deduced as particular cases of a very general limit theorem, which can even be supplied with o-rates or O-rates of convergence, as will be shown in Section 5 and 6.

For the sake of clarity let us present a particular case of the "conditional Trotter operator" tailored to MDS. If $(X_k)_{k\in\mathbf{N}}$ is a MDS, i.e., $E[X_k|\mathfrak{F}_{k-1}]=0$ a.s., $k\in\mathbf{N}$, where $\mathfrak{F}_{k-1}=\mathfrak{A}(X_1, \ldots, X_{k-1})$ is the σ-algebra generated by the r.v.'s $X_1, \ldots, X_{k-1}$, then the conditional Trotter operator $V_{X_k}^{\mathfrak{F}_{k-1}}$ is defined for each $f\in C_B$ and $y\in\mathbf{R}$ as the conditional expectation of the r.v. $f(X_k+y)$ relative to $\mathfrak{F}_{k-1}$, i.e.,

$$(V_{X_k}^{\mathfrak{F}_{k-1}}f)(y):=E[f(X_k+y)|\mathfrak{F}_{k-1}]\quad(k\in\mathbf{N}).$$

If the r.v.'s X_k are independent, then the properties associated with conditional expectation yield that

$$V_{X_k}^{\mathfrak{F}_{k-1}}f=E[f(X_k+\cdot)|\mathfrak{F}_{k-1}]=E[f(X_k+\cdot)]=V_{X_k}f,$$

so that the conditional Trotter operator coincides with the classical Trotter operator. Furthermore, the operator $V_{X_k}^{\mathfrak{F}_{k-1}}$ has all of the basic characteristics of V_{X_k}, so that it is possible to establish with its help the counterparts of the properties (1.1) and (1. 2) for dependent r.v.'s (see (3.6) and (3.7)). For this reason it is not only possible to extend all of the limit theorems established by means of Trotter operators for independent r.v.'s to the case of arbitrary dependent r.v.'s — whereby the dependency structure just depends upon moment conditions of type (4.1) — but also to extend them to particular types of restrictedly dependent r.v.'s, namely to MDS and MDA, without having to modify the proofs as has been necessary so far (see e.g. [7, 8]).

Concerning a comparison with the literature existing in the field, let us first note that apart from the paper [12] cited for the WLLN as well as another [13] by P. Gudynas, no further papers are known to the authors that deal with assertions on convergence in distribution without restricting the dependency structure in some way or other. The r.v.'s are either assumed to be independent or dependent in the sense of MDS, MDA, or inverse martingales. Whereas the WLLN without rates is also a particular case of our results (see Theorem 3), direct comparisons with the results of Gudynas are hardly possible since he is concerned with inequalities for metrics of vector-valued r.v.'s. Points of comparison with other papeıs devoted to independent r.v.'s or to MDS or MDA will be gone into in the course of the paper.

Part I of this paper consists of five sections, the second of which is concerned with the preliminary results needed from approximation and probability theory. Section 3 deals with the definition of the conditional Trotter operator and its basic properties, while Section 4 is devoted to the general limit theorem, namely Theorem 1, which is then applied to give the CLT and WLLN. Section 5 contains the general approximation with o-rates, Theorem 4, together with applications. The second and last part of the paper, covering Sections 6 to 8, begins with two general approximation theorems with O-rates for convergence in distribution (Theorems 7 and 8) which are applied to yield to O-error estimates for assertions of Berry—Esséen-type, i.e. for the uniform convergence of distribution functions (Theorem 11 and 12), dealt with in Section 7. Section 8 is concerned with the particular case of MDA as well as with the existing literature in the matter.

2. Notations and preliminaries. In the following, $C_B=C_B(\mathbf{R})$ will denote the vector space of all real-valued, bounded, uniformly continuous functions defined on the reals $\mathbf{R}$, endowed with norm $\|f\|_{C_B}$: $\sup_{x\in\mathbf{R}}|f(x)|$. For $r\in\mathbf{P}:=\mathbf{N}\cup\{0\}$ we set

$$C_B^0:=C_B,\quad C_B^r:=\{g\in C_B;\ g^{(j)}\in C_B,\ 1\leqq j\leqq r\},$$

the seminorm on C_B^r being given by $|g|_{C_B^r}:=\|g^{(r)}\|_{C_B}$. For any $f\in C_B$ and $t\geqq 0$ the K-functional, needed in Part II, is defined by

$$K(t;\ f;\ C_B,\ C_B^r):=\inf_{g\in C_B^r}\{\|f-g\|_{C_B}+t|g|_{C_B^r}\}.$$

This functional is equivalent to the rth modulus of continuity, defined for $f\in C_B$ by

$$\omega_r(t;\ f;\ C_B):=\sup_{|h|\leqq t}\Big\|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(u+kh)\Big\|_{C_B},$$

in the sense that there are constants $c_{1,r};c_{2,r}>0$, independent of f and $t\geqq 0$, such that (see [4, pp. 192, 258])

$$c_{1,r}\omega_r(t^{1/r};\ f;\ C_B)\leqq K(t;\ f;\ C_B,\ C_B^r)\leqq c_{2,r}\omega_r(t^{1/r};\ f;\ C_B). \tag{2.1}$$

Lipschitz classes of index $r\in\mathbf{N}$ and order $\alpha, 0<\alpha\leqq r$ will be needed in Part I. They are defined for $f\in C_B$ by

$$\text{(2.2)} \qquad \mathrm{Lip}\,(\alpha; r; C_B) := \{\omega_r(t; f; C_B) \leqq L_f t^\alpha\},$$

L_f being the so-called Lipschitz constant. Note that for $\alpha=r'+\beta$, $r'\leqq r-1$, $0<\beta\leqq 1$ (see [14])

$$\text{(2.3)} \qquad f^{(r')}\in \mathrm{Lip}\,(\beta;\ r-r';\ C_B) \Rightarrow f\in \mathrm{Lip}\,(r'+\beta;\ r;\ C_B).$$

Several preliminaries from probability theory will be noted. Let $(\Omega, \mathfrak{A}, P)$ denote a probability space with set Ω, σ-algebra $\mathfrak{A}$ and probability measure P, $\mathfrak{B}$ the σ-algebra of Borel sets in $\mathbf{R}$, $\mathfrak{Z}(\Omega, \mathfrak{A}):=\{X\colon\ \Omega\to\mathbf{R},\ X \text{ is } \mathfrak{A}, \mathfrak{B}\text{-measurable}\}$ the set of all real r.v.'s on Ω, and $\mathfrak{L}(\Omega, \mathfrak{A}, P):=\{X\in\mathfrak{Z}(\Omega, \mathfrak{A});\ X \text{ is } P\text{-integrable}\}$ the set of all real P-integrable r.v.'s on Ω.

The general convergence theorems of this paper will be formulated, as indicated in the introduction, for φ-decomposable r.v.'s. If $\varphi\colon \mathbf{N}\to\mathbf{R}^+$ is a positive normalizing function, then $Z\in\mathfrak{Z}(\Omega, \mathfrak{A})$ is called φ-decomposable, if for each $n\in\mathbf{N}$ there exist n independent r.v.'s $Z_k=Z_{k,n}$, $1\leqq k\leqq n$, such that the distributions of the r.v. Z and the normalized sums $\varphi(n)\sum_{k=1}^{n} Z_k$ coincide, i.e., if

$$\text{(2.4)} \qquad P_Z = P_{\varphi(n)\sum_{k=1}^{n} Z_k}.$$

An important concept needed for the proofs will be the conditional expectation (see e.g. [3, p. 292]), to be denoted for $X\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$ and each sub-σ-algebra $\mathfrak{G}\subset\mathfrak{A}$ by $E[X|\mathfrak{G}]$. If Y also belongs to $\mathfrak{L}(\Omega, \mathfrak{A}, P)$, and $\mathfrak{G}'$ is a further sub-σ-algebra of $\mathfrak{A}$, then there hold the properties (see e.g. [3, p. 293f.])

$$\text{(2.5)} \qquad E\big[E[X|\mathfrak{G}]\big] = E[X];$$

$$\text{(2.6)} \qquad E[X|\mathfrak{G}_0] = E[X] \quad \text{a.s. for} \quad \mathfrak{G}_0 = \{\Phi, \Omega\};$$

$$\text{(2.7)} \qquad X \leqq Y \quad \text{a.s. implies} \quad E[X|\mathfrak{G}] \leqq E[Y|\mathfrak{G}] \quad \text{a.s.};$$

$$\text{(2.8)} \qquad X = c \quad \text{a.s., some} \quad c\in\mathbf{R}, \quad \text{implies} \quad E[X|\mathfrak{G}] = c \quad \text{a.s.};$$

$$\text{(2.9)} \qquad E[\alpha X+\beta Y|\mathfrak{G}] = \alpha E[X|\mathfrak{G}]+\beta E[Y|\mathfrak{G}] \quad \text{a.s.} \quad (\alpha, \beta\in\mathbf{R});$$

(2.10) $E[X|\mathfrak{G}] = E[X]$ a.s. provided the σ-algebra $\mathfrak{A}(X)$, generated by X, is independent of $\mathfrak{G}$;

$$\text{(2.11)} \qquad E\big[E[X|\mathfrak{G}]|\mathfrak{G}'\big] = E\big[E[X|\mathfrak{G}']|\mathfrak{G}\big] = E[X|\mathfrak{G}] \quad \text{a.s.}$$

The aim now is to represent the conditional expectation as an integral. For this purpose two concepts need be recapitulated. If $\mathfrak{G}\subset\mathfrak{A}$ is a σ-algebra and $X\in\mathfrak{Z}(\Omega, \mathfrak{A})$, a function $\hat{P_X}\colon\ \Omega\times\mathfrak{A}\to\mathbf{R}$ is said to be a *regular conditional probability distribution* of X relative to $\mathfrak{G}$, if it satisfies the conditions (see e.g. [16, p. 372ff.]): (i) For every fixed $\omega \in \Omega$, the set function $\hat{P_X}(\omega, \cdot)$, defined on $\mathfrak{A}$, is a probability measure; (ii) for every fixed $A \in \mathfrak{A}$, $\hat{P_X}(\cdot, A) \in \mathfrak{Z}(\Omega, \mathfrak{G})$; (iii) for every $A \in \mathfrak{A}$ and $G \subset \mathfrak{G}$, there holds

$$\int_G \hat{P_X}(\omega, X^{-1}(A))\, dP = P(G \cap X^{-1}(A)).$$

The function F_Z: $\mathbf{R} \times \Omega \to \mathbf{R}$, defined by

$$F_X(x|\mathfrak{G}) = F_X(x|\mathfrak{G})(\omega) = P_{\hat{X}}(\omega, (-\infty, x]) \quad \text{a.s.} \quad (x \in \mathbf{R});$$

is called a *conditional distribution function* of X with respect to $\mathfrak{G}$. [Note that if $(\Omega, \mathfrak{A}, P)$ is an arbitrary probability space, and $\mathfrak{G}$ an arbitrary sub-σ-algebra of $\mathfrak{A}$, then for each $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$ there always exists a regular conditional distribution (and so also a conditional distribution function) of X with respect to $\mathfrak{G}$ (see e.g. [16, p. 373])].

Now to the integral representation. Let $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$, $\mathfrak{G}$ be a sub-σ-algebra of $\mathfrak{A}$, g: $\mathbf{R} \to \mathbf{R}$ a Borel-measurable function with $E[g(X)] < \infty$, and $F_X(X|\mathfrak{G})$ be a conditional distribution function of X relative to $\mathfrak{G}$. Then there exists a $G \in \mathfrak{G}$ with $P(G)=0$ such that for all $\omega \in \Omega \setminus G$ (sdee [16, p. 375])

$$E[g(X)|\mathfrak{G}](\omega) = \int g(x)\, d(F_X(x|\mathfrak{G})(\omega)). \tag{2.12}$$

For the proofs an (ordinary) Lindeberg condition of order s, $s>0$ — generalized to the situation of a φ-decomposable limiting r.v. (cf. [5]) — and sometimes the usual Feller condition will be needed. Both will be formulated for $X_k \in \mathfrak{Z}(\Omega, \mathfrak{A})$. If $X_k^s \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ for some $s \in (0, \infty)$ and all $k \in \mathbf{N}$, then the sequence $(X_k)_{k \in \mathbf{N}}$ satisfies a *Lindeberg condition of order* s, if for every $\delta > 0$

$$\Big(\sum_{k=1}^{n} \int_{|x| \geq \delta/\varphi(n)} |x|^s\, dF_{X_k}(x)\Big) \Big/ \Big(\sum_{k=1}^{n} E[|X_k|^s] \to 0 \quad (n \to \infty). \tag{2.13}$$

If $0 < \sigma_k^2 < \infty$, where $\sigma_k^2 := E[X_k^2]$, $k \in \mathbf{N}$, and $s_n = \big(\sum_{k=1}^{n} \sigma_k^2\big)^{1/2}$, then $(X_k)_{k \in \mathbf{N}}$ satisfies a *Feller-condition*, if

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} \frac{\sigma_k^2}{s_n} = 0. \tag{2.14}$$

3. A generalization of the Trotter-operator for dependent r.v.'s. As already mentioned in the introduction, the Trotter-operator plays an important role in establishing rates of convergence for independent r.v.'s. For the development of corresponding assertions in the instance of dependent r.v.'s a new operator concept — closely related to the usual Trotter-operator — will be introduced in this paper. To elucidate the connections, let us first recall the definition of the Trotter-operator and its most important properties.

For any $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$ having distribution function F_X the associated Trotter-operator $V_X\colon C_B \to C_B$ is defined for $f \in C_B$ by

$$V_X f(y) := \int_{\mathbf{R}} f(x+y)\, dF_X(x) = E[f(X+y)], \quad (y \in \mathbf{R}). \tag{3.1}$$

Lemma 1. *Let $X, Y \in \mathfrak{Z}(\Omega, \mathfrak{A})$. Let $X_1, \ldots, X_n, Z_1, \ldots, Z_n$, $n \in \mathbf{N}$, be independent r.v.'s belonging to $\mathfrak{Z}(\Omega, \mathfrak{A})$. Then*

a) *V_X is a positive, linear operator satisfying inequality*

$$\|V_X f\|_{C_B} \leqq \|f\|_{C_B} \quad (f \in C_B); \tag{3.2}$$

b) *$V_X = V_Y$ provided X and Y are identically distributed;*

c) *V_X and V_Y are commutative provided X and Y are independent;*

(3.3) d)
$$V_{S_n} f = V_{X_1} V_{X_2} \ldots V_{X_n} f \quad (f \in C_B);$$

(3.4) e)
$$\|V_{S_n} f - V_{\sum_{k=1}^n Z_k} f\|_{C_B} \leqq \sum_{k=1}^n \|V_{X_k} f - V_{Z_k} f\|_{C_B} \quad (f \in C_B).$$

The Trotter operator may be generalized as follows by using the concept of conditional expectation.

Definition 1. Let $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ and $\mathfrak{G}$ be an arbitrary sub-σ-algebra of $\mathfrak{A}$. The conditional Trotter operator $V_X^{\mathfrak{G}}\colon C_B \to C_B \times (\mathfrak{Z}(\Omega, \mathfrak{G}))$ of X relative to $\mathfrak{G}$ is defined for $f \in C_B$ by

$$V_X^{\mathfrak{G}} f(y) := E[f(X+y) \mid \mathfrak{G}] \quad (y \in \mathbf{R}).$$

The most important properties of this operator, which is uniquely determined up to a set of measure zero by definition, are collected in the following lemma; below one has set $((V_X^{\mathfrak{G}} f)(y))(\omega) = (V_X^{\mathfrak{G}} f)(y, \omega)$.

Lemma 2. *Let $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$, $\mathfrak{G}$ be an arbitrary sub-σ-algebra of $\mathfrak{A}$, and f and g belong to C_B. Then*

a) $(V_X^{\mathfrak{G}} f)(y, \cdot) \in \mathfrak{Z}(\Omega, \mathfrak{G})$ $(y \in \mathbf{R})$;

b) *there exists a set $G_1 \in \mathfrak{G}$ with $P(G_1) = 0$ such that*

$$\sup_{y \in \mathbf{R}} |(V_X^{\mathfrak{G}} f)(y, \omega)| \leqq \|f\|_{C_B} \quad (\omega \in \Omega \setminus G_1;\ f \in C_B);$$

c) *there exists a set $G_2 \in \mathfrak{G}$ with $P(G_2) = 0$ such that $(V_X^{\mathfrak{G}} f)(\cdot, \omega) \in C_B$ for all $\omega \in \Omega \setminus G_2$;*

d) *there exists a set $G_3 \in \mathfrak{G}$ with $P(G_3) = 0$ such that $(V_X^{\mathfrak{G}}(\alpha f + \beta g))(\cdot, \omega) = \alpha (V_X^{\mathfrak{G}} f)(\cdot, \omega) + \beta (V_X^{\mathfrak{G}} g)(\cdot, \omega)$ for all $\omega \in \Omega \setminus G_3$ and $\alpha, \beta \in \mathbf{R}$;*

e) $(V_X^{\mathfrak{G}} f)(y) = E[f(X+y) \mid \mathfrak{G}] = (V_X f)(y)$ *a.s. provided $\mathfrak{A}(X)$ is independent of $\mathfrak{G}$.*

Proof a). An immediate consequence of Definition 1.

b) In view of (2.7), (2.8) one has

$$\sup_{y\in\mathbf{R}} |V_X^{\mathfrak{G}} f(y,\omega)| \leqq \sup_{y\in\mathbf{R}} |E[f|(X+y)||\mathfrak{G}](\omega)| \leqq E[\|f\|_{C_B}|\mathfrak{G}] = \|f\|_{C_B} \quad \text{a.s.}$$

c) Since $V_X^{\mathfrak{G}} f(y)$ is bounded a.s. by part b), it remains to show that $V_X^{\mathfrak{G}} f$ is uniformly continuous a.s. Let $\varepsilon>0$ be arbitrary. Since $f\in C_B(\mathbf{R})$, there exists a $\delta>0$ such that $|f(y_1)-f(y_2)|<\varepsilon$ for all $y_1, y_2\in\mathbf{R}$ with $|y_1-y_2|<\delta$, so that $\sup_{x\in\mathbf{R}} |f(x+y_1)-f(x+y_2)|<\varepsilon$. But (2.9) and (2.7) yield

$$|V_X^{\mathfrak{G}} f(y_1,\omega)-V_X^{\mathfrak{G}} f(y_2,\omega)| = |E[f(X+y_1)|\mathfrak{G}](\omega)-E[f(X+y_2)|\mathfrak{G}](\omega)| \leqq$$

$$\leqq E[|f(X+y_1)-f(X+y_2)||\mathfrak{G}](\omega) \leqq \sup_{x\in\mathbf{R}} |f(x+y_1)-f(x+y_2)| < \varepsilon \quad \text{a.s.}$$

establishing c).

d) and e) follow directly from (2.9), (2.10), respectively.

From Lemma 2 b)—d) one obtains

Corollary 1. *Let $(\Omega, \mathfrak{A}, P)$, $\mathfrak{G}$, X and f be given as in Lemma 2. There exists a set $G\in\mathfrak{G}$ with $P(G)=0$ such that $(V_X^{\mathfrak{G}} f)(\cdot,\omega)$ is a linear operator of C_B into itself for all $\omega\in\Omega\setminus G$ satisfying $\|(V_X f)(\cdot,\omega)\|_{C_B}\leqq\|f\|_{C_B}$.*

Proof. With G_1, G_2, G_3 given as in Lemma 2 b)—d), then $(V_X^{\mathfrak{G}} f)(\cdot,\omega)$ is a contraction endomorphism on C_B for each $\omega\in\Omega\setminus G$, where $G:=G_1\cup G_2\cup G_3$ with $P(G)=0$.

Basic for the main convergence theorem of this paper is the counterpart of inequality (3.4) for the operator $V_{S_n}^{\mathfrak{G}}$ for partial sums S_n of not necessarily independent r.v.'s. For this purpose two lemmas will be needed.

Lemma 3. *Given $(\Omega, \mathfrak{A}, P)$ and any $X\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, there exists a set $A=A(X)\in\mathfrak{A}$ with $P(A)>0$ such that*

$$|E[X]| \leqq |X(\omega)| \quad (\omega\in A).$$

Take $A:=\{\omega\in\Omega;\ |E[X]|\leqq|X(\omega)|\}$ and show that assumption $P(A)=0$ leads to a contradiction.

Lemma 4. *Let $X, Y\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, $f\in C_B$ and $\mathfrak{G}$ be any sub-σ-algebra of $\mathfrak{A}$. To each $y\in\mathbf{R}$ there exists a set $G^y=G^y(f,X,Y)\in\mathfrak{G}$ with $P(G^y)>0$ such that*

$$|V_{X+Y} f(y)| \leqq |V_X(V_Y^{\mathfrak{G}} f)(y,\omega)| \quad (\omega\in G^y). \tag{3.5}$$

Proof. According to Lemma 2 c) there is a set $G_1^y\in\mathfrak{G}$ with $P(G_1^y)=0$ such that $(V_Y^{\mathfrak{G}} f)(\cdot,\omega)\in C_B$ for all $\omega\in\Omega\setminus G_1^y$. Since $E[f(X+Y+y)|\mathfrak{G}]\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, on account of Lemma 3 to each $y\in\mathbf{R}$ there exists a set $G_2^y=G_2^y(f,X,Y)\in\mathfrak{G}$ with

$P(G_2^y)>0$ such that, noting (3.1), Definition 1 and (2.5),

$$|V_X((V_Y^{\mathfrak{G}}f)(y,\omega))(y) = \Big|\int_{\mathbf{R}}(V_Y^{\mathfrak{G}}f)(x+y,\omega)\,dF_X(x)\Big| =$$

$$= \big|E\big[E[f(X+Y+y)|\mathfrak{G}](\omega)\big]\big| =$$

$$= \big|[E[f(X+Y+y)]|\mathfrak{G}](\omega)\big| \geqq \big|E\big[E[f(X+Y+y)|\mathfrak{G}]\big]\big| =$$

$$= |E[f(X+Y+y)]| = |V_{X+Y}f(y)|$$

for all $\omega\in G^y := \Omega\setminus(G_1^y\cap G_2^y)$. Since $G^y\in\mathfrak{G}$ and $P(G^y)=0$ by definition of G^y, the proof is complete.

Now to the fundamental lemma of the paper, namely the counterpart of assertions (3.3) and (3.4) for the operator $V_X^{\mathfrak{G}}$.

Lemma 5. *Given* $\mathfrak{L}(\Omega,\mathfrak{A},P)$, *let* $(X_n)_{n\in\mathbf{N}}$ *be a sequence of r.v.'s from* $\mathfrak{L}(\Omega,\mathfrak{A},P)$, $(\mathfrak{G}_n)_{n\in\mathbf{N}}$ *a sequence of sub-σ-algebras from* $\mathfrak{A}$, $\mathfrak{G}_0=\{\Phi,\Omega\}$.

a) *For each* $f\in C_B$ *one has*

$$(V_{X_1}^{\mathfrak{G}_0}V_{X_2}^{\mathfrak{G}_1}\dots V_{X_n}^{\mathfrak{G}_{n-1}}f)(y) = V_{S_n}f(y) \quad a.s. \quad (y\in\mathbf{R};\ n\in\mathbf{N}). \tag{3.6}$$

b) *If* $(Z_n)_{n\in\mathbf{N}}$ *is a further sequence from* $\mathfrak{L}(\Omega,\mathfrak{A},P)$ *it being assumed that the* Z_n, *for each* $n\in\mathbf{N}$, *are independent amongst themselves as well as of the* X_n, *then there exist for each* $y\in\mathbf{R}$, $n\in\mathbf{N}$ *and* $1\leqq k\leqq n$ *sets*

$$G_{n,k-1}^y\in\mathfrak{G}_{k-1} \quad with \quad P(G_{n,k-1}^y)>0$$

such that for each $\omega=\omega(n,k,y)\in G_{n,k-1}^y$

$$\|V_{S_n}f - V_{\sum_{k=1}^n Z_k}f\| \leqq \sum_{k=1}^{n}\sup_{y\in\mathbf{R}}|(V_{X_k}^{\mathfrak{G}_{k-1}}f)(y,\omega) - V_{Z_k}f(y)| \quad (n\in\mathbf{N}). \tag{3.7}$$

Proof. Now $E[X_k|\mathfrak{G}_0]=E[X_n]$ a.s., all $k\in\mathbf{N}$ by (2.6). So a repeated application of (2.11) as well as (3.3) yield for $n\in\mathbf{N}$ and $y\in\mathbf{R}$

$$(V_{X_1}^{\mathfrak{G}_0}V_{X_2}^{\mathfrak{G}_1}\dots V_{X_n}^{\mathfrak{G}_{n-1}}f)(y) =$$

$$= E\big[E\dots E[f(X_1+\dots+X_n+y)|\mathfrak{G}_{n-1}]\dots|\mathfrak{G}_1]|\mathfrak{G}_0\big] =$$

$$= E[f(X_1+\dots+X_n+y)|\mathfrak{G}_0] = E[f(X_1+\dots+X_n+y)] =$$

$$= (V_{X_1}\dots V_{X_n}f)(y) = (V_{S_n}f)(y) \quad \text{a.s.},$$

establishing (3.6). Concerning part b), one has by (3.6), (3.2) and Lemma 1 c),

$$\|V_{S_n}f - V_{\Sigma_{k=1}^n Z_k} f\|_{C_B} = \tag{3.8}$$

$$= \sup_{y\in\mathbf{R}} \Big|\sum_{k=1}^{n} (V_{X_1}^{\mathfrak{G}_0} V_{X_2}^{\mathfrak{G}_1} \ldots V_{X_{k-1}}^{\mathfrak{G}_{k-2}} [V_{X_k}^{\mathfrak{G}_{k-1}} - V_{Z_k}] V_{\Sigma_{j=k+1}^n Z_j} f)(y)\Big| \leqq$$

$$\leqq \sum_{k=1}^{n} \sup_{y\in} |(V_{X_1}^{\mathfrak{G}_0} V_{X_2}^{\mathfrak{G}_1} \ldots V \text{ [illegible] } V_{X_k}^{\mathfrak{G}_{k-1}} - V_{Z_k}] f)(y)|.$$

According to Lemma 4 applied to the r.v.'s S_{k-1} and X_k, there exists to each $y\in\mathbf{R}$ a set $G_{k-1}^y \in \mathfrak{G}_{k-1}$ with $P(G_{k-1}^y) > 0$. Associating to each $y\in\mathbf{R}$ a fixed $\omega_{n,k}\in G_{k-1}^y$ for which inequality (3.5) holds, one deduces by applying (3.6) and (3.2) the estimate

$$|(V_{X_1}^{\mathfrak{G}_0} V_{X_2}^{\mathfrak{G}_1} \ldots V_{X_{k-1}}^{\mathfrak{G}_{k-2}} [V_{X_k}^{\mathfrak{G}_{k-1}} - V_{Z_k}] f)(y)| \leqq$$

$$\leqq |V_{S_{k-1}}(V_{X_k}^{\mathfrak{G}_{k-1}} f)(y, \omega_{n,k}) - (V_{S_{k-1}} V_{Z_k} f)(y)| \leqq$$

$$\leqq \sup_{y\in\mathbf{R}} |(V_{X_k}^{\mathfrak{G}_{k-1}} f)(y, \omega_{n,k}) - V_{Z_k} f(y)|.$$

If one now takes the supremum over all $y\in\mathbf{R}$ on the left side of this inequality and then sums over k, the proof of (3.7) follows in conjunction with (3.8).

4. Convergence theorems for dependent random variables. This section is concerned with weak convergence theorems in the case of arbitrary dependent r.v.'s. The basis is a general limit theorem which yields both the CLT and WLLN by specializing the limit r.v. Since the results of this section deal with convergence without rates, it is possible to formulate them also for uniform convergence of distribution functions or for stochatic convergence. The hypotheses are, apart from the usual Lindeberg conditions for the sequences of r.v.'s $(X_k)_{k\in\mathbf{N}}$ and the decomposition components $(Z_k)_{k\in\mathbf{N}}$, the positivity and the uniform boundedness of the second moments of the X_k, as well as the moment condition (4.1). The latter reduces to the coincidence of the first and second moments of X_k and Z_k provided the r.v.'s are independent. Since the σ-algebras $\mathfrak{G}_k$, $k\in\mathbf{P}$, occurring in (4.1) may, apart from $\mathfrak{G}_0$, be chosen freely, distinct forms of dependency are admitted.

4.1. General limit theorem.

Theorem 1. *Let $(X_k)_{k\in\mathbf{N}}$ be a sequence of dependent r.v.'s such that $0 < m \leqq E[X_k^2] \leqq M < \infty$ for $k\in\mathbf{N}$, and some constants m, $M > 0$. Let $(\mathfrak{G}_k)_{k\in\mathbf{N}}$ be a sequence of sub-σ-algebras of $\mathfrak{A}$, $\mathfrak{G}_0 = \{\Phi, \Omega\}$, and Z a φ-decomposable r.v. with decomposition components Z_k, $k\in\mathbf{N}$. If*

$$E[X_k^j \mid \mathfrak{G}_{k-1}] = E[Z_k^j] \quad (k\in\mathbf{N};\ j\in\{1, 2\}) \tag{4.1}$$

and the sequences $(X_k)_{k\in\mathbf{N}}$, $(Z_k)_{k\in\mathbf{N}}$ both satisfy Lindeberg conditions of order 2 (*cf.*

(2.13)), *then there holds for each* $f \in C_B^2$ *in case*

(4.2) $$\varphi(n) = O(n^{-1/2}) \quad (n \to \infty)$$

(4.3) $$\|V_{\varphi(n)S_n} f - V_Z f\|_{C_B} = o_f(1) \quad (n \to \infty).$$

If the distribution function F_Z *of* Z *is continuous, one has in addition*

(4.4) $$\sup_{x \in \mathbf{R}} |F_{\varphi(n)S_n}(x) - F_Z(x)| = o(1) \quad (n \to \infty).$$

Proof. Firstly, one can ensure that the r.v.'s Z_k, $k \in \mathbf{N}$, are independent of the X_k as well as of the sub-σ-algebras $\mathfrak{G}_k$, $k \in \mathbf{N}$ by means of an appropriate choice of the probability space. According to Lemma 5 b) to each $y \in \mathbf{R}$, $n \in \mathbf{N}$ and $1 \leqq k \leqq n$ there exists sets $G^y_{n,k-1} \in \mathfrak{G}_{k-1}$ with $P(G^y_{n,k-1}) > 0$ such that for each $\omega = \omega(n, k, y) \in G_{n,k-1}$ by (2.4)

(4.5) $$\|V_{\varphi(n)S_n} f - V_Z f\|_{C_B} = \|V_{\varphi(n)S_n} f - V_{\varphi(n)\sum_{k=1}^n Z_k} f\|_{C_B} \leqq$$

$$\leqq \sum_{k=1}^n \sup_{y \in \mathbf{R}} |(V^{\mathfrak{G}_{k-1}}_{\varphi(n)X_k} f)(y, \omega) - (V_{\varphi(n)Z_k} f)(y)|.$$

Now choose $\bar{\omega} \in G^y_{n,k-1}$ such that condition (4.1) is satisfied for it. An application of (2.12) plus Taylor's formula to both $f(\varphi(n)X_k + y)$ and $f(\varphi(n)Z_k + y)$ then gives

$$|V^{\mathfrak{G}_{k-1}}_{\varphi(n)X_k} f(y, \bar{\omega}) - V_{\varphi(n)Z_k} f(y)| =$$

(4.6) $$= \Big| \int_{\mathbf{R}} f(x+y)\, d\big(F_{\varphi(n)X_k}(x|\mathfrak{G}_{k-1})\big)(\bar{\omega}) - \int_{\mathbf{R}} f(x+y)\, dF_{\varphi(n)X_k}(x) \Big| =$$

$$= \Big| \int_{\mathbf{R}} \Big\{ \sum_{j=0}^{2} \frac{\varphi(n)^j x^j}{j!} f^{(j)}(y) + \frac{1}{2} \varphi(n)^2 x^2 [f^2(\eta) - f^2(y)] \Big\} d(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar{\omega})) -$$

$$- \int_{\mathbf{R}} \Big\{ \sum_{j=0}^{2} \frac{\varphi(n)^j x^j}{j!} f^{(j)}(y) + \frac{1}{2} \varphi(n)^2 x^2 [f^{(2)}(\eta) - f^{(2)}(y)] \Big\} dF_{Z_k}(x) \Big|$$

where $|\eta - y| \leqq \varphi(n)|x|$. Since $f^{(2)} \in C_B$, to any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that $|f^{(2)}(\eta) - f^{(2)}(y)| < \varepsilon$ for $|\eta - y| < \delta$. But by (4.2) to each $\delta > 0$ and $x \in \mathbf{R}$ there is an $n \in \mathbf{N}$ with $|\eta - y| \leqq \varphi(n)|x| < \delta$. So splitting up the range of integration in (4.6) into $\{x \in \mathbf{R};\ |x| < \delta/\varphi(n)\}$ and its complementary set, one obtains by (2.12) and (4.1) the expression

$$\Big| \Big(\int_{|x| < \delta/\varphi(n)} + \int_{|x| \geqq \delta/\varphi(n)} \Big) \frac{1}{2} \varphi(n)^2 x^2 [f^{(2)}(\eta) - f^{(2)}(y)]\, d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar{\omega})\big) -$$

$$- \Big(\int_{|x| < \delta/\varphi(n)} + \int_{|x| \geqq \delta/\varphi(n)} \Big) \frac{1}{2} \varphi(n)^2 x^2 [f^{(2)}(\eta) - f^{(2)}(y)]\, dF_{Z_k}(x) \Big|$$

which can in turn be estimated by

$$\text{(4.7)}\qquad \frac{\varphi(n)^2}{2}\Big\{\varepsilon\big(E[|X_k|^2|\mathfrak{G}_{k-1}](\bar\omega)+E[|Z_k|^2]\big)+$$
$$+\|f^{(2)}\|_{C_B}\Big(\int\limits_{|x|\geqq\delta/\varphi(n)} x^2\,d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar\omega)\big)+\int\limits_{|x|\geqq\delta/\varphi(n)} x^2\,dF_{Z_k}(x)\Big)\Big\}.$$

Since $E[X_k^2]\leqq M$, (2.5) and (4.1) yield that $E[Z_k^2]\leqq M$ as well as $E[X_k^2|\mathfrak{G}_{k-1}]\leqq M$ a.s. for all $k\in\mathbf{N}$. Since further $E[X_k^2]\geqq m>0$, there are constants $M_1, M_2>0$ such that $E[X_k^2|\mathfrak{G}_{k-1}]\leqq M_1E[X_k^2]$ a.s., and

$$E[X_k^2\mathbf{1}_{\{|X_k|\geqq\delta/\varphi(n)\}}|\mathfrak{G}_{k-1}]\leqq M_2E[X_k^2\mathbf{1}_{\{|X_k|\geqq\delta/\varphi(n)\}}]\quad\text{a.s.,}$$

$\mathbf{1}_A$, $A\in\mathfrak{A}$ being the indicator function. Then one deduces from (4.6) and (4.7) that for each $y\in\mathbf{R}$, $n\in\mathbf{N}$ and $1\leqq k\leqq n$

$$\text{(4.8)}\qquad |(V^{\mathfrak{G}_{k-1}}_{\varphi(n)X_k}f)(y,\bar\omega)-V_{\varphi(n)Z_k}f(y)|\leqq$$
$$\leqq\frac{\varphi(n)^2}{2}\big[\varepsilon(M_1E[X_k^2]+E[Z_k^2])+$$
$$+\|f^{(2)}\|_{C_B}\Big(M_2\int\limits_{|x|\geqq\delta/\varphi(n)} x^2\,dF_{X_k}(x)+\int\limits_{|x|\geqq\delta/\varphi(n)} x^2\,dF_{Z_k}(x)\Big)\Big].$$

Taking now the supremum on the left side of this inequality for all $y\in\mathbf{R}$, summing up over k from 1 to n, and finally dividing the result by the strictly positive expression $(\varphi(n)^2/2)\sum_{k=1}^{n}(M_1E[X_k^2]+E[Z_k^2])$, one obtains from (4.5)

$$\text{(4.9)}\qquad 2\|V_{\varphi(n)S_n}f-V_Zf\|_{C_B}\Big/\Big(\varphi(n)^2\sum_{k=1}^{n}(M_1E[X_k^2]+E[Z_k^2])\Big)\leqq$$
$$\leqq\varepsilon+\|f^{(2)}\|_{C_B}\left\{\frac{M_2\sum\limits_{k=1}^{n}\int\limits_{|x|\geqq\delta/\varphi(n)}x^2\,dF_{X_k}(x)}{M_1\sum\limits_{k=1}^{n}E[X_k^2]}+\frac{\sum\limits_{k=1}^{n}\int\limits_{|x|\geqq\delta/\varphi(n)}x^2\,dF_{Z_k}(x)}{\sum\limits_{k=1}^{n}E[Z_k^2]}\right\}.$$

Since the sequences $(X_k)_{k\in\mathbf{N}}$, $(Z_k)_{k\in\mathbf{N}}$ are assumed to satisfy Lindeberg conditions of order 2, the term in square brackets converges to zero for $n\to\infty$ by (4.2). Since $\varepsilon>0$ was arbitrary, assertion (4.3) follows by noting that the denominator on the left side of (4.9) is uniformly bounded in n because of (4.2) and the uniform boundedness of $E[X_k^2]$ and $E[Z_k^2]$. This in turn yields (4.4) since F_Z is continuous (cf. [11, p. 140]).

4.2. The central limit theorem. A particular case of Theorem 1 is the following version of the CLT.

Theorem 2. *Let* $(X_k)_{k\in\mathbf{N}}$ *and* $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ *be given as in Theorem* 1, *denote* $\sigma_k:=E[X_k^2]$, $k\in\mathbf{N}$, *and* $s_n:=\left(\sum_{k=1}^{n}\sigma_k^2\right)^{1/2}$.

a) *If there holds*

$$E[X_k^j|\mathfrak{G}_{k-1}]=\sigma_k^jE[X^{*j}] \quad \text{a.s.} \quad (k\in\mathbf{N};\quad j\in\{1,2\}), \tag{4.10}$$

and if the sequences $(X_k)_{k\in\mathbf{N}}$ *and* $(\sigma_k)_{k\in\mathbf{N}}$ *satisfy a Lindeberg condition of order 2 and the Feller condition* (2.14), *respectively, then one has*

$$\|V_{s_n^{-1}S_n}f-V_{X^*}f\|_{C_B}=o_f(1) \quad (f\in C_B^2;\ n\to\infty) \tag{4.11}$$

or, equivalently,

$$\sup_{x\in\mathbf{R}}|F_{s_n^{-1}S_n}(x)-F_{X^*}(x)|=o(1) \quad (n\to\infty). \tag{4.12}$$

b) *If the r.v.'s* X_k *are in addition idetically distributed,* (4.10) *is valid with* $\sigma_k=1$, *then one has for* $f\in C_B^2$

$$\|V_{n^{-1/2}S_n}f-V_{X^*}f\|_{C_B}=o_f(1) \quad (n\to\infty) \tag{4.13}$$

or, equivalently,

$$\sup_{x\in\mathbf{R}}|F_{n^{-1/2}S_n}(x)-F_{X^*}(x)|=o(1) \quad (n\to\infty). \tag{4.14}$$

Proof. a) Choosing for the decomposition components Z_k of Theorem 1 the independent r.v.'s σ_kX^*, then condition (2.4) is satisfied with $\varphi(n):=s_n^{-1}$. Further, $Z=\sigma_kX^*$ implies that hypothesis (4.10) corresponds to (4.1). Since also $E[X_k^2]\leq M<\infty$ for all $k\in\mathbf{N}$, $\varphi(n)=s_n^{-1}\leq M_n^{-1/2}$, and so (4.2) is satisfied. It can be shown (cf. [3, p. 268]) that the Lindeberg condition for $(X_k)_{k\in\mathbf{N}}$ plus the Feller condition for $(\sigma_k)_{k\in\mathbf{N}}$ yields the Lindeberg condition for $(Z_k)_{k\in\mathbf{N}}$. So assertion (4.11) follows from (4.3). Finally, (4.12) is a derivation of (4.4) in view of the continuity of F_{X^*}.

b) Assertions (4.13) and (4.14) are immediate consequences of (4.11) and (4.12), noting that conditions (2.13) and (2.14) are always automatically satisfied for identically distributed r.v.'s.

4.3. The weak law of large numbers. Since the partial sums in the WLLN are normalized by n^{-1} and not $n^{-1/2}$ as for the CLT, just a Lindeberg condition of order one need be assumed for $(X_k)_{k\in\mathbf{N}}$ while the moment condition (4.1) reduces to the condition that the conditional moments of the X_k with respect to $\mathfrak{G}_{k-1}$ be zero.

Theorem 3. *Let* $(X_k)_{k\in\mathbf{N}}$ *and* $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ *be defined as in Theorem* 1, *let* X_0 *be a r.v. taking on the value zero with probability* 1, *and let*

$$E[X_k|\mathfrak{G}_{k-1}]=0 \quad a.s. \quad (k\in\mathbf{N}).$$

a) *If the sequence* $(X_k)_{k\in\mathbf{N}}$ *satisfies a Lindeberg condition of order* 1 *with* $\varphi(n)=$

$=n^{-1}$, *then* $(X_k)_{k\in\mathbf{N}}$ *satisfies the weak law of large numbers, i.e., for each* $\varepsilon>0$

$$\lim_{n\to\infty} P(\{|n^{-1}S_n| \geqq \varepsilon\}) = 0. \tag{4.15}$$

b) *If the r.v.'s* X_k *are just identically distributed, then* (4.15) *again holds.*

Proof. a) If one chooses the decomposition components Z_k such that $P_{Z_k}=P_{X_0}$ for all $k\in\mathbf{N}$, then (2.4) is satisfied with $\varphi(n)=n^{-1}$. An application of the Taylor expansion of $f\in C_B^1$ up to the order 1 yields, just as in the proof of Theorem 1,

$$\|V_{n^{-1}S_n}f-V_{X_0}f\|_{C_B} = o_f(1) \quad (n\to\infty). \tag{4.16}$$

Since convergence in distribution is equivalent to stochastic convergence for the limit r.v. X_0 (cf. e.g. [3, 220]), (4.16) implies assertion (4.15). Part b) is a particular case of a), Lindeberg's condition being satisfied automatically.

5. Convergence theorems for dependent random variables with o-rates. It is possible to equip the limit theorems of Section 4 with rates without any larger modifications of the proofs; just stronger assumptions upon the moments and higher-order Lindeberg conditions will be needed. However, the assertions will now be restricted to the convergence in distribution of the normalized partial sums, since the equivalence of convergence in distribution with uniform convergence of the distribution functions in case of the CLT and with stochastic convergence in the case of the WLLN is only valid for convergence *without* rates.

5.1. A general approximation theorem.

Theorem 4. *Let* $(X_k)_{k\in\mathbf{N}}$ *be a sequence of r.v.'s,* $r\in\mathbf{N}\setminus\{1\}$, *and* m, M *be two positive constants with* $0<m\leqq E[|X_k|^r]\leqq M<\infty$ *for* $k\in\mathbf{N}$. *Let* $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ *be a sequence of sub-σ-algebras of* $\mathfrak{A}$ *with* $\mathfrak{G}_0=\{\Phi, \Omega\}$. *If* Z *is a* φ*-decomposable r.v. with decomposition components* Z_k, $k\in\mathbf{N}$, *condition* (4.1) *is fulfilled for* $j\in\mathbf{N}$, $1\leqq j\leqq r$, *and the sequence* $(X_k)_{k\in\mathbf{N}}$ *as well as* $(Z_k)_{k\in\mathbf{N}}$ *satisfy Lindeberg conditions of order* r, *then for* $f\in C_B^r$

$$\|V_{\varphi(n)S_n}f-V_Zf\|_{C_B} = o_f(n[\varphi(n)]^r) \quad (n\to\infty). \tag{5.1}$$

Proof. The proof of this theorem is based upon that of Theorem 1. Just as there one has inequality (4.5). For a suitable $\bar\omega$ (cf. the proof of Theorem 1), an application of Taylor's expansion, this time up to the order r for $f\in C_B^r$, yields

$$|(V^{\mathfrak{G}_{k-1}}_{\varphi(n)X_k}f)(y,\bar\omega)-(V_{\varphi(n)Z_k}f)(y)| = \tag{5.2}$$

$$= \Bigg|\int_{\mathbf{R}}\Bigg\{\sum_{j=0}^{r}\frac{\varphi(n)^j x^j}{j!}f^{(j)}(y)+\frac{\varphi(n)^r x^r}{r!}[f^{(r)}(\eta)-f^{(r)}(y)]\Bigg\}\,d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar\omega)\big)-$$

$$-\int_{\mathbf{R}}\Bigg\{\sum_{j=0}^{r}\frac{\varphi(n)^j x^j}{j!}f^{(j)}(y)+\frac{\varphi(n)^r x^r}{r!}[f^{(r)}(\eta)-f^{(r)}(y)]\Bigg\}\,dF_{Z_k}(x)\Bigg|.$$

Following the arguments in the proof of Theorem 1 with $f^{(2)}$ replaced by $f^{(r)}$, one obtains after the range of integration has been split up and estimates analogous to (4.7) and (4.8) have been carried out, that for $1 \leqq k \leqq n$, $y \in \mathbf{R}$, the right side of (5.2) is bounded from above by

$$\frac{\varphi(n)^r}{r!}\Big\{\varepsilon\big(M_1^* E[|X_k|^r \,|\, \mathfrak{G}_{k-1}](\bar\omega) + E[|Z_k|^r]\big) +$$

$$+\|f^{(r)}\|_{C_B}\Big(M_2^* \int\limits_{|x| \geqq \delta/\varphi(n)} |x|^r \, d\big(F_{X_k}(x \,|\, \mathfrak{G}_{k-1})(\bar\omega)\big) + \int\limits_{|x| \geqq \delta/\varphi(n)} |x|^r \, dF_{Z_k}(x)\Big)\Big\},$$

where M_1^* and M_2^* are the constants corresponding to M_1 and M_2 in inequality (4.8), noting that the remaining terms of the Taylor expansion up to the order r vanish on account of (4.1). The Lindeberg conditions of order r for the sequences $(X_k)_{k \in \mathbf{N}}$ and $(Z_k)_{k \in \mathbf{N}}$ then yield, as in Theorem 1,

$$r!\,|V_{\varphi(n)S_n} f - V_Z f| \Big/ \Big(\varphi(n)^r \sum_{k=1}^{n} \big(M_1^* E[|X_k|^r] + E[|Z_k|^r]\big)\Big) = o_f(1) \quad (n \to \infty). \tag{5.3}$$

Since $E[|X_k|^r]$ is uniformly bounded by hypothesis, and so also $E[|Z_k|^r]$ by (4.1), there exists a constant $M_3 > 0$ such that $\big(\sum_{k=1}^{n} (M_1^* E[|X_k|^r] + E[Z_k]^r]\big) \leqq nM_3$. Inserting this estimate into (5.3) gives statement (5.1).

5.2. Applications to the CLT and WLLN with o-rates. By specializing the limit r.v. in Theorem 4 one obtains

Theorem 5. *Let $(X_k)_{k \in \mathbf{N}}$ and $(\mathfrak{G}_k)_{k \in \mathbf{P}}$ be given as in Theorem* 1, σ_k, s_n *as defined in Theorem* 2, *and let* $r \in \mathbf{N}$.

a) *If* (4.10) *is satisfied for* $1 \leqq j \leqq r$, *and the sequences $(X_k)_{k \in \mathbf{N}}$ and $(\sigma_k)_{k \in \mathbf{N}}$ satisfy a Lindeberg condition of order r and the Feller condition* (2.14), *respectively, then $f \in C_B^r$ implies*

$$\|V_{s_n^{-1}S_n} f - V_{X^*} f\|_{C_B} = o_f(n s_n^{-r}) \quad (n \to \infty). \tag{5.4}$$

b) *If the r.v.'s X_k are identically distributed,* (4.10) *holds for* $\sigma_k = 1$ *and* $1 \leqq j \leqq r$, *then for* $f \in C_B^r$

$$\|V_{n^{-r/2}S_n} f - V_{X^*} f\|_{C_B} = o_f(n^{(2-r)/2}) \quad (n \to \infty). \tag{5.5}$$

Concerning the proof, assertion (5.4) follows from (5.1) just as does (4.13) from (4.3); (5.5) is immediate by (5.4).

If one compares the rate in (5.5) with that known for independent r.v.'s and MDS (cf. [5] or [7]), it will be seen that the same approximation order could be achieved even though the X_k are now dependent.

Now to the WLLN. Since a moment condition corresponding to (4.1) for $r \geqq 2$, i.e., a condition of form $E[X_k^j | \mathfrak{G}_{k-1}] = E[X_0^j]$ a.s., $1 \leqq j \leqq r$ would now mean that only those r.v.'s X_k can be admitted that take on the value zero with probability 1 just as does X_0, since $E[E[X_k^2|\mathfrak{G}_{k-1}]]=E[X_k^2]=E[X_0^2]=0$, such a condition will now be replaced by the weaker (5.6).

Theorem 6. *Let $(X_k)_{k\in\mathbf{N}}$ and $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ be defined as in Theorem 4, X_0 as in Theorem 3. If $(X_k)_{k\in\mathbf{N}}$ satisfies condition*

$$(5.6)\qquad n^{r-j}\sum_{k=1}^{n}|E[X_k^j|\mathfrak{G}_{k-1}]|=o\Big(\sum_{k=1}^{n}E[|X_k|^r]\Big)\quad a.s.\quad (1\leqq j\leqq r;\ n\to\infty),$$

for some $r\in\mathbf{N}$ as well as a Lindeberg condition of order r, then for $f\in C_B^2$, $n\to\infty$

$$\|V_{n^{-1}S_n}f-V_{X_0}f\|_{C_B}=o_f\Big(n^{-r}\sum_{k=1}^{n}E[|X_k|^r]\Big).$$

Proof. Choosing the decompostion components Z_k such that $P_{Z_k}=P_{X_0}$ and sets $\varphi(n)=n^{-1}$ as in the proof of Theorem 3, a Taylor expansion up to the order r yields, by taking into account that $E[|Z_k|^j]=0$, $1\leqq j\leqq r$, that for $y\in\mathbf{R}$ and suitable $\bar{\omega}$ (see (4.6) and (4.8))

$$|(V_{n^{-1}X_k}^{\mathfrak{G}_{k-1}}f)(y,\bar{\omega})-V_{n^{-1}Z_k}f(y)|=$$

$$=\Big|\int_{\mathbf{R}}\sum_{j=0}^{r}\frac{n^{-j}x^j}{j!}f^{(j)}(y)+\frac{n^{-r}x^r}{r!}[f^{(r)}(\eta)-f^{(r)}(y)]\,d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar{\omega})\big)-f(0)\Big|\leqq$$

$$\leqq\sum_{j=1}^{r}\frac{n^{-j}}{j!}\|f^{(j)}\|_{C_B}E[|X_k|^j|\mathfrak{G}_{k-1}](\bar{\omega})+$$

$$+\frac{n^{-r}}{r!}\Big\{\varepsilon M_1^{**}E[|X_k|^r]+\|f^{(r)}\|_{C_B}\Big(M_2^{**}\int_{|x|\geqq\delta n}|x|^r\,dF_{X_k}(x)\Big)\Big\},$$

M_1^{**} and M_2^{**} being the constants corresponding to M_1 and M_2 from (4.8). As in the proof of Theorem 1 one has in view of (5.6)

$$(5.7)\qquad \|V_{n^{-1}S_n}f-V_{X_0}f\|_{C_B}\Big/\Big(n^{-r}\sum_{k=1}^{n}E[|X_k|^r]\Big)=o_f(1)\quad(n\to\infty).$$

Note that the rate of approximation in (5.7) is a good as that given in [5] and [7] for the WLLN for independent r.v.'s and MDS, respectively. For $r=3$ the rate is $o(n^{-2})$, provided the r.v.'s are identically distributed. Though the r.v.'s X_k are now arbitrarily dependent, no additional assumption was needed to obtain this rate of convergence. So Theorem 6 can be regarded as a true generalization of the corresponding assertions in [5] and [7].

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Norm convergence of generalized martingales in L^p-spaces over von Neumann algebras

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Dedicated to Professor Károly Tandori on his 60th birthday

Introduction

After scattered partial results the norm convergence of martingales in L_p-spaces over von Neumann algebras has been proved by GOLDSTEIN [10]. The main difference between his approach and our one is twofold. While in [10] (as well as in [3], [6], [7], [15]) the martingale sequence is formed by means of conditional expectations (i.e. state preserving projections of norm one onto subalgebras) we use ω-conditional expectations introduced in [1] (which are not projections in general but they always exist). On the other hand, the L^p-norm we shall use is different from the L_p-norm used in [10] when restricted to L^∞. So [10] does not cover our results even in the case in which all the conditional expectations involved are norm one projections.

All the theorems are proved for a von Neumann algebra with a faithful normal state on it. The framework is the theory of $L(p)$ spaces as complex forms rather than operators developed in [4] which are, very roughly speaking, representations of the spaces of TERP [18], and so closely connected to the spaces of CONNES and HILSUM [5], [14].

The results of this paper are contained in Theorem 9 and Theorem 10. Their forerunner (the strong convergence of bounded martingales with ω-conditional expectations) was obtained in [16], [17] and independently in [13].

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Preliminaries

Let M be a von Neumann algebra acting on a Hilbert space H. We denote by M' the commutant of M and by ω' a faithful normal state on M'. The triple $(\pi_{\omega'}, H_{\omega'}, \Omega')$ is the result of the GNS-construction with ω'.

We summarize some results and notations contained in [5]. As usually we set $D(H, \omega')=\{\xi\in H\colon \|a\xi\|\leqq c\omega'(a^*a)^{1/2}$ for all $a\in M'$ and some $c>0\}$. The space $D(H, \omega')$ is a dense vector space in H and for each $\xi\in D(H, \omega')$ there is a unique bounded linear operator $R_{\omega'}(\xi)\colon H_{\omega'}\to H$ such that

$$R_{\omega'}(\xi)\pi_{\omega'}(a)\Omega' = a\xi.$$

The correspondence $\xi\mapsto R_{\omega'}(\xi)$ is linear and for all $\xi, \eta\in D(H, \omega')$ the operator $R_{\omega'}(\xi)R_{\omega'}(\eta)^*$ is in M. If $\varphi\in M_*^+$ then the equality

$$q_\varphi(\xi) = \varphi\big(R_{\omega'}(\xi)R_{\omega'}(\xi)^*\big)$$

defines a lower semicontinuous positive form on $D(H, \omega')$ to which a positive self-adjoint operator $(d\varphi)/(d\omega')$ (the spatial derivative of φ with respect to ω') is associated ([5]).

Now we are in a position to define the spaces $L^p(M, \omega')$ for $1\leqq p<\infty$ as in [14]. $L^p(M, \omega')$ is the set of all closed densely defined operators on H with polar decomposition $T=u|T|$ such that

$$u\in M \quad \text{amd} \quad |T|^p = \frac{d\varphi}{d\omega'}$$

for some $\varphi\in M_*^+$. If $\psi\in M_*$ has a polar decomposition $\psi=u|\psi|$ then we define

$$T_{\omega'}(\psi) = u\frac{d|\psi|}{d\omega'} \quad \text{and} \quad T_{\omega'}(\psi)\, d\omega' = \psi(1).$$

The spaces $L^p(M, \omega')$ $(1\leqq p<\infty)$ are Banach spaces endowed with the norm

$$\|T\|_p = \left(\int |T|^p\, d\omega'\right)^{1/p}$$

if by sum (and later by product) of unbounded operators we take the strong sum (and strong product).

Let us now fix a faithful normal state ω on M and shorten $(d\omega)/(d\omega')$ in d. For $1\leqq p<\infty$ we define $H(p, \omega, \omega')$ as the Hilbert space completion of the domain of $d^{-1/2p}$ under the inner product

$$\langle\xi, \eta\rangle_p = \langle d^{-1/2p}\xi, d^{-1/2p}\eta\rangle$$

and $H(\infty, \omega, \omega')=H$. There is a unique unitary operator $V(\omega, \omega', P_2, P_1)$: $H(p_1, \omega, \omega')\to H(p_2, \omega, \omega')$ such that

$$V(\omega, \omega', P_2, P_1)\xi = d^{-(p_1^{-1}-p_2^{-1})/2}\xi$$

for $\xi \in D(H, \omega)$ and for $1 \leqq p_1 < p_2 \leqq \infty$. (Here $D(H, \omega)$ is defined and has the same properties as $D(H, \omega')$ above by reversing the roles of M and M'.)

When ω and ω' are fixed we shall shorten our notation to $H(p)$ for the Hilbert spaces and to $V(p_2, p_1)$ for the unitaries introduced above.

Let $1 \leqq p < \infty$. We set $L(p, M, \omega, \omega')$ for the set of all complex forms (i.e. complex linear combinations of positive forms) defined on $D(H, \omega)$ and having the form

$$q(T)(\xi) = \langle |T|^{1/2} V(p, \infty)^* u^* V(p, \infty)\xi, |T|^{1/2}\xi \rangle_p$$

when T is a closed densely defined operator on $H(p)$ with a polar decomposition

$$V(\infty, p)^* u V(\infty, p)|T|$$

such that u is a partial isometry in M and

$$V(\infty, p) T V(\infty, p)^*$$

is in $L^p(M, \omega')$.

For $p = \infty$ we set $L(\infty, M, \omega, \omega') = \{q(a)\colon a \in M\}$ where $q(a)(\xi) = \langle \xi, a\xi \rangle$ $(\xi \in D(H, \omega))$.

We define a norm on $L(p, M, \omega, \omega')$ by requiring the linear bijection $\lambda_p\colon L(p, M, \omega, \omega') \to L^p(M, \omega')$, $\lambda_p\colon q(T) \mapsto V(\infty, p) T V(\infty, p)^*$ to be an isometry for $1 \leqq p \leqq \infty$. In [4] it was shown that the spaces $L(p, M, \omega, \omega')$ do not depend on the auxiliarly state ω' used in their construction (ω' can even be taken to be a normal semifinite weight).

We note so that $L(1, M, \omega)$ is isometrically isomorphic to M_* and we denote this isomorphism by ι_ω. Explicitely,

$$\iota_\omega(\psi)(\xi) = \psi(|R_{\omega'}(d^{-1/2}\xi)^*|^2) \qquad (\psi \in M_*, \xi \in D(H, \omega)),$$

since $d^{-1/2}\xi \in D(H, \omega')$.

If $1 \leqq p_1 < p_2 \leqq \infty$ then $L(p_2, M, \omega) \subset L(p_1, M, \omega)$ and $L(p_2, M, \omega)$ is norm dense in $L(p_1, M, \omega)$. For $q \in L(p_2, M, \omega)$ we have

$$\|q\|_{L(p_2, M, \omega)} \geqq \|q\|_{L(p_1, M, \omega)}.$$

These properties will be used without reference.

Let M_0 be a subalgebra of M and $\omega_0 = \omega | M_0$. The ω-conditional expectation $E^\omega\colon M \to M_0$ defined in [1] is an ω-preserving completely positive contraction and it turns out to be the dual of the embedding of M_0 into M when suitable embeddings of the algebras into their preduals are considered (see [2] and [17]). In [4] it was proved that there exists a contraction $\varepsilon^\omega\colon L(1, M, \omega) \to L(1, M_0, \omega_0)$ such that

$$\varepsilon^\omega q(a)(\xi) = \langle \xi, E^\omega(a)\xi \rangle$$

($\xi\in D(H, \omega_0)$, $a\in M$). Interpolation techniques give that the restriction of ε^ω to $L(p, M, \omega)$ is also a contraction into $L(p, M_0, \omega_0)$ ($1<p<\infty$, see [4] and [18]). Later we define a natural mapping $\varkappa$: $L(p, M_0, \omega_0)\to L(p, M, \omega)$ and we form the composition $\varkappa\circ\varepsilon^\omega$ in order to have a selfmapping of $L(p, M, \omega)$.

Results

The elements of the spaces $L(p, M, \omega)$ are complex forms on $D(H, \omega)$ so the pointwise convergence of forms can be defined in a natural way. We deal with the relation of this convergence to the norm convergence in $L(p, M, \omega)$. We need also the connection between the strong operator topology on M and the norm topology of $L(p, M, \omega)$.

Lemma. *Let* $(q_n)\subset L(1, M, \omega)$. *If* $\iota_\omega^{-1}(q_n)\to 0$ *weakly then for any* $\xi\in D(H, \omega)$

$$q_n(\xi)\to 0.$$

Moreover, if (q_n) *is bounded then the converse also holds.*

Proof. Since

$$q_n(\xi) = (\iota_\omega^{-1} q_n)(|R_{\omega'}(d^{-1/2}\xi)^*|^2)$$

the first part of the statement follows immediately. To get the converse it suffices to note that the linear hull of the set

$$\{|R_{\omega'}(d^{-1/2}\xi)^*|^2 : \xi\in D(H, \omega)\}$$

is dense in M.

Proposition 2. *Let* $(q_n)\subset L(p, M, \omega)$ *and* $1\leqq p<\infty$. *If* $q_n\to q$ *in norm of* $L(p, M, \omega)$ *then*

$$q_n(\xi)\to q(\xi)$$

for every $\xi\in D(H, \omega)$.

Proof. $q_n\to q$ in the norm of $L(1, M, \omega)$ and so in the weak topology. Lemma 1 can be applied.

Now we prove technical lemmas on different norms. To simplify formulas we shall shorten $d^{1/2s}$ in D.

Lemma 3. *Let* $a\in M$ *and* s, k *be integers such that* $s\geqq 3$ *and* $0\leqq k\leqq s-3$. *Then*

$$\|q(a)\|_{L(2^s, M, \omega)}^{2^s} \leqq \|a\|^{2^s+2^{s-2-k}+1}\|(Da^*Da)^{2^{s-2}-2^k} d^{2^{-s+k+1}}\|_{L^2(M, \omega')}^{2^{-k-1}}.$$

Proof. We apply induction on k. First let $k=0$.

$$\|q(a)\|_{L(2^s,M,\omega)}^{2^s}=\|D^{1/2}aD^{1/2}\|_{L(2^s,M,\omega')}^{2^s}=$$
$$=\int(d^{1/2^{s+1}}a^*Dad^{1/2^{s+1}})^{2^s-1}\,d\omega'=\int(a^*DaD)^{2^{s-1}}\,d\omega'\leqq$$
$$\leqq\|(a^*DaD)^{2^{s-2}}a^*\|_{L^2(M,\omega')}\|DaD(a^*DaD)^{2^{s-2}-1}\|_{L^2(M,\omega')}\leqq$$
$$\leqq\|a\|^{2^{s-1}+1}\Big[\int(Da^*Da)^{2^{s-2}-1}Da^*D^2aD(a^*DaD)^{2^{s-2}-1}\,d\omega'\Big]^{1/2}\leqq$$
$$\leqq\|a\|^{2^{s-1}+1}\Big[\int a^*Da(Da^*Da)^{2^{s-2}-2}Da^*D^2aD(a^*DaD)^{2^{s-2}-1}\,d\omega'\Big]^{1/2}\leqq$$
$$\leqq\|a\|^{2^{s-1}+1}\Big[\int a^*Da(Da^*Da)^{2^{s-2}-2}Da^*D^2aD(a^*DaD)^{2^{s-2}-1}\,d\omega'\Big]^{1/2}\leqq$$
$$\|a\|^{2^{s-1}+1}\|a^*Da(Da^*Da)^{2^{s-2}-2}Da^*D^2a\|_{L^2(M,\omega')}^{1/2}\|D(a^*DaD)^{2^{s-2}-1}D\|_{L^2(M,\omega')}^{1/2}\leqq$$
$$\leqq\|a\|^{2^{s-1}+1}\|a\|^{2^{s-2}}\|D(a^*DaD)^{2^{s-2}-1}D\|_{L^2(M,\omega')}^{1/2}=$$
$$=\|a\|^{2^s-2^{s-2}+1}\|(Da^*Da)^{2^{s-2}-1}D^2\|_{L^2(M,\omega')}^{1/2}.$$

Here we have used the Hölder inequality repetedly. Now we carry out the induction step. We have:

$$\|(Da^*Da)^{2^{s-2}-2^k}d^{2^{-s+k+1}}\|_{L^2(M,\omega')}=$$
$$=\Big(\int d^{2^{-s+k+1}}(a^*DaD)^{2^{s-2}-2^k}(Da^*Da)^{2^{s-2}-2^k}d^{2^{-s+k+1}}\,d\omega'\Big)^{1/2}=$$
$$=\Big(\int(a^*DaD)^{2^{s-2}-2^k}(Da^*Da)^{2^{s-2}-2^k}d^{2^{-s+k+2}}\,d\omega'\Big)^{1/2}\leqq$$
$$\leqq\|(a^*DaD)^{2^{s-2}-2^k}(Da^*Da)^{2^k}\|_{L^2(M,\omega')}^{1/2}$$
$$\|(Da^*Da)^{2^{s-2}-2^{k+1}}d^{2^{-s+k+1}}\|_{L^2(M,\omega')}^{1/2}\leqq$$
$$\|a\|^{2^{s-2}}\|(Da^*Da)^{2^{s-2}-2^{k+1}}d^{-s+k+2}\|_{L^2(M,\omega')}^{1/2}.$$

So our hypothesis on k implies our claim for $k+1$.

Lemma 4. *Let a and s be as in the previous Lemma. Then*

$$\|q(a)\|_{L(2^s,M,\omega)}^{2^s}\leqq\|a\|^{m(s)}\|ad^{1/2}\|_{L^2(M,\omega')}^{2^{-s+1}}$$

where $m(s)=2^s-1+(2^{s-1}-1)2^{-s+3}$.

Proof. Using Lemma 3 with $k=s-3$ we can majorize as follows.

$$\|(Da^*Da)^{2^{s-3}}d^{1/4}\|_{L^2(M,\omega')}=$$
$$=\Big[\int d^{1/4}(a^*DaD)^{2^{s-3}}(Da^*Da)^{2^{s-3}}d^{1/4}\,d\omega'\Big]^{1/2}=$$
$$=\Big[\int(a^*DaD)^{2^{s-3}}(Da^*Da)^{2^{s-3}}d^{1/2}\,d\omega'\Big]^{1/2}\leqq$$
$$\|(a^*DaD)^{2^{s-3}}(Da^*Da)^{2^{s-3}-1}Da^*D\|_{L^2(M,\omega')}^{1/2}\|ad^{1/2}\|_{L^2(M,\omega')}^{1/2}\leqq$$
$$\|a\|^{2^{s-2}-1/2}\|ad^{1/2}\|_{L^2(M,\omega')}^{1/2}$$

Proposition 5. *Let $(a_n)\subset M$ be a bounded sequence. If $a_n \to a$ strongly then $q(a_n)\to q(a)$ in the norm of $L(p, M, \omega)$ for $1\leqq p<\infty$.*

Proof. We may assume that $a=0$ and $p=2^s$. For arbitrary $a\in M$ we have

$$\|ad^{1/2}\|^2_{L^2(M,\omega')} = \int d^{1/2}a^*a\, d^{1/2}\, d\omega' = \int da^*a\, d\omega' = \omega(a^*a).$$

Now an application of Lemma 4 completes the proof.

Let M_0 be a subalgebra of M. We denote by ω_0 the restriction of ω to M_0. Clearly, $D(H,\omega)\subset D(H,\omega_0)$ and if q is a form on $D(H,\omega_0)$ then $\varkappa(q)$ will stand for $q|D(H,\omega)$.

Lemma 6. *Let M_0, M, ω_0, ω and $\varkappa$ be as above. Then $\varkappa|L(1, M_0, \omega_0)$ is a linear contraction from $L(1, M_0, \omega_0)$ to $L(1, M_0, \omega)$.*

Proof. Denote by H_ω (H_{ω_0}) the Hilbert space for the standard representation $\pi_\omega(M)$ $\big(\pi_{\omega_0}(M_0)\big)$ with respect to $\omega(\omega_0)$ and Ω its cyclic and separating vector defining ω (and also ω_0 since H_{ω_0} is considered as a subspace of H_ω). Let J_ω (J_{ω_0}) be the usual canonical conjugation of the Tomita—Takesaki theory for the couple $(M,\omega)\big((M_0,\omega_0)\big)$ and P the projection from H_ω onto H_{ω_0}. We define a partial isometry V as it was denote in [1].

$$VJ_{\omega_0}\pi_\omega(a)\Omega = J_\omega\pi_\omega(a)\Omega \quad \text{for} \quad a\in M_0$$
$$V\xi = 0 \quad \text{for} \quad \xi\perp H_{\omega_0}$$

From [4] we know that $J_\omega|R_\omega(\xi)|^2 J_\omega\in\pi_\omega(M)$ $\big(\xi\in D(H,\omega)\big)$ $\big($and, for $\xi\in D(H,\omega_0)$, $J_{\omega_0}|R_{\omega_0}(\xi)|^2J_{\omega_0}\in\pi_{\omega_0}(M_0)\big)$. Now if E^ω is the ω-conditional expectation from M to M_0 then

$$E^\omega\big(\pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2 J\omega)\big) = \pi_{\omega_0}^{-1}\big(V^*J_\omega|R_\omega(\xi)|^2J_\omega V\big) =$$
$$= \pi_{\omega_0}^{-1}\big(J_{\omega_0}|R_{\omega_0}(\xi)|^2J_{\omega_0}\big).$$

The last equality follows from: $R_\omega(\xi)P\pi_\omega(a)\Omega = R_\omega(\xi)\pi_\omega(a)\Omega = a\xi = R_{\omega_0}(\xi)\pi_{\omega_0}(a)\Omega$ for $a\in M_0$ and $\xi\in D(H,\omega)$, which implies $R_\omega(\xi)P|H_{\omega_0} = R_{\omega_0}(\xi)$.

Let nos $\varphi\in M_*$. It is proved in [4] that $\iota_\omega(\varphi)(\xi) = \pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2J_\omega)$ for $\xi\in D(H,\omega)$ and the similar equality holds also for ι_{ω_0}. We have therefore, for $\xi\in D(H,\omega)$ and $\varphi\in(M_0)_*$,

$$\varkappa\big(\iota_{\omega_0}(\varphi)(\xi) = \iota_{\omega_0}(\varphi)(\xi) = \varphi\big(\pi_{\omega_0}^{-1}(J_{\omega_0}|R_{\omega_0}(\xi)|^2J_{\omega_0})\big) =$$
$$= \varphi\big(E^\omega(\pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2J_\omega))\big) = \iota_\omega(\varphi\circ\varepsilon),$$

and

$$\|\varkappa\circ\iota_{\omega_0}(\varphi)\|_{L(1,M,\omega)} = \|\iota_\omega(\varphi\circ\varepsilon)\|_{L(1,M,\omega)} = \|\varphi\circ\varepsilon\| \leqq$$
$$\leqq \|\varphi\|_{(M_0)_*} = \|\iota_{\omega_0}(\varphi)\|_{L(1,M,\omega)},$$

which proves our statement.

From the above Lemma, it is clear that $\varkappa \circ \iota_{\omega_0}(\varphi)$ depends only on the value of φ on the range of E^ω. This implies that $\varkappa$ in general is not injective on $L(1, M, \omega)$. More precisely, $\varkappa \circ \iota_{\omega_0}(\varphi)=0$ if $\varphi|E^\omega(M)\equiv 0$. This implies that $\varkappa$ is injective if and only if $E^\omega(M)$ is weak-operator dense in M_0, which is not the case in general (cf. [1], section 4).

Proposition 7. *Let M, M_0, ω, ω_0 and $\varkappa$ be as above. If $q\in L(p, M_0, \omega_0)$ then $\varkappa(q)\in L(p, M, \omega)$ for $1<p<\infty$. Moreover, $\varkappa$ is a contraction with respect to the $L(p)$ norms.*

Proof. It is straightforward that for $a\in M_0$ we have $\varkappa(q(a))\in L(\infty, M, \omega)$ and

$$\|\varkappa(q(a))\|_{L(\infty, M, \omega)} = \|q(a)\|_{L(\infty, M_0, \omega_0)}$$

where $q(a)(\xi)=\langle \xi, a\xi\rangle$ $(\xi\in D(H, \omega_0))$. On the other hand the statement has been proved in Lemma 6 for $p=1$. By the Calderon—Lions interpolation theorem ([4], [18]) for $1<p<\infty$ we have $\varkappa(q)\in L(p, M, \omega)$ and

$$\|\varkappa(q)\|_{L(p, M, \omega)} \leqq \|q\|_{L(p, M_0, \omega_0)}$$

whenever $q\in L(p, M, \omega)$.

Let us fix a von Neumann algebra M with a faithful normal state ω and an increasing sequence (M_n) of von Neumann subalgebras. Assume that M is generated by $\bigcup_{n=1}^{\infty} M_n$. We denote by ω_n the restriction of ω to M_n and E_n^ω will stand for the ω-conditional expectation $M\to M_n$. It is porved in [16], [17] and independently in [13] that $E_n^\omega(a)\to a$ strongly for every $a\in M$. As above we write ε_n^ω for the extension of E_n^ω to $L(1, M, \omega)$ and $\varkappa_n\colon L(1, M_n, \omega_n)\to L(1, M, \omega)$ is the restriction mapping.

Theorem 8. *With the notation above, for every $q\in L(p, M, \omega)$*

$$\varkappa_n \circ \varepsilon_n^\omega(q) \to q$$

in the norm of $L(p, M, \omega)$ $(1\leqq p<\infty)$.

Proof. Since the sequence $(\varkappa_n\circ\varepsilon_n^\omega)$ is uniformly bounded it is sufficient to prove our statement on a dense set. We shall assume that $q\in L(\infty, M, \omega)$, that is $q=q(a)$ for some $a\in M$. So $E_n^\omega(a)\to a$ strongly and by Proposition 5 $q(E_n^\omega(a))\to q(a)$ in the norm of $L(p, M, \omega)$. However, $q\circ E_n^\omega=K_n\circ\varepsilon_n^\omega$ and the proof is complete.

Let $(q_n)\subset L(p, M, \omega)$ be a sequence such that

$$\varkappa_k\cdot\varepsilon_k^\omega(q_n) = q_k \quad (n>k).$$

Such a sequence (q_n) will be called (generalized) martingale (adapted to the sequence (M_n) of subalgebras). The martingale (q_n) is called regular if there is a $q\in L(p, M, \omega)$ such that $q_n=K_n\circ\varepsilon_n^\omega(q)$.

Theorem 9. *Let* $(q_n)\subset L(p, M, \omega)$ *be a martingale (adapted to the sequence* (M_n)*) and* $1<p<\infty$. *Then the following conditions are equivalent.*

(i) (q_n) *is regular.*

(ii) (q_n) *converges in the norm of* $L(p, M, \omega)$.

(iii) $\sup_n \|q_n\|_{L(p,M,\omega)}<\infty$

Proof. (i)→(ii) is just the previous Theorem. (ii)→(iii) is trivial. If (iii) holds then due to the reflexivity of $L(p, M, \omega)$ (see [4], [14], [18]) we can find a weakly convergent subsequence of (q_n), say $q_{k(n)}\to q$ weakly. If n is large enough then

$$\varkappa_m \varepsilon_m^\omega(q_{k(n)}) = q_m$$

and we have $q_m=\varkappa_m\varepsilon_m^\omega(q)$.

Theorem 10. *Let* $(q_n)\subset L(1, M, \omega)$ *be a martingale (adapted to the sequence* (M_n)*). Then the following conditions are equivalent.*

(i) (q_n) *is regular,*

(ii) (q_n) *converges in the norm of* $L(1, M, \omega)$.

(iii) $\{q_n\colon n\in \mathrm{N}\}$ *is relatively* $\sigma(L(1), L(\infty))$ *compact in* $L(1, M, \omega)$.

Proof. We can follow the proof of Theorem 9 but instead of reflexivity we may apply the Eberlein—Smulian theorem ([8]).

The reversed martingale convergence theorem does not hold if the sequence is formed with ω-conditional expectations. A counter example is contained in [1].

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Approximation by polynomials and extension of Parseval's identity for Legendre polynomials to the L^p case

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To Professor Károly Tandori on his 60th birth anniversary

In this note we look at the B-spline polynomial basis in the space $\mathscr{P}_k$ of real algebraic polynomials of order k i.e. of degree not exceeding $k-1$, $k \geqq 1$, over the interval $I=\langle -1, 1\rangle$. For a given sequence $t_0, \ldots, t_k$ with $t_0 \leqq \ldots \leqq t_k$ and $t_0 < t_k$ the corresponding B-spline of order k (cf. [6]) is the function

$$N(t_0, \ldots, t_k;\ t) = (t_k - t_0)[t_0, \ldots, t_k](\cdot - t)_+^{k-1},$$

where the square bracket denotes the devided difference taken at $t_0, \ldots, t_k$ and $s_+ = \max(s, 0)$. In particular, for $i=0, \ldots, k-1$, the spline

$$N_{i,k}(t) = N(\underbrace{-1, \ldots, -1}_{i+1}, \underbrace{1, \ldots, 1}_{k-i};\ t)$$

is a polynomial of degree $k-1$ in I and

$$N_{i,k}(t) = \binom{k-1}{i}\left(\frac{1+t}{2}\right)^{k-1-i}\left(\frac{1-t}{2}\right)^{i}. \tag{1.1}$$

Clearly, we have the following properties:

$$N_{i,k} \geqq 0 \quad \text{for} \quad i = 0, \ldots, k-1. \tag{1.2}$$

$$\sum_{i=0}^{k-1} N_{i,k}(t) = 1 \quad \text{for} \quad t \in I. \tag{1.3}$$

$$\mathscr{P}_k = \operatorname{span}[N_{0,k}, \ldots, N_{k-1,k}]. \tag{1.4}$$

$$\int_I N_{i,k} = \frac{2}{k}. \tag{1.5}$$

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For later convenience introduce

$$(f, g) = \int_I fg,$$

$$\|f\|_p = \Big(\int_I |f|^p\Big)^{1/p} \quad \text{if} \quad 1 \leqq p \leqq \infty,$$

$$\|\underline{a}\|_p = \Big(\sum_{i=0}^{k-1} |a_i|^p\Big)^{1/p} \quad \text{for} \quad \underline{a} \in R^k \quad \text{and} \quad 1 \leqq p \leqq \infty,$$

$$N_{i,k,p} = N_{i,k}/(N_{i,k}, 1)^{1/p} = N_{i,k} \cdot (k/2)^{1/p},$$

$$M_{i,k} = N_{i,k,1}.$$

Clearly, $N_{i,k} = N_{i,k,\infty}$. Now, Jensen's inequality and (1.3) imply for $\underline{a} \in R^k$

$$\Big\|\sum_{i=0}^{k-1} a_i N_{i,k,p}\Big\|_p \leqq \|\underline{a}\|_p. \tag{1.6}$$

The kernel for our approximation method is defined as follows

$$R_k(s, t) = \sum_{i=0}^{k-1} N_{i,k,p}(s) N_{i,k,q}(t) = \sum_{i=0}^{k-1} M_{i,k}(s) N_{i,k}(t) \tag{1.7}$$

with $1/p + 1/q = 1$. Clearly, R_k is independent of p and

$$R_k(s, t) = R_k(t, s). \tag{1.8}$$

$$R_k(s, t) \geqq 0 \quad \text{for} \quad s, t \in I. \tag{1.9}$$

For $f \in L^1(I)$ we also define

$$(R_k f)(t) = \int_I f(s) R_k(s, t)\, ds.$$

It now follows by (1.5) and (1.3) that

$$R_k 1 = 1. \tag{1.10}$$

Thus, the standard argument with Hölder's or Jensen's inequality and (1.10) give for the norm of R_k: $L^p(I) \to L^p(I)$

$$\|R_k\|_p = 1 \quad \text{for} \quad 1 \leqq p \leqq \infty. \tag{1.11}$$

Theorem 1. *Let* $f \in L^p(I)$ *if* $1 \leqq p < \infty$ *and let* $f \in C(I)$ *if* $p = \infty$. *Then*

$$\|f - R_k f\|_p \to 0 \quad \text{as} \quad k \to 0. \tag{1.12}$$

Proof. Since we have (1.11) it is sufficient to check (1.12) in a dense set. For $f \in \mathscr{P}_n$ we have

$$f = \sum_{j=0}^{n-1} c_j P_j \tag{1.13}$$

with some coefficients c_j, where P_0, P_1, ... are the orthonormal on I Legendre polynomials. In [3] we have proved

$$R_k(s, t) = \sum_{j=0}^{k-1} m_{j,k} P_j(s) P_j(t), \tag{1.14}$$

where

$$m_{j,k} = \frac{(k-1)\dots(k-j)}{(k+1)\dots(k+j)} \quad \text{for} \quad j = 1, \dots, k-1; \quad m_{0,k} = 1. \tag{1.15}$$

Thus for f as given in (1.13) and for $k \geqq n$ we have

$$R_k f = \sum_{j=0}^{n-1} m_{j,k} c_j P_j.$$

However (1.15) implies that for each j, $0 \leqq j \leqq n-1$, $m_{j,k} \to 1$ as $k \to \infty$ and therefore

$$R_k f \to \sum_{j=0}^{n-1} c_j P_j = f \quad \text{as} \quad k \to \infty,$$

and this completes the proof.

Remark. If we start with (1.14) and (1.15) as the definition of R_k, then Theorem 1 can be proved by a different method. Namely, extending the definition (1.15) by letting $m_{j,k} = 0$ for $j \geqq k$ we see that R_k: $L^p(I) \to L^p(I)$ is a multiplier operator i.e. for

$$f \sim \sum_{j=0}^{\infty} c_j P_j$$

we have

$$R_k f = \sum_{j=0}^{\infty} m_{j,k} c_j P_j.$$

Now, the theory developed in [7] can be applied to obtain (1.12). The disadvantage of this approach is that it does not seem to imply (1.7).

To state our next result we need some more definitions. In $\mathscr{P}_k$ we introduce the descrete scalar product

$$\langle f, g \rangle = \sum_{i=0}^{k-1} f(i) g(i).$$

With respect to this scalar product the orthogonal Chebyshev polynomials $u_k^{(j)} \in \mathscr{P}_{j+1}$, $j = 0, \dots, k-1$, are determined by the condition $u_k^{(j)}(0) = (j+1/2)^{1/2}$ (see e.g. [5]). Here $u_k^{(j)}(i)$ denotes the same value as $u_{i,k}^{(j)}$ in [3].

Theorem 2. *Let $f \in L^p(I)$ if $1 \leqq p < \infty$ and let $f \in C(I)$ if $p = \infty$. Then for $f \sim \sum_{j=0}^{\infty} c_j P_j$ we have*

$$\left(\frac{2}{k}\sum_{i=0}^{k-1}\left|\sum_{j=0}^{k-1} m_{j,k} u_k^{(j)}(i) c_j\right|^p\right)^{1/p} \nearrow \|f\|_p \quad as \quad k \nearrow \infty, \quad p < \infty, \tag{1.16}$$

$$\max_{0 \leqq i \leqq k-1}\left|\sum_{j=0}^{k-1} m_{j,k} u_k^{(j)}(i) c_j\right| \nearrow \|f\|_\infty \quad as \quad k \nearrow \infty.$$

Moreover, for $p=2$ (1.16) gives

$$\Big(\sum_{j=0}^{k-1} m_{j,k} c_j^2\Big)^{1/2} \nearrow \|f\|_2 \quad as \quad k \nearrow \infty. \tag{1.17}$$

Remark. The relation (1.17) is equivalent to

$$\sum_{j=0}^{\infty} c_j^2 = \|f\|_2^2, \tag{1.18}$$

and therefore (1.16) can be regarded as an extension to the L^p case of (1.18).

Proof. In [2] the following relation is established

$$N_{i,k} = \frac{k-i}{k} N_{i,k+1} + \frac{i+1}{k} N_{i+1,k+1},$$

which rewritten for the $M_{i,k}$'s gives for $i = 0, \dots, k-1$

$$M_{i,k} = \frac{k-i}{k+1} M_{i,k+1} + \frac{i+1}{k+1} M_{i+1,k+1}. \tag{1.19}$$

It is important that the right hand side is a convex combination. Introducing

$$\mathfrak{M}_{k,p}(f) = \left(\frac{2}{k}\sum_{i=0}^{k-1} |(f, M_{i,k})|^p\right)^{1/p}$$

we get by (1.19) and Jensen's inequality that

$$\mathfrak{M}_{k,p}(f) \leqq \mathfrak{M}_{k+1,p}(f). \tag{1.20}$$

Moreover Jensen's inequality and (1.3) give

$$\mathfrak{M}_{k,p}(f) \leqq \|f\|_p. \tag{1.21}$$

Now, letting in (1.6) $a_i = (f, M_{i,k})$ we obtain

$$\|R_k f\|_p \leqq \mathfrak{M}_{k,p}(f). \tag{1.22}$$

The combination of the inequalities (1.20)—(1.22) gives

$$\|R_k f\|_p \leqq \mathfrak{M}_{k,p}(f) \leqq \mathfrak{M}_{k+1,p}(f) \leqq \|f\|_p. \tag{1.23}$$

The next step is to identify $\mathfrak{M}_{k,p}(f)$ by means of the Fourier—Legendre coefficients. It is proved in [3] that

$$M_{i,k}=\sum_{j=0}^{k-1}(M_{i,k},P_j)P_j=\sum_{j=0}^{k-1}m_{j,k}u_k^{(j)}(i)P_j$$

and therefore

$$(f,M_{i,k})=\sum_{j=0}^{k-1}m_{j,k}u_k^{(j)}(i)c_j, \tag{1.24}$$

whence we infer

$$\mathfrak{M}_{k,p}(f)=\left(\frac{2}{k}\sum_{i=0}^{k-1}\Big|\sum_{j=0}^{k-1}m_{j,k}u_k^{(j)}(i)c_j\Big|^p\right)^{1/p}. \tag{1.25}$$

To get (1.16) it remains to insert (1.25) into (1.23) and apply Theorem 1. Formula (1.17) we obtain from (1.16) by the following orthogonality relation (cf. [3])

$$\langle u_k^{(j)},u_k^{(i)}\rangle=\delta_{i,j}\frac{k}{2}m_{i,k}^{-1},$$

and this completes the proof of Theorem 2.

Corollary. *Let* $f\in L^p(I)$ *and* $f\sim\sum_{j=0}^{\infty}c_jP_j$. *Then* $\|f\|_p$ *can be numerically evaluated by means of* (c_j). *Indeed, we at first evaluate* (1.24) *and then* (1.25). *Moreover, inequalities* (1.23) *imply the following error estimate*

$$0\leqq\|f\|_p-\mathfrak{M}_{k,p}(f)\leqq\|f-R_kf\|_p.$$

In particular, for $f=P_j$ *we get*

$$0\leqq\|P_j\|_p-\mathfrak{M}_{k,p}(P_j)\leqq(1-m_{j,k})\|P_j\|_p.$$

Comments. Inequalities (1.20) and (1.21) are proved already in [4]. Theorem 2 is related to the L^p moment problem on finite interval by the formula

$$(f,N_{i,k})=(-1)^{k-1-i}\binom{k-1}{i}\Delta^{k-1-i}\mu_i,$$

where

$$\mu_i=\int_{-1}^{1}f(s)\left(\frac{1+s}{2}\right)^i ds.$$

For more details we refer to [4] (see also [1]).

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On the product of certain permutable subgroups

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Dedicated to Professor K. Tandori on his 60th birthday

It is well-known that the finite p-nilpotent groups form a Fitting class; in particular, for $N_1, N_2 \lhd G$ and N_1, N_2 p-nilpotent, $\langle N_1, N_2 \rangle$ is also p-nilpotent. In [1] there was defined $\mathcal{N}(p, q)$ (generalizing the concept of p-nilpotence) as the class of finite groups, in which for every p-subgroup P, $|N(P)/C(P)|$ is not divisible by the prime q. By the theorem in [1], $\mathcal{N}(p, q)$ is a Fitting class for any primes $p \neq q$. In this paper we prove a stronger result:

Theorem. *Let G be a finite group and $H_1, H_2 \leqq G$. Assume that $H_1 H_2 \leqq G$ (i.e. $H_1 H_2 = H_2 H_1$) and $H_t M \leqq G (t=1, 2)$ for every q-subgroup M. Then $H_1, H_2 \in \mathcal{N}(p, q)$ implies $H_1 H_2 \in \mathcal{N}(p, q)$.*

For the proof we need the following lemmas, dealing with the permutability of subgroups of a group G. (Throughout in the text, p and q are distinct fixed primes.)

Lemma 1. *Suppose that $H \leqq G$ and $HM = MH$ for any q-subgroup M. Let S be a subgroup of G then $(H \cap S)D = D(H \cap S)$ for any q-subgroup D in S.*

Proof. $(H \cap S)D = HD \cap S = S \cap DH = D(S \cap H)$.

Lemma 2. *Assume $H, K, L, T \leqq G$ and $L \leqq H \cap K$. If $G = HK = LT$ then $T = (T \cap H)(T \cap K)$.*

Proof. $H = G \cap H = LT \cap H = L(T \cap H)$, similarly $K = L(T \cap K)$, hence $G = HK = L(T \cap H)L(T \cap K) = L(T \cap H)(T \cap K)$, thus $T = T \cap L(T \cap H)(T \cap K) = (T \cap L)(T \cap H)(T \cap K) = (T \cap H)(T \cap K)$.

Lemma 3. *If $R < G$, $|G:R| = q$ then $RD = DR$ for any q-subgroup D.*

Proof. It can be assumed that $D \not\leqq R$. Let z be an element in $D \setminus R$ then $R\langle z \rangle = G = \langle z \rangle R$, hence $RD = G = DR$.

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Lemma 4. (see KEGEL [4] or [2, p. 677]). *Let A and B be subgroups of the finite group G. Suppose that for all $x \in G$, $AB^x = B^x A$; if $AB \neq G$, then at least one of A and B is contained in a normal subgroup of G, different from G.*

Proof of the theorem. By induction; suppose it were false and let G be a counterexample for which $f(G, H_1, H_2) := |G| + |H_1 : H_1 \cap H_2|_q + |H_2 : H_1 \cap H_2|_q + |G : H_1| + |G : H_2|$ is minimal. So $G = H_1 H_2$ and there is a subgroup $U = U_p U_q$ in G with a normal Sylow p-subgroup U_p and cyclic Sylow q-subgroups like U_q such that all subgroups of U except U are (p-)nilpotent. (For the standard properties of such U-s we will use see [2, chapter IV.] or [3]). As $U = \langle U_r^q : r \in U \rangle$, each $H_t U = U H_t$.

($*$) For any subgroup $X \leqq G$ and $H_t \leqq X$ (for at least one t) $X = X \cap G = X \cap H_t H_{t'} = H_t(X \cap H_{t'})$, hence by Lemma 1, $X = G$ or $X \in \mathcal{N}(p, q)$. In particular, $H_t U = G \quad (t = 1, 2)$.

Suppose $U_q \leqq H_t$, then $U_q^{H_t \cap U} := \langle U_q^s : s \in H_t \cap U \rangle \leqq H_t \cap U$, so $H_t \cap U \leqq Z(U) U_q$; thus for a suitable $u \in U$ we get $U_q^u \not\leqq H_t$, hence $H_t < H_t U_q^u$, yielding $1 \neq |H_t U_q^u : H_t| \,|\, (|U_q|, |G : H_t|) \,|\, (|U_q|, |U_p|) = 1$, a contradiction, which gives

$$(1) \qquad U_q \not\leqq H_t \quad (t = 1, 2).$$

(2) Either (i) $q = |G : H_t|$ and $U_p \leqq \bigcap_{y \in G} H_t^y$ (for at least one t), or

(ii) $T := \langle Q : Q \in \mathrm{Syl}_q(G) \rangle \neq G$ and $(H_1 \cap H_2) T \neq G$:

If $T < G$ then $(H_1 \cap H_2) T = G$ would yield by Lemma 2 that

$$T = (T \cap H_1)(T \cap H_2) \in \mathcal{N}(p, q)$$

by the minimality of G and Lemma 1, contrary to $U \leqq T$; so $(H_1 \cap H_2) T < G$ in this case.

Now assume $T = G$. Suppose $H_t Q < G$ for both t and all $Q \in \mathrm{Syl}_q(G)$, then $H_t^G < G$ for each t by Lemma 4, hence $H_1^G, H_2^G \in \mathcal{N}(p, q)$ by ($*$); so

$$G = H_1^G H_2^G \in \mathcal{N}(p, q)$$

by [1], a contradiction. Thus we can assume $H_1 Q = G$ (with a $Q \in \mathrm{Syl}_q(G)$). Then with a suitable $Q_1 < Q$ we get $|G : H_1 Q_1| = q$. By Lemma 3 and $f(G, H_1 Q_1, H_2) \leqq f(G, H_1, H_2) - (|G : H_1| - q)$ we see that $|G : H_1| = q$. Let $x \in G$, then $G = H_1 U^x$ by (1), thus $|U^x : H_1 \cap U^x| = |G : H_1| = q$, so $U_p^x \leqq H_1$, as required.

(3) If $H_t U_q < G$ then $H_t \cap U = 1$ and $|U_q| = q$: $H_t U_q < G$ implies $H_t U_q \in \mathcal{N}(p, q)$, thus $H_t \cap U \leqq Z(U)$ by (1). $D := (H_t \cap U)^G = (H_t \cap U)^{U H_t} = (H_t \cap U)^{H_t} \leqq H_t$, hence $D \cap U = H_t \cap U \leqq Z(U)$, thus $U/D \cap U \notin \mathcal{N}(p, q)$. If $D > 1$, then — all conditions of the theorem remaining valid for G/D, $H_1 D/D$, $H_2 D/D$ — $G/D \in \mathcal{N}(p, q)$, contrary to $UD/D \leqq G/D$; so $D = 1$. Let $D_1 = (\Phi(U_q))^G$, then $D_1 =$

$=(\Phi(U_q))^{UH_t}=(\Phi(U_q))^{H_t}\leqq H_t\Phi(U_q)$, hence $D_1\cap U\leqq H_t\Phi(U_q)\cap U=(H_t\cap U)\Phi(U_q)=$ $=\Phi(U_q)\leqq Z(U)$. Thus we get (factorizing by D_1) $D_1=1$.

Now, by (2), we separate two cases.

Case 1: $U_p\leqq N:=\bigcap_{x\in G} H_1^x$, $|G: H_1|=q$.

Case 1/a: $NH_2<G$. As $G=H_2U=NH_2U_q$, $1\neq|G: NH_2|$ is a power of q and $NH_2\in\mathcal{N}(p,q)$ by Lemma 1. Then $f(G,H_1,NH_2)\leqq f(G,H_1,H_2)$ with equality iff $N\leqq H_2$. Thus $|G: H_2|$ is a power of q, consequently $H_2\leqq\tilde{H}_2<G$ with $|G: H_2|=q$. As $f(G,H_1,\tilde{H}_2)\leqq f(G,H_1,H_2)$, $H_2=\tilde{H}_2$ is of index q. So $U_p\leqq M:=N\cap\bigcap_{x\in G}H_2^x$. Let $U_p\leqq R\in\mathrm{Syl}_p(M)$, then $G=MN_G(R)$, so by Lemma 2, $N_G(R)=N_{H_1}(R)N_{H_2}(R)$. Thus $N_G(R)\in\mathcal{N}(p,q)$ by Lemma 1, if $N_G(R)<G$. If so, then for $Q_1\in\mathrm{Syl}_q(C_G(R))$ there exists a Sylow q-subgroup Q_2 of M such that (Q_1 normalizes Q_2, hence) $Q_1Q_2\in\mathrm{Syl}_q(G)$. Let $U_q=\langle b\rangle$, then $b\in T=\langle (Q_1Q_2)^x\colon x\in G\rangle\leqq(C_G(R)M)^G=C_G(R)M$, thus $b=b_Cb_M$ with $b_C\in C_G(R)\leqq C_G(U_p)$ and $b_M\in M$. So $b_M\in N_M(U_p)\setminus C_G(U)_p$ and for any u in U_p, $u^b=u^{b_M}$. Hence $1=u^{b^q}=u^{b_M^q}$, yielding with a suitable power b_M^k a q-subgroup $\langle b_M^k\rangle$, that normalizes but does not centralize the p-subgroup U_p, contrary to $M\leqq H_1\in\mathcal{N}(p,q)$; thus $R\lhd G$. For $t=1,2$ let $S_t\in\mathrm{Syl}_q(H_t)$, then $S_1^G, S_2^G\leqq C_G(R)$. Let $S\in\mathrm{Syl}_q(G)$; there exist elements e,f in G with $S_1^e, S_1^f\leqq S$. $S\not\leqq C_G(R)$ and $|S: S_1^e|=q=|S: S_2^f|$ (because of $|G: H_1|=q=|G: H_2|$), so $S_1^e=S_2^f$. $ef^{-1}=g_1g_2$ (with $g_t\in H_t$) and $S_1^{g_1g_2}\leqq H_1^{g_2}\cap H_2$; as $f(G,H_1^{g_2}; H_2^{g_2}=H_2)\leqq$ $\leqq f(G,H_1,H_2)-\sum_{t=1,2}|H_t: H_1\cap H_2|_q$, we get that $|H_1\cap H_2|_q=|H_1|_q=|H_2|_q$, contrary to $|H_1\cap H_2|=|H_1||H_2||G|^{-1}|=q^{-1}|H_1|$.

Case 1/b: $NH_2=G$. As $NU=NU\cap NH_2=N(NU\cap H_2)$, $NU=G$ by Lemma 1. Thus $NU_q=G$, G/N is cyclic, so $H_1\lhd G$. Suppose $H_2U_q=G$, then $H_2\leqq\hat{H}_2<G$ with a $|G: \hat{H}_2|=q$, so by induction, $H_2=\hat{H}_2$. Let $E=\bigcap_{x\in G}H_2^x$, then $G\neq H_1E$ by [1] and $U_p\leqq E$, producing Case 1/a with (H_2,E,H_1) instead of (H_1,N,H_2). Thus $H_2U_q\neq G$. $H_2 \lhd\!\lhd G$ by [1], so $L:=U_q^G<G$ by Lemma 4.

$H_1\cap L\lhd G$, $|L: H_1\cap L|=q$, $L\notin\mathcal{N}(p,q)$, hence $L\neq(H_1\cap L)(H_2\cap L)$ by Lemma 1, which yields $H_2\cap L\leqq H_1\cap L$. Suppose $(L\cap H_1)H_2<G$, then (as $G=H_2U=$ $=H_2L$), $|G: (L\cap H_1)H_2|=q$, $f(G,H_1,(L\cap H_1)H_2)\leqq f(G,H_1,H_2)$. Thus H_2 is of index q in G, $G=H_2U_q$, which is not the case; so $G=(L\cap H_1)H_2$. We get $G/L\cap H_1\simeq H_2/L\cap H_1\cap H_2=H_2/L\cap H_2\simeq G/L$, $L\leqq H_1$, a contradiction.

Case 2: $T=\langle Q\colon Q\in\mathrm{Syl}_q(G)\rangle\neq G$. Having eliminated Case 1 we may assume by (2) and (3) that $H_t\cap\hat{U}=1$ $(t=1,2)$ and $|\hat{U}_q|=q$ for *any* $\hat{U}$, being of the same type as U. Also by (2), $(H_1\cap H_2)T<G$.

As $\hat{U}\cap H_t=1$, $|T: T\cap H_t|=|\hat{U}|$; let $Q\in\mathrm{Syl}_q(G)$, then by Lemma 1, $(T\cap H_t)Q^x=Q^x(T\cap H_t)$ for any $x\in G$. $T\neq(T\cap H_t)Q$, hence by Lemma 4, there exist $W_t \lneqq T$ $(t=1,2)$ with $T\cap H_t \leqq W_t$. $\hat{U}_q \not\leqq W_t$ yields $W_t \in \mathcal{N}(p,q)$ and the existence of $V_t \lhd T$ with $W_t \leqq V_t$ and $|T\colon V_t|=q$ $(t=1,2)$. Still $\hat{U}_q \not\leqq V_t$, so $V_1, V_2 \in \mathcal{N}(p,q)$. By $|T\colon V_t|=q$, $T \notin \mathcal{N}(p,q)$ and [1], $V_1=V_2$.

On the other hand, $V_t=V_t\cap T=V_t\cap(T\cap H_t)\hat{U}=(T\cap H_t)(V_t\cap\hat{U})=(T\cap H_t)\hat{U}_p$; thus

(4) $(T\cap H_1)\hat{U}_p=(T\cap H_2)\hat{U}_p$, consequently $|T\cap H_1|=|T\cap H_2|$.

(5) $|G\colon T|$ is a power of p, hence for any $A\leqq B\leqq G$, $|B\colon A|_q=|B\cap T\colon A\cap T|_q$:

Let P be a Sylow p-subgroup of G then $(|G\colon TP|, |G\colon H_1\cap H_2|)=(|G\colon TP|, |\hat{U}|^2)=1$, thus $G=(H_1\cap H_2)TP$, so $TP=(TP\cap H_1)(TP\cap H_2)$ by Lemma 2. By $U\leqq TP$, $TP=G$.

Let $Q_t\in \mathrm{Syl}_q(H_t\cap T)$ for $t=1,2$, then by (4), $Q_2=Q_1^g$ for some g. Let $g=g_1g_2$ with $g_t\in H_t$; as $Q_2\leqq(T\cap H_1^{g_2})\cap(T\cap H_2)$, $f(G, H_1^{g_2}, H_2^{g_2}=H_2)=|G|+|G\colon H_1|+|G\colon H_2|+|H_1^{g_2}\colon H_1^{g_2}\cap H_2|_q+|H_2\colon H_1^{g_2}\cap H_2|_q=|G|+|G\colon H_1|+|G\colon H_2|+|T\cap H_1^{g_2}\colon T\cap H_1^{g_2}\cap H_2|_q+|T\cap H_2\colon T\cap H_1^{g_2}\cap H_2|_q$ by (5). But $|T\cap H_1^{g_2}\colon T\cap H_1^{g_2}\cap H_2|_q=1=|T\cap H_2\colon T\cap H_1^{g_2}\cap H_2|_q$ as $Q_2\leqq T\cap H_1^{g_2}\cap H_2$, so by the minimality of $f(G, H_1, H_2)$, we have $|H_1\colon H_1\cap H_2|_q=|H_2\colon H_1\cap H_2|_q=1$.

On the other hand, $|H_1\colon H_1\cap H_2|_q=|G\colon H_2|_q=|U|_q=q=|G\colon H_1|_q=|H_2\colon H_1\cap H_2|_q$, the final contradiction.

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Completeness in coalgebras

B. CSÁKÁNY*

To Professor Károly Tandori on his sixtieth birthday

1. Preliminaries. For a set A and n positive integer, denote by $A^{(n)}$ the n'th copower (i.e. the union of n disjoint copies) of A. Dualizing the notion of an n-ary operation we obtain that of an *n-ary co-operation on* A: this is a mapping $f\colon A \to A^{(n)}$. The corresponding notion may be introduced in any well-copowered category, cf. [4], [6], [10]. For A non-empty and F a set of co-operations on A, the pair $\langle A;\ F\rangle$ is called a *coalgebra.* Coalgebras were considered by DRBOHLAV [2]; he introduced the common algebraic notions and proved the Birkhoff variety theorem for them. Here we shall study completeness of sets of co-operations on finite sets.

Let $\mathbf{n}$ stand for $\{0, \ldots, n-1\}$. One can introduce $A^{(n)}$ as $\mathbf{n}\times A$, and so each co-operation $f\colon A \to A^{(n)}$ is uniquely determined by a pair of mappings $\langle f_0, f_1\rangle$ where $f_0\colon A \to \mathbf{n}$ and $f_1\colon A \to A$. We call f_0 and f_1 *the labelling* and *the mapping of f,* respectively. We can imagine co-operations — as well as other mappings — by means of graphs, e.g. Fig. 1 displays the ternary co-operation on $\mathbf{3}$ having the cycle (012) as labelling and the transposition (01) as mapping.

The *n-ary coprojections* may be defined by dualizing the notion of the n-ary projection. We write $p^{n,i}$ for the i'th n-ary coprojection $(i=0, \ldots, n-1)$; then $p_0^{n,i}(a)=i$ and $p_1^{n,i}(a)=a$ for each $a\in A$.

The superposition $f(g_0, \ldots, g_{n-1})$ of an operation $f\colon A^n \to A$ and n operations $g_i\colon A^k \to A$ $(i=0, \ldots, n-1)$ may be considered as follows. There exists a (unique) $g\colon A^n \to A^n$ such that $g_i=ge_i^n$ for each $i\in\mathbf{n}$. Then $f(g_0, \ldots, g_{n-1})=gf$. Dually, for arbitrary co-operations $f\colon A \to A^{(n)}$, $g^{(i)}\colon A \to A^{(k)}$ $(i=0, \ldots, n-1)$ there exists a (unique) mapping $g\colon A^{(n)} \to A^{(k)}$ such that $g^{(i)}=p^{n,i}g$ for each $i\in\mathbf{n}$. The co-operation $fg\colon A \to A^{(k)}$ is called the *superposition* of f and $g^{(i)}$; we denote it by $f(g^{(0)}, \ldots, g^{(n-1)})$. Fig. 2 and 3 display $f(g^{(0)}, g^{(1)}, g^{(2)})$ with $f, g^{(0)}, g^{(1)}$ the co-operation on Fig. 1, and $g^{(2)}=p^{3,2}$. For the labelling and mapping of a superposition $s=f(g^{(0)}, \ldots, g^{(n-1)})$

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we have

$$s_0(a) = g_0^{(f_0(a))}(f_1(a)), \qquad (1)$$
$$s_1(a) = g_1^{(f_0(a))}(f_1(a)).$$

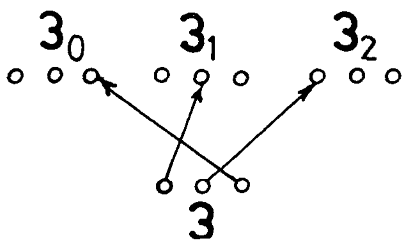

Fig. 1

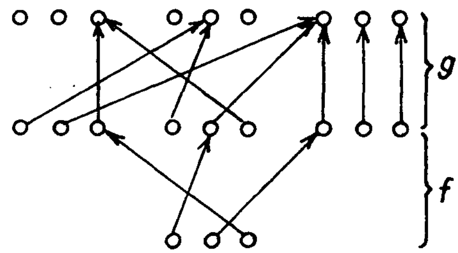

Fig. 2

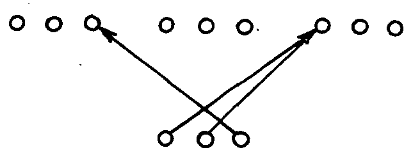

Fig. 3

Analogously to the case of operations, a set of co-operations on a set A is called a *clone* if it is closed under superpositions and contains all coprojections. A clone of co-operations is also an abstract clone, i.e. it is a heterogeneous clone in the sense of TAYLOR [13]. Indeed, it satisfies the identities (2.8.1)—(2.8.3) in the definition of heterogeneous clone in [13]; they may be written in the form

$$f(g^{(0)}(h^{(0)}, \ldots, h^{(k-1)}), \ldots, g^{(n-1)}(h^{(0)}, \ldots, h^{(k-1)})) = \tag{2.1}$$
$$= (f(g^{(0)}, \ldots, g^{(n-1)}))(h^{(0)}, \ldots, h^{(k-1)})$$

for arbitrary $f, g^{(i)}, h^{(j)}$ of appropriate arities;

$$f(p^{n,0}, \ldots, p^{n,n-1}) = f \tag{2.2}$$

for f n-ary; and

$$p^{n,i}(f^{(0)}, \ldots, f^{(n-1)}) = f^{(i)} \tag{2.3}$$

for $f^{(0)}, \ldots, f^{(n-1)}$ of the same arity. Denote, e.g., the left and right side of (2.1) by p and q, and let $\bar f$ and $\bar g^{(i)}$ stand for $f(g^{(0)}, \ldots, g^{(n-1)})$ and $g^{(i)}(h^{(0)}, \ldots, h^{(k-1)})$, respectively. Then, for every $a \in A$, the equations (1) give

$$p_0(a) = \bar g_0^{(f_0(a))}(f_1(a)) = h_0^{(g_0^{(f_0(a))}(f_1(a)))}(g_1^{(f_0(a))}(f_1(a))) =$$
$$= h_0^{(\bar f_0(a))}(\bar f_1(a)) = q_0(a),$$

and similarly we obtain $p_1(a) = q_1(a)$. One can verify also (2.2) and (2.3).

We shall denote the clone of all co-operations on A by $\mathscr{C}_A$, and the set of all n-ary co-operations of A by $\mathscr{C}_A^n$.

An n-ary operation f on A depends on its i'th variable iff there is an n-ary g on A such that $f(e_1^n, \ldots, e_{i-1}^n, g, e_{i+1}^n, \ldots, e_n^n) \neq f$. Accordingly, an $f \in \mathscr{C}_A^n$ *depends on its i'th variable* if there exists a $g \in \mathscr{C}_A^n$ with $f(p^{n,0}, \ldots, p^{n,i-1}, g, p^{n,i+1}, \ldots, p^{n,n-1}) \neq f$. It is easy to verify that f depends on its i'th variable iff $f_0^{-1}(i)$ is not void. We say that $f(\in \mathscr{C}_A^n)$ is *essentially k-ary* if there exist exactly k elements $i \in \mathbf{n}$ such that f depends on its i'th variable, i.e. if f_0 has k-element range.

2. Complete sets of co-operations. We shall study co-operations on finite sets $\mathbf{n}$ $(n>1)$. For $C \subseteq \mathscr{C}_{\mathbf{n}}$, the least clone in $\mathscr{C}_{\mathbf{n}}$ containing C will be denoted by $[C]$ and called *the clone generated by C*. If $[C] = \mathscr{C}_{\mathbf{n}}$ (i.e. every co-operation on n may be obtained from those in C and coprojections using superposition) then C is said to be *complete*. In this case we call also the coalgebra $\langle \mathbf{n}; C \rangle$ *primal*.

We shall need terms and notations for special co-operations. The *diagonal co-operation d* on $\mathbf{n}$ is n-ary with d_0, d_1 identical. The n-ary *(i,j)-constant co-operation* $i^{n,j}$ is determined by $i_0^{n,j}(k)=j$, $i_1^{n,j}(k)=i$ for each $i,j \in \mathbf{n}$ and for each $k \in \mathbf{n}$. An *(i,j)-translation* is a co-operation t with $t_1(i)=j$. If such a t is m-ary then $t(p^{n,l_0}, \ldots, \ldots, p^{n,l_{m-1}})$ is an n-ary (i,j)-translation (which is essentially $|\{l_0, \ldots, l_{m-1}\}|$-ary). Similarly, from an (i,j)-constant we can get an (i,j)-constant of arbitrary arity. We call a co-operation g *(i,j)-gluing* if $g_k(i)=g_k(j)$ for all $k \in \mathbf{2}$; g is *gluing* if it is (i,j)-gluing for some $i,j \in \mathbf{n}$. Thus, g is not gluing iff the mapping $i \mapsto \langle g_0(i), g_1(i) \rangle$ is 1—1 on $\mathbf{n}$.

The following observations are trivial:

Proposition 0. *An essentially k-ary co-operation is a superposition of a k-ary co-operation and some coprojections. If a co-operation on* $\mathbf{n}$ *is essentially k-ary then $k \leqq n$.*

This implies

Proposition 1. *The set of all at most n-ary co-operations on* $\mathbf{n}$ *is complete.*

Thus, studying completeness on $\mathbf{n}$, we can restrict ourselves to co-operations with arity $\leqq n$.

The mappings of a set C of co-operations on n generate a semigroup $\mathscr{S}(C)$ of self-mappings of $\mathbf{n}$, called *the semigroup of C*. We call C *transitive* if $\mathscr{S}(C)$ is transitive. Note that *each self-mapping in $\mathscr{S}(C)$ is the mapping of some (unary) co-operation in* $[C]$, *i.e.*, $\mathscr{S}(C) \subseteq \mathscr{S}[C]$. Indeed, for co-operations f and g of arbitrary arities, let $h=f(g(p^{1,0}, \ldots, p^{1,0}), \ldots, g(p^{1,0}, \ldots, p^{1,0}))$. Then for each $i \in \mathbf{n}$, $h_1(i)=g_1(f_1(i))$, proving that $\mathscr{S}[C]$ is closed under products of mappings, whence the assertion follows.

Proposition 2. *A transitive set of co-operations on* $\mathbf{n}$ *is complete provided it contains an essentially n-ary co-operation.*

Proof. By Proposition 1, we have to prove that, for a set of co-operations C satisfying the conditions of Proposition 2, every at most n-ary co-operation g on $\mathbf{n}$ is a composition of some co-operations in C. Let $f\in C$ be essentially n-ary. Then the labelling of f is onto, hence it is a permutation of $\mathbf{n}$. Form $f(p^{k,g_0(f_0^{-1}(0))}, \dots, p^{k,g_0(f_0^{-1}(n-1))})=f'$; then, for each $i\in\mathbf{n}$, we have $f_0'(i)=p_0^{k,g_0(i)}(f_1(i))=g_0(i)$, i.e., the arity and labelling of f are the same as those of g, while its mapping is the mapping of f: for $i\in\mathbf{n}$, $f_1'(i)=p_1^{k,g_0(i)}(f_1(i))=f_1(i)$.

On the other hand, as C is transitive, for every $k, l\in\mathbf{n}$ there exists a (k, l)-translation $t^{k,l}$; we can assume that $t^{k,l}$ is unary. Then $t^{k,l}(p^{n,j})$ is an n-ary (k, l)-translation whose labelling is the constant function with value j. Now form

$$f\left(t^{f_1(f_0^{-1}(0)),g_1(f_0^{-1}(0))}(p^{n,0}), \dots, t^{f_1(f_0^{-1}(n-1)),g_1(f_0^{-1}(n-1))}(p^{n,n-1})\right) = f^*.$$

Then, for each $i\in\mathbf{n}$, $f_0(i) = \left(t^{f_1(i),g_1(i)}(p^{n,f_0(i)})\right)_0(f_1(i)) = f_0(i)$, and $f_1(l) = = \left(t^{f_1(i),g_1(i)}(p^{n,f_0(i)})\right)=g_1(i)$, i.e., we have an essentially n-ary f^* whose labelling coincides with that of f, while its mapping is the mapping of g.

Finally, $g=(f^*)'\in[C]$.

Corollary 2.1. *If f is diagonal and the mapping of g is a cycle on* $\mathbf{n}$ *then* $\{f, g\}$ *is complete.*

Indeed, the diagonal co-operation is essentially n-ary, and a cycle on $\mathbf{n}$ generates a transitive group on $\mathbf{n}$.

Corollary 2.2. *The set* $\mathscr{C}_\mathbf{n}^2$ *of all binary co-operations is complete.*

This is the coalgebraic version of Sierpiński's completeness theorem [11]. As clearly there are binary co-operations whose mapping is cyclic (hence generates a transitive semigroup), we have to show only that there is an essentially n-ary co-operation in the clone generated by the superposition of binary co-operations on $\mathbf{n}$. Define $b^{n,i}\in\mathscr{C}_\mathbf{n}^2$ $(i\in\mathbf{n}-1)$ by $b_0^{n,i}(k)=0$ if $k\leqq i$ and $b_0^{n,i}(k)=1$ otherwise, while $b_1^{n,i}$ is identical. Then $b^{n,0}(p^{n,0}, n^{n,1}(p^{n,1}, \dots, b^{n,n-2}(p^{n,n-2}, p^{n,n-1})\dots))=d$, the diagonal (i.e., any essentially n-ary) co-operation on $\mathbf{n}$.

Next we determine the Sheffer c operations: a co-operation on $\mathbf{n}$ is *Sheffer* if it generates the clone of all co-operations on $\mathbf{n}$ (cf. [7]). Consider a partition π of $\mathbf{n}$. We say that a co-operation f on $\mathbf{n}$ *preserves* π if π is a refinement of the partition induced on $\mathbf{n}$ by f_0 (i.e. f_0 is constant on each block of π), and is compatible with f_1 (i.e. on each block of π all the values are in the same block of π). A set C of co-operations preserves π if each $f\in C$ preserves π. Every co-operation preserves the least partition (the one with 1-element blocks) and exactly the essentially unary co-operations preserve the greatest partition (with one block). Further, let S be a non-empty subset of $\mathbf{n}$. We say that a set C of co-operations on $\mathbf{n}$ *preserves* S if S is closed under f_1 for every $f \in C$.

Proposition 3. *A co-operation f on $\mathbf{n}$ is Sheffer if and only if it preserves neither non-least partitions nor non-empty proper subsets of $\mathbf{n}$.*

Proof. Sufficiency. The second condition means that $[\{f\}]$ is transitive. By Proposition 2, it is enough to prove that f contains an essentially n-ary co-operation.

Suppose that f is m-ary. Then $m \geqq 2$, and f is essentially at least binary, since it does not preserve the partition of n consisting of one block. Further, f_1 is cyclic, since f is transitive; hence f is not gluing.

We show that, for each pair i, j of different elements from $\mathbf{n}$, there exists a non-negative integer k such that $f_0(f_1^k(i)) \neq f_0(f_1^k(j))$. Write i^0 for i, and i^k for $f_1(i^{k-1})$. Suppose that $f_0(i^k) = f_0(j^k)$ for every integer $k \geqq 0$, contrary to the claim; in particular, $f_0(i) = f_0(j)$. As f_1 is cyclic, there is a least natural number $t(<n)$ such that $j = i^t$, and hence $j^k = i^{t+k}$. It follows $j^{(r-1)t} = i^{rt}$, and thus $f_0(i^{rt}) = f_0(i)$ for every non-negative integer r. If $(t, n) = 1$ then $\{i^{rt}\colon r \geqq 0\} = \mathbf{n}$, so f_0 is constant, a contradiction, because f is at least binary. Hence $1 < (t, n) < n$. Now we see that $f_0(i^u) = f_0(i^v)$ whenever $u \equiv v \pmod{(t, n)}$. Define an equivalence $\sim$ on $\mathbf{n}$ by $i^u \sim i^v$ iff $u \equiv v$ $\pmod{(t, n)}$; this is a refinement of the equivalence induced by f_0. Also, clearly, $\sim$ is preserved by f_1. Hence f preserves the (non-trivial) partition of this equivalence, a contradiction again.

Given an integer $k \geqq 0$, there exists a unary co-operation h in $[f]$ such that, for each $i \in \mathbf{n}$, $h_1(i) = f_1^k(i)$. Hence for the m-ary co-operation $s^{i,j} = h(f)$ we have $s_0^{i,j}(i) = f_0(h_1(i)) = f_0(f_1^k(i)) \neq f_0(f_1^k(j)) = s_0^{i,j}(j)$.

Now, if $2 \leqq k < n$, for every non-gluing essentially k-ary co-operation $c \in [f]$ we construct a non-gluing essentially at least $(k+1)$-ary co-operation $c' \in [f]$ as follows:

Since $k < n$, and c is not gluing, there exist $i, j \in \mathbf{n}$ such that $c_0(i) = c_0(j)$, and $c_1(i) \neq c_1(j)$. Let c be (formally) l-ary. Put

$$c' = c\big(p^{l+1,0}, \dots, p^{l+1,c_0(i)-1}, s^{c_1(i),c_1(j)}(\underbrace{p^{l+1,l}}_{0}, \dots, p^{l+1,l},$$

$$\underbrace{p^{l+1,c_0(i)}}_{s_0^{c_1(i),c_1(j)}(c_1(i))}, \quad p^{l+1,l}, \dots, \underbrace{p^{l+1,l}}_{m-1}), p^{l+1,c_0(i)+1}, \dots, p^{l+1,l-1}\big).$$

Assume that c depends on its q'th variable. Then there is an $r \in \mathbf{n}$ such that $c_0(r) = q$. If $q \neq c_0(i)$ then $c_0'(r) = p_0^{l+1,q}(c_1(r)) = q$, and if $q = c_0(i)$ then $c_0'(i) = = p_0^{l+1,c_0(i)}(s_1^{c_1(i),c_1(j)}(c_1(i))) = c_0(i) = q$, i.e., c' also depends on its q'th variable. In addition, c' depends on its l'th variable, too: $c_0(j) = {}_0^{l+1,l}(s_1^{c_1(i),c_1(j)}(c_1(j))) = l$.

We have shown that c' is essentially at least $(k+1)$-ary. It remains to show that c' is not gluing. Observe that, for $a\in\mathbf{n}$, $c_0'(a)=l$ if $a\neq i$ and $c_0(a)=c_0(i)$, while $c_0'(a)=c_0(a)$ otherwise; further $c_1'(a)=s_1^{c_1(i),c_1(j)}(c_1(a))$ if $c_0(a)=c_0(i)$, and $c_1'(a)=c_1(a)$ otherwise. Since $s_1^{c_1(i),c_1(j)}$ is a permutation of $\mathbf{n}$, we obtain that, for $a,b\in\mathbf{n}$ with $c_0'(a)=c_0'(b)$, $c_1'(a)\neq c_1'(b)$ whenever $c_1(a)\neq c_1(b)$. This means that c' is (a, b)-gluing only if c is (a, b)-gluing. Thus, c' is not gluing, as required.

Using this construction, from f we get an essentially n-ary co-peiation in f in a finite number of steps, proving the sufficiency.

Necessity. We have to show that if a co-operation f preserves a non-trivial partition π of $\mathbf{n}$ then every co-operation in $[f]$ also preserves π, and the same holds for non-empty subsets instead of non-trivial partitions. As the coprojections preserve everything, it is enough to show that any composition $f(g^0, \dots, g^{k-1})$ preserves the partition π provided $f, g^0, \dots, g^{k-1}$ preserve it.

Put $h=f(g^0, \dots, g^{k-1})$, and let $a\equiv b(\pi)$. Then $h_0(a)=g_0^{f_0(a)}(f_1(a))$, $h_0(b)= =g_0^{f_0(b)}(f_1(b))$. Here $f_1(a)\equiv f_1(b)(\pi)$ and $f_0(a)=f_0(b)$, hence $g_0^{f_0(a)}(f_1(a))= =g_0^{f_0(b)}(f_1(b))$, as needed. Also we have $h_1(a)=h_1(b)$, again by (1) and the definition of preservation. The case of subsets is even simpler. Thus, Proposition 3 is proved.

Consider the case when n is a prime number. Then the non-preserving of non-empty proper subsets by f means that f_1 is a prime-order cycle, hence it preserves no non-trivial partition with more than one blocks. Thus we have to exclude the preservation of the one-block partition only. This can be done by requiring that f is essentially at least binary. Hence it follows:

Corollary 3.1. *Let n be a prime number. A co-operation f on $\mathbf{n}$ is Sheffer tf and only if it is essentially at least binary and f_1 is a cyclic permutation of* $\mathbf{n}$.

Introducing some natural algebraic notions for coalgebras, we can given a more familiar form to Proposition 3. Let $\mathbf{A}=\langle A;\ F\rangle$ be a coalgebra. If the subset B of A is preserved by F, we can obtain a subcoalgebra $\mathbf{B}=\langle B;\ F'\rangle$ of $\mathbf{A}$ by putting $F'=\{f'\colon f\in F\}$ where f_i' $(i\in\mathbf{2})$ are the restrictions of f_i to B. A subcoalgebra $\mathbf{B}$ of $\mathbf{A}$ is *proper* if B is a proper subset of A.

Furthermore, if the partition π of A is preserved by F, we can obtain a coalgebra $\bar{\mathbf{A}}=\langle\bar{A};\ \bar{F}\rangle$, where $\bar{A}=\{\bar{a}\colon a\in A\}$ is the set of blocks of π, while $\bar{F}=\{\bar{f}\colon f\in F\}$ and $\bar{f}$ is defined by $\bar{f}_0(\bar{a})=f_0(a)$, $\bar{f}_1(\bar{a})=\overline{f_1(a)}$ for each $a\in A$. Coalgebras $\bar{\mathbf{A}}$ arising in such a way are called *factorcoalgebras* of $\langle A;\ F\rangle$; $\bar{\mathbf{A}}$ is *proper* if it is induced by a partition with at least one non-trivial block. As it is usual for algebras, a coalgebra $\mathbf{B}$ which may be obtained from another coalgebra $\mathbf{A}$ by forming a subcoalgebra of a factorcoalgebra is called a *factor* of A. A factor of $\mathbf{A}$ is *proper* if in the process of its formation we take a proper sub- or factoralgebra. Using the just introduced notions, Proposition 3 states:

Proposition 3. *A finite coalgebra with one co-operation is primal if and only if it has no proper factors.*

This is the coalgebraic version of Rousseau's theorem (a finite algebra with one operation is primal iff it has no proper factors and is rigid [8], [7]).

The following proposition corresponds to Słupecki's completeness criterion for operations [12], [7]. Call a co-operation *essential* if it is essentially at least binary and non-gluing.

Proposition 4. *The set consisting of all unary co-operations and an arbitrary essential co-operation is complete on any* $\mathbf{n}$.

Proof. Denote the set and the essential co-operation in the proposition by S and f, respectively. We show that there is a Sheffer co-operation in $[S]$. For this aim we prove the following two claims:

(α) There exists a Sheffer co-operation g on $\mathbf{n}$ such that $g_0=f_0$.

(β) If g is a co-operation on $\mathbf{n}$ such that $g_0=f_0$, then $g\in[S]$.

Proof of (α). A co-operation g on $\mathbf{n}$ with $g_0=f_0$ is fully determined by its mapping g_1. We have to define a g_1 such that neither non-empty proper subsets nor non-least partitions would be preserved by g. Concerning the subsets, it is sufficient to choose g_1 a cyclic permutation of $\mathbf{n}$. As for the partitions, the co-operation g may preserve only refinements of the partition λ induced by its labelling. Thus, we have to show that under appropriate choice of the cycle g_1, no non-trivial refinement of λ will be preserved by (the unary operation) g_1. We can suppose that λ itself is not least, else we are done.

Given a cyclic permutation g_1 of $\mathbf{n}$ and an element $i\in\mathbf{n}$, each element of $\mathbf{n}$ may be written in the form $g_1^r(i)$; for this element, we write shortly i^r. Partitions preserved by g_1 are the same as congruences of the algebra $\langle\mathbf{n};\ g_1\rangle$. Each such non-trivial and proper congruence is uniquely determined by a divisor $d(1<d<n)$ of n (and hence it may be denoted by π_d) in the following way: $i^r\equiv i^s(\pi_d)$ if and only if $r\equiv s\ (\mathrm{mod}\ d)$.

Let $\bar{\imath}$ be a block of λ with minimal number of elements, and $i\in\bar{\imath}$. Then $|\bar{\imath}|\leqq n/2$. On the other hand, the number of non-trivial proper divisors of n is less than $n/2$; hence we can define g_1 so that for each non-trivial proper divisor d of n $i^d\notin\bar{\imath}$. Now if, for some s, $\pi_d\leqq\lambda$ then from $i^d\equiv i^0=i\ (\pi_d)$ it follows $i^d\equiv i(\lambda)$, i.e., $i^d\in\bar{\imath}$; a contradiction.

Proof of (β). Let f be l-ary, $k\in\mathbf{l}$, $i\in\mathbf{n}$. As f is not gluing, the system of equations

$$f_0(x)=k,\quad f_1(x)=i$$

has at most one solution $x^{k,i}$ in $\mathbf{n}$. Clearly, each element of $\mathbf{n}$ may be written in form

$x^{k,i}$ with uniquely determined k and i. Define the unary co-operation t^k by

$$t_1^k(i) = \begin{cases} g_1(x^{k,i}) & \text{if } x^{k,i} \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Define $f' \in [S]$ by $f' = f(t^0(p^{l,0}), \dots, t^{l-1}(p^{l,l-1}))$. Then

$$f_0'(x^{k,i}) = p_0^{l,f_0(x^{k,i})}(t_1^{f_0(x^{k,i})}(f_1(x^{k,i}))) = p_0^{l,k}(t_1^k(i)) = k = f_0(x^{k,i}) =$$
$$= g_0(x^{k,i}), \quad \text{and} \quad f_1'(x^{k,i}) = p_1^{l,k}(t_1^k(i)) = g_1(x^{k,i}).$$

Thus, $g = f' \in [S]$, as required, and the proposition is proved.

Call a co-operation f *sharp* if it is k-ary and essentially k-ary for some k. From Proposition 0 if follows that the number of sharp co-operations is finite on every $\mathbf{n}$, and a clone of co-operations is uniquely determined by the sharp co-operations it contains. Hence we infer that the number of clones of co-operations i finite for each $\mathbf{n}$, i.e. the clones of co-operations on $\mathbf{n}$ form a finite lattice. For $n=2$, there is as few as 12 sharp co-operations, and even this number decreases to 8 if we do not distinguish between $f = f(p^{2,0}, p^{2,1})$ and $f(p^{2,1}, p^{2,0})$ (as they are the same „up to a permutation of variables"). Fig. 4 shows the

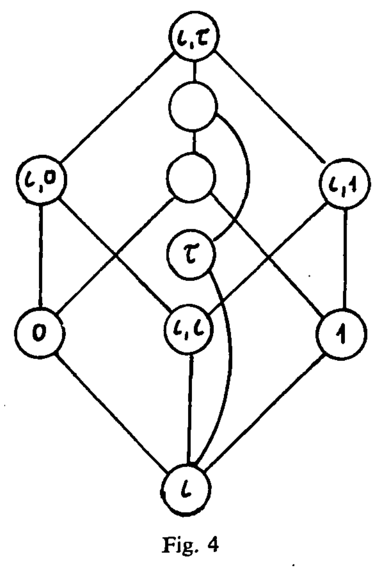

Fig. 4

lattice of clones of co-operations on **2** (the coalgebraic version of the Post diagram; cf. [5]). Circles standing for clones contain pairs or single signs; the denote the labelling-mapping pair or the mapping of the co-operation generating the given clone (if it is generated by one co-operation). We write ι and τ for the identical and non-identical permutation of **2**, and $i(\in \mathbf{2})$ for the constant mapping with value i.

3. Co-operations and selective operations. Given arbitrary non-empty sets P and M, a natural number k, and mappings f_0: $P \to \mathbf{k}$, f_1: $P \to P$, we define a k-ary operation f on M^P by agıeeing that, for every $p \in P$, the p-component of the result of f is the f_1-component of the f_0'th operand. Operations obtained in this way are called regular selective operations (see [1]). The mappings f_0 and f_1 are referred to as the first and second selectors of f. Observe that they can be considered as the labelling and the mapping of a co-operation (of the same arity as f) on P. Moreover, for any nontrivial M and nonempty P, there is a bijection between the regular selective operations on M^P and the co-operations on P assigning to a selective operation f a co-operation whose labelling and mapping are the first and second selectors of f, respectively. This bijection is a clone isomorphism, i.e. it sends a projection into the coprojection with appropriate indices, and a superposition of operations into the superposition of co-operations being the images thereof. This follows immediately from (2) in [1] and (1) in this paper. Hence the study of clones (including lattices of clones) of regular selective operations on a finite power of a set reduces to the study of clones of co-operations on a finite set.

E.g., Corollary 2.1. implies that the basic operations of a k-dimensional die D (see [3]) generate the clone of all selective operations on the base set M^k of D. Hence it follows that *the variety of k-dimensional dice is equivalent to the* k'th *power-variety of sets,* an observation due to TAYLOR [15] (see also [14]).

Further, we can reformulate Corollary 3.1., using the following consequence of Corollary 2.2.: a co-operation f on $\mathbf{n}$ is Sheffer iff $\mathscr{C}_{\mathbf{n}}^2 \subseteq [f]$, and translating it into the language of selective operations, we obtain the following fact: *For p prime, all binary selective operations on* M^p $(|M|>1)$ *are term functions of the given binary selective operation f if and only if f is essentially binary and the second selector of f is a cyclic permutation of* $\mathbf{p}$. Formulated in different terms, this is the main result in [9].

Finally, Fig. 4 may be considered as the lattice of clones of selective operations on M^2 $(|M|>1)$.

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On the stability of the local time of a symmetric random walk

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. rv with $P\{X_1=-1\}=P\{X_1=1\}=1/2$, and consider the symmetric random walk $S_0=0$, $S_n=X_1+\ldots+X_n$ $(n=1, 2, \ldots)$. Define the local time of $\{S_k\}$ by

$$\xi(x, n) := \text{No. } \{k:\ 0 < k \leqq n,\ S_k = x\} \quad (n = 1, 2, \ldots;\ x = 0,\ \pm 1,\ \pm 2, \ldots),$$

i.e., $\xi(x, n)$ is the number of visits of $\{S_k\}$ at x up to time n. The properties of $\xi(x, n)$ have been studied by a number of authors for a long time now. Here we present some well known and important results.

Theorem A.

$$P\{\xi(0, 2n) = k\} = 2^{k-2n}\binom{2n-k}{n} \, (k = 0, 1, 2, \ldots, n;\ n = 1, 2, \ldots),$$

$$\lim_{n\to\infty} P\{n^{-1/2}\xi(x, n) \leqq u\} = \left(\frac{2}{\pi}\right)^{1/2} \int_0^u e^{-t^2/2}\, dt \quad (u > 0;\ x = 0, \pm 1, \pm 2, \ldots).$$

Theorem B (*Kesten*, 1965). *For any* $x=0, \pm 1, \pm 2, \ldots$ *we have*

$$\limsup_{n\to\infty} \frac{\xi(x, n)}{(2n \log\log n)^{1/2}} = \limsup_{n\to\infty} \frac{\sup_{-\infty<x<\infty} \xi(x, n)}{(2n \log\log n)^{1/2}} = 1 \quad a.s.,$$

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right) \xi(x, n) = \gamma_1 \quad a.s.$$

where γ_1 *is a positive absolute constant.*

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Remark 1. The actual value of γ_1 was not given by Kesten. It was recently evaluated by E. Csáki (oral communication).

Remark 2. Roughly speaking the above two theorems say that $\xi(x, n)$ (for any fixed $x=0, \pm 1, \pm 2, \ldots$) goes to infinity like $n^{1/2}$ does.

Intuitively it is clear that $\xi(x, n)$ is close to $\xi(y, n)$ if x is close to y. This paper is devoted to studying this problem.

Here we present the main results.

Theorem 1. *For any* $k=\pm 1, \pm 2, \ldots$ *we have*

$$\limsup_{N\to\infty} \frac{\xi(k, N)-\xi(0, N)}{(\xi(0, N)\log\log N)^{1/2}} = \limsup_{N\to\infty} \frac{|\xi(k, N)-\xi(0, N)|}{(\xi(0, N)\log\log N)^{1/2}} =$$

$$= \limsup_{N\to\infty} \sup_{n\le N} \frac{\xi(k, n)-\xi(0, n)}{(\xi(0, N)\log\log N)^{1/2}} = \limsup_{N\to\infty} \sup_{n\le N} \frac{|\xi(k, n)-\xi(0, n)|}{(\xi(0, N)\log\log N)^{1/2}} =$$

$$= 2(2k-1)^{1/2} \quad a.s.$$

Theorem 2.

$$\limsup_{N\to\infty} \frac{\xi(1, N)-\xi(0, N)}{N^{1/4}(\log\log N)^{3/4}} = \limsup_{N\to\infty} \frac{|\xi(1, N)-\xi(0, N)|}{N^{1/4}(\log\log N)^{3/4}} =$$

$$= \limsup_{N\to\infty} \sup_{n\le N} \frac{\xi(1, n)-\xi(0, n)}{N^{1/4}(\log\log N)^{3/4}} = \limsup_{N\to\infty} \sup_{n\le N} \frac{|\xi(1, n)-\xi(0, n)|}{N^{1/4}(\log\log N)^{3/4}} = \left(\frac{128}{27}\right)^{1/4} \quad a.s.$$

Theorem 3. *For any* $\varepsilon>0$ *we have*

$$\lim_{n\to\infty} \sup_{|k|\le a_n} \left|\frac{\xi(k, n)}{\xi(0, n)}-1\right| = 0 \quad a.s.$$

where $a_n=n^{1/2}(\log n)^{-(2+\varepsilon)}$.

Remark 3. Theorems 1 and 2 essentially say that for any fixed k the distance between $\xi(k, n)$ and $\xi(0, n)$ for large n behaves like $n^{1/4}$. Since $\xi(0, n)$ is about $n^{1/2}$ asymptotically, this means that $\xi(k, n)$ is relatively close asymptotically to $\xi(0, n)$. The meaning of Theorem 3 is about the same. However in the latter theorem we claim that for large n $\xi(k, n)$ is close to $\xi(0, n)$ whenever $|k|\le a_n$, but the meaning of "close" is not as precise as in Theorems 1 and 2. Theorem 3 is nearly the best possible in the following sense.

Theorem 4.

$$\limsup_{n\to\infty} \sup_{|k|\le b_n} \left|\frac{\xi(k, n)}{\xi(0, n)}-1\right| \ge 1 \quad a.s.,$$

where $b_n=n^{1/2}(\log n)^{-1}$.

2. Proof of Theorem 1.

Among the statements of Theorem 1 we only prove

$$\limsup_{N\to\infty} \frac{\xi(k, N)-\xi(0, N)}{(\xi(0, N)\log\log N)^{1/2}} = 2(2k-1)^{1/2} \quad \text{a.s. for any} \quad k = \pm 1, \pm 2, \ldots.$$

The proofs of its other statements can be obtained without any further difficulty along the same lines.

Let $A_{ij}(m)$ be the event that a symmetric random walk starting form m hits i before j $(i \leqq m \leqq j)$. Then

Lemma 2.1.

$$\mathrm{P}\{A_{ij}(m)\} = (j-m)/(j-i).$$

Proof is trivial.

For any $x=0, \pm 1, \pm 2, \ldots$ define

$$\tau_0(x) := 0,$$

$$\tau_1(x) := \inf\{l\colon\ l > 0,\ S_l = x\},$$

$$\ldots$$

$$\tau_{i+1}(x) := \inf\{l\colon\ l > \tau_i(x),\ S_l = x\} \quad (i = 0, 1, 2, \ldots),$$

$$\tau_i := \tau_i(0) \quad (i = 0, 1, 2, \ldots),$$

and let

$$\alpha_1(k) := \xi(k, \tau_1) - \xi(0, \tau_1) = \xi(k, \tau_1) - 1,$$

$$\ldots$$

$$\alpha_i(k) := (\xi(k, \tau_i) - \xi(k, \tau_{i-1})) - (\xi(0, \tau_i) - \xi(0, \tau_{i-1}))$$
$$= (\xi(k, \tau_i) - \xi(k, \tau_{i-1})) - 1 \quad (k = \pm 1,\ \pm 2, \ldots;\ i = 1, 2, \ldots).$$

Clearly then $\alpha_1(k), \alpha_2(k), \ldots$ is a sequence of i.i.d. rv for any $k = \pm 1, \pm 2, \ldots$. Now we evaluate the distribution of $\alpha_1(k)$. We have

Lemma 2.2.

$$\mathrm{P}\{\alpha_1(k) = -1\} = \mathrm{P}\{\xi(k, \tau_1) = 0\} = \frac{2|k|-1}{2|k|}, \tag{2.1}$$

$$\mathrm{P}\{\alpha_1(k) = l\} = \mathrm{P}\{\xi(k, \tau_1) = l+1\} = \left(\frac{1}{2|k|}\right)^2 \left(\frac{2|k|-1}{2|k|}\right)^l \quad (l = 0, 1, 2, \ldots). \tag{2.2}$$

Proof. Without loss of generality we assume that $k > 0$. Then

$$\{\xi(k, \tau_1) = 0\} = \{X_1 = -1\} \cup \{X_1 = 1,\ S_2 \neq k,\ S_3 \neq k, \ldots, S_{\tau_1 - 1} \neq k\}.$$

Hence by Lemma 2.1.

$$\mathrm{P}\{\xi(k,\tau_1)=0\}=\frac{1}{2}+\frac{1}{2}\frac{k-1}{k}=\frac{2k-1}{2k},$$

and (2.1) is proven. Similarly, in case of $m>0$ we have

$$\begin{aligned}\{\xi(k,\tau_1)=m\}=[\{X_1=1\}\cap\{S_2\neq 0,\ S_3\neq 0,\ \ldots,S_{\tau_1(k)-1}\neq 0,\ S_{\tau_1(k)}=k\}]\cap\\ \cap[\{X_{\tau_1(k)+1}=1\}\cup(\{X_{\tau_1(k)+1}=-1\}\cap\{S_{\tau_1(k)+1}\neq 0,\ S_{\tau_1(k)+2}\neq 0,\ \ldots,\\ \ldots,S_{\tau_2(k)-1}\neq 0,S_{\tau_2(k)}=k\})]\cap\ldots\cap[\{X_{\tau_{m-1}(k)+1}=1\}\cup(\{X_{\tau_{m-1}(k+1)}=-1\}\cap\\ \cap\{S_{\tau_{m-1}(k)+1}\neq 0,\ S_{\tau_{m-1}(k)+2}\neq 0,\ \ldots,\ S_{\tau_m(k)-1}\neq 0,\ S_{\tau_m(k)}=k\})]\cap\\ \cap[\{X_{\tau_m(k)+1}=-1\}\cup\{S_{\tau_m(k)+2}\neq k,\ S_{\tau_m(k)+3}\neq k,\ \ldots,\ S_{\tau_1(0)-1}\neq k,\ S_{\tau_1(0)}=0\}].\end{aligned}$$

(Note that in case of $\xi(k,\tau_1)=m$ we have $0<\tau_1(k)<\tau_2(k)<\ldots<\tau_m(k)<\tau_1(0)<$ $<\tau_{m+1}(k)$). Hence, again by Lemma 2.1

$$\mathrm{P}\{\xi(k,\tau_1)=m\}=\frac{1}{2}\frac{1}{k}\left(\sum_{j=0}^{m-1}\binom{m-1}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\frac{k-1}{2}\right)^{m-1-j}\right)\frac{1}{2}\frac{1}{k}=$$
$$=\left(\frac{1}{2k}\right)^2\left(\frac{2k-1}{2k}\right)^{m-1},$$

and (2.2) is also proven. This also completes the proof of Lemma 2.2.

Lemma 2.2 implies

Lemma 2.3.

(2.3) $$E\alpha_1=0,\quad E\alpha_1^2=4k-2,$$

(2.4) $$\lim_{n\to\infty}\mathrm{P}\{n^{-1/2}(\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_n(k))\leq x(4k-2)^{1/2}\}=$$
$$=(2\pi)^{-1/2}\int_{-\infty}^{x}e^{-u^2/2}\,du,\quad -\infty<x<\infty,$$

(2.5) $$\lim_{n\to\infty}\mathrm{P}\{n^{-1/2}\sup_{j\leq n}(\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_j(k))\leq x(4k-2)^{1/2}\}=$$
$$=\left(\frac{2}{\pi}\right)^{1/2}\int_0^x e^{-u^2/2}\,du,\quad x>0,$$

and

(2.6) $$\limsup_{n\to\infty}\frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_n(k)}{(n\log\log n)^{1/2}}=2(2k-1)^{1/2}\quad a.s.$$

The following two lemmas are simple consequences of (2.6).

Lemma 2.4. *Let* $\{\mu_n\}$ *be any sequence of positive integer valued rv with* $\lim_{n\to\infty} \mu_n=\infty$ *a.s. Then*

$$\limsup_{n\to\infty} \frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\mu_n}(k)}{(\mu_n \log\log \mu_n)^{1/2}} \leqq 2\sqrt{2k-1} \quad a.s.$$

Lemma 2.5. *Let* $\{\nu_n\}$ *be a sequence of positive integer valued rv with the following properties:*

(i) $\lim_{n\to\infty} \nu_n=\infty$ *a.s.*

(ii) *there exists a set* $\Omega_0\subset\Omega$ *such that* $\mathrm{P}(\Omega_0)=0$ *and for each* $\omega\notin\Omega_0$ *and* $k=1, 2, \ldots$ *there exists an* $n=n(\omega, k)$ *for which* $\nu_{n(\omega,k)}=k$.

Then

$$\limsup_{n\to\infty} \frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\nu_n}(k)}{(\nu_n \log\log \nu_n)^{1/2}} = 2\sqrt{2k-1} \quad a.s.$$

Utilizing Lemma 2.5. with $\nu_n=\xi(0, n)$, Theorem 3 and the trivial inequality

$$\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\xi(0,n)}(k) \leqq \xi(k, n)-\xi(0, n) \leqq \alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\xi(0,n)+1}(k)+1,$$

we obtain Theorem 1.

3. Proof of Theorem 2

Here we only present a proof of the statement

$$\limsup_{N\to\infty} \frac{\xi(1, N)-\xi(0, N)}{N^{1/4}(\log\log N)^{3/4}} = \left(\frac{128}{27}\right)^{1/4} \quad \text{a.s.}$$

The other statements of Theorem 2 are proven along similar lines.

The proof of Theorem 2 is based on the following result of Dobrushin (1955).

Theorem C.

$$\lim_{n\to\infty} \mathrm{P}\left\{n^{-1/4}(\xi(1, n)-\xi(0, n)) \leqq 2^{1/2}x\right\} = \frac{2}{\pi} \int_{-\infty}^{x}\int_{0}^{\infty} \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right) dz\, dy.$$

Dobrushin also notes that the density function g of $|N_1|^{1/2}N_2$, where N_1 and N_2 are independent normal (0, 1) rv, is

$$g(y)=\frac{2}{\pi}\int_0^{\infty} \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right) dz.$$

Hence Theorem C can be reformulated via saying that

$$2^{-1/2}n^{-1/4}(\xi(1, n)-\xi(0, n)) \xrightarrow{\mathscr{D}} |N_1|^{1/2}N_2 \quad (n\to\infty). \tag{3.1}$$

In fact this statement is not very surprising since on replacing n by $\xi(0,n)$ and k by 1 in (2.4), *intuitively* it is clear that

$$\frac{\alpha_1(1)+\alpha_2(1)+\ldots+\alpha_{\xi(0,n)}(1)}{\sqrt{2\xi(0,n)}}\sim\frac{\xi(1,n)-\xi(0,n)}{\sqrt{2\xi(0,n)}}\xrightarrow{\mathcal{D}} N_2 \quad (n\to\infty). \tag{3.2}$$

(We must emphasize that we do not know any proof of this intuitively clear statement.)

Also, by Theorem A

$$n^{-1/4}(\xi(0,n))^{1/2}\xrightarrow{\mathcal{D}} |N_1|^{1/2} \quad (n\to\infty). \tag{3.3}$$

Intuitively it is again clear (however not yet proved) that

$$\frac{\xi(1,n)-\xi(0,n)}{\sqrt{2\xi(0,n)}} \text{ and } n^{-1/4}(\xi(0,n))^{1/2} \text{ are asymptotically independent rv.} \tag{3.4}$$

"Hence" (3.2), (3.3) and (3.4) together imply (3.1). The proof of Dobrushin is not based on this idea. Following his method however, a slightly stronger version of his Theorem C can be obtained.

Theorem C*. *Let $\{x_n\}$ be any sequence of positive numbers such that $x_n=o(\log n)$. Then*

$$\mathrm{P}\{n^{-1/4}(\xi(1,n)-\xi(0,n))<-2^{1/2}x_n\}\approx\frac{2}{\pi}\int_{-\infty}^{-x_n}\int_0^{\infty}\exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right)dz\,dy$$

and

$$\mathrm{P}\{n^{-1/4}(\xi(1,n)-\xi(0,n))>2^{1/2}x_n\}\approx\frac{2}{\pi}\int_{x_n}^{\infty}\int_0^{\infty}\exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right)dz\,dy.$$

We have also

Lemma 3.1. *There exists a positive constant C such that*

$$g(y)\leqq Cy^{1/3}\exp\left(-(3/2^{5/3})y^{4/3}\right). \tag{3.5}$$

Proof. Substituting $z=xy^{1/3}$ we obtain

$$g(y)=\int_0^{\infty}\exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right)dz=y^{1/3}\int_0^{\infty}\exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx.$$

Note that the function

$$f(x)=\frac{1}{x^2}+x^4$$

attains its maximum at $x_0=2^{-1/6}$ and $f(2^{-1/6})=3/2^{2/3}$. Let $x_1=(3/2^{2/3})^{1/4}$. Then

$$g(y) = y^{1/3}\left[\int_0^{x_1} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx + \int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx\right] \leqq$$

$$\leqq x_1 y^{1/3}\exp\left(-\frac{y^{4/3}}{2}3\cdot 2^{-2/3}\right)+y^{1/3}\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx.$$

For $y>2^{-3/4}x_1^{-1/4}$ we also have

$$\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx \leqq \int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}x^4\right)dx \leqq 2x_1^3 y^{4/3}\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}x^4\right)dx \leqq$$

$$\leqq 2y^{4/3}\int_{x_1}^{\infty} x^3\exp\left(-\frac{y^{4/3}}{2}x^4\right)dx = \exp\left(-\frac{y^{4/3}}{2}x_1^4\right).$$

Hence we have (3.5).

Lemma 3.2. *For any $\varepsilon>0$ there exists a $C=C(\varepsilon)>0$ such that*

$$g(y) \geqq C\exp\left(-\frac{y^{4/3}}{2-\varepsilon}3\cdot 2^{-2/3}\right).$$

Proof. With $x_0=2^{-1/6}$ and $\delta>0$ we have

$$g(y) \geqq y^{1/3}\int_{x_0-\delta}^{x_0+\delta} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx \geqq$$

$$\geqq 2\delta y^{1/3}\exp\left(-\frac{y^{4/3}}{2}3\cdot 2^{-2/3}\frac{1}{1-\varepsilon^*}\right),$$

where ε^* is defined by

$$\max\left(\frac{1}{(x_0-\delta)^2}+(x_0-\delta)^4,\ \frac{1}{(x_0+\delta)^2}+(x_0+\delta)^4\right) = \frac{3/2^{2/3}}{1-\varepsilon^*}.$$

Hence Lemma 3.2 is proved.

Lemmas 3.1, 3.2 and some standard calculus imply

Lemma 3.3. *Let $\{a_n\}$ be a sequence of positive numbers with $a_n\uparrow\infty$. Then for any $\varepsilon>0$ there exist a $C_1=C_1(\varepsilon)>0$ and a $C_2=C_2(\varepsilon)>0$ such that*

$$C_1\exp\left(-\frac{a_n^{4/3}}{2-\varepsilon}(3/2^{2/3})\right) \leqq \int_{a_n}^{\infty} g(y)\,dy \leqq C_2\exp\left(-\frac{a_n^{4/3}}{2+\varepsilon}(3/2^{2/3})\right).$$

By Theorem C* and Lemma 3.3. we have

Lemma 3.4. *For any* $\varepsilon>0$ *there exist a* $C_1=C_1(\varepsilon)>0$ *and a* $C_2=C_2(\varepsilon)>0$ *such that*

$$P\left\{n^{-1/4}(\xi(1,n)-\xi(0,n)) \geqq (1+2\varepsilon)\left(\frac{128}{27}\right)^{1/4}(\log\log n)^{3/4}\right\} \leqq C_2(\log n)^{-(1+\varepsilon)}$$

and

$$P\left\{n^{-1/4}(\xi(1,n)-\xi(0,n)) \geqq (1-2\varepsilon)\left(\frac{128}{27}\right)^{1/4}(\log\log n)^{3/4}\right\} \geqq C_1(\log n)^{-(1-\varepsilon)}.$$

Next we prove

Lemma 3.5.

$$\limsup_{n\to\infty}\frac{\xi(1,n)-\xi(0,n)}{n^{1/4}(\log\log n)^{3/4}} \geqq \left(\frac{128}{27}\right)^{1/4} \quad a.s.$$

Proof. Let

$$n_k := [\exp(k\log k)], \quad b_k := \left(\frac{128}{27}\right)^{1/4} n_k^{1/4}(\log\log n_k)^{3/4},$$

$$\zeta(n) := \xi(1,n)-\xi(0,n), \quad \xi(x,(m,n)) := \xi(x,n)-\xi(x,m) \quad (m<n),$$

$$\zeta(m,n) := \xi(1,(m,n))-\xi(0,(m,n)), \; A_k := \{\zeta(n_k) \geqq (1-2\varepsilon)b_k\}.$$

By Lemma 3.4

$$P\{A_k\} \geqq C(k\log k)^{-(1-\varepsilon)}. \tag{3.6}$$

Let $j<k$ and consider

$$P\{A_kA_j\} = \sum_{l=(1-2\varepsilon)b_j}^{\infty} P\{A_k,\zeta(n_j)=l\} =$$

$$= \sum_x \sum_{l=(1-2\varepsilon)b_j}^{\infty} P\{A_k,\zeta(n_j)=l,\ S_{n_j}=x\} =$$

$$= \sum_x \sum_{l=(1-2\varepsilon)b_j}^{\infty} P\{A_k|\zeta(n_j)=l,\ S_{n_j}=x\}P\{\zeta(n_j)=l,\ S_{n_j}=x\} =$$

$$= \sum_x \sum_{l=(1-2\varepsilon)b_j}^{\infty} P\{\zeta(n_j,n_k) \geqq (1-2\varepsilon)b_k-l\,|\,S_{n_j}=x\}P\{\zeta(n_j)=l,\ S_{n_j}=x\} \leqq$$

$$\leqq \sum_{l=(1-2\varepsilon)b_j}^{\infty} \sup_x P\{\zeta(n_j,n_k) \geqq (1-2\varepsilon)b_k-l\,|\,S_{n_j}=x\}\sum_x P\{\zeta(n_j)=l,\ S_{n_j}=x\} =$$

$$= \sum_{l=(1-2\varepsilon)b_j}^{\infty} P\{\zeta(n_k-n_j) \geqq (1-2\varepsilon)b_k-l\}P\{\zeta(n_j)=l\} \leqq$$

$$\leqq \sum_{l=(1-2\varepsilon)b_j}^{\infty} P\{\zeta(n_k) \geqq (1-2\varepsilon)b_k-l\}P\{\zeta(n_j)=l\} \approx$$

$$\approx \int_{(1-2\varepsilon)2^{-1/2}n_j^{-1/4}b_j} P\{\zeta(n_k) \geqq (1-2\varepsilon)b_k-2^{1/2}n_j^{1/4}y\}P\{\zeta(n_j)=2^{1/2}n_j^{1/4}y\}\,dy =$$

$$= \int_A^{\infty} g(y)\int_{B(y)}^{\infty} g(z)\,dz\,dy,$$

where

$$A := (1-2\varepsilon)2^{-1/2}n_j^{-1/4}b_j = (1-2\varepsilon)2^{-1/2}\left(\frac{128}{27}\right)^{1/4}(\log\log n_j)^{1/4}$$

and

$$B(y) := (1-2\varepsilon)b_k 2^{-1/2}n_j^{-1/4}b_j - 2^{1/2}n_j^{1/4}y2^{-1/2}n_k^{-1/4} =$$

$$= (1-2\varepsilon)2^{-1/2}\left(\frac{128}{27}\right)^{1/4}(\log\log n_k)^{1/4} - y\left(\frac{n_j}{n_k}\right)^{1/4}.$$

Now a simple but tedious calculation yields that for any $\varepsilon>0$ there exists a j_0 such that if $j_0<j<k$ then

$$\mathrm{P}\{A_jA_k\} \leqq (1+\varepsilon)\mathrm{P}\{A_j\}\mathrm{P}\{A_k\}. \tag{3.7}$$

Here we omit the details of the proof of this fact, and sketch only the main idea behind it. Since $(n_j/n_k)^{1/4} \leqq k^{-1/4}$ $(j=1, 2, \ldots, k-1)$, the lower limit of integration $B(y)$ above is nearly equal to

$$(1-2\varepsilon)2^{1/2}\left(\frac{128}{27}\right)^{1/4}(\log\log n_k)^{1/4} \quad \text{if} \quad y \leqq k^{1/4}, \quad \text{say.}$$

Hence for the latter y values the integral $\int_{B(y)}^{\infty} g(z)\,dz$ is nearly equal to $P\{A_k\}$. Similarly, the integral $\int_A^{\infty} g(y)\,dy$ gives $P\{A_j\}$, and our claim (3.7) follows, for in the case of $y>k^{1/4}$ the value of $g(y)$ is very small.

Now (3.6), (3.7) and the Borel—Cantelli lemma combined give Lemma 3.5. We have also

Lemma 3.6. *Let* $m_k := [\exp(k/\log^2 k)]$ *and*

$$B_k :=$$

$$= \left\{\xi(0, (m_k, m_{k+1})) \geqq (1+\varepsilon)\left[(m_{k+1}-m_k)\left(\log\frac{m_{k+1}}{m_{k+1}-m_k} + 2\log\log m_{k+1}\right)\right]^{1/2}\right\}.$$

Then of the events B_k only finitely many occur with probability one.

Proof. This lemma is an immediate consequence of Theorem 1 of Csáki—Csörgő—Földes—Révész (1983), where the corresponding statement is formulated in terms of Wiener process instead of symmetric random walk. The analogue statement is easily obtained.

Lemma 3.7. *Let*

$$M_{k+1} := ((2+\varepsilon) m_{k+1} \log\log m_{k+1})^{1/2},$$

$$a_{k+1} := (1+\varepsilon)\left[(m_{k+1}-m_k)\left(\log \frac{m_{k+1}}{m_{k+1}-m_k} + 2\log\log m_{k+1}\right)\right]^{1/2}$$

and

$$D_k := \left\{ \sup_{l \leqq M_{k+1}-a_{k+1}} \sup_{j \leqq a_{k+1}} |\alpha_l + \alpha_{l+1} + \dots + \alpha_{l+j}| \geqq \right.$$
$$\left. \geqq \left[(2+\varepsilon) a_{k+1}\left(\log \frac{M_{k+1}}{a_{k+1}} + \log\log M_{k+1}\right)\right]^{1/2} \right\}.$$

Then of the events D_k only finitely many occur with probability one.

Proof. Cf. Theorem 3.11 of Csörgő—Révész (1981).

A simple consequence of Lemmas 3.6, 3.7 and Theorem B is

Lemma 3.8. *Let*

$$E_k := \left\{ \sup_{m_k \leqq n \leqq m_{k+1}} |\zeta(m_k, n)| \geqq \left[(2+\varepsilon) a_{k+1}\left(\log \frac{M_{k+1}}{a_{k+1}} + \log\log M_{k+1}\right)\right]^{1/2} \right\}.$$

Then of the events E_k only finitely many occur with probability one.

Lemma 3.9.

$$\limsup_{n\to\infty} \frac{\xi(1, n) - \xi(0, n)}{n^{1/4} (\log\log n)^{3/4}} \leqq \left(\frac{128}{27}\right)^{1/4} \quad a.s. \tag{3.8}$$

Proof. Let

$$c_k := \left(\frac{128}{27}\right)^{1/4} m_k^{1/4} (\log\log m_k)^{3/4}, \quad E_k := \{\zeta(m_k) \geqq (1+2\varepsilon) c_k\}.$$

Then by Lemma 3.4 only finitely many of the events E_k occur with probability one. Now observing that

$$\left[(2+\varepsilon) a_{k+1}\left(\log \frac{M_{k+1}}{a_{k+1}} + \log\log M_{k+1}\right)\right]^{1/2} = o(c_k),$$

we have (3.8) by Lemma 3.8, and Lemma 3.9 is proved.

Also Lemmas 3.5 and 3.9 combined give Theorem 2.

4. Proof of Theorem 3.

A simple calculation and Lemma 2.2 imply

Lemma 4.1. *For any* $k=1,2,\dots,n;\ n=1,2,\dots$ *we have*

$$E\exp\left(-\frac{\alpha_1(k)+\alpha_2(k)+\dots+\alpha_n(k)}{((4k-2))n^{1/2}}\right)\leqq C,$$

where C is an absolute positive constant.

The above lemma together with the Chebishev inequality and the Borel—Cantelli lemma imply

Lemma 4.2. *For any* $\varepsilon>0$

$$\lim_{n\to\infty}\sup_{|k|\leqq n}\frac{\alpha_1(k)+\alpha_2(k)+\dots+\alpha_n(k)}{(kn)^{1/2}(\log n)^{1+\varepsilon}}=0\quad a.s.$$

Consequently, on replacing n by $\xi(0,n)$, *we get*

$$\lim_{n\to\infty}\sup_{|k|\leqq\xi(0,n)}\frac{\xi(k,n)-\xi(0,n)}{(k\xi(0,n))^{1/2}(\log n)^{1+\varepsilon}}=0\quad a.s.$$

and

$$\lim_{n\to\infty}\sup_{|k|<\xi(0,n)(\log n)^{-(2+3\varepsilon)}}\frac{\xi(k,n)-\xi(0,n)}{\xi(0,n)(\log n)^{-\varepsilon/2}}=0\quad a.s. \tag{4.1}$$

By (4.1) we have also Theorem 3.

5. Proof of Theorem 5.

A theorem of Hirsch (1965) (cf. p. 124 in Csörgő—Révész (1981)) says:

$$\max_{1\leqq k\leqq n}S_k\leqq n^{1/2}(\log n)^{-1}\quad i.o.$$

with probability one. This, in turn, implies Theorem 4.

6. A problem

To fill in the gap between Theorems 3 and 4 is an interesting enough problem. The following conjecture, however, is even more challenging.

Conjecture.

$$\lim_{n\to\infty}\sup_{m_n\leqq k\leqq M_n}\left|\frac{\xi(k,n)}{\xi(0,n)}-1\right|=0\quad a.s.,$$

where

$$m_n := \frac{\inf_{1 \le k \le n} S_k}{\log \log n}, \quad M_n := \frac{\sup_{1 \le k \le n} S_k}{\log \log n}.$$

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Rates of uniform convergence for the empirical characteristic function

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In honour of Professor Károly Tandori on his sixtieth birthday

Introduction, results, and discussion

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed d-dimensional random vectors, $d \geqq 1$, defined on a probability space $(\Omega, \mathcal{A}, P)$, with common distribution function $F(x)$, $x \in \mathbf{R}^d$, and characteristic function

$$C(t) = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF(x), \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbf{R}^d$. The nth, empirical characteristic function of the sequence is

$$C_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{i\langle t, X_j\rangle} = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF_n(x), \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d,$$

where $F_n(x)$, $x \in \mathbf{R}^d$, denotes the empirical distribution function of $X_1, \ldots, X_n$. For any extended number $0 < T \leqq \infty$ consider the random variable

$$\Delta_n(T) = \sup_{|t| < T} |C_n(t) - C(t)|.$$

It is trivial that $\Delta_n(T) \to 0$ almost surely as $n \to \infty$ for any fixed $T < \infty$, but CSÖRGŐ and TOTIK [2] have pointed out that $\Delta_n(\infty) \to 0$ almost surely if and only if F is purely discrete. These two facts lead to considering the quantities $\Delta_n(T_n)$ for some sequences $\{T_n\}$ of finite positive numbers converging to infinity at an intermediate rate. In Theorem 1 of [2] we have shown in a simple elementary fashion that $\Delta_n(\exp\{n/G_n\}) \to 0$ almost surely for any sequence $\{G_n\}$ such that $G_n \to \infty$. More interesting is the fact that this result is optimal in general. We proved in Theorem 2 of [2] that for any characteristic function C which vanishes at infinity along at least one

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path, $\Delta_n(\exp\{n/G_n\})$ does not converge to zero even in probability if G_n does not converge to infinity. It is natural to expect that if we specify the rate at which G_n goes to infinity then we should be able to derive rates at which $\Delta_n(\exp\{n/G_n\})$ converges to zero almost surely. The present note adresses exactly this problem.

It should be pointed out that later but independently YUKICH [4] also proved, at least in the univariate special case $d=1$, that $\Delta_n(\exp\{n/G_n\})\to 0$ almost surely whenever $G_n\to\infty$. He derives this result, necessarily in a more complicated way, from a general theorem of his concerning the law of large numbers for empirical measures on general measurable state spaces and indexed by classes of functions. He does not attempt the optimality of the result. However, the fact that he obtained an apparently unimprovable corollary shows the strength of his general theorem. It was YUKICH [4] who first obtained a special rate result on moving intervals. Again in the univariate case $d=1$ he deduced from his general theorem what is Example (liii) below, with a larger constant on the right side.

Our approach is direct and elementary. The proof of the following result, presented in the next section, appears as a straightforward extension of the proof of Theorem 1 in [2]. In order to avoid trivialities we assume that $X=X_1$ is nondegenerate.

Theorem. *Let* $K_n=\inf\{x>0\colon P\{|X|>x\}\leqq R_n\}$ *where* $\{R_n\}$ *is a nonincreasing sequence of positive numbers. If*

$$\sum_{n=n_0}^{\infty} e^{-M_1R_n^2n}+\sum_{n=n_0}^{\infty}(K_nT_n/R_n)^d e^{-M_2R_n^2n}<\infty$$

for some $M_1, M_2>0$ *such that*

$$n_0=n_0(M_1,M_2)=\inf\{n\colon R_n\leqq 1/(4\sqrt{\max(M_1,M_2)})\}<\infty,$$

then

$$\limsup_{n\to\infty}\frac{1}{R_n}\Delta_n(T_n)\leqq 2+2\sqrt{M_1}+4\sqrt{M_2}$$

almost surely.

Setting $R_n\equiv\varepsilon$ and $K_n\equiv K(\varepsilon)$ with any small $\varepsilon>0$ and $T_n=\exp\{o(n)\}$, we see that this theorem gives Theorem 1 of CSÖRGŐ and TOTIK [2]. On the other hand, it is not surprising that on smaller balls the rates depend on the tail behaviour of the underlying distribution. Even in the case of a fixed ball when $T_n\equiv T<\infty$, the ideal almost sure rate result $\Delta_n(T)=O\big(\sqrt{(\log\log n)/n}\big)$ can be achieved only under some tail condition. Improving an earlier result of CSÖRGŐ [1], who required $\varepsilon>1$, the presently available weakest such condition of LEDOUX [3] is satisfied if $Eg_\varepsilon(|X|)<\infty$ for any $\varepsilon>0$, where

$$g_\varepsilon(u)=\begin{cases}(\log u)(\log_2 u)^2(\log_3 u)^{1+\varepsilon}, & \text{if } u\geqq\exp\{\exp\{e\}\},\\ 1, & \text{if } 0\leqq u<\exp\{\exp\{e\}\}.\end{cases}$$

Here and in what follows, for any integer $k \geqq 1$, $\log_k$ denotes the k times iterated logarithm.

In view of this circumstance, it is perhaps tolerable that the Theorem above, designed for growing balls, can at best give the rate $R_n = \sqrt{(\log n)/n}$ even when $T_n \equiv T < \infty$ and X is a bounded random variable. The following version of the Theorem is more restrictive because it can be applied only for balls with radius $T_n \geqq n^A$, where $A > 0$. Nevertheless, in this domain it is more suggestive.

Corollary. *Let* $K_n = \inf\{x > 0\colon P\{|X| > x\} \leqq 1/\sqrt{G_n}\}$, *where* $\{G_n\}$ *is a sequence of positive numbers such that* $G_n \to \infty$ *as* $n \to \infty$. *If*

$$\sum_{n=1}^{\infty} e^{-M_1 n/G_n} + \sum_{n=1}^{\infty} G_n^{d/2} K_n^d e^{-M_3 n/G_n} < \infty$$

for some $M_1, M_3 > 0$, *then*

$$\limsup_{n\to\infty} \sqrt{G_n}\, \Delta_n(\exp\{n/G_n\}) \leqq 2 + 2\sqrt{M_1} + 4\sqrt{d + M_3}$$

almost surely.

We illustrate our result by way of natural examples. Note that if $P\{|X| > x\} \leqq h(x)$ for large enough $x > 0$, where h is a nonincreasing function and if $V_n = \inf\{x > 0\colon h(x) \leqq R_n\}$ then $K_n \leqq V_n$ for large enough n. Therefore, if the condition of the Theorem or the Corollary is satisfied with V_n replacing K_n, then the corresponding conclusion is applicable. The following examples follow by elementary calculation. All the limsup statements are meant to hold almost surely.

Example (1). Suppose that $P\{|X| > x\} \leqq L x^{-\alpha}$ for all large enough x, where L and α are arbitrary positive constants. Then

(i) for any $A > 0$ and integer $k \geqq 1$,

$$S_1(A, k) = \limsup_{n\to\infty} (\log_k n)^{A/2} \Delta_n(\exp\{n/(\log_k n)^A\}) \leqq 2 + 4\sqrt{d},$$

(ii) for any $0 < A < 1$,

$$S_2(A) = \limsup_{n\to\infty} n^{(1-A)/2} \Delta_n(\exp\{n^A\}) \leqq 2 + 4\sqrt{d},$$

(iii) for any $A > 0$

$$S_3(A) = \limsup_{n\to\infty} \sqrt{\frac{n}{\log n}}\, \Delta_n(n^A) \leqq 2 + 2\sqrt{\min(A, 1)} + 4\sqrt{1 + \left(A + \frac{1+\alpha}{2\alpha}\right) d},$$

(iv) for any $A > 0$ and integer $k \geqq 1$,

$$S_4(A, k) = \limsup_{n\to\infty} \sqrt{\frac{n}{\log n}}\, \Delta_n((\log_k n)^A) \leqq 4 + 4\sqrt{1 + d\,\frac{1+\alpha}{2\alpha}}.$$

Note that in case (iii) the right-side constant is obtained by applying both the Theorem and the Corollary. When $d=1$, the result in (iii) was obtained by YUKICH [4], as mentioned above, with the greater upper bound $12\{1+A+(1+\alpha)/(2\alpha)\}$.

Example (2). Suppose that $P\{|X|>x\}\leqq L_1 \exp\{-L_2x^\alpha\}$ for all large enough x and positive constants L_1, L_2 and α. Then we obtain $S_2(A, k)\leqq 2+4\sqrt{d}$, $S_2(A)\leqq \leqq 2+4\sqrt{d}$ and

$$S_3(A) \leqq 2+2\sqrt{\min(A, 1)}+4\sqrt{1+\left(A+\frac{1}{2}\right)d}, \quad S_4(A, k) \leqq 4+4\sqrt{1+\frac{d}{2}}.$$

This means that the Theorem cannot distinguish the exponential decrease of a tail from power decrease. Of course, the latter inequalities follow directly from those in Example (1) by taking the limit as $\alpha\to\infty$. If we assume that X is a bounded random variable, the Theorem and Corollary give nothing better than the latter four inequalities.

Example (3). Suppose that $P\{|X|>x\}\leqq 1/(\log_k x)^\alpha$ for all large enough x, a positive constant α and an integer $k\geqq 1$. Then

$$\limsup_{n\to\infty} (\log_k n)^\alpha \Delta_n\left(\exp\{n/(\log_k n)^{2\alpha}\}\right) \leqq 2+4\sqrt{d},$$

and even on a fixed ball of radius $T<\infty$,

$$\limsup_{n\to\infty} (\log_k n)^\alpha \Delta_n(T) \leqq 2.$$

We know from the mentioned results in [1] and [3] that the Theorem is not optimal when $T_n\equiv T<\infty$. It is tempting to beleive that the above rates cannot essentially be improved upon on large balls with $T_n\geqq n^A$. However, this remains an open question at this writing.

Proof of Theorem

Introducing the truncated integrals

$$B_n(t) = \int\limits_{|x|\leqq K_n} e^{i\langle t,x\rangle} dF_n(x) = \frac{1}{n}\sum_{j=1}^{n} e^{i\langle t, X_j\rangle}\chi(|X_j|\leqq K_n),$$

$$\tilde{B}_n(t) = \int\limits_{|x|\leqq K_n} e^{i\langle t,x\rangle}\, dF(x),$$

where $\chi(A)$ denotes the indicator of the event A, and writing $D_n(t)=B_n(t)-\tilde{B}_n(t)$, we have just as in [2] that

$$\Delta_n(T_n) \leqq \sup_{|t|\leqq T_n} |D_n(t)| + \sup_{|t|\leqq T_n} |B_n(t)-C_n(t)| + \sup_{|t|\leqq T_n} |\tilde{B}_n(t)-C(t)|.$$

The third term is obviously less than or equal to R_n. The second term is

$$\frac{1}{n}\sup_{|t|\leqq T_n}\Big|\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j|>K_n)\Big| \leqq \frac{1}{n}\sum_{j=1}^{n}\chi(|X_j|>K_n),$$

and hence

$$q_n = P\Big\{\sup_{|t|\leqq T_n}|B_n(t)-C_n(t)| > (1+2\sqrt{M_1})R_n\Big\} \leqq$$

$$\leqq P\Big\{\frac{1}{n}\sum_{j=1}^{n}\big(\chi(|X_j|>K_n)-P\{|X|>K_n\}\big) > 2\sqrt{M_1}\,R_n\Big\} \leqq$$

$$\leqq \begin{cases} 2\exp\{-M_1R_n^2 n/\sigma_n^2\}, & \text{if } \sqrt{M_1}<\sigma_n^2/4,\\ 2\exp\{-\sqrt{M_1}\,R_n n/4\}, & \text{if } \sqrt{M_1}\geqq\sigma_n^2/4,\end{cases}$$

by Bernstein's inequality, where $\sigma_n^2=\operatorname{Var}\chi(|X|>K_n)\leqq 1$. Thus, if $n\geqq n_0$, then $q_n\leqq 2\exp\{-M_1R_n^2 n\}$.

Let $\varepsilon>0$ be arbitrarily small, $\varepsilon\leqq 4d^{3/2}$, and let us cover the cube $[-T_n, T_n]^d$, and hence our ball $\{t\colon |t|\leqq T_n\}$ by $N_n=\big([(4d^{3/2}K_nT_n)/(\varepsilon R_n)]+1\big)^d$ disjoint small cubes $\Lambda_1, \dots, \Lambda_{N_n}$, the edges of each of which are of length $(\varepsilon R_n)/(2d^{3/2}K_n)$. If $t_1, \dots, t_{N_n}$ denote the centres of these cubes then

$$\sup_{|t|\leqq T_n}|D_n(t)| \leqq \max_{1\leqq k\leqq N_n}|D_n(t_k)| + \max_{1\leqq k\leqq N_n}\sup_{t\in\Lambda_k}|D_n(t)-D_n(t_k)| \leqq$$

$$\leqq \max_{1\leqq k\leqq N_n}|D_n(t_k)|+\varepsilon R_n,$$

for, exactly as in [2], $|D_n(s)-D_n(t)|\leqq 2d^{3/2}K_n|s-t|$. On the other hand, proceeding exactly as in [2] again, we obtain via another application of the Bernstein inequality that

$$P\Big\{\max_{1\leqq k\leqq N_n}|D_n(t_k)| > 4\sqrt{M_2}\,R_n\Big\} \leqq N_n\max_{1\leqq k\leqq N_n}P\big\{|D_n(t_k)|>4\sqrt{M_2}\,R_n\big\} \leqq$$

$$\leqq Q(K_nT_n/R_n)^d\sup_{t\in\mathbf{R}^d}P\big\{|D_n(t)|>4\sqrt{M_2}\,R_n\big\} \leqq$$

$$\leqq 4Q(K_nT_n/R_n)^d\exp\{-M_2R_n^2 n\}$$

whenever $n\geqq n_0$, where $Q=(4d^{3/2}/\varepsilon)^d$. Summing up,

$$\sum_{n=n_0}^{\infty}P\big\{\Delta_n(T_n) > (2+2\sqrt{M_1}+4\sqrt{M_2}+\varepsilon)R_n\big\} \leqq$$

$$\leqq 4Q\Big(\sum_{n=n_0}^{\infty}\exp\{-M_1R_n^2 n\}+\sum_{n=n_0}^{\infty}(K_nT_n/R_n)^d\exp\{-M_2R_n^2 n\}\Big),$$

and the Borel—Cantelli lemma yields the desired result.

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Rate of approximation of linear processes

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

The well-known Korovkin theorem [8] established that a positive linear process, (or a sequence of positive linear operators), on $C[a, b]$ that approximates the functions 1, x and x^2 (for instance) also approximates any continuous function. An off-spring of that result is the MOND-SHISHA [11] theorem that yields the rate of approximation of a function with certain smoothness to the rate of approximation of 1, x and x^2. The rate of approximation in the Mond-Shisha theorem and in many results that followed were forcibly uniform, that is independent of the point at which the function was approximated. In other words, the rate of approximation prescribed does not take into account that $L_n(\varphi_i, t)$ could tend to φ_i (φ_i being $1, x$ and x^2) at different rates for different points t. Recently, ESSER [6] and STRUKOV and TIMAN [13] proved that if L_n are positive linear operators on $C(I)$, $L_n(1, t)=1$, $L_n(x, t)=t$ and $L_n(x^2, t)=t^2+D_n(t)$ (in which case $D_n(t)\geqq 0$), we have $|L_n(f, t)-f(t)|\leqq 15\omega_2(f, 1/2\sqrt{D_n}(t))$ where

$$\omega_2(f, h)\equiv \operatorname*{Sup}_{\eta\leqq h}\{\operatorname*{Sup}_{x}(\Delta_\eta^2 f(x)|;\ [x-\eta, x+\eta]\subset I)\}, \tag{1.1}$$

I is $[a, b]$ or R^+ or R, $\Delta_h^r f\equiv\Delta_h(\Delta_h^{r-1}f)$ and $\Delta_h f(x)=f(x+h/2)-f(x-h/2)$. (The result mentioned here is that of STRUKOV and TIMAN [13]; ESSER [6] proved somewhat less but earlier).

Examining the situation on the particular but significant example of the Bernstein polynomials given by

$$B_n(f, t)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.2}$$

we have the following two results which do not imply each other:

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a) For $0<\alpha<2$, $\operatorname{Sup}_{h<x<1-h} |\Delta_h^2 f(x)| \leq Mh^\alpha$ if and only if $|B_n(f,t)-f(t)| \leq \leq M_1\left(\frac{t(1-t)}{n}\right)^{\alpha/2}$, BERENS and LORENTZ [1].

b) For $0<\alpha<2$, $\operatorname{Sup}_{h<x<1-h} |(x(1-x))^{\alpha/2}\Delta_h^2 f(x)| \leq Mh^\alpha$ if and only if

$$\|B_n(f,\cdot)-f(\cdot)\| \leq Mn^{-\alpha/2},$$

[2]. (In [5] it was shown that $\operatorname{Sup}_{h^2<x<1-h^2} |\Delta_{h\varphi(x)}^2 f(x) \leq Mh^\alpha$ for $\varphi(x)=\sqrt{x}$ when $x\in[0, 1/2]$ and, $\varphi(x)=\sqrt{1-x}$ when $x\in(1/2, 1]$, is also equivalent to the above).

In this paper we will be concerned only with the direct theorem, that is, in the particular cases (a) and (b) with the "only if" aspect. For positive operators for which $L_n(x, t)=t$ Strukov and Timan settled the "only if" question analogous to (a) and for positive operators with several conditions on $D_n(t)$ TOTIK [16] settled the question analogous to (b).

We will not have the restriction $L_n(x, t)=t$ (see application in §6) nor will we require that the operators be positive, and, therefore, higher moduli of smoothness can enter into the discussion (see applications in §7 and §8). We will impose relatively simple conditions on the moments and the result will be applied to many operators. The present result will provide some new applications even for positive operators. Its main additional strength, however, will be its uses for some non-positive operators, for instance combinations of "Exponential-type operators" introduced by C. P. MAY [10]. We will be aided by results using interpolation of spaces and Peetre K functionals and will introduce those concepts when needed.

2. Rate of convergence using moduli of continuity. In this section we will establish a direct theorem analogous to example (a) in the introduction. For a positive operator satisfying $L_n(1, x)=1$ we have the representation $L_n(f, x)=\int_I f(t)\,d\alpha_{n,x}(t)$ where $\alpha_{n,x}(t)$ is increasing and $\int_I d\alpha_{n,x}(t)=1$. The operators which we will treat in this paper will have the representation

$$L_n(f, x) = \int_I f(t)\, d\alpha_{n,x}(t) \quad \text{where} \quad \int_I d\alpha_{n,x}(t) = 1, \tag{2.1}$$

$$\int_{u\leq t} |d\alpha_{n,x}(u)| \equiv v_{n,x}(t) \quad \text{and} \quad v_{n,x}(t) \leq M.$$

The domain I represents a finite interval, semi-infinite ray or the whole real line and with no loss of generality we may assume that I is [0, 1], R^+ or R.

We can now state our first result.

Theorem 2.1. *Let* $I=[0, 1]$, $I=R^+$ or $I=R$, $L_n(f, t)$ *defined by* (2.1), $\int_I t^i d\alpha_{n,x}(t)=x^i$ *for* $i=0, 1, \dots, 2m-1$, and $\int_I (t-x)^{2m} dv_{n,x}(t) \leqq D_n(x)$. *Then*

$$|L_n(f, x)-f(x)| \leqq \left(M+1+\frac{(4m)^{2m}}{2m!}\right) \omega_{2m}(f, D_n^{1/2m}(x))$$

for $I=R^+$ *or* $I=R$ *and*

$$|L_n(f, x)-f(x)| \leqq \left(M+1+\frac{(4m)^{2m}}{2m!}+LD_n^{1/2m}(x)\right) \omega_{2m}(f, D_n^{1/2m}(x))$$

for $I=[0, 1]$ *where L depends only on m.*

Remark 2.2. The theorem is interesting only when $D_n(x)=o(1)$, $n\to\infty$. Obviously, $D_n(x)\geqq 0$ and at a point x_0 at which $D_n(x_0)=0$, $L_n(f, x_0)=f(x_0)$. In applications commonly met $D_n(x_0)=0$ only at the boundary of I. One can construct a sequence of linear (and even positive) operators for which $D_n(x)$ has a zero at a point internal to I, but in general those examples seem contrived and not very interesting.

Proof. We define for $h>0$ the Steklov-type averages

$$f_h(x) = \left(\frac{2m}{h}\right)^{2m} \int_0^{h/2m} \cdots \int_0^{h/2m} \left\{\sum_{k=1}^{2m} \binom{2m}{k} (-1)^{k+1} f(x+k(u_1+\dots+u_{2m}))\, du_1 \dots du_{2m}\right.$$

and for x such that $[x, x+2mh]$ is in I we have $|f(x)-f_h(x)|\leqq\omega_{2m}(f, h)$ and f_h has $2m$ continuous derivatives ($f_h^{(j)}$ is absolutely continuous for $j<2m$) and

$$|f_h^{(2m)}(x)| \leqq \left(\frac{2m}{h}\right)^{2m} \sum_{k=1}^{2m} \binom{2m}{k} \omega_{2m}\left(f, \frac{k}{2m}h\right) \leqq \left(\frac{4m}{h}\right)^{2m} \omega_{2m}(f, h).$$

We now show that for g with $2m$ derivatives we have

$$|L_n(g, x)-g(x)| \leqq \frac{D_n(x)}{2m!} \|g^{(2m)}\|.$$

Using Taylor's expansion we have

$$|L_n(g, x)-g(x)| \leqq \sum_{i=1}^{2m-1} \frac{1}{i!} |g^{(i)}(x)| |L_n((t-x)^i, x)+$$

$$+\frac{1}{2m!} |g^{(2m)}(\xi)| \int_I (x-t)^{2m} dV_{n,x}(t)$$

$$\leqq \frac{D_n(x)}{2m!} \|g^{(2m)}\|_{C(I)}.$$

For $\overset{\circ}{I}=R^+$ or R we write

$$|L_n(f,x)-f(x)| \leqq |L_n(f-f_h,x)-(f(x)-f_h(x))+|L_n(f_h,x)-f_h(x)| \leqq$$

$$\leqq (M+1)\omega_{2m}(f,h)+\frac{D_n(x)}{h^{2m}}\frac{(4m)^{2m}}{2m!}\omega_{2m}(f,h)$$

and choose

$$h=(D_n(x))^{1/2m}.$$

For $I=[0,1]$ we choose a function $\psi\in C^\infty$ such that $\psi(x)=1$ on $[0,1/3]$, $\psi(x)=0$ on $[1/3,1]$ and $\psi(x)$ is decreasing. We define $F_h(x)=f_h(x)\psi(x)+f_{-h}(x)(1-\psi(x))$ where f_{-h} is the same as f_h but using $-h$ instead of h. We have $|F_h(x)-f(x)|\leqq\omega_{2m}(f,h)$ and $F_h^{(2m)}(x)=f_h^{(2m)}(x)$ in $[0,1/3]$ while $F_h^{(2m)}(x)=f_{-h}^{(2m)}$ in $(2/3,1]$. To calculate $F_h^{(2m)}(x)$ in $[1/3,2/3]$ we write

$$F_h^{(2m)}(x)=f_h^{(2m)}(x)-\{(f_h(x)-f_{-h}(x))(1-\varphi(x))\}^{(2m)}$$

and

$$\{(f_h(x)-f_{-h}(x))(1-\psi(x))\}^{(2m)}=\{(f_h^{(2m)}(x)-f_{-h}^{(2m)}(x))\}(1-\psi(x))-$$

$$-\sum_{k=0}^{2m-1}\binom{2m}{k}(f_h(x)-f_{-h}(x))^{(k)}\psi^{(2m-k)}(x)\equiv I_1+I_2.$$

By earlier consideration $|f_h^{(2m)}(x)-I_1|\leqq((4m)/h)^{2m}\omega_{2m}(f,h)$. To estimate I_2 we write $K\equiv\max_{x,i\leqq 2m}|\psi^{(i)}(x)|$, and so $|I_2|\leqq K\sum_{k=0}^{2m-1}\binom{2m}{k}|(f_h(x)-f_{-h}(x))^{(k)}|$, and therefore, it is enough to estimate for $k<2m$

$$|(f_h(x)-f_{-h}(x))^{(k)}(x)|=\left|\left(\frac{2m}{h}\right)^{2m}\left(\frac{d}{dx}\right)^k\int_0^{h/2m}\cdots\int_0^{h/2m}\bar{\Delta}^{2m}_{u_1+\ldots+u_{2m}}f(x)\,du_1\ldots du_{2m}-\right.$$

$$\left.-\left(\frac{2m}{h}\right)^{2m}\left(\frac{d}{dx}\right)^k\int_0^{h/2m}\cdots\int_0^{h/2m}\bar{\Delta}^{2m}_{-u_1\ldots-u_{2m}}f(x)\,du_1\ldots du_{2m}\right|\equiv J(k;h),$$

where $\bar{\Delta}_\eta f(x)\equiv f(x+\eta)-f(x)$ and $\bar{\Delta}^l_\eta f(x)\equiv\bar{\Delta}(\bar{\Delta}^{l-1}_\eta f(x))$. We can now estimate $J(k,h)$ by estimating $J_+(k,h)$ and $J_-(k,h)$ being the first and second terms in the sum defining $J(k,h)$ respectively.

$$J_+(k,h)=\left|\left(\frac{2m}{h}\right)^{2m}\frac{d^k}{dx^k}\int_0^{h/2m}\cdots\int_0^{h/2m}\bar{\Delta}^{2m}_{u_1+\ldots+u_{2m}}f(x)\,du_1\ldots du_{2m}=\right.$$

$$=\left|\left(\frac{2m}{h}\right)^{2m}\int_0^{h/2m}\cdots\int_0^{h/2m}\bar{\Delta}^k_{h/2m}\bar{\Delta}^{2m}_{u_{k+1}+\ldots+u_{2m}}f(x)\,du_{k+1}\ldots du_{2m}\right|\leqq$$

$$\leqq\left(\frac{2m}{h}\right)^k 2^k\omega_{2m}\left(f,\frac{2m-k}{2m}h\right)$$

and $J_-(k,h)$ is evaluated similarly. Therefore

$$|I_2| \leqq 2h^{-2m+1}K\sum_{k=0}^{2m-1}\binom{2m}{k}(4m)^k\omega_{2m}(f,h) \leqq h^{-2m+1}2K(4m+1)^{2m}\omega_{2m}(f,h),$$

and choosing h as in the earlier case, we obtain our results.

Remark 2.3. We could have used Whitney's extension theorem (see STEIN [12, Ch. VI]) to find a function $F(x)$ defined on R that is identical with f in the domain I and whose $2m$ modulus of continuity $\omega_{2m}(F, h)$ in R is bounded by $K\omega_{2m}(f, h)$ in I. However, this method would still leave us with the need to estimate K and we will still need the Steklov averaging functions and almost all the steps of the present proof.

Remark 2.4. For $m=1$ and positive opeartors Strukov and Timan have a better estimate for the constant in the theorem, as extension F of f defined on [0, 1] or R is easily shown to satisfy $\omega_2(F,h)$ on R is smaller than $5\omega_2(f,h)$ on I (see TIMAN [14, p. 122]). That method is valid for $m=1$ even without the positivity but will yield a somewhat different constant.

It is obvious that instead of $\int_I t^i d\alpha_{n,x}(t)=x^i$ for $i=0, 1, \dots, 2m-1$ we could have imposed $\int_I (t-x)^i d\alpha_{n,\mu}(t)=0$ for $i=1, \dots, 2m-1$ and $\int d\alpha_{n,x}(t)=1$. We can now derive from Theorem 2.1 a generalization relaxing the conditions on the moments that would be useful for applications. We note that the next theorem would yield an estimate of the rate of convergence for positive operators for which $\int_I t\,d\alpha_{n,x}(t)\neq x$.

Theorem 2.2. *Suppose* $I=[0,1]$, $I=R^+$ *or* $I=R$, $\int d\alpha_{n,x}(t)=1$ *and* $\int_I (t-x)^i d\alpha_{n,x}(t) \equiv R_{n,i}(x)$, $i=1, \dots, 2m-1$ *where* $R_{n,i}(x)=o(1)$ $n\to\infty$, $\int_{u\leqq t}|d\alpha_{n,x}(u)|=v_{n,x}(t)\leqq M$ *and* $\int_I (t-x)^{2m}dv_{n,x}(t)\leqq D_n(x)$, *then*

$$|L_n(f,x)-f(x)| \leqq C\sum_{i=1}^{2m-1}\omega_i(f,|R_{n,i}(x)|^{1/i})+C\omega_{2m}(f,R_n(x)^{1/2m})$$

where $R_n(x)\equiv D_n(x)+K\sum_{i=1}^{2m-1}|R_{n,i}(x)|^{2m/i}$ *and where* C *and* K *depend only on* m.

Proof. To prove our theorem we will construct a new operator $A_n(f,x)$ that will satisfy the assumptions on the operator in Theorem 2.1. For that we add operators to obtain a new operator $A_n(f,x)$ such that on the one hand $A_n((t-x)^j, x)=0$ for $j\leqq 2m-1$ and on the other hand $\int_I (t-x)^{2m}dV_{n,x}(t)\leqq R_n(x)$ where $V_{n,x}(t)$ is the vaiiation up to t of $\beta_{n,x}(u)$, the measure describing $A_n(f,x)$, that is $A_n(f,x)\equiv$

$\equiv \int_I f(u)\, d\beta_{n,x}(u)$, and $R_n(x)$ would be as stated in the theorem. The function $R_n(x)$ will replace $D_n(x)$ and the operator $A_n(f, x)$ will replace $L_n(f, x)$ when we apply Theorem 2.1 to the present situation. We will write $\bar{\Delta}_h f(x) \equiv f(x+h)-f(x)$ and $\bar{\Delta}_h^m \equiv \bar{\Delta}_h(\bar{\Delta}_h^{m-1})$ and define $L_{n,i}(f, x) = ((-1)/i!) \operatorname{Sgn} R_{n,i}(x) \bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f(x)$ for all x in case $I=R_+$ or $I=R$, and for $0 \leqq x \leqq 1/2$ in case $I=[0, 1]$, in which case $L_{n,i}(f, x) = (-1)^{i+1}/i! \operatorname{Sgn} R_{n,i}(x) \bar{\Delta}^i_{-|R_{n,i}(x)|^{1/i}} f(x)$ for $1/2 < x < 1$. Since a simple calculation will yield $L_{n,i}((t-x)^i, x) = -R_{n,i}(x)$, we can add this operator to eliminate the i moment. However, for $i < j \leqq 2m-1$

$$L_{n,i}((t-x)^j, x) = c_{i,j} |R_{n,i}|^{j/i}(x) \operatorname{Sgn} R_{n,i}(x)$$

where $c_{i,j}$ is a constant that depends on i and j but not on n and x. To cancel that effect for $j=j_1$ we add the operator

$$L_{n,i,j_1}(f, x) \equiv \frac{-c_{i,j_1}}{j_1!} \operatorname{Sgn} R_{n,i}(x) \bar{\Delta}^{j_1}_{|R_{n,i}(x)|^{1/i}} f(x)$$

(and a similar version for $1/2 < x \leqq 1$ in case $I=[0, 1]$). Of course for $j_1 < j \leqq 2m-1$ we still have $L_{n,i,j_1}((t-x)^{j_2}, x) = c_{i,j_1,j_2} |R_{n,i}(x)|^{j_2/i} \operatorname{Sgn} R_{n,i}(x)$, the effect of which we cancel by adding $L_{n,i,j_1,j_2}(f, x)$ given in a similar way. In general we will define $L_{n,i,j_1,\dots j_k}(f, x)$ by induction. We have

$$L_{n,i,j_1,\dots j_{k-1}}((t-x)^{j_k}, x) = c_{i,j_1,\dots j_k}(\operatorname{Sgn} R_{n,i}(x)) |R_{n,i}(x)|^{j_k/i}$$

and for $i < j_1 < \dots < j_k \leqq 2m-1$ we define

$$L_{n,i,j_1\dots j_k}(f, x) = \frac{-c_{i,j_1\dots j_k}}{j_k!} \operatorname{Sgn} R_{n,i}(x) \bar{\Delta}^{j_k}_{|R_{n,i}(x)|^{1/i}} f(x)$$

together with an appropriate modification for $1/2 < x \leqq 1$ in case $I=[0, 1]$. The operator $A_n(f, x)$ is given by

$$A_n(f, x) = L_n(f, x) + \sum_{i=1}^{2m-1} \Big\{ L_{n,i}(f, x) + \\ + \sum_{1 \leqq i < j_1 < \dots < j_k \leqq 2m-1} L_{n,i,j_1\dots j_k}(f, x) \Big\}$$

where the second sum is taken on all finite sequences $j_1, \dots, j_k$ for which $1 < j_1 < \dots \dots < j_k \leqq 2m-1$. To calculate the variation of the measure defining $A_n(f, x)$, we simply estimate its norm as an operator on $C(I)$. We can write

$$\|A_n\| \leqq M + \sum_{i=1}^{2m-1} \left\{ \frac{2^i}{i!} + \sum_{1 \leqq i < j_1 < \dots < j_k \leqq 2m-1} |c_{i,j_1,\dots,j_k}| \frac{2^{j_k}}{j_k!} \right\},$$

and since $c_{i,j_1,\dots,j_k}$ are just constants that do not depend on our operator at all just on the i and j's (in case $R_{n,i}(x)=0$ they do not count at all being multiplied by 0), we have a bound for the variation $V_{n,x}(t)$ of the measure $\beta_{n,x}(t)$ given for $A_n(f,x)\equiv$ $\equiv \int_I f(t)\,d\beta_{n,x}(t)$. To estimate $J(n,x)=\int_I (t-x)^{2m}dV_{n,x}(t)$, we write

$$J(n,x) \leqq D_n(x)+\sum_{i=1}^{2m-1}\{|R_{n,i}(x)|^{2m/i}m_i+$$

$$+\sum_{1\leqq i<j_1<\ldots<j_k\leqq 2m-1}|c_{i,j_1\ldots j_k}||R_{n,i}(x)|^{2m/i}m_{i,j_1,\ldots,j_k}\leqq$$

$$\leqq D_n(x)+K\sum_{i=1}^{2m-1}|R_{n,i}(x)|^{2m/i}$$

since m_i and $m_{i,j_1,\ldots,j_k}$ are numbers independent of the particular operator.

Finally, we use theorem 2.1 and obtain $|A_n(f,x)-f(x)|\leqq C\omega_{2m}(f, R_n(x)^{1/2m})$ where an estimate for C is given in that theorem. To obtain the estimate for $|L_n(f,x)-f(x)|$, we estimate $L_{n,i}$ and $L_{n,i,j_1,\ldots,j_k}$ by $|L_{n,i}(f,x)|\leqq(1/i!)\,\omega_i(f, |R_{n,i}(x)|^{1/i})$ and

$$|L_{n,i,j_1,\ldots,j_k}(f,x)| \leqq \frac{|c_{i,j_1,\ldots,j_k}|}{j_k!}\,\omega_{j_k}(f, |R_{n,i}(x)|^{1/i})\leqq$$

$$\leqq \frac{|c_{i,j_1,\ldots,j_k}|}{j_k!}\,2^{j_k-i}\,\omega_i(f, |R_{n,i}(x)|^{1/i})$$

and this completes the proof.

3. Some preliminary Lemmas. For the result involving interpolation spaces we will need a few preliminary Lemmas which may be of interest by themselves.

Lemma 3.1. *Suppose* $f\in C[0,3/4]$, $f^{(i)}(x)$ *for* $0\leqq i<2m-1$ *is locally absolutely continuous in* $(0,3/4)$ *and* $\|x^{2m\gamma}f^{(2m)}(x)\|_{C[0,(3/4)]}=\Phi(f)<\infty$ *where* $0<\gamma<1$, *then we have*

$$\|x^{2m\gamma-2m+i}f^{(i)}(x)\|_{C\left[0,\frac{3}{4}\right]}\leqq K\left(\Phi(f)+\|f\|_{C\left[0,\frac{3}{4}\right]}\right) \quad \text{for} \quad 2m\gamma-2m+i>0 \tag{3.1}$$

and

$$\|f^{(i)}(x)\|_{C\left[0,\frac{3}{4}\right]}\leqq K\left(\Phi(f)+\|f\|_{C\left[0,\frac{3}{4}\right]}\right) \quad \text{for} \quad 2m\gamma-2m+i<0. \tag{3.2}$$

Proof of Lemma 3.1. The proof follows to some extent a proof of a special case proved earlier by the author [2, p. 280]. We have $|x^{2m\gamma}f^{(2m)}(x)|\leqq\Phi(f)$ in $(0,3/4]$ and therefore $|f^{(2m)}(x)|\leqq M\Phi(f)$ in $[1/4,3/4]$. Consequently, $|f^{(2m-r)}(x)|\leqq$ $\leqq M_1(M\Phi(f)+\|f\|_{C[0,(3/4)]})$ for $1/4\leqq x\leqq 3/4$ and in particular for $x=1/2$. (This result follows a Kolmogorov-type inequality in a finite interval where the best constant is not known.) Assuming by induction on i $|x^{2m\gamma-i}f^{(2m-i)}(x)|\leqq K(\Phi(f)+$

$+\|f\|)$ $i=0, \dots, j-1$ and $2m\gamma - j > 0$, then

$$\left|f^{(2m-j)}(x)-f^{(2m-j)}\left(\frac{1}{2}\right)\right| \leqq \left|\int_x^{1/2} f^{(2m-j-1)}(u)\,du\right| \leqq$$

$$\leqq K(\Phi(f)+\|f\|)\left|\int_x^{1/2} u^{-2m\gamma+j-1}\,du\right| \leqq K_1(\Phi(f)+\|f\|)x^{j-2m\gamma}$$

or

$$f^{(2m-j)}(x) \leqq K_1(\Phi(f)+\|f\|)x^{j-2m\gamma}+M_1(M\Phi(f)+\|f\|) \leqq$$
$$\leqq K_2(\Phi(f)+\|f\|)x^{j-2m\gamma}$$

which concludes the proof for $2m\gamma-j>0$. For j satisfying $2m\gamma-j=0$ we obtain the estimate $|f^{(2m-j)}(x)| \leqq K_1(\Phi(f)+\|f\|)\,|\log x|$. Since $\int_x^{1/2} |\log u|\,du \leqq M$ and $\int_x^{1/2} \frac{du}{u^\alpha} \leqq M$ for $\alpha<1$, we obtain the estimate $|f^{(2m-j_0)}(x)| \leqq K_2(\Phi(f)+\|f\|)$ in $[0, 3/4]$, for the first j satisfying $2m\gamma-2m+j<0$ which we denote by j_0. Therefore, also for $j_0 \leqq j < 2m$, we have

$$|f^{(2m-j)}(x)| \leqq \left(\|f^{(2m-j_0)}\|^{\frac{j-j_0}{2m-j_0}}\,\|f\|^{\frac{2m-j}{2m-j_0}}\right) \leqq K_3\left(\Phi(f)+\|f\|_{C\left[0,\frac{3}{4}\right]}\right).$$

We will now define the interpolation space which we will need in this paper. A will be the space of functions with $2m$ continuous derivatives in the interior of I whose i derivative for $i<2m$ is locally absolutely continuous, and for which the seminorm $\Phi_A(f)=\|(\varphi(x))^{2m}f^{(2m)}(x)\|_{C(I)}<\infty$ for some fixed weight function $\varphi(x)$. Recall that the Peetre K functional for the pair of spaces (C, A) is $K(f, t)=$ $=\inf_{f=f_1+f_2}\{\|f_1\|_{C(I)}+t\Phi_A(f_2)\}$ and the interpolation space $(C, A)_\alpha$ (or $(C, A)_{\alpha,\infty}$) is the collection of all functions for which $\sup_t \frac{K(f, t)}{t^\alpha} \leqq M_f$ for some constant M_f.

Lemma 3.2. *Let A_i and A be the spaces for which*

$$\|x^{2m\gamma-2mi}f^{(i)}(x)\|_{C\left[0,\frac{3}{4}\right]}<\infty \quad \text{and} \quad \|x^{2m\gamma}f^{(2m)}(x)\|_{C\left[0,\frac{3}{4}\right]}<\infty$$

respectively (the derivatives of lower order being absolutely continuous locally), then for $0<\gamma<1$ and $2m\gamma-2m+i>0$ we have $f\in(C[0, 3/4], A)_\beta$ implies $f\in(C[0, 3/4], A_i)_\beta$.

Proof. The Lemma follows immediately from (3.1).

Lemma 3.3. *Let $A(j+1)$ be the space of functions whose derivatives up to $f^{(j)}$ are locally absolutely continuous in $(0, 3/4)$ and for which $\|x^\sigma f^{(j+1)}(x)\|_{C[0,3/4]}=$ $=\Phi_j(f)<\infty$ for some $0<\sigma<j+1$, then $f\in(C, A(j+1))_\beta$ where $C=C[0, 3/4]$*

for some $0<\beta<1$ *implies* $|\Delta_{hx^{\sigma/j+1}}^{j+1}f(x)|\leqq Mh^{(j+1)\beta}$ *and in case* $\sigma\leqq 1$ *it also implies* $f\in \mathrm{Lip}^*\, j\beta$.

Actually for $\sigma<1$ the second part of the Lemma is an immediate corollary of Lemma 3.1 and is not too interesting. The interesting part is when $\sigma=1$. The first part of the Lemma for even $j+1$ was proved in [5] and in fact the proof is very similar.

Proof of Lemma 3.3. For $f\in(C, A(j+1))_\beta$ we have for any τ functions f_1 and f_2 such that $f_1\in C[0, 3/4]$, $\|f_1\|_{C[0,3/4]}\leqq M\tau^\beta$ and $f_2\in A(j+1)$ such that $\Phi_j(f_2)\leqq \leqq M\tau^{\beta-1}$ and $f=f_1+f_2$ and M does not depend on τ. For

$$\left[x-\frac{j+1}{2}hx^{\sigma/j+1},\ x+\frac{j+1}{2}hx^{\sigma/j+1}\right]\subset\left[0,\frac{3}{4}\right]$$

we have

$$|\Delta_{hx^{\sigma/j+1}}^{j+1}f(x)|\ \leqq\ |\Delta_{hx^{\sigma/j+1}}^{j+1}f_1(x)|+|\Delta_{hx^{\sigma/j+1}}^{j+1}f_2(x)|\ =\ I_1+I_2.$$

Choosing $\tau=h^{j+1}$, $I_1\leqq 2^{j+1}Mh^{(j+1)\beta}$. Using Taylor's formula, we have

$$|I_2|\leqq M_1\max_{0\leqq l\leqq j+1}\left|\int\limits_{x-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}}^{x}\left(u-x+\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}\right)^j|f_2^{(j+1)}(u)|\,du\right.\equiv$$

$$\equiv M_1\max_{0\leqq l\leqq j+1}I(l),$$

and we will estimate $I(l)$ for $l<(j+1)/2$ and $l>(j+1)/2$ separately. For $l>(j+1)/2$ we have

$$I(l)\leqq\int\limits_{x}^{x-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}}\frac{\left|u-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}\right|^j}{u^\sigma}|u^\sigma f_2^{(j+1)}(u)|\,du\leqq$$

$$\leqq M\Phi_j(f_2)h^{j+1}x^{-\sigma}x^\sigma\leqq M\Phi_j(f_2)\tau.$$

For $l<(j+1)/2$ we have the same estimate provided $x/2>((j+1)/2)hx^{\sigma/j+1}$. If however $x/2<((j+1)/2)hx^{\sigma/j+1}$ then

$$I(l)\leqq\int\limits_{x-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}}^{x}\frac{\left(u-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}\right)^j}{u^\sigma}|u^\sigma f_2^{(j+1)}(u)|\,du\leqq$$

$$\leqq M\Phi_j(f_2)\int\limits_0^x u^{j-\sigma}\,du\leqq M\Phi_j(f_2)X^{j+1-\sigma}\leqq M\Phi_j(f_2)(hx^{\sigma/j+1})^{j+1}x^{-\sigma}=M\Phi_j(f_2)\tau.$$

This completes the proof that $|\Delta_{hx^{\sigma/j+1}}^{j+1}f(x)|\leqq Mh^{(j+1)\beta}$.

We are now ready to prove the second (and probably the more important) contention of our Lemma. Using a well-known result [14, p. 105], we have

$$\bar{\Delta}^j_{2\eta} f(x) - 2^j \bar{\Delta}^j_\eta f(x) = \sum_{\nu=0}^{j-1} \sum_{\mu=\nu+1}^{j} \binom{j}{\mu} \bar{\Delta}^{j+1}_\eta f(x+\nu\eta)$$

where $\bar{\Delta}_\eta f(x)$ are forward differences. We now use the former estimate on $\bar{\Delta}^{j+1}_\eta f(x) \equiv \equiv \bar{\Delta}^{j+1}_\eta f(x+(j+1)\eta/2)$ and obtain

$$|\bar{\Delta}^j_\eta f(x)| \leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} \max_{0 \leqq \nu \leqq j-1} |\bar{\Delta}^{j+1}_\eta f(x+\nu\eta)| \leqq$$

$$\leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} M \left(\frac{\eta}{\left[x + \left(\nu + \frac{j+1}{2}\right)\eta\right]^{\sigma/j+1}} \right)^{(j+1)\beta} \leqq$$

$$\leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} M \eta^{(1-\sigma/j+1)(j+1)\beta},$$

and for $\sigma \leqq 1$ we have

$$|\bar{\Delta}^j_\eta f(x)| \leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} M \eta^{j\beta}.$$

Repeating the above process l times,

$$|\bar{\Delta}^j_\eta f(x)| \leqq \frac{1}{2^{jl}} |\bar{\Delta}^j_{2^l\eta} f(x)| + \frac{j}{2} M \sum_{k=0}^{l-1} 2^{-jk} (2^k \eta)^{j\beta} \leqq \frac{1}{2^{jl}} |\bar{\Delta}^j_{2^l\eta} f(x)| + \frac{j}{2} M_1 \eta^{j\beta}.$$

Choosing l such that $1/8 \leqq 2^l \eta \leqq 1/4$ and using the elementary estimate $|\bar{\Delta}^j_{2^l\eta} f(x)| \leqq 2^j \|f\|$, we complete the proof. (Actually we proved the Lemma for $0 < x < 1/4$ but apart from the singularity near zero the Lemma and an even better estimate are well-known).

Lemma 3.4. For $\|x^{2m\beta} f^{(2m)}(x)\|_{C[1/2,\infty)} < \infty$ and $\|f\|_{C[1/2,\infty]} < \infty$ we have

$$\|x^{i\beta} f^{(i)}(x)\|_{C\left[\frac{1}{2},\infty\right]} \leqq C\left(\|x^{2m\beta} f^{(2m)}\|_{C\left[\frac{1}{2},\infty\right]} + \|f\|_{C\left[\frac{1}{2},\infty\right]}\right).$$

Proof. In $[A, A+A^\beta]$ the Lemmas follows [5, p. 311] with $b-a=A^\beta$. Our Lemma follows patching together pieces of this type.

4. Rate of convergence for the intermediate space. In this section we will be interested in the analogue to the direct theorem that shows that for $0<\alpha<2$ $((x(1-x))^{\alpha/2} \Delta^2_h f(x)) \leqq Mh^\alpha$ or $|\Delta^2_{h\varphi} f(x)| \leqq Mh^\alpha$ where $\varphi(x) = (x(1-x))^{1/2}$ which is equivalent (see [5, p. 312]) we have $\|B_n(f,x) - f(x)\| = O(1/n^{\alpha/2})$. The results here are not corollaries of the results in section 2 and this is best illustrated by the fact that in the particular case Bernstein polynomials after proving the analogue of Theorem 2.1 for positive operators, Strukov and Timan [13] show with a relatively lengthy computation that $\|B_n(x^\gamma, t)-t^\gamma\|=0(1/n^\gamma)$ for $0<\gamma<1$ which would have followed from $((x(1-x))^\gamma \Delta_h^2(x^\gamma))\leqq Mh^{2\gamma}$ (a result which follows observing that for $x(1-x)\leqq 5h$ the estimate is obvious and for $x(1-x)\geqq 5h$ the mean value theorem yields the estimate).

Definition 4.1 (a) A function $\varphi(x)$ defined on $I=[0, 1]$ satisfies the γ condition for some $0\leqq\gamma\leqq 1$ if $0<Ax^\gamma\leqq\gamma(x)\leqq Bx^\gamma$ for $0<x\leqq 1/2$ and $0<A(1-x)^\gamma\leqq\leqq\varphi(x)\leqq B(1-x)^\gamma$ for $1/2\leqq x<1$. (b) A function on $I=R^+$ or R satisfies the (γ, β) condition for some $0\leqq\gamma\leqq 1$, $0\leqq\beta\leqq 1$ if $0<Ax^\gamma\leqq\varphi(x)\leqq Bx^\gamma$ for $0<|x|\leqq 1/2$ and $0<A|x|^\beta\leqq\varphi(x)\leqq B|x|^\beta$ elsewhere.

Theorem 4.1. *Suppose for a sequence of linear operators on* $C(I)$ *where* $I=[0, 1]$, $I=R^+$ or R *given by* $L_n(f, x)=\int_I f(t)\,d\alpha_{n,x}(t)$ *we have*

a)
$$\int_I t^i\, d\alpha_{n,x}(t) = x^i \quad \textit{for} \quad i = 0, 1, \ldots, 2m-1,$$
$$v_{n,x}(t) \equiv \int_{u\leqq t} |d\alpha_{n,x}(u)| \leqq M$$

and

b)
$$\int_I (t-x)^{2m}\, dv_{n,x}(t) \leqq \sigma_n^{2m}(\varphi(x)+\eta_n)^{2m}$$

where $\sigma_n=o(1)$, $\varphi(x)$ *satisfies condition* γ *or condition* (γ, β) *(Definition 4.1(a) and (b)),* $\eta_n=O(\sigma_n^{\gamma/1-\gamma})$ *if* $0<\gamma<1$ *while* $\eta_n=0$ *for* $\gamma=0$ *and* $\gamma=1$. *Then for* f $f\in(C, A)_\alpha$ *we have* $\|f(\cdot)-L_n(f, \cdot)\|_{C(I)}=O(\sigma_n^{2m\alpha})$.

The spaces $(C, A)_\alpha$ were characterized in [5] for $\varphi(x)$ given here. (See also Lemma 3.3).

Remark 4.1 (a) The addition of the term η_n is important for some applications, though it looks at first glance somewhat artificial. Of course the theorem is valid (and easier to prove) with $\eta_n=0$.

(b) One could have different γ near 0 and 1 which we call γ_0 and γ_1 respectively in which case the theorem would still be valid provided that $0\leqq\gamma_i\leqq 1$ and $\int_I (t-x)^{2m}dv_{n,x}(t)\leqq\sigma_n^{2m}(\varphi(x)+\eta_n(i))^{2m}$ with $\eta_n(0)=O(\sigma_n^{\gamma_0/1-\gamma_0})$ for $0\leqq x\leqq 1/2$ and $\eta_n(1)=O(\sigma_n^{\gamma_1/1-\gamma_1})$ for $1/2\leqq x\leqq 1$ if $\gamma_i\neq 0, 1$, and $\eta_n(i)=0$ otherwise. Since the proof will concentrate at the boundary points one at a time, no other change will be required.

(c) The special case of Theorem 4.1 dealing with positive operators was treated by V. Totik who had somewhat different (and more involved) conditions on $\varphi(x)$ [16]. It can be noticed that differentiability, convexity etc. of $\varphi(x)$ are not the issue here. However, it should be noted that Totik treated, the inverse as well as the direct theorem for positive operators.

(d) For $I=R$ generally $\gamma=0$ (in applications).

Proof. For $f\in(C,A)_\alpha$ there exists for each τ f_1 and f_2 such that $f=f_1+f_2$ and $\|f_1\|_{C(I)}+\tau\Phi(f_2)\leqq K\tau^\alpha$ or $\|f_1\|\leqq K\tau^\alpha$ and $\tau\Phi(f_2)\leqq K\tau^\alpha$. Choosing $\tau=\sigma_n^{2m}$, we have $\|L_n(f,x)-f(x)\|\leqq\|L_n(f_1,x)-f_1(x)\|+\|L_n(f_2,x)-f_2(x)\|\leqq(M+1)K\sigma_n^{2m\alpha}+$ $+\|L_n(f_2,x)-f_2(x)\|$.

We will now show that for $f_2\in A$ $\|L_n(f_2,x)-f_2(x)\|\leqq N\sigma_n^{2m}\Phi(f_2)$ which is the crucial step in the proof and which with the estimates above will complete the proof our theorem. For $g(t)\in A$ we write the Taylor expansion

$$g(t)=g(x)+(t-x)g'(x)+\ldots+\frac{(t-x)^{(2m-1)}}{(2m-1)!}g^{(2m-1)}(x)+$$
$$+\frac{1}{(2m-1)!}\int_x^t(u-t)^{2m-1}g^{(2m)}(u)\,du.$$

For $0<\gamma<1$, $B\sigma_n^{1/1-\gamma}\leqq x\leqq 1/2$ and for $I=R^+$ (or R) $x<t$, (while in case $I=[0,1]$ $x<t\leqq 3/4$) we have

$$\left|\int_x^t(u-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq\frac{c(x-t)^{2m}}{\varphi(x)^{2m}}\Phi(g)\leqq\frac{c_1(x-t)^{2m}}{(\varphi(x)+\eta_n)^{2m}}\Phi(g).$$

For $t<x$, $0<\gamma<1$ and $B\sigma_n^{1/1-\gamma}\leqq x\leqq 1/2$ we have

$$\frac{x}{2^k}\leqq t<\frac{x}{2^{k-1}}\quad\text{and therefore for } k=1,\text{ or }\quad\frac{x}{2}\leqq t<x,$$

$$\left|\int_t^x(u-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq C\Phi(g)\int_t^x\frac{(u-t)^{2m-1}}{u^{2m\gamma}}\,du\leqq C\frac{2^{2m\gamma}}{2m}(x-t)^{2m}\Phi(g)\leqq$$

$$\leqq C_1\frac{\Phi(g)}{(\varphi(x)+\eta_n)^{2m}}(x-t)^{2m}.\quad\text{For}\quad k>1,\quad\frac{x}{2^k}\leqq t<\frac{x}{2^{k-1}}\quad\text{we have,}$$

$$\left|\int_t^x(x-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq C\Phi(g)\left|\int_t^x\frac{(u-t)^{2m-1}}{u^{2m\gamma}}\,du\right|\leqq$$

$$\leqq C\Phi(g)\int_{x/2^k}^x\frac{(u-x/2^k)^{2m-1}}{u^{2m\gamma}}\,du\leqq C\Phi(g)\sum_{l=0}^{k-1}\int_{x/2^{l+1}}^{x/2^l}\frac{|u-x/2^k|^{2m-1}}{u^{2\gamma m}}\,du\leqq$$

$$\leqq C\Phi(g)\sum_{l=0}^{k-1}\frac{\left(x\left|\frac{1}{2^l}-\frac{1}{2^k}\right|\right)^{2m-1}\frac{x}{2^{l+1}}}{(x/2^{l+1})^{2m\gamma}}\leqq C_1\Phi(g)\frac{(x-t)^{2m}}{x^{2m\gamma}}\leqq C_2\Phi(g)\frac{(x-t)^{2m}}{(\varphi(x)+\eta_n)^{2m}}.$$

Therefore, we use the estimate

$$L_n\big(g(t)-g(x), x\big) = \frac{1}{(2m-1)!} L_n\Big(\int_x^t (u-t)^{2m-1} g^{(2m)}(u)\,du,\ x\Big)$$

for $I=R^+$ or R and $B\sigma_n^{1/1-\gamma} \leqq x \leqq 1/2$ to get

$$L_n\big(g(t)-g(x), x\big) \leqq C\Phi(g)\frac{\big(\varphi(x)+\eta_n\big)^{2m}}{\big(\varphi(x)+\eta_n\big)^{2m}}\sigma_n^{2m} \leqq C\Phi(g)\sigma_n^{2m}.$$

For $I=[0,1]$, $B\sigma_n^{1/1-\gamma} \leqq x \leqq 1/2$ we have

$$|L_n\big(g(t)-g(x), x\big) \leqq \int_0^{3/4} \Big|\int_x^t (u-t)^{2m-1} g^{(2m)}(u)\Big|\,dv_{n,x}(t) +$$

$$+\sum_{i=0}^{2m-1}\frac{1}{i!}|g^{(i)}(x)|\int_{3/4}^{1}|t-x|^i\,dv_{n,x}(t) \leqq$$

$$\leqq C\Phi(g)\sigma_n^{2m} + C\sum_{i=0}^{2m-1}|g^{(i)}(x)|\int_0^1 |t-x|^{2m}\,dv_{n,x}(t)$$

which, using Lemma 3.1, implies for the x in question $|L_n\big((g(t)-g(x)), x\big)| \leqq \leqq C\Phi(g)\sigma_n^{2m}$. For $x<B\sigma_n^{1/1-\gamma}$ we observe that $2m(1-\gamma)$ is either an integer or not. If $2m(1-\gamma)=i$, then $2m\gamma-2m+i=0$ and therefore, using Lemma 3.3, $f\in \mathrm{Lip}^*\,\alpha i$ in $[0, 3/4]$. Therefore, for a given $\tau, f=f_1+f_2$ such that $\|f_1\| \leqq M\tau^\alpha$ and $\|f_2^{(i)}\| \leqq \leqq M\tau^{\alpha-1}$, and we write again $\tau=\sigma_n^{2m}$. We observe now that

$$|L_n(f_2, x)-f_2(x)| \leqq \frac{1}{i!}\|f_2^{(i)}\|\int_I |t-x|^i\,dv_{n,x}(t) \leqq$$

$$\leqq \frac{1}{i!}\|f_2^{(i)}\|\Big\{\int_I (t-x)^{2m}\,dv_{n,x}(t)\Big\}^{i/2m}\Big\{\int_I dv_{n,x}(t)\Big\}^{1-(i/2m)} \leqq K\|f_2^{(i)}\|\big(\varphi(x)+\eta_n\big)^i\sigma_n^i \leqq$$

$$\leqq K_1\|f_2^{(i)}\|\,\sigma_n^{\gamma i/1-\gamma}\sigma_n^i \leqq K_1\|f_2^{(i)}\|\,\sigma_n^{2m} \leqq K_2\sigma_n^{2m\alpha}.$$

For $2m(1-\gamma)$ not an integer, choose i such that $0<2m\gamma-2m+i<1$ and, using Lemma 3.2, $f\in(C,A)_\alpha$ implies $f\in(C,A_i)_\alpha$ where $A_i=\{f, \|x^{2m\gamma-2+i}f^{(i)}(x)\|_{C[0,3/4]}<\infty\}$.

We write $f=f_1+f_2$ where $\|f_1\| \leqq M\tau^\alpha$ and $\Phi_i(f_2) \leqq M\tau^{\alpha-1}$ and set $\tau=\sigma_n^{2m}$. Now

$$|L_n(f_2, x)-f_2(x)| \leqq \frac{1}{(i-1)!}\Phi_i(f_2)\int_I\Big|\int_t^x \frac{(u-t)^{i-1}}{u^{2m\gamma-2m+i}}\,du\Big|\,dv_{n,x}(t).$$

For $x<B\sigma_n^{1/1-\gamma}$ we have for $t>x$

$$\Big|\int_x^t \frac{(u-t)^{i-1}}{u^{2m\gamma-2m+i}}\,du\Big| \leqq C|t-x|^{i-1}\int_0^t u^{2m-2m\gamma-i}\,du \leqq C_1|t-x|^{i-1}t^{2m-2m\gamma-i+1} \leqq$$

$$\leqq C_1|t-x|^{2m-2m\gamma}$$

and for $t<x$

$$\left|\int_t^x \frac{(u-t)^{i-1}}{u^{2m\gamma-2m+i}}\,du\right| \leqq \left|\int_t^x \frac{|u-t|^{i-1}}{|u-t|^{2m\gamma-2m+i}}\,du\right| \leqq C|x-t|^{2m-2m\gamma}.$$

But, using Hölder's inequality,

$$J=\int_I |x-t|^{2m-2m\gamma}\,dv_{n,x}(t) \leqq \Big\{\int_I |x-t|^{2m}\,dv_{n,x}(t)\Big\}^{1-\gamma}\Big\{\int_I dv_{n,x}(t)\Big\}^{\gamma} \leqq$$
$$\leqq C\big[\sigma_n\big(\varphi(x)+\eta_n\big)\big]^{2m-2m\gamma}$$

which for $x\leqq B\sigma_n^{1/1-\gamma}$ implies $J\leqq C_1\sigma_n^{2m-2m\gamma}(\sigma_n^{\gamma/1-\gamma})^{2m-2m\gamma}=C_1\sigma_n^{2m}$. With the above choice of f_2 and τ, we have our estimate for $0<x\leqq 1/2$ and $0<\gamma<1$. For $\gamma=0$ the estimate is actually trivial. For $\gamma=1$ we write

$$\left|\int_t^x (u-t)^{2m-1}g^{(2m)}(u)\,du\right| \leqq \Phi(g)\frac{4^m}{x^{2m}}\frac{1}{2m}(u-t)^{2m}$$

for $t\geqq x/2$ and therefore

$$|L_n(g,x)-g(x)| \leqq \left|\int_0^{x/2} g(t)\,d\alpha_{n,x}(t)-g(x)\int_0^{x/2} d\alpha_{n,x}(t)\right|+$$
$$+\left|\frac{1}{(2m-1)!}\int_{I\cap\{t>(x/2)\}}\left\{\int_t^x (u-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq 2\|g\|_C\int_0^{x/2} dv_{m,x}(t)+$$
$$+\Phi(g)\frac{4^m}{x^{2m}}\frac{1}{(2m)!}\int_{I\cap\{t>(x/2)\}}(u-t)^{2m}\,dv_{n,x}(t)\leqq$$
$$\leqq\left(\|g\|_C 2\left(\frac{2}{x}\right)^{2m}\int_I (u-t)^{2m}\,dv_{n,x}(t)\right)+$$
$$+\Phi(g)\frac{4^m}{x^{2m}}\frac{1}{2m!}\int_I (t-x)^{2m}\,dv_{n,x}(t)\Big\}\leqq C(\|g\|_C+\Phi(g))\sigma_n^{2m}.$$

One can note that $\|g\|_C\leqq\|f\|_C+1$ and therefore the estimate follows). For $I=[0,1]$ near $x=1$ the estimate is similar to the above. We now have to estimate the rate for x bounded away from 0 for R^+ or R. For $t>x\geqq 1/2$ (in R^+ say)

$$\left|\int_x^t (u-t)^{2m-1}g^{(2m)}(u)\,du\right| \leqq \frac{C}{\varphi(x)^{2m}}(x-t)^{2m}\Phi(g).$$

Otherwise we distinguish the two cases $x-x^\beta/4<t\leqq x$, $x\geqq 1/2$ and $t<x-x^\beta/4$, $x\geqq 1/2$. In the first case we have $\left|\int_x^t |u-t|^{2m}g^{(2m)}(u)\,du\right|\leqq\dfrac{C}{\varphi(x)^{2m}}(x-t)^{2m}\Phi(g)$

and in the second case we just write $g(t)$. Consequently

$$|L_n(g,x)-g(x)| \leqq \frac{C}{\varphi(x)^{2m}} \int (x-t)^{2m} \Phi(g)\, dv_{n,x}(t)+$$

$$+C\sum_{i=0}^{2m-1} |g^{(i)}(x)| \int_{t\leqq x-(1/4)} |t-x|^i\, dv_{n,x}(t) \leqq \frac{C}{\varphi(x)^{2m}}\, \varphi(x)^{2m}\sigma_n^{2m}\Phi(g)+$$

$$+C_1\sum_{i=0}^{2m-1} |g^{(i)}(x)|(x^{\beta})^{-2m+i} \int_I |t-x|^{2m}\, dv_{n,x}(t) \leqq$$

$$\leqq C\sigma_n^{2m}\Phi(g)+C_2\sum_{i=0}^{2m-1} |g^{(i)}(x)|x^{\beta i}\sigma_n^{2m},$$

and using Lemma 3.4, we complete the proof of our theorem.

5. Rate of convergence, continued. In this section we will deal with the situation in which moments of lower order are different from 0. We denote

$$(5.1) \qquad \int_I (t-x)^i\, d\alpha_{n,x}(t) = R_{n,i}(x).$$

A different result for approximation operators for which $R_{n,i}(x)\neq 0$ for some of the i's was given in theorem 2.2. We will first itemize what conditions the functions $R_{n,i}(x)$ have to satisfy and while these conditions are not very simple to state, they are relatively simple to verify in applications.

Definition 5.1. For $I=[0,1]$, $0<\gamma<1$, $R_{n,i}(x)$ satisfies the $(\gamma, 2m, i, \sigma_n)$ condition if

$$|R_{n,i}(x)| \leqq M\sigma_n^{2m} \min\{\max(x(1-x), \sigma_n^{1/1-\gamma})^{2m\gamma-2m+i}, 1\};$$

for $\gamma=0$ $|R_{n,i}(x)|\leqq M\sigma_n^{2m}$ and for $\gamma=1$ $|R_{n,i}(x)|\leqq M\sigma_n^{2m}(x(1-x))^i$.

Definition 5.2. For $I=R^+$ (or R) $R_{n,i}(x)$ satisfies the $(\gamma,\beta,2m,i,\sigma_n)$ condition if for $|x|\leqq 1/2$ it satisfies the condition in Definition 5.1 (x may replace $x(1-x)$ but that would not change the situation) and for other x, $|R_{n,i}(x)|\leqq$ $\leqq M\sigma_n^{2m}|x|^{\beta i}$.

We are now ready to state and prove our theorem about rate of convergence.

Theorem 5.1. *Suppose a sequence of linear operators on* $C(I)$, $L_n(f,x)=$ $=\int_I f(t)\,d\alpha_{n,x}(t)$ *satisfy the conditions of Theorem* 4.1 *except that* $R_{n,i}(x)$ *are not necessarily* 0 *but satisfy the conditions in definition* 5.1 *and* 5.2 *with the same* γ $0\leqq\gamma\leqq 1$ and β $0\leqq\beta\leqq 1$ *given in theorem* 4.1, *then for* $f\in(C,A)_\alpha$, $0<\alpha<1$, $\|L_n(f,k)-f(\cdot)\|=$ $=0(\sigma_n^{2m\alpha})$, *where* $A=\{f;\ \Phi(f)=\|\varphi(x)^{2m}f^{(2m)}(x)\|_{C(I)}<\infty$ *and* f *has* $2m-1$ *absolutely continuous derivatives locally in the interior of* $I\}$.

Proof. The process that we use is the same as that of Theorem 2.2 and we construct a new operator $A_n(f, x)=\int f(t)\,d\beta_{n,x}(t)$. In order to complete the proof we have to show two things: (a) that the behaviour of $\int_I (t-x)^{2m}dV_{n,x}(t)$, where $V_{n,x}(t)$ is the variation of $\beta_{n,x}(t)$, is the same as $\int (t-x)^{2m}dv_{n,x}(t)$ (required in Theorem 4.1) where $v_{n,x}(t)$ is the variation of $\alpha_{n,x}(t)$; (b) that for $f\in(C, A)_\alpha$ the operators we added contribute at most $M\sigma_n^{2m\alpha}$.

To prove (a) let us recall that we have introduced the operators $C\bar{\Delta}^j_{|R_{n,i}(x)|^{1/i}}$ $i\leqq j<2m$ for $x<1/2$ or in general in case I is not $[0, 1]$ and $C\bar{\Delta}^j_{-|R_{n,i}(x)|^{1/i}}$ for $x>1/2$ and $I=[0, 1]$. Each term of this kind will add to the variaton of $\alpha_{n,x}(t)$, that is to $v_{n,x}(t)$, to produce eventually (after the process is completed) the operator $A_n(f, x)=\int_I f(t)\,d\beta_{n,x}(t)$ where we denote the variation of $\beta_{n,x}(t)$ by $V_{n,x}(t)$. The amount added to $\int_I (t-x)^{2m}dv_{n,x}(x)$ to get $\int_I (t-x)^{2m}dV_{n,x}(t)$ is for each i a constant times $|R_{n,i}(x)|^{2m/i}$. We will now show that these additions with $R_{n,i}(x)$ restricted as in the conditions of our theorem will leave us with a new operator $\int_I f(t)\,d\beta_{n,x}(t)$ that satisfies the restriction in Theorem 4.1.

For $\gamma=0$ or for $0<\gamma<1$ and $2m\gamma-2m+i\leqq 0$ we have

$$|R_{n,i}(x)|^{2m/i}\leqq (M\sigma_n^{2m})^{2m/i}\leqq M\sigma_n^{(2m)^2/i}=M\sigma_n^{2m}\sigma_n^{2m((2m/i)-1)}\leqq M\sigma_n^{2m}(\sigma_n^{\gamma/1-\gamma})^{2m}$$

since $\gamma/(1-\gamma)\leqq 2m/i-1=(2m-i)/i$ which follows $2m-i\geqq 2m\gamma$ and $i\leqq 2m(1-\gamma)$. For $0<\gamma<1$ and $2m\gamma-2m+i>0$ we have to distinguish two possibilities: (I) $x(1-x)\leqq A\sigma_n^{1/1-\gamma}$ for $I=[0, 1]$ (and $x\leqq A\sigma_n^{1/1-\gamma}$ for $I=R^+$ or $I=R$); (II) $x(1-x)\geqq A\sigma_n^{1/1-\gamma}$ for $I=[0, 1]$ (and $x\geqq A\sigma_n^{1/1-\gamma}$ for $I=R^+$ or $I=R$). For the situation (I) we have

$$|R_{n,i}(x)|^{2m/i}\leqq M\big(\sigma_n^{2m/i}(\sigma_n^{1/1-\gamma})^{(2m\gamma-2m+i)/i}\big]^{2m}=M\sigma_n^{2m/1-\gamma}=M\sigma_n^{2m}\sigma_n^{2m(\gamma/1-\gamma)}.$$

For the estimate in case II we will be concerned with the case $0<x<1/2$ (the other case being similar) and obtain

$$|R_{n,i}(x)|^{2m/i}\leqq M\sigma_n^{(2m)^2/i}x^{2m(2m\gamma-2m+i)/i}=M\sigma_n^{2m}[\sigma_n^{2m(2m-i)/i}x^{2m(2m\gamma-2m+i)/i}]\leqq$$
$$\leqq M_1\sigma_n^{2m}[x^{(1-\gamma)2m(2m-i)/i}x^{2m(2m(\gamma-1)+i)/i}]=M_1\sigma_n^{2m}x^{2m\gamma}.$$

For $\gamma=1$ we have near $x=0$ $|R_{n,i}(x)|^{2m/i}\leqq M(\sigma_n^{i/i}|x|^{i/i})^{2m}\leqq M\sigma_n^{2m}x^{2m}$ (and for x near 1 in case $I=[0, 1]$ the same type of estimate follows too). We are left with the estimate for other x but there $|R_{n,i}(x)|^{2m/i}\leqq M\sigma_n^{(2m)^2/i}|x|^{2m\beta i/i}\leqq M\sigma_n^{2m}|x|^{2m\beta}$.

We will now prove (b), that is, we will show that near 0 $f\in(C, A)_\alpha$ implies for $i\leqq j<2m$ $|\bar{\Delta}^j_{|R_{m,i}(x)|^{1/i}}f(x)|\leqq M\sigma_n^{2m\alpha}$. It is a similar situation near $x=1$ in case $I=[0, 1]$ and for other x it is substantially simpler. It is enough to prove the above contention for $j=i$. First we see that for $f\in(C, A)_\alpha$ we have $f(x)=f_1(x)+f_2(x)$

where $\|f_1\|_{C(I)} \leqq K\sigma_\alpha^{2m\alpha}$ and $\Phi(f_2) \leqq K\sigma_n^{2m(\alpha-1)}$ where K does not depend on n. We can now write

$$|\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f(x)| \leqq |\bar{\Delta}^i_{|R_{m,i}(x)|^{1/i}} f_1(x)| + |\bar{\Delta}^i_{|R_{m,i}(x)|^{1/i}} f_2(x)| \equiv I_1 + I_2.$$

Obviously, $I_1 \leqq 2^i K\sigma_n^{2m\alpha}$ and $\|f_2\|_C \leqq \|f\|_C + 1$. For $2m\gamma - 2m + i < 0$ Lemma 3.1, yields $\|f_2^{(i)}\| \leqq M(\Phi(f_2) + \|f_2\|_C)$ and $|R_{n,i}(x)|^{1/i} \leqq M\sigma_n^{2m/i}$ or

$$|\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f_2(x)| \leqq |R_{n,i}(x)|\, \|f_2^{(i)}(x)\| \leqq M'\sigma_n^{2m}\sigma_n^{2m(\alpha-1)} = M'\sigma_n^{2m\alpha}.$$

For $2m\gamma - 2m + i > 0$ we estimate first for $x \geqq A\sigma_n^{1/1-\gamma}$ or for $\gamma = 1$, and $x \leqq 1/2$ and write using Lemma 3.1

$$|I_2| = |R_{n,i}(x)|\, |f_2^{(i)}(\xi)| \leqq M\sigma_n^{2m} x^{2m\gamma-2m+i} |f_2^{(i)}(\xi)| \leqq M\sigma_n^{2m}(\xi^{2m\gamma-2m+i}|f_2^{(i)}(\xi)|) \leqq$$
$$\leqq M_1\sigma_n^{2m}(\Phi(f_2) + \|f_2\|) \leqq M_2\sigma_n^{2m}\sigma_n^{2m(\alpha-1)} = M_2\sigma_n^{2m\alpha}.$$

For $x \leqq A\sigma_n^{1/1-\gamma}$ we observe that $|R_{n,i}(x)| \leqq M\sigma_n^{2m}(\sigma_n^{1/1-\gamma})^{2m\gamma-2m+i} = M\sigma_n^{i/1-\gamma}$ or $|R_{n,i}(x)|^{1/i} \leqq M_1\sigma_n^{1/1-\gamma}$. Writing $\theta = |R_{n,i}(x)|^{1/i}$, we have $|\bar{\Delta}^i_\theta f_2(x)| \leqq \Delta^i_\theta f_2(x + (i\theta/2))|$ and we can use the Taylor formula with integral remainder to expand around $x + (i\theta)/2$ and obtain

$$|\bar{\Delta}^i_\theta f_2(x)| \leqq M \max_{0 \leqq l \leqq i} \Big| \int_{x+l\theta}^{x+(i/2)\theta} (u - x - l\theta)^{i-1} \Big| f_2^{(i)}(u)\, |du| = M \max_{0 \leqq l \leqq i} J(l).$$

For $l > i/2$ we have

$$J(l) \leqq \int_{x+(i/2)\theta}^{x+l\theta} (x + l\theta - u)^{i-1} |f_2^{(i)}(u)|\, du \leqq$$
$$\leqq \int_{x+(i/2)\theta}^{x+l\theta} \frac{(x + l\theta - u)^{i-1}}{\left(x + \frac{i}{2}\theta\right)^{2m\gamma-2m+i}} |u^{2m\gamma-2m+i} f_2^{(i)}(u)|\, du \leqq$$
$$\leqq M\frac{\Phi(f_2) + \|f_2\|}{\theta^{2m\gamma-2m+i}}\theta^i \leqq M'(\Phi(f_2) + \|f_2\|)\sigma_n^{2m} \leqq M''\sigma_n^{2m\alpha}.$$

For $l < i/2$ we have

$$J(l) \leqq M \int_{x+l\theta}^{x+(i/2)\theta} \frac{(u - x - l\theta)^{i-1}}{u^{2m\gamma-2m+i}} |u^{2m\gamma-2m+i} f_2^{(i)}(u)|\, du \leqq$$
$$\leqq M_1\Big(\int_0^{x+(i/2)\theta} u^{2m-2m\gamma+i}\, du\Big)(\Phi(f_2) + \|f_2\|)$$

and since $2m - 2m\gamma > 0$, as we already treated $\gamma = 1$, we have

$$J(l) \leqq M_2\theta^{2m-2m\gamma}\sigma_n^{2m(\alpha-1)} \leqq M_3\sigma_n^{2m}\sigma_n^{2m(\alpha-1)} = M_3\sigma_n^{2m\alpha}.$$

We now turn our attention to the case $2m - 2m\gamma + i = 0$ by first observing that $f \in (C, A)_\alpha$ implies quite easily $f \in (C, A_{i+1})_\alpha$ where $A_{i+1} = \{f;\ f, \ldots, f^{(i)}$ are locally absolutely continuous in (0, 3/4) and $\|xf^{(i+1)}\|<\infty\}$. Using Lemma 3.3 with $i=j$, $\sigma=1$ and $\beta=\alpha$, we have

$$|\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f(x)| \leqq M|R_{n,i}(x) \leqq M_1\sigma_n^{2m\alpha}.$$

One need now only observe that near $x=1$ (in case $I=[0, 1]$) the proof is similar and for other x we actually just use $|\bar{\Delta}^i_\eta f(x)|\leqq\eta^i f^{(i)}(\xi)$ and obtain our result.

6. Application, some positive operators. **(a)** The Kantorovich operator given by

$$K_n(f, t) = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k}(n+1) \int_{k/n+1}^{k+1/n+1} f(u)\,du \tag{6.1}$$

or by $K_n(f, t)=(d/dt)B_{n+1}(F, t)$ where $F(u)=\int_0^u f(v)dv$ and $B_n(f, t)$ are then Bernstein polynomials. It is known that $K_n(1, t)=1$, $K_n(\cdot-t, t)=\dfrac{1-2t}{2(n+1)}$ and $K_n((\cdot-t)^2, t)=\dfrac{t(1-t)}{n}+0(n^{-2})$. Using Theorem 2.2. with

$$R_n(t) = \frac{t(1-t)}{n}+\frac{1}{4}\left(\frac{1-2t}{n}\right)^2+O\left(\frac{1}{n^2}\right) = \frac{t(1-t)}{n}+O\left(\frac{1}{n^2}\right)$$

and $R_{n,1}(t)=\dfrac{1-2t}{2(n+1)}$, we have:

Theorem 6.1. *For $f\in C[0, 1]$ and $K_n(f, t)$ defined by* (6.1), *we have*

$$|K_n(f, t)-f(t)| \leqq M\omega_2\left(f, \left(\frac{t(1-t)}{n}+\frac{L}{n^2}\right)^{1/2}\right)+\omega_1\left(f, \frac{|1-2t|}{n}\right) \tag{6.2}$$

(and the theorems 2.1 *and* 2.2 *can yield a reasonable estimate on M while L can be estimated by* 1*).*

Using Theorems 4.1 and 5.1, we obtain with $\gamma=1/2$, $\sigma_n=1/\sqrt{n}$, $\eta_n=1/\sqrt{n}$ and $m=2$ the following result.

Theorem 6.2. For $f\in(C[0, 1], A)_\alpha$, $0<\alpha<1$, where $A=\{f; t(1-t)f''(t)\in \in C[0, 1]$ and f, f' are locally absolutely continuous in the interior of $(0, 1\}$, then $\|K_n(f, \cdot)-f(\cdot)\|_{C[0,1]}=0(1/n^\alpha)$.

Remarks. (I) We cannot omit the second term in Theorem 6.1 as is obvious when we observe the effect of the function x. (II) In [15, p. 54], in an added in proof remark, V. Totik indicated that the analogous result (to Theorem 6.2 is valid for L_p, $1<p<\infty$.

(b) The integral version of the Szász and Baskakov operators are given by

$$S_n^*(f, x) = \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} n \int_{k/n}^{k+1/n} f(u)\,du \tag{6.2}$$

and

$$V_n^*(f, x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k(1+x)^{-n-k} n \int_{k/n}^{k+1/n} f(u)\,du \tag{6.3}$$

which can also be given by

$$S_n^*(f, x) = \frac{d}{dx} S_n(F, x) \quad V_n^*(f, x) = \frac{d}{dx} V_{n-1}(F, x) \quad \text{and} \quad F(u) = \int_0^u f(u)\,dv \tag{6.4}$$

where $S_n(f, x)$ and $V_n(f, x)$ are the Szász and Baskakov operators given respectively by

$$S_n(f, x) = \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{6.5}$$

and

$$V_n(f, x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k(1+x)^{-n-k} f\left(\frac{k}{n}\right). \tag{6.6}$$

Theorem 6.3. *For* $f \in C(R^+)$

$$|S_n^*(f, x) - f(x)| \leqq M\omega_2\left(f, \sqrt{\frac{x}{n}+\frac{1}{n^2}}\right) + \omega_1\left(f, \frac{1}{2n}\right) \tag{6.7}$$

and

$$|V_n^*(f, x) - f(x)| \leqq M\omega_2\left(f, 2\sqrt{\frac{x(1+x)}{n}+\frac{1}{n^2}}\right) + \omega_1\left(f, \frac{x}{n}+\frac{1}{2n}\right). \tag{6.8}$$

Proof. One can calculate $D_n(x)$ $R_{n,1}(x)$ and $R_n(x)$ as $D_n(x) = x/n + 1/3n^2$, $R_{n,1}(x) = 1/2n$ and $R_n(x) = x/n + (7/12)(1/n^2)$ for the Szász operator.

We will calculate in detail for the integral version $V_n^*(f, x)$ of the Baskakov operator $D_n(x)$, $R_n(x)$ and $R_{n,1}(x)$,

$$D_n(x) = V_n^*\left((t-x)^2, x\right) =$$

$$= \sum_{k=0}^{\infty} \binom{n-k-1}{k} x^k x^{-n-k} \left[\frac{n}{3}\left(\frac{k+1}{n}\right)^3 - \frac{n}{3}\left(\frac{k}{n}\right)^3 - 2x\left(\frac{n}{2}\left(\frac{k+1}{n}\right)^2 - \frac{n}{2}\left(\frac{k}{n}\right)^2\right) + x^2\right] =$$

$$= V_n(t^2, x) + \frac{1}{n} V_n(t, x) + \frac{1}{3n^2} V_n(1, x) - 2xV_n(t, x) - \frac{x}{n} V_n(1, x) + x^2 V_n(1, x) =$$

$$= \frac{x(1+x)}{n} + \frac{1}{3n^2},$$

$R_{n,1}(x)=V_n^*((t-x),x)=x/n+1/(2n)$ and therefore $R_n(x)\leqq D_n(x)+R_{n,1}(x)^2\leqq \leqq 2x(1+x)/n+1/n^2$. Substituting the above in Theorem 2.1, we have Theorem 6.3.

Remarks. One cannot omit the second term from formulae (6.7) and (6.8) as the result would fail then for the function x. The corresponding results for the operators $S_n(f,x)$ and $V_n(f,x)$ would already follow the theorem of Strukov and Timan [13] and therefore are not stated here. Similarly, one can prove the following corollary of Theorems 4.1 and 5.1.

Theorem 6.4. *For* $f\in(C(R^+),A)_\alpha$ *where* $A=\{f;\ x\cdot f\in C(R^+)\}$

$$\|S_n^*(f,x)-f(x)\|_{C(R^+)}\leqq M\frac{1}{n^\alpha} \tag{6.9}$$

and

$$\|S_n(f,x)-f(x)\|_{C(R^+)}\leqq M\frac{1}{n^\alpha}. \tag{6.10}$$

Theorem 6.5. *For* $f\in(C(R^+),A)_\alpha$ *where* $A=\{f;\ x(1+x)f''\in C(R^+)\}$

$$\|V_n^*(f,x)-f(x)\|_{C(R^+)}\leqq M\frac{1}{n^\alpha} \tag{6.11}$$

and

$$\|V_n(f,x)-f(x)\|_{C(R^+)}\leqq M\frac{1}{n^\alpha}. \tag{6.12}$$

Proof. We simply adjust the moments already calculated to the moments and functions in Theorems 4.1 and 5.1. We observe that $\gamma=1/2$, $\sigma_n=1/\sqrt{n}$ and $\eta_n=1/\sqrt{n}$ in both Theorems, but $\beta=1/2$ in Theorem 6.4 and $\beta=1$ in Theorem 6.5.

Theorems 6.4 and 6.5 could be adjusted to exponential behaviour as x tends to infinity following the treatment in [3] for instance but it is the goal here to get corollaries of the general theorems preceding this section rather than deal with particular behaviour.

One should note that in Theorems 6.2 and 6.4 and 6.5 we have $\eta_n\neq 0$ and while it looked redundant to allow such η_n in the beginning, from the point of view of the applications it would appear quite important.

(c) The Post-Widder Laplace transform inversion formula.

The Post-Widder Laplace transform inversion formula is in face an approximation operator given by [17, Ch. 7]

$$P_n(f,t)=\frac{1}{n!}\left(\frac{n}{t}\right)^{n+1}\int_0^\infty e^{-nu/t}u^n f(u)\,du. \tag{6.13}$$

It is an inversion of the Laplace transform given by

$$(6.14)\quad P_n(f, t) = (-1)^n F^{(n)}\left(\frac{n}{t}\right)\left(\frac{n}{t}\right)^{n+1}\frac{1}{n!} \quad \text{where} \quad F(u) = \int_0^\infty e^{-ut} f(t)\,dt.$$

The difference between this and earlier examples is that here γ (that corresponds to Theorem 4.1) is equal to 1 rather than 1/2 in (a) and (b). Since $P_n(1, t)=1$, $P_n((\cdot -t), t)=0$ and $P_n((\cdot -t)^2, t)=t^2/n$, we have:

Theorem 6.6. *For $P_n(f, t)$ defined by* (5.13) *on* $C(R^+)$

$$(6.15)\qquad |P_n(f, t)-f(t)| \leqq 15\omega_2\left(f, \frac{t}{\sqrt{n}}\right)$$

and for $f\in (C(R^+), A)_\alpha$ *where* $A=\{f;\ t^2 f''(t)\in C(R^+)\}$

$$(6.16)\qquad \|P_n(f, t)-f(t)\|_{C(R^+)}\leqq M\,\frac{1}{n^\alpha}.$$

Again one can modify the result for exponential growth.

(d) the Meier—König and Zeller operator given by

$$(6.17)\qquad M_n(f, t) = (1-t)^n \sum_{k=0}^{k}\binom{k+n}{k} t^k f\left(\frac{k}{n+k}\right)$$

can also be treated using theorem 4.1 and it can be shown that:

Theorem 6.7. *For* $f\in(C[0, 1], A)_\alpha$, $0<\alpha<1$ *where*

$$A = \{f;\ \|x(1-x)^2 f''(x)\|_{C[0,1]}<\infty \quad f, f'\in AC_{\mathrm{loc}}(0, 1)\},$$

we have

$$\|M_n(f, t)-f(t)\| \leqq M\frac{1}{n^{\alpha/2}}.$$

Proof. This immediately follows the calculation of the moments.

The interesting part about this operator is that the γ's near zero and near one are different (1/2 and 1 respectively), a possibility mentioned in Remark 4.1(b).

For a similar operator

$$M_n^+(f, t) = (1-t)^n \sum_{k=0}^{\infty}\binom{n+k-1}{k} t^k f\left(\frac{k}{k+n}\right),$$

we have $M_n^+(f, t)=V_n(f_1, t(1-t))$ where $f_1(u)=f(u/(1+u))$ and V_n is the Baskakov operator given in (6.6). It is not easy to translate the behaviour of V_n to that of M_n^+ or M_n and it better done directly.

7. Non-positive approximation processes, combinations of Bernstein polynomials. In sections 7 and 8 approximation processes that are not positive but that converge faster depending on higher degrees of smoothness, will be discussed. In particular in section 8 we apply our theorems to combinations of „Exponential-type" operators introduced by C. P. May in [9] and [10]. In [2] the author proved a global direct and inverse theorem for combinations of Bernstein polynomials. We will see first how the direct part of [2] follows from the general theorems of this paper. Actually the results in Theorems 4.1 and 5.1 were motivated by the result on Bernstein polynomials and it seems interesting how those general theorems apply.

Combinations of Bernstein polynomials that would yield faster rates of convergence are given by

$$(2^r-1)B_n(f,r,x) \equiv 2^r B_{2n}(f,r-1,x)-B_n(f,r-1,x) \tag{7.1}$$

and $B_n(f,0,x)\equiv B_n(f,x)$. Other combinations are possible (see [2, p. 278]) but these seem to be the simplest form with the given rate of convergence. To establish results as corollaries of the theorems of this paper we have to compute moments of $B_n(f,r-1,x)$. (We choose $B_n(f,r-1,x)$ with $r-1$ use the same notation used in [2]). First we observe that

$$B_n(f,r-1,x) = \sum_{j=0}^{r-1} C_j B_{2^j n}(f,x) \tag{7.2}$$

and C_i are constants independent of n which among other properties satisfy

$$\sum_{i=0}^{r-1} C_j = 1 \quad \text{and} \quad \sum_{l=0}^{r-1} C_j n^{-l} = 0 \quad \text{for} \quad l=1,\dots,r-1. \tag{7.3}$$

We set $2m=2r$ and calculate $D_n(x)$. Using (6.2) and [2, (4.2) p. 285], we have

$$D_n(x) \leqq \sum_{i=0}^{r-1} |C_i| B_{2^i n}((t-x)^{2r},x) \leqq \Big(\sum_{i=1}^{r-1}|C_i|\Big)\max_i \big|B_{2^i n}((t-x)^{2r},x)\big| \leqq$$

$$\leqq \Big(\sum_{i=0}^{r-1}|C_i|\Big)B_n((t-x)^{2r};\ x) =$$

$$= \Big(\sum_{i=1}^{r-1}|C_i|\Big)\left(\frac{1}{n^r}\left(A_1(x(1-x))^r + A_2(x)\frac{(x(1-x))^{r-1}}{n}+\dots+A_r(x)\frac{x(1-x)}{n^{r-1}}\right)\right) \leqq$$

$$\leqq Kn^{-r}\left[(x(1-x))+\frac{1}{n}\right]^r \leqq K_1 n^{-r}\left[\sqrt{x(1-x)}+n^{-1/2}\right]^{2r}.$$

To calculate $R_{n,i}(x)$ we use formulae (4.2) and (4.3) of [2, p. 285] together with formula (7.3) here to obtain first $R_{n,i}(x)=0$ for $i=1,\dots,r$ and then for $i\geqq r+1$ we have

$$R_{n,i}(x) = \frac{1}{n^r}\sum_{j=1}^{i-r}\frac{(x(1-x))^j}{n^{i-r-j}}B_{j,i}(x)$$

or for $i \geqq r+1$,

$$|R_{n,i}(x)| \leqq B\frac{1}{n^r}\{(x(1-x))+n^{-1}\}^{i-r} \leqq B_*\frac{1}{n^r}[\max(x(1-x), n^{-1})]^{r-2r+i}.$$

We can estimate $R_n(x)$ by $R_n(x) \leqq K_2 n^{-r}[\sqrt{x(1-x)}+n^{-1/2}]^{2r}$ and this implies the following theorem.

Theorem 7.1. *For* $B_n(f, r-1, x)$ *defined by* (6.1) *and* $f(x) \in C[0, 1]$ *we have*

$$(7.4)\qquad |B_n(f, r-1, x)-f(x)| \leqq K\left\{\omega_{2r}\left(f, \left(\frac{x(1-x)}{n}+n^{-2}\right)^{1/2}\right)+\right.$$

$$\left.+\sum_{j=r+1}^{2r-1}\omega_j\left(f, n^{-r/j}\left[x(1-x)+\frac{1}{n}\right]^{(j-r)/j}\right)\right\}.$$

This result is new and was not proved in [2]. In particular for $B_n(f, 1, x) \equiv \equiv 2B_{2n}(f, x)-B_n(f, x)$ we have

$$(7.5)\qquad |2B_{2n}(f, x)-B_n(f, x)-f(x)| \leqq K\left\{\omega_4\left(f, \left(\frac{x(1-x)}{n}+n^{-2}\right)^{1/2}+\right.\right.$$

$$\left.+\omega_3\left(f, n^{-2/3}\left[x(1-x)+\frac{1}{n}\right]^{1/3}\right)\right\}.$$

We recall that for x^3 $\omega_3(f, h) \sim Kh^3$ which will fit exactly here in view of the fact that, as we observed in [2, p. 279],

$$|2B_n(f, x)-B_n(x)-f(x)| \leqq M\left(\frac{x(1-x)}{n}\right)^{\alpha/2}$$

is not equivalent to $f \in \mathrm{Lip}^* \alpha$. As a corollary of Theorem 5.1 we have:

Theorem 7.2. *For* $f \in (C, A_{2r})_\alpha$ *where*

$$A_{2r} = \{f;\ f, \dots, f^{2r-1} \in A.C._{\mathrm{loc}}(0, 1) \quad and \quad \|(x(1-x))^r f^{(2r)}(x)\| < \infty\}$$

and for $B_n(f, r-1, x)$ *given by* (6.1), *we have*

$$(6.6)\qquad \|B_n(f, r-1, x)-f(x)\|_{C[0,1]} \leqq M\frac{1}{n^{r\alpha}}.$$

This theorem is the direct theorem proved in [2, p. 284].

8. Combinations of exponential-type operators. Exponential-type operators were defined first by C. P. May in [9] and [10] by

$$(8.1)\qquad S_\lambda(f, x) = \int_A^B W(\lambda, t, u) f(u)\, du$$

where A and B may be infinite and $W(\lambda, t, u)$ is a measure in u satisfying

$$\frac{\partial}{\partial t} W(\lambda, t, u) = \frac{\lambda}{p(\lambda)} W(\lambda, t, u)(u-t) \tag{8.2}$$

(where the derivative is taken in the distribution sense). C. P. May restricted himself to $p(t) \geqq 0$ being a polynomial of degree less than or equal to two for which many well-known applications are valid (Bernstein, Baskakov, Szász, Post-Widder and Gauss-Weierstrass). Later ISMAIL and MAY [7] showed that if $p(t) \geqq 0$ is analytic in (A, B), we still have some of the properties and results of [10]. MAY [9], [10] proved that for combinations of exponential-type operators local, direct and inverse theorems are valid and ISMAIL and MAY [7] showed that a local direct and inverse theorem is valid for $S_\lambda(f, x) - f(x)$ (no combinatons). We will show that in those cases global direct theorems follows Theorems 4.1 and 5.1. (The global result in this case is new.) The result in section 6(c) and the result in section 7 about Bernstein polynomials are included in this but the result in section 7 is important, being the motivating result for much of this paper; and in fact Bernstein polynomials were the motivation for exponential-type operators.

We are now ready to define the combinations of $S_\lambda(f, x)$ for finite, fixed but arbitrary constants $d_0, \ldots, d_k$:

$$S_\lambda(f, k, x) = \sum_{j=0}^{k} C(j, k) S_{d_j \lambda}(f, x) \tag{8.3}$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i} \quad k \neq 0 \quad \text{and} \quad C(0, 0) = 1. \tag{8.4}$$

We are now in a position to state and prove our result.

Theorem 8.1. *For* $f \in C[A, B]$ *abd* $f \in (C, A(k+1))_\alpha$ *where* $A(k+1) = \{f; f, \ldots, f^{(2k+1)}$ *are absolutely continuous locally in* (A, B) *and* $\Phi_k(f) = \|p(x)^{k+1} f^{(2k+2)}(x)\|_{C[A,B]} < \infty\}$, *when* $p(x) \geqq 0$ *is a polynomial of degree* 2, *we have*

$$\|S_\lambda(f, k, x) - f(x)\|_{C[A,B]} \leqq M \lambda^{-(k+1)\alpha}. \tag{8.5}$$

For other analytic posititve $p(x)$ where $\sqrt{p(x)}$ behaves like the $\varphi(x)$ of definition 4.1 near the boundary points A and B, we have (8.5) for $k=0$.

Remark 8.2. (a) ISMAIL and MAY [7] do not deal with the convergence of combinations of the operators there but, following their properties 2.2 of [7, p. 448], some of these results will still be valid. Here we just want to show the applicability of our earlier result and not get involved in various generalizations of particular situations.

(b) As MAY [10] and ISMAIL and MAY [7] observed, and as was also observed earlier in this paper, exponential behaviour of the functions is allowed in case A or B (or both) are not finite.

Proof. The key to the proof is proposition 3.2 of May's paper [10], p. 227]. The moments

$$A_m(\lambda, t) = \lambda^m \int_A^B W(\lambda, t, u)(u-t)^m \, du \tag{8.6}$$

are studied and, using the recursion relation

$$A_{m+1}(\lambda, t) = \lambda m p(t) A_{m-1}(\lambda, t) + p(t) \frac{d}{dt} A_m(\lambda, t), \tag{8.7}$$

May showed that $A_m(\lambda, t)$ are polynomials in λ (and in t when $p(t)$ is a polynomial of degree less than or equal to 2) of degree [m/2] in λ and that the coefficient of λ^m in $A_{2m}(\lambda, t)$ is $cp(t)^m$ and in $A_{2m+1}(\lambda, t)$ is $c(t)p(t)^m$. What is not exactly stated but still follows from (8.7) is that $\lambda^{-2k} A_{2k}(\lambda, t)$ is a sum of the type

$$\frac{p(t)^k}{\lambda^k} + c_1(t) \frac{p(t)^{k-1}}{\lambda^{k-1}} + c_2(t) \frac{p(t)^{k-2}}{\lambda^{k-2}} + \dots$$

and that $\lambda^{-2k-1} A_{2k}(\lambda, t)$ is a sum of the type $\frac{p(t)^k}{\lambda^{k+1}} + c_1(t) \frac{p(t)^{k-1}}{\lambda^{k-1}} + \dots$ where if the corresponding γ in Theorem 4.1 is 1, $c_1(t)$ may have a zero at the boundary ($c_2(t)$ a double zero, etc.). Observing that the combinations in (8.3) will cause $\sum_{j=0}^{k} c(j, k) \frac{1}{(d_j k)^l} = 0$ for $l \leqq k$, we will following the Bernstein polynomials case, obtain the correct estimate on the moments. For $p(t)$ analytic we just claim that if $\sqrt{p(t)}$ satisfies the condition on $\varphi(x)$ in theorem 4.1, then the result is valid, which is obvious as $\lambda^{-1} p(t) = D_\lambda(t)$ and the first moment is equal to 0.

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Contractions as restricted shifts

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Dedicated to Professzor K. Tandori on the occasion of his 60th birthday

In what follows T is a contraction (i.e. linear operator with $\|T\| \leqq 1$) on a Hilbert space $\mathscr{H}$ and "c.n.u." stands for "completely nonunitary".

A familiar result (cf. [2], [1]) guarantees that, if $T^n \to 0$ strongly as $n \to \infty$, then T is unitarily equivalent to a restriction of a backward shift. If T is c.n.u., on a complex separable space, then its functional model [3, Ch. VI, Sec. 2] shows that T is unitarily equivalent to a restriction of the orthogonal sum of a backward shift and a bilateral shift.

The purpose of this note is to generalize Rota's construction [2] to the case of an arbitrary c.n.u. contraction. We shall give the shift operators in question, the space of the restriction as well as the operator which provides the unitary equivalence to a certain extent explicitely in terms of T. Maybe Lemma 1 is of own independent interest. Our method is elementary and self-containing in the sense that it uses only very standard facts of operator theory.

Since $\{T^{*n}T^n\}_{n=0}^{\infty}$ is a decreasing sequence of selfadjoint operators, its strong limit exists, is a positive contraction and therefore

$$A = \left(\lim_{n\to\infty} T^{*n}T^n\right)^{1/2}$$

exists, A is selfadjoint and $0 \leqq A \leqq I$. Similarily, the selfadjoint operator

$$\hat{A} = \left(\lim_{n\to\infty} T^n T^{*n}\right)^{1/2}$$

exists and $0 \leqq \hat{A} \leqq I$. We define an operator V on $\overline{A\mathscr{H}}$ by

$$VAh = ATh \quad (h \in \mathscr{H})$$

and then by taking closure. The definition of A shows that $T^*A^2T = A^2$ and thus

$$\|ATh\| = \|Ah\| \quad (h \in \mathscr{H}),$$

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consequently V is an isometry on $\overline{A\mathscr{H}}$.

Lemma 1. $\|AV^n g\|\uparrow\|g\|$ *and* $\|AV^{*n}g\|\downarrow\|\hat{A}Ag\|$ *for* $g\in\overline{A\mathscr{H}}$ *as* $n\to\infty$.

Proof. For $h\in\mathscr{H}$ we have

$$\|AVAh\| = \|A^2Th\| \geqq \|T^*A^2Th\| = \|A^2h\|$$

and this implies that

$$\|AVf\| \geqq \|Af\| \quad \text{for} \quad f\in\overline{A\mathscr{H}}.$$

Substituting $f=V^n g$, we obtain

$$\|AV^{n+1}g\| \geqq \|AV^n g\| \quad (g\in\overline{A\mathscr{H}}, \quad n=0,1,\ldots).$$

This shows that $\{\|AV^n g\|\}_{n=0}^{\infty}$ is an increasing sequence.

Now let P_λ denote the spectral projection of A belonging to $[0, \lambda]$. If $0\leqq\lambda<1$ and $h\in\mathscr{H}$, then

$$\begin{aligned}\|Ah\|^2 &= \|AT^n h\|^2 = \|P_\lambda AT^n h\|^2+\|(I-P_\lambda)AT^n h\|^2 = \\ &= \|AP_\lambda T^n h\|^2+\|A(I-P_\lambda)T^n h\|^2 \leqq \lambda^2\|P_\lambda T^n h\|^2+\|(I-P_\lambda)T^n h\|^2 = \\ &= (\lambda^2-1)\|P_\lambda T^n h\|^2+\|T^n h\|^2.\end{aligned}$$

So we get

$$0 \leqq (1-\lambda^2)\|P_\lambda T^n h\|^2 \leqq \|T^n h\|^2-\|Ah\|^2 \to 0 \quad \text{as} \quad n\to\infty.$$

This implies that for each $h\in\mathscr{H}$ and $0\leqq\lambda<1$,

$$\|P_\lambda T^n h\| \to 0 \quad \text{as} \quad n\to\infty.$$

Thus

$$\|P_\lambda V^n Ah\| = \|P_\lambda AT^n h\| \leqq \|A\|\,\|P_\lambda T^n h\| \to 0 \quad \text{as} \quad n\to\infty.$$

So we have

$$\begin{aligned}\|AV^n Ah\|^2 &\geqq \|(I-P_\lambda)AV^n Ah\|^2 = \|A(I-P_\lambda)V^n Ah\|^2 \geqq \lambda^2\|(I-P_\lambda)V^n Ah\|^2 = \\ &= \lambda^2\|V^n Ah\|^2-\lambda^2\|P_\lambda V^n Ah\|^2 \to \lambda^2\|Ah\|^2 \quad (n\to\infty).\end{aligned}$$

This means that, for each $h\in\mathscr{H}$ and $\varepsilon>0$,

$$\|Ah\| \geqq \|AV^n Ah\| \geqq (1-\varepsilon)\|Ah\|$$

if n is sufficiently large, i.e.:

$$\|AV^n Ah\| \to \|Ah\| \quad (h\in\mathscr{H}, \quad n\to\infty).$$

Now for $\varepsilon>0$ and $g\in\overline{A\mathscr{H}}$ there exists $h\in\mathscr{H}$ such that $\|g-Ah\|<\varepsilon$ and for sufficiently large n we have

$$0 \leqq \|g\|-\|AV^n g\| \leqq \big|\|g\|-\|Ah\|\big|+\big|\|Ah\|-\|AV^n Ah\|\big|+\big|\|AV^n Ah\|-\|AV^n g\|\big| <$$
$$< \|g-Ah\|+\varepsilon+\|AV^n\|\,\|Ah-g\| < 3\varepsilon.$$

So the first statement of the lemma is proved. In order to prove the second one, let $g\in\overline{A\mathscr{H}}$ and $h\in\mathscr{H}$. Then we have

$$(AV^{*n}g, h) = (g, V^n Ah) = (g, AT^n h) = (T^{*n}Ag, h)$$

and consequently

$$(1) \qquad AV^{*n}g = T^{*n}Ag \quad (g\in\overline{A\mathscr{H}}, \quad n = 1, 2, \dots).$$

Thus, using the definition of $\hat{A}$ we obtain

$$\|AV^{*n}g\| = \|T^{*n}Ag\|\downarrow\|\hat{A}Ag\| \quad (g\in\overline{A\mathscr{H}}, \quad n\to\infty),$$

and so the lemma is proved.

Let us introduce an operator B: $\overline{A\mathscr{H}}\to\overline{A\mathscr{H}}$ by

$$B = (I-A\hat{A}^2A)^{1/2}|\overline{A\mathscr{H}}$$

and a linear manifold $\mathscr{H}_0$ by

$$\mathscr{H}_0 = \{h\in\mathscr{H}\colon\ Ah\in\operatorname{Ran} B\}.$$

Lemma 2. *B commutes with V and V^*, and $T\mathscr{H}_0\subset\mathscr{H}_0$. If T is c.n.u., then* $\operatorname{Ker} B=\{0\}$ *and $\mathscr{H}_0$ is dense in $\mathscr{H}$.*

Proof. Using the fact that $\hat{A}^2=T\hat{A}^2T^*$ (which is an easy consequence of the definition of $\hat{A}$) and (1), for $g\in\overline{A\mathscr{H}}$ we obtain

$$V^*A\hat{A}^2Ag = V^*AT\hat{A}^2T^*Ag = V^*VA\hat{A}^2AV^*g = A\hat{A}^2AV^*g.$$

This shows that $A\hat{A}^2A|\overline{A\mathscr{H}}$ commutes with V^*. Therefore B, being the limit of a sequence of polynomials of $A\hat{A}^2A|\overline{A\mathscr{H}}$, also commutes with V^*. Since B is self-adjoint, it commutes with V, too.

If $h\in\mathscr{H}_0$, then

$$ATh = VAh\in V(\operatorname{Ran} B) = \operatorname{Ran}(BV)\subset\operatorname{Ran} B$$

and therefore $T\mathscr{H}_0\subset\mathscr{H}_0$.

Suppose now that T is c.n.u. and $f\in\operatorname{Ker} B$. Then

$$0 = \|Bf\|^2 = \big((I-A\hat{A}^2A)f, f\big) = \|f\|^2-\|\hat{A}Af\|^2,$$

i.e.:

$$(2) \qquad \|\hat{A}Af\| = \|f\|.$$

Since A and $\hat{A}$ are contractions, this implies that

$$\|Af\| = \|f\| \tag{3}$$

and, since $0 \leqq A \leqq I$, $Af=f$ follows. Thus (2) implies

$$\|\hat{A}f\| = \|f\|. \tag{4}$$

Using the definitions of A and $\hat{A}$, (3) and (4) imply

$$\|T^n f\| = \|f\| = \|T^{*n} f\| \quad (n = 1, 2, \ldots).$$

Since by assumption T is c.n.u., $f=0$ follows [3, Ch. I, Th. 3.2].

In order to prove the last statement of the lemma, suppose that T is c.n.u. and $h \in \operatorname{Ker}(I - A^2\hat{A}^2)$. In this case

$$0 = \big((I - A^2\hat{A}^2)h, h\big) = \|h\|^2 - (\hat{A}^2 h, A^2 h)$$

and thus

$$\|h\|^2 = |(\hat{A}^2 h, A^2 h)| \leqq \|\hat{A}^2 h\| \|A^2 h\| \leqq \|\hat{A}h\| \|Ah\| \leqq \|h\|^2.$$

This shows that $\|\hat{A}h\| \|Ah\| = \|h\|^2$ and therefore

$$\|Ah\| = \|h\| = \|\hat{A}h\|$$

which implies

$$\|T^n h\| = \|h\| = \|T^{*n} h\| \quad (n = 1, 2, \ldots)$$

and consequently $h=0$.

So we have proved that, if T is c.n.u., then $\operatorname{Ker}(I - A^2\hat{A}^2) = \{0\}$ and consequently $\operatorname{Ran}(I - \hat{A}^2 A^2)$ is dense in $\mathscr{H}$. Since

$$A(I - \hat{A}^2 A^2) = (I - A\hat{A}^2 A)A = B^2 A,$$

we obtain

$$A \operatorname{Ran}(I - \hat{A}^2 A^2) = B^2 A \mathscr{H},$$

i.e.: $\mathscr{H}_0 \supset \operatorname{Ran}(I - \hat{A}^2 A^2)$ and so $\mathscr{H}_0$ is also dense in $\mathscr{H}$. This completes the proof of the lemma.

Let us introduce the following notations:

$$D = (I - T^*T)^{1/2}, \quad \hat{D} = (I - TT^*)^{1/2},$$

$$V_n = \begin{cases} V^n & \text{if} \quad n \geqq 0 \\ V^{*-n} & \text{if} \quad n < 0. \end{cases}$$

Clearly $V_n^* = V_{-n}$, and (1) implies that

$$T^* A V_n g = A V_{n-1} g \quad (g \in \overline{A\mathscr{H}}, \quad n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

Lemma 3. *If T is c.n.u., then*

$$\sum_{n=-\infty}^{\infty} \|\hat{D}AB^{-1}V_n f\|^2 = \|f\|^2 \quad \textit{for} \quad f \in \operatorname{Ran} B.$$

Proof. If $f \in \operatorname{Ran} B$, i.e. $f = Bg$ with a $g \in \overline{A\mathscr{H}}$ then, by Lemma 2 and the definition of V_n, we have

$$V_n f = V_n Bg = BV_n g \quad (n = 0, \pm 1, \ldots).$$

Since by assumption T is c.n.u., Lemma 2 implies the existence of B^{-1} on Ran B. Let $N, M \geqq 0$. Using (5) we obtain

$$\sum_{n=-M}^{N} \|\hat{D}AB^{-1}V_n f\|^2 = \sum_{n=-M}^{N} \|\hat{D}AV_n g\|^2 =$$

$$= \sum_{n=-M}^{N} [(V_{-n}A^2 V_n g, g) - (V_{-n}ATT^* AV_n g, g)] =$$

$$= \sum_{n=-M}^{N} [\|AV_n g\|^2 - \|AV_{n-1} g\|^2] = \|AV^N g\|^2 - \|AV^{*M+1} g\|^2.$$

Now Lemma 1 implies that

$$\sum_{n=-\infty}^{\infty} \|\hat{D}AB^{-1}V_n f\|^2 = \|g\|^2 - \|\hat{A}Ag\|^2 = \|Bg\|^2 = \|f\|^2,$$

and so the lemma is proved.

Now define $\mathscr{K}$ by

$$\mathscr{K} = [\bigoplus_{m=0}^{\infty} \overline{D\mathscr{H}}] \oplus [\bigoplus_{n=-\infty}^{\infty} \overline{\hat{D}A\mathscr{H}}].$$

Let S denote the operator on $\mathscr{K}$ defined by

$$S\{[\bigoplus_{m=0}^{\infty} h_m] \oplus [\bigoplus_{n=-\infty}^{\infty} h_n']\} = [\bigoplus_{m=0}^{\infty} h_{m+1}] \oplus [\bigoplus_{n=-\infty}^{\infty} h_{n+1}'].$$

This S is the orthogonal sum of a backward shift and a bilateral shift. An easy computation shows that

$$\sum_{m=0}^{\infty} \|DT^m h\|^2 = \sum_{m=0}^{\infty} [\|T^m h\|^2 - \|T^{m+1} h\|^2] = \|h\|^2 - \|Ah\|^2 \quad (h \in \mathscr{H}).$$

In what follows suppose that T is c.n.u. Then the above formula and Lemma 3 imply that

$$\text{(6)} \qquad \sum_{m=0}^{\infty} \|DT^m h\|^2 + \sum_{n=-\infty}^{\infty} \|\hat{D}AB^{-1}V_n Ah\|^2 = \|h\|^2 \quad \text{for} \quad h \in \mathscr{H}_0.$$

Thus the linear manifold $\mathscr{H}_0'$, defined by

$$\mathscr{H}_0' = \{[\bigoplus_{m=0}^{\infty} DT^m h] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_n Ah] \colon h \in \mathscr{H}_0\},$$

exists and is contained in $\mathscr{K}$. Let $\mathscr{H}'$ denote the closure of $\mathscr{H}_0'$ in $\mathscr{K}$. Define a mapping $U_0 \colon \mathscr{H}_0 \to \mathscr{H}_0'$ by

$$U_0 h = [\bigoplus_{m=0}^{\infty} DT^m h] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_n Ah] \quad (h \in \mathscr{H}_0).$$

Our main result is the following

Theorem. *If T is c.n.u. then, using the above notations, U_0 extends to a unitary operator $U \colon \mathscr{H} \to \mathscr{H}'$, $\mathscr{H}'$ is an invariant subpace for S and $UT = (S|\mathscr{H}')U$.*

Proof. The definition of U_0 and (6) show that U_0 is linear and isometric. Since, by Lemma 2, Dom U_0 is dense in $\mathscr{H}$ and, by the definition of $\mathscr{H}'$, Ran U_0 is dense in $\mathscr{H}'$, U_0 extends by continuity to a unitary operator $U \colon \mathscr{H} \to \mathscr{H}'$.

If $h \in \mathscr{H}_0$ then, by Lemma 2, $Th \in \mathscr{H}_0$ and so we have

$$SUh = SU_0 h = [\bigoplus_{m=0}^{\infty} DT^{m+1} h] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_{n+1} Ah] =$$

$$= [\bigoplus_{m=0}^{\infty} DT^m (Th)] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_n A(Th)] = U_0 Th = UTh.$$

Therefore, by continuity, for every element h of $\mathscr{H}$ we have $SUh = UTh$ and, since $U\mathscr{H} = \mathscr{H}'$, we can conclude that $\mathscr{H}'$ is invariant for S and $UTh = (S|\mathscr{H}')Uh$ for $h \in \mathscr{H}$. So the theorem is proved.

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Homomorphically complete classes of automata with respect to the α_2-product

Z. ÉSIK

Dedicated to Professor K. Tandori on his 60th birthday

Homomorphically complete classes of automata with respect to the general product were characterized by A. A. LETIČEVSKIĬ in [8]. In order to decrease the complexity of the general product F. Gécseg introduced the concept of α_i-products in [5]. The notion of α_0-product coincides with that of the loop-free product used by J. HARTMANIS (cf. [7]). It is known that there exists no homomorphically complete finite class of automata for the α_0- or α_1-product (cf. [4]). Using a result in [3], P. DÖMÖSI (cf. [1]) succeeded in proving that there is a single automaton homomorphically complete with respect to the α_2-product. In the present paper we show that a class of automata is homomorphically complete with respect to the α_2-product if and only if it is homomorphically complete with respect to the general product. Thus, Letičevskiĭ's criterion can be used to describe those classes which are homomorphically complete with respect to the α_2-product. Our result can also be used to show that for every $i\geqq 2$, the α_i-product is homomorphically as general as the general product (cf. [2]).

By an automaton we shall always mean a finite automaton. Given a finite system $\mathbf{A}_t=(A_t, X_t, \delta_t)$ $(t=1, \ldots, n,\ n\geqq 1)$ of automata together with a finite set of input signs X and a family of feedback functions $\varphi_t\colon A_1\times\ldots\times A_n\times X\to X_t$ $(t=1, \ldots, n)$ we can form the general product (cf. [6]) $\prod(\mathbf{A}_1, \ldots, \mathbf{A}_n|\varphi)=(A_1\times\ldots\times A_n, X, \delta)$ where $\delta((a_1, \ldots, a_n), x)=(\delta_1(a_1, x_1), \ldots, \delta_n(a_n, x_n))$, provided that $a_t\in A_t$, $x\in X$, $x_t=\varphi_t(a_1, \ldots, a_n, x)$ $(t=1, \ldots, n)$. If $i\geqq 0$ is a given integer and none of the feedback functions φ_t depends on the states a_s with $t+i\leqq s\leqq n$, then we come to the notion of α_i-products introduced in [5]. Further, if for each t, φ_t only depends on its last variable (the input sign) then we get the concept of the quasi-direct product. If all the $\mathbf{A}_t$-s coincide then we speak about a general, α_i- or quasi direct power

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according to the cases described above. We say that an automaton $\mathbf{A}=(A, X, \delta)$ homomorphically realizes an automaton $\mathbf{B}=(B, X, \delta')$ if $\mathbf{B}$ is a homomorphic image of a subautomaton of $\mathbf{A}$. A class $\mathscr{K}$ of automata is called homomorphically complete with respect to the general product (homorphically complete, for short) if every automaton can be homomorphically realized by a general product of automata belonging to $\mathscr{K}$. Homorphically α_i-complete classes are similarly defined. By Letičevskiĭ's result in [8], a class of automata is homomorphically complete if and only if it contains an automaton $\mathbf{A}=(A, X, \delta)$ having states $a_0, a_1, a_1' \in A$ such that $a_1 \neq a_1'$, further, $\delta(a_0, x)=a_1$, $\delta(a_0, x')=a_1'$, $\delta(a_1, p)=a_0$ and $\delta(a_1', p')=a_0$ hold for some input signs $x, x' \in X$ and strings $p, p' \in X^*$.

We are going to show that homomorphically complete classes with respect to the α_2-product are exactly the homomorphically complete classes. For this reason we have to prove that if an automaton satisfies Letičevskiĭ's criterion then it is homomorphically complete with respect to the α_2-product. Let us denote by $\mathbf{U}=(U, \{x_1, x_2\}, \delta)$ an automaton with the following properties:

(i) $U=\{u_0, \ldots, u_{k_1-1}\} \cup \{u_0', \ldots, u_{k_2-1}'\}$ where $k_1, k_2 \geqq 1$, $k_1>1$ or $k_2>1$, further, $u_0=u_0'$, $u_i \neq u_j$ if $i \neq j$ $(0 \leqq i, j<k_1)$ and $u_i' \neq u_j'$ if $i \neq j$ $(0 \leqq i, j<k_2)$,

(ii) $\delta(u_0, x_1)=u_1$, $\delta(u_0', x_2)=u_1'$, $\delta(u_i, x_j)=u_{i+1}$ $(i=1, \ldots, k_1-1,\ j=1, 2)$, $\delta(u_i', x_j)=u_{i+1}'$ $(i=1, \ldots, k_2-1, j=1, 2)$ where we have used the notations $u_{k_1}=u_0$ and $u_{k_2}'=u_0$,

(iii) $u_1 \neq u_1'$.

It is obvious that if an automaton $\mathbf{A}$ satisfies Letičevskiĭ's criterion then for some k_1 and k_2 an automaton $\mathbf{U}$ having properties (i), (ii) and (iii) above can be isomorphically embedded into an α_1-power of $\mathbf{A}$ with a single factor. Therefore, if each automaton $\mathbf{U}$ is homomorphically complete with respect to the α_2-product then so is $\mathbf{A}$. In this way it is enough to show that any automaton $\mathbf{U}$ is homomorphically complete with respect to the α_2-product.

In the next two lemmas we fix an automaton $\mathbf{U}$ and denote by k the l.c.m. of k_1 and k_2. For every integer i we shall denote by u_i and u_i' the states u_r resp. u_s' with $r \in \{0, \ldots, k_1-1\}$, $s \in \{0, \ldots, k_2-1\}$ and such that $i \equiv r \pmod{k_1}$ and $i \equiv s \pmod{k_2}$. First we prove that all the automata $\mathbf{S}_m=(\{1, \ldots, mk\}, \{x_1, x_2\}, \delta)$ can be homomorphically realized by α_2-powers of $\mathbf{U}$, where the transitions in $\mathbf{S}_m$ are defined by $\delta(i, x_1)=j$ if and only if $j \equiv i+1 \pmod{mk}$ and

$$\delta(i, x_2)=\begin{cases} j & \text{where } j \equiv i+1 \pmod{mk} \quad \text{if} \quad i \not\equiv 0 \pmod{k}, \\ 1 & \text{if} \quad i \equiv 0 \pmod{k}. \end{cases}$$

Lemma 1. *Each automaton* $\mathbf{S}_m$ $(m \geqq 1)$ *can be homomorphically realized by an* α_2*-power of* $\mathbf{U}$.

Proof. Let $\mathbf{C}=(\{1, \ldots, k\}, \{x\}, \delta_C)$ be a counter, i.e. $\delta_C(i, x) \equiv i+1 \pmod{k}$. It is quite obvious that **C** is isomorphic to a quasi-direct power of **U** with two factors. We define an α_2-product $\mathbf{A}=(A, \{x_1, x_2\}, \delta)=\prod(\mathbf{C}, \underbrace{\mathbf{U}, \dots, \mathbf{U}}_{mk \text{ times}}|\varphi)$ with

$$\varphi_1(i, v_1, x_j) = x,$$

$$\varphi_{1+t}(i, v_1, \dots, v_t, v_{t+1}, x_j) = \begin{cases} x_2 & \text{if } v_{t+1} = u_1', \\ x_1 & \text{otherwise,} \end{cases}$$

$$\varphi_{1+mk}(i, v_1, \dots, v_{mk}, x_1) = \begin{cases} x_2 & \text{if} \quad i\in\{1, \dots, k-1\} \quad \text{and} \quad v_{mk-i+1} = u_1', \\ & \text{or} \quad i = k, \quad (v_1, \dots, v_{mk}) = (u_1', \dots, u_k', u_1, \dots, u_{(m-1)k}), \\ x_1 & \text{otherwise,} \end{cases}$$

$$\varphi_{1+mk}(i, v_1, \dots, v_{mk}, x_2) = \begin{cases} x_2 & \text{if} \quad i\in\{1, \dots, k-1\} \quad \text{and} \quad v_{mk-i+1} = u_1', \\ & \text{or} \quad i = k, \\ x_1 & \text{otherwise,} \end{cases}$$

where $1\leqq t<mk$, $i\in\{1, \dots, k\}$, $v_1, \dots, v_{mk}\in U$ and $j=1$ or $j=2$.

Let B consist of those elements $(i, v_1, \dots, v_{mk})\in A$ for which there exist an integer $j\in\{1, \dots, mk\}$ and $v_1', \dots, v_{(m+1)k}'\in U$ satisfying the following three conditions:

(i) $i \equiv j \pmod k$, $v_{mk-j+1} = u_1'$,

(ii) $(v_{tk+1}', \dots, v_{(t+1)k}') = (u_1, \dots, u_k)$ or $(v_{tk+1}', \dots, v_{(t+1)k}') = (u_1', \dots, u_k')$,

(iii) $(v_1, \dots, v_{mk}) = (v_{i+1}', \dots, v_{i+mk}')$.

It is not difficult to check that $\mathbf{B}=(B, \{x_1, x_2\}, \delta)$ is a subautomaton of **A** and $\mathbf{S}_m$ is a homomorphic image of **B** under the mapping ψ: $B\to\{1, \dots, mk\}$ defined by $\psi((i, v_1, \dots, v_{mk}))=\min\{j|1\leqq j\leqq mk, i\equiv j \pmod k, v_{mk-j+1}=u_1'\}$.

Note that φ_1 was independent of its variables, therefore our contruction gives rise to a homomorphic realization of $\mathbf{S}_m$ by an α_2-power of **U**. As none of the functions φ_{1+t} $(1\leqq t<mk)$ depended on the input sign $\mathbf{S}_m$ can also be homomorphically realized by an α_2-power of **U** in such a way that the feedback functions, except the last one, are independent of the input sign.

Next we show that all shift-registers can be homomorphically realized by an α_2-power of **U**. As usual, by a shift-register on a fixed alphabet $X=\{x_1, \dots, x_n\}$ we shall mean any automaton isomorphic to one of the automata $\mathbf{R}_m=(X^m, X, \delta)$ $(m\geqq 1)$, where X^m denotes the set of all strings $y_1\dots y_m$ of length m on X, and $\delta(y_1\dots y_m, y)=y_2\dots y_m y$ $(y_1, \dots, y_m, y\in X)$.

Lemma 2. *Every shift-register can be homomorphically realized by an α_2-power of* **U**.

Proof. As $\mathbf{R}_{m_1}$ is a homomorphic image of $\mathbf{R}_{m_2}$ whenever $m_1\leqq m_2$, it is enough to show that shift-registers $\mathbf{R}_{mk}$ with $m\geqq n$ can be homomorphically realized by an α_2-power of **U**.

Let $\mathbf{C}=(\{1, \dots, mk\}, \{x\}, \delta_C)$ be a counter having mk states. We shall define an α_2-product $\mathbf{A}=(A, X, \delta)=\prod(\mathbf{C}, \underbrace{\mathbf{U}, \dots, \mathbf{U}}_{(m+n)mk^2 \text{ times}}|\varphi)$ where the last $(m+n)mk^2$ components will be treated as mk buffers b_i of length $(m+n)k$. The counter will point to the buffer used last. That is, if $i\in\{1, \dots, mk\}$ is the first component of a state of $\mathbf{A}$ then b_i contains the input sign arrived for the last time. Buffers are used in a circular way: if $i<mk$ then b_{i+1}, otherwise b_1 is the buffer available next. Consequently, the mk signs arrived last will be contained by buffers $b_{i+1}, \dots, b_{mk}, b_1, \dots, b_i$ in this order. We shall use the states $u_1\neq u_1'$ to encode a sign by the fixed mapping $\tau\colon X\to\{u_1, u_1'\}^n$, $\tau(x_j)=u_1^{j-1}u_1'u_1^{n-j}$ $(j=1, \dots, n)$. Therefore, in order to store a sign x_j into the next available buffer we shall set the $(m+j)k$-th component of this buffer to u_1', and set all the $(m+j')k$-th components for $j'=1, \dots, j-1, j+1, \dots, n$ to u_1. During this transition all already stored input signs will be shifted with one place to the left, the values of the first components of the buffers underflow.

Now we put this into a precise form by defining the feedback functions of the product. For every $i\in\{1, \dots, mk\}$, $v_1, \dots, v_{mk}\in U$, $j\in\{1, \dots, n\}$ and $t(1\leqq t<(m+n)mk^2)$ we put $\varphi_1(i, v_1, x_j)=x$,

$$\varphi_{1+t}(i, v_1, \dots, v_t, v_{t+1}, x_j)=\begin{cases} x_2 & \text{if } v_{t+1}=u_1', \\ x_1 & \text{otherwise,}\end{cases}$$

provided that $t\not\equiv 0 \pmod k$, and if $t\equiv 0 \pmod k$ then

$$\varphi_{1+t}(i, v_1, \dots, v_t, v_{t+1}, x_j)=$$

$$=\begin{cases} x_2 \text{ if } t=i'(m+n)k-(n-j)k \text{ where } i'\in\{1, \dots, mk\} \text{ is determined by } i+1\equiv i' \\ \quad \pmod{mk}, \\ \quad \text{or } v_{t+1}=u_1' \text{ and there exists an integer } i'\in\{1, \dots, mk\} \text{ with } i+1\equiv i' \\ \quad \pmod k,\ (i'-1)(m+n)k<t<i'(m+n)k, \\ \quad \text{or there exist } i'\in\{1, \dots, mk\},\ r\in\{2, \dots, k\} \text{ such that } i+r\equiv i' \pmod k, \\ \quad v_{t-k+r}=u_1' \text{ and } (i'-1)(m+n)k<t\leqq i'(m+n)k, \\ x_1 \text{ otherwise,}\end{cases}$$

and similarly,

$$\varphi_{1+m(m+n)k^2}(i, v_1, \dots, v_{m(m+n)k^2}, x_j)=$$

$$=\begin{cases} x_2 \text{ if } i=mk-1;\ j=n, \\ \quad \text{or there exists } r\in\{2, \dots, k\} \text{ with } i+r\equiv mk \pmod k \\ \quad \text{and } v_{m(m+n)k^2-k+r}=u_1', \\ x_1 \text{ otherwise.}\end{cases}$$

Next we give a subautomaton $\mathbf{B}=(B, X, \delta)$ of $\mathbf{A}$ and a homomorphism of $\mathbf{B}$ onto $\mathbf{R}_m$. This will be accomplished by the help of the auxiliary functions $\varrho_j\colon A\to U^n$ $(j=1, \dots, mk)$ and $\varrho\colon A\to U^{mnk}$. Suppose that $a=(i, v_1^1, \dots, v_{(m+n)k}^1, \dots, v_1^{mk}, \dots,$

$\ldots, v^{mk}_{(m+n)k})$, $j \in \{1, \ldots, mk\}$. If $j > i$ then we put

$$\varrho_j(a) = v^j_{j-i+k} \ldots v^j_{j-i+nk} \in U^n$$

else

$$\varrho_j(a) = v^j_{mk-(i-j)+k} \ldots v^j_{mk-(i-j)+nk} \in U^n.$$

By $\varrho(a)$ we shall denote the string

$$\varrho(a) = \varrho_{i+1}(a) \ldots \varrho_{mk}(a) \varrho_1(a) \ldots \varrho_i(a) \in U^{mnk}.$$

Now let B consist of all those elements $a = (i, v^1_1, \ldots, v^1_{(m+n)k}, \ldots, v^{mk}_1, \ldots, v^{mk}_{(m+n)k})$ which satisfy the following conditions:

(i) There exists a string $y_1 \ldots y_{mk} \in X^{mk}$ with $\varrho(a) = \tau(y_1) \ldots \tau(y_{mk})$,

(ii) If $j \in \{1, \ldots, mk\}$, $r \in \{1, \ldots, k\}$ and $j \equiv i + r \pmod k$ then $\{v^j_1 \ldots v^j_k, \ldots, \ldots, v^j_{(m+n-1)k+1} \ldots v^j_{(m+n)k}\} \subseteq U_r$ where U_r denotes a set of four strings:

$$U_r = \{\underbrace{u_{2-r} \ldots u_{-1} u_0}_{r-1} \underbrace{u_1 \ldots u_{k-r+1}}_{k-r+1}, \underbrace{u_{2-r} \ldots u_{-1} u_0}_{r-1} \underbrace{u'_1 \ldots u'_{k-r+1}}_{k-r+1},$$

$$\underbrace{u'_{2-r} \ldots u'_{-1} u'_0}_{r-1} \underbrace{u_1 \ldots u_{k-r+1}}_{k-r+1}, \underbrace{u'_{2-r} \ldots u'_{-1} u'_0}_{r-1} \underbrace{u'_1 \ldots u'_{k-r+1}}_{k-r+1}\},$$

(iii) If $j \in \{1, \ldots, mk\}$ then $v^j_t \ldots v^j_{(m+n)k} = u_1 \ldots u_{(m+n)k-t+1}$ where $t = j - i + nk + k$ if $j > i$ and $t = mk - (i-j) + nk + k$ if $j \leqq i$.

It can be seen that with the definition above **B** becomes a subautomaton of **A** and the mapping ψ: $B \to X^m$ determined by $\psi(a) = y_1 \ldots y_{mk}$ if and only if $\varrho(a) = \tau(y_1) \ldots \tau(y_{mk})$ is a homomorphism of **B** onto $\mathbf{R}_{mk}$. As the counter **C** is an X-subautomaton of $\mathbf{S}_m$, by Lemma 1 and the fact that φ_1 is a constant mapping, we obtain a homomorphic realization of $\mathbf{R}_{mk}$ by an α_2-power of U.

Now we are ready to state our

Theorem. *Every homomorphically complete class of automata is homomorphically complete with respect to the α_2-product.*

Proof. Given a homomorphically complete class of automata, by the result of A. Letičevskiĭ in [8], there is an automaton $\mathbf{U}_0$ in this class such that for some k_1, k_2 ($k_1, k_2 \geqq 1, k_1 \neq 1$ or $k_2 \neq 1$) the automaton **U** can be isomorphically embedded into an α_1-power of $\mathbf{U}_0$ with a single factor. Therefore it is enough to show that every automaton $\mathbf{A} = (A = \{a_1, \ldots, a_m\}, X = \{x_1, \ldots, x_n\}, \delta)$ can be homomorphically realized by an α_2-power of this automaton **U**. In order to prove this statement we form an α_2-product $\mathbf{B} = (B, X, \delta') = \prod(\mathbf{R}_{mk}, \mathbf{S}_m, \underbrace{\mathbf{U}, \ldots, \mathbf{U}}_{mk \text{ times}} | \varphi)$ where $\mathbf{R}_{mk}$ and $\mathbf{S}_m$ are the automata described previously, and for any $y_1 \ldots y_{mk} \in X^{mk}$, $i \in \{1, \ldots, mk\}$, $v_1, \ldots, v_{mk} \in U$,

$j \in \{1, \ldots, n\}$ and $t (1 \leqq t < mk)$

$$\varphi_1(y_1 \ldots y_{mk}, i, x_j) = x_j,$$

$$\varphi_2(y_1 \ldots y_{mk}, i, v_1, x_j) = \begin{cases} x_2 & \text{if } v_1 = u_1' \text{ and } i \equiv 0 \pmod{k}, \\ x_1 & \text{otherwise,} \end{cases}$$

$$\varphi_{2+t}(y_1 \ldots y_{mk}, i, v_1, \ldots, v_t, v_{t+1}, x_j) =$$

$$= \begin{cases} x_2 & \text{if } \quad v_{t+1} = u_1', \\ & \text{or } \quad t \equiv 0 \pmod{k} \text{ and } v_{t-r+1} = u_1' \text{ for an integer} \\ & \qquad r \in \{1, \ldots, k-1\} \text{ with } i \equiv r \pmod{k}, \\ & \text{or } \quad v_1 = u_1', \; i \equiv 0, \; t \equiv 0 \pmod{k} \text{ and } \delta(a_{i/k}, y_{mk-i+2} \ldots y_{mk} x_j) = a_{t/k}, \\ x_1 & \text{otherwise,} \end{cases}$$

and similarly,

$$\varphi_{2+mk}(y_1 \ldots y_{mk}, i, v_1, \ldots, v_{mk}, x_j) =$$

$$= \begin{cases} x_2 & \text{if } \quad v_{mk-r+1} = u_1' \text{ for an } r \in \{1, \ldots, k-1\} \text{ satisfying } i \equiv r \pmod{k}, \\ & \text{or } \quad v_1 = u_1', \; i \equiv 0 \pmod{k} \text{ and } \delta(a_{i/k}, y_{mk-i+2} \ldots y_{mk} x_j) = a_m, \\ x_1 & \text{otherwise.} \end{cases}$$

Let $C \subseteq B$ contain all states $b = (y_1 \ldots y_{mk}, i, v_1, \ldots, v_{mk}) \in B$ with the following property: there are $r \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, m\}$ such that $i \equiv r \pmod{k}$, $tk + i - r \leqq \leqq mk$ and

(i) $v_{tk-r+1} \ldots v_{mk} v_1 \ldots v_{tk-r} = u_1' \ldots u_k' u_{k+1} \ldots u_{mk}$ if $i \geqq k$, and

(ii) $v_{tk-r+1} \ldots v_{tk} v_1 \ldots v_{tk-r} v_{(t+1)k-r+1} \ldots v_{mk} v_{tk+1} \ldots v_{(t+1)k-r} = u_1' \ldots u_k' u_{k+1} \ldots u_{mk}$ if $i < k$. It is easy to show that $\mathbf{C} = (C, X, \delta')$ is a subautomaton of $\mathbf{B}$. Indeed, assume that $b \in C$ and the integers t and r are determined as previously, and let $y \in X$. Then $\delta'(b, y) = (y_2 \ldots y_{mk} y, i', v_1', \ldots, v_{mk}')$ where i' and $v_1', \ldots, v_{mk}'$ are determined according to the three cases below:

Case 1. If $r \neq k$ and $i > k$; or $r = k$ and $t \neq 1$ then $i' = i + 1$, $v_1' = v_2, \ldots, \ldots, v_{mk-1}' = v_{mk}$, $v_{mk}' = v_1$. (Observe that now $k \leqq i < mk$.)

Case 2. If $r \neq k$ and $i < k$ then $i' = i + 1$, $v_1' = v_2, \ldots, v_{tk-1}' = v_{tk}$, $v_{tk}' = v_1$, $v_{tk+1}' = v_{tk+2}, \ldots, v_{mk-1}' = v_{mk}$, $v_{mk}' = v_{tk+1}$.

Case 3. If $r = k$ and $t = 1$ then $i' = 1$, $v_1' = v_2, \ldots, v_{sk-1}' = v_{sk}$, $v_{sk}' = v_1$, $v_{sk+1}' = = v_{sk+2}, \ldots, v_{mk-1}' = v_{mk}$, $v_{mk}' = v_{sk+1}$ where $s \in \{1, \ldots, m\}$ is determined by $\delta(a_{i/k}, y_{mk-i+2} \ldots y_{mk} y) = a_s$. It can be checked that $b' \in C$ in all the three cases above.

In order to complete the proof we have to give a homomorphism ψ of $\mathbf{C}$ onto $\mathbf{A}$. Let $b = (y_1 \ldots y_{mk}, i, v_1, \ldots, v_{mk}) \in C$ be artibtrary. Then there are uniquely determined integers $r \in \{1, \ldots, k\}$ and $t \in \{1, \ldots, m\}$ fulfilling $i \equiv r \pmod{k}$, $tk + i - r \leqq mk$ and such that either condition (i) or (ii) holds according to $i \geqq k$ or $i < k$. Put

$\psi(b)=\delta(a_{(tk+i-r)/k}, y_{mk-i+1}\dots y_{mk})$. Then, corresponding with the previously listed three cases, one can easily verify that ψ is a homomorphism. On the other hand ψ is obviously surjective.

We have seen that **A** is homomorphically realized by **B**. From this the result follows by the lemmas, the fact that $\mathbf{S}_m$ was homomorphically realized by an α_2-power of **U** in such a way that with the exception of the last feedback function none of the feedback functions depended on the input sign, further, by observing that in our construction of **B**, φ_1 only depends on the input sign. This ends the proof of the Theorem.

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Representation of functions in the space φ(L) by Vilenkin series

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To Professor K. Tandori on his 60th birthday

1. Let Φ be the set of all even real functions, which are nondecreasing on $[0, +\infty)$ and have the following properties:

$$\text{(i)} \qquad \varphi(0)=\varphi(+0)=0$$

$$\text{(ii)} \qquad \varphi(x)>0 \quad (x>0)$$

$$\text{(iii)} \qquad \varphi(2x)=O(\varphi(x)) \quad (x\to+\infty) \quad (\varphi\in\Phi).$$

(The last property is called "Δ_2-condition".) For every $\varphi\in\Phi$ let us define the space $\varphi(L)$ as the set of measurable and almost everywhere finite functions f defined on $[0, 1]$, for which

$$\|f\|_\varphi := \int_0^1 \varphi(f(x))\,dx < +\infty$$

holds. If the functions f, g belong to $\varphi(L)$, then let their φ-distance be defined as $\|f-g\|_\varphi$, which determines the φ-convergence in the usual way. It is well-known [1] that $\varphi(L)$ is a linear space if and only if the Δ_2-condition holds. Furthermore, as special cases we get the L_p spaces for $0<p<+\infty$ $\big(\varphi(x):=|x|^p\ (x\in\mathbf{R})\big)$, the Orlicz spaces (if φ is convex), the space of a.e. finite functions with the convergence in measure $\left(\varphi(x):=\dfrac{|x|}{1+|x|}\ (x\in\mathbf{R})\right)$.

The system of functions $g_n\in\varphi(L)$ $(n\in\mathrm{N}:=\{0, 1, \ldots\})$ is called a system of representation in $\varphi(L)$, if for every $f\in\varphi(L)$ there exists a series $\sum a_k g_k$ with coefficients a_n $(n\in\mathrm{N})$ such that $\lim\limits_{n\to\infty}\|f-\sum\limits_{k=0}^{n} a_k g_k\|_\varphi=0$. We remark that the uniqueness of such series for all f is not assumed. If this holds too, then the system is a Schauder basis. The following problem is due to P. L. ULJANOV [1]: by what means can be

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characterized the spaces $\varphi(L)$, in which the classical systems of functions are systems of representation? He himself gave in [2] a necessary and sufficient condition for this with respect to the Faber—Schauder system. The analogous question was answered by P. Oswald [3], [4] for $\varphi(L)\subset L_1:=L_1[0, 1]$ and for the trigonometric, resp. the Haar system. In [5] we formulated whithout proof the next statement.

Theorem 1. *If* $\varphi\in\Phi$, $\varphi(L)\not\subset L_1$ $\left(\textit{i.e.}\ \liminf_{x\to+\infty}\frac{\varphi(x)}{x}=0\right)$, $p\geq 1$ *and* $\limsup_{x\to+\infty}\frac{\varphi(x)}{x^p}<+\infty$, *then every orthogonal basis in* L_p *is a system of representation in* $\varphi(L)$, *whereas the representation is not unique.*

The aim of this work is to solve the above mentioned Uljanov's problem with respect to the Vilenkin systems [6]. To the definition of these systems we fix a sequence of natural numbers $m=(m_0, m_1, \dots)$ for which $m_k\geq 2$ $(k\in\mathbf{N})$ holds. Define the group G_m as the set of all sequences $x=(x_0, x_1, \dots)$ $(0\leq x_k<m_k,\ x_k\in\mathbf{N},\ k\in\mathbf{N})$ with the group-operation $x+y:=((x_0+y_0)(\mathrm{mod}\, m_0),\ (x_1+y_1)(\mathrm{mod}\, m_1), \dots)$ $(x, y\in G_m)$. The topology of G_m is given by the neighborhoods $I_n(x):=\{y\in G_m:\ y_0=x_0, \dots, y_{n-1}=x_{n-1}\}$ $(x\in G_m, n\in\mathbf{N})$, thus G_m forms a compact Abelian group. Let us introduce in G_m the normalized Haar measure. If $M_0:=1$, $M_{k+1}:=m_kM_k$ $(k\in\mathbf{N})$, then the group G_m can be transformed in the interval $[0, 1]$ by means of the following mapping

$$G_m\ni x\mapsto \sum_{j=0}^{\infty}\frac{x_j}{M_{j+1}}\in[0, 1].$$

It is easy to see that this correspondence is almost one-to-one and measure-preserving.

The system of characters of G_m can be given in the following way. For $k\in\mathbf{N}$ define the function r_k as

$$r_k(x):=\exp\frac{2\pi i x_k}{m_k}\quad (x\in G_m, i:=\sqrt{-1})$$

and arrange the finite products of r_k's as follows. If $n\in\mathbf{N}$, then there exists a unique representation

$$n=\sum_{k=0}^{\infty}n_kM_k\quad (0\leq n_k<m_k,\ n_k\in\mathbf{N},\ k\in\mathbf{N}).$$

Let $\psi_n:=\prod_{k=0}^{\infty}r_k^{n_k}$, then the functions ψ_n are uniformly bounded and form a complete orthonormal system in L_1, which is called Vilenkin system (generated by the sequence m).

It is known, [7] [8], [16] that every Vilenkin system is a Schauder basis in L_p $(1<p<+\infty)$, from which it follows by means of interpolation the same statement for all reflexive Orlicz spaces. Taking into account Theorem 1 and the fact that the Vilenkin systems are bases in L_p $(1<p<\infty)$ we get

Theorem 2. *The assumptions* $\varphi\in\Phi$, $\varphi(L)\not\subset L_1$ *imply that all Vilenkin systems are systems of representation in* $\varphi(L)$. *(The representation is not unique.)*

In the case $\varphi(L)\subset L_1$ the Vilenkin systems may be at most Schauder bases in $\varphi(L)$, since they are uniformly bounded systems of functions. In this connection P. Oswald [9] showed that if a complete orthonormal system of uniformly bounded functions is basis in $\varphi(L)$ (for some $\varphi\in\Phi$), then $\varphi(L)$ is equivalent to an Orlicz space. (We consider L_1 as Orlicz space too.) It remains to answer only the question, in what Orlicz spaces are the Vilenkin systems bases? We know that the reflexivity of the space is sufficient for this. The next theorem shows that this condition is also necessary.

Theorem 3. *The Vilenkin systems are Schauder bases in a separable Orlicz space if and only if the space is reflexive.*

Furthermore, it follows from Theorem 2 and 3 the next statement.

Theorem 4. *If* $\varphi\in\Phi$, *then the Vilenkin systems are systems of representation in* $\varphi(L)$ *if and only if either* $\varphi(L)\not\subset L_1$ *or* $\varphi(L)$ *is equivalent to a reflexive Orlicz space.*

2. To the proof of Theorem 1 we need the following lemma.

Lemma 1. *Let* $1\leqq p<\infty$ *and the orthogonal system* $(g_n, n\in\mathbf{N})$ *be basis in* L_p *and* $\varphi\in\Phi$ *such that* $\liminf\limits_{x\to+\infty}\dfrac{\varphi(x)}{x}=0$, $\limsup\limits_{x\to+\infty}\dfrac{\varphi(x)}{x^p}<+\infty$. *Then for all* $f\in\varphi(L)$, $\varepsilon>0$ *and* $N\in\mathbf{N}$ *there exist* $R\in\mathbf{N}$ *and a polynomial* $P=\sum\limits_{k=N}^{R}a_kg_k$ *with respect to the system* $(g_n, n\in\mathbf{N})$, *for which*

$$\|f-P\|_\varphi\leqq\varepsilon \tag{1}$$

and

$$\Big\|\sum_{k=N}^{M}a_kg_k\Big\|_\varphi\leqq A_\varphi\|f\|_\varphi+\varepsilon\quad(N\leqq M\leqq R), \tag{2}$$

where the constant $A_\varphi>0$ *depends only on* φ.

Proof. It suffices to show that the statement is valid for the function $f=\alpha\chi$, where $\alpha\in\mathbf{R}$ and χ is the characteristic function of an arbitrary closed subinterval $[a, b]$ of $[0, 1]$. To this end define the functions u_n $(n\in\mathbf{N})$ as follows

$$u_n(x):=\begin{cases}-n & \left(x\in\left[0,\dfrac{1}{n+1}\right)\right)\\ 1 & \left(x\in\left[\dfrac{1}{n+1},1\right)\right)\end{cases}$$

and let $u_n(x+1)=u_n(x)$ $(x\in\mathbf{R})$. Thus

$$\int_0^1 u_n=0,\ \|u_n\|_p=\Big(\int_0^1 |u_n|^p\Big)^{1/p}\leqq 2n^{1-1/p}. \tag{3}$$

We can suppose (see [10]) that

$$\varphi(x+y)\leqq C_\varphi(\varphi(x)+\varphi(y)),\ \int_0^1\varphi(f(x))\,dx\leqq\psi(\|f\|_p)\quad (x,y\geqq 0,\ f\in L_p), \tag{4}$$

where $\psi\in\Phi$ is a suitable function and $C_\varphi>0$ is a constant depending only on φ. Since $\liminf\limits_{x\to+\infty}\dfrac{\varphi(x)}{x}=0$, thus for all $\varepsilon>0$ there exits $n\in\mathbf{N}$ such that

$$\frac{\varphi(\alpha(n+1))}{n+1}\leqq\frac{\varepsilon}{4C_\varphi}. \tag{5}$$

Denote C_p the Banach constant in L_p with respect to the system $(g_n, n\in\mathbf{N})$, i.e. for all series $\sum\alpha_k g_k$ we have that

$$\Big\|\sum_{k=0}^{M}\alpha_k g_k\Big\|_p\leqq C_p\Big\|\sum_{k=0}^{\infty}\alpha_k g_k\Big\|_p\quad (M\in\mathbf{N}).$$

Choose $j\in\mathbf{N}$ so that

$$\psi\left(4^{1/p}C_p\alpha\frac{n^{1-1/p}}{j^{1/p}}\right)\leqq\varepsilon. \tag{6}$$

Let $\bigcup\limits_{k=1}^{j}\Delta_k$ be a decomposition of $[a,b]$, where Δ_k's are disjoint intervals and the length of Δ_k $(k=1,\dots,j)$ is $\dfrac{b-a}{j}$. Furthermore, denote χ_k the characteristic function of Δ_k $(k=1,\dots,j)$. If $t_1,\dots,t_j$ are natural numbers having the property $t_k\geqq T(j)$ $(k=1,\dots,j)$ with some $0<T(j)\in\mathbf{N}$, then applying the Fejér lemma [11] (p. 77) we get from (5) that

$$\begin{aligned}\int_0^1\varphi\Big(\alpha\chi(x)-\sum_{k=1}^{j}\alpha\chi_k(x)u_n(t_kx)\Big)\,dx &=\sum_{k=1}^{j}\int_{\Delta_k}\varphi(\alpha(1-u_n(t_kx)))\,dx\leqq\\ \leqq 2\sum_{k=1}^{j}\frac{b-a}{j}\int_0^1\varphi(\alpha(1-u_n(x)))\,dx &\leqq 2\frac{\varphi(\alpha(n+1))}{n+1}\leqq\frac{\varepsilon}{2C_\varphi}.\end{aligned} \tag{7}$$

In virtue of (3) we have for fixed $s\in\mathbf{N}$ that

$$\lim_{t_k\to+\infty}\int_0^1\chi_k u_{nk}g_s=0\quad (1\leqq k\leqq j),$$

where the function u_{nk} is defined by $u_{nk}(x):=u_n(t_kx)$ $(x\in\mathbf{R})$. Because of this and since the system $(g_k, k\in\mathbf{N})$ is a basis in L_p, there exist natural numbers R_k; N_k and polynomials

$$p_k := \sum_{s=N_k}^{R_k} a_s g_s \quad (N_k < R_k < N_{k+1}, k\in\mathbf{N})$$

such that if $N_k \leqq M \leqq R_k$ $(M\in\mathbf{N})$, then

$$\begin{gathered}\|\sum_{s=N_k}^{M} a_s g_s\|_p^p \leqq C_p^p \|\alpha\chi_k u_{nk}\|_p^p \leqq \\ \leqq C_p^p |\alpha|^p \int_{\Delta_k} |u_{nk}|^p \leqq 2C_p^p |\alpha|^p \frac{b-a}{j} \int_0^1 |u_n|^p \leqq 4C_p^p |\alpha|^p \frac{n^{p-1}}{j}\end{gathered} \tag{8}$$

and

$$\|\alpha\chi_k u_{nk} - p_k\|_\varphi \leqq \frac{\varepsilon}{2jC_\varphi^j}.$$

We shall show that

$$P := \sum_{k=1}^{j} p_k = \sum_{k=1}^{j} \sum_{s=N_k}^{R_k} a_s g_s$$

is the desired polynomial. Indeed, in virtue of (4), (7) and (8) we have that

$$\|\alpha\chi - \sum_{k=1}^{j} p_k\|_\varphi \leqq C_\varphi \big(\|\alpha\chi - \sum_{k=1}^{j} \alpha\chi_k u_{nk}\|_\varphi + C_\varphi^{j-1} \sum_{k=1}^{j} \|\alpha\chi_k u_{nk} - p_k\|_\varphi\big) \leqq \varepsilon \tag{10}$$

and thus inequality (1) is proved.

Let S_M be the Mth $(M\in\mathbf{N})$ partial sum of P, i.e.

$$S_M := \sum_{k=1}^{q-1} p_k + \sum_{s=N_q}^{M} a_s g_s \quad (2 \leqq q \leqq j, N_q \leqq M \leqq R_q, q\in\mathbf{N}).$$

Then

$$\|S_M\|_\varphi \leqq C_\varphi \big(\|\sum_{k=1}^{q-1} p_k\|_\varphi + \|\sum_{s=N_q}^{M} a_s g_s\|_\varphi\big) =: C_\varphi (J_1 + J_2). \tag{11}$$

As in the proof of (1) we obtain

$$\begin{gathered} J_1 = \|\sum_{k=1}^{q-1} p_k\|_\varphi \leqq \\ \leqq C_\varphi^2 \big(\|\alpha \sum_{k=1}^{q-1} \chi_k\|_\varphi + \|\alpha \sum_{k=1}^{q-1} \chi_k (1-u_{nk})\|_\varphi + \|\sum_{k=1}^{q-1} \alpha\chi_k u_{nk} - p_k\|_\varphi\big) \leqq C_\varphi^2 (\|\alpha\chi\|_\varphi + 2\varepsilon).\end{gathered} \tag{12}$$

From (4), (6) and (9) it follows that

$$J_2 = \|\sum_{s=N_q}^{M} a_s g_s\|_\varphi \leqq \psi\big(\|\sum_{s=N_q}^{M} a_s g_s\|_p\big) \leqq \varepsilon. \tag{13}$$

Using the estimations (11), (12) and (13) we get (2), which completes the proof of Lemma 1.

Proof of Theorem 1. Let $f\in\varphi(L)$. Applying Lemma 1 we consider the series $\sum a_s g_s=\sum p_k$, where

$$\|f-\sum_{k=0}^{n} p_k\|_\varphi \leqq 2^{-n} \quad (n\in\mathbf{N}) \quad\text{and}\quad \|\sum_{s=N_n}^{M} a_s g_s\|_\varphi \leqq A_\varphi(\|f-\sum_{k=0}^{n-1} p_k\|_\varphi+2^{-n})$$

$$(N_n \leqq M \leqq R_n,\ M\in\mathbf{N}).$$

It is not hard to see that this series converges to f in $\varphi(L)$. Theorem 1 is proved.

3. Let n be a natural number not less than 2. Denote Z_n the discrete cyclic group of order n, i.e. $Z_n:=\{0, 1, \ldots, n-1\}$. Furthermore, let

$$p_{s,n}(t) = \sum_{j=0}^{s} c_j \exp\frac{2\pi ijt}{n} \quad (t, s\in Z_n)$$

be a discrete trigonometric polynomial of order s defined on Z_n (c_j's are arbitrary complex numbers) and $\|p_{s,n}\|_\infty:=\max_{t\in Z_n}|p_{s,n}(t)|$. We introduce the discrete measure on Z_n, i.e. let mes $\{t\}:=1/n$ $(t\in Z_n)$.

Lemma 2. *For all* $0<\alpha<1$ *and for all discrete trigonometric polynomials* $p_{s,n}$ $(0<s\in Z_n,\ n\in\mathbf{N})$ *the inequality*

$$\text{mes}\{t\in Z_n\colon |p_{s,n}(t)| \geqq \alpha\|p_{s,n}\|_\infty\} \geqq \frac{1-\alpha}{2\pi s}$$

is true.

Proof. We denote by $P_{s,n}$ the following trigonometric polynomial

$$P_{s,n}(t):= \sum_{j=0}^{s} c_j \exp\frac{2\pi ijt}{n} \quad (t\in\mathbf{R}),$$

where

$$p_{s,n}(t)= \sum_{j=0}^{s} c_j \exp\frac{2\pi ijt}{n} \quad (t\in Z_n,\ n\in\mathbf{N},\ 0<s\in Z_n)$$

is a given discrete trigonometric polynomial. Let

$$\|P_{s,n}\|_\infty:= \max_{t\in\mathbf{R}} |P_{s,n}(t)|.$$

On account of the well-known Bernstein inequality we have for the derivative of $P_{s,n}$ that

$$\|P'_{s,n}\|_\infty \leqq \frac{2\pi s}{n}\,\|P_{s,n}\|_\infty.$$

If $t_0\in[0, n)$ is a point for which $|P_{s,n}(t_0)|=\|P_{s,n}\|_\infty$, then

$$|P_{s,n}(t)| = \left|P_{s,n}(t_0)+\int_{t_0}^{t} P'_{s,n}\right| \geqq \|P_{s,n}\|_\infty\left(1-\frac{2\pi s}{n}|t-t_0|\right) \quad (t\in[0, n)).$$

Hence there exists an interval $\Delta\subset[0, n]$, the measure of which is not less than $(1-\alpha)n/\pi s$ such that

$$|P_{s,n}(t)| \geqq \|P_{s,n}\|_\infty\alpha \quad (t\in\Delta).$$

The number of the integers being in Δ is at least $[(1-\alpha)n/\pi s]$ (where $[x]$ denotes the integer part of the real number x) and since $\|P_{s,n}\|_\infty\geqq\|p_{s,n}\|_\infty$, therefore

$$\text{mes}\,\{t\in Z_n\colon\ |p_{s,n}(t)| \geqq \alpha\,\|p_{s,n}\|_\infty\} \geqq \max\left\{\frac{1}{n},\ \frac{1}{n}\left[\frac{(1-\alpha)n}{\pi s}\right]\right\} \geqq \frac{1-\alpha}{2\pi s}.$$

Thus Lemma 2 is proved.

We shall show that the analogue of Lemma 2 is true for the Vilenkin systems too.

Lemma 3. *For all* $0<\alpha<1$ *and for all Vilenkin polynomials*

$$p_n = \sum_{k=0}^{n} c_k\psi_k$$

of order $(0<)n\in\mathbf{N}$ *(c_k's are arbitrary complex numbers) the inequality*

$$\text{mes}\,\{x\in G_m\colon\ |p_n(x)| \geqq \alpha\|p_n\|_\infty\} \geqq \frac{1-\alpha}{2\pi n}$$

is true, where $\|p_n\|_\infty:=\max_{x\in G_m}|p_n(x)|$.

Proof. If p_n is the above Vilenkin polynomial and $jM_s\leqq n<(j+1)M_s$ ($n\in\mathbf{N}$, $j\in Z_{m_s}$), then

$$p_n = \sum_{k=0}^{M_s-1} c_k\psi_k+\sum_{t=1}^{j-1}\sum_{k=tM_s}^{(t+1)M_s-1} c_k\psi_k+\sum_{k=jM_s}^{n} c_k\psi_k =$$

$$= \sum_{k=0}^{M_s-1} c_k\psi_k+\sum_{t=1}^{j-1} r_s^t\sum_{k=0}^{M_s-1} c_{tM_s+k}\psi_k+r_s^j\sum_{k=0}^{n-jM_s} c_{k+jM_s}\psi_k =:\ P_0+\sum_{t=1}^{j} r_s^t P_t,$$

where the Vilenkin polynomial P_t ($t=0, \ldots, j$) depends only on the first s coordinates of the argument. Let $z\in G_m$ such that $|p_n(z)|=\|p_n\|_\infty$, then $|p_n(x)|=\|p_n\|_\infty$ $(x\in I_{s+1}(z))$ and $p_n(y)=P_0(z)+\sum_{t=1}^{j}\exp\frac{2\pi ity_s}{m_s}P_t(z)$ $\left((y\in I_s(z)\right)$. Denote p_{j,m_s} the following discrete trigonometric polynomial

$$p_{j,m_s}(t):= P_0(z)+\sum_{t=1}^{j} P_t(z)\exp\frac{2\pi itv}{m_s} \quad (v\in Z_{m_s}),$$

then $\|p_{j,m_s}\|_\infty=\|p_n\|_\infty$ and $|p_{j,m_s}(z_s)|=\|p_{j,m_s}\|_\infty$. On the other hand we have by Lemma 2 that

$$\text{mes}\,\{v\in Z_{m_s}\colon\ |p_{j,m_s}(v)|\geqq\alpha\|p_{j,m_s}\|_\infty\}\geqq\frac{1-\alpha}{2\pi j}.$$

Hence

$$\text{mes}\,\{x\in G_m\colon\ |p_n(x)|\geqq\alpha\|p_n\|_\infty\}\geqq\text{mes}\,\{x\in I_s(z)\colon\ |p_n(x)|\geqq\alpha\|p_n\|_\infty\}=$$

$$=\frac{m_s}{M_{s+1}}\text{mes}\,\{v\in Z_{m_s}\colon\ |p_{j,m_s}(v)|\geqq\alpha\|p_{j,m_s}\|_\infty\}\geqq\frac{1-\alpha}{2\pi jM_s}\geqq\frac{1-\alpha}{2\pi n},$$

which proves our lemma.

We get by standard argument from Lemma 3 the next

Corollary. *If* $0<q\leqq p\leqq+\infty$, *then for all Vilenkin polynomials* p_n *of order* $(0<)n\in\mathbf{N}$ *the following inequality is valid*

$$\|p_n\|_p\leqq C_{p,q}n^{\frac{1}{q}-\frac{1}{p}}\|p_n\|_q,$$

where $C_{p,q}>0$ *depends only on* p *and* q.

We remark that the special case $1\leqq q\leqq p\leqq+\infty$ can be found in [12].

Let $n,s\in\mathbf{N}$, $n\geqq 2$, $1\leqq s<n$ and

$$K_{s,n}(t):=\sum_{j=1}^{s}\exp\frac{2\pi ijt}{n}\quad(t\in Z_n).$$

Since for $0\neq t\in Z_n$ we have $|K_{s,n}(t)|=\dfrac{\left|\sin\dfrac{\pi ts}{n}\right|}{\sin\dfrac{\pi t}{n}}$ and $(2/\pi)x\leqq\sin x\leqq x$ $(0\leqq x\leqq\pi/2)$, therefore by $|K_{s,n}(t)|=|K_{s,n}(n-t)|$ it follows that

$$\text{(14)}\qquad \text{card}\left\{t=1,\dots,n-1\colon\ |K_{s,n}(t)|\geqq\frac{2}{\pi}s\right\}\geqq2\left[\frac{n}{2s}\right]-1\quad\left(1\leqq s\leqq\left[\frac{n}{2}\right]\right).$$

A simple calculation shows the existence of an absolute constant $A\geqq1$, such that

$$\text{(15)}\qquad \text{card}\,\{t=1,\dots,n-1\colon\ |K_{s,n}(t)|\geqq y\}\geqq\frac{n}{Ay}\quad\left(1\leqq s\leqq\left[\frac{n}{2}\right],\ 1\leqq y\leqq\frac{s}{A}\right).$$

Define the numbers α_k $(k\in\mathbf{N})$ as follows. If $m_k\geqq 6A$, then let $\alpha_k=1$. If k,h are natural numbers such that $m_k\geqq6A$, $m_{k+h}\geqq6A$ but $m_{k+j}<6A$ $(0<j<h)$, then let $\alpha_{k+j}=0$ (if j is even) and $\alpha_{k+j}=1$ (if j is odd). Let us consider now the set of natural

numbers having the form

$$(16)\qquad N_n := \sum_{k=0}^{n-1} \alpha_k \left[\frac{m_k}{2}\right] M_k + a_n M_n \quad \left(1 \leqq n\in \mathbf{N},\ 1 \leqq a_n \leqq \left[\frac{m_n}{2}\right]\right),$$

where in the case $m_n \geqq 6A$ let $a_n \geqq 3A$. Thus $a_n M_n \leqq N_n < (a_n+1) M_n$ and

$$(17)\qquad \frac{N_{n+1}}{N_n} \leqq \max_{k\in \mathbf{N}} \frac{(3A+1)M_{k+1}}{\left[\frac{m_k}{2}\right] M_k} \leqq 3(3A+1).$$

Let $D_n := \sum_{k=0}^{n-1} \psi_k$ $(n\in\mathbf{N})$ the nth Dirichlet kernel with respect to the Vilenkin system. To the proof of Theorem 3 we need the following lemma.

Lemma 4. *If N_n $(n\in\mathbf{N})$ is of the form as in* (16), *then*

$$\lambda_{N_n}(x) := \operatorname{mes}\{z\in G_m\colon |D_{N_n}(z)| \geqq x\} \geqq \frac{C}{x} \quad \left(1 \leqq x \leqq \frac{N_n}{\pi}\right).$$

(Here and later on $C>0$ denotes an absolute constant.)

Proof. If $z\in G_m$, then (see e.g. [7])

$$(18)\qquad |D_{N_n}(z)| = \left| \sum_{k=0}^{n} \sum_{j=1}^{t_k} \exp \frac{2\pi i j z_k}{m_k} D_{M_k}(z) \right|,$$

where $t_k := \alpha_k \left[\frac{m_k}{2}\right]$ $(0 \leqq k \leqq n-1)$ and $t_n := a_n$. It is also known [6] that

$$(19)\qquad D_{M_k}(z) = \begin{cases} M_k & (z\in I_k) \\ 0 & (z\in G_m \setminus I_k) \end{cases} \quad (k\in\mathbf{N}).$$

(I_k stands for $I_k(0) = \{y\in G_m\colon y_0=0, \dots, y_{k-1}=0\}$.) Let $jM_s \leqq x < (j+1)M_s$ $\left(s=0, 1, \dots, n,\ 1 \leqq j \leqq \frac{1}{A}\left[\frac{m_s}{2}\right[-2 \text{ for } s<n \text{ and } 1 \leqq j \leqq \frac{a_n}{A} - 2 \text{ for } s=n,\ j\in\mathbf{N}\right)$, where we assume as the first case that $m_s \geqq 6A$. Then by (15), (18) and (19) it follows for suitable $z\in I_s \setminus I_{s+1}$ that

$$|D_{N_n}(z)| \geqq M_s \left| \sum_{t=1}^{t_s} \exp \frac{2\pi i t z_s}{m_s} \right| - \sum_{k=0}^{s-1} \alpha_k \left[\frac{m_k}{2}\right] M_k \geqq (j+2)M_s - M_s \geqq (j+1)M_s \geqq x$$

and

$$(20)\quad \lambda_{N_n}(x) \geqq \operatorname{mes}\{z\in I_s \setminus I_{s+1}\colon |D_{N_n}(z)| \geqq (j+1)M_s\} \geqq \frac{1}{M_{s+1}} \frac{m_s}{A(j+2)} \geqq \frac{C}{x}.$$

Now, let $jM_n \leqq x < (j+1)M_n$, $x \leqq \frac{N_n}{\pi}$, $m_n \geqq 6A$ and $\frac{a_n}{A} - 1 \leqq j \leqq a_n$. Then for suitable

$z \in I_n \setminus I_{n+1}$ we get by (14), (18) and (19) that

$$|D_{N_n}(z)| \geqq \frac{2}{\pi} a_n M_n - M_n \geqq \frac{a_n+1}{\pi} M_n \geqq \frac{N_n}{\pi} \geqq x$$

and

$$\lambda_{N_n}(x) \geqq \operatorname{mes}\left\{z \in I_n \setminus I_{n+1}\colon |D_{N_n}(z)| \geqq \frac{N_n}{\pi}\right\} \geqq \left(2\left[\frac{m_n}{2a_n}\right]-1\right)\frac{1}{M_{n+1}} \geqq \frac{C}{x}.$$

If $M_n \leqq x \leqq \frac{N_n}{\pi}$ and $m_n < 6A$, then

$$\lambda_{N_n}(x) \geqq \operatorname{mes}\{z \in G_m\colon |D_{N_n}(z)| \geqq N_n\} \geqq \frac{1}{M_{n+1}} \geqq \frac{C}{x}. \tag{21}$$

Finally, let $jM_s \leqq x < (j+1)M_s$, $s \leqq n-1$, $m_s < 6A$ and $1 \leqq j \leqq m_s - 1$. If $\alpha_s = 0$, then there exist five cases: 1) $s \leqq n-1$ and $m_{s+1} \geqq 6A$, 2) $s \leqq n-2$, $m_{s+1} < 6A$ and $m_{s+2} \geqq 6A$, 3) $s = n-1$ and $m_n < 6A$, 4) $s \leqq n-3$, $m_{s+1} < 6A$ and $m_{s+2} < 6A$, 5) $s = n-2$, $m_{n-1} < 6A$ and $m_n < 6A$. In the case 1) we get by (20)

$$\lambda_{N_n}(x) \geqq \operatorname{mes}\{z \in I_{s+1} \setminus I_{s+2}\colon |D_{N_n}(z)| \geqq 2M_{s+1}\} \geqq \frac{1}{M_{s+1}} \geqq \frac{C}{x}.$$

The case 2) follows by same argument. In the case 3) it follows from (21) that $\lambda_{N_n}(x) \geqq 1/M_{n+1} \geqq C/x$. We get similarly the case 5). Hence it remains only the case $\cdot$). Since $\alpha_{s+1} \neq 0$, $\alpha_{s+2} = 0$ and for $z \in I_{s+2} \setminus I_{s+3}$

$$D_{N_n}(z) = \sum_{k=0}^{s+1} \alpha_k \left[\frac{m_k}{2}\right] M_k \geqq M_{s+1} \geqq x$$

is true, therefore it follows $\lambda_{N_n}(x) \geqq 1/2M_{s+2} \geqq C/x$.

If $\alpha_s = 1$, then $\alpha_{s+1} = 0$ or $m_{s+1} \geqq 6A$ and these cases can be examined as above. Since we showed already that $\lambda_{N_n}(M_s) \geqq C/M_s$ $(0 \leqq s \leqq n)$, therefore for $jM_s \leqq x < (j+1)M_s$, $0 \leqq s \leqq n-1$, $m_s \geqq 6A$ and $\frac{1}{A}\left[\frac{m_s}{2}\right] - 1 \leqq j \leqq m_s - 1$ we get $\lambda_{N_n}(x) \geqq \lambda_{N_n}(M_{s+1}) \geqq C/x$. This completes the proof of Lemma 4.

Proof of Theorem 3. It is well-known that the Vilenkin systems are not bases in L_1. (This follows from Lemma 4 too.) Let L_M be a separable Orlicz space generated by the N-function M and let $p := M'$. Furthermore, let N be the conjugate function of M in Young's sense and

$$\|f\|_M := \sup \int_0^1 fg \quad (f \in L_M)$$

where the supremum is taken over all g, for which $\int_0^1 N(g) \leqq 1$ is true. (For more details see e.g. [13].) If the Vilenkin system is a basis in the space L_M, then applying

Lemma 3 for $\alpha:=1/2$ it can be shown by same argument as in [14] that for the Dirichlet kernels the following estimation holds

$$\|D_n\|_M \leqq C \inf_x \frac{n+M(x)}{x} \leqq \tilde{C} \frac{n}{M^{-1}(n)} \quad (n\in\mathbf{N}).$$

On the other hand, we get by Lemma 4 (as in [14] again) for the indices N_n $(n\in\mathbf{N})$

$$\|D_{N_n}\|_M \geqq Cp(x) \ln \frac{N_n}{xp(x)} \quad (xp(x) \geqq 1).$$

Therefore $xp(x)\geqq 1$ implies $p(x)\ln \frac{N_n}{xp(x)} \leqq C \frac{N_n}{M^{-1}(N_n)}$ $(n\in\mathbf{N})$. In virtue of the Δ_2-condition and (17) this estimation holds for all $n\in\mathbf{N}$, from which the reflexivity of L_M follows by similar method as in [15]. Thus Theorem 3 is proved.

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On the everywhere divergence of Vilenkin—Fourier series

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction

In this paper we are concerned with everywhere divergence of Vilenkin—Fourier series. The Vilenkin systems are generalizations of the Walsh system. It is well known that A. N. Kolmogoroff gave the first example for integrable function with everywhere divergent Fourier series. The corresponding question for Wals system wash solved by F. SCHIPP [4], [5], and the construction for arbitrary Vilenkin system due to P. SIMON [7]. There are many interesting new results in this theme. For example S. SH. GALSTIAN [2] proved the existence of an integrable function with everywhere divergent Fourier series, the Fourier coefficients of which tend to zero so rapid as possible. It is natural to ask whether the analogue theorem is true for Vilenkin systems or not. We show that a similar theorem is valid for the so called bounded Vilenkin systems. Our theorem is sharper than the theorem concerning the trigonometric system [2], because we construct an appropriate function, which is not only integrable, but belongs to a function class connected with a Hardy type space, too. In our construction the Vilenkin polynomials introduced by P. Simon [6], [7] play important role. We give proof only for the case of Vilenkin groups, but this proof can be easily transferred for the case of [0, 1).

2. Preliminaries

Let $m:=(m_k, k\in\mathrm{N})$ $(\mathrm{N}=0, 1, \ldots)$ be a sequence of natural numbers, the terms of which are not less than 2. Denote by Z_{m_k} $(k\in\mathrm{N})$ the discrete cyclic group of order m_k. We define the so called Vilenkin group G_m as the direct product of Z_{m_k}'s $(k\in\mathrm{N})$. Thus G_m is a compact Abelian group, the elements of which are represented in the

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form $x=(x_0, x_1, ..., x_k, ...)$ $(0 \leqq x_k < m_k,\ x_k,\ k \in \mathbf{N})$. μ denotes the normalized Haar measure on G_m.

Introduce the next notations:

$$M_0 := 1, \quad M_{k+1} := \prod_{i=1}^{k} m_i \quad (k \in \mathbf{N}).$$

It is clear that every $n \in \mathbf{N}$ can be uniquely rewritten in the form

$$n = \sum_{k=0}^{\infty} n_k M_k \quad (0 \leqq n_k < m_k, \quad n_k \in \mathbf{N}).$$

We shall need the following subsets of G_m:

$$I_n(x) := \{y \in G_m | y_k = x_k, \quad k < n\} \quad (n \in \mathbf{N}, \quad x \in G_m).$$

Obviously $\mu(I_n(x)) = M_n^{-1}$. Let $\hat{G}_m := \{\psi_n, n \in \mathbf{N}\}$ the character system of G_m ordered in the Walsh—Paley sense, i.e.

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k},$$

where

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (k \in \mathbf{N}, x \in G_m, i := \sqrt{-1}).$$

It is known that $\hat{G}_m$ is a complete orthonormal system with respect to μ. The Vilenkin system is said to be bounded if $\lim\sup m < \infty$.

Denote $L(G_m)$ the space of μ-integrable functions and define the norm of $f \in L(G_m)$ as $\|f\|_1 := \int_{G_m} |f| \, d\mu$. If $f \in L(G_m)$ then let

$$\hat{f}(k) := \int_{G_m} f \bar{\psi}_k \, d\mu \quad (k \in \mathbf{N})$$

the k-th Vilenkin—Fourier coefficient of f,

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in \boldsymbol{P} := 1, 2, ...)$$

he n-th partial sum of Vilenkin—Fourier series of f,

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \boldsymbol{P})$$

the n-th Dirichlet kernel with respect to the Vilenkin system $\hat{G}_m$. Define the functiont $\tau_h f$ $(h \in G_m,\ f \in L(G_m))$ as follows

$$\tau_h f(x) := f(x \dot{-} h) \quad (x \in G_m),$$

where $\dot{-}$ is the inverse of the group operation which is denoted by $\dot{+}$.

It is known [8] that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \in G_m \setminus I_n(0), \end{cases} \tag{1}$$

where $\mathbf{0} := (0, 0, \ldots) \in G_m$.

We need the following notations:

$$\Delta f := \{k \in \mathbf{N} \mid \hat{f}(k) \neq 0\},$$

$$\Omega f := \sup_{n, l} |S_n f - S_l f| \quad (f \in L(G_m)).$$

Now we define a Hardy type space $H(G_m)$. Let us denote by f^* the following maximal function of $f \in L(G_m)$

$$f^* := \sup_n |S_{M_n} f| \quad (n \in \mathbf{N}).$$

We say that $f \in L(G_m)$ belongs to $H(G_m)$ if $f^* \in L(G_m)$, and let $\|f\|_H := \|f^*\|_1$. $H(G_m)$ is a Banach space with this norm. If Φ: $[0, \infty) \to [0, \infty)$ is an increasing function, then we denote by $H\Phi(H)$ the class of μ-integrable functions for which

$$f^* \Phi \circ f^* \in L(G_m)$$

($\circ$ stands for composition of functions). In this paper we prove the next statement for bounded Vilenkin systems.

Theorem. *Let Φ: $[0, +\infty) \to [0, +\infty)$ be an increasing function with $\Phi(n) = o(\log \log n)$ $(n \to +\infty)$ and $(\alpha_n, n \in \mathbf{N})$ a decreasing sequence tending to zero for which*

$$\sum_{n=0}^{\infty} \alpha_n^2 = \infty.$$

Then there exists a function $f \in H\Phi(H)$ such that

$$|\hat{f}(k)| \leqq \alpha_k \quad (k \in \mathbf{N})$$

and the Vilenkin—Fourier series of f diverges everywhere.

Remark. The Vilenkin systems are orthonormal systems with respect to the Lebesgue measure on $[0, 1)$, therefore all the concepts like Vilenkin—Fourier series, maximal function etc. can be introduced also for functions of $L[0, 1)$. The above Theorem can be formulated for this case too. It is not hard to check, that all the considerations used in the proof of Theorem can be transferred for this case. This is based on the fact, that there is a close connection between G_m and $[0, 1)$, namely

$$\lambda\colon G_m \to [0, 1), \quad \lambda(x) := \sum_{k=0}^{\infty} \frac{x_k}{M_{k+1}}$$

is an almost one-one and measure preserving mapping. C will denote an absolute positive, but not always the same constant in this paper.

3. Two lemmas

In order to prove Theorem we need two lemmas. Let us denote by P_n $(n \in \mathbf{P})$ the set of Vilenkin polynomials of order n, i.e.

$$P_n := \{g \in L(G_m) | \sup \Delta g < n\}.$$

Let $\hat{G}_m$ be an arbitrary Vilenkin system. Then the following lemma is true.

Lemma 1. *For all* $n, p \in \mathbf{N}$ *and* $0 \leqq j < M_p$ $(j \in \mathbf{N})$ *there exists a Vilenkin polynomial* $Q_{n,p,j}$ *such that*

(i) $Q_{n,p,j} \in P_{M_{M_{n+1}+p}}$,

(ii) $\|Q_{n,p,j}\|_1 = 1$,

(iii) $\Omega Q_{n,p,j}(x) > CM_p n \quad (x \in I_p(e_j))$,

$\left(\textit{where } e_j := (j_0, j_1, \ldots, j_{p-1}, 0, \ldots) \in G_m,\ jM_p^{-1} = \sum_{k=0}^{p-1} j_k M_{k+1}^{-1}\right)$,

(iv) $\operatorname{supp} Q_{n,p,j} \subset I_p(e_j)$.

Proof. Let the numbers n, p, j and the Vilenkin system $\hat{G}_m$ be fixed. Define the sequence $m' = (m'_k, k \in \mathbf{N})$ as follows

$$m'_k := m_{k+p} \quad (k \in \mathbf{N}).$$

m' generates the Vilenkin group $G_{m'}$. We shall denote by Q_n the same Vilenkin polynomial as in [7] (pp. 361—362). The corresponding polynomial with respect to $\hat{G}_{m'}$ is denoted by Q'_n. It is shown in [7] that

$$\|Q'_n\|_1 = 1, \quad \Omega Q'_n(x) > Cn, \quad Q'_n \in P_{M'_{M'_{n+1}}}$$

where

$$M'_l := \prod_{i=0}^{l-1} m'_i = \prod_{i=0}^{l-1} m_{p+i} (x \in G_m, l \in \mathbf{N}).$$

By means of Q'_n we introduce the Vilenkin polynomial $Q_{n,p}$ on G_m as follows

$$Q_{n,p}(x) := Q'_n(y), \quad \text{where} \quad y_k = x_{k+p} \quad (x \in G_m, y \in G_m, \quad k \in \mathbf{N}).$$

It is clear from the definition of $Q_{n,p}$, that if $Q'_n = \sum a_i \psi'_i$ (ψ'_i is the i-th element of $\hat{G}_{m'}$), then $Q_{n,p} = \sum a_i \psi_{iM_p}$, and $\|Q_{n,p}\|_1 = 1$, $\Omega Q_{n,p}(x) > Cn$, $Q_{n,p} \in P_{M_{M_{n+1}+p}}$ $(x \in G_m)$. Define $Q_{n,p,j} := \tau_{e_j} D_{M_p} Q_{n,p}$. Applying (1) it is easy to check that $Q_{n,p,j}$ has the desired properties. Lemma 1 is proved.

The second lemma is a modification of a lemma of Stečkin [1].

Lemma 2. *If* $(\alpha_n, n\in\mathbf{N})$ *tends monotonely to zero and* $\sum_{n=1}^{\infty}\alpha_n^2=\infty$, *then for all* $s, q\in\mathbf{N}$ *there exist* $t\in\mathbf{N}$ *and* $p_1\leqq p_2\leqq\ldots\leqq p_t$ $(p_i\in\mathbf{N},\ i=1, \ldots, t)$ *such that*

(i) $\alpha_{q+2\sum_{i=1}^{k}M_{s+p_i}} > \dfrac{1}{M_{p_k}}$ $\quad (k = 1, \ldots, t)$,

(ii) $\sum_{i=1}^{t}\dfrac{1}{M_{p_i}} = 1.$

Proof. Let s, q and $(\alpha_n,\ n\in\mathbf{N})$ be fixed. First we verify the following statement: For all $v, u\in\mathbf{N}$ there exists $r\in\mathbf{N}$ such that $\alpha_{v+2M_{u+r}}>\dfrac{1}{M_r}$. Suppose indirectly that $\alpha_{v+2M_{u+r}}\leqq\dfrac{1}{M_r}$ $(r\in\mathbf{N})$. Since $(\alpha_n,\ n\in\mathbf{N})$ is monotone, and $\limsup m<\infty$, therefore

$$\sum_{n=v+2M_u}^{\infty}\alpha_n^2=\sum_{r=0}^{\infty}\sum_{i=v+2M_{u+r}}^{v+2M_{u+r+1}}\alpha_i^2\leqq\sum_{r=0}^{\infty}2(M_{u+r+1}-M_{u+r})\frac{1}{M_r^2}<\infty,$$

but this is a contradiction, since $\sum_{n=1}^{\infty}\alpha_n^2=\infty$. Part (i) of Lemma 2 is a simple consequence of this statement.

In order to prove part (ii) let

$$p_1:=\min\left\{n\in\mathbf{N}|\alpha_{q+2M_{s+n}}>\frac{1}{M_n}\right\}$$

and

$$p_{k+1}:=\min\left\{n\in\mathbf{N}|\alpha_{q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+n}}>\frac{1}{M_n},\ n\geqq p_k\right\}\quad (k\in\mathbf{N}).$$

The existence of p_k's $(k\in\mathbf{N})$ follows from the above statement. Since $(\alpha_n,\ n\in\mathbf{N})$ is monotone, therefore from the minimum property of p_k's and from $\limsup m<\infty$ we have

$$\sum_{n=q+2\sum_{i=1}^{k}M_{s+p_i}}^{q+2\sum_{i=1}^{k+1}M_{s+p_i}-1}\alpha_n^2=\sum_{n=q+2\sum_{i=1}^{k}M_{s+p_i}}^{q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+p_k}-1}\alpha_n^2+\sum_{l=p_k}^{p_{k+1}-1}\sum_{n=q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+l}}^{q+2\sum_{i=1}^{k}M_{s+p_i}+2M_{s+l+1}-1}\alpha_n^2\leqq$$

$$\leqq 2M_{s+p_k}\frac{1}{M_{p_k-1}^2}+\sum_{l=p_k}^{p_{k+1}-1}2(M_{s+l+1}-M_{s+l})\frac{1}{M_l^2}<C_{s,m}\frac{1}{M_{p_k}}\quad (k\in\boldsymbol{P})$$

(where $C_{s,m}>0$ depends only on s and m), whence by $\sum_{n=1}^{\infty}\alpha_n^2=\infty$

$$\sum_{k=1}^{\infty}\frac{1}{M_{p_k}}=\infty.$$

On account of the monotonicity of $(p_k, k\in\mathbf{N})$ $M_{p_{k+1}}$ is divisible by M_{p_k} $(k\in\mathbf{N})$, namely there exists $l_k\in\mathbf{N}$ for which $\sum_{i=1}^{k}\frac{1}{M_{p_i}}=\frac{l_k}{M_{p_k}}$. If $l_k<M_{p_k}$ then $\sum_{i=1}^{k+1}\frac{1}{M_i}=$ $=\frac{l_k}{M_{p_k}}+\frac{1}{M_{p_{k+1}}}\leqq 1$. Thus the divergence of $\sum_{k=1}^{\infty}\frac{1}{M_{p_k}}$ implies the existence of $t\in\mathbf{N}$ such that $\sum_{k=1}^{t}\frac{1}{M_{p_k}}=1$. This completes the proof of Lemma 2.

4. Proof of Theorem

Let us fix a bounded Vilenkin system $\hat{G}_m$. Denote $(n_l,\ l\in\mathbf{N})$ a sequence of indices for which

$$\sum_{l=1}^{\infty}\frac{\Phi(2^{M_{n_l}})}{n_l}<\infty. \tag{2}$$

The existence of such a sequence follows from $\Phi(u)=o(\log\log u)$ $(u\to\infty)$. By means of $(n_l,\ l\in\mathbf{N})$ we define $(q_l,\ l\in\mathbf{N})$ inductively. Let $q_0:=0$. If q_l is given, then apply Lemma 2 for $q:=M_{q_l}$ and for $s:=M_{M_{n_l}+1}$. Thus we get the existence of $t_l\in\mathbf{N}$ and $p_{1,l}\leqq\ldots\leqq p_{t_l,l}$ $(p_{i,l}\in\mathbf{N},\ i=1,\ldots,t_l)$ for which

$$\alpha_{M_{q_l}+2\sum_{i=1}^{k}M_{M_{n_l}+1+p_{i,l}}}>\frac{1}{M_{p_{k,l}}}\quad (k=1,\ldots,t_l), \tag{3}$$

$$\sum_{k=1}^{t_l}\frac{1}{M_{p_{k,l}}}=1. \tag{4}$$

Let us see the polynomials (see Lemma 1) $Q_{n_l,p_{k,l},j_{k,l}}$ where $j_{k,l}:=M_{p_{k,l}}\sum_{i=1}^{k-1}\frac{1}{M_{p_{i,l}}}$ $(k=1,\ldots,t)$. Obviously

$$\bigcup_{k=1}^{t_l}I_{p_{k,l}}(e_{j_{k,l}})=G_m. \tag{5}$$

Define the numbers $s_{k,l}$ $(k\in\mathbf{P})$ by induction:

$$s_{1,l}:=\max\{M_{q_l},\ M_{M_{n_l}+1+p_{1,l}}\}$$

$$s_{k+1,l}:=\min\{n\in\mathbf{N}\,|\,M_{M_{n_l}+1+p_{k+1,l}}\ \text{is a divisor of } n,\quad n>\max\Delta(\psi_{s_{k,l}}Q_{n_l,p_{k,l},j_{k,l}})\}.$$

It is easy to see, that if M_t is a divisor of u $(t,u\in\mathbf{N})$ then $\psi_u\psi_v=\psi_{u+v}$ $(v<M_t)$. By reason of this the Vilenkin polynomial $F_{k,l}:=\psi_{s_{k,l}}Q_{n_l,p_{k,l},j_{k,l}}$ can be given by shift of the spectrum of $Q_{n_l,p_{k,l},j_{k,l}}$ $(k=1,\ldots,t_l)$. It is clear from the definition of $s_{k,l}$'s that

$$\min\Delta F_{1,l}>M_{q_l} \tag{6}$$

$$\Delta F_{k,l}\cap\Delta F_{h,l}=\emptyset\quad (k,h=1,\ldots,t_l,\ k\neq h). \tag{7}$$

Furthermore, since $\max \Delta F_{1,l} < M_{q_l} + 2M_{M_{n_l+1}+p_{1,l}}$ and $s_{k+1,l} - \max \Delta F_{k,l} \leqq \leqq M_{M_{n_l+1}+p_{k+1,l}}$ therefore we have by induction

(8) $$\max \Delta F_{k,l} < M_{q_l} + 2\sum_{i=1}^{k} M_{M_{n_l+1}+p_{i,l}} \quad (k = 1, \dots, t_l).$$

Obviously there exists $h_k \in \mathrm{N}$ such that

(9) $$\Delta F_{k,l} \subset [M_{h_k}, M_{h_k+1}) \quad (k = 1, \dots, t_l).$$

$F_{k,l}$ preserves evidently the nice properties of $Q_{n_l, p_{k,l}, j_{k,l}}$, i.e.

(10) $$\|F_{k,l}\|_1 = 1, \quad \Omega F_{k,l}(x) > CM_{p_{k,l}} n_l, \quad \operatorname{supp} F_{k,l} \subset I_{p_{k,l}}(e_{j_{k,l}})$$
$$\big(x \in I_{p_{k,l}}(e_{j_{k,l}}), \quad k = 1, \dots, t_l\big).$$

Let

$$f_l := \sum_{k=1}^{t_l} \frac{1}{M_{p_{k,l}} n_l} F_{k,l}.$$

If $s \in \Delta f_l$ then by (7) there exists a number k uniquely determined ($1 \leqq k \leqq t_l$) such that $s \in \Delta F_{k,l}$. Thus (3), (8) and (10) imply that

(11) $$|\hat{f}_l(s)| = \frac{1}{M_{p_{k,l}} n_l} \hat{F}_{k,l}(s) \leqq \frac{1}{M_{p_{k,l}} n_l} \|F_{k,l}\|_1 =$$
$$= \frac{1}{M_{p_{k,l}} n_l} < \alpha_s \quad (s \in \Delta F_{k,l}), \quad \text{viz.} \quad |\hat{f}_l(s)| < \alpha_s \quad (s \in \mathrm{N}).$$

By reason of (5) there exists for all $x \in G_m$ $1 \leqq k \leqq t_l$ such that $x \in I_{p_{k,l}}(e_{j_{k,l}})$ and then by (7), (10) we have

(12) $$\Omega f_l(x) \geqq \frac{1}{M_{p_{k,l}} n_l} \Omega F_{k,l}(x) > C.$$

On the other hand it follows from (9) that

$$f_l^* \leqq \sum_{k=1}^{t_l} \frac{1}{M_{p_{k,l}} n_l} F_{k,l}^* = \sum_{k=1}^{t_l} \frac{1}{M_{p_{k,l}} n_l} |F_{k,l}|.$$

The estimation

(13) $$\max |Q_n| \leqq 2^{M_{n_l}}$$

is trivial from the definition of Q_n (see [7] p. 362). Taking into consideration the construction of $F_{k,l}$'s and of f_l we have by (10), (13) that

$$\max_{x \in I_{p_{k,l}}(e_{j_{k,l}})} f_l^*(x) \leqq \frac{1}{M_{p_{k,l}} n_l} \max |F_{k,l}| \leqq 2^{M_{n_l}} \quad (k = 1, \dots, t_l).$$

Thus

$$(f_l^*\Phi(f_l^*))(x) \leqq \frac{1}{M_{p_{k,l}} n_l} |F_{k,l}(x)|\,\Phi(2^{M_{n_l}}) \quad (x\in I_{p_{k,l}}(e_{j_{k,l}})), \quad (k=1, \dots t_l),$$

consequently by reason of (4) and (10)

$$\|f_l^*\Phi(f_l^*)\|_1 \leqq \frac{\Phi(2^{M_{n_l}})}{n_l} \tag{14}$$

is valid. Now we define the sequence $(q_l,\ l\in\mathbf{N})$ inductively. Let $q_{l+1}:= \\ :=\min\{n\in\mathbf{N}|M_n>\max\Delta f_l\}$, and define f_{l+1} in the same manner as f_l $(l\in\mathbf{N})$. Denote f the following function

$$f:=\sum_{l=0}^{\infty} f_l.$$

Since $\Delta f_k\cap\Delta f_j=\emptyset$ $(k,j\in\mathbf{N},\ k\neq j)$ and $\Delta f=\bigcup_{l=0}^{\infty}\Delta f_l$, therefore by (12) we have the everywhere divergence of the Vilenkin—Fourier series of f. (11) yields

$$|\hat{f}(s)|<\alpha_s \quad (s\in\mathbf{N}),$$

furthermore (2) and (14) furnish

$$\|f^*\Phi(f^*)\|_1 \leqq \sum_{l=0}^{\infty}\|f_l^*\Phi(f_l^*)\|_1 \leqq \sum_{l=0}^{\infty}\frac{\Phi(2^{M_{n_l}})}{n_l}<\infty,$$

i.e. $f\in H\Phi(H)$. This completes the proof of Theorem.

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Metric equivalence of tree automata

FERENC GÉCSEG

To Professor K. Tandori on his 60th birthday

In [2] and [3] it has been shown that for both frontier-to-root and root-to-frontier tree automata the general product and the α_0-product have the same power from the point of view of metric completeness. In this paper we strengthen these results by showing that for both classes of tree automata mentioned above the α_0-product is metrically equivalent to the general product.

For all the notions and notations not defined in this paper we refer the reader to [2], [3] and [4].

1. Frontier-to-root tree automata

Throughout this section we use a fixed rank type R. To exclude trivial cases, it will be supposed that for an $m>0$, $m\in R$.

Let Σ and Σ^i $(i\in I)$ be ranked alphabets of rank type R, and consider the algebars $\mathcal{A}_i=(A_i, \Sigma^i)$ $(i\in I)$. Furthermore, let

$$\varphi = \{\varphi^m\colon (\prod(A_i|i\in I))^m\times\Sigma_m \to \prod(\Sigma^i_m|i\in I)|m\in R\}$$

be a family of mappings. Then by the *general product* of $\mathcal{A}_i$ $(i\in I)$ with respect to Σ and φ we mean the Σ-algebra

$$\mathcal{A} = (A, \Sigma) = \prod(\mathcal{A}_i|i\in I)[\Sigma, \varphi]$$

with $A=\prod(A_i|i\in I)$, and for arbitrary m, $\sigma\in\Sigma_m$ and $a_1, \dots, a_m\in A$

$$\mathrm{pr}_i(\sigma^{\mathcal{A}}(a_1, \dots, a_m)) = \sigma_i(\mathrm{pr}_i(a_1), \dots, \mathrm{pr}_i(a_m)),$$

where pr_i is the ith projection operator and $\sigma_i=pr_i(\varphi^m(a_1, \dots, a_m, \sigma))$. In the sequel we assume that I is given together with a linear ordering $\leqq$.

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We now define a special type of the general product. To this take the mappings φ_i^m $(m\in R, i\in I)$ given by $\varphi_i^m(a, \sigma)=\mathrm{pr}_i(\varphi^m(a, \sigma))$ $(a\in A, \sigma\in\Sigma_m)$. We say the product $\mathscr{A}$ above is an *α_0-product* if for every $i\in I$ and $m\in R$, $\varphi_i^m(a_1, \ldots, a_m, \sigma)$ $(a_1, \ldots, a_m\in A,\ \sigma\in\Sigma_m)$ is independent of $\mathrm{pr}_j(a_1), \ldots, \mathrm{pr}_j(a_m)$ $(j\in I)$ whenever $i\leqq j$.

Let K be a class of algebras of rank type R. Then the operators $\mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P}_g$, $\mathbf{P}_{gf}$, $\mathbf{P}_{\alpha_0}$ and $\mathbf{P}_{\alpha_0 f}$ are defined in the following way.

$\mathbf{H}(K)$: homomorphic images of algebras from K.

$\mathbf{S}(K)$: subalgebras of algebras from K.

$\mathbf{P}(K)$: direct products of algebras from K.

$\mathbf{P}_g(K)$: general products of algebras from K.

$\mathbf{P}_{gf}(K)$: products from $\mathbf{P}_g(K)$ with finitely many factors.

$\mathbf{P}_{\alpha_0}(K)$: α_0-products of algebras from K.

$\mathbf{P}_{\alpha_0 f}(K)$: α_0-products from $\mathbf{P}_{\alpha_0}(K)$ with finitely many factors.

Next we define the metric equivalence of the general product and the α_0-product. We say that the α_0-product is *metrically equivalent* to the general product if for arbitrary class K of finite algebras with rank type R, integer $m\geqq 0$ and DFT-transducer $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, P, A')\in\mathrm{tr}\,(\mathbf{A})$ with $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_u, A')$ and $\mathscr{A}=(A, \Sigma)\in P_{gf}(K)$ there are a $\mathscr{B}=(B, \Sigma)\in\mathbf{P}_{\alpha_0 f}(K)$, $\mathbf{B}=(\mathscr{B}, \mathbf{b}, X_u, B')$ $(\mathbf{b}\in B^u, B'\subseteqq B)$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, P', B')\in\mathrm{tr}\,(\mathbf{B})$ such that $\tau_{\mathfrak{A}}\overset{m}{=}\tau_{\mathfrak{B}}$.

Before showing that the α_0-product is metrically equivalent to the general product we recall the following result from [1].

Theorem 1. *For arbitrary class K of algebras with rank type R the equality*

$$\mathbf{HSP}_g(K) = \mathbf{HSP}_{\alpha_0}(K) = \mathbf{HSPP}_{\alpha_0}(K)$$

holds.

Using Theorem 1 we prove

Theorem 2. *The α_0-product is metrically equivalent to the general product.*

Proof. It is enough to show that for arbitrary ranked alphabet Σ of rank type R, integers $m, n\geqq 0$, Σ-algebra $\mathscr{A}=(A, \Sigma)$ in $\mathbf{P}_{gf}(K)\cap K_\Sigma$ and vector $\mathbf{a}=(a_1, \ldots, a_n)\in A^n$ there are a $\mathscr{B}=(B, \Sigma)$ in $\mathbf{P}_{\alpha_0 f}(K)\cap K_\Sigma$ and a vector $\mathbf{b}=(b_1, \ldots, b_n)\in B^n$ such that $(\mathscr{B}, \mathbf{b})$ can be mapped m-homomorphically onto $(\mathscr{A}, \mathbf{a})$. If $\mathscr{A}\in\mathbf{P}_{gf}(K)\cap K_\Sigma$ then, by Theorem 1, $\mathscr{A}$ is in $\mathbf{HSPP}_{\alpha_0 f}(K)\cap K_\Sigma$. Therefore, there is a $\mathscr{C}=(C, \Sigma)\in\mathbf{PP}_{\alpha_0 f}(K)\cap K_\Sigma$ such that a subalgebra $\mathscr{C}'=(C', \Sigma)$ of $\mathscr{C}$ can be mapped homomorphically onto $\mathscr{A}$ under a homomorphism ψ. Let us write $\mathscr{C}$ in the form $\mathscr{C}=\prod(\mathscr{A}_i|i\in I)$ $(\mathscr{A}_i=(A_i, \Sigma)\in\mathbf{P}_{\alpha_0 f}(K), i\in I)$, and for every j $(=1, \ldots, n)$ take a $c_j\in\psi^{-1}(a_j)$. Set $\mathbf{c}=(c_1, \ldots, c_n)$. Denote by J a minimal subset of I such that for arbitrary $c, c' \in C_{\mathbf{c}}^{(m)}$ there is a $j \in J$ with $\mathrm{pr}_j(c) \neq \mathrm{pr}_j(c')$. Let $\mathscr{B} = (B, \Sigma) = \prod(\mathscr{A}_j | j \in J)$, and define $b_i \in B$ by $\mathrm{pr}_j(b_i) = \mathrm{pr}_j(c_i)$ $(j \in J, i = 1, \ldots, n)$. Moreover, set $\mathbf{b} = (b_1, \ldots, b_n)$. Then $\mathscr{B} \in \mathbf{P}_{\alpha_0 f}(K) \cap K_\Sigma$ and $(\mathscr{B}, \mathbf{b})$ is m-isomorphic to $(\mathscr{C}, \mathbf{c})$. Therefore $(\mathscr{B}, \mathbf{b})$ can be mapped m-homomorphically onto $(\mathscr{A}, \mathbf{a})$.

2. Root-to-frontier tree automata

First of all we fix a finite rank type R such that $0 \notin R$. Moreover, $F_\Sigma(X_n \cup \xi)$ will denote the set of all trees from $F_\Sigma(X_n \cup \xi)$ whose frontiers contain the auxiliary variable ξ exactly once.

Let us define the path, path (p), leading from the root of a tree $p \in F_\Sigma(X_n \cup \xi)$ to the leaf ξ in the following way.

(i) If $p = \xi$ then $\mathrm{path}(p) = \xi$.

(ii) If $p = \sigma(p_1, \ldots, p_m)$ $(\sigma \in \Sigma_m, m \in R)$ and $p_j \in F_\Sigma(X_n \cup \xi)$ then $\mathrm{path}(p) = (\sigma, j)(\mathrm{path}(p_j))$.

Next we recall some concepts concerning ascending algebras which are not so well known (cf. [3]).

Let Σ be an operator domain with $\Sigma_0 = \emptyset$. A (deterministic) *ascending Σ-algebra* $\mathscr{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an m-ary ascending operation $\sigma^{\mathscr{A}}: A \to A^m$, where m is the arity of σ. The mapping $\sigma \to \sigma^{\mathscr{A}}$ will not be mentioned explicitly, but we write $\mathscr{A} = (A, \Sigma)$. If Σ is not specified then we speak about an ascending algebra. $\mathscr{A}$ is *finite* if A is finite and Σ is a ranked alphabet. Moreover, $\mathscr{A}$ has rank type R if Σ is of rank type R.

Take two ascending Σ-algebras $\mathscr{A} = (A, \Sigma)$ and $\mathscr{B} = (B, \Sigma)$. $\mathscr{B}$ is a *subalgebra* of $\mathscr{A}$ if

(i) $B \subseteq A$, and

(ii) $\sigma^{\mathscr{B}}(b) = \sigma^{\mathscr{A}}(b)$ for arbitrary $\sigma \in \Sigma$ and $b \in B$.

Again consider the ascending algebras $\mathscr{A}$ and $\mathscr{B}$ above. Moreover, let $\psi: A \to B$ be a mapping. ψ is a *homomorphism* of $\mathscr{A}$ into $\mathscr{B}$ if the equality

$$\sigma^{\mathscr{B}}(\psi(a)) = (\psi(a_1), \ldots, \psi(a_m))$$

holds for arbitrary $\sigma \in \Sigma$ and $a \in A$, where $(a_1, \ldots, a_m) = \sigma^{\mathscr{A}}(a)$. If there is a homomorphism of $\mathscr{A}$ onto $\mathscr{B}$ then $\mathscr{B}$ is a *homomorphic image* of $\mathscr{A}$.

Next we define the concept of the product of ascending algebras.

Let Σ and Σ^i $(i \in I)$ be ranked alphabets of rank type R, and consider the ascending Σ^i-algebras $\mathscr{A}_i = (A_i, \Sigma^i)$ $(i \in I)$. Furthermore, let

$$\varphi = \{\varphi^m: \prod(A_i | i \in I) \times \Sigma_m \to \prod(\Sigma_m^i | i \in I) | m \in R\}$$

be a family of mappings. Then by the *general product* of $\mathcal{A}_i$ $(i\in I)$ with respect to Σ and φ we mean the ascending Σ-algebra

$$\mathcal{A} = (A, \Sigma) = \prod (\mathcal{A}_i|i\in I)[\Sigma, \varphi]$$

with $A=\prod(A_i|i\in I)$ and for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $a\in A$

$$(\mathrm{pr}_i(\sigma^{\mathcal{A}}(a)_1), \ldots, \mathrm{pr}_i(\sigma^{\mathcal{A}}(a)_m) = \sigma_i^{\mathcal{A}_i}(\mathrm{pr}_i(a)) \quad (i\in I)$$

where $\sigma^{\mathcal{A}}(a)_j$ is the j^{th} component of $\sigma^{\mathcal{A}}(a)$ and $\sigma_i=\mathrm{pr}_i(\varphi^m(a, \sigma))$ $(i\in I)$. In the sequel we shall assume that I is given together with a linear ordering $\leqq$. (If we have more than one index set then the same notations $\leqq$ will be used for the linear ordering of each of them. This will not cause any confusion.)

To define the concept of the α_0-product of ascending algebras let us introduce the notation $\varphi_i(a, \sigma)=\mathrm{pr}_i(\varphi^m(a, \sigma))$ for arbitrary $i\in I$, $a\in A$ and $\sigma\in\Sigma_m$. We say that the product $\mathcal{A}$ above is an α_0*-product* if for arbitrary $i\in I$, φ_i is independent of its j^{th} component $(j\in I)$ whenever $i\leqq j$.

In this section the symbols $\mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P}_{gf}, \mathbf{P}_{\alpha_0}$ and $\mathbf{P}_{\alpha_0 f}$ introduced in Section 1 will be used in their original sense and they also denote the corresponding operators for ascending algebras. This double use will not cause any difficulties since their concrete meaning will be clear from the context.

We say that (regarding ascending algebras) the α_0-product is *metrically equivalent* to the product if for arbitrary class K of finite ascending algebras with rank type R, integer $m\geqq 0$, uniform deterministic root-to-frontier transducer $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a_0, P)\in\mathrm{tr}\,(\mathbf{A})$ with $\mathbf{A}=(\mathcal{A}, a_0, X_u, \mathbf{a})$ and $\mathcal{A}\in\mathbf{P}_{gf}(K)$ there are a $\mathcal{B}=(B, \Sigma)\in\mathbf{P}_{\alpha_0 f}(K)$, $\mathbf{B}=(\mathcal{B}, b_0, X_u, \mathbf{b})$ $(b_0\in B, \mathbf{b}\in P(B)^u)$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, b_0, P')\in\mathrm{tr}\,(\mathbf{B})$ such that $\tau_{\mathfrak{A}}\overset{m}{=}\tau_{\mathfrak{B}}$.

We introduce some more terminology.

For every operator domain Σ (of rank type R), $\bar{\Sigma}$ will denote the operator domain $\{(\sigma, k)|\sigma\in\Sigma_m,\ 1\leqq k\leqq m,\ m\in R\}$ of unary operators.

Take a Σ-algebra $\mathcal{A}=(A, \Sigma)$ of rank type R. Correspond to $\mathcal{A}$ the $\bar{\Sigma}$ algebra $s(\mathcal{A})=(A, \bar{\Sigma})$ given by $(\sigma, k)^{s(\mathcal{A})}(a)=\mathrm{pr}_k(\sigma^{\mathcal{A}}(a))$ $(\sigma\in\Sigma_m,\ 1\leqq k\leqq m,\ a\in A)$.

Obviously, s is a one-to-one mapping of K_Σ onto $K_{\bar{\Sigma}}$, where K_Σ is the class of all ascending Σ-algebras. Moreover, we have

Statement 1. *For arbitrary operator domain Σ of rank type R and Σ-algebras $\mathcal{A}, \mathcal{B}$ and $\mathcal{B}_i$ $(i\in I)$ we have*

(i) $\mathcal{A}=\prod(\mathcal{A}_i|i\in I)$ *if and only if* $s(\mathcal{A})=\prod(s(\mathcal{A}_i)|i\in I)$,

(ii) $\mathcal{B}$ *is a subalgebra of* $\mathcal{A}$ *if and only if* $s(\mathcal{B})$ *is a subalgebra of* $s(\mathcal{A})$,

(iii) $\mathcal{B}$ *is a homomorphic image of* $\mathcal{A}$ *if and only if* $s(\mathcal{B})$ *is a homomorphic image of* $s(\mathcal{A})$.

Next we define a restricted form of products for the above $\bar{\Sigma}$-algebras. Take a family $\mathcal{A}_i=(\mathcal{A}_i, \bar{\Sigma}^i)$ $(i\in I)$ of $\bar{\Sigma}^i$-algebras, where every Σ^i $(i\in I)$ is an operator domain of rank type R. Moreover, let Σ be an operator domain with rank type R. Then a general product (α_0-product)

$$\mathcal{A} = (A, \bar{\Sigma}) = \prod (\mathcal{A}_i|i\in I)[\bar{\Sigma}, \varphi]$$

is a *restricted product (restricted α_0-product)* if for arbitrary $i\in I$, $a\in A$ and (σ, k), $(\sigma, l)\in\bar{\Sigma}$, $\varphi_i(a, (\sigma, k))=(\sigma_i, k)$ and $\varphi_i(a, (\sigma, l))=(\sigma_i, l)$ $(\sigma_i\in\Sigma^i)$.

The following result is also obvious.

Statement 2. *The formations of the restricted product and the restricted α_0-product are transitive. Moreover, the direct product preserves both the restricted product and the restricted α_0-product.*

For arbitrary $\mathbf{Q}\in\{\mathbf{P}_g, \mathbf{P}_{\alpha_0}\}$ denote $\overline{\mathbf{Q}}$ the restricted version of $\mathbf{Q}$. Moreover, $\mathbf{Q}_f$ will stand for $\mathbf{P}_{gf}$ if $\mathbf{Q}=\mathbf{P}_g$, and $\mathbf{Q}_f=\mathbf{P}_{\alpha_0 f}$ if $\mathbf{Q}=\mathbf{P}_{\alpha_0}$. We use the notation $\overline{\mathbf{Q}}_f$ for the restricted form of $\mathbf{Q}_f$. Take a set K of ascending algebras with rank type R. Then $\bar{K}$ is defined by $\bar{K}=\{s(\mathcal{A})|\mathcal{A}\in K\}$.

Statement 3. *For arbitrary class K of ascending algebras with rank type R, algebra $\mathcal{A}$ of rank type R and $\mathbf{Q}\in\{\mathbf{P}_g, \mathbf{P}_{\alpha_0}\}$ the following conditions hold.*

(i) $\mathcal{A}\in\mathbf{Q}(K)$ *if and only if* $s(\mathcal{A})\in\overline{\mathbf{Q}}(\bar{K})$.

(ii) $\mathcal{A}\in\mathbf{Q}_f(K)$ *if and only if* $s(\mathcal{A})\in\overline{\mathbf{Q}}_f(\bar{K})$.

Next we prove

Lemma 1. *Let K be a class of ascending algebras with rank type R, and take a $\mathbf{Q}\in\{\mathbf{P}_g, \mathbf{P}_{\alpha_0}\}$. Then* $\mathbf{HSQ}(K)=\mathbf{HSPQ}_f(K)$.

Proof. The inclusion $\mathbf{HSPQ}_f(K)\subseteq\mathbf{HSQ}(K)$ is obvious.

Let us show the converse inclusion. By Statements 1 and 3, $s(\mathbf{HSQ}(K))=$ $=\mathbf{HS}\overline{\mathbf{Q}}(\bar{K})$ and $s(\mathbf{HSPQ}_f(K))=\mathbf{HSP}\overline{\mathbf{Q}}_f(\bar{K})$, where s is extended to classes of ascending algebras in an obvious way. We show that $\mathbf{HS}\overline{\mathbf{Q}}(\bar{K})\cap K_{\bar{\Sigma}}\subseteq\mathbf{HSP}\overline{\mathbf{Q}}_f(\bar{K})\cap K_{\bar{\Sigma}}$ for every operator domain Σ of rank type R. This will imply $\mathbf{HSQ}(K)=$ $=s^{-1}(\mathbf{HS}\overline{\mathbf{Q}}(\bar{K}))\subseteq s^{-1}(\mathbf{HSP}\overline{\mathbf{Q}}_f(\bar{K}))=\mathbf{HSPQ}_f(K)$.

By Statement 2, $\mathbf{HS}\overline{\mathbf{Q}}(\bar{K})\cap K_{\bar{\Sigma}}$ is an equational class. Assume that an equation $p(x)=q(x)$ $(p, q\in F_{\bar{\Sigma}}(x))$ does not hold in $\mathbf{HS}\overline{\mathbf{Q}}(\bar{K})\cap K_{\bar{\Sigma}}$. Let us write p and q in a more detailed form $p=\sigma_k(\ldots(\sigma_1(x))\ldots)$, $q=\omega_l(\ldots(\omega_1(x))\ldots)$ $(\sigma_i, \omega_j\in\bar{\Sigma}$, $i=1, \ldots, k$, $j=1, \ldots, l)$, and assume that $l\leq k$. Moreover, set $p_i=\sigma_i(\ldots(\sigma_1(x))\ldots)$ $(i=0, \ldots, k)$ and $q_i=\omega_i(\ldots(\omega_1(x))\ldots)$ $(i=1, \ldots, l)$, where $p_0=q_0=x$. There are an

$$\mathcal{A} = (A, \bar{\Sigma}) = \prod (\mathcal{A}_i|i\in I)[\bar{\Sigma}, \varphi]\in\overline{\mathbf{Q}}(\bar{K})$$

and an $a_0 \in A$ such that $p(a_0) \neq q(a_0)$. (Here and in the sequel the above notation means that $\mathscr{A}$ is formed by the product represented by $\bar{\mathbf{Q}}$ and every $\mathscr{A}_i$ $(i \in I)$ is in $\bar{K}$.) Set $A' = \{a_0 p_i | i=1, \ldots, k\} \cup \{a_0 q_i | i=1, \ldots, l\}$. Denote by I_1 a minimal subset of I such that for arbitrary two distinct elements $a, b \in A'$ there is an $i \in I_1$ with $\mathrm{pr}_i(a) \neq \mathrm{pr}_i(b)$. Moreover, let I_{j+1} $(1 \leqq j \leqq k)$ be a minimal extension of I_j under which for arbitrary $i \in I_j$, $a, b \in A'$ and $\sigma \in \bar{\Sigma}$ if $\varphi_i(a, \sigma) \neq \varphi_i(b, \sigma)$ then there is a $t \in I_{j+1}$ such that φ_i depends on its tth component and $\mathrm{pr}_t(a) \neq \mathrm{pr}_t(b)$. (We write φ_i for φ_i^1.) Set $J = I_{k+1}$ and restrict the ordering of I to J. Obviously, J is finite. Take the product

$$\bar{\mathscr{A}} = (\bar{A}, \bar{\Sigma}) = \prod(\mathscr{A}_j | j \in J)[\bar{\Sigma}, \varphi']$$

where φ' is given as follows. For arbitrary $\bar{a} \in \bar{A}$, $1 \leqq j \leqq k$, $i \in I_j$ and $\sigma \in \bar{\Sigma}$, $\varphi_i'(\bar{a}, \sigma) = \varphi_i(a, \sigma)$ if there is an $a \in A'$ such that $\mathrm{pr}_{I_{j+1}}(\bar{a}) = \mathrm{pr}_{I_{j+1}}(a)$. (Here and in the sequel if $I' \subseteq I$ and $a_1, a_2 \in \prod(A_i | i \in I)$ then $\mathrm{pr}_{I'}(a_1) = \mathrm{pr}_{I'}(a_2)$ means that $\mathrm{pr}_i(a_1) = \mathrm{pr}_i(a_2)$ for every $i \in I'$.) In all other cases φ' is given arbitrarily in accordance with the definition of the product represented by $\bar{\mathbf{Q}}$. φ' is obviously well defined. It is also clear that $\bar{\mathscr{A}} \in \bar{\mathbf{Q}}_f(\bar{K})$.

For every $m=1, \ldots, k+1$ introduce the relation $\bar{a} \sim_m a$ $(\bar{a} \in \bar{A}, a \in A')$ if and only if $\mathrm{pr}_{I_m}(\bar{a}) = \mathrm{pr}_{I_m}(a)$, and let $\bar{a}_0 \in \bar{A}$ satisfy $\bar{a}_0 \sim_{k+1} a_0$. Then $p_i(\bar{a}_0) \sim_{k+1-i} p_i(a_0)$ and $q_j(\bar{a}_0) \sim_{k+1-j} q_j(a_0)$ for arbitrary i $(=0, \ldots, k)$ and j $(=0, \ldots, l)$. In particular, $p(\bar{a}_0) \sim_1 p(a_0)$ and $q(\bar{a}_0) \sim_1 q(a_0)$. Therefore $p(\bar{a}_0) \neq q(\bar{a}_0)$, that is $p(x) = q(x)$ does not hold in $\mathbf{HSP}\bar{\mathbf{Q}}_f(\bar{K}) \cap K_{\bar{\Sigma}}$.

The case when an equation $p(x) = q(y)$ $\big(p \in F_{\bar{\Sigma}}(x),\ q \in F_{\bar{\Sigma}}(y)\big)$ is not valid in $\mathbf{HS}\bar{\mathbf{Q}}(\bar{K}) \cap K_{\bar{\Sigma}}$ can be treated similarly.

Lemma 2. *For arbitrary class K of ascending algebras with rank type R the equality $\mathbf{HSP}_g(K) = \mathbf{HSP}_{\alpha_0}(K)$ holds.*

Proof. The inclusion $\mathbf{HSP}_{\alpha_0}(K) \subseteq \mathbf{HSP}_g(K)$ is obviously valid.

To prove $\mathbf{HSP}_g(K) \subseteq \mathbf{HSP}_{\alpha_0}(K)$, by Statements 1 and 3 it is enough to show $\mathbf{HS}\bar{\mathbf{P}}_g(\bar{K}) \subseteq \mathbf{HS}\bar{\mathbf{P}}_{\alpha_0}(\bar{K})$. Take an operator domain Σ of rank type R, and consider an equation $p(x) = q(x)$ $\big(p, q \in F_{\bar{\Sigma}}(x), h(q) \leqq h(p) = k\big)$ which does not hold in $\mathbf{HS}\bar{\mathbf{P}}_g(\bar{K}) \cap K_{\bar{\Sigma}}$. Then there are an

$$\mathscr{A} = (A, \bar{\Sigma}) = \prod(\mathscr{A}_i = (A_i, \bar{\Sigma}^i) | i \in I)[\bar{\Sigma}, \psi] \in \bar{\mathbf{P}}_g(\bar{K})$$

and an $a \in A$ such that $p(a) \neq q(a)$. Take $J = \{1, \ldots\ k+1\}$ with the natural ordering and order $J \times I$ in the following way: for arbitrary two $(j_1, i_1), (j_2, i_2) \in J \times I$, $(j_1, i_1) \leqq (j_2, i_2)$ if and only if $j_1 < j_2$, or $j_1 = j_2$ and $i_1 \leqq i_2$. Consider the restricted α_0-product

$$\mathscr{B} = (B, \bar{\Sigma}) = \prod(\mathscr{A}_{(j,i)} | (j, i) \in J \times I)[\bar{\Sigma}, \varphi'],$$

where $\mathscr{A}_{(j,i)}=\mathscr{A}_i$ $((j,i)\in J\times I)$, and for arbitrary $b\in B$ and $\sigma\in\bar{\Sigma}$, $\varphi_{(1,i)}(b,\sigma)$ $(i\in I)$ is arbitrary, and $\varphi_{(j,i)}(b,\sigma)=\varphi_i(b_{j-1},\sigma)$ $(1<j\leqq k+1, i\in I)$, where $b_t\in\prod(A_i|i\in I)$ is given by the equality $\mathrm{pr}_{(t,i)}(b)=\mathrm{pr}_i(b_t)$ $(t=1,\ldots,k+1, i\in I)$. Introduce the notation $b=(b_1,\ldots,b_{k+1})$ where $b_1,\ldots,b_{k+1}$ are defined by the previous equalities. Taking $b=(a,\ldots,a)$ and an $r\in F_{\bar{\Sigma}}(x)$ with $h(r)\leqq k$, one can show easily by induction on $h(r)$, that the equality

$$r^{\mathscr{B}}(b)=(c_1,\ldots,c_t,r^{\mathscr{A}}(a),\ldots,r^{\mathscr{A}}(a))$$

holds, where $t=h(r)$ and $c_1,\ldots,c_t\in\prod(A_i|i\in I)$. Especially, $p^{\mathscr{B}}(b)=(c_1,\ldots,\ldots,c_k,p^{\mathscr{A}}(a))$ and $q^{\mathscr{B}}(b)=(c_1',\ldots,c_k',q^{\mathscr{A}}(a))$ $(c_1,\ldots,c_k,c_1',\ldots,c_k'\in\prod(A_i|i\in I))$. Therefore, $p^{\mathscr{B}}(b)\neq q^{\mathscr{B}}(b)$, that is $p(x)=q(x)$ does not hold in $\mathbf{HS\bar{P}}_{\alpha_0}(\bar{K})\cap K_{\bar{\Sigma}}$.

The case when an equation of form $p(x)=q(y)$ $(p(x)\in F_{\bar{\Sigma}}(x),\ q(y)\in F_{\bar{\Sigma}}(y))$ is not valid in $\mathbf{HS\bar{P}}_g(\bar{K})\cap K_{\bar{\Sigma}}$ can be treated similarly. Thus we got that $\mathbf{HS\bar{P}}_g(\bar{K})\cap K_{\bar{\Sigma}}\subseteq\mathbf{HS\bar{P}}_{\alpha_0}(\bar{K})\cap K_{\bar{\Sigma}}$, which implies the inclusion $\mathbf{HSP}_g(K)\cap K_\Sigma\subseteq\subseteq\mathbf{HSP}_{\alpha_0}(K)\cap K_\Sigma$. This ends the proof of Lemma 2.

For arbitrary class K of ascending algebras with rank type R let $\mathbf{1}(K)$ denote the subclass consisting of all ascending algebras from K generated by single elements. The members of $\mathbf{1}(K)$ will be written as systems $(\mathscr{A},a)$ where $\mathscr{A}\in K$ and a is a generating element of $\mathscr{A}$.

Theorem 1. *The general product is metrically equivalent to the α_0-product if and only if for arbitrary class K of finite ascending algebras with rank type R the equality*

$$(*)\qquad \mathbf{1HSP}_g(K)=\mathbf{1HSP}_{\alpha_0}(K)$$

holds.

Proof. Assume that $(*)$ is valid. Take a system $(\mathscr{A},a)\in\mathbf{1SP}_{gf}(K)$ with $\mathscr{A}=(A,\Sigma)$. By $(*)$ and Lemma 1, $(\mathscr{A},a)\in\mathbf{1HSPP}_{\alpha_0 f}(K)$. Then there are a $\mathscr{B}=(B,\Sigma)\in\mathbf{PP}_{\alpha_0 f}(K)$ and a $b\in B$ such that for the subalgebra $\mathscr{B}'=(B',\Sigma)$ of $\mathscr{B}$ generated by b the system $(\mathscr{B}',b)$ can be mapped homomorphically onto $(\mathscr{A},a)$. (This terminology means that b is mapped into a under the given homomorphism of $\mathscr{B}'$ onto $\mathscr{A}$.) Let us write $\mathscr{B}$ in the form

$$\mathscr{B}=\prod(\mathscr{B}_i|\in I)\ (\mathscr{B}_i=(\mathscr{B}_i,\Sigma^i)\in\mathbf{P}_{\alpha_0 f}(K),\quad i\in I).$$

Take an integer $m\geqq 0$, and consider $B_b^{(m)}$. Denote by J a minimal subset of I such that for arbitrary two distinct elements $b_1,b_2\in B_b^{(m)}$ there is a $j\in J$ with $\mathrm{pr}_j(b_1)\neq\neq\mathrm{pr}_j(b_2)$. Obviously, J is finite. Define $\bar{b}\in\prod(B_j|j\in J)$ by $\mathrm{pr}_J(\bar{b})=\mathrm{pr}_J(b)$. Let $\bar{\mathscr{B}}=(\bar{B},\Sigma)$ be the ascending subalgebra of $\prod(\mathscr{B}_j|j\in J)$ generated by $\bar{b}$. Then $(\bar{\mathscr{B}},\bar{b})\in\mathbf{1SP}_{\alpha_0 f}(K)$ and it is m-isomorphic to $(\mathscr{B}',b)$. Thus $(\bar{\mathscr{B}},\bar{b})$ can be mapped m-homomorphically onto $(\mathscr{A},a)$. This ends the proof of the sufficiency.

In order to prove the necessity assume that the α_0-product is metrically equivalent to the product. Take a class K of finite ascending algebras with rank type R. Set $L=\mathbf{HSP}_g(K)$ and $\bar{L}=\mathbf{HSP}_{\alpha_0}(K)$. We show that $(*)$ holds, i.e., $\mathbf{1}L=\mathbf{1}\bar{L}$. To this, by Statements 1 and 3 it is enough to prove that for arbitrary operator domain Σ of rank type R if an equation $\bar{p}(x)=\bar{q}(x)$ $\big(\bar{p},\bar{q}\in F_{\bar{\Sigma}}(x)\big)$ does not hold in $s(L)\cap K_{\bar{\Sigma}}$ then it is not valid in $s(\bar{L})\cap K_{\bar{\Sigma}}$ since this implies that the free algebras in the equational classes $s(L)\cap K_{\bar{\Sigma}}$ and $s(\bar{L})\cap K_{\bar{\Sigma}}$ generated by single elements are isomorphic.

Thus assume that $\bar{p}(x)=\bar{q}(x)$ $\big(\bar{p},\bar{q}\in F_{\bar{\Sigma}}(x)\big)$ does not hold in $s(L)\cap K_{\bar{\Sigma}}$. Then, by Lemma 1, there is an $(\bar{\mathcal{A}},a_0)\in\mathbf{1S\bar{P}}_{gf}(\bar{K})$ $\big(\bar{\mathcal{A}}=(A,\bar{\Sigma}),\,a_0\in A\big)$ such that $\bar{p}(a_0)\neq\bar{q}(a_0)$. Take $\mathcal{A}=(A,\Sigma)$ with $s(\mathcal{A})=\bar{\mathcal{A}}$. Then $(\mathcal{A},a_0)\in\mathbf{1SP}_{gf}(K)$. Consider the transducer $\mathfrak{A}=(\Sigma,X_n,A,\Omega,A\times X_n,a_0,P)$ where $n>1$ is an arbitrary natural number, $\Omega_l=A\times\Sigma_l$ $(l\in R)$ and P consists of the following productions:

$$ax_i\to(a,x_i)\quad(a\in A,\ x_i\in X_n),\tag{1}$$

$$a\sigma\to(a,\sigma)(a_1\xi_1,\ldots,a_l\xi_l)\quad\big(a\in A,\,\sigma\in\Sigma_l,\,l\in R,\quad\sigma^{\mathcal{A}}(a)=(a_1,\ldots,a_l)\big).\tag{2}$$

Take two trees $p,q\in F_\Sigma(X_n\cup\xi)$ such that $\bar{p}=\mathrm{path}\,(p)$ and $\bar{q}=\mathrm{path}\,(q)$. Let $m\geq h(p),h(q)$. Then, by our assumptions, there is a $(\mathcal{B},b_0)\in\mathbf{1SP}_{\alpha_0 f}(K)$ $\big(\mathcal{B}=(B,\Sigma),\,b_0\in B\big)$ such that for a $\mathfrak{B}=(\Sigma,X_n,B,\Omega,A\times X_n,b_0,P')\in\mathrm{tr}\,(\mathbf{B})$ $\big(\mathbf{B}=(\mathcal{B},\dot{B}_0;X_n,\mathbf{b})\big)$ we have $\tau_{\mathfrak{A}}\overset{m}{=}\tau_{\mathfrak{B}}$. One can easily show by induction on the height of a tree that for every $r\in F_\Sigma(X_n\cup\xi)$ with $h(r)\leq m$ and $\mathrm{path}\,(r)=\bar{r}$ the derivations

$$a_0r\Rightarrow^*_{\mathfrak{A}}r'(a\xi)\quad\text{and}\quad b_0r\Rightarrow^*_{\mathfrak{B}}r''(b\xi)$$

hold, where $r',r''\in F_\Omega(A\times X_n\cup\xi)$, $a=\bar{r}^{\bar{\mathcal{A}}}(a_0)$, $b=\bar{r}^{\bar{\mathcal{B}}}(b_0)$ $\big(\bar{\mathcal{B}}=s(\mathcal{B})\big)$ and $\mathrm{path}\,(r'')$ is a subword of $\mathrm{path}\,(r')$. In particular,

$$a_0p\Rightarrow^*_{\mathfrak{A}}p'(a_1\xi),\quad b_0p\Rightarrow^*_{\mathfrak{B}}p''(b_1\xi),$$

and

$$a_0q\Rightarrow^*_{\mathfrak{A}}q'(a_2\xi),\quad b_0q\Rightarrow^*_{\mathfrak{B}}q''(b_2\xi)$$

where $p',p'',q',q''\in F_\Omega(A\times X_n\cup\xi)$, $a_1=\bar{p}^{\bar{\mathcal{A}}}(a_0)$, $b_1=\bar{p}^{\bar{\mathcal{B}}}(b_0)$, $a_2=\bar{q}^{\bar{\mathcal{A}}}(a_0)$ and $b_2=\bar{q}^{\bar{\mathcal{B}}}(b_0)$. By our assumptions, $a_1\neq a_2$. Assume that $b_1=b_2$. Take the trees $p(x_1)$ and $q(x_1)$. Then

$$a_0p(x_1)\Rightarrow^*_{\mathfrak{A}}p'\big((a_1,x_1)\big),\quad a_0q(x_1)\Rightarrow^*_{\mathfrak{A}}q'\big((a_2,x_1)\big)$$

and

$$b_0p(x_1)\Rightarrow^*_{\mathfrak{B}}p''(s),\quad b_0q(x_1)\Rightarrow^*_{\mathfrak{B}}q''(s)$$

where s is the right side of the rule $b_1x_1\to s$ in P'. Therefore, at least one of the equalities $p'\big((a_1,x_1)\big)=p''(s)$ and $q'\big((a_2,x_1)\big)=q''(s)$ does not hold contradicting the choice of $\mathfrak{B}$. Thus we got that $\bar{p}^{\bar{\mathcal{B}}}(b_0)\neq\bar{q}^{\bar{\mathcal{B}}}(b_0)$, that is the equality $\bar{p}(x)=\bar{q}(x)$ is not valid in $s(\bar{L})\cap K_{\bar{\Sigma}}$, which ends the proof of Theorem 1.

From Theorem 1, by Lemma 2, we obtain

Theorem 2. *Regarding ascending algebras the α_0-product is metrically equivalent to the general product.*

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The mixture of probability distribution functions by absolutely continuous weight functions

B. GYIRES

Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction

A function $F(x)$ of the real variable X is called a probability distribution function, if it satisfies the following conditions:

a) $F(x)$ is non-decreasing.

b) $F(x)$ is right continuous.

c) $F(\infty)=1$, $F(-\infty)=0$.

If a probability distribution function $F(x)$ is absolutely continuous with respect to the Lebesgue measure then $f(x)=F'(x)$ is called its probability density function.

We say $G(z, x)$ to be a family of probability distribution functions with parameter x, if the following properties are satisfied:

a) For each value of x the function $G(z, x)$ is a probability distribution function in z.

b) $G(z, x)$ is a measurable function of x.

Let $q(x)$ be an arbitrary probability distribution function. We form the expression

$$F(z) = \int_{-\infty}^{\infty} G(z, x)\, dq(x). \tag{1.1}$$

It is not difficult to show ([4], p. 199) that $F(z)$ is a probability distribution function, which is called the mixture of the family of probability distribution functions $G(z, x)$ with the weight function $q(x)$. An important question of the mixture theory of probability distribution functions is the following: Let the probability distribution function $F(z)$, and a family of probability distribution functions $G(z, x)$ with parameter x be given. What is the necessary and sufficient condition of having a probability dis-

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tribution function $q(x)$, which satiesfies equation (1.1)? Or in other words, what is the necessary and sufficient condition for the Fredholm's integral equation of first kind (1.1) to have such a solution, which is a probability distribution function?

This problem was solved by the author [2] in whole generality under the assumption that $q(x)$ is a discrete probability distribution function.

In this paper we give an answer to the raised question in case $q(x)$ is an absolutely continuous probability distribution function. The method in force makes necessary to introduce additional assumptions.

Let a and b, $a<b$, be given real numbers, where $a=-\infty$, $b=\infty$ are permitted, too. Denote by $E(a, b)$ the set of continuous probability distribution functions, which are strictly monotone increasing in $[a, b]$ and have values 0 and 1 at the points a and b, respectively. The inverse of $F\in E(a, b)$ is denoted by F^{-1}.

Without loss of generality we can assume that $G(z, x)$ is a family of probability distribution functions with parameter $x\in[0, 1]$.

The problem, which will be solved in this paper, is the following:

Let $F(z)\in E(a, b)$ and the family of probability distribution functions $G(z, x)\in E(a, b)$ with parameter $x\in[0, 1]$ be given. Assume that $q(x)$ is an absolutely continuous probability distribution function with probability density function $f(x)$, $x\in[0, 1]$. What is the necessary and sufficient condition for the Fredholm's integral equation of first kind

$$\int_0^1 G(z, x) f(x)\, dx = F(z) \tag{1.2}$$

to have a solution with square integrable probability density function $f(x)$?

To solve this integral equation, it will be traced back to the solution of the minimum problem of a symmetric positive definite Hilbert—Schmidt's kernel, using a distance concept between two probability distribution functions ([2], Chapter II).

Besides the Introduction the paper consists of three chapters. In the second one the problem will be traced back to the minimum problem of the above mentioned Hilbert—Schmidt's kernel. We deal with the eigenvalues and with the eigenfunctions of this kernel too. In the third chapter the answer will be given to the above raised question in two theorems. In the fourth one we give a family of probability distribution functions, by which the given probability distribution function is not representable as their mixture.

2. Preliminary

2.1. Let $F\in E(a, b)$, and let

$$F(x_{Nk}) = \frac{k}{N} \quad (k = 0, 1, \dots, N),$$

where $a=x_{N0}<x_{N1}<\ldots<x_{NN-1}<x_{NN}=b$ are the Nth quantiles of F. Let $G\in E(a, b)$, and let us form the expression

$$D_N(G|F) = N\sum_{k=1}^{N}[G(x_{Nk})-G(x_{Nk-1})]^2.$$

We say

$$D(G|F) = \sup_{N\to\infty} D_N(G|F)$$

to be the discrepancy of $G\in E(a, b)$ with respect to $F\in E(a, b)$.

This discrepancy idea was investigated more generally by the author ([2]; Chapter II.) and the following statement shows that this concept is a measure of the distance of two probability distribution functions:

$$D(G|F) \geqq 1$$

with equality if and only if $G=F$ ([2], Theorem 2.1.).

Denote by $H(F)\subset E(a, b)$ the set of the probability distribution functions with finite discrepancy with respect to $F\in E(a, b)$.

It can be shown ([2], Theorems 2.4. and 2.5.) that if $F\in E(a, b)$, $G\in H(F)$, then the probability distribution function $G(F^{-1}(z))$, $z\in[0, 1]$ is absolutely continuous with respect to the Lebesgue measure, and

$$D(G|F) = \int_0^1 \left[\frac{d}{dz} G(F^{-1}(z))\right]^2 dz.$$

Let $G_j\in H(F)$ $(j=1, 2)$ be given. The quantity

$$(G_1, G_2)_F = \int_0^1 \left[\frac{d}{dz} G_1\, F^{-1}(z)\right]\left[\frac{d}{dz} G_2((F^{-1}(z))\right] dz > 0$$

is said to be the common discrepancy of the probability distribution functions G_j $(j=1, 2)$ with respect to F. It is obvious that

$$(G, G)_F = D(G|F)$$

with $G_j=G$, $j=1, 2$. It follows from the Schwartz's inequality that

$$(G_1, G_2)_F \leqq [(G_1, G_1)_F(G_2, G_2)_F]^{1/2}.$$

Let $F\in E(a, b)$, and let

$$G(z, x)\in H(F), \quad x\in[0, 1]. \tag{2.1}$$

Let us introduce the quantity

$$K(x, y) = (G(z, x), \quad G(z, y))_F =$$

$$= \int_0^1 \left[\frac{d}{dz} G(F^{-1}(z), x)\right]\left[\frac{d}{dz} G(F^{-1}(z), y)\right] dz > 0, \quad x, y\in[0, 1], \tag{2.2}$$

and suppose that $K(x, y)$ is continuous in x and y. Moreover, let

$$\int_0^1 \int_0^1 K^2(x, y)\, dx\, dy < \infty, \tag{2.3}$$

i.e. $K(x, y)$ is a continuous Hilbert—Schmidt's kernel ([6], p. 135.).

Let the functions $f, g \in L_2(0, 1)$ be probability density functions. Then the probability distribution functions

$$G_f(z) = \int_0^1 G(z, x) f(x)\, dx,$$

$$G_g(z) = \int_0^1 G(z, x) g(x)\, dx$$

are mixtures of probability distribution functions (2.1) with respect to the weights f and g, respectively. Using Fubini's theorem we obtain

$$(G_f, G_g)_F = \int_0^1 \int_0^1 K(x, y) f(x) g(y)\, dx\, dy > 0.$$

Based on the foregoing we obtain

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx\, dy \geqq 1 \tag{2.4}$$

with equality if and only if

$$F(z) = \int_0^1 G(z, x) f(x)\, dx. \tag{2.5}$$

By the help of the kernel (2.2) our problem formulated in the Introduction can be expressed in the following way, too. Let $F \in E(a, b)$ and (2.1) be given. What is the necessary and sufficient condition of having such a probability density function $f \in L_2(0, 1)$ by which equation (2.5) is satisfied, or of having equality in the inequality (2.4).

2.2. Let the kernel (2.2) be given. It is well-known that if $h \in L_2(0, 1)$ is arbitrary, then the function

$$f(x) = \int_0^1 K(x, y) h(y)\, dy \in L_2(0, 1). \tag{2.6}$$

It is obvious that (2.2) is a symmetric kernel. It is well-known too that the integral equation

$$\varphi(x) - \lambda \int_0^1 K(x, y) \varphi(y)\, dy = 0, \quad x \in [0, 1] \tag{2.7}$$

has a solution different from $\varphi(x)=0$ if and only if λ satisfies the equation

$$f(-\lambda) = 0, \tag{2.8}$$

where the entire function

$$f(z) = 1+\sum_{n=1}^{\infty} a_n z^n \tag{2.9}$$

is the Fredholm's determinant of the function $zK(x, y)$, i.e. the coefficients of (2.9) are given by the following way ([6], p. 159). Let

$$x_k, y_k \in [0, 1] \quad (k = 1, \ldots, n), \tag{2.10}$$
$$x_k \neq x_l, \quad y_k \neq y_l, \quad k \neq l.$$

Moreover, let

$$K\begin{pmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{pmatrix} = \begin{pmatrix} K(x_1, y_1) \ldots K(x_1, y_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K(x_n, y_1) \ldots K(x_n, y_n) \end{pmatrix}.$$

It can be shown ([2], Theorem 3.1.) that the matrix

$$K\begin{pmatrix} x_1 \ldots x_m \\ x_1 \ldots x_n \end{pmatrix} - M$$

is positive definite or semidefinite, where M is the $n \times n$ matrix with entries 1. Thus

$$\operatorname{Det} K\begin{pmatrix} x_1 \ldots x_n \\ x_1 \ldots x_n \end{pmatrix} > 0$$

on the n-dimensional unit-cube, except a subset of this cube with zero measure, where inequalities (2.10) are not satisfied. From here we obtain

$$a_k = \frac{1}{k!} \int_0^1 \ldots \int_0^1 \operatorname{Det} K\begin{pmatrix} x_1 \ldots x_k \\ x_1 \ldots x_k \end{pmatrix} dx_1 \ldots dx_k > 0 \quad (k = 1, 2, 3, \ldots).$$

The zeros of equation (2.8) are the eigenvalues of kernel (2.2). Since all zeros of equation (2.8) are different from zero, and since kernel (2.2) is symmetric and positive definite, the eingevalues of kernel (2.2) are positive numbers.

By the help of the inequality concerning to the determinants of positive definite matrices ([1], Chapter 2, Theorem 7), we have

$$a_k \leqq \frac{k_1! \ldots k_s!}{k!} a_{k_1} \ldots a_{k_s}, \quad k_1 + \ldots + k_s = k \quad (s = 2, 3, \ldots; \ k = 2, 3, \ldots).$$

In particular, we obtain

$$a_k \leqq \frac{1}{k} a_{k-1} a_1 \quad (k = 1, 2, \ldots),$$

where a_1 is the trace of the kernel (2.2) satisfying the inequality

$$a_1 = \int_0^1 K(x, x)\,dx \geqq \int_0^1 \int_0^1 K(x, y)\,dx\,dy \geqq 1.$$

Since the inequality

$$\frac{a_{k-1}}{a_k} \geqq \frac{k}{a_1} \quad (k = 1, 2, \ldots) \tag{2.11}$$

holds, based on the theorem of Kakeya ([5], p. 25) the moduli of zeros of the polinomial

$$f_n(z) = \sum_{k=0}^{n} a_k z^k \quad (n = 1, 2, \ldots) \tag{2.12}$$

lie in the closed interval the endpoints of which are the minimum and the maximum of the numbers

$$\frac{a_{k-1}}{a_k} \quad (k = 1, \ldots, n),$$

respectively. Thus the moduli of the zeros of polynomials (2.12) are in the interval $[1/a, \infty)$ based on (2.11). According to the theorem of Hurwitz ([3], p. 78) the zeros of (2.8) are equal to the limit points of the zeros of polynomials (2.12). Therefore, and since the zeros of equation (2.8) are positive numbers, we get that the eigenvalues of the kernel (2.2) lie on the interval $(1/a, \infty)$.

The eigenvectors of kernel (2.2) are the solutions of the integral equation (2.7) if λ runs over the eigenvalues of kernel (2.2).

In what follows let ω be the number of the eigenvalues of $K(x, y)$, i.e. ω is a positive integer, or equal to infinity accordingly to $K(x, y)$ is degenerate, or non-degenerate, respectively.

Let $1/a_1 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots$ be the eigenvalues of $K(x, y)$, and let $\varphi_1(x), \varphi_2(x), \ldots$ be the sequence of the corresponding orthonormal eigenfunctions.

Denote by $E_2(0, 1)$ the set of the functions defined on $[0, 1]$, which can be represented by the formula (2.6) with square integrable functions. Assume that the integrals of the functions of $E_2(0, 1)$ are equal to one. Let $E_2^+(0, 1)$ be the subset of $E_2(0, 1)$ with non-negative elements. It is obvious that these sets are convex.

Let $g, h \in L_2(0, 1)$. In what follows we apply the usual notation $(g, h) = \int_0^1 g(x)h(x)\,dx$.

Lemma 2.1. *Let the kernel* (2.2) *be given. Then the function* (2.4) *is concave on* $E_2(0, 1)$ *and on* $E_2^+(0, 1)$, *respectively.*

Proof. Let

$$f_j \in E_2(0,1)(E_2^+(0,1)) \quad (j=1,\dots,s), \quad q_j \geqq 0, \quad \sum_{j=1}^{s} q_j = 1.$$

In this case

$$\sum_{j=1}^{s} q_j f_j \in E_2(0,1)(E_2^+(0,1)).$$

Using the theorem of Mercer ([6], p. 230) we get partly that

$$(2.13)\quad \int_0^1\int_0^1 K(x,y)\Big(\sum_{j=1}^{s} q_j f_j(x)\Big)\Big(\sum_{j=1}^{s} q_j f_j(y)\Big)\,dx\,dy = \sum_{k=1}^{\infty}\frac{1}{\lambda_k}\Big[\sum_{j=1}^{s} q_j(f_j,\varphi_k)\Big]^2,$$

and partly that

$$(2.14)\qquad \sum_{j=1}^{s} q_j \int_0^1\int_0^1 K(x,y) f_j(x) f_j(y)\,dx\,dy = \sum_{j=1}^{s} q_j \sum_{k=1}^{\infty}\frac{1}{\lambda_k}(f_j;\ \varphi_k)^2.$$

Based on the Cauchy's inequality we obtain

$$\Big(\sum_{j=1}^{s} q_j(f_j,\varphi_k)\Big)^2 \leqq \sum_{j=1}^{s} q_j \sum_{j=1}^{s} q_j(f_j,\varphi_k)^2.$$

This inequality and expressions (2.13) and (2.14) give us the statement of the Lemma.

Lemma 2.2. *Let* $f\in E_2(0,1)$. *Then*

$$\int_0^1\int_0^1 K(x,y) f(x) f(y)\,dx\,dy \geqq 1.$$

Proof. First of all we mention that

$$\varphi(z) = \frac{d}{dz}\int_0^1 G\big(F^{-1}(z),x\big) f(x)\,dx \in E_2(0,1).$$

Since $\varphi(z)\in E_2(0,1)$, it is sufficient to show that the integral of $\varphi(z)$ is equal to one. But this is obvious. Namely, using Fubini's theorem, we get

$$\int_0^1 \varphi(z)\,dz = \int_0^1 f(x)[G(b)-G(a)]\,dx = 1.$$

Accordingly

$$(2.15)\qquad \int_0^1\int_0^1 K(x,y) f(x) f(y)\,dx\,dy = \int_0^1 \varphi^2(z)\,dz \geqq \int_0^1 \varphi(z)\,dz = 1,$$

and this is the statement of our Lemma.

We have equality in (2.15) if and only if $\varphi(z)=1$, $z\in[0,1]$.

3. The solution of the problem

In this chapter we give a solution in connection with the problem mentioned in the Introduction in the following sense. Let $F\in E(a, b)$ and the family of probability distribution functions (2.1) be given. We look for a necessary and sufficient condition of having a solution of (2.5) or having equality in (2.4) with a function $f\in E_2^+(0, 1)$.

Let us define the following quantites:

$$\mu_0 = \inf_{f\in E_2^+(0,1)} \int_0^1 \int_0^1 K(x, y)f(x)f(y)\,dx\,dy, \tag{3.1}$$

$$\mu_1 = \inf_{f\in E_2(0,1)} \int_0^1 \int_0^1 K(x, y)f(x)f(y)\,dx\,dy.$$

Based on Lemma 2.2 we have

$$\mu_0 \geqq \mu_1 \geqq 1.$$

It follows from here that $\mu_1=1$ is the necessary condition for the mixibility of $F\in E(a, b)$ by the family of probability distribution functions (2.1). This condition is sufficient too if $f\in E_2(0, 1)$ satisfying (3.1) is an element of the set $E_2^+(0, 1)$. Accordingly we can proceed on the following way. We calculate μ_1 and a function $f\in E_2(0, 1)$ satisfying equation (3.1). If $\mu_1=1$ and $f\in E_2^+(0, 1)$ then $F\in E(a, b)$ can be mixed by the family of probability distribution functions (2.1) with weight f. If $\mu_1>1$, or if $\mu_1=1$ but $f\notin E_2^+(0, 1)$, the F cannot be mixed by this family of probability distribution functions.

The number μ_1 can regard as the measure of the mixibility of $F\in E(a, b)$ by the family of probability distribution functions (2.1).

Theorem 3.1. *Let $\{\lambda_k\}$ be the set of eigenvalues of the kernel* (2.2) *enumerated in an increasing way, and let the suitable orthonormal eigenfunctions be the elements of the sequence $\{\varphi_k(x)\}$. Let*

$$\alpha_k = (1, \varphi_k) \quad (k = 1, 2, \ldots). \tag{3.2}$$

Then

$$\mu_1 = \frac{1}{\sum_{k=1}^{\omega} \alpha_k^2 \lambda_k}, \tag{3.3}$$

where

$$0 < \sum_{k=1}^{\omega} \alpha_k^2 \leqq 1. \tag{3.4}$$

Moreover

$$f(x) = \mu_1 \sum_{k=1}^{\infty} \alpha_k \lambda_k \varphi_k(x) \in E_2(0, 1) \tag{3.5}$$

is the only solution, which satisfies equality (3.1).

Proof. Let

$$f(x) = \int_0^1 K(x, y) h(y)\, dy = \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} \varphi_k(x) \in E_2(0, 1), \tag{3.6}$$

where $h \in L_2(0, 1)$ and

$$\chi_k = (h, \varphi_k) \quad (k = 1, 2, \ldots). \tag{3.7}$$

Considering the Hilbert—Schmidt's theorem ([6], p. 227) we have

$$\int_0^1 f(x)\, dx = \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} \alpha_k = 1 \tag{3.8}$$

where the numbers α_k are defined by (3.2). In this case

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx dy = \int_0^1 \int_0^1 K_3(x, y) h(x) h(y)\, dx dy,$$

where $K_3(x, y)$ is the third iterated of kernel (2.2) ([6], p. 144). On the basis of Mercer's theorem ([6], p. 230) we get that the series

$$K_3(x, y) = \sum_{k=1}^{\infty} \frac{\varphi_k(x) \varphi_k(y)}{\lambda_k^3} \quad (x, y \in [0, 1]) \tag{3.9}$$

converges uniformly. Thus

$$\int_0^1 \int_0^1 K_3(x, y) h(x) h(y)\, dx dy = \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^3}. \tag{3.10}$$

Our next duty is to calculate the minimum of (3.10) under condition (3.8).
Since (3.9) is a concave function on the set $E_2(0, 1)$, using the method of Lagrange's multipliers, (3.10) has an absolute minimum inside of $E_2(0, 1)$ if and only if

$$\frac{\partial \Phi}{\partial x_k} = 0 \quad (k = 1, 2, \ldots)$$

with

$$\Phi(x_1, x_2, \ldots) = \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^3} - 2\lambda \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} \alpha_k,$$

i.e. if the conditions

$$x_k = \lambda \alpha_k \lambda_k^2 \quad (k = 1, 2, \ldots) \tag{3.11}$$

are satisfied. On the basis of (3.8) we have

$$\lambda \sum_{k=1}^{\omega} \alpha_k^2 \lambda_k = 1. \tag{3.12}$$

Considering partly that the elements of the sequence $\{\alpha_k\}$ are the Fourier's coefficients of $f(x)=1$, $x\in[0, 1]$ related to the orthonormal system $\{\varphi_k(x)\}$ and partly that

$$\int_0^1 \int_0^1 K(x, y)\, dx dy = \sum_{k=1}^{\omega} \frac{\alpha_k^2}{\lambda_k} \leqq a_1 \sum_{k=1}^{\omega} \alpha_k^2,$$

we obtain

$$\frac{\int_0^1 \int_0^1 K(x, y)\, dx dy}{\int_0^1 K(x, x)\, dx} \leqq \sum_{k=1}^{\omega} \alpha_k^2 \leqq 1,$$

i.e. the inequality (3.4) holds.

Substituting (3.11) into (3.10), we get that

$$\mu_1 = \lambda^2 \sum_{k=1}^{\omega} \alpha_k^2 \lambda_k = \lambda,$$

thus we obtain formula (3.3) on the basis of (3.12).

However substituting (3.11) into (3.6), we get that (3.10) touches his minimum by the function (3.5) of the set $E_2(0, 1)$.

We must mention separately the case if each element of the orthonormal system $\varphi_k(x)$, say $\varphi_s(x)$, is equal to one. In this case the minimum of (3.10) does not fall into the inside of $E_2(0, 1)$, since $\alpha_s=1$ and $\alpha_k=0$, if $k\neq s$. On the basis of (3.8) $x_s=\lambda_s$. Evidently (3.10) has minimum, if $x_k=0$, $k\neq s$, and

$$\mu_1 = \frac{1}{\lambda_s}; \quad f(x) = 1, \ x\in[0, 1].$$

Accordingly, (3.6) gives us the minimum of (3.10) in case the function (3.7) falls to the boundary of $E_2(0, 1)$, too. Thus the proof of Theorem 3.1 is finished.

Corollary 3.1. *Under the assumptions and notations of Theorem 3.1 the representation*

$$G(z) = \int_0^1 G(z, x) f(x)\, dx = \mu_1 \sum_{k=1}^{\omega} \alpha_k \lambda_k \int_0^1 G(x, z) \varphi_k(x)\, dx$$

holds uniformly in $z\in[a, b]$, *where* $(G, G)_F=1$.

Proof. Namely

$$G(z) = \int_0^1 G(z, x)\big(\mu_1 \sum_{k=1}^{\omega} \alpha_k \lambda_k \varphi_k(x)\big)\, dx.$$

Using the Schwartz's inequality

$$\Big|G(z) - \mu_1 \sum_{k=1}^{n} \alpha_k \lambda_k \int_0^1 G(z, x)\varphi_k(x)\, dx\Big| = \Big|\int_0^1 G(z, x)\big(\mu_1 \sum_{k=n+1}^{\omega} \alpha_k \lambda_k \varphi_k(x)\big)\, dx\Big| \leqq$$

$$\leqq \Big(\int_0^1 G^2(z, x)\, dx\Big)^{1/2} \big(\mu_1^2 \sum_{k=n+1}^{\omega} \alpha_k^2 \lambda_k^2\big)^{1/2} \leqq \big(\mu_1^2 \sum_{k=n+1}^{\omega} \alpha_k^2 \lambda_k^2\big)^{1/2}.$$

Since

$$\mu_1^2 \sum_{k=1}^{\omega} \alpha_k^2 \lambda_k^2 = \int_0^1 f^2(x)\, dx < \infty,$$

the sequence

$$\Big\{\mu_1^2 \sum_{k=n+1}^{\omega} \alpha_k^2 \lambda_k^2\Big\}_{n=0}^{\infty}$$

converges to zero.

The chief result of this paper is the following theorem, which arises directly from Theorem 3.1 and from Corollary 3.1.

Theorem 3.2. *Let $\{\lambda_k\}$ be the set of eigenvalues of the kernel* (2.2.) *enumerated in an increasing way, and let the suitable orthonormal eigenfunctions be the elements of the sequence $\{\varphi_k(x)\}$. Let*

$$\alpha_k = (1, \varphi_k) \quad (k = 1, 2, \ldots).$$

The probability distribution function $F \in E(a, b)$ can be mixed by the family of probability distribution functions (2.1) *with weight functions from the set $E_2^+(0, 1)$ if and only if the conditions*

$$\sum_{k=1}^{\omega} \alpha_k^2 \lambda_k = 1,$$

and

$$f(x) = \sum_{k=1}^{\omega} \alpha_k \lambda_k \varphi_k(x) \geqq 0, \quad x \in [0, 1]$$

are satisfied. In this case the represantion

$$F(z) = \sum_{k=1}^{\omega} \alpha_k \lambda_k \int_0^1 G(z, x)\varphi_k(x)\, dx$$

holds uniformly in $z \in [a, b]$.

4. Construction such a family of probability distribution functions by which a given probability distribution function cannot be represented as their mixture

The aim of this chapter is to prove the following theorem.

Theorem 4.1. *Let* $\varphi(x) \geqq 1$, $x \in [0, 1]$ *be a continuous function different from a constant. Then* $F(z) \in E(a, b)$ *cannot be represented as the mixture of the family of probability distribution functions*

$$G(z, x) = \big(F(z)\big)^{\varphi(x)}, \quad x \in [0, 1]$$

with a weight function of set $L_2(0, 1)$.

Proof. In this case

$$K(x, y) = \frac{\varphi(x)\varphi(y)}{\varphi(x)+\varphi(y)-1}; \quad x, y \in [0, 1]$$

on the basis of (2.2). Taking into account that the identity

$$K(x, y) = \frac{\varphi(x)\varphi(y)}{\varphi(x)\varphi(y)-[\varphi(x)-1][\varphi(y)-1]}$$

holds, we get that

$$K(x, y) = \sum_{k=0}^{\infty} \left(\frac{\varphi(x)-1}{\varphi(x)} \frac{\varphi(y)-1}{\varphi(y)} \right)^k, \quad x, y \in [0, 1]. \tag{4.1}$$

Let

$$\max_{x \in [0, 1]} \frac{\varphi(x)-1}{\varphi(x)} = \Theta. \tag{4.2}$$

Using (4.2) and

$$0 < \Theta < 1, \tag{4.3}$$

we obtain that the series (4.1) converges uniformly and

$$1 \leqq \int_0^1 \int_0^1 K^2(x, y)\, dx\, dy < \frac{1}{1-\Theta^2}.$$

Moreover, if $h \in L_2(0, 1)$ then representation

$$\int_0^1 \int_0^1 K(x, y) h(x) h(y)\, dx\, dy = \sum_{k=1}^{\infty} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)} \right)^k h(x)\, dx \right]^2 \tag{4.4}$$

holds.

Namely let m, n be arbitrary positive integers. Let

$$\Delta(n, m) = \int_0^1 \int_0^1 \sum_{k=n}^{n+m} \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k \left(\frac{\varphi(y)-1}{\varphi(y)}\right)^k h(x)h(y)\,dx\,dy =$$

$$= \sum_{k=n}^{n+m} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k h(x)\,dx\right]^2.$$

Using the Schwartz inequality, moreover the relations (4.2) and (4.3),

$$\Delta(n, m) \leqq \sum_{k=n}^{n+m} \int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^{2k} dx \int_0^1 h^2(x)\,dx < \frac{\Theta^{2(n+1)}}{1-\Theta^2} \int_0^1 h^2(x)\,dx \quad (m = 1, 2, \ldots).$$

Thus

$$\left|\int_0^1 K(x, y)h(x)h(y)\,dx\,dy - \sum_{k=0}^{n-1} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k h(x)\,dx\right]^2\right| <$$

$$< \frac{\Theta^{2(n+1)}}{1-\Theta^2} \int_0^1 h^2(x)\,dx,$$

which gives us the representation (4.4).

Let now $f \in L_2(0, 1)$ be a probability density function. Using representation (4.4)

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\,dx\,dy = 1 + \sum_{k=1}^{\infty} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k f(x)\,dx\right]^2.$$

From here we get that the identity

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\,dx\,dy = 1$$

holds if and only if the conditions

$$\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k f(x)\,dx = 0 \quad (k = 1, 2, \ldots) \tag{4.5}$$

are satisfied. Since f is a probability density function, neither of (4.5) can be satisfied. This completes the proof.

In particular, let $\varphi(x) = 1 + x$, $x \in [0, 1]$. Then

$$K(x, y) = \frac{(1+x)(1+y)}{x+y+1} = \sum_{k=0}^{\infty} \left(\frac{x}{1+x}\frac{y}{1+y}\right)^k,$$

and $\Theta = 1/2$. Using Theorem 4.1 we obtain the following result.

Corollary 4.1. *Under the assumption of Theorem* 4.1 $F\in E(a, b)$ *cannot be represented as the mixture of family of probability distribution functions*

$$G(z, x) = \big(F(x)\big)^{1+x}, \quad x\in[0, 1]$$

with a weight function from the set $L_2[0, 1]$.

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Stability properties of the equilibrium under the influence of unbounded damping

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction

It is a well-known phenomenon that damping can make mechanical equilibria asymptotically stable. However, as is to be expected, in the presence of too large damping the system can remain far from the equilibrium position. For example, the equation $\ddot{x}+(2+e^t)\dot{x}+x=0$ due to LASALLE [7] admits nonvanishing solutions $x=a(1+e^{-t})$ ($a=$const.). Considering the second order nonlinear differential equation

$$\ddot{x}+f(t,x,\dot{x})|\dot{x}|^\alpha\dot{x}+g(x)=0,$$

BALLIEU and PEIFFER [1] investigated which conditions on f assure attractivity or nonattractivity of the origin. They have proved that if $f(\cdot,x,\dot{x})$ is "not too large" then the equilibrium is attractive, and if it is "large enough" then the equilibrium is not attaractive. Now the following question arises: what happens in the second case? Experience suggests (see also LaSalle's example) that the deviation x tends to a finite limit (possibly different from zero) and the velocity $\dot{x}$ tends to zero as $t\to\infty$. In other words, the point asymptotically stops (possibly far from the original equilibrium position).

In this paper we study the conditions of the asymptotic stop by Lyapunov's direct method and differential inequalities. In [12] the second author gave conditions assuring x-stability of the equilibrium state and the convergence of the deviation $x(t)$ as $t\to\infty$. Recently [4] the first author got conditions for the convergence to zero of the velocities in a mechanical system. Here it will be pointed out that the two methods can be combined to get conditions for the asymptotic stop.

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After some preliminaries (Section 2) we present a theorem for general differential systems which guarantees the stability of the zero solution with respect to a part of the variables, the convergence of this part to a finite limit and the convergence to zero of the further variables along the solutions as $t\to\infty$ (Section 3). In the final two sections we apply this result to establish stability properties of equilibria of dissipative mechanical systems and of the zero solution of nonlinear second order differential equations. The paper is concluded by the example of the mathematical plain pendulum with changing length.

2. Preliminaries

Consider the system of differential equations

$$\dot{x} = X(t, x), \tag{2.1}$$

where $t\in R_+:=[0, \infty)$ and $x\in R^k$ with a norm $|x|$. Let a partition $x=(y, z)$ ($y\in R^m$, $z\in R^n$; $1\leqq m\leqq k$, $n=k-m$) be given. Assume that the function X is defined and continuous on the set $\Gamma:=R_+\times R^m\times D$, where $D\subset R^n$ is open and contains the origin, and $X(t, 0)\equiv 0$, i.e. $x=0$ is a solution of (2.1). We denote by $x(t)=x(t; t_0, x_0)$ any solution with $x(t_0)=x_0$. We always assume the solutions to be *y-continuable* which means that if $x(t)=(y(t), z(t))$ is a solution of (2.1) and $|z(t)|$ is bounded in $[t_0, T)$, then $x(t)$ can be continued to the closed interval $[t_0, T]$.

The zero solution of (2.1) is said to be:

z-stable if for every $\varepsilon>0$, $t_0\in R_+$ there exists a $\delta(\varepsilon, t_0)>0$ such that $|x_0|<\delta(\varepsilon, t_0)$ implies $|z(t; t_0, x_0)|<\varepsilon$ for $t\geqq t_0$;

asymptotically z-stable if it is z-stable and, in addition, for every $t_0\in R_+$ there exists a $\sigma(t_0)>0$ such that $|x_0|<\sigma(t_0)$ implies $|z(t; t_0, x_0)|\to 0$ as $t\to\infty$.

Instead of x-stability we will say simply stability.

With a continuously differentiable function V: $\Gamma\to R$ we associate a function $\dot V$: $\Gamma\to R$ by the definition

$$\dot V(t, x) := \frac{\partial V(t, x)}{\partial x} X(t, x)+\frac{\partial V(t, x)}{\partial t}$$

which is called the *derivative of V with respect to* (2.1). Here as well as in the sequel, for two vectors $a, b\in R^k$ by ab we denote the scalar product of a and b.

Denote by L^γ the class of the Lebesgue measurable functions f: $R_+\to R$ with

$$\|f\|_\gamma:= \begin{cases} \left[\int_0^\infty |f|^\gamma\right]^{1/\gamma}<\infty & (0<\gamma<\infty) \\ \sup\limits_{s\in R_+} \operatorname{ess} |f(s)| & (\gamma=\infty). \end{cases}$$

For a continuously differentiable function $f\colon R_+ \to R$, by the *function of the positive* and *negative variation* of the function f we mean

$$\int_0^t [\dot f(s)]_+ ds, \quad \int_0^t [\dot f(s)]_- ds,$$

respectively, where

$$[a]_+ := \max\{a, 0\} \quad [a]_- := \max\{-a, 0\} \quad (a \in R).$$

One of the basic notion of the main theorem will be the integral positivity. A continuous function $f\colon R_+ \to R_+$ is called *integrally positive* if $\int_I f = \infty$ whenever $I = \bigcup_{i=1}^{\infty} [a_i, b_i]$, and $a_i < b_i < a_{i+1}$, $b_i - a_i \geqq \delta > 0$ hold for all $i = 1, 2, \ldots$ with some positive constant δ.

In the proofs of the theorem we will need the following

Lemma 2.1. *If the functions $f\colon R_+ \to R_+$, $g\colon R_+ \to (0, \infty)$ are continuous, f is integrally positive, and there exists an α $(0 < \alpha \leqq \infty)$ such that*

$$\frac{f^{1+1/\alpha}}{g} \in L^\alpha,$$

then g is integrally positive.

Proof. Suppose that the statement is not true, i.e. there exists a sequence of intervals $[a_i, b_i]$ possessing the properties in the definition of the integral positivity, and such that

$$\sum_{i=1}^{\infty} \int_{a_i}^{b_i} g < \infty. \tag{2.2}$$

Suppose that $\alpha < \infty$ and introduce the notations $p := 1 + 1/\alpha$, $q := \alpha + 1$. By Hölder's inequality we get the estimate

$$\int_{a_i}^{b_i} f = \int_{a_i}^{b_i} g^{1/p} \frac{f}{g^{1/p}} \leqq \Big[\int_{a_i}^{b_i} g\Big]^{1/p} \Big[\int_{a_i}^{b_i} \frac{f^q}{g^{q/p}}\Big]^{1/q} \tag{2.3}$$

for all $i = 1, 2, \ldots$. For every fixed natural number N the application of the Cauchy inequality yields

$$\sum_{i=1}^{N} \int_{a_i}^{b_i} f \leqq \Big[\sum_{i=1}^{N} \int_{a_i}^{b_i} g\Big]^{1/p} \Big[\sum_{i=1}^{N} \int_{a_i}^{b_i} \frac{f^{\alpha+1}}{g^\alpha}\Big]^{1/q}.$$

In consequence of (2.2) we have

$$\sum_{i=1}^{\infty} \int_{a_i}^{b_i} f < \infty,$$

in contradiction to the fact that f is integrally positive.

In the case of $\alpha=\infty$, instead of (2.3) we start from the estimate

$$\int_{a_i}^{b_i} f = \int_{a_i}^{b_i} g\frac{f}{g} \leqq \left\|\frac{f}{g}\right\|_\infty \int_{a_i}^{b_i} g,$$

which leads to a contradiction because of (2.2). The lemma is proved.

3. The main theorem

Consider the system of the differential equations

$$\dot{y} = Y(t, y, z), \quad \dot{z} = Z(t, y, z), \tag{3.1}$$

where $y \in R^m$, $z \in R^n$, $t \in R_+$, and right-hand sides Y, Z satisfy the assumptions in Section 2.

Theorem 3.1. *Suppose that there are continuous functions $V_1, V_2\colon \Gamma \to R$; $c, r\colon R_+^2 \to R_+$; $b\colon R_+^3 \to R_+$; $a, \varphi, \psi, \chi\colon R_+ \to R_+$ and real numbers α, β $(0 \leqq \beta \leqq \alpha)$ satisfying the following conditions on the set Γ:*

(i) *functions V_1, V_2 are continuously differentiable and $V_1(t, x) \geqq 0$, $V(t, x) := := V_1(t, x) + V_2(t, x) \geqq 0$, $V(t, 0) \equiv 0$;*

(ii) *$\dot{V}(t, x) \leqq -\varphi(t) V_1^\alpha(t, x) + r(t, V(t, x))$, where φ is integrally positive; the function $r(t, \cdot)$ is nondecreasing for every $t \in R_+$ and the zero solution of the equation $\dot{u} = r(t, u)$ is stable;*

(iii)
$$|Z(t, x)| \leqq \psi(t) V_1^\beta(t, x) a(V(t, x))$$

and the function a is nondecreasing;

(iv)
$$\left[\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial y} Y\right]_+ (t, y, z) \leqq b(t, |z|, V(t, y, z)),$$

so that for every $t \in R_+$ the function $b(t, \cdot, \cdot)$ is nondecreasing in its both variables, and for every $r_1, r_2 > 0$ the primitive $\int_0^t b(s, r_1, r_2)\, ds$ is uniformly continuous on R_+;

(v)
$$\left|\frac{\partial V_2(t, y, z)}{\partial z}\right| \leqq \chi(t) c(|z|, V(t, y, z)),$$

where c is nondecreasing in its both variables;

(vi) $\psi/\varphi^{\beta/\alpha}\in L^{\alpha/(\alpha-\beta)}$, *and the function*

$$\int_t^{t+1}\left(\frac{\chi^\alpha\psi^\alpha}{\varphi^\beta}\right)^{1/(\alpha-\beta)} \quad \text{if} \quad \alpha>\beta$$

$$\frac{\chi\psi}{\varphi} \quad \text{if} \quad \alpha=\beta$$

is bounded on R_+.

Then every solution $(y(t), z(t))$ *of* (3.1) *with sufficiently small initial values exists for large* t, $z(t)\to$const., $V_1(t, y(t), z(t))\to 0$ *as* $t\to\infty$, *and the zero solution is z-stable.*

Proof. If the initial value u_0 is sufficiently small then the maximal solution of the initial value problem

$$\dot{u}=r(t,u) \quad u(t_0)=u_0\geqq 0$$

exists for all $t\geqq t_0$, and being nondecreasing it has a finite limit $u_\infty(t_0, u_0)$ as $t\to\infty$. Since the zero solution of the equation $\dot{u}=r(t,u)$ is stable, we have

$$\lim_{u_0\to 0} u_\infty(t_0, u_0)=0.$$

For any solution $x\colon [t_0, T)\to R^k$ $(t_0<T\leqq\infty)$ of (3.1) introduce the notations

$$v_1(t):=V_1(t, x(t)), \quad v_2(t):=V_2(t, x(t))$$

$$v(t)=v_1(t)+v_2(t), \quad w(t):=v(t)+u_\infty(t_0, u_0)-u(t; t_0, u_0),$$

where $u_0:=v(t_0)$.

In view of condition (ii) function v satisfies the estimate $\dot{v}(t)\leqq r(t, v(t))$, hence by the theory of differential inequalities ([6], Theorem 1.4.1) we have $v(t)\leqq$ $\leqq u(t; t_0, u_0)$ for all $t\in[t_0, T)$. Therefore,

$$\dot{w}(t)\leqq-\varphi(t)v_1^\alpha(t)\leqq 0 \quad (t_0<t<T), \tag{3.3}$$

which together with $w(t)\geqq 0$ implies

$$\int_{t_0}^{T}\varphi(t)v_1^\alpha(t)\,dt<\infty.$$

Besides, the function w and, consequently, v as well, is bounded also above.

In order to establish z-stability and the existence of the limit of $z(t)$ we estimate the variation of $z(t)$ over an interval $[A, B]$ making use of the Hölder inequality:

$$(3.4)\qquad |z(A)-z(B)| \leqq \int_A^B |Z(t, x(t))|\, dt \leqq \int_A^B \psi(t) v_1^\beta(t) a(v(t))\, dt \leqq$$

$$\leqq a(u_\infty(t_0, u_0)) \int_A^B \frac{\psi}{\varphi^{\beta/\alpha}} \varphi^{\beta/\alpha} v_1^\beta \leqq a(u_\infty(t_0, u_0)) \left\| \frac{\psi}{\varphi^{\beta/\alpha}} \right\|_{\alpha/(\alpha-\beta)} \Big[\int_A^B \varphi v_1^\alpha\Big]^{\beta/\alpha} \leqq$$

$$\leqq a(u_\infty(t_0, u_0)) \left\| \frac{\psi}{\varphi^{\beta/\alpha}} \right\|_{\alpha/(\alpha-\beta)} (w(A)-w(B)).$$

First we apply (3.4) to prove the continuability of the solutions and the z-stability of the zero solution. Let $\varepsilon>0$, $t_0 \in R_+$ be given such that $|z|<\varepsilon$ implies $z \in D$. Fix an $x_0=(y_0, z_0)$ $(|z_0|<\varepsilon)$ and denote

$$T^* := \sup \{T\colon\ t_0 < T,\ |z(t;\ t_0, x_0)| < \varepsilon \quad \text{for} \quad t \in [t_0, T]\}.$$

We prove $T^*=\infty$ provided $|x_0|$ is sufficiently small. By (3.4)

$$|z(t_0)-z(t)| \leqq c_1 a(u_\infty(t_0, V(t_0, x_0)))u_\infty(t_0, V(t_0, x_0))$$

for every $t \in (t_0, T)$ with an appropriate constant $c_1>0$ independent of the solution. Because of

$$\lim_{x_0 \to 0} V(t_0, x_0) = 0, \quad \lim_{u_0 \to 0} u_\infty(t_0, u_0) = 0$$

we can choose a $0<\delta(\varepsilon, t_0)<\varepsilon/3$ such that $|x_0|<\delta$ implies $|z_0-z(t;\, t_0, x_0)|<\varepsilon/3$. Consequently, $|z(t;\, t_0, x_0)|<2\varepsilon/3$ for all $t \in [t_0, T)$. By y-continuability of the solutions and the definition of T^* this means that $T^*=\infty$, i.e. the solutions $x(t;\ t_0, x_0)$ with $|x_0|<\delta$ can be continued to all $t \geqq t_0$, and the zero solution of (3.1) is z-stable.

On the other hand, function $w(t)$ has a finite limit as $t \to \infty$, thus $w(A)-w(B) \to 0$ as $A, B \to \infty$. Hence, by (3.4), $|z(A)-z(B)| \to 0$ as $A, B \to \infty$, i.e. $z(t)$ has also a finite limit as $t \to \infty$.

It remains to prove that $V_1(t, x(t)) \to 0$ as $t \to \infty$. To this end, take an $\varepsilon_0>0$, $t_0 \in R_+$ and consider a solution $\xi(t)=(\eta(t), \zeta(t))$ of (3.1) with $|\xi(t_0)|<\delta(\varepsilon, t_0)$. We know that $|\zeta(t)| \leqq c_2$, $v(t) \leqq c_3$ with appropriate constants c_2, c_3 and

$$(3.5)\qquad \int_{t_0}^\infty \varphi(t) v_1^\alpha(t)\, dt < \infty.$$

Suppose now that $v_1(t) \not\to 0$ as $t \to \infty$. This assumption together with inequality (3.5) and the fact that φ is integrally positive imply the existence of a $\gamma>0$ such that for every $T \geqq 0$ there are $A=A(T)$, $B=B(T)$ $(T<A<B)$ with the properties

$$v_1(A)=2\gamma/3, \quad v_1(B)=\gamma/3$$

$$\gamma/3 \leqq v_1(t) \leqq 2\gamma/3 \quad (A(T) \leqq t \leqq B(T))$$

(3.6) $$B(T)-A(T) \to 0 \quad (T \to \infty).$$

On the other hand, the sum $v_1(t)+v_2(t)$ has a finite limit; consequently, there exists a $\tau>0$ such that the positive variation of $v_2(t)$ over $[A(T), B(T)]$ is greater than τ for every T. But, by conditions (iii)—(v) of the theorem we have

(3.7) $$\tau \leqq \int_A^B [\dot v_2(t)]_+ dt = \int_A^B [\dot V_2(t, \xi(t))]_+ dt \leqq \int_A^B b(t, |\xi(t)|, v(t))\, dt +$$

$$+ \int_A^B \chi(t)\psi(t)v_1^\beta(t)\, c(|\xi(t)|, v(t))\, a(v(t))\, dt \leqq$$

$$\leqq \int_A^B b(t, c_2, c_3)\, dt + c(c_2, c_3)\, a(c_3) \int_A^B \chi\psi v_1^\beta.$$

Using the Hölder inequality, for the last integral we get the estimate

(3.8) $$\int_A^B \frac{\chi\psi}{\varphi^{\beta/\alpha}} \varphi^{\beta/\alpha} v_1^\beta \leqq \left[\int_A^B \frac{(\chi\psi)^{\alpha/(\alpha-\beta)}}{\varphi^{\beta/(\alpha-\beta)}}\right]^{(\alpha-\beta)/\alpha} \left[\int_A^B \varphi v_1^\alpha\right]^{\beta/\alpha}.$$

(3.5)—(3.8) and condition (iv) imply

$$0<\tau \leqq \int_{A(T)}^{B(T)} [\dot v_2(t)]_+ \, dt \to 0 \quad (T \to \infty),$$

which is a contradiction proving that $v_1(t) \to 0$ as $t \to \infty$. The theorem is proved.

Remark 3.1. It can be proved that the zero solution of the equation $\dot u = r(t, u)$ is stable (see condition (ii) in Theorem 3.1) if and only if

$$\int_0^\infty r(t, u)\, dt < \infty$$

for every sufficiently small $u \geqq 0$, and the function $u(t) \equiv 0$ is the unique solution of the initial value problems

$$\dot u = r(t, u), \quad u(t_0)=0 \quad (t_0 \geqq 0).$$

Remark 3.2. It is clear from the proof that in condition (iv) the "positive part" $[\cdot]_+$ on the left-hand side can be replaced by the "negative part" $[\cdot]_-$, the theorem remains true.

Remark 3.3. By Lemma 2.1, in consequence of condition (vi) in the theorem we can require of function ψ to be integrally positive instead of φ.

Remark 3.4. Analysing the proof of the theorem one can easily see that estimates (3.7)—(3.8) are not needed if we know the function $v_2(t)$ to be uniformly continuous on $[t_0, \infty)$. Consequently, if it is a priori known that the function $V_2(t, y(t), z(t))$ is uniformly continuous on $[t_0, \infty)$ for every solution $(y(t), z(t))$ of (3.1), such that $z(t) \to$ const., $V(t, y(t), z(t))$ is bounded as $t \to \infty$ and

$$\int_0^\infty \varphi(t) V_1^\alpha(t, y(t), z(t))\, dt < \infty,$$

then after dropping conditions (iv)—(v) and (3.2) the theorem remains true. In the next section we show how this condition can be checked directly in a special case.

4. Applications to damped mechanical systems

Consider a holonomic scleronomic mechanical system of r degrees of freedom. Assume that there act upon the system potential and dissipative forces depending also on the time. Let the motions be described by the Lagrangian equation ([10], Appendix II)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} + Q, \tag{4.1}$$

where $q \in D \subset R^r$ is the vector of the generalized coordinates (D is open and contains the origin of R^r), $\dot q \in R^r$ is the vector of the generalized velocities; $T = T(q, \dot q)$ denotes the kinetic energy which is a quadratic form of the velocities; $\Pi = \Pi(t, q)$ is the potential energy, the vector $Q = Q(t, q, \dot q)$ is the resultant of frictional and gyroscopic forces, i.e. $Q(t, q, \dot q)\dot q \leqq 0$ for all $t \geqq 0$, $q \in D$, $\dot q \in R^r$. Suppose that

$$\Pi(t, 0) \equiv 0, \quad \frac{\partial \Pi}{\partial q}(t, 0) \equiv 0,$$

which means that $q = \dot q = 0$ is an equilibrium state of the system.

Many authors have investigated the conditions of the asymptotic stability of the equilibrium state (see [10, 11, 9, 2]). As the simple example in the Introduction shows, for this property it is necessary to bound above the damping in some way.

In this section we examine what happens under the action of damping not restricted above at all. It will turn out that if the damping is sufficiently large then the system asymptotically stops, i.e. for every motion of (4.1) $q(t) \to$ const. (maybe different from the origin), $\dot q(t) \to 0$ as $t \to \infty$.

Theorem 4.1. *Suppose that there exist constants* $h, \gamma > 0$ *and continuous functions* $\varphi\colon R_+ \to R_+$, $r\colon R_+^2 \to R_+$ *such that the following conditions are satisfied on the set* $\Gamma := \{(t, q, \dot q)\colon t \in R_+,\ |q| \leqq h,\ |\dot q| \leqq h\}$:

(i)
$$\Pi(t, q) \geqq 0;$$

(ii)
$$\left[\frac{\partial \Pi}{\partial t}(t, q)\right]_+ \leqq r\big(t, \Pi(t, q)\big),$$

where the function $r(t, \cdot)$ *is nondecreasing for every* $t \in R_+$, *and the zero solution of the equation* $\dot u = r(t, u)$ *is stable;*

(iii) *the dissipation is complete and large enough in the sense that the inequality*

$$Q(t, q, \dot q)\dot q \leqq -\varphi(t)|\dot q|^{1+\gamma}$$

holds so that

$$\int_0^\infty \varphi^{-1/\gamma} < \infty; \tag{4.2}$$

(iv) *the function*

$$\int_t^{t+1} \max\left\{\left|\frac{\partial \Pi}{\partial q}(s, q)\right| : |q| \leqq h\right\}^{(1+\gamma)/\gamma} \varphi^{-1/\gamma}(s)\, ds$$

is bounded on R_+.

Then the equilibrium state $q = \dot q = 0$ *of* (4.1) *is stable, asymptotically* $\dot q$*-stable, and along every motion with sufficiently small initial values* $|q(t_0)|$, $|\dot q(t_0)|$ *the vector of the generalized coordinates has a finite limit as* $t \to \infty$ *(i.e. the system asymptotically stops).*

Proof. It is known ([10], Appendix II) that the matrix in the quadratic form of the kinetic energy is positive definite, so by introducing the new variable $y := \dot q$ system (4.1) can be rewritten into the explicit form

$$\dot y = Y(t, y, q), \quad \dot q = y. \tag{4.3}$$

Take the auxiliary functions

$$V_1(y, q) := T(q, y) \quad V_2(t, y) := \Pi(t, q).$$

An easy computation shows (see [10]) that the derivative of $V(t, y, q) := V_1(y, q) + V_2(t, y)$ with respect to (4.3) reads

$$\dot V(t, y, q) = Q(t, q, y)y + \frac{\partial \Pi}{\partial t}(t, q) \leqq -\varphi(t)|y|^{1+\gamma} + r\big(t, \Pi(t, q)\big) \quad \big((t, y, q) \in \Gamma\big). \tag{4.4}$$

Because the kinetic energy T is a positive definite quadratic form, there are $0 < \lambda < \Lambda$ such that

$$\lambda^2|y|^2 = V_1(y, q) \leqq \Lambda^2|y|^2 \quad (|q| \leqq h, y \in R^r).$$

Therefore, from (4.4) we got the estimate

$$\dot V(t, y, q) \leqq -\frac{\varphi(t)}{\lambda} V_1^{(1+\gamma)/2}(y, q)+r\big(t, V(t, y, q)\big) \tag{4.5}$$

on the set Γ. By setting $\alpha:=(1+\gamma)/2$, $\beta:=1/2$,

$$\psi(t):= 1/\lambda, \quad b(t, r_1, r_2):= r(t, r_2), \quad c(r_1, r_2):= 1$$

$$\chi(t):= \max\left\{\left|\frac{\partial\Pi}{\partial q}(t, q)\right|: |q| \leqq h\right\}$$

all the conditions of Theorem 3.1 are met on the set Γ (for the integral positivity of φ see Remark 3.3). Thus if we have proved y-stability of the zero solution of (4.3), the further statements of the theorem follows from Theorem 3.1.

Function V is positive definite with respect to y and in view of (4.5) it satisfies the differential inequality $\dot u \leqq r(t, u)$. The zero solution of the associated differential equation $\dot u = r(t, u)$ is stable; therefore, by C. Corduneanu's theorem [5] (see also [6]) the equilibrium state $q=\dot q=0$ is q-stable. The theorem is proved.

Remark. It is worth noticing that either the stronger version

$$\left|\frac{\partial\Pi}{\partial t}(t, q)\right| \leqq r\big(t, \Pi(t, q)\big) \tag{ii'}$$

of condition (ii) or the condition that $\Pi(t, q)$ is uniformly continuous for $t\in R_+$, $|q|\leqq h$ can replace condition (iv) in the theorem.

Indeed, according to Remark 3.4 it is enough to prove that for every motion $q(t)$ with sufficiently small initial values $|q(t_0)|$, $|\dot q(t_0)|$ the function $\Pi\big(t, q(t)\big)$ is uniformly continuous on R_+, provided that $q(t)\to$const. as $t\to\infty$. This is obviously satisfied if $\Pi(t, q)$ is uniformly continuous. On the other hand, for any q fixed sufficiently small, from condition (ii') we get the estimate

$$|\Pi(A, q)-\Pi(B, q)| \leqq \left|\int_A^B r\big(t, u(t;\ t_0, \Pi(t_0, q_0)\big)\, dt\right| =$$

$$= \big|u\big(A;\ t_0, \Pi(t_0, q_0)\big)-u\big(B;\ t_0, \Pi(t_0, q_0)\big)\big| \to 0 \quad (A, B \to\infty).$$

Therefore, $\Pi(t, q)\to\Pi^*(q)$ uniformly in a small ball around the origin as $t\to\infty$, hence $\Pi\big(t, q(t)\big)$ also has a finite limit, which is sufficient for the uniform continuity of $\Pi\big(t, q(t)\big)$.

5. Application to second order equations

In this section we apply our main theorem to study of asymptotic behaviour of the motions of a rheonomic mechanical system of one degree of freedom. In differential equation language, consider the equation

(5.1) $$(p(t)\dot{x})^{\cdot}+g(t,x,\dot{x})\dot{x}+q(t)f(x)=0 \quad (x\in R),$$

where $p, q\colon R_+\to(0,\infty)$ are continuously differentiable, and $g\colon R_+\times R^2\to R$, $f\colon R\to R$ are continuous functions, and

$$xf(x)\geqq 0 \quad (t\in R_+,\, x\in R).$$

The following two theorems illuminate how to get different conditions for the same asymptotic property of the solutions of the same equation by different choices of the auxiliary functions. The first theorem concerns the case of bounded q, the second one can be applied also to unbounded q.

Theorem 5.1. *Suppose that*

(i) *there exists a function* $\gamma\colon R_+\to R_+$ *such that*

$$\gamma(t)\leqq g(t,u,v) \quad (t\in R_+;\, u,v\in R)$$

$$2\gamma(t)+\dot{p}(t)>0, \quad \int_0^\infty \frac{1}{2\gamma+p^{\cdot}}<\infty;$$

(ii) $$\int_0^\infty \left[\frac{\dot{q}}{q}\right]_+<\infty;$$

(iii) *either* $(2\gamma+\dot{p})/p$ *or* $1/\sqrt{p}$ *is integrally positive.*

Then the zero solution of (5.1) *is x-stable and for every solution* $x(t)$ *with sufficiently small initial values* $x(t)\to\text{const.}$ $p(t)\dot{x}^2(t)\to 0$ *as* $t\to\infty$.

Proof. Introducing the notations $y=p(t)\dot{x}$, $z=x$, we can write equation (5.1) in the form

(5.2) $$\begin{cases}\dot{y}=-q(t)f(z)-g(t,z,y/p(t))\,y/p(t)\\ \dot{z}=y/p(t).\end{cases}$$

Define

$$F(z):=\int_0^z f(r)\,dr\geqq 0,\quad V_1(t,y):=y^2/2p(t),\quad V_2(t,z):=q(t)F(z).$$

The derivative of the total mechanical energy $V:=V_1+V_2$ can be estimated as follows:

$$\dot{V}=-\left(\frac{2g}{p}+\frac{\dot{p}}{p}\right)V_1+\dot{q}F\leqq-\frac{2\gamma+\dot{p}}{p}V_1+\left[\frac{\dot{q}}{q}\right]_+V.$$

By Lemma 2.1, condition (iii) implies the function $(2\gamma+\dot{p})/p$ to be integrally positive. Because of (ii) the function q is bounded, so the conditions of Theorem 3.1 are met by the choices $\beta:=1/2$, $\psi:=\sqrt{2/p}$, $\chi:=q$ and

$$c(r, s) := \max\{|f(z)|: |z| \leqq r\}.$$

The theorem is proved.

Theorem 5.2. *Suppose that*

(i) *there exist a constant* $h>0$ *and a function* $\gamma: R_+ \to R_+$ *such that*

$$\gamma(t) \leqq g(t, u, v) \quad (t\in R_+, |u| \leqq h, v\in R),$$

$$(\ln(pq))^{\cdot}(t)+2\frac{\gamma(t)}{p(t)} > 0;$$

(ii)
$$\int_0^\infty \frac{q}{p(\ln(pq))^{\cdot}+2\gamma} < \infty;$$

(iii) *either* $\sqrt{q/p}$ *or* $(\ln(pq))^{\cdot}+2\gamma/p$ *is integrally positive.*

Then the zero solution of (5.1) *is x-stable, and for every solution* $x(t)$ *with sufficiently small initial values* $x(t)\to$const., $p(t)\dot{x}^2(t)/q(t)\to 0$ *as* $t\to\infty$.

Proof. After setting

$$V_1(t, y) := \frac{1}{p(t)q(t)}y^2, \quad V_2(z) := 2F(z)$$

and computing

$$(V_1+V_2)^{\cdot} = -\left[(\ln(pq))^{\cdot}+\frac{2g}{p}\right]V_1$$

the proof of the theorem can be concluded in the same way as in Theorem 5.1.

Finally, in order to illuminate these results we give sufficient conditions for the asymptotic stop of a mathematical plain pendulum whose length changes by the law $l=l(t)$ (see [3]). Assume that there acts viscous friction on the material point such that the damping force is proportionate to the velocity. Let the position of the material point in the plain be described by the length $l(t)$ of the thread and the angle φ between the axis directed vertically downwards and the thread. Then the kinetic energy T, the potential energy Π and the dissipative force Q are

$$T = \frac{1}{2}m[l^2(t)\dot{\varphi}^2\dot{l}^2(t)], \quad \Pi = mgl(t)(1-\cos\varphi)+\dot{l}, \quad Q = -h(t)l^2(t)\dot{\varphi},$$

where m is the mass of the material point, g denotes the constant of gravity and $h(t)$ is the frictional coefficient at the moment t. The motions are described by the Lagrange's equation

$$(ml^2(t)\dot{\varphi})^{\cdot}+hl^2(t)\dot{\varphi}+mgl(t)\sin\varphi = 0.$$

Corollary 5.1. *If*

(i)
$$h(t)+m(\ln l(t))^{\cdot}>0 \quad (t\in R_+),$$
$$\int_0^\infty \frac{1/l^2}{h+m(\ln l)^{\cdot}}<\infty;$$

(ii)
$$\int_0^\infty [(\ln l)^{\cdot}]_+<\infty,$$

then the equilibrium state $\varphi=\dot\varphi=0$ *is* φ*-stable; every motion* $\varphi(t)$ *has a finite limit, and* $l(t)\dot\varphi(t)\to 0$ *as* $t\to\infty$.

Corollary 5.2. *If*

(i)
$$3m(\ln l(t))^{\cdot}+2h(t)>0 \quad (t\in R_+);$$

(ii)
$$\int_0^\infty \frac{1/l}{3m(\ln l)^{\cdot}+2h}<\infty;$$

(iii) *either* $1/\sqrt{l(t)}$ *or* $3m(\ln l)^{\cdot}+2h$ *is integrally positive,*
then the equilibrium state $\varphi=\dot\varphi=0$ *is* φ*-stable, every motion* $\varphi(t)$ *has a finite limit and* $\sqrt{l(t)}\dot\varphi(t)\to 0$ *as* $t\to\infty$.

Proof. They immediately follow from Theorems 5.1 and 5.2, respectively.

One can observe that Corollary 5.2 concerns the case of nonincreasing length (perhaps $l(t)\to 0$ as $t\to\infty$), and Corollary 5.2 works mainly if $l(t)$ is nondecreasing (possibly, $l(t)\to\infty$ as $t\to\infty$).

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Empirical kernel transforms of parameter-estimated empirical processes

LAJOS HORVÁTH

In honour of Professor Károly Tandori on his 60th birthday

1. Introduction. Let $d \geqq 1$ be an integer and let $X_1, X_2, \ldots$ be a sequence of independent d-dimensional random vectors with common distribution function $F(x)$, $x \in R^d$. We assume that a parametric family of d-variate distribution functions is given,

$$\mathscr{F} = \{F(x, \theta) \colon x \in R^d;\ \theta \in \Theta \subset R^p\},$$

and the common distribution function of the $X_1, X_2, \ldots$ belongs to this family, i.e., there is a parameter $\theta_0 \in \Theta$ so that $F(x) = F(x; \theta_0) \equiv F_0(x)$. The true value θ_0 is unknown. Consider the estimated empirical process defined by

$$\beta_n(x) = n^{1/2}(F_n(x) - F(x;\ \theta_n)), \quad x \in R^d,$$

where F_n is the empirical distribution function of $X_1, \ldots, X_n$ and $\theta_n = (\theta_{n1}, \ldots, \theta_{np})$ is some estimator of θ_0 based on the random sample $X_1, \ldots, X_n$.

The weak convergence of the estimated empirical process was studied by several authors. We will use the general strong approximation result of BURKE *et al* [1] in this note. Introduce the notations $\theta = (\theta_1, \ldots, \theta_p)$ and

$$\nabla F(x;\ \theta^*) = \nabla_\theta F(x; \theta)|_{\theta=\theta^*} = \left(\frac{\partial}{\partial\theta_1} F(x;\ \theta), \ldots, \frac{\partial}{\partial\theta_p} F(x;\ \theta)\right)\bigg|_{\theta=\theta^*},$$

and let

$$\langle x, y\rangle = \sum_{j=1}^{p} x_j y_j, \quad x = (x_1, \ldots, x_p), \quad y = (y_1, \ldots, y_p),$$

stand for the inner product in R^p. Let $a^{\mathrm{T}} = (a_1, \ldots, a_p)^{\mathrm{T}}$ denote the column vector corresponding to the row vector $a = (a_1, \ldots, a_p)$. The norm of a vector $x = (x_1, \ldots, x_p)$ and a matrix $M = \{m_{ij}\}_{i,j=1}^{p}$ is defined by $\|x\| = \max\{|x_i| \colon 1 \leqq i \leqq p\}$ and

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$\|M\| = \max\{|m_{ij}|:\ 1 \leq i,j \leq p\}$. If ξ_n converges to zero in probability we will use the notation $\xi_n \xrightarrow{P} 0$ $(n \to \infty)$. A Brownian bridge $B^{F_0}(x)$, $x \in R^d$, associated with the distribution function F_0 is a d-variate Gaussian random field such that $EB^{F_0}(x)=0$ and $EB^{F_0}(x)B^{F_0}(y) = F_0(x \wedge y) - F_0(x)F(y)$, where $x \wedge y = (\min(x_1, y_1), \dots, \min(x_d, y_d))$.

Theorem A (BURKE, M. CSÖRGŐ, S. CSÖRGŐ and RÉVÉSZ [1], and S. CSÖRGŐ [2]). *Suppose that the sequence θ_n satisfies the following conditions:*

$$\text{(i)} \qquad n^{1/2}(\theta_n - \theta_0) = n^{-1/2} \sum_{j=1}^{n} l(X_j; \theta_0) + \varepsilon_n,$$

where $l(\cdot\,; \theta_n)$ is a measurable d-dimensional (row) vector-valued function and $\varepsilon_n \xrightarrow{P} 0$ $(n \to \infty)$.

$$\text{(ii)} \qquad El(X_1; \theta_0) = 0.$$

(iii) $M(\theta_0) = El^{\mathrm{T}}(X_1;\ \theta_0)l(X_1;\ \theta_0)$ *is a finite and nonnegative definite matrix.*

(iv) *The vector $\nabla_\theta F(x; \theta)$ is uniformly continuous in x and $\theta \in \Lambda$, where Λ is the closure of a given neighbourhood of θ_0.*

(v) $d=1$; *Each component of the vector function $l(x;\ \theta_0)$ is of bounded variation on each finite interval.*

$d>1$; *The partial derivatives of each component of the vector function l, with order not exceeding d, exist almost everywhere (with respect to the d-dimensional Lebesgue measure) on R^d, and for any $u>0$*

$$\sup_{\|x\| \leq u} \sum_{j=1}^{d} \sum_{\substack{j_1, \dots, j_d \geq 0 \\ j_1 + \dots + j_d = j}} \left\| \frac{\partial^j l(x;\ \theta_0)}{\partial x_1^{j_1} \dots \partial x_d^{j_d}} \right\| < \infty.$$

If the underlying probability space is rich enough, one can define a sequence of d-dimensional Brownian bridges $\{B_n^{F_0}(x)\}$ associated with the distribution function F_0 such that

$$\sup_{x \in R^d} |\beta_n(x) - D_n(x;\ \theta_0)| \xrightarrow{P} 0 \quad (n \to \infty),$$

where

$$D_n(x;\ \theta_0) = B_n^{F_0}(x) - \Big\langle \int_{R^d} l(y;\ \theta_0)\, dB_n^{F_0}(y),\ \nabla_\theta F(x;\ \theta_0) \Big\rangle$$

is a sequence of copies of the Durbin process.

The limiting Gaussian process of this theorem depends, in general, not only on F but also on θ_0, the true, unknown value of parameter. On the other hand, the distributions of the functionals of $D_1(x;\ \theta_0)$ (supremum functional, square integral functional) as functions of θ_0 are unknown. According to the references below, Bolshev asked whether there is a kernel k such that the random variable

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x, y)\,dD_1(x;\theta_0)D_1(y;\theta_0), \quad d=1, \quad x, y\in R^1$$

have a prescribed distribution. This problem was investigated by HMALADZE [6], [7] and DZAPARIDZE and NIKULIN [5]. The methods of Dzaparidze and Nikulin are based on the orthogonal expansion of the limiting Gaussian process. In the case $d=1$ and when only shift and scale parameters are estimated they proposed statistics whose limit distributions are independent of the unknown parameters and depend only on F, but these limit distributions are usually complicated and therefore it would be hard to compute percentage points for these statistics. Using the martingale property of $\beta_n(x)$ if $d=1$, Hmaladze proved some weak convergence results in L_2 sense. Analogous results were obtained earlier by NEUHAUS [10], [11]. He proved the weak convergence of $\beta_n(x)$, $x\in[0, 1]^d$, $d\geqq 1$, in supremum metric under contiguous alternatives.

The above question was generalized by S. CSÖRGŐ [2], who introduced

$$\int_{R^d} k(x, y)\,d\beta_n(x), \quad x\in R^d, \quad y\in R^q,$$

a kernel transform of the parameter-estimated empirical process. Here $q\geqq 1$ is an arbitrary integer. Assuming some regularity conditions on k, he proved that

$$\sup_{y\in R^q}\Big|\int_{R^d} k(x, y)\,d\beta_n(x)-\int_{R^d} k(x, y)\,dD_n(x;\theta_0)\Big| \xrightarrow{P} 0, \quad n\to\infty.$$

Unfortunately, he cannot choose a kernel k and a functional h on the space of continuous functions on R^q such that the random variable

$$h\Big(\int_{R^d} k(x, y)\,dD_1(x;\theta_0)\Big)$$

has a distribution not depending on θ_0.

In this note we are interested in a sequence of kernel type transformations of $\beta_n(x)$, where the kernel will also depend on the sample. We are able to choose a sequence of kernels $\{k^N(x, y;\theta_0)\}$ such that

$$\int_{R^d} k^N(x, y)\beta_n(x)\,dx, \quad y\in I^q, \tag{1.1}$$

converges weakly to a q-dimensional standard Wiener process or to the standard Brownian bridge on I^q (or, for that matter, to any prescribed Gaussian process) if N and $n(N)$ go to infinity, where I^q is the unit cube of R^q. The transformations (1.1) depend, in general, on the unknown parameter θ_0, therefore we will prove, that there is a sequence of random kernels $\{k^n_{N,m}(x, y)\}$ based on the sample $X_1, \ldots, X_n$ such that

$$\int_{R^d} k^n_{N,m}(x, y)\,\beta_n(x)\,dx, \quad y\in I^q,$$

converges weakly to a prescribed process, if $N, m(N), n(N, m)$ go to infinity. Our methods may be extended to some more general parametric models. Section 2 presents a general result, where the role of $\beta_n(x)$ is played by an arbitrary sequence of processes $X_n(x)$. Section 3 will then specify the result for $\beta_n(x)$.

2. Main theorem. The reproducing kernel Hilbert space $H(R)$ generated by a covariance function $R(t, s)$ plays a fundamental role in our note. Suppose that $X(t)$, $t\in I^d$, is a centered Gaussian process having continuous paths on I^d a.s. and continuous covariance function:

$$EX(t) = 0, \quad EX(t)X(s) = R(t, s), \quad t, s\in I^d.$$

It is well known, that the space $\mathscr{C}$ of all continuous functions $I^d\to R$ with the topology induced by the supremum norm is a separable Banach space and the collection $\mathscr{C}^*$ of all linear and bounded functionals on $\mathscr{C}$ can be identified with the space of all (regular) measures ν on the Borel subsets of I^d. If ν^+ and ν^- denote the Hahn decomposition of ν, then $\|\nu\| = \nu^+(I^d)+\nu^-(I^d)$ is a norm on $\mathscr{C}^*$. SATO [12] has shown, that for some complete orthonormal sequence (CONS) $\{e_i(t), i\geqq 1\}$ in $H(R)$ one can write

$$e_i(t) = \int_{I^d} R(t, s)\,\nu_i(ds),$$

where $\nu_i\in\mathscr{C}^*$ and

(2.1) $$\nu_i = \mu_i/\sigma_i, \quad \mu_i\in\mathscr{C}^*,$$

(2.2) $$\|\mu_i\| = 1,$$

(2.3) $$\sigma_i = \Big[\int_{I^d}\int_{I^d} R(t, s)\,\nu_i(ds)\,\nu_i(dt)\Big]^{1/2} > 0,$$

(2.4) $$\int_{I^d}\int_{I^d} R(t, s)\,\nu_i(ds)\,\nu_j(dt) = 0, \quad i\neq j.$$

MANGANO [9] proved that

(2.5) $$\int_{I^d} e_i(t)\,\nu_j(dt) = \delta_{ij},$$

where $\delta_{ii}=1$, $\delta_{ij}=0$, $i\neq j$. The following lemma is a simple variant of Lemma 2.2 in [9].

Lemma 2.1. *Let $R(t, s)$ and $G(t, s)$ be continuous covariance functions of two centered Gaussian processes with a.s. continuous paths on I^d. Let N be a positive integer* *and* $\{e_i(t),\ i\geqq 1,\ t\in I^d\}$ *be a CONS in* $H(R)$ *generated by measures* $\{\nu_i,\ i\geqq 1\}$. *If*

$$\sup_{(t,s)\in I^{2d}} |G(t,s)-R(t,s)| = \Delta \leqq 1/K_1, \tag{2.6}$$

then there exists an orthonormal set $\{f_i,\ 1\leqq i\leqq N\}$ *of functions in* $H(G)$ *generated by the measures* $\{\varkappa_i,\ 1\leqq i\leqq N\}$ *such that*

$$\|\varkappa_i-\nu_i\| \leqq \Delta K_2,\quad 1\leqq i\leqq N \tag{2.7}$$

and

$$\sup_{t\in I^d} |e_i(t)-f_i(t)| \leqq \Delta K_3,\quad 1\leqq i\leqq N, \tag{2.8}$$

where K_1, K_2, K_3 *are suitably chosen polynomials of* $N, M, \|\nu_i\|, 1\leqq i\leqq N,$ *with positive coefficients and* $M=\max_{(t,s)\in I^{2d}} |R(t,s)|$.

Proof. The proof follows from the construction of Mangano. He constructed measures $\varkappa_i$, $1\leqq i\leqq N$, which are linear combinations of the measures ν_i, $1\leqq i\leqq N$. It is not too difficult to check that the measures and functions $f_i(t)$, $1\leqq i\leqq N$, constructed by Mangano satisfy (2.7) and (2.8) with suitably chosen functions K_1, K_2, K_3.

Let $\{X_n(t),\ t\in I^d\}$ be a sequence of stochastic processes such that

$$\sup_{t\in I^d} |X_n(t)-Y_n(t)| \xrightarrow{P} 0 \quad (n\to\infty), \tag{2.9}$$

where $\{Y_n(t),\ t\in I^d\}$ is a sequence of copies of a Gaussian process $\{Y(t;\theta),\ t\in I^d\}$ depending on a parameter θ. We suppose, that the process $Y(t;\theta)$ has continuous paths on I^d a.s., its covariance function is continuous for every $\theta\in\Theta$, where $\Theta\subset R^p$ is a compact parameter set, and

$$\sup_{(t,s)\in I^{2d}} |R(t,s;\theta)-R(t,s;\theta^*)| \to 0, \quad \text{if} \quad \theta\to\theta^*. \tag{2.10}$$

It is well known from functional analysis, that the kernel function $R(t,s;\theta)$ has a sequence of eigenvalues $\{\lambda_i(\theta),\ i\geqq 1\}$ and eigenfunctions $\{\varphi_i(t;\theta), i\geqq 1\}$, that is

$$\lambda_i(\theta)\varphi_i(t;\theta) = \int_{I^d} R(s,t;\theta)\varphi_i(s;\theta)\,ds,$$

$$\int_{I^d} \varphi_i(t;\theta)\varphi_j(t;\theta)\,dt = \delta_{ij},\quad \lambda_i(\theta)>0.$$

The sequence of eigenvalues and eigenfunctions determines a CONS in $H(R(\theta))$:

$$e_i(t;\theta) = (\lambda_i(\theta))^{-1/2}\varphi_i(t;\theta). \tag{2.11}$$

It follows from (2.11) that in this case

$$\nu_i(ds) = (\lambda_i(\theta))^{-3/2}\varphi_i(s;\theta)\,ds, \tag{2.12}$$

so we have

$$\|v_i(\theta)\| \leqq (\lambda_i(\theta))^{-3/2} M(\theta), \tag{2.13}$$

where $M(\theta) = \sup_{(t,s)\in I^{2d}} |R(t, s; \theta)|$.

Let N be given. Then the polynomials K_1, K_2 and K_3 of Lemma 2.1 which depend only on N, $M(\theta)$ and $\|v_i(\theta)\|$, $1 \leqq i \leqq N$, are continuous positive functions of θ. Let $0 < \varepsilon < 2 \inf_{\theta\in\Theta} \{1/(K_1 \min (1/K_2, 1/K_3))\}$ and define $\delta(\theta) > 0$ for every $\theta\in\Theta$ in the following way: if $|\theta^* - \theta| < \delta(\theta)$, then the inequality

$$\sup_{(t,s)\in I^{2d}} |R(t, s; \theta) - R(t, s; \theta^*)| \leqq \frac{\varepsilon}{2} \operatorname{mix}\left(\frac{1}{K_2}, \frac{1}{K_3}\right) \leqq \frac{1}{K_1} \tag{2.14}$$

holds. The existence of $\delta(\theta)$ follows from (2.10). Let $A(\varepsilon, \theta)$ denote the open ball with centre θ and radius $\delta(\theta)$. Then the union $\bigcup_{\theta\in\Theta} A(\varepsilon, \theta)$ covers the set Θ. Using the compactness of Θ, we have that a finite sequence $A_1(\varepsilon, \theta_1), \ldots, A_l(\varepsilon, \theta_l)$ also covers Θ. If $\theta\in A_i(\varepsilon, \theta_i)$, $1 \leqq i \leqq l$, then we can define with Mangano's method an orthogonal set $\{f_{j,i}(\theta),\ 1 \leqq j \leqq N\}$ in $H(R(\theta))$ generated by the measures $\{\varkappa_{j,i}(\theta),\ 1 \leqq j \leqq N\}$. As we said in the proof of Lemma 2.1, these functions and measures are linear combinations of $\{e_{j,i},\ 1 \leqq j \leqq N\}$ and $\{v_{j,i},\ 1 \leqq j \leqq N\}$, where $\{e_{j,i},\ 1 \leqq j \leqq N\}$ is an orthonormal set in $H(R(\theta_i))$. If the measures $\{v_{j,i},\ 1 \leqq j \leqq N\}$ are generated by the eigenfunctions $\{\varphi_{j,i},\ 1 \leqq j \leqq N\}$ of $R(t, s; \theta_i)$ then $\varkappa_{j,i}$ can be written in the form

$$\varkappa_{j,i}(dt) = c_{j,i} \varphi_{j,i}(t)\, dt, \tag{2.15}$$

where the $c_{j,i}$, $1 \leqq i \leqq l$, $1 \leqq j \leqq N$, are constants. It follows from the definition of A_i and from Lemma 2.1, that if $\theta, \theta^* \in A_i$, then

$$\sup_{t\in I^d} |f_{j,i}(t; \theta) - f_{j,i}(t; \theta^*)| < \varepsilon, \quad 1 \leqq j \leqq N, \tag{2.16}$$

$$\|\varkappa_{j,i}(\theta) - \varkappa_{j,i}(\theta^*)\| < \varepsilon, \quad 1 \leqq j \leqq N. \tag{2.17}$$

The covariance function of the limiting process $\{Y(t; \theta_0),\ t\in I^d\}$ depends on an unknown parameter θ_0, which will be estimated with a sequence of random variables θ_n, such that

$$|\theta_n - \theta_0| \xrightarrow{P} 0 \quad (n \to \infty). \tag{2.18}$$

Let $\varepsilon = 2^{-m}$ and define the following random functions and measures: if $\theta_n \in A_i(2^{-m}; \theta_i)$, then $\{f^n_{j,i},\ 1 \leqq j \leqq N\}$ denotes the corresponding orthonormal sequence and $\{\varkappa^n_{j,i},\ 1 \leqq j \leqq N\}$ denotes the measures corresponding to $\{v_{j,i}(\theta_i),\ 1 \leqq j \leqq N\}$.

Jain and Kallianpur [8] proved that if $\{Y(t),\ t\in I^d\}$ is a Gaussian process with continuous sample paths a.s., having mean zero and continuous covariance $R(t, s)$, then the sums

$$\sum_{j=1}^{N} \xi_j \varphi_j(t) \tag{2.19}$$

converge uniformly in $t \in I^d$ to $Y(t)$ a.s. as $N \to \infty$, where $\{\varphi_j, j \geqq 1\}$ is a CONS in $H(R)$ and $\{\xi_j, j \geqq 1\}$ is a suitable sequence of independent standard normal random variables. On the other hand, in case $d=1$; if G is a covariance such that there exists a Gaussian process with this covariance and with almost all sample paths continuous, $\{\xi_j^*, j \geqq 1\}$ is a sequence of independent standard normal variables, and if $\{\psi_j, j \geqq 1\}$ is a CONS in $H(G)$, then the sums

$$\sum_{j=1}^{N} \xi_j^* \psi_j(t) \tag{2.20}$$

converge uniformly a.s. in $t \in [0, 1]$ to a Gaussian process whose covariance is G and almost all of whose sample paths are continuous, as $N \to \infty$ (Theorem 2 in [8]). If $d > 1$, then some further conditions on G are needed to retain this statement.

We will assume in this section, that Z is a Gaussian process with almost all sample paths continuous,

$$EZ(t) = 0, \quad EZ(t)Z(s) = G(t, s), \quad t, s \in I^q, \; q \geqq 1,$$

and for every sequence of standard normal random variables $\{\xi_j^*, j \geqq 1\}$ there is a centered Gaussian process Z^* having continuous sample paths a.s. and covariance G such that

$$\sup_{t \in I^q} \Big| \sum_{j=1}^{N} \xi_j^* \psi_j(t) - Z^*(t) \Big| \xrightarrow{P} 0 \quad (N \to \infty), \tag{2.21}$$

where $\{\psi_j(t), j \geqq 1\}$ is a CONS in $H(G)$.

Let $X_n^*(s)$ denote the empirical kernel transform of $X_n(t)$

$$X_n^*(s) = \sum_{j=1}^{N} \int_{I^d} X_n(t) \psi_j(s) \varkappa_{j,i}^n(dt), \quad \text{if} \quad \theta_n \in A_i(2^{-m}, \theta_i) \; s \in I^q. \tag{2.22}$$

If the sequence $\{v_{j,i}, j \geqq 1\}$ is generated the eigenfunctions $\{\varphi_{j,i}, j \geqq 1\}$ of $R(\cdot, \theta_i)$, then the transform can be written in the form

$$X_n^*(s) = \int_{I^d} X_n(t) k_{N,m}^n(t, s)\, dt, \quad s \in I^q, \tag{2.23}$$

where

$$k_{N,m}^n = \sum_{j=1}^{N} c_{j,i}^n \varphi_{j,i}(t) \psi_j(s), \quad \text{if} \quad \theta_n \in A_i(2^{-m}, \theta_i), \tag{2.24}$$

is a random kernel function.

Theorem 2.2. *If the underlying probability space is rich enough, then we can define a sequence $\{Z_n(s)\}$ of copies of $Z(s)$ such that we have*

$$\sup_{s\in I^q} |X_n^*(s)-Z_n(s)| \xrightarrow{P} 0,$$

if N, $m(N)$ and $n(N, m)$ go to infinity.

Proof. Let ε and δ be arbitrary positive constants. The distribution of $\sup_{t\in I^d} |Y_n(t)|$ is independent of n, so there is a constant M_1 such that we have

$$P\{\sup_{t\in I^d} |Y_n(t)| > M_1\} \leqq \delta/8. \tag{2.25}$$

Using condition (2.21) we have that

$$P\Big\{\sup_{s\in I^q} \Big|\sum_{j=N+1}^{\infty} \xi_j \psi_j(s)\Big| > \varepsilon/3\Big\} < \delta/4, \tag{2.26}$$

if $N \geqq N_0$ for every sequence of independent standard normal random variables. Set

$$M_2 = \max_{1\leqq j\leqq N} \sup_{s\in I^q} |\psi_j(s)|.$$

Let $m=m(N)$ be so large that

$$2^{-m} < \varepsilon/(3NM_1M_2). \tag{2.27}$$

The sequence θ_n goes to θ_0 in probability, therefore there is a paramater subset $A_i(2^{-m}, \theta_i)$ such that

$$P\{\theta_n \in A_i(2^{-m}, \theta_i)\} > 1-\delta/2, \tag{2.28}$$

if $n \geqq n_1(N, m)$.

The transformed process X_n^* can be decomposed as

$$X_n^*(s) = \int_{I^d} (X_n(t)-Y_n(t)) \sum_{j=1}^{N} \psi_j(s) \varkappa_{j,i}^n(dt) +$$

$$+\int_{I^d} Y_n(t) \sum_{j=1}^{N} \psi_j(s)\big(\varkappa_{j,i}^n(dt)-\varkappa_{j,i}(\theta_0)(dt)\big) + \int_{I^d} Y_n(t) \sum_{j=1}^{N} \psi_j(s)\varkappa_{j,i}(\theta_0)(dt) =$$

$$= a_{1n}(s)+a_{2n}(s)+a_{3n}(s),$$

say. We assume that $\theta_0 \in A_i(2^{-m}, \theta_i)$. Using (2.9) we have that

$$P\{\sup_{s\in I^q} |a_{1n}(s)| > \varepsilon/3,\ \theta_n \in A_i(2^{-m}, \theta_i)\} \leqq$$

$$\leqq P\{\sup_{t\in I^d} |X_n(t)-Y_n(t)| NM_2(\max_{1\leqq j\leqq N} \|v_{j,i}\| + 2^{-m}) > \varepsilon/3,\ \theta_n \in A_i(2^{-m}, \theta_i)\} \leqq \delta/8,$$

if $n \geqq n_0 = \max\big(n_1(N, m), n_2(N, m)\big)$. The second term also goes to zero in probability, because it follows from (2.25), and (2.27) that

$$P\{\sup_{s\in I^q}|a_{2n}(s)|>\varepsilon/3,\ \theta_n\in A_i(2^{-m},\theta_i)\}\leqq$$
$$\leqq P\{M_1M_2N\max_{1\leqq j\leqq N}(\|\varkappa^n_{j,i}-\varkappa_{j,i}(\theta_0)\|)>\varepsilon/3\}=0.$$

The orthonormal set $\{f_{j,i}(\theta_0),\ 1\leqq j\leqq N\}$ corresponding to the measures $\{\varkappa_{j,i}(\theta_0),\ 1\leqq j\leqq N\}$ can be completed to a CONS $\{f_{j,i}(\theta_0),\ j\geqq 1\}$ in $H(R(\theta_0))$. The sequence $\sum_{j=1}^{M}\xi^n_{j,i}f_{j,i}(t)$ converges uniformly in $t\in I^d$ to $Y_n(t)$, as $M\to\infty$, a.s. with a suitably chosen sequence of independent standard normal variables $\{\xi^n_{j,i},\ j\geqq 1\}$ (see Theorem 1 of JAIN and KALLIANPUR [8]). So by (2.5), $a_{3n}(s)$ can be decomposed as a finite sum

$$a_{3n}(s)=\sum_{j=1}^{N}\xi^n_{j,i}\psi_j(s).$$

Using the condition (2.21), the partial sums

$$\sum_{j=1}^{N}\xi^n_{j,i}\psi_j(s)$$

converge (as $N\to\infty$) uniformly in $s\in I^q$ to a separable Gaussian process denoted by $Z_n(s)$. On the other hand, we have that

$$\{(Z_n(s),\ \sum_{j=1}^{N}\xi^n_{j,i}\psi_j(s)),\ s\in I^q\}\stackrel{\mathscr{D}}{=}\{(Z(s),\ \sum_{j=1}^{N}\xi_j\psi_j(s)),\ s\in I^q\},$$

where

$$Z(s)=\sum_{j=1}^{\infty}\xi_j\psi_j(s)\quad\text{a.s.},$$

and $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution. So it follows form (2.26) that

$$P\{\sup_{s\in I^q}|a_{3n}(s)-Z_n(s)|>\varepsilon/3\}\leqq\delta/4.$$

Summing up, we proved that if $N\geqq N_0(\varepsilon,\delta)$, $m\geqq m_0(N,\varepsilon,\delta)$ and $n\geqq n_0(N,m,\varepsilon,\delta)$, then

$$P\{\sup_{s\in I^q}|X^*_n(s)-Z_n(s)|>\varepsilon\}\leqq$$
$$\leqq P\{\sup_{s\in I^q}|X^*_n(s)-Z_n(s)|>\varepsilon,\ \theta_n\in A_i(2^{-m},\theta_i)\}+P\{\theta_n\notin A_i(2^{-m},\theta_i)\}<\delta,$$

which is the desired conclusion.

3. Applications. So far $q\geqq 1$ was arbitrary, and from now on we choose $q=1$ since univariate limit processes are more convenient to handle. First we study the estimated empirical process when $d=1$. Let $F^{-1}(t;\theta)$ denote the inverse function to $F(t;\theta)$,

$$F^{-1}(t;\theta)=\inf\{s:\ F(s;\theta)\geqq t\}.$$

It follows from Theorem A, that

$$\sup_{0\le t\le 1} |\beta_n(F^{-1}(t;\theta_n))-D_n(F^{-1}(t;\theta_0);\theta_0)| \xrightarrow{P} 0 \quad (n\to\infty)$$

and

$$ED_n(F^{-1}(t;\theta_0);\theta_0)=0,$$

$$ED_n(F^{-1}(t;\theta_0);\theta_0)D_n(F^{-1}(s;\theta_0);\theta_0)=R_1(t,s;\theta_0)=t\wedge s-ts-$$
$$-J(F^{-1}(t;\theta_0);\theta_0)\nabla_\theta^T F(F^{-1}(s;\theta_0);\theta_0)-J(F^{-1}(s;\theta_0);\theta_0)\nabla_\theta^T F(F^{-1}(t;\theta_0);\theta_0)+$$
$$+\nabla_\theta F(F^{-1}(t;\theta_0);\theta_0)M(\theta_0)\nabla_\theta^T F(F^{-1}(s;\theta_0);\theta_0),$$

where

$$J(t;\theta_0)=\int_{-\infty}^{t} l(u;\theta_0)\,dF(u;\theta_0).$$

The processes $D_n(F^{-1}(t;\theta_0);\theta_0)$ have continuous sample functions a.s. if $\nabla_\theta F(F^{-1}(t;\theta_0);\theta_0)$ and $J(F^{-1}(t;\theta_0))$ are continuous functions of t. The covariance function $R_1(t,s;\theta^*)$ will be continuous in θ^* if $M(\theta^*)$, $J(F^{-1}(t;\theta^*);\theta^*)$ and $\nabla_\theta F(F^{-1}(t;\theta^*);\theta^*)$ are continuous functions of $\theta^*\in\Theta$. The random function $R_1(t,s;\theta_n)$ is an estimate of the covariance function of the limit process. So we can define β_n^*, the empirical transform of $\beta_n(F^{-1}(t;\theta_n))$ as it was defined by (2.22) or (2.23). The sample $X_1,\dots,X_n$ from a distribution belonging to the parametric family $\mathscr{F}$ determine only the random measures (and functions) in the definition of the empirical transform, so we can choose the eigenfunctions $\{\psi_j, j\ge 1\}$ of the limit process without restriction. For example, if

$$\psi_k(s)=(\sqrt{2}/k\pi)\sin k\pi s,\quad 0\le s\le 1, \tag{3.1}$$

then the limit process will be the Brownian bridge. If

$$\psi_1(s)=s, \tag{3.2}$$

$$\psi_{k+1}(s)=(\sqrt{2}/k\pi)\sin k\pi s,\quad 0\le s\le 1 \tag{3.3}$$

then $\beta_n^*(s)$, $0\le s\le 1$, will converge weakly to the Wiener process.

Theorem 3.1. *We suppose, that the conditions* (i), (ii), (v) *of Theorem A are satisfied and*

(iii)* $M(\theta^*)=El^T(X^*;\theta^*)l(X^*;\theta^*)$ *is a finite, nonnegative definite matrix and* $M(\theta^*)$ *is continuous in* $\theta^*\in\Theta$, *where* $P(X^*<t)=F(t;\theta^*)$,

(iv)* $J(F^{-1}(t;\theta^*);\theta^*)$ *and* $\nabla_\theta F(F^{-1}(t;\theta^*);\theta^*)$ *are uniformly continuous in* t, $0\le t\le 1$, *and* $\theta^*\in\Theta$, *where* $\Theta\subset R^p$ *is a compact parameter set and the true value* θ_0 *is an interior point of* Θ.

Then we can define a sequence $\{Z_n(s)\}$ *of copies of* $Z(s)$ *on the probability space of Theorem 2.2 such that we have*

$$\sup_{0\le s\le 1}|\beta_n^*(s)-Z_n(s)| \xrightarrow{P} 0,$$

if N, m, n *go to infinity.*

Proof. It follows from the conditions of the theorem that $R_1(t, s; \theta)$, $0 \leqq t, s \leqq 1$, $\theta \in \Theta$, satisfies (2.10) and $|\theta_n - \theta_0| \xrightarrow{P} 0$, as $n \to \infty$. The processes $D_n(F^{-1}(t; \theta); \theta)$ have continuous sample path functions a.s., so this theorem is a consequence of Theorem 2.2.

The most important special case of this theorem is when we estimate shift and location parameters only, i.e., the parametric family can be written in the form

$$\mathscr{F}_s = \left\{F\left(\frac{t-\theta_0^1}{\theta_0^2}\right), \ -\infty < \theta_0^1 < \infty, \theta_0^2 > 0, \ t \in R^1\right\}.$$

The covariance function of the limit process for the shift and location estimated empirical process was computed by DARLING [3] and DURBIN [4] (cf. [5]). They proved that the covariance function of $D_n(F^{-1}(t; \theta_0^1, \theta_0^2); \theta_0^1, \theta_0^2)$ does not depend on (θ_0^1, θ_0^2):

$$ED_n(F^{-1}(t; \theta_0^1, \theta_0^2); \theta_0^1, \theta_0^2) D_n(F^{-1}(s; \theta_0^1, \theta_0^2); \theta_0^1, \theta_0^2) = R_2(t, s) =$$
$$= t \wedge s - ts - [I_{11} I_{22} - I_{12}^2]^{-1} [I_{22} w_1(t) w_1(s) +$$
$$+ I_{11} w_2(t) w_2(s) - I_{12}(w_1(t) w_2(s) + w_2(t) w_1(s))],$$

where

$$w_1(t) = f(F^{-1}(t)), \quad w_2(t) = F^{-1}(t) f(F^{-1}(t)),$$

$$I_{11} = \int_{-\infty}^{\infty} \left[\frac{f'(x)}{f(x)}\right]^2 f(x)\,dx, \quad I_{12} = -\int_{-\infty}^{\infty} x \left[\frac{f'(x)}{f(x)}\right]^2 f(x)\,dx,$$

$$I_{22} = \int_{-\infty}^{\infty} x^2 \left[\frac{f'(x)}{f(x)}\right]^2 f(x)\,dx - 1,$$

and f, f' are the first and second derivatives of F assumed to exists. In this special case we do not have to estimate R_2 from the sample, so the transformation of $\beta_n(F^{-1}(t; \theta_{n1}, \theta_{n2}))$ will be non-random. If $\{\varphi_j^*, j \geqq 1\}$ is a CONS in $H(R_2)$ generated by the measures $\{v_j^*, j \geqq 1\}$ then the transformation of $\beta_n(F^{-1}(t; \theta_n^1, \theta_n^2))$

$$\beta_n^*(s) = \int_0^1 \beta_n(F^{-1}(t; \theta_{n1}, \theta_{n2})) \sum_{j=1}^N \psi_j(s) v_j^*(dt)$$

is also non-random, and

$$\sup_{0 \leqq s \leqq 1} |\beta_n^*(s) - Z_n(s)| \xrightarrow{P} 0 \quad (n \to \infty),$$

where $\{Z_n\}$ is a sequence of copies of Z.

Finally we study the general case, when d is an arbitrary positive integer. The transformation of the parameter estimated empirical process into the unit interval was very simple in the one dimensional case, but in the general case it is a bit more complicated. Let $F_j(x_j; \theta)$ denote the j^{th} marginal distribution of $F(x; \theta)$, $x = (x_1, \ldots, x_d)$. There is a d-variate distribution function $H(x; \theta)$, all the univariate marginals of which being uniformly distributed on [0, 1], such that

$$F(x;\theta) = H\big(F_1(x_1;\theta), \dots, F_d(x_d;\theta);\theta\big), \quad x=(x_1,\dots,x_d).$$

Let F_j^{-1} denote the left-continuous inverse of F_j and define the following function

$$\bar F(t;\theta) = H\big(F_1^{-1}(t_1;\theta), \dots, F_d^{-1}(t_d;\theta);\theta\big), \quad t=(t_1,\dots,t_d).$$

So the process $\beta_n(\bar F(t;\theta_n))$ is defined on I^d, and it follows from Theorem A, that

$$\sup_{t\in I^d} \big|\beta_n(\bar F(t;\theta_n)) - D_n(\bar F(t;\theta_0);\theta_0)\big| \xrightarrow{P} 0 \quad (n\to\infty).$$

The covariance function of $D_n(\bar F(t;\theta_0);\theta_0))$ can be computed from the representation of the Durbin process. We have, for $t=(t_1,\dots,t_d)$, $s=(s_1,\dots,s_d)$, that

$$ED_n(\bar F(t;\theta_0);\theta_0) = 0,$$

$$ED_n(\bar F(t;\theta_0);\theta_0)\, D_n(\bar F(s;\theta_0);\theta_0) = R_3(t,s;\theta_0) =$$
$$= F(\bar F(t\wedge s;\theta_0);\theta_0) - F(\bar F(t;\theta_0);\theta_0)F(\bar F(s;\theta_0);\theta_0) -$$
$$- J(\bar F(t;\theta_0);\theta_0)\nabla_\theta^T F(\bar F(s;\theta_0);\theta_0) - J(\bar F(s;\theta_0);\theta_0)\nabla_\theta^T F(\bar F(t;\theta_0);\theta_0) +$$
$$+ \nabla_\theta F(\bar F(t;\theta_0);\theta_0)M(\theta_0)\nabla_\theta^T F(\bar F(s;\theta_0);\theta_0).$$

The following theorem is a generalization of Theorem 3.1 for arbitrary d. Let β_n^* denote the empirical kernel-transformed empirical process defined by (2.22) or (2.23).

Theorem 3.2. *We suppose, that the conditions* (i), (ii), (v) *of Theorem A are satisfied and*

(iii)* $M(\theta^*) = El^T(X^*;\theta^*)l(X^*;\theta^*)$ *is a finite, nonnegative definite matrix and* $M(\theta^*)$ *is continuous in* $\theta^*\in\Theta$, *where* $P\{X^{*1}<t_1,\dots,X^{*d}<t_d\} = F(t;\theta^*)$, $t=(t_1,\dots,t_d)$, $X^*=(X^{*1},\dots,X^{*d})$,

(iv)* $J(\bar F(t;\theta^*);\theta^*)$ *and* $\nabla_\theta F(\bar F(t;\theta^*);\theta^*)$ *are uniformly continuous in* $t\in I^d$ *and* $\theta^*\in\Theta$, *where* $\Theta\subset R^p$ *is a compact parameter set, and* θ_0 *is an interior point of* Θ.

Then we can define a sequence $\{Z_n(s), 0\le s\le 1\}$ *of copies of* $Z(s)$ *on the probability space of Theorem 2.2 such that we have*

$$\sup_{0\le s\le 1} |\beta_n^*(s) - Z_n(s)| \xrightarrow{P} 0,$$

if N, m, n *go to infinity.*

The proof of this theorem is very similar to the proof of Theorem 3.1, therefore it is omitted. These conditions are stronger than the conditions of Theorem A in order to guarantee the applicability of Theorem 3.1.

We proved only the existence of the empirical kernel transform with nice limiting properties, but so far we said nothing about the decomposition $A_i(\varepsilon;\theta_i)$, $i=1,\dots,l$, of the compact parameter set Θ and hence about the concrete choice of the $\varphi_{j,i}$ functions and the quantities $c_{j,i}$ in (2.24). We noticed in Section 2, that the eigenvalues and eigenfunctions of $R(t,s;\theta)$ determine a CONS in $H(R(\theta))$. Let

$\lambda_1(\theta), \ldots, \lambda_N(\theta)$ denote the first N largest eigenvalues of $R(\theta)$. We can choose a Θ^* neighbourhood of θ_0, such that there is a positive lower bound of $\lambda_j(\theta)$, $1 \leqq j \leqq N$, $\theta \in \Theta^*$. It follows from (2.12) and from the continuity of K_i, $i=1, 2, 3$, that we have

$$K_i(N, M(\theta), \|v_1(\theta)\|, \ldots, \|v_N(\theta)\|) \leqq L_i, \quad i = 1, 2, 3,$$

if $\theta \in \Theta^*$. Using (2.13) we see that every $\theta \in \Theta^*$ can be the centre of the balls $A_i(\varepsilon; \theta)$ and the radii of $A_i(\varepsilon; \theta)$ does not already depend on θ. Therefore an arbitrary devision of Θ^* will suffice if this devision is fine enough (for example, the common radius of the balls is small enough). But because $\theta_n \xrightarrow{P} \theta_0$, in practice we may assume that $\theta_n \in \Theta^*$, and hence the devision of $\Theta \setminus \Theta^*$ is completely arbitrary. This choice of the $A(\varepsilon; \theta_i)$ will be suitable for us, if we use the first N largest eigenvalues and the corresponding eigenfunctions of $R(\theta)$ to make the empirical kernel transformation.

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On non-modular n-distributive lattices: The decision problem for identities in finite n-distributive lattices

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To Professor K. Tandori on his sixtieth birthday

1. Introduction. It was proved in [1] that the lattice $\mathfrak{C}(R^{n-1})$ of all convex sets of the $n-1$ dimensional Euclidean space R^{n-1} is a member of the lattice variety $\boldsymbol{D}_n^f$ generated by the finite n-distributive lattices. It is an open question whether this variety equals $\boldsymbol{D}_n$, the class of all n-distributive lattices. An answer might be based on a solution to the word problem for free lattices in $\boldsymbol{D}_n$. In this paper we accomplish a slightly different task and solve the word problem for free lattices in $\boldsymbol{D}_n^f$. Besides, we give a new example of a lattice in this variety, namely we show that the dual of $\mathfrak{C}(R^{n-1})$ is a member of $\boldsymbol{D}_n^f$, too.

We need some notions of universal algebra and lattice theory. By an n-distributive lattice we mean a lattice satisfying the identity

$$x\wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n} \Big(x\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big).$$

A lattice variety is a class of lattices that can be characterized by a set of identities. The variety generated by a class $\boldsymbol{K}$ of lattices is the smallest lattice variety containing $\boldsymbol{K}$. The decision problem for identities in a class $\boldsymbol{K}$ of lattices is the problem of finding an algorithm which, given any identity $p=q$, decides whether $p=q$ holds in every member of $\boldsymbol{K}$ or not. It is equivalent to the word problem for free lattices in the variety generated by $\boldsymbol{K}$.

We are going to use the following concepts concerning convex sets. Let $\boldsymbol{a}, \boldsymbol{r}_0\in R^{n-1}$. Then the set of all $\boldsymbol{r}\in R^{n-1}$ such that the scalar product $(\boldsymbol{a}, \boldsymbol{r}-\boldsymbol{r}_0)$ equals 0, is called a hyperplane. The set of all $\boldsymbol{r}$ with $(\boldsymbol{a}, \boldsymbol{r}-\boldsymbol{r}_0)\geqq 0$ is called a (closed) halfspace. A finite intersection of halfspaces is a convex polyhedron. The convex closure of a finite number of points is a convex polytope. It is well-known that convex polytopes,

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convex polyhedra and convex sets of R^{n-1} all form lattices, and that in all these three lattices the operations are the intersection and the convex closure of two convex sets. (See [2].) Convex polytopes are exactly the bounded convex polyhedra, thus, in the above list, the former lattice is always a proper sublattice of the latter one.

2. On the dual of $\mathfrak{C}(R^{n-1})$. We prove the following theorem.

Theorem 2.1. *The dual of $\mathfrak{C}(R^{n-1})$ is a member of the variety D_n^f.*

Proof. In [1], Lemma 3.1, it was shown that $\mathfrak{C}(R^{n-1})$ is a member of the variety generated by the lattice $\mathfrak{C}_{\text{fin}}(R^{n-1})$ of all $n-1$ dimensional convex polytopes, therefore, it is also a member of the variety generated by $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ of all $n-1$ dimensional convex polyhedra. Thus it is sufficient to show that the latter lattice is a member of the variety generated by all finite dually n-distributive lattices. By Theorem 1.1 of [1], $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ is dually n-distributive and its meet-irreducible elements are exactly the halfspaces of R^{n-1}. Let K be a finite set of halfspaces and let $\mathfrak{C}^-(K)$ consist of all those convex polyhedra that are intersections of elements of K. $\mathfrak{C}^-(K)$ is a lattice ordered by the inclusion relation, in fact, it is a meet-sublattice of $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$. Let $\mathscr{K}$ denote the set of all finite subsets of the set of all halfspaces of R^{n-1}. The following two facts obviously include Theorem 2.1.

Lemma 2.2. *For any $K \in \mathscr{K}$, $\mathfrak{C}^-(K)$ is dually n-distributive.*

Lemma 2.3. *$\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ is a member of the variety generated by all $\mathfrak{C}^-(K)$; $K \in \mathscr{K}$.*

Proof of Lemma 2.2. The dual n-distributivity of $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ and the meet-irreducibility of halfspaces in it imply that whenever a halfspace contains the intersection of a finite number of other halfspaces, then it contains the intersection of n of these halfspaces. In fact, let $h, h_1, \ldots, h_m$, $m > n$, be halfspaces and assume that h contains the intersection of the h_i, $i = 1, 2, \ldots, m$. Then (denoting by $\vee$ the convex closure)

$$h = h \vee \bigcap_{i=1}^{m} h_i = \bigcap_{\substack{L \subseteq \{1, \ldots, m\} \\ |L| = n}} \left(h \vee \bigcap_{i \in L} h_i\right),$$

and, by the irreducibility of h, there is an L, $|L| = n$ with

$$h = h \vee \bigcap_{i \in K} h_i, \quad \text{i.e.,} \quad h \supseteq \bigcap_{i \in K} h_i.$$

Clearly, the lattices $\mathfrak{C}^-(K)$ also satisfy this property, as it refers only to inclusion and intersection, which coincide in $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ with those in $\mathfrak{C}^-(K)$. This, in turn, implies that the lattices $\mathfrak{C}^-(K)$ are also dually n-distributive. To prove this, let $a, b_0, \ldots$

$\ldots, b_n \in \mathfrak{C}^-(K)$. Let $h \in K$, and assume that

$$h \supseteq a \vee_H \bigcap_{i=0}^{n} b_i.$$

Then h contains a and h also contains n of the halfspaces occurring in the meet-representations of the b_i's. Thus h contains n of the b_i's, too, that is,

$$h \supseteq \bigcap_{j=0}^{n} (a \vee_H \bigcap_{\substack{i=0 \\ i \neq j}}^{n} b_i).$$

Thus the meet-representations of $a \vee_H \bigcap_{i=0}^{n} b_i$ and of $\bigcap_{j=0}^{n} (a \vee_H \bigcap_{\substack{i=0 \\ i \neq j}} b_i)$ coincide.

Proof of Lemma 2.3. Let $p \geqq q$ be an m-ary lattice inequality holding in all the lattices $\mathfrak{C}^-(K)$, $K \in \mathscr{K}$. Let $a_1, \ldots, a_m \in \mathfrak{C}^-_{\text{fin}}(R^{n-1})$. Let A be the set of subpolynomials of p, that is, (i) let $p \in A$, (ii) for $p_1 \wedge p_2 \in A$ or $p_1 \vee p_2 \in A$ let $p_1, p_2 \in A$, and (iii) let A be minimal relative to (i) and (ii). Let B be the set of subpolynomials of q. Finally let C be the set of all polyhedra $r(a_1, \ldots, a_m)$, $r \in A \cup B$, and let K be the set of all halfspaces occurring in the irredundant meet-representation of one of the elements of C. (A polyhedron can be represented as an intersection of halfspaces in different ways, however, the irredundant meet-representation is unique.) Let the realization of a polynomial r in the lattice $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ be also denoted by r and let its realization in $\mathfrak{C}^-(K)$ be denoted by r^K. Then

$$p(a_1, \ldots, a_m) = p^K(a_1, \ldots, a_m) \geqq q^K(a_1, \ldots, a_m) = q(a_1, \ldots, a_m),$$

as K was chosen exactly to satisfy the two equalities in the above calculation.

3. On the variety D_n^f. Here we deal with the word problem for free lattices of D_n^f, in other words with the decision problem for identities in D_n^f.

Theorem 3.1. *The word problem for free lattices in D_n^f is solvable.*

Before the proof we introduce some notations. Clearly, every lattice polynomial p can be written in the form

$$p = \bigvee_{i_1 \in I} \bigwedge_{i_2 \in I_{i_1}} \bigvee_{i_3 \in I_{i_1 i_2}} \cdots \bigwedge_{i_{2k-2} \in I_{i_1 \ldots i_{2k-3}}} \bigvee_{i_{2k-1} \in I_{i_1 \ldots i_{2k-2}}} x_{i_1 \ldots i_{2k-1}} \tag{1}$$

if we allow I and the $I_{i_1 \ldots i_r}$'s to consist of one element. We define the depth $d(p)$ of p by $d(p) := k$. $m(p)$ denotes the length (that is, the number of components) in the longest meet:

$$m(p) = \max \{\max_{i_1 \in I} |I_{i_1}|, \max_{\substack{i_1 \in I \\ i_2 \in I_{i_1} \\ i_3 \in I_{i_1 i_2}}} |I_{i_1 i_2 i_3}|, \ldots\}.$$

Now define

$$c_n(p):=1+n+n^2\cdot m(p)+n^3\cdot(m(p))^2+\ldots+n^{d(p)}\cdot(m(p))^{d(p)-1}.$$

We are ready to formulate the following lemma.

Lemma 3.2. *Let $p\leqq q$ be a lattice inequality holding in all finite n-distributive lattices containing at most $c_n(p)$ join-irreducible elements. Then $p\leqq q$ holds in every finite n-distributive lattice.*

To decide whether $p\leqq q$ holds in $\boldsymbol{D}_n^f$ requires now to check those finite n-distributive lattices having at most $c_n(p)$ join-irreducibles. This can be carried out in finite time, hence Lemma 3.2 implies Theorem 3.1.

Proof of the lemma. Let L be a finite lattice, let p and q be lattice polynomials in m variables and let $a_1, \ldots, a_m \in L$. Let K denote the set of join-irreducible elements of L. For a lattice polynomial r, let r^L denote the realization of r on L. Let $b\in K$ and let $b\leqq p^L(a_1, \ldots, a_m)$. Under the hypotheses of the lemma, we shall prove that $b\leqq q^L(a_1, \ldots, a_m)$. Let us introduce the following notations for subpolynomials of p. (p is defined by (1).)

$$p_{i_1}=\bigwedge_{i_2\in I_{i_1}}\ \bigvee_{i_3\in I_{i_1 i_2}}\cdots\bigwedge_{i_{2k-2}\in I_{i_1\ldots i_{2k-3}}}\ \bigvee_{i_{2k-1}\in I_{i_1\ldots i_{2k-2}}} x_{i_1\ldots i_{2k-1}},\quad i_1\in I,$$

$$p_{i_1 i_2}=\bigvee_{i_3\in I_{i_1 i_2}}\cdots\bigwedge_{i_{2k-2}\in I_{i_1\ldots i_{2k-3}}}\ \bigvee_{i_{2k-1}\in I_{i_1\ldots i_{2k-2}}} x_{i_1\ldots i_{2k-1}},\quad i_1\in I,\quad i_2\in I_{i_1},$$

etc. Now, by the assumption on b, we have

$$b\leqq\bigvee_{i_1\in I} p_{i_1}^L(a_1,\ldots,a_m).$$

Each $p_{i_1}^L(a_1, \ldots, a_m)$ is a join of join-irreducibles. By the n-distributivity of L, we may choose n of these join-irreducibles, say $b_1, \ldots, b_n$ such that

$$b\leqq\bigvee_{j=1}^{n} b_j.$$

(A detailed proof of this fact can be given by dualizing and generalizing the first part of the proof of Lemma 2.2.) Now assign to each b_j one (and only one) $p_{i_1}^L(a_1, \ldots, a_m)$ such that

$$b_j\leqq p_{i_1}^L(a_1,\ldots,a_m).$$

Then, for every b_j, if $p_{i_1}^L(a_1, \ldots, a_m)$ is assigned to b_j, we have

$$b_j\leqq p_{i_1 i_2}^L(a_1,\ldots,a_m),\quad i_2\in I_{i_1},$$

that is,

$$b_j\leqq\bigvee_{i_3\in I_{i_1 i_2}} p_{i_1 i_2 i_3}^L(a_1,\ldots,a_m),\quad i_2\in I_{i_1}.$$

Now we carry out the same construction in these $|I_{i_1}|$ different cases on b_j and on $\bigvee_{i_3 \in I_{i_1 i_2}} p^L_{i_1 i_2 i_3}(a_1, a_2, \ldots, a_m)$, with which we started on b and on $\bigvee_{i_1 \in I} p^L_{i_1}(a_1, \ldots, a_m)$: For arbitrary fixed $i_2 \in I_{i_1}$ choose join-irreducibles $b_{ji_21}, \ldots, b_{ji_2n}$ of L such that

$$b_j \leqq \bigvee_{l=1}^{n} b_{ji_2l}.$$

Again, each b_{ji_2l} is less than or equal to one of the $p^L_{i_1 i_2 i_3}(a_1, \ldots, a_m)$'s. Assign a $p^L_{i_1 i_2 i_3}(a_1, \ldots, a_m)$ to b_{ji_2l} such that

$$b_{ji_2l} \leqq p^L_{i_1 i_2 i_3}(a_1, \ldots, a_m),$$

etc. Let K_0 be the set of join-irreducibles defined during this procedure, that is,

$$K_0 = \{b\} \cup \{b_1, \ldots, b_n\} \cup \bigcup_{j=1}^{n} \bigcup_{\substack{i_2 \in I_{i_1} \\ i_1 \text{ is} \\ \text{assigned} \\ \text{to } j}} \{b_{ji_21}, \ldots, b_{ji_2n}\} \cup \ldots .$$

Clearly, $|K_0| \leqq c_n(p)$. Let $\tilde{a}_i = \bigvee_{\substack{c \in K_0 \\ c \leqq a_i}} c$.

Let, furthermore, L_0 consist of all joins of elements of K_0. Then, by the definitions of K_0, L_0 and of $\tilde{a}_i$, $b \leqq p^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m)$. By the hypotheses, $p^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m) \leqq q^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m)$. (Here we need the n-distributivity of L_0, which is a consequence of the fact that whenever a join-irreducible element in L_0 is less than or equal to a join of elements of L_0, then it is less than or equal to an n-element subjoin of that join.) We obviously have $q^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m) \leqq q^L(\tilde{a}_1, \ldots, \tilde{a}_m) \leqq q^L(a_1, \ldots, a_m)$. Hence $b \leqq q^L(a_1, \ldots, a_m)$, as claimed.

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Multiplicative functions with nearly integer values

I. KÁTAI and B. KOVÁCS

Dedicated to Professor Károly Tandori on the occasion of his 60th birthday

We shall say that a realvalued arithmetical function $f(n)$ is completely multiplicative if $f(mn)=f(m)\cdot f(n)$ holds for each pairs of integers. Let $\|z\|$ denote the distance of z to the nearest integer, and $[z]$ denote the integer part of z.

We are interested in to determine the class of those completely multiplicative functions for which

$$\|f(n)\| \to 0 \quad (n \to \infty). \tag{1}$$

It is obvious that the validity of (1) does not depend on the sign of $f(n)$, since $\|z\|=\|-z\|$, so we may assume that $f(n)\geqq 0$.

We shall say that a real number θ is a Pisot-number, if it is an algebraic integer, $\theta>1$, and if all conjugates $\theta_2, \ldots, \theta_r$, are in the domain $|z|<1$. It is well known for a Pisot-number the relation

$$\|\theta^n\| \to 0 \tag{2}$$

holds. (See [1].)

Let now the whole set of primes $\mathcal{P}$ be divided into two disjoint subsets $\mathcal{P}_1, \mathcal{P}_2$ and $f(n)$ be defined for $p\in\mathcal{P}$ as follows:

$$f(p) = \begin{cases} 0 & \text{if } p\in\mathcal{P}_1 \\ \theta^{x(p)} & \text{if } p\in\mathcal{P}_2, \end{cases}$$

where $x(p)$ is a positive integer for each $p\in\mathcal{P}_2$ and $x(p)\to\infty$ if $p\to\infty$, furthermore θ is a Pisot-number. Then the completely multiplicative $f(n)$ determined by these values satisfies the relation (1).

For an algebraic α let $Q(\alpha)$ denote the simple extension of the rational number field generated by α.

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Lemma 1. *Let β be an algebraic number, $f(n)$ be completely multiplicative with values in $Q(\beta)$. Let $\beta_2, \ldots, \beta_r$ be the conjugates of β over Ω. Let $\varphi_j(n)$ denote the conjugate of $f(n)$ defined by the substitution $\beta \to \beta_j$. Then $\varphi_j(n)$ are completely multiplicative functions as well.*

Proof. Let $f(n)=r_n(\beta)$. Then $\varphi_j(n)=r_n(\beta_j)$. Since $r_{mn}(\beta)=f(mn)= =f(m)\cdot f(n)=r_m(\beta)\cdot r_n(\beta)$, therefore $\varphi_j(mn)=r_{mn}(\beta_j)=r_m(\beta_j)r_n(\beta_j)=\varphi_j(m)\varphi_j(n)$.

Lemma 2. *Let β be an algebraic number and $f(n)$ a completely multiplicative function the values $f(n)$ of which are integers in $Q(\beta)$. Assume that*

$$\varphi_j(p) \to 0 \quad \text{as} \quad p \to \infty, \quad (j = 2, \ldots r), \tag{3}$$

where p runs over the set of primes. Then (1) *holds.*

Proof. It is obvious that (3) involves that $\varphi_j(n) \to 0$ $(n \to \infty)$. Furthermore $\varphi_j(n)$ are algebraic integers, and so

$$f(n)+\varphi_2(n)+\ldots+\varphi_r(n) = E_n = \text{rational integer},$$

whence (1) follows immediately.

To give a partial answer for our problem we shall use the following known theorems [1] as Lemma 3 and 4.

Lemma 3. *Let $\alpha>1$ be an algebraic number, $\lambda \neq 0$ be a real number and*

$$\|\lambda\alpha^n\| \to 0 \quad (n \to \infty). \tag{4}$$

Then α is a Pisot-number, $\lambda=\alpha^{-N}\mu$, where $N \geqq 0$ is a suitable integer, $\mu \in Q(\alpha)$.

Lemma 4. *Let $\alpha>1$, $\lambda \neq 0$ be a real number and*

$$\sum_{0 \leqq n < \infty} \|\lambda\alpha^n\|^2 < \infty. \tag{5}$$

Then α is an algebraic number, consequently the assertion stated in Lemma 3 holds.

Lemma 5. *Let $f(n) \geqq 0$ be a completely multiplicative function for which* (1) *holds. If $f(n_0)>1$ for at least one n_0, then $f(n) \geqq 1$ or $f(n)=0$ for each values of n.*

Proof. Assume in contrary that $0<f(m_0)<1$. Let $b=f(n_0)$, $a=f(m_0)$, $a=f(m_0)$, $x_0=[-3\log a]+1$. For infinitely many k, l pairs of positive integers we have

$$\frac{-2x_0}{\log a} > k + l\frac{\log b}{\log a} > \frac{-x_0}{\log a},$$

since the length of the interval $\left(\frac{-x_0}{\log a}, \frac{-2x_0}{\log a}\right)$ is at least three. For such pairs

k, l we have $2^{-2x_0} < a^k b^l < 2^{-x_0}$, consequently

$$2^{-2x_0} < \|f(m_0^k n_0^l)\| = \|a^k b^l\| < 2^{-x_0}.$$

But this contradicts to (1).

Lemma 6. *Let $f \geqq 0$ be a completely multiplicative function satisfying* (1). *Assume that there exists an m for which $f(m) > 1$ and $f(m)$ is algebraic over Q. Let $\mathcal{P}_1$ be the set of those primes p for which $f(p) \neq 0$. Then the values $f(p)$ are Pisot-numbers for each $p \in \mathcal{P}_1$, and for every $p_1, p_2 \in \mathcal{P}_1$ we have $Q(\alpha_{p_1}) = Q(\alpha_{p_2})$, $\alpha_{p_1} = f(p_1)$, $\alpha_{p_2} = f(p_2)$.*

Proof. Let $f(m) = \alpha$. Since $\alpha > 1$, α algebraic; and $\|f(m^k)\| = \|\alpha^k\| \to 0$ $(k \to \infty)$, by Lemma 3 we get that $f(m)$ is a Pisot-number.

Let now n be an arbitrary natural number for which $f(n) \neq 0$. Since $\|f(nm^k)\| = \|f(n)\alpha^k\| \to 0$ $(k \to \infty)$, from Lemma 3 we deduce that $f(n) = \alpha^{-N}\gamma$, $N \geqq 0$, integer, $\gamma \in Q(\alpha)$. Hence $\beta = f(n) = \alpha^{-N}\gamma \in Q(\alpha)$. Since $\beta \neq 0$, from Lemma 3 we get that $\beta > 1$, and so by repeating the above argument with β instead of α, we deduce that β is a Pisot-number and $\alpha \in Q(\beta)$. The assertion is proved.

Corollary. Let $f(n) \geqq 0$ be a completely multiplicative function satisfying (1). If $1 < f(n) \in Q$ holds for at least one n, then $f(n)$ takes on integer values for every n.

Lemma 7. *Let $f(n) \geqq 0$ be a completely multiplicative function satisfying the relation*

$$\|f(n)\| \leqq \varepsilon(n), \tag{6}$$

where $\varepsilon(n)$ is a monotonically decreasing function, with

$$\sum_{k=1}^{\infty} \varepsilon^2(2^k) < \infty. \tag{7}$$

Then the following possibilities are:

a) *f takes on integer values for every n.*

b) *For a suitable n $0 < f(n) < 1$. Then $f(n) \to 0$ as $n \to \infty$.*

c) *For a suitable m $f(m) > 1$. Let $\mathcal{P}_1$ denote the whole set of those primes p for* which $f(p) \neq 0$. *Then there exists a Pisot-number Θ such that $Q(f(p)) = Q(\Theta)$ for each $p \in \mathcal{P}_1$.*

Proof. The relation (6) involves (4). If $0 < f(n) < 1$ then from Lemma 5 $f(m) \leqq 1$ for every m. If $f(m) = 1$, then $\|f(nm^k)\| = \|f(n)\|$ as $k \to \infty$; that contradicts to (1). Consequently $f(m) < 1$ for each $m > 1$. Assume that there exists a subsequence $n_1 < n_2 < \ldots$ such that $f(n_j) \to 1$. Then $f(nn_j) \to f(n)$ $(j \to \infty)$ that contradicts to (1). Consequently $f(m) \to 0$ as $m \to \infty$.

Let us consider the case c). Taking into account (6) and (7), the conditions of Lemma 4 are satisfied with $\lambda=1, \alpha=f(m)>1$. Consequently α is an algebraic number, and the assertion is an immediate consequence of Lemma 6.

Theorem 1. *Let $f(n)\geqq 0$ be a completely multiplicative function that takes on at least one algebraic value $f(n_0)=\alpha>1$. Let $\mathcal{P}_1$ denote those set of primes p for which $f(p)\neq 0$.*

If (1) holds, then the values $f(p)=\alpha_p$ are Pisot-numbers, for each $p_1, p_2\in\mathcal{P}_1$ we have $Q(\alpha_{p_1})=Q(\alpha_{p_2})$. Let Θ denote one of the values α_p $(p\in\mathcal{P}_1)$, $\Theta_2, \ldots, \Theta_r$ its conjugates, $\varphi_2(n), \ldots, \varphi_r(n)$ be defined as in Lemma 1. Then

$$\varphi_j(n)\to 0 \quad as \quad n\to\infty, \quad j=2, \ldots r. \tag{8}$$

In contrary, let us assume that the values $f(p)$ are zeros or Pisot-numbers from a given algebraic number field $\Omega(\Theta)$. If

$$\varphi_j(p)\to 0 \quad as \quad p\to\infty \quad (j=2, \ldots, r) \tag{9}$$

then (1) holds.

Proof. Let us assume that (1) holds. From Lemma 6 we get that the values $f(n)$ are zeros or Pisot-numbers taken from a given number field $\Omega(\Theta)$. Let us consider the vector

$$\Psi(n)=(\varphi_2(n), \ldots, \varphi_r(n)),$$

and denote by X the set of the limit points of $\psi(n)$ $(n\to\infty)$. Let $(x_2, \ldots, x_r)\in X$. Since

$$f(n)+\varphi_2(n)+\ldots+\varphi_r(n)=\text{rational integer},$$

$\|f(n)\|\to 0$, we get that $x_2+\ldots+x_r=$rational integer. Let m_j be such a sequence for which

$$\Psi(m_j)\to(x_2, \ldots, x_r).$$

Then $\Psi(m_j^h)\to(x_2^h, \ldots, x_r^h)$, $x_2^h+\ldots+x_r^h=$rational integer, consequently $0<|x_j|<1$ is impossible, that is $x_j=0$ or $|x_j|=1$. Let now n be fixed such that $f(n)\neq 0$. Then $\varphi_j(n)\neq 0$, $0<|\varphi_j(n)|<1$,

$$\Psi(nm_j)\to(\varphi_2(n)\,x_2, \ldots, \varphi_r(n)x_r)\in X.$$

If $x_l\neq 0$ for a suitable l, then $0\neq|\varphi_l(n)x_l|<1$, which is impossible. Consequently we have (8).

The converse assertion is an immediate consequence of Lemma 2.

Theorem 2. *Let $f(n)\geqq 0$ be a completely multiplicative function satisfying the conditions* (6), (7). *Let us assume that $f(n)\not\to 0$, and that $f(n)$ takes on at least one nonintegral value. Then $f(n)$ takes on algebraic values, and the first assertion, stated in Theorem* 1, *holds.*

Proof. This is an immediate consequence of Lemma 7 and Theorem 1.

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Approximation and quasisimilarity

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Dedicated to Professor Károly Tandori on the occasion of his 60th birthday

1. Introduction

For an arbitrary complex Hilbert space $\mathfrak{H}$ let $\mathscr{L}(\mathfrak{H})$ denote the Banach algebra of all bounded linear operators acting on $\mathfrak{H}$. For any $T\in\mathscr{L}(\mathfrak{H})$ let Alg T denote the weakly closed subalgebra of $\mathscr{L}(\mathfrak{H})$ generated by T and the identity I, while $\{T\}'$ stands for the commutant of T. We call a subspace $\mathfrak{M}$ of $\mathfrak{H}$ to be cyclic for T if $\bigvee_{n\geq 0} T^n\mathfrak{M}=\mathfrak{H}$; $\mathfrak{M}$ is a minimal cyclic subspace if it contains no proper subspace which is also cyclic for T. The number disc T is defined as the supremum of the dimensions of all finite dimensional minimal cyclic subspaces for T.

In this paper we are going to investigate the problems of quasisimilarity invariances of the approximating property "Alg $T=\{T\}'$" and the number "disc T" in the class of cyclic C_{11}-contractions. We remark that the commutant of a C_{11}-contraction T is commutative if and only if T is cyclic. So to consider only cyclic contractions does not mean the restriction of the generality in connection with the first problem.

Our paper is organized as follows. In section 2 we discuss the approximating property "Alg $T=\{T\}'$" and describe its connection with the reflexivity problem of C_{11}-contractions. In section 3 the question of quasisimilarity invariance of "disc T" is performed. Our main goal is formulated in section 4: to construct contractions with special properties, whose study may give hope to solve the previous problems. Our construction is given in section 7 and is based on the results of sections 5 and 6. Section 5 deals with injective contractions, while in section 6 it is proved that a large amount of cyclic C_{11}-contractions possesses 0 as an "approximate reducing eigenvalue".

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2. The approximating property "Alg $T=\{T\}'$"

Let us consider the normalized Lebesgue measure m on the unit circle $\mathbf{T}$ of the complex plane $\mathbf{C}$ and let α be a Borel subset of $\mathbf{T}$ with positive measure: $m(\alpha)>0$. For any function φ from $L^\infty(\alpha)$ let $M_{\alpha,\varphi}$ denote the operator of multiplication by φ in $L^2(\alpha)$. (The spaces $L^p(\alpha)$, $p=2, \infty$, are defined with respect to the measure m on α.) In the case of the identical function $\varphi(\zeta)\equiv\zeta$ we will use the notation M_α for $M_{\alpha,\varphi}$, moreover $M_{\mathbf{T}}$ will simply be denoted by M. We remark that for two Borel subsets α, β of $\mathbf{T}$ the operators M_α, M_β are unitarily equivalent, $M_\alpha\cong M_\beta$, if and only if the symmetric difference $\alpha\triangle\beta$ of these sets is of measure 0, in notation $\alpha=\beta$ $[m]$ (cf. [6]).

Let us denote by $\mathcal{L}^\infty(\alpha)$ the set of multiplication operators on $L^2(\alpha)$, i.e.

$$\mathcal{L}^\infty(\alpha) = \{M_{\alpha,\varphi}\colon \varphi\in L^\infty(\alpha)\}.$$

It is known (cf. e.g. [16, Theorem 1]) that the commutant of M_α coincides with $\mathcal{L}^\infty(\alpha)$.

Let $P^\infty(\alpha)$ be the w^*-closure of the set of polynomials in $L^\infty(\alpha)$. It can be easily seen that the corresponding set of multiplication operators is exactly the closure of the set of polynomials of M_α in the weak operator topology, i.e.

$$\{M_{\alpha,\varphi}\colon \varphi\in P^\infty(\alpha)\} = \operatorname{Alg} M_\alpha.$$

Furthermore, it is a remarkable fact that $P^\infty(\alpha)=L^\infty(\alpha)$ if and only if m is not absolutely continuous with respect to the measure $\chi_\alpha dm$, where χ_α stands for the characteristic function of α (cf. [16, p. 17]). Hence, it follows that

$$\operatorname{Alg} M_\alpha = \{M_\alpha\}',$$

i.e. every operator in the commutant of M_α can be approximated by polynomials of M_α in the weak operator topology, exactly when $\alpha\neq\mathbf{T}[m]$.

Let T be a contraction acting on a complex separable Hilbert space $\mathfrak{H}$, i.e. $T\in\mathcal{L}(\mathfrak{H})$ and $\|T\|\leq 1$. Let us assume that T is of class C_{11}, that is $\lim\limits_{n\to\infty}\|T^nh\|\neq 0\neq \lim\limits_{n\to\infty}\|T^{*n}h\|$ for every $0\neq h\in\mathfrak{H}$, and that T has a cyclic vector f, which means that every vector $h\in\mathfrak{H}$ can be approximated in the norm of $\mathfrak{H}$ by vectors of the form $p(T)f$, where $p(\lambda)$ is a complex polynomial. Moreover, for the sake of simplicity, we assume that the unitary part of T (cf. [17, Theorem I.3.2]) is absolutely continuous with respect to the Lebesgue measure. Let C_1 denote the class of such contractions.

It is known (cf. [11, p. 15]) that for any operator $T\in C_1$ there exists a unique Borel subset α of $\mathbf{T}$ such that $m(\alpha)>0$ and T is quasisimilar to M_α: $T\sim M_\alpha$, which means that appropriate quasiaffinities X, Y (i.e. operators with zero kernels and dense ranges) interwine T and M_α: $XT=M_\alpha X$, $YM_\alpha=TY$. (In connection with the theory of contractions we refer to our main reference [17].)

Now the question naturally arises:

Problem A. Is the approximating property $\mathrm{Alg}\, T=\{T\}'$ a quasisimilarity invariant in the class C_1?

The answer for this question is *negative,* as it was pointed out for me by Hari Bercovici. Indeed, let T be a completely non-unitary (c.n.u.) contraction from C_1 which is quasisimilar to M_α, where $\alpha\neq \mathbf{T}[m]$. We can choose T such that its spectrum covers the whole unit disc: $\sigma(T)=D^-$ (cf. [4, Proposition 3.1]). Since T is a C_{11}-contraction, its essential spectrum coincides with $\sigma(T)$. So we can infer that the Sz.-Nagy, Foiaş functional calculus for T is an isometry and $\mathrm{Alg}\, T$ coincides with the w^*-closure of the set of polynomials of T (cf. [1, Corollary 1]), consequently $\mathrm{Alg}\, T=H^\infty(T):=\{u(T): u\in H^\infty$ (the Hardy space)$\}$.

On the other hand, the commutant $\{T\}'$ of T never coincides with $H^\infty(T)$, and so $\mathrm{Alg}\, T\neq\{T\}'$ contrary to the fact that $\mathrm{Alg}\, M_\alpha=\{M_\alpha\}'$.

The following argumentation proving the inequality $\{T\}'\neq H^\infty(T)$ is slightly different from the one given by Bercovici.

Let us say that the subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathfrak{H})$ has the property (P^*) if every non-zero operator in $\mathscr{A}$ is a quasiaffinity. Since every non-zero function $u\in H^\infty$ differs from 0 a.e., it is immediate that $H^\infty(M_\alpha)=\{u(M_\alpha)\colon u\in H^\infty\}$ has the property (P^*). Let us assume that $T\in C_1$ is quasisimilar to M_α, i.e. $XT=M_\alpha X$ and $TY=YM_\alpha$ with some quasiaffinities X and Y. Then for any non-zero function $u\in H^\infty$ the relations $Xu(T)=u(M_\alpha)X$ and $u(T)Y=Yu(M_\alpha)$ imply that $u(T)$ is a quasiaffinity. (The first one implies that $u(T)$ is injective, while the second one implies that its range is dense.) So we infer that $H^\infty(T)$ possesses the property (P^*).

Let us consider now an arbitrary operator S from $\{M_\alpha\}'$. Then $YSX\in\{T\}'$, XY, $XYSXY\in\{M_\alpha\}'$ and since $\{M_\alpha\}'=\mathscr{L}^\infty(\alpha)$ is commutative, it follows that $X(YSX)Y=(XY)S(XY)=S(XY)^2=(XY)^2S$. These relations show that if $S\neq 0$, then $YSX\neq 0$, and if YSX is a quasiaffinity, then so is S also. Since $\{M_\alpha\}'=\mathscr{L}^\infty(\alpha)$ does not have obviously property (P^*), the statements above result that $\{T\}'$ does not possess (P^*) too.

Consequently, for every contraction $T\in C_1$ we have $H^\infty(T)\subsetneqq\{T\}'$.

Owing to the negative answer for Problem A we introduce the operator class C_1' consisting of those elements of C_1 for which $\mathrm{Alg}\, T\neq H^\infty(T)$. For example every contraction $T\in C_1$ whose spectrum does not include $\mathbf{T}$ belongs to C_1' (cf. [2, p. 337]). Now we formulate our question in the following form:

Problem A′. Is the property $\mathrm{Alg}\, T=\{T\}'$ a quasisimilarity invariant in C_1'?

This problem seems to be relevant in connection with the *reflexivity* problem of C_{11}-contractions. (As for the notion of reflexivity see for instance [8, chapter 9].) Indeed, the operators in some subclasses of $C_1\setminus C_1'$, e.g. operators with dominating essential spectrum (so-called (BCP)-operators) belonging to C_1 are reflexive by a recent result of BERCOVICI, FOIAŞ, LANGSAM and PEARCY [1]. On the other hand, it is well-known that quasisimilarity preserves the reflexivity of the commutant (cf. [3, Proposition 4.1]). So an affirmative answer for Problem A′ would reduce the reflexivity problem to certain subclasses of $C_1 \setminus C_1'$ at least in the case of contractions from C_1 which are quasisimilar to some M_α with $\alpha \neq \mathbf{T}[m]$. Conversely, a counterexample for Problem A' might be a candidate for a non-reflexive C_{11}-contraction.

3. The number "disc T"

The second problem we are interested in was posed by Nikolskii and Vasjunin. In connection with questions concerning controllable systems they have introduced the number "disc" of an arbitrary Hilbert space operator (cf. [13]). Namely, for an operator $T \in \mathcal{L}(\mathfrak{H})$ let Cyc T be the set of finite dimensional cyclic subspaces:

$$\operatorname{Cyc} T := \Big\{\mathfrak{M} \text{ subspace of } \mathfrak{H}\colon \dim \mathfrak{M} < \infty,\ \bigvee_{n \geqq 0} T^n \mathfrak{M} = \mathfrak{H}\Big\}.$$

Then disc T denotes the number

$$\operatorname{disc} T := \sup \big\{\min \{\dim \mathfrak{N}\colon \mathfrak{N} \in \operatorname{Cyc} T,\ \mathfrak{N} \subset \mathfrak{M}\}\colon \mathfrak{M} \in \operatorname{Cyc} T\big\}.$$

Nikolskii and Vasjunin posed the question of quasisimilarity invariance of disc T in general (cf. [13, p. 330]). In particular, it would be interesting to know the answer for the following problem:

Problem B. Is the number disc T a quasisimilarity invariant in the class C_1?

This question seems to be of considerable interest, because disc M_α takes on different values according to the case that $\alpha \neq \mathbf{T}[m]$ or not. Namely, the following is true:

Proposition 1. *disc* $M_\alpha = 1$ *if* $\alpha \neq \mathbf{T}[m]$, *while* disc $M = 2$.

Proof. If $\alpha \neq \mathbf{T}[m]$ then $\operatorname{Alg} M_\alpha = \{M_\alpha\}' = \mathcal{L}^\infty(\alpha)$, and so $\operatorname{Lat} M_\alpha = \{\chi_\beta L^2(\alpha)\colon \beta \subset \alpha\}$, where χ_β is the characteristic function of β. This implies that $f \in L^2(\alpha)$ is cyclic for M_α if and only if $f(x) \neq 0$ a.e. On the other hand by Szegő's theorem (cf. [10]) we know that the cyclic vectors of M are the functions f such that $f(x) \neq 0$ a.e. and $\int_{\mathbf{T}} \log |f|\, dm = -\infty$.

Now, if $\alpha \neq \mathbf{T}[m]$, $\mathfrak{M} \in \operatorname{Cyc} M_\alpha$ and $\{f_i\}_{i=1}^n$ is a basis in $\mathfrak{M}$, then $\sum_{i=1}^n |f_i(x)| \neq 0$ a.e.. An elementary argumentation shows that $f(x) \neq 0$ a.e. on α for a suitable linear combination $f = \sum_{i=1}^n c_i f_i$.

Indeed, proceeding by induction on n, we can reduce the proof to the case $n=2$. So let us assume that $|f_1(x)|+|f_2(x)|\neq 0$ a.e., and let $\alpha_j=\{x\in\alpha\colon f_j(x)=0\}$ $(j=1, 2)$ and $\alpha'=\alpha\setminus(\alpha_1\cup\alpha_2)$. Then $m(\alpha_1\cap\alpha_2)=0$ and so it is enough to show that $f_1(x)+$ $+cf_2(x)\neq 0$ a.e. on α' with some non-zero complex number c. Taking different numbers c and d the sets $E_c=\{x\in\alpha'\colon f_1(x)+cf_2(x)=0\}$ and $E_d=\{x\in\alpha'\colon f_1(x)+df_2(x)=0\}$ will be disjoint. Therefore, for all but countably many points c of $\mathbf{C}\setminus\{0\}$ the set E_c will be of measure 0, but for such a number c we have $f_1(x)+cf_2(x)\neq 0$ a.e. on α.

Therefore, we conclude that $\operatorname{disc} M_\alpha=1$.

Let us determine now $\operatorname{disc} M$! Let $\mathbf{T}^+$ and $\mathbf{T}^-$ denote the upper and lower semicircle, respectively, and let $\mathfrak{M}$ be the 2-dimensional subspace spanned by $\chi_{\mathbf{T}^+}$ and $\chi_{\mathbf{T}^-}$. Then $\mathfrak{M}\in\operatorname{Cyc} M$, but $\mathfrak{M}$ does not contain any cyclic vector of M. Therefore $\operatorname{disc} M\geqq 2$.

Now let $\mathfrak{M}\in\operatorname{Cyc} M$ be a subspace with $\dim\mathfrak{M}=n\geqq 2$. We want to show that $\min\{\dim\mathfrak{N}\colon \mathfrak{N}\in\operatorname{Cyc} T,\ \mathfrak{N}\subset\mathfrak{M}\}\leqq 2$. As before, we can infer that $f(x)\neq 0$ a.e. for some $f\in\mathfrak{M}$. It can be assumed that $\int_{\mathbf{T}}\log|f|\,dm>-\infty$ for these functions. It is well-known (cf. [9, Theorems II. 2 and 3]) that the invariant subspace lattice of M has the form:

$\operatorname{Lat} M=\{L^2(\alpha)\colon \alpha\subset\mathbf{T}\}\cup\{qH^2\colon q$ is a unimodular function on $\mathbf{T}\}$.

Let us assume that there exists a function $g\in\mathfrak{M}$ such that $0<m(\{x\in\mathbf{T}\colon g(x)\neq 0\})<1$. Then $\mathfrak{M}_g:=\bigvee_{k\geqq 0} M^k g=L^2(\alpha)$ with $\alpha\neq\mathbf{T}[m]$, while $\mathfrak{M}_f:=\bigvee_{k\geqq 0} M^k f=qH^2$ with a unimodular q. Since, for every $h\in H^2$, $\int_{\mathbf{T}}\log|h|\,dm>-\infty$ (cf. [17, Sec. III. 1]) we infer that $\mathfrak{M}_g\vee\mathfrak{M}_f=L^2(\alpha)\vee qH^2=L^2(\mathbf{T})$.

Hence we have only to deal with the case when for every nonzero $f\in\mathfrak{M}$ we have $\int_{\mathbf{T}}\log|f|\,dm>-\infty$. We prove by induction on n that $\mathfrak{M}_{i_1}\vee\mathfrak{M}_{i_2}=L^2(\mathbf{T})$ for some $1\leqq i_1,\ i_2\leqq n,\ i_1\neq i_2$, where $\mathfrak{M}_i=\mathfrak{M}_{f_i}$ for $i=1,\dots,n$. For $n=2$ this is obvious. Let us assume that it is true for $n-1$ $(n\geqq 3)$. If $\bigvee_{i=1}^{n-1}\mathfrak{M}_i=L^2(\mathbf{T})$, then we can apply our assumption. If $\bigvee_{i=1}^{n-1}\mathfrak{M}_i\neq L^2(\mathbf{T})$, then $\bigvee_{i=1}^{n-1}\mathfrak{M}_i=qH^2$ for a unimodular function q. On account of Beurling's theorem there are inner functions $u_i\in H^\infty$ such that $\mathfrak{M}_i=qu_iH^2$ for $i=1,\dots,n-1$ (cf. [9, Sec. II. 4]). Since $\left(\bigvee_{i=1}^{n-1}\mathfrak{M}_i\right)\vee\mathfrak{M}_n=$ $=(qH^2)\vee(q_nH^2)=L^2(\mathbf{T})$ (q_n unimodular), and the multiplication by an inner function is a unitary operator on $L^2(\mathbf{T})$, we conclude that $L^2(\mathbf{T})=u_1L^2(\mathbf{T})=$ $=(u_1qH^2)\vee(u_1q_nH^2)=\mathfrak{M}_1\vee\bigl(q_n(u_1H^2)\bigr)\subset\mathfrak{M}_1\vee\mathfrak{M}_n\subset L^2(\mathbf{T})$, and so $\mathfrak{M}_1\vee\mathfrak{M}_n=L^2(\mathbf{T})$.

Consequently we obtain that $\operatorname{disc} M=2$.

Of course the values of disc M_α $(\alpha\neq\mathbf{T}[m])$ and disc M can be computed from the general formula of the Theorem of [13]. We have given the above simple proof for the sake of the reader's convenience.*

4. Our programme

First of all we remark that in virtue of Wu's results the answers for Problems A and B are affirmative, if we assume that the defect indeces of the contraction $T\in C_1$ are finite. (Cf. [18, Corollary 4.6], [11, Corollary 1] and [13, p. 330].)

We have seen in section 2 that the answer for Problem A is negative in general. Another fact which points out that the general case is more complicated is the following.

For an arbitrary unitary operator U let us denote by $\mathrm{Lat}_1\, U$ the lattice of the reductive subspaces of U. It follows by section 2 that if $\alpha\neq\mathbf{T}[m]$, then $\mathrm{Lat}\, M_\alpha=\mathrm{Lat}_1\, M_\alpha$, while $\mathrm{Lat}\, M\neq\mathrm{Lat}_1\, M$. A natural generalization of a reductive subspace for a C_{11}-contraction T is an invariant subspace $\mathfrak{L}\in\mathrm{Lat}\, T$ such that $T|\mathfrak{L}\in C_{11}$. $\mathrm{Lat}_1\, T$ stands for the set of "C_{11}-invariant subspaces": $\mathrm{Lat}_1\, T=\{\mathfrak{M}\in\mathrm{Lat}\, T\colon T|\mathfrak{M}\in C_{11}\}$. It was shown in [4] (cf. Remark 3.4) that the property of reductivity "$\mathrm{Lat}\, T=\mathrm{Lat}_1\, T$" is not a quasisimilarity invariant in C_1.

In order to study Problems A′ and B in the general setting it would be very useful to have contractions which are close in a certain sense to some M_α with $\alpha\neq\mathbf{T}[m]$ and to M at the same time. The aim of the present paper is to provide such operators, which may clarify the real situation, perhaps they can be candidates to be counterexamples.

Our construction is based on theorems concerning injective contractions and the approximate reducing point spectrum, which will be proved in the next two sections.

5. Injective contractions

We begin by proving two lemmas which are refinements of [4, Lemma 3.2].

Lemma 2. *Let $T\in\mathcal{L}(\mathfrak{H})$ be a contraction. Then for every $g, u\in\mathfrak{H}$ we have*

$$\|Tg+D_{T^*}^2u\|^2\leqq\|g\|^2+\|D_{T^*}u\|^2.$$

$(D_T=(I-T^*T)^{1/2}$ and $D_{T^*}=(I-TT^*)^{1/2}$ are the defect operators of T.)

* After this paper had been submitted, prof. Vasjunin informed me that a direct proof of this proposition can be found in their paper "Control subspaces of minimal dimension, and spectral multiplicities" published in the Proceedings of the 6[th] Operator Theory Conference, held in Romania.

Proof. Using the identity $D_{T^*}T=TD_T$, we obtain $\langle Tg, D_{T^*}^2u\rangle=\langle D_{T^*}Tg, D_{T^*}u\rangle=\langle TD_Tg, D_{T^*}u\rangle=\langle D_Tg, T^*D_{T^*}u\rangle$. Applying this and the Schwartz inequality, it follows that

$$\|Tg+D_{T^*}^2u\|^2=\|Tg\|^2+2\operatorname{Re}\langle Tg, D_{T^*}^2u\rangle+\|D_{T^*}^2u\|^2=\|Tg\|^2+$$
$$+2\operatorname{Re}\langle D_Tg, T^*D_{T^*}u\rangle+\|D_{T^*}^2u\|^2\leqq\|Tg\|^2+2\|D_Tg\|\,\|T^*D_{T^*}u\|+$$
$$+\|D_{T^*}^2u\|^2\leqq\|Tg\|^2+\|D_Tg\|^2+\|T^*D_{T^*}u\|^2+\|D_{T^*}D_{T^*}u\|^2=\|g\|^2+\|D_{T^*}u\|^2.$$

We recall that, for an operator $T\in\mathcal{L}(\mathfrak{H})$ and a vector $f\in\mathfrak{H}$, $(f, T)\in\mathcal{L}(\mathbf{C}\oplus\mathfrak{H})$ denotes the operator defined by

$$(f, T)(\lambda\oplus g):=0\oplus(\lambda f+Tg)\qquad(\lambda\in\mathbf{C},\ g\in\mathfrak{H})\quad(\text{cf. [4]}).$$

Lemma 3. *Let $T\in\mathcal{L}(\mathfrak{H})$ be a non-ivertible injective contraction. Then for every $\varepsilon>0$ there exists a vector $f\in\mathfrak{H}$ such that $\|f\|>1-\varepsilon$ and (f, T) is an injective contraction.*

Proof. If $u\in\mathfrak{H}\setminus\operatorname{ran}T$, then $f=D_{T^*}^2u=(I-TT^*)u\in\mathfrak{H}\setminus\operatorname{ran}T$, and so (f, T) is an injective operator. On account of Lemma 2 (f, T) is a contraction if $\|D_{T^*}u\|\leqq1$, so if $\|u\|\leqq1$. Since $\|u\|^2=\|T^*u\|^2+\|D_{T^*}u\|^2=\|T^*u\|^2+\|T^*D_{T^*}u\|^2+\|D_{T^*}^2u\|^2$, it follows that $\|f\|^2=\|u\|^2-\|T^*u\|^2-\|D_TT^*u\|^2\geqq\|u\|^2-2\|T^*u\|^2$.

The injectivity of T implies that $(\operatorname{ran}T^*)^-=\mathfrak{H}$. Taking into account that T^* is not invertible we infer that T^* is not bounded from below. Therefore for an arbitrary $0<\eta<1$ there exists a unit vector $u_0\in\mathfrak{H}$ such that $\|T^*u_0\|<\eta$.

Since T is injective, non-invertible, the closed graph theorem implies that $\operatorname{ran}T\neq\mathfrak{H}$. It follows that $\mathfrak{H}\setminus\operatorname{ran}T$ is dense in $\mathfrak{H}$. Hence for any $1>\delta>0$ there exists a vector $u\in\mathfrak{H}\setminus\operatorname{ran}T$ such that $\|u-(1-\delta)u_0\|<\delta$. Then $\|u\|\leqq1$, so with $f=D_{T^*}^2u$ the operator (f, T) will be an injective contraction. On the other hand $\|f\|^2\geqq\|u\|^2-2\|T^*u\|^2\geqq\|u\|^2-2(\|T^*u_0\|+\|u-u_0\|)^2$, and since $\|u\|\geqq1-2\delta$, $\|T^*u_0\|<\eta$ and $\|u-u_0\|<2\delta$, it follows that

$$\|f\|^2>(1-2\delta)^2-2(\eta+2\delta)^2\geqq1-16\delta-2\eta^2.$$

We infer that $\|f\|>1-\varepsilon$ if $\eta>0$ and $\delta>0$ are chosen to be small enough, and the proof is complete.

To any operator $T\in\mathcal{L}(\mathfrak{H})$ let us correspond the number

$$\nu(T):=\inf\{\max\{\|Tx\|, \|T^*x\|\}:\ x\in\mathfrak{H},\ \|x\|=1\}.$$

Proposition 4. *Let $T\in\mathcal{L}(\mathfrak{H})$ be a non-invertible quasiaffine contraction with $\nu(T)<1/2$. Then for any $\varepsilon>0$ there exist vectors $f, g\in\mathfrak{H}$ such that $\|f\|, \|g\|>1-2\nu(T)-\varepsilon$, $\|f-g\|<2\nu(T)+\varepsilon$ and (f, T), $(g, T^*)\in\mathcal{L}(\mathbf{C}\oplus\mathfrak{H})$ are injective contractions.*

Proof. Let $\nu(T)<\eta<1$ be arbitrary. By the assumption there exists a unit vector $u_0\in\mathfrak{H}$ such that $\|Tu_0\|$ and $\|T^*u_0\|<\eta$. The proof of Lemma 3 shows that for any $1>\delta>0$ we can find vectors $u\in\mathfrak{H}\setminus\operatorname{ran} T$ and $v\in\mathfrak{H}\setminus\operatorname{ran} T^*$ such that $\|u-(1-\delta)u_0\|<\delta$ and $\|v-(1-\delta)u_0\|<\delta$. Then, considering the vectors $f=D_{T^*}^2u$ and $g=D_T^2v$, the operators (f, T) and (g, T^*) will be injective contractions while $\|f\|^2, \|g\|^2\geqq 1-16\delta-2\eta$. Furthermore, we have

$$\|f-g\| = \|(I-TT^*)u-(I-T^*T)v\| \leqq \|u-v\|+\|T^*u\|+$$
$$+\|Tv\| \leqq \|u-v\|+\|T^*u_0\|+\|u-u_0\|+\|Tu_0\|+\|v-u_0\| \leqq 6\delta+2\eta.$$

Consequently, we conclude that $\|f\|, \|g\|>1-2\nu(T)-\varepsilon$ and $\|f-g\|<2\nu(T)+\varepsilon$ if η is close enough to $\nu(T)$ and δ is small enough.

Remark 5. Note that if $\nu(T)=0$, that is if 0 is an "approximate reducing eigenvalue" of T, then f and g can be chosen to be arbitrarily close to each other, with norms arbitrarily close to 1.

6. C_1-contractions with approximate reducing eigenvalue 0

In this section we shall show that there is an abundance of C_1-contractions T with $\nu(T)=0$. In proving our results we need the following lemma which will be used in the next section too.

Lemma 6. *Let α be a Borel set on $\mathbf{T}$ such that $m(\alpha)>0$. Then there exists a sequence $\{\beta_n\}_{n=1}^\infty$ of closed arcs of $\mathbf{T}$ such that $\beta_n\cap\beta_{n+1}$ consists of exactly one point and $m(\alpha_n)>0$, where $\alpha_n=\beta_n\cap\alpha$, for all n, and $\bigcup_{n=1}^\infty \beta_n=\mathbf{T}$.*

Proof. Since $m(\alpha)>0$, it follows that α contains a point C of density 1. Moreover, there exists a sequence $\{C_n\}_{n=1}^\infty$ of different density points of α converging to C. For every n, let γ_n be one of the two closed arcs of $\mathbf{T}$ determined by C_n and C. With an appropriate choice of these arcs and passing on to a subsequence, if it is necessary, we can achieve that $\{\gamma_n\}_{n=1}^\infty$ be a decreasing sequence of sets. For every n, let B_n be an arbitrary point of the set $\gamma_n\setminus(\gamma_{n+1}\cup\{C_n\})$. Now we define β_n to be the closed arc with endpoints B_{n-1}, B_n and containing C_n, if $n\geqq 2$; while β_1 is the arc with endpoints B_1, C and containing C_1.

Since each β_n contains in its interior a density point of α, it follows that $m(\alpha_n)>0$ for every n. It is easy to see that the sequence $\{\beta_n\}_{n=1}^\infty$ possesses the other properties of the statement also.

Theorem 7. *For every Borel set $\alpha\subset\mathbf{T}$ with positive Lebesgue measure there exists a C_1-contraction T such that T is quasisimilar to M_α and $\nu(T)=0$.*

Proof. We first show that for every $\varepsilon>0$ there is a C_1-contraction T such that $T\sim M_\alpha$ and $\nu(T)<\varepsilon$.

By Lemma 6 there exist closed arcs β', β'' of $\mathbf{T}$ such that $\beta'\cup\beta''=\mathbf{T}$, $\beta'\cap\beta''$ has two points and $m(\alpha')>0$, $m(\alpha'')>0$, where $\alpha'=\beta'\cap\alpha$ and $\alpha''=\beta''\cap\alpha$. On account of [4, Proposition 3.1] we can find non-invertible C_1-contractions $T'\in\mathscr{L}(\mathfrak{H}')$ and $T''\in\mathscr{L}(\mathfrak{H}'')$, T' being quasisimilar to $M_{\alpha'}$ and T'' being quasisimilar to $M_{\alpha''}$. Then Lemma 3 ensures us vectors $f\in\mathfrak{H}'$ and $g\in\mathfrak{H}''$ such that $(f, T'^*)\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H}')$ and $(g, T'')\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H}'')$ are injective contractions. The matrices of these operators are

$$(f, T'^*)=\begin{bmatrix}0 & 0\\ f & T'^*\end{bmatrix} \quad\text{and}\quad (g, T'')=\begin{bmatrix}0 & 0\\ g & T''\end{bmatrix},$$

where f denotes also the operator of rank 1: $f\colon \mathbf{C}\to\mathfrak{H}'$, $f\colon \lambda\mapsto\lambda f$; its adjoint is $f^*\colon \mathfrak{H}'\to\mathbf{C}$, $f^*\colon h\mapsto\langle h, f\rangle$.

Let us consider the Hilbert space $\mathfrak{H}=\mathfrak{H}'\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathfrak{H}''$ and define the operator $T\in\mathscr{L}(\mathfrak{H})$ by the matrix

$$T=\begin{bmatrix}T' & 0 & 0 & 0 & 0\\ f^* & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\varepsilon & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\varepsilon & 0 & 0\\ 0 & 0 & 0 & g & T''\end{bmatrix}.$$

It is easy to see that T is a C_{11}-contraction. Since $\|T(0\oplus0\oplus1\oplus0\oplus0)\|=\|T^*(0\oplus0\oplus1\oplus0\oplus0)\|=(1/2)\varepsilon$, it follows that $\nu(T)<\varepsilon$. Moreover, by [4, Theorem 1.7] the residual part R_T of T (that is the residual part of its unitary dilation) is unitarily equivalent to $R_{T'}\oplus R_{T''}$: $R_T\cong R_{T'}\oplus R_{T''}$. This implies that $R_T\cong M_{\alpha'}\oplus M_{\alpha''}\cong M_\alpha$, and so T is quasisimilar to M_α (cf. [4, Proposition 1.3]).

Let us now prove the existence of a C_1-contraction T with $T\sim M_\alpha$ and $\nu(T)=0$. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of Borel sets corresponding to α by Lemma 6. Then, on account of the first part of this proof, for every n there exists an operator $T_n\in\mathscr{L}(\mathfrak{H}_n)$ such that $T_n\sim M_{\alpha_n}$ and $\nu(T_n)<1/n$. It follows that the direct sum $T=\bigoplus_{n=1}^\infty T_n$ of these operators is quasisimilar to M_α and $\nu(T)=0$. The proof is finished.

Now we prove that it can be achieved that the spectrum of the contraction T in the previous theorem be rather thin. We shall need the following:

Lemma 8. *Let $A\in\mathscr{L}(\mathfrak{H}')$, $B\in\mathscr{L}(\mathfrak{H}'')$ be invertible operators. Then the operator T acting on $\mathfrak{H}=\mathfrak{H}'\oplus\mathfrak{H}''$ and defined by the matrix* $T=\begin{bmatrix}A & 0\\ C & B\end{bmatrix}$ *is also invertible, and* $\|T^{-1}\|\leqq\max\{\|A^{-1}\|, \|B^{-1}\|\}+\|A^{-1}\|\|B^{-1}\|$ *if* $\|C\|\leqq1$.

Proof. It is easy to verify that the inverse of T is $T^{-1}=\begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1}\end{bmatrix}$. Moreover, for the norm of T^{-1} we have

$$\|T^{-1}\| \leqq \left\|\begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1}\end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & 0 \\ -B^{-1}CA^{-1} & 0\end{bmatrix}\right\| = \max\{\|A^{-1}\|, \|B^{-1}\|\}+\|B^{-1}CA^{-1}\| \leqq$$

$$\leqq \max\{\|A^{-1}\|, \|B^{-1}\|\}+\|B^{-1}\|\|A^{-1}\|.$$

Furthermore we shall use the following notation. If $\alpha\subset\mathbf{T}$ is a Borel set, $m(\alpha)>0$, then $\alpha^=$ stands for the support of the measure $\chi_\alpha dm$ and, for any $\zeta\in\alpha^=$, $D(\alpha,\zeta):=$ $:=\alpha^=\cup\{r\zeta\colon 0\leqq r\leqq 1\}$.

Theorem 9. *Let α be a Borel subset of $\mathbf{T}$ such that $m(\alpha)>0$ and let ζ be any point of $\alpha^=$. Then there exists a C_1-contraction T such that $T\sim M_\alpha$, $\nu(T)=0$ and $\sigma(T)=D(\alpha,\zeta)$.*

We remark that if $\nu(T)=0$, then $0\in\sigma(T)$, and that for every C_1-contraction T each closed and open part of $\sigma(T)$ intersects $\mathbf{T}$, and that $T\sim M_\alpha$ implies $\sigma(T)\supset\alpha^=$ (cf. [5]). In the light of these facts the spectrum of T in the previous theorem can not be thinner.

Proof. Let $\{\beta_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ be sequences corresponding to α by Lemma 6 such that the cluster point of the endpoints of the arcs β_n is the given ζ.

For every $n\in\mathbf{N}$, the set of natural numbers, let Ω_n denote the domain $\Omega_n=\{r\lambda\colon \lambda$ belongs to the interior of β_n and $1/(n+1)<r<1\}$, and let $\mu_n\in\Omega_n$ be a point such that $|\mu_n|<1/n$. By [5] we can find for every n a C_{11}-contraction $S_n\in\mathscr{L}(\mathfrak{K}_n)$ such that S_n is quasisimilar to M_{α_n}, $\sigma(S_n)=\alpha_n^=$, $\|(S_n-\lambda I)^{-1}\|\leqq\operatorname{dist}(\lambda,\Omega_n^-)^{-1}$ for all $\lambda\in\mathbf{C}\setminus\Omega_n^-$, and $\|(S_n-\mu_n I)x_n\|<1/n$ for a suitable unit vector $x_n\in\mathfrak{K}_n$.

Let us decompose $\mathbf{N}$ into the union of pairwise disjoint sets

$$\mathbf{N}=(\bigcup_{i\in\mathbf{N}} N_i')\cup(\bigcup_{i\in\mathbf{N}} N_i''),$$

each of which contains infinitely many points. For every $i\in\mathbf{N}$, let us define $T_i'\in\mathscr{L}(\mathfrak{H}_i')$ and $T_i''\in\mathscr{L}(\mathfrak{H}_i'')$ by $T_i'=\bigoplus_{n\in N_i'} S_n$ and $T_i''=\bigoplus_{n\in N_i''} S_n$, respectively.

Since $\inf_{n\in N_i'}\|(S_n-\mu_n I)x_n\|=\inf_{n\in N_i''}\|(S_n-\mu_n I)x_n\|=0$, it follows that T_i' and T_i'' are not invertible. So we infer by Lemma 3 that there exist vectors $f_i\in\mathfrak{H}_i'$ and $g_i\in\mathfrak{H}_i''$ such that $(f_i, T_i'^*)$ and (g_i, T_i'') are injective contractions. Now we define the operator T_i acting on the Hilbert space $\mathfrak{H}_i=\mathfrak{H}_i'\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathfrak{H}_i''$ by the matrix

$$T_i=\begin{bmatrix} T_i' & 0 & 0 & 0 & 0 \\ f_i^* & 0 & 0 & 0 & 0 \\ 0 & (i+1)^{-1} & 0 & 0 & 0 \\ 0 & 0 & (i+1)^{-1} & 0 & 0 \\ 0 & 0 & 0 & g_i & T_i'' \end{bmatrix}.$$

Finally, our operator $T\in\mathscr{L}(\mathfrak{H})$ is defined as the orthogonal sum of these operators: $T=\bigoplus_{i\in\mathbf{N}} T_i$.

It is immediate that T is a contraction. We can show as in the proof of Theorem 7 that for every i

$$T_i\sim T_i'\oplus T_i''\cong\bigoplus_{n\in N_i'\cup N_i''} S_n\sim\bigoplus_{n\in N_i'\cup N_i''} M_{\alpha_n},$$

and so

$$T\sim\bigoplus_{i\in\mathbf{N}}\Big(\bigoplus_{n\in N_i'\cup N_i''} M_{\alpha_n}\Big)\cong\bigoplus_{n\in\mathbf{N}} M_{\alpha_n}\cong M_\alpha,$$

i.e. T is quasisimilar to M_α. It follows that T belongs to the class C_1.

Since $v(T_i)<1/i$ for every i, we infer that $v(T)=0$.

Let us assume now that $\lambda\in\mathbf{C}\diagdown D(\alpha,\zeta)$. Then, for every $n\in\mathbf{N}$, the operator $S_n-\lambda I$ is invertible. Moreover, for all but at most two n, λ belongs to $\mathbf{C}\diagdown\Omega_n^-$. For these indeces $\|(S_n-\lambda I)^{-1}\|\leqq\operatorname{dist}(\lambda,\Omega_n^-)^{-1}$, and the sequence on the right side is bounded. Hence we conclude that $\{\|(S_i-\lambda I)^{-1}\|\}_{i\in\mathbf{N}}$ is bounded, and so applying Lemma 8 we obtain that $\{\|(T_i-\lambda I)^{-1}\|\}_{i\in\mathbf{N}}$ is bounded too. Therefore, we infer that $T-\lambda I$ is invertible, i.e. $\lambda\notin\sigma(T)$.

On the other hand $v(T)=0$ implies that $0\in\sigma(T)$, moreover on account of [5] we know that $\alpha^-\subset\sigma(T)$ and that every closed and open subset of $\sigma(T)$ intersects the unit circle. Consequently, we obtain that $\sigma(T)=D(\alpha,\zeta)$.

We remark that with the additional assumption that the defect number of T is 1 the statement of Theorem 7 becomes false. Namely, the following holds:

Proposition 10. *The number* $v_1=\inf\{v(T)\colon\ T\in C_1$ *with defect index* $1\}$ *is strictly positive.*

Proof. It is evident that we can restrict our attention to c.n.u. C_1-contractions with defect index 1. We shall consider the functional models of these contractions (cf. [17, chapter VI]).

So let H_e^∞ denote the set of outer functions ϑ in the (scalar) Hardy space H^∞ such that $|\vartheta(e^{it})|\leqq 1$ a.e. and ϑ is not a constant of absolute value 1. To any $\vartheta\in H_e^\infty$ there corresponds a Hilbert space

$$\mathfrak{H}(\vartheta)=[H^2\oplus(\Delta L^2)^-]\ominus\{\vartheta w\oplus\Delta w\colon\ w\in H^2\},$$

where $\Delta(e^{it})=(1-|\vartheta(e^{it})|^2)^{1/2}$, and an operator $S(\vartheta)\in\mathcal{L}(\mathfrak{H}(\vartheta))$ defined by $S(\vartheta)=P_{\mathfrak{H}(\vartheta)}U|\mathfrak{H}(\vartheta)$, where U denotes the operator of multiplication by e^{it} on $H^2\oplus L^2$ and $P_{\mathfrak{H}(\vartheta)}$ is the orthogonal projection in $H^2\oplus L^2$ onto the subspace $\mathfrak{H}(\vartheta)$. The operator $S(\vartheta)$ is a C_1-contraction of defect index 1 and being quasisimilar to M_α for $\alpha=\{e^{it}\in\mathbf{T}:\ |\vartheta(e^{it})|<1\}$ (cf. [17, Proposition VI. 3.5] and [11, Corollary 1]). Moreover, in this way we obtain all c.n.u. C_1-contractions of defect index 1 up to unitary equivalence.

So we have to prove that the infimum

$$\nu_1=\inf\{\nu(S(\vartheta)):\ \vartheta\in H_e^\infty\}$$

is not equal to zero.

Let $\vartheta\in H_e^\infty$ be an arbitrary function. The Hilbert space $\mathfrak{H}(\vartheta)$ can be decomposed into the orthogonal sums

$$\mathfrak{H}(\vartheta)=\mathfrak{D}_{S(\vartheta)}\oplus\mathfrak{D}_{S(\vartheta)}^{\perp}=\mathfrak{D}_{S(\vartheta)^*}\oplus\mathfrak{D}_{S(\vartheta)^*}^{\perp},$$

where $\mathfrak{D}_{S(\vartheta)}=(\ker D_{S(\vartheta)})^\perp$ and $\mathfrak{D}_{S(\vartheta)^*}=(\ker D_{S(\vartheta)^*})^\perp$ are 1-dimensional subspaces, the so-called defect subspaces of $S(\vartheta)$. Since $S(\vartheta)|\mathfrak{D}_{S(\vartheta)}^{\perp}:\ \mathfrak{D}_{S(\vartheta)}^{\perp}\to\mathfrak{D}_{S(\vartheta)^*}^{\perp}$ is an isometric surjection with inverse $S(\vartheta)^*|\mathfrak{D}_{S(\vartheta)^*}^{\perp}:\ \mathfrak{D}_{S(\vartheta)^*}^{\perp}\to\mathfrak{D}_{S(\vartheta)}^{\perp}$, and for appropriate unit vectors $g_0\in\mathfrak{D}_{S(\vartheta)}$ and $h_0\in\mathfrak{D}_{S(\vartheta)^*}$ we have $S(\vartheta)g_0=\vartheta(0)h_0$ and $S(\vartheta)^*h_0=\overline{\vartheta(0)}g_0$ (cf. [17, Sec. VI. 4]), it follows that $\nu_1=0$ if and only if for a sequence $\{\vartheta_n\}_{n=1}^\infty$ in H_e^∞ $\vartheta_n(0)$ and the distance of the subspaces $\mathfrak{D}_{S(\vartheta_n)}$ and $\mathfrak{D}_{S(\vartheta_n)^*}$ tend simultaneously to zero. Under the latter distance we mean the distance of the unit spheres of these subspaces, i.e.

$$\operatorname{dist}(\mathfrak{D}_{S(\vartheta_n)},\mathfrak{D}_{S(\vartheta_n)^*})=\inf\{\|x-y\|:\ x\in\mathfrak{D}_{S(\vartheta_n)},\ \|x\|=1,\ y\in\mathfrak{D}_{S(\vartheta_n)^*},\ \|y\|=1\}.$$

An easy computation shows that for any $u\oplus v\in\mathfrak{H}(\vartheta)$ $(\vartheta\in H_e^\infty)$

$$(I-S(\vartheta)S(\vartheta)^*)(u\oplus v)=u(0)P_{\mathfrak{H}(\vartheta)}(1\oplus 0),$$

and again a usual computation yields that

$$P_{\mathfrak{H}(\vartheta)}(1\oplus 0)=(1-\overline{\vartheta(0)}\vartheta)\oplus-\overline{\vartheta(0)}\Delta=:h.$$

The norm of h is $\|h\|=(1-|\vartheta(0)|^2)^{1/2}\neq 0$ and $\{h_0=h/\|h\|\}$ forms an orthonormal basis in $\mathfrak{D}_{S(\vartheta)^*}$. Then

$$g=S(\vartheta)^*h=e^{-it}\overline{\vartheta(0)}(\vartheta(0)-\vartheta)\oplus-e^{-it}\overline{\vartheta(0)}\Delta$$

belongs to $\mathfrak{D}_{S(\vartheta)}$, $\|g\|=|\vartheta(0)|\ \|h\|$ and $\{g_0=g/\|g\|\}$ is a basis in $\mathfrak{D}_{S(\vartheta)}$.

The distance of the subspaces $\mathfrak{D}_{S(\vartheta)}$ and $\mathfrak{D}_{S(\vartheta)^*}$ is by our definition

$$d(\vartheta):=\operatorname{dist}(\mathfrak{D}_{S(\vartheta)},\mathfrak{D}_{S(\vartheta)^*})=\inf\{\|\alpha h_0-g_0\|:\ \alpha\in\mathbf{C},\ |\alpha|=1\}.$$

Since $\|\alpha h_0-g_0\|^2=2\left(1-\dfrac{\operatorname{Re}(\alpha\langle h,g\rangle)}{\|h\|\,\|g\|}\right)$, it follows that $d(\vartheta)=\sqrt{2}\left(1-\dfrac{|\langle h,g\rangle|}{\|h\|\,\|g\|}\right)^{1/2}$.

A direct computation shows that $\langle h, g\rangle = -\vartheta(0)\vartheta'(0)$, and so we obtain that

$$d(\vartheta) = \sqrt{2}\left(1 - \frac{|\vartheta'(0)|}{1-|\vartheta(0)|^2}\right)^{1/2}.$$

Being an outer function ϑ has the form

$$\vartheta(\lambda) = \varkappa \exp\left[\int_{\mathbf{T}} \frac{e^{it}+\lambda}{e^{it}-\lambda} \log|\vartheta(e^{it})|\,\mathrm{dm}(t)\right] \quad (\lambda\in D),$$

where $\varkappa\in\mathbf{T}$ is a constant of absolute value 1, and D denotes the open unit disc in $\mathbf{C}$. We infer that

$$|\vartheta(0)| = \exp\left[\int_{\mathbf{T}} \log|\vartheta(e^{it})|\,dm(t)\right] \text{ and } |\vartheta'(0)| = 2|\vartheta(0)|\left|\int_{\mathbf{T}} e^{-it}\log|\vartheta(e^{it})|\,dm(t)\right|.$$

Let us assume now that $\vartheta_n(0)$ tends to 0 for a sequence $\{\vartheta_n\}_{n=1}^{\infty}$ in H_e^{∞}. Then $y_n = -\int_{\mathbf{T}} \log|\vartheta_n(e^{it})|\,dm(t)$ tends to infinity and so in virtue of $|\vartheta_n'(0)| \leqq 2\exp(-y_n)y_n$ it follows that $\vartheta_n'(0)$ converges to 0. We conclude that $\lim_{n\to\infty} \vartheta_n(0)=0$ implies $\lim_{n\to\infty} d(\vartheta_n)=\sqrt{2}$. Therefore, the number ν_1 is not 0.

7. The construction

Let α be an arbitrary subset of $\mathbf{T}$ such that $m(\alpha)>0$. Applying Lemma 6 we can find a sequence $\{\beta_n\}_{n=-\infty}^{\infty}$ of closed arcs of $\mathbf{T}$ such that $\beta_n\cap\beta_{n+1}$ consists of exactly one point, $m(\alpha_n)>0$, where $\alpha_n=\beta_n\cap\alpha$, for every $n\in\mathbf{Z}$ (the set of integers), and $\bigcup_{n=-\infty}^{\infty} \beta_n$ covers the whole $\mathbf{T}$ except one point.

For every $n\in\mathbf{Z}$, Theorem 7 ensures us a C_1-contraction $T_n\in\mathscr{L}(\mathfrak{H}_n)$ with $\nu(T_n)=0$ and being quasisimilar to M_{α_n}. Then the orthogonal sum $T' = \bigoplus_{n=-\infty}^{\infty} T_n \in \mathscr{L}(\mathfrak{H} = \bigoplus_{n=-\infty}^{\infty} \mathfrak{H}_n)$ is quasisimilar to $\bigoplus_{n=-\infty}^{\infty} M_{\alpha_n} \cong M_\alpha$.

Let us given a sequence $\{\varepsilon_n\}_{n=-\infty}^{\infty}$ of positive numbers such that $0<\varepsilon_n<1$ for every $n\in\mathbf{Z}$ and $\lim_{|n|\to\infty} \varepsilon_n=0$. Such sequences will be called *admissible*. On account of Proposition 4 we can find, for every $n\in\mathbf{Z}$, vectors $f_n, g_n\in\mathfrak{H}_n$ such that $\|f_n\|$, $\|g_n\|>1-\varepsilon_n$, $\|f_n-g_n\|<\varepsilon_n$ and (f_n, T_n), $(g_n, T_n^*)\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H}_n)$ are injective contractions. Now we define the operator $T''\in\mathscr{L}(\mathfrak{H})$ to be the sum $T'' = \sum_{n=-\infty}^{\infty} f_{n+1}\otimes g_n$, where for any n $f_{n+1}\otimes g_n\in\mathscr{L}(\mathfrak{H}_n, \mathfrak{H}_{n+1})$ denotes the operator of rank 1: $(f_{n+1}\otimes g_n)h = \langle h, g_n\rangle f_{n+1}$ $(h\in\mathfrak{H}_n)$, and the partial sums of the series converge in the strong operator topology.

Definition. We call the operator $S\in\mathcal{L}(\mathfrak{H})$ to be a *quasibilateral shift*, if there exists a sequence $\{\mathfrak{L}_n\}_{n=-\infty}^{\infty}$ of pairwise orthogonal subspaces and for every $n\in\mathbf{Z}$ there exist vectors $f_n, g_n\in\mathfrak{L}_n$ such that $\{\|f_n\|\}_{n=-\infty}^{\infty}$, $\{\|g_n\|\}_{n=-\infty}^{\infty}$ are bounded sequences and $S=\sum_{n=-\infty}^{\infty} f_{n+1}\otimes g_n$.

We say that the quasibilateral shift S *assymptotically approximates the bilateral shift in order* ε, where $\varepsilon=\{\varepsilon_n\}_{n=-\infty}^{\infty}$ is an admissible sequence, if the sequences $\{f_n\}_{n=-\infty}^{\infty}$, $\{g_n\}_{n=-\infty}^{\infty}$ fulfill the following relation:

$$\max\{|\|f_n\|-1|,\ |\|g_n\|-1|,\ \|f_n-g_n\|\}<\varepsilon_n$$

for every $n\in\mathbf{Z}$.

We note that every (simple) bilateral shift is unitarily equivalent to M.

It is evident that the sum $T=T'+T''$ of the contractions T', T'' obtained before is an injective contraction with dense range. With the notions introduced above our result can be formulated as follows:

Theorem 11. *Let α be a subset of* $\mathbf{T}$ *such that $m(\alpha)>0$ and let $\varepsilon\in\{\varepsilon_n\}_{n=-\infty}^{\infty}$ be an admissible sequence. Then there exist a C_1-contraction $T'\in\mathcal{L}(\mathfrak{H})$ which is quasisimilar to M_α and a contractive quasibilateral shift $T''\in\mathcal{L}(\mathfrak{H})$ which assymptotically approximates the bilateral shift in order ε such that their sum $T=T'+T''$ is a quasiaffine contraction on $\mathfrak{H}$.*

The contraction T is close, in different senses, both to M_α and to M. Unfortunately, we are not able to prove yet that T can be a C_{11}-contraction. However, by modifying our construction and assuming that some subspaces of $\mathfrak{H}$ are C_{11}-semiinvariant for T, we can show that T is quasisimilar to M_α.

Namely, taking into account Theorem 9 we can achieve that the spectrum of every contraction $T_n\in\mathcal{L}(\mathfrak{H}_n)$ considered be $D(\alpha_n, \zeta_n)$, where ζ_n is an arbitrary fixed point of α_n^-. Let $f_n, g_n\in\mathfrak{H}_n$ be as before, and let us introduce the operators $T_1'\in\mathcal{L}(\mathfrak{K}_1)$, $T_2'\in\mathcal{L}(\mathfrak{K}_2)$, $T''\in\mathcal{L}(\mathfrak{K}_1)$ as follows:

$$T_1'=\bigoplus_{k=-\infty}^{\infty} T_{2k},\ T_2'=\bigoplus_{k=-\infty}^{\infty} T_{2k+1},\ T''=\sum_{k=-\infty}^{\infty} f_{2(k+1)}\otimes g_{2k},$$

where $\mathfrak{K}_1=\bigoplus_{k=-\infty}^{\infty}\mathfrak{H}_{2k}$ and $\mathfrak{K}_2=\bigoplus_{k=-\infty}^{\infty}\mathfrak{H}_{2k+1}$. Now we define the quasiaffine contraction $T\in\mathcal{L}(\mathfrak{H})$ by

$$T=(T_1'+T'')\oplus T_2'.$$

Let $\mathfrak{M}$ and $\mathfrak{N}$ denote the subspaces $\mathfrak{M}=\bigoplus_{k=0}^{\infty}\mathfrak{H}_{2k}$, $\mathfrak{N}=\bigoplus_{k=-\infty}^{-1}\mathfrak{H}_{2k}$. It is evident that $T|\mathfrak{M}\in C_{1\cdot}$ and $P_{\mathfrak{N}}T|\mathfrak{N}\in C_{\cdot 1}$. Moreover, we shall prove:

Theorem 12. *If $T|\mathfrak{M}$ and $P_{\mathfrak{N}}T|\mathfrak{N}$ are C_{11}-contractions, then T is quasisimilar to M_α.*

Proof. We have to show that $T|\mathfrak{K}_1$ is quasisimilar to $\bigoplus_{k=-\infty}^{\infty} M_{\alpha_{2k}}$.

Cosidering the matrix $T|\mathfrak{K}_1=\begin{bmatrix} T|\mathfrak{M} & * \\ 0 & P_{\mathfrak{N}}T|\mathfrak{N}\end{bmatrix}$ of $T|\mathfrak{K}_1$ in the decomposition $\mathfrak{K}_1=\mathfrak{M}\oplus\mathfrak{N}$ the assumption $T|\mathfrak{M}$, $P_{\mathfrak{N}}T|\mathfrak{N}\in C_{11}$ immediately implies that $T\in C_{11}$. Therefore, T is quasisimilar to its residual part R_T (cf. [4, Proposition 1.3]). On account of [4, Theorem 1.7] R_T is unitarely equivalent to $R_{T|\mathfrak{M}}\oplus R_{P_{\mathfrak{N}}T|\mathfrak{N}}\oplus R_{T|\mathfrak{K}_2}$.

We shall prove that $R_{T|\mathfrak{M}}\cong\bigoplus_{k=0}^{\infty} M_{\alpha_{2k}}$. Applying [4, Theorem 1.7] several times we obtain that $R_{T|\mathfrak{M}}\cong\big(\bigoplus_{k=0}^{n} R_{T_{2k}}\big)\oplus R_{T|\bigoplus_{k=n+1}^{\infty}\mathfrak{H}_{2k}}\cong\big(\bigoplus_{k=0}^{n} M_{\alpha_{2k}}\big)\oplus R_{T|\bigoplus_{k=n+1}^{\infty}\mathfrak{H}_{2k}}$ for every $n\in\mathbf{N}$. Taking into account the functional model of $R_{T|\mathfrak{M}}$ (cf. [6, Theorem X.10]), we infer that $M'=\bigoplus_{k=0}^{\infty} M_{\alpha_{2k}}$ can be injected into $R_{T|\mathfrak{M}}$: $M'\overset{i}{\prec}R_{T|\mathfrak{M}}$, that is some injective operator X intertwines these operators: $XM'=R_{T|\mathfrak{M}}X$.

Next we want to show that $T|\mathfrak{M}\overset{i}{\prec}M'$. We are looking for an injection X such that $X(T|\mathfrak{M})=M'X$. Let us consider the matrix $[X_{ij}]_{i,j=0}^{\infty}$ of X with respect to the decompositions $\mathfrak{M}=\bigoplus_{k=0}^{\infty}\mathfrak{H}_{2k}$ and $\mathfrak{E}'=\bigoplus_{k=0}^{\infty}\mathfrak{E}_k$, where $\mathfrak{E}'$, $\mathfrak{E}_k$ are the domains of M' and $M_k:=M_{\alpha_{2k}}$, respectively. The commuting relation above can be expressed by the equations

$$(*)\qquad M_iX_{ij}-X_{ij}T_{2j}=(X_{i,j+1}f_{2(j+1)})\otimes g_{2j}\quad (i,j\in\mathbf{N}).$$

Since T_{2i} is quasisimilar to M_i ($i\in\mathbf{N}$ is arbitrary), we can find an intertwining quasiaffinity $X_i\in\mathscr{L}(\mathfrak{H}_{2i},\mathfrak{E}_i)$ such that $M_iX_i=X_iT_{2i}$. Let us define X_{ij} to be zero if $j>i$, and $X_{ii}=X_i$ for every $i\in\mathbf{N}$. Then equality $(*)$ holds, whenever $i\leqq j$.

Let us now assume that $0\leqq j<i$. Since $\sigma(M_i)=\alpha_{2i}^{=}$, $\sigma(T_{2j})=D(\alpha_{2j},\zeta_{2j})$ and $\alpha_{2i}^{=}\cap D(\alpha_{2j},\zeta_{2j})=\emptyset$, it follows by Rosenblum's theorem (cf. [14, Theorem 3.1]) that X_{ij} can be expressed from $(*)$ by the integral formula:

$$X_{ij}=\frac{1}{2\pi i}\int_{\Gamma_{ij}}(M_i-\lambda)^{-1}\big((X_{i,j+1}f_{2(j+1)})\otimes g_{2j}\big)(\lambda-T_{2j})^{-1}\,d\lambda,$$

where Γ_{ij} is a rectifiable Jordan curve surrounding $\alpha_{2i}^{=}$ and containing $D(\alpha_{2j},\zeta_{2j})$ in its exterior. We obtain X_{ij} for $j<i$ successively from X_i by this formula.

It is easy to see that for every $i\in\mathbf{N}$ there exists a constant K_i such that $\sum_{j=0}^{\infty}\|X_{ij}\|\leqq K_i\|X_i\|$. Since X_i's can be chosen with arbitrary small norms, we can achieve that $\sum_{i,j=0}^{\infty}\|X_{ij}\|<\infty$ hold for the operators defined above. Then the matrix $[X_{ij}]_{i,j=0}^{\infty}$ actually defines an operator $X\in\mathscr{L}(\mathfrak{M},\mathfrak{E}')$ (cf. [7, Sec. 36]), which will intertwine $T|\mathfrak{M}$ and M', and is evidently injective. Therefore, $T|\mathfrak{M}$ can be injected into M'.

Being a C_{11}-contraction, $T|\mathfrak{M}$ is quasisimilar to its residual part $R_{T|\mathfrak{M}}$, and we conclude by the chain of relations

$$R_{T|\mathfrak{M}} \sim T|\mathfrak{M} \overset{i}{\prec} M' \overset{i}{\prec} R_{T|\mathfrak{M}}$$

that $R_{T|\mathfrak{M}}$ is unitarily equivalent to M'. (Cf. [12, Lemma 6] and [17, Proposition II.3.4].)

An analogous argumentation yields that $R_{P_{\mathfrak{N}}T|\mathfrak{N}} \cong \bigoplus_{k=-\infty}^{-1} M_{\alpha_{2k}}$. Consequently $R_T \cong R_{T|\mathfrak{M}} \oplus R_{P_{\mathfrak{N}}T|\mathfrak{N}} \oplus R_{T|\mathfrak{K}_2} \cong (\bigoplus_{k=0}^{\infty} M_{\alpha_{2k}}) \oplus (\bigoplus_{k=-\infty}^{-1} M_{\alpha_{2k}}) \oplus (\bigoplus_{k=-\infty}^{\infty} M_{\alpha_{2k+1}}) \cong \bigoplus_{n=-\infty}^{\infty} M_{\alpha_n} \cong$

$\cong M_\alpha$, and so T is quasisimilar to M_α. The proof is finished.

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Local upper estimates for the eigenfunctions of a linear differential operator

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Dedicated to Professor Károly Tandori on his 60th birthday

Let $G\subset\mathbf{R}$ be an arbitrary open interval, $n\in\mathbf{N}$, $q_1, \dots, q_n\in L^1_{\mathrm{loc}}(G)$ arbitrary complex functions, and consider the differential operator

$$Lu = u^{(n)}+q_1u^{(n-1)}+\dots+q_nu.$$

We recall the definition of the eigenfunctions of higher order:

Given a complex number λ, the function $u\colon G\to\mathbf{C}$, $u\equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. A function $u\colon G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of order m $(m=0, 1, \dots)$ of the operator L with the eigenvalue λ if the following two conditions are satisfied:

— u, together with its first $n-1$ derivatives is absolute continuous on every compact subinterval of G;

— there exists an eigenfunction u^* of order $m-1$ of the operator L with the eigenvalue λ such that for almost all $x\in G$

$$(Lu)(x) = \lambda u(x)+u^*(x). \tag{1}$$

Let u be an eigenfunction of order m $(m=0, 1, \dots)$ of the operator L with some eigenvalue λ. Let us index the n-th roots of λ such that

$$\operatorname{Re}\mu_1 \geqq \dots \geqq \operatorname{Re}\mu_n. \tag{2}$$

It is known (see the references below) that to any compact subinterval K of G there exists a constant $G=G_m$ such that

$$\|u\|_{L^\infty(K)} \leqq C\bigl(1+\max_{1\leqq p\leqq n}|\operatorname{Re}\mu_p|\bigr)\|u\|_{L^1(K)} \tag{3}$$

and for $|\lambda|$ sufficiently large

$$\|u^{(i)}\|_{L^\infty(K)} \leqq C\bigl(1+\max_{1\leqq p\leqq n}|\operatorname{Re}\mu_p|\bigr)\|u^{(i)}\|_{L^1(K)} \quad (i = 1, \dots, n-1). \tag{4}$$

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Furthermore, if $q_1; \dots, q_n \in L^p_{loc}(G)$ for some $p \in [1, \infty]$ then

(5) $$\|u^*\|_{L^{p'}(K)} \leqq C(1+|\mu_1|)^{n-1}(1+\max_{1\leqq p\leqq n} |\operatorname{Re}\mu_p|)\|u\|_{L^{p'}(K)}.$$

The constant C does not depend on the choice of u.

Remarks. (i) If $n \geqq 3$ then the quantity $\max_{1\leqq p\leqq n} |\operatorname{Re}\mu_p|$ may be replaced by $|\mu_1| = |\lambda|^{1/n}$: they are equivalent. One can see easily by counterexamples that the above estimates are the best possible.

(ii) The estimates (3), (4), (5) were proved in [3] for the case $n=2$ and $q_1 \equiv 0$ (see also [2]), in [5] for the case $n \geqq 3$ and $q_1 \equiv 0$; in the general case $q_1 \not\equiv 0$, using the results of the paper [5], they were proved in [6] for the case $n \geqq 3$ and in [7] for the case $n \leqq 2$.

The aim of this paper is to show that if we replace the compact interval K on the right side of the estimates (3), (4), (5) by another compact interval, strictly containing K, then the terms $(1+\max_{1\leqq p\leqq n} |\operatorname{Re}\mu_p|)$ can be omitted. This phenomenon plays an important role in the local investigation of spectral expansions.

Remarks. (i) The first results of this type were proved by V. A. Il'in [1] and were used to prove a general local basis theorem. For the proof he used the following condition: putting

(6) $$\mu = \begin{cases} [(-1)^{n/2}\lambda]^{1/2} & \text{if } n \text{ is even,} \\ [i\lambda]^{1/n} & \text{if } n \text{ is odd and } \operatorname{Im}\lambda \leqq 0, \\ [-i\lambda]^{1/n} & \text{if } n \text{ odd and } \operatorname{Im}\lambda > 0 \end{cases}$$

where

$$[re^{i\varphi}]^{1/n} = r^{1/n}e^{i\varphi/n}, \quad -\frac{\pi}{2} < \varphi \leqq \frac{3\pi}{2},$$

the existence of a constant C was proved for any fixed band

(7) $$|\operatorname{Im}\mu| \leqq C_1 \quad (C_1 \text{ is constant}).$$

Also, the coefficients of the differential operator were assumed to be sufficiently smooth. As we shall see, the above conditions can be omitted.

(ii) In the proof of Theorem 2 of this paper we shall use a formula obtained in [5] for the coefficients of which very simple explicit formulas were found by by Joó [4]. This will play an important role in the proof.

In the sequel we shall use the following notations:

(8) $$n' = \left[\frac{n+1}{2}\right], \; N' = n'(m+1), \; N = n(m+1), \; \mu = \mu_n, \; \varrho = |\operatorname{Re}\mu|.$$

Obviously

(9) $$\varrho = \min \{|\mathrm{Re}\,\mu_p|\colon\, p = 1, \dots, n\}$$

and

(10) $$\varrho = \mathrm{Re}\,\mu \quad \text{if} \quad n \quad \text{is} \quad \text{even}.$$

Remark. Suppose n is odd and consider the operator

$$\tilde{L}u = u^{(n)} + \tilde{q}_1 u^{(n-1)} + \dots + \tilde{q}_n u$$

on the interval $\tilde{G} := -G$ where $\tilde{q}_s(y) := (-1)^s q_s(-y)$. Then, for any eigenfunction u of order m of the operator L with some eigenvalue λ, the function $\tilde{u}(y) := u(-y)$ is an eigenfunction of order m of the operator $\tilde{L}$ with the eigenvalue $-\lambda$. This correspondence makes us possible to consider always the case $\mathrm{Re}\,\mu \geqq 0$ i.e. $\varrho = \mathrm{Re}\,\mu$.

In the sequel $u = u_m$ will denote an arbitrary eigenfunction of order m of the operator L with some eigenvalue λ. Let us introduce recursively the continuous functions

$$u_j\colon\, G \to \mathbf{C}, \quad u_j = Lu_{j+1} - \lambda u_{j+1} \quad \text{a.e. on } G$$

for $0 \leqq j \leqq m-1$. Then u_j is an eigenfunction of order j of the operator L with the eigenvalue λ and $u_{m-1} = u^*$.

1. Local "anti a priori" estimates. In this section we shall prove the following result:

Theorem 1. *Assume* $q_1 \equiv 0$ *and* $q_2, \dots, q_n \in L^p_{\mathrm{loc}}(G)$ *for some* $p \in [1, \infty]$. *Then to any* $m \in \{0, 1, \dots\}$ *and to arbitrary compact intervals* $K_1, K_2 \subset G$, $K_1 \subset \mathrm{int}\, K_2$, *there exists a constant C such that for any eigenfunction u of order m of the operator L with some eigenvalue* $\lambda = \mu^n$,

(11) $$\|u^*\|_{L^{p'}(K_1)} \leqq C(1+|\mu|)^{n-1} \|u\|_{L^{p'}(K_2)}.$$

The proof will be based on the following assertion:

Proposition 1. *Given* $0 \neq \mu \in \mathbf{C}$ *and* $t \in \mathbf{R}$ *arbitrarily, there exist numbers* $d(\mu, t)$, $d_k(\mu, t)$ *and continuous functions* $D_r(\mu, t, \cdot)$ *such that for any eigenfunctions u of order m of the operator L with the eigenvalue* $\lambda = \mu^n$,

(12) $$t^n d(\mu, t) u_{m-1}(x) =$$

$$= \sum_{k=N'-N+1}^{N'} d_k(\mu, t) u_m(x+kt) + \sum_{r=0}^{m} \sum_{s=1}^{n} \int_{x+(N'-N+1)t}^{x+N't} D_r(\mu, t, x-\tau)\, q_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau$$

whenever $x + (N'-N+1)t \in G$ *and* $x + N't \in G$. *Furthermore, introducing the no-*

tation

(13)
$$P(\mu, t) = (\mu t)^{n(1+\dots+m)} \exp\Big(\sum_{i=n'-n+1}^{n'} \big(i(m+1)+\dots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t\Big),$$

there exist positive constants C_1, C_2 *and to any fixed positive number* A *a positive constant* C *such that*

$$|d_k(\mu, t)| \leqq C|\mu|^{n-1}|P(\mu, t)|e^{-|k|\varrho t} \quad \textit{for all } k, \tag{14}$$

$$|D_r(\mu, t, x-\tau)| \leqq C|\mu|^{r(1-n)}|P(\mu, t)|e^{-\varrho|x-\tau|} \tag{15}$$

for all $x+(N'-N+1)t \leqq \tau \leqq x+N't$,

$$\sup_{t/2 \leqq t_0 \leqq t} |d(\mu, t_0)| > C_1|P(\mu, t_0)| \tag{16}$$

whenever

$$\operatorname{Re}\mu \geqq 0, \quad 0 \leqq t \leqq A \quad \textit{and} \quad |\mu t| \geqq C_2. \tag{17}$$

First we deduce Theorem 1 from Proposition 1. For $m=0$ the theorem is obvious because $u^* \equiv 0$. Assume $m \geqq 1$ and that the theorem is true for $m-1$. Let us fix a compact interval $K \subset G$ such that

$$K_1 \subset \operatorname{int} K \quad \text{and} \quad K \subset \operatorname{int} K_2$$

and put

$$\varepsilon = (N')^{-1} \operatorname{dist}(K_1, \partial K).$$

It suffices to consider the case $\operatorname{Re}\mu \geqq 0$ in view of (10) and the Remark after (10). For $|\mu|$ sufficiently large we can fix a number $t \in [\varepsilon/2, \varepsilon]$ by Proposition 1 such that

$$|d_k(\mu, t)| \leqq C|\mu|^{n-1}|d(\mu, t)|, \quad |D_r(\mu, t, x-\tau)| \leqq C|\mu|^{r(1-n)}|d(\mu, t)|.$$

Fixing t by this manner, we have by (12) for any $x \in K_1$

$$|u_{m-1}(x)| \leqq C|\mu|^{n-1} \sum_{k=N'-N+1}^{N'} |u_m(x+kt)| + C\sum_{r=0}^{m}\sum_{s=2}^{n} |\mu|^{r(1-n)} \|q_s\|_{L^p(K)} \|u_{m-r}^{(n-s)}\|_{L^{p'}(K)}$$

whence

$$\|u_{m-1}\|_{L^{p'}(K_1)} \leqq C|\mu|^{n-1}\|u_m\|_{L^{p'}(K)} + C\sum_{r=0}^{m}\sum_{s=2}^{n} |\mu|^{r(1-n)} \|u_{m-r}^{(n-s)}\|_{L^{p'}(K)}. \tag{18}$$

(Here and in the sequel C denotes diverse constants which do not depend on the choice of u.) Being $|\mu|$ large, by Theorem 2 of [5] we have

$$\|u_{m-r}^{(n-s)}\|_{L^{p'}(K)} \leqq C|\mu|^{n-s}\|u_{m-r}\|_{L^{p'}(K)} \leqq C|\mu|^{n-2}\|u_{m-r}\|_{L^{p'}(K)}.$$

On the other hand, using the inductive hypothesis, for $r \geqq 2$

$$\|u_{m-r}\|_{L^{p'}(K)} \leqq C|\mu|^{(r-1)(n-1}\|u_{m-1}\|_{L^{p'}(K_2)}.$$

Finally, using again Theorem 2 of [5] we obtain

$$\|u_{m-1}\|_{L^{p'}(K_2)} \leqq C|\mu|^n \|u_m\|_{L^{p'}(K_2)}.$$

Therefore we obtain from (18) the estimate

$$\|u_{m-1}\|_{L^{p'}(K_1)} \leqq C|\mu|^{n-1} \|u_m\|_{L^{p'}(K_2)}$$

i.e. (11) is proved for $|\mu|$ sufficiently large. But for $|\mu|$ bounded (11) follows immediately from (5) and the theorem is proved.

Let us now turn to the proof of Proposition 1. Putting

$$K_0(\mu, y) = \sum_{p=1}^{n} \frac{\mu_p}{n\lambda} e^{\mu_p y}, \quad K_r(\mu, y) = \int_0^y K_0(\mu, \xi) K_{r-1}(\mu, y-\xi)\, d\xi \quad (r = 1, 2, \ldots)$$

and for any fixed $x \in G$

(19)
$$v_m(y) = u_m(y) + \sum_{r=0}^{m} \int_x^y K_r(\mu, y-\tau) \sum_{s=0}^{n} q_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau, \quad v_{m-1}(y) = v_m^{(n)}(y) - \lambda v_m(y),$$

it follows from the results of the paper [5] that v_m is an eigenfunction of order m of the operator $L_0 v = v^{(n)}$ (defined on G) with the eigenvalue λ, and $v_{m-1}(x) = u_{m-1}(x)$.

Consequently the function $v_m(y)$ is a linear combination of the functions

$$y \mapsto (r!)^{-1}(\mu_p(y-x))^r e^{\mu_p(y-x)} \quad (r = 0, \ldots, m,\ p = 1, \ldots, n);$$

therefore the determinant

$$\begin{vmatrix} \ldots & v_m(x+kt) & \ldots & t^n u_{m-1}(x) \\ & \vdots & & \vdots \\ \ldots & \dfrac{(k\mu_p t)^r}{r!} e^{k\mu_p t} & \ldots & C_r(\mu_p t)^n \\ & \vdots & & \vdots \end{vmatrix}$$

$$\left(r = 0, \ldots, m, \quad p = 1, \ldots, n, \quad k = N'-N+1, \ldots, N', \quad C_r = (1-\delta_{r0})\binom{n}{r}\right)$$

vanishes whenever $x+(N'-N+1)t \in G$ and $x+N't \in G$. Developing the determinant according to its first row, in view of (19) we obtain (with obvious notations) the formula (12).

Let us set for brevity

$$z_1^* = \sum_{i=1}^{n'}\big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t,$$

$$(21)\qquad z_2^* = \sum_{i=n'-n+1}^{0}\big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t,$$

$$z^* = z_1^*+z_2^*.$$

We shall also use the notation

$$w_1 \overset{r}{\leqq} w_2 \Leftrightarrow \operatorname{Re} w_1 \leqq \operatorname{Re} w_2.$$

First we prove (14). One can see easily that each term of the development of the minor defining $d_k(\mu, t)$ can be estimated by an expression of type

$$C|\mu|^{n-1+n(1+\ldots+m)}|e^z|.$$

In view of (13) and (21) it suffices to show that we can always choose z such that

$$(22)\qquad \operatorname{Re}(z-z^*) \leqq -|k|\varrho t.$$

Introducing the notation $k=(m+1)l_1-l_2$, $l_1\in\{n'-n+1, \ldots, n'\}$, $l_2\in\{0, \ldots, m\}$, we can choose

$$z = z_2^*+\sum_{i=l_1+1}^{n'}\big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t+$$

$$+\big(l_1(m+1)+\ldots+(k+1)+(k-1)+\ldots+(l_1-1)(m+1)\big)\mu_{n'+1-l_1t}+$$

$$+\sum_{=1}^{l_1-1}\big((i(m+1)-1)+\ldots+(i-1)(m+1)\big)\mu_{n'+1-i}t$$

is $k\geqq 1$, and

$$z = z_1^*+\sum_{i=n'-n+1}^{l_1-1}\big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$

$$+\big((l_1(m+1)+1)+\ldots+(k+1)+(k-1)+\ldots+((l_1-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{i=l_1+1}^{0}\big((i(m+1)+1)+\ldots+((i-1)(m+1)+2)\big)\mu_{n'+1-i}t-\mu_{n'+1}t$$

if $k\leqq 0$. Using (2) hence we obtain

$$z-z^* = \big((l_1-1)(m+1)-k\big)\mu_{n'+1-l_1}-(m+1)\sum_{i=1}^{l_1-1}\mu_{n'-1-i}t \overset{r}{\leqq}$$

$$\overset{r}{\leqq} \big((l_1-1)(m+1)-k\big)(\mu_{n'+1-l_1}-\mu_{n'})t-k\mu_{n'}t \overset{r}{\leqq} -k\mu_{n'}t = -|k|\mu_{n'}t$$

if $k\geqq 1$, and

$$z-z^*=(l_1(m+1)+1-k)\mu_{n'+1-l_1}t+(m+1)\Big(\sum_{i=l_1-1}^{0}\mu_{n'+1-i}t\Big)-\mu_{n'+1}t\overset{r}{\leqq}$$

$$\overset{r}{\leqq}(l_1(m+1)+1-k)(\mu_{n'+1-l_1}-\mu_{n'+1})t-k\mu_{n'+1}t\overset{r}{\leqq}-k\mu_{n'+1}t=|k|\mu_{n'+1}t$$

if $k\leqq 0$. In view of (9) hence (22) follows in both cases and (14) is proved.

Now we prove (15). Let us fix $r\in\{0,\dots,m\}$ arbitrarily and let $l\in\{N'-N+2,\dots,\dots,N'\}$ be such that

(23) $$x+(l-1)t\leqq\tau\leqq x+lt.$$

Then $D_r(\mu,t,x-\tau)$ is defined by the determinant which differs from the determinant (20) in the first row:

in case $\tau\geqq x$ the element $v_m(x+kt)$ is replaced by $K_r(\mu;x-\tau+kt)$ if $l\leqq k\leqq N'$, all the other elements are replaced by 0;

in case $\tau\leqq x$ the element $v_m(x+kt)$ is replaced by, $-K_r(\mu,x-\tau+kt)$ if $N'-N+1\leqq k\leqq l-1$, all the other elements are replaced by 0.

One can see easily by induction on r that with some constants $c_{rp\alpha}$

$$K_r(\mu,x-\tau+kt)=\sum_{p=1}^{n}\mu_p^{1-rn-n}\sum_{\alpha=0}^{r}c_{rp\alpha}(\mu_p(x-\tau+kt))^\alpha e^{\mu_p(x-\tau+kt)}.$$

In view of (17) it suffices to show that for any fixed $q\in\{1,\dots,n\}$ and $\beta\in\{0,\dots,r\}$, *if we replace in the first row of the determinant* (20)

in case $\tau\geqq x$ the element $v_m(x+kt)$ by $k^\beta e^{\mu_q(x-\tau+kt)}$ if $l\leqq k\leqq N'$, all the other element by 0;

in case $\tau\leqq x$ the element $v_m(x+kt)$ by $-k^\beta e^{\mu_q(x-\tau+kt)}$ if $N'-N+1\leqq k\leqq l-1$, all the other elements by 0,

then this new determinant can be estimated by

$$C|\mu|^{n-1}|P(\mu,t)|e^{-\varrho|x-\tau|}.$$

One can see esaily that those terms of this determinant the factors of which choosen from the first row and from the row corresponding to $p=q$ and $r=\beta$ are in case $\tau\geqq x$ in one of the l-th, ..., N'-th columns, in case $\tau\leqq x$ in one of the $(N'-N+1)$-th, ..., $(l-1)$-th columns, pairwise eliminate each other. All the other terms can be estimated by

$$C|\mu|^{n-1+n(1+\dots+m)}|e^z|;$$

it suffices to show that here one can always choose z such that

(24) $$\operatorname{Re}(z-z^*)\leqq-\varrho|x-\tau|.$$

Let us consider first the case $\tau\geqq x$. Putting

$$l=(m+1)l_1-l_2,\ l_1\in\{1,\dots,n'\},\quad l_2\in\{0,\dots,m\},$$

we can take

$$z = z_2^* + \sum_{i=l_1+1}^{n'} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t+\mu_q(x-\tau)+(l-1)\mu_q t+$$
$$+\big(l_1(m+1)+\ldots+l+(l-2)+\ldots+(l_1-1)(m+1)\big)\mu_{n'+1-l_1}t+$$
$$+\sum_{i=1}^{l_1-1} \big((i(m+1)-1)+\ldots+(i-1)(m+1)\big)\mu_{n'+1-i}t$$

if $q \leqq n'+1-l_1$, and

$$z = z_2^* + \sum_{i=l_1+1}^{n'} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t+\mu_q(x-\tau+lt)+$$
$$+\big(l_1(m+1)+\ldots+(l+1)+(l-1)+\ldots+(l_1-1)(m+1)\big)\mu_{n'+1-l_1}t+$$
$$+\sum_{i=1}^{l_1-1} \big((i(m+1)-1)+\ldots+(i-1)(m+1)\big)\mu_{n'+1-i}t$$

if $q > n'+1-l_1$. Now using (2) and (23), in both cases

$$z-z^* \overset{r}{\leqq} \mu_{n'+1-l_1}\big(x-\tau+(l_1-1)(m+1)t\big)-(m+1)\sum_{i=1}^{l_1-1}\mu_{n'+1-it} \overset{r}{\leqq}$$

$$\overset{r}{\leqq} (\mu_{n'+1-l_1}-\mu_{n'})\big(x-\tau+(l_1-1)(m+1)t\big)+\mu_{n'}(x-\tau) \overset{r}{\leqq} \mu_{n'}(x-\tau) = -\mu_{n'}|x-\tau|$$

whence (24) follows.

Let us now consider the case $\tau \leqq x$. Putting $l-1=(m+1)l_1-l_2$, $l_1 \in \{n'-n+1, \ldots, 0\}$, $l_2 \in \{0, \ldots, m\}$, we can take

$$z = z_1^* + \sum_{i=n'-n+1}^{l_1-1} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)_{n'+1-1}t+\mu_q(x-\tau)+l\mu_q t+$$
$$+\big((l_1(m+1)+1)+\ldots+(l+1)+(l-1)+\ldots+((l_1-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$
$$+\sum_{i=l_1+1}^{0} ((i(m+1)+1)+\ldots+((i-1)(m+1)+2))\mu_{n'+1-i}t-\mu_{n'+1}t$$

if $q \geqq n'+1-l_1$, and

$$z = z_1^* + \sum_{i=n'-n+1}^{l_1-1} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{\cdot+1-i}t+\mu_q\big(x-\tau+(l-1)t\big)+$$
$$+\big((l_1(m+1)+1)+\ldots+l+(l-2)+\ldots+((l_1-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$
$$+\sum_{i=l_1+1}^{0} \big((i(m+1)+1)+\ldots+((i-1)(m+1)+2)\big)\mu_{n'+1-i}t-\mu_{n'+1}t$$

if $q<n'+1-l_1$. Using again (2) and (23), in both cases

$$z-z^* \overset{r}{\leqq} \mu_{n'+1-l_1}(x-\tau+(l_1(m+1)+1)t)-(m+1)\Big(\sum_{i=l_1+1}^{0}\mu_{n'+1-i}t\Big)-\mu_{n'+1}t \overset{r}{\leqq}$$

$$\overset{r}{\leqq} (\mu_{n'+1-l_1}-\mu_{n'+1})(x-\tau+(l_1(m+1)+1)t)+\mu(x-\tau) \overset{r}{\leqq} \mu_{n'+1}(x-\tau) = \mu_{n'+1}|x-\tau|$$

whence (24) follows and (15) is proved.

Finally we prove (16). One can see by induction on m that

$$|d(\mu,t)| = |\mu t|^{m(m+1)/2}|e^{\mu t}|^{(m+1)(m+2)/2}$$

if $n=1$, and

$$|d(\mu,t)| = |\mu t|^{m(m+1)n/2}\prod_{1\leqq p<q\leqq n}|e^{\mu_p t}-e^{\mu_q t}|^{(m+1)^2}$$

if $n\geqq 2$. In case $n=1$ (16) hence follows at once because $|d(\mu,t)|=|P(\mu,t)|$. In case $n\geqq 2$, taking into account that $e^{(\mu_1+\dots+\mu_n)t}=1$, we obtain

$$|d(\mu,t)| = |P(\mu,t)|\prod_{1\leqq p<q\leqq n}|1-e^{(\mu_q-\mu_p)t}|^{(m+1)^2}.$$

Taking into account that

$$\operatorname{Re} z \leqq -1/2 \Rightarrow |1-e^z| \geqq 1-e^{-1/2},$$

we have for any $t_0\in[t/2,t]$

$$|d(\mu,t_0)| \geqq |P(\mu,t_0)|(1-e^{-1/2})^{n(n-1)/2}\prod_{\substack{1\leqq p<q\leqq n\\ \operatorname{Re}(\mu_q-\mu_p)t>-1}}|1-e^{(\mu_q-\mu_p)t_0}|^{(m+1)^2}.$$

If we choose C_2 sufficiently large, the condition (17) implies for all the pairs (p,q) in this product

$$|\operatorname{Im}(\mu_q-\mu_p)t| > 2\pi$$

and then, in view of the inequality

$$\operatorname{Re} z > -1 \Rightarrow |1-e^z| \geqq e^{-1}|\sin(\operatorname{Im} z)|$$

(16) reduces to the following lemma:

Lemma. *Given $a_1,\dots,a_{k_0}\in\mathbf{R}$, $k_0\in\mathbf{N}$ such that $|a_k|>2\pi$ for all $k=1,\dots,k_0$, we have*

$$\sup_{1/2\leqq b\leqq 1}\min_{k=1}^{k_0}|\sin(ba_k)| \geqq \sin(\pi/(12k_0)).$$

Indeed, for any $k\in\{1,\dots,k_0\}$ the measure of the set

$$\{b\in[1/2,1]\colon\ |\sin(ba_k)| < \sin(\pi/(12\,k_0))\}$$

is less than or equal to $(3k_0)^{-1}$ whence the lemma follows.

The proof of Proposition 1 (and also of Theorem 1) is finished.

Remark. In case $n\leqq 2$ Theorem 1 remains valid under the weaker condition $q_1\in L^p_{\text{loc}}(G)$, too. Indeed, we proved in [7] that in case $n\leqq 2$ there exists a positive constant R such that for all the eingenfunctions u of order m of the operator L with some eigenvalue λ,

$$\|u\|_{L^\infty(K_1)} \leqq Ce^{-R|\mathrm{Re}\,\mu_1|}\|u\|_{L^\infty(K_2)}. \tag{25}$$

Using (3), (5) and (25),

$$\begin{aligned}\|u^*\|_{L^{p'}(K_1)} &\leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)\|u\|_{L^{p'}(K_1)} \leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)\|u\|_{L^\infty(K_1)}\leqq\\ &\leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)e^{-R|\mathrm{Re}\,\mu_1|}\|u\|_{L^\infty(K_2)} \leqq\\ &\leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)^2e^{-R|\mathrm{Re}\,\mu_1|}\|u\|_{L_1(K_2)} \leqq\\ &\leqq C(1+|\mu|)^{n-1}\|u\|_{L^1(K_2)} \leqq C(1+|\mu|)^{n-1}\|u\|_{L^{p'}(K_2)}.\end{aligned}$$

Conjecture. *The condition $q_1\equiv 0$ in Theorem 1 can be replaced by the weaker condition $q_1\in L^p_{\text{loc}}(G)$ in case $n\geqq 3$, too.*

2. Local uniform estimates. We shall prove the following result:

Theorem 2. *Assume $q_1\equiv 0$. Then to any $m\in\{0, 1, \dots\}$ and to any compact intervals $K_1, K_2\subset G$, $K_1\subset\operatorname{int} K_2$, there exists a constant C such that for any eigenfunction u of order m of the operator L with some eigenvalue λ,*

$$\|u\|_{L^\infty(K_1)} \leqq C\|u\|_{L^1(K_2)}. \tag{26}$$

For $|\lambda|$ sufficiently large we have also

$$\|u^{(i)}\|_{L^\infty(K_1)} \leqq C\|u^{(i)}\|_{L^1(K_2)} \quad (i = 1, \dots, n-1). \tag{27}$$

We need the following assertion:

Proposition 2. *There exist continuous functions f_k, F_r such that for any eigenfunction u_m of order m of the operator L with some eigenvalue $\lambda=\mu^n$,*

$$\begin{aligned}&\sum_{k=N'-N}^{N'} f_k(\mu, t)u_m^{(i)}(x+kt) =\\ &= \sum_{r=0}^{m}\int_{x+(N'-N)t}^{x+N't} D_3^iF_r(\mu, t, x-\tau)\sum_{s=1}^{n} q_s(\tau)u_{m-r}^{(n-s)}(\tau)\,d\tau \quad (i = 0, \dots, n-1)\end{aligned} \tag{28}$$

whenever $x+(N'-N)t\in G$ and $x+N't\in G$. Furthermore, introducing the notation

$$Q(\mu, t) = \exp\big((m+1)(\mu_1+\dots+\mu_{n'})t\big), \tag{29}$$

to any fixed positive number A there exists a constant C such that

(30) $$|f_0(\mu, t)-Q(\mu, t)| \leqq C|Q(\mu, t)|e^{\mathrm{Re}(\mu_{n'+1}-\mu_{n'})t},$$

(31) $$|f_{m+1}(\mu, t)-e^{-(m+1)\mu_{n'}t}Q(\mu, t)| \leqq C|e^{-(m+1)\mu_{n'}t}Q(\mu, t)|e^{\mathrm{Re}(\mu_{n'}-\mu_{n'-1})t}$$

(32) $$|f_k(\mu, t)| \leqq C|Q(\mu, t)|e^{-|k|\varrho t},$$

(33) $$|D_3^i F_r(\mu, t, x-\tau)| \leqq C|\mu|^{i+(r+1)(1-n)}|Q(\mu, t)|e^{-\varrho|x-\tau|}$$

whenever

(34) $$\mathrm{Re}\,\mu \geqq 0, \quad 0 \leqq t \leqq A \quad \textit{and} \quad |\mu| \geqq 1.$$

First we deduce Theorem 2 from Proposition 2. As in Theorem 1, it suffices to consider the case $\mathrm{Re}\,\mu \geqq 0$. Let us fix a compact interval $K \subset G$ such that

$$K_1 \subset \mathrm{int}\, K \quad \text{and} \quad K \subset \mathrm{int}\, K_2$$

and put

$$R = (m+1+N')^{-1} \text{ dist } (K_1, \partial K)$$

Let us fix $B_1 > 0$ such that

(35) $$\mathrm{Re}\,\mu \geqq B_1 \text{ and } t \geqq R/2 \Rightarrow |f_0(\mu, t)| \geqq 2^{-1}|Q(\mu, t)|$$

and then $B_2, B_3 > 0$ such that

(36) $$|\mu| \geqq B_2 \Rightarrow \|u\|_{L^\infty(K_2)} \leqq B_3|\mu|^{1-i}\|u^{(i)}\|_{L^1(K_2)}$$

and

(37) $$|\mu| \geqq B_2, \mathrm{Re}\,\mu \leqq B_1 \quad \text{and} \quad t \geqq R/2 \Rightarrow |f_{m+1}(\mu, t)| \geqq 2^{-1}|e^{-(m+1)\mu_{n'}t}Q(\mu, t)|.$$

This is possible by (30), (31) and by Theorems 3, 4 in [6] (if we are interested only in the estimate (26), it suffices to use Theorem 2 in [5] instead of the results of the paper [6]). Now we distinguish three cases.

If $|\mu| \leqq B_2$ then (26) follows from (3).

If $|\mu| > B_2$ and $\mathrm{Re}\,\mu \geqq B_1$ then we apply the formula (28) with any $x \in K_1$ and $R/2 \leqq t \leqq R$; in view of (22), (33) and (35) we obtain

$$|u_m^{(i)}(x)| \leqq C \sum_{\substack{N'-N \leqq k \leqq N' \\ k \neq 0}} |u_m^{(i)}(x+kt)| + C \sum_{r=0}^{m} \sum_{s=2}^{n} |\mu|^{i+(r+1)(1-n)} \|q_s\|_{L^1(K)} \|u_{m-r}^{(n-s)}\|_{L^\infty(K)}.$$

Using Theorem 2 in [5], Theorem 1 from the preceeding section and (36),

$$|\mu|^{i+(r+1)(1-n)}\|u_{m-r}^{(n-s)}\|_{L^\infty(K)} \leqq C|\mu|^{i+1-s}\|u_m\|_{L^\infty(K_2)} \leqq C|\mu|^{2-s}\|u_m^{(i)}\|_{L^1(K_2)} \leqq C\|u_m^{(i)}\|_{L^1(K_2)};$$

therefore

$$|u_m^{(i)}(x)| \leqq C \sum_{\substack{N'-N \leqq k \leqq N' \\ k \neq 0}} |u_m^{(i)}(x+kt)| + C\|u_m^{(i)}\|_{L^1(K_2)}.$$

Applying the transformation $\int\limits_{R/2}^{R} dt$ we obtain

$$|u_m^{(i)}(x)| \leqq C\|u_m^{(i)}\|_{L^1(K_2)}$$

whence

$$\|u_m^{(i)}\|_{L^\infty(K_1)} \leqq C\|u_m^{(i)}\|_{L^1(K_2)}$$

and (26), (27) are proved.

If $|\mu|>B_2$ and $\operatorname{Re}\mu<B_1$, then we apply the formula (28) with any $x\in K_1$ and $R/2\leqq t\leqq R$ (we put x in place of $x+(m+1)t$); using (32), (33) and (37) we obtain

$$|u_m^{(i)}(x)| \leqq C \sum_{\substack{N'-N-m-1\leqq k\leqq N'-m-1\\ k\neq 0}} |u_m^{(i)}(x+kt)|+$$

$$+C\sum_{r=0}^{m}\sum_{s=2}^{n}|\mu|^{i+(r+1)(1-n)}\|q_s\|_{L^1(K)}\|u_{m-r}^{(n-s)}\|_{L_\infty(K)};$$

hence we can conclude (26), (27) similarly as in the preceeding case. The theorem is proved.

Now we prove Proposition 2. Let us denote by $S_k(\mu, t)$ the elementary symmetric polynomial of order k of $e^{\mu_1 t}, \ldots, e^{\mu_n t}$ with the main coefficient $(-1)^{n-k}$ if $k\in\{0, \ldots, n\}$; otherwise we put $S_k(\mu, t)=0$. Define

$$f_{k+N'-N}(\mu, t) = S_{n-k}(\mu, t)$$

if $m=0$, and

$$f_{k+N'-N}(\mu, t) = \sum_{r_1\in \mathbf{Z}} \cdots \sum_{r_m\in \mathbf{Z}} S_{n-r_1}(\mu, t)\ldots S_{n-r_m}(\mu, t)S_{n-k+r_1+\ldots r_m}(\mu, t)$$

if $m\geqq 1$. It was shown by Joó [4] that for any eigenfunction v_m of order m of the operator $L_0 v=v^{(n)}$ with some eigenvalue $\lambda=\mu^n$,

$$\sum_{k=N'-N}^{N'} f_k(\mu, t)v_m(x+kt) \equiv 0;$$

hence for $i\in\{0, \ldots, n-1\}$

$$\sum_{k=N'-N}^{N'} f_k(\mu, t)v_m^{(i)}(x+kt) \equiv 0. \tag{38}$$

Using the notations of the preceeding section, let us define v_m by the formula (19). Then we have (see also [5])

$$v_m^{(i)}(y) = u_m^{(i)}(y)+\sum_{r=0}^{m}\int\limits_{x}^{y} D_2^i K_r(\mu, y-\tau)\sum_{s=1}^{n} q_s(\tau)u_{m-r}^{(n-s)}(\tau)\,d\tau. \tag{39}$$

(38) and (39) imply (28) (with obvious notations).

The estimates (30), (31), (32) follow easily from the explicit expressions of the functions f_k. To prove (33) we note that the formula (38) can be obtained if we develop the determinant

$$\begin{vmatrix} \dots & v_m(x+kt) & \dots \\ & \vdots & \\ \dots & \dfrac{(k\mu_p t)^r}{r!} e^{k\mu_p t} & \dots \\ & \vdots & \end{vmatrix} \tag{40}$$

$$(r = 0, \dots, m, \quad p = 1, \dots, n, \quad k = N'-N, \dots, N')$$

according to the first row and then we simplify the obtained formula by a suitable expression $R(\mu, t)$. Repeating the proof of the estimate (15) in Proposition 1, we obtain (33) under the condition $R(\mu, t)\neq 0$. But this condition can be omitted because for any fixed $\mu\neq 0$, both sides of (33) are continuous in t and the set

$$\{t\in\mathbf{R}\colon\ R(\mu, t) = 0\}$$

is discrete.

The proposition (and also the theorem) is proved.

Remark. In case $n\leqq 2$ the condition $q_1\equiv 0$ in Theorem 2 can be omitted. Indeed, using (25) and (3),

$$\|u\|_{L^\infty(K_1)} \leqq Ce^{-R|\operatorname{Re}\mu_1|}\,\|u\|_{L^\infty(K_2)} \leqq Ce^{-R|\operatorname{Re}\mu_1|}(1+|\operatorname{Re}\mu_1|)\,\|u\|_{L^1(K_2)} \leqq C\|u\|_{L^1(K_2)}.$$

Conjecture. *The condition $q_1\equiv 0$ in Theorem 2 can be omitted in case $n\geqq 3$, too.*

Finally we note another version of Theorem 2 which is a little weaker than the above conjecture:

Theorem 3. *Assume $q_1, \dots, q_n\in L^p_{\mathrm{loc}}(G)$ for some $p\in[1, \infty]$. Then to any compact intervals $K_1, K_2\subset G$, $K_1\subset \operatorname{int} K_2$, there exists a constant C such that for any eigenfunction u of order 0 of the operator L with some eigenvalue λ,*

$$\|u\|_{L^\infty(K_1)} \leqq C\|u\|_{L^{p'}(K_2)}. \tag{41}$$

For $|\lambda|$ sufficiently large we have also

$$\|u^{(i)}\|_{L^\infty(K_1)} \leqq C\|u^{(i)}\|_{L^{p'}(K_2)} \quad (i = 1, \dots, n-1). \tag{42}$$

Proof. We repeat the proof of Theorem 2 with the following changes:

In case $|\mu|>B_2$ and $\operatorname{Re}\mu\geqq B_1$ we have now

$$|u_0^{(i)}(x)| \leqq C \sum_{\substack{n'-n\leqq k\leqq n' \\ k\neq 0}} |u_0^{(i)}(x+kt)| + C\sum_{s=1}^{n} |\mu|^{i+1-n}\|q_s\|_{L^p(K)}\|u_0^{(n-s)}\|_{L^{p'}(K)}.$$

Using Theorem 2 in [5] and (36),

$$|\mu|^{i+1-n}\|u_0^{(n-s)}\|_{L^{p'}(K)} \leqq C|\mu|^{i+1-s}\|u_0\|_{L^{p'}(K)} \leqq C|\mu|^i\|u_0\|_{L^{p'}(K)} \leqq C\|u_0^{(i)}\|_{L^{p'}(K)};$$

therefore

$$|u_0^{(i)}(x)| \leqq C \sum_{\substack{n'-n\leqq k\leqq n' \\ k\neq 0}} |u_0^{(i)}(x+kt)| + C\|u_0^{(i)}\|_{L^{p'}(K)},$$

$$|u_0^{(i)}(x)| \leqq C\|u_0^{(i)}\|_{L^1(K)} + C\|u_0^{(i)}\|_{L^{p'}(K)},$$

and

$$\|u_0^{(i)}\|_{L^\infty(K_1)} \leqq C\|u_0^{(i)}\|_{L^{p'}(K)} \leqq C\|u_0^{(i)}\|_{L^{p'}(K_2)}.$$

The case $|\mu|>B_2$ and $\operatorname{Re}\mu<B_1$ is similar.

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On the unitary representations of compact groups

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

Let G be a compact group and let $g \to U_g$ be a continuous (strongly or weakly, since one implies the other) unitary representation of G on a complex Hilbert space $\mathfrak{H}$. By definition, the dimensionality of the representation is the Hilbert dimension of $\mathfrak{H}$. Denote by $\mathscr{B}(\mathfrak{H})$ the von Neumann algebra of all bounded linear operators of $\mathfrak{H}$. Furthermore, let $\mathscr{U}$ be the von Neumann algebra generated by $\{U_g: g \in G\}$. The representation $g \to U_g$ is said to be *irreducible* if $\mathscr{U} = \mathscr{B}(\mathfrak{H})$.

One of the fundamental theorems of representation theory is that a continuous irreducible unitary representation of a compact group G is finite-dimensional. Several methods have been introduced in the literature to prove this theorem. Most of them use the analytic geometry of Hilbert spaces and their compact operators, and some of them use the elegant but highly intricate machinery of Hilbert algebras or von Neumann algebras ([1], [3], [4], [5]). All of them use, however, invariant integration on G in one way or another. In [4], invariant integration served also for the basis of a general theory to create a mapping $T \to \tilde{T}$ of $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{U}'$, the algebraic commutant of $\mathscr{U}$, which yielded, in final analysis, a non-elementary, but an easy access to the fundamental facts of the representation theory of compact groups on Hilbert spaces, including the above mentioned theorem (cf. [4], 2, §4). Later, Karl H. Hofmann, apparently unaware of [4], came up with an explicit form of $\tilde{T}$ as

$$\tilde{T} = \int_G U_g^{-1} T U_g \, dg \quad (T \in \mathscr{B}(\mathfrak{H})), \tag{1}$$

where dg is the normalized Haar measure on G, and (1) is taken, for instance, in the sense of the weak operator topology of $\mathscr{B}(\mathfrak{H})$. Furthermore, using certain considerations in topological vector spaces, he observed that the mapping $T \to \tilde{T}$ carried compact operators into compact operators, a fact which rendered the proof of the theorem elegant, elementary, and easy (cf. [2]).

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We are going to give a proof here which is technically different from Hofmann's. In fact, denote by $\mathscr{B}^+(\mathfrak{H})$ the convex cone of the non-negative elements of $\mathscr{B}(\mathfrak{H})$ and consider an orthonormal basis $(e_i)_{i\in I}$ of $\mathfrak{H}$. For $T\in\mathscr{B}^+(\mathfrak{H})$, put

$$\text{(2)} \qquad \operatorname{tr}(T) = \sum_{i\in I}(Te_i|e_i),$$

where the symbol $(\cdot|\cdot)$ stands for the inner product of $\mathfrak{H}$. For further references, we precise that the summation in (2) is understood in the following manner. Let $\mathscr{F}$ be the family of all finite subsets J of I[1]. Then, by definition,

$$\text{(3)} \qquad \sum_{i\in I}(Te_i|e_i) = \sup_{J\in\mathscr{F}}\Big\{\sum_{i\in J}(Te_i|e_i)\Big\}.$$

The extended valued function $\operatorname{tr}(\cdot)$ which does not depend on the particular choice of (e_i) is linear, unitarily invariant, and is known as the canonical trace of $\mathscr{B}^+(\mathfrak{H})$. If P is an orthogonal projection of $\mathfrak{H}$, then $\operatorname{tr}(P)=\dim P\mathfrak{H}$. This implies that $\mathfrak{H}$ is finite-dimensional if and only if $\operatorname{tr}(I)<\infty$, where I is the identity operator of $\mathfrak{H}$. Now, choose, for instance, an arbitrary one-dimensional projection P of $\mathfrak{H}$, i.e. $\operatorname{tr}(P)=1$, and observe that for every $g\in G$ we have

$$\text{(4)} \qquad \sum_{i\in I}(U_g^{-1}PU_ge_i|e_i) = \operatorname{tr}(U_g^{-1}PU_g) = \operatorname{tr}(P) = 1.$$

For every $J\in\mathscr{F}$, let $f_J(g)=1-\sum_{i\in J}(U_g^{-1}PU_ge_i|e_i)$. Then, $(f_J)_{J\in\mathscr{F}}$ forms a downward directed family of continuous functions on G such that $\inf_{J\in\mathscr{F}} f_J=0$. Then, an elementary property of Radon integrals tells us that (4) can be termwise integrated (cf. [5], III, 2, §6):

$$\text{(5)} \qquad \operatorname{tr}(P)=1=\int_G \operatorname{tr}(P)\,dg = \sum_{i\in I}\int_G (U_g^{-1}PU_ge_i|e_i)\,dg = \sum_{i\in I}(\tilde{P}e_i|e_i) = \operatorname{tr}(\tilde{P}).$$

From this, we conclude that $\tilde{P}\neq 0$. Furthermore, the translation invariance of the Haar measure and (1) imply that $\tilde{P}\in\mathscr{U}'$ and $\tilde{P}>0$. Now, if the representation $g\to U_g$ is irreducible, then $\mathscr{U}'=(\mathscr{B}(\mathfrak{H}))'=(cI)_{c\in\mathbf{C}}$ ($\mathbf{C}$ is the complex number field), hence $\tilde{P}=c_0I$ with $c_0\neq 0$. Then this and (5) imply $\operatorname{tr}(I)=1/c_0<\infty$, i.e., $\mathfrak{H}$ is finite-dimensional. The proof is complete.

[1]) With respect to the inclusion of the elements of $\mathscr{F}$ as a partial ordering, $\mathscr{F}$ is an upward directed set.

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A Bohr type inequality on abstract normed linear spaces and its applications for special spaces

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction. BOHR [1] proved (in another form) that if a 2π-periodic integrable function g is orthogonal to every trigonometric polynomial of order at most n then the following inequality is true

$$\left|\int_0^x g(t)\,dt\right| \leqq \frac{c_1}{n}|g(x)| \quad (-\infty < x < \infty,\ n = 1, 2, \ldots) \tag{1}$$

where c_1 (and later c_k, $k=2, 3, \ldots$) denotes an absolute constant. Later an inequality of type (1) was discussed by many authors (see e.g. [2], [3], [4], [6], [9]).

Let $L^p_{2\pi}$ $(1 \leqq p \leqq \infty)$ be the Banach space of all 2π-periodic functions with the usual norm

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p\,dx\right\}^{1/p} \quad (1 \leqq p < \infty),$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{-\infty < x < \infty} |f(x)|.$$

We denote by T_n the set of all trigonometric polynomials of order at most n $(n=0, 1, \ldots)$. For $f \in L^p_{2\pi}$ let

$$E^p_n(f) = \inf_{t_n \in T_n} \|f - t_n\|_p \quad (n = 0, 1, 2, \ldots). \tag{2}$$

Let $D^p_{2\pi}$ be the set of all 2π-periodic functions f which are absolutely continuous on $(-\infty, \infty)$ and for which $f' \in L^p_{2\pi}$. It is vell known that

$$E^p_n(f) \leqq \frac{c_2}{n}\|f'\|_p \quad (1 \leqq p \leqq \infty,\ f \in D^p_{2\pi},\ n = 1, 2, \ldots). \tag{3}$$

Using the inequality (3) (case $p=1$) we can prove the inequality (1) and conversely.

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In this paper we prove this statement in abstract normed linear spaces and we give applications for special spaces.

2. A Bohr type inequality in abstract spaces. Let X be an arbitrary normed linear space. The norm in X is denoted by $\|\cdot\|$. Let furthermore X^* be the dual space of X (the space of all continuous linear functionals defined on X). The norm in X^* is denoted by $\|\cdot\|^*$. Let L be a subspace of X and

$$L^\perp = L^\perp(X) := \{g \in X^*: g(x) = 0 \quad \forall x \in L\}.$$

We can prove that $L^\perp$ is a subspace of X^*. We define the best approximation of an element $x \in X$ by elements of L:

$$E_L(x) = \inf_{y \in L} \|x-y\|.$$

Let T be the following operator:

$$T\colon D(T) \to X \quad \text{linear and} \quad T(D) = X \tag{4}$$

where $D = D(T) (\subseteqq X)$ denotes the domain of T.

Suppose that there exists an operator I which has domain $D(I) \subseteqq X^*$,

$$I\colon D(I) \to X^* \quad \text{is linear,} \tag{5}$$

I and T satisfy the following relation

$$g(x) = I_g(Tx) \quad (\forall x \in D(T),\ \forall g \in D(I)). \tag{6}$$

Then the following statement is true:

Theorem 1. *Let T and I be two operators satisfying* (4), (5), (6). a) *If $D(I) = L^\perp$ then the following statements are equivalent for $\lambda > 0$:*

$$E_L(x) \leqq \lambda \|Tx\| \quad (\forall x \in D(T)), \tag{7}$$

$$\|Ig\|^* \leqq \lambda \|g\|^* \quad (\forall g \in D(I)). \tag{8}$$

b) *In the case $D(I) \subset L^\perp$ the inequality* (7) *implies* (8).

Proof. a) (7)→(8): We have by the duality principle of Nikolskii (see e.g. Singer [8, p. 22])

$$\sup_{\substack{g \in L^\perp \\ \|g\|^* \leqq 1}} |Ig(Tx)| = \sup_{\substack{g \in L^\perp \\ \|g\|^* \leqq 1}} |g(x)| = E_L(x) \leqq \lambda \|Tx\|.$$

So for any fixed $g \in D(I) \subseteqq L^\perp (\|g\|^* \leqq 1)$ we have

$$|Ig(Tx)| \leqq \lambda \|Tx\| \quad (\forall x \in D(T)).$$

Hence by (4) we obtain

$$|Ig(y)| \leqq \lambda \|y\| \quad (\forall y \in X)$$

therefore we get (8) from the definition of norm in X^*.

b) (8)→(7): We have by duality principle and by (8)

$$E_L(x) = \sup_{\substack{g\in L^\perp \\ \|g\|^*\le 1}} |g(x)| = \sup_{\substack{g\in L^\perp \\ \|g^*\|\le 1}} |I_g(Tx)| \leqq \sup_{\substack{g\in L^\perp \\ \|g\|^*\le 1}} \|Ig\|^*\|Tx\| \leqq \lambda\|Tx\|.$$

3. Applications. a) Let $X=L^p_{2\pi}$ $(1\leqq p<\infty)$ and let $L=T_n$ $(n=1, 2, \dots)$. Then we have $X^*=L^q_{2\pi}$ $(1/p+1/q=1,\ 1\leqq p<\infty)$ and $(L^\infty_{2\pi})^*\supset L^1_{2\pi}$. Let

$$T_n^\perp(L^p_{2\pi}) = \Big\{g\in L^q_{2\pi}\colon \int_0^{2\pi} gt_n\,dx = 0,\ \forall t_n\in T_n\Big\} \quad (1\leqq p<\infty).$$

$$T_n^\perp(L^\infty_{2\pi}) \supset \Big\{g\in L^1_{2\pi};\ \int_0^{2\pi} gt_n\,dx = 0,\ \forall t_n\in T_n\Big\} := \Omega(L^1_{2\pi}).$$

Let $Tf:=f'$ $\big(f\in D(T):=D^p_{2\pi}\big)$ and

$$Ig(x) := \int_0^x g(t)\,dt \quad \left[g\in D_{q,n}(I) := \begin{cases} T_n^\perp(L^p_{2\pi}) & (1\leqq p<\infty), \\ \Omega(L^1_{2\pi}) & (p=\infty) \end{cases}\right].$$

It is easy to see that T and I satisfy the conditions (4), (5), (6) $\big($with $D(T)=L^\perp$ in the case $1\leqq p<\infty$, $D(T)\subset L^\perp$ in the case $p=\infty\big)$. So by Theorem 1 we have

Theorem 2. *Let* $1\leqq q\leqq\infty$, $n=1, 2, \dots$. *For every* $g\in D_{q,n}(I)$ *we have*

$$\Big\|\int_0^x g(t)\,dt\Big\|_{L^q_{2\pi}} \leqq \frac{c_3}{n}\|g\|_{L^q_{2\pi}}.$$

b) Let $X=L^p(w)$ $1\leqq p\leqq\infty$ be the Banach space of all measurable functions defined on $[-1, 1]$ with norm

$$\|f\|_{p,w} = \Big\{\int_{-1}^1 |f|^p w\,dx\Big\}^{1/p} \quad (1\leqq p<\infty),$$

$$\|f\|_{\infty,w} = \|f\|_\infty = \operatorname*{ess\,sup}_{x\in[-1,1]} |f(x)|$$

where

$$w(x) = (1-x)^\alpha(1+x)^\beta \quad (\alpha, \beta > -1,\ x\in[-1, 1]).$$

We have $X^*=[L^p(w)]^*=L^q(w)$ $(1\leqq p<\infty,\ 1/p+1/q=1)$ and $[L^\infty(w)]^*\supset L^1(w)$.

Let Π_n be the set of all algebraic polynomials of degree at most n $(n=0, 1, 2, \dots)$ and let $L=\Pi_n$. Then we have

$$L^\perp = \Pi_n^\perp(L^p(w)) = \Big\{g\in L^q(w)\colon \int_{-1}^1 gp_n w\,dx = 0,\quad \forall p_n\in\Pi_n\Big\}$$

(9) $$(1\leqq p<\infty,\quad 1/p+1/q=1),$$

$$\Pi_n^\perp[L^\infty(w)] \supset \Big\{g\in L^1(w)\colon \int_{-1}^1 gp_n w\,dx = 0,\quad \forall p_n\in\Pi_n\Big\} := \Omega_n(w).$$

For any $f \in L^p(w)$ $(1 \leqq p \leqq \infty)$ we define

$$E_n^p(w,f) = \inf_{p_n \in \Pi_n} \|f - p_n\|_{p,w} \quad (n = 0, 1, 2, \ldots).$$

The following class of functions was defined in [7]:

$$M_p(w) := \{f \in L^p(w) \colon f \text{ is absolutely continuous in } (-1, 1), \sqrt{1-x^2}\, f'(x) \in L^p(w)\}.$$

In [7] we proved that

$$(10) \quad E(w,f) \leqq \frac{c_4}{n} \left\|\sqrt{1-x^2}\, f'(x)\right\|_{p,w} \quad (1 \leqq p \leqq \infty,\ f \in M_p(w),\quad n = 1, 2, \ldots).$$

Now, let us define the operators T and I as follows:

$$Tf(x) = T_p f(x) := \sqrt{1-x^2}\, f'(x) \quad (f \in D(T_p) := M_p(w)),$$

$$Ig(x) = I_{q,n}\, g(x) := \frac{1}{\sqrt{1-x^2}\, w(x)} \int_{-1}^{x} w(t) g(t)\, dt \quad (g \in D(I_{q,n}))$$

where $D(I_{q,n})$ denotes the domain of $I = I_{q,n}$ which is defined by

$$(11) \qquad D(I_{q,n}) := \Pi_n^{\perp}[L^p(w)] \quad (2 < q \leqq \infty,\ 1/p + 1/q = 1),$$

$$(12) \qquad \begin{aligned} &D(I_{q,n}) := \{g \in \Pi_n^{\perp}[L^p(w)] \colon g \text{ satisfies condition (13)}\} \quad (1 < q \leqq 2) \\ &D(I_{q,n}) := \{g \in \Omega_n(w) \colon g \text{ satisfies condition (13)}\} \quad (q = 1), \end{aligned}$$

where

$$(13) \qquad \int_{-1}^{x} w(t) g(t)\, dt = o\left[w^{1-1/q}(x) \sqrt{1-x^2}\right] \quad (|x| \to 1).$$

We prove that the operators T and I satisfy the conditions in Theorem 1.

Let $f \in D(T_p)$, $g \in D(I_{q,n})$ and let $G(x) = \int_{-1}^{x} w(t) g(t)\, dt$.

In the case $(1 \leqq p < 2$, so $2 < q \leqq \infty)$ we have for $-1 < x < 0$

$$|G(x)| = \left|\int_{-1}^{x} w(t) g(t)\, dt\right| \leqq \left(\int_{-1}^{x} |g(t)|^q w(t)\, dt\right)^{1/q} \left(\int_{-1}^{x} w(t)\, dt\right)^{1/p} \leqq$$

$$\leqq \|g\|_{q,w}\, O[(x+1)^{1/p} w^{1/p}(x)] =$$

$$= (x+1)^{1/p-1/p}\, O\left[w^{1/2}(x) \sqrt{1-x^2}\right] = o\left[w^{1/p}(x) \sqrt{1-x^2}\right] \quad (x \to -1).$$

For $0 < x < 1$, using the relation

$$G(x) = \int_{-1}^{x} w(t) g(t)\, dt = -\int_{x}^{1} w(t) g(t)\, dt$$

(which follows from the fact that $\int_{-1}^{1} wg\,dt=0$ since $g\in D(I_{q,n})$). By a similar method we obtain

$$G(x) = o\left[w^{1/p}(x)\sqrt{1-x^2}\right] \quad (x \to 1).$$

So relation (13) is true for every $g\in D(I_{q,n})$ $(1\leqq q\leqq \infty,\ n=1, 2, \ldots)$. Therefore by integration by part we have

$$\int_{-1}^{1} f(x)g(x)w(x)\,dx = \int_{-1}^{1} f'(x)G(x)\,dx =$$

$$= \int_{-1}^{1} \sqrt{1-x^2}\,f'(x)\frac{1}{\sqrt{1-x^2}\,w(x)}G(x)\,w(x)\,dx = \int_{-1}^{1} Tf(x)\,Ig(x)w(x)\,dx.$$

Since this integral exists for every $Tf\in L^p(w)$ and $T[D(T)]=L^p(w)$, we have by a well known theorem of functional analysis that $Ig\in L^q(w)$ and the last formula proves (6).

By Theorem 1, using (10) we have

Theorem 3. *Let* $1\leqq q\leqq\infty$, $n=1, 2, \ldots$. *For every* $g\in D(I_{q,n})$ *we have*

$$\left\|\frac{1}{\sqrt{1-x^2}\,w(x)}\int_{-1}^{x} w(t)g(t)\right\|_{q,w} \leqq \frac{c_5}{n}\|g\|_{q,w}.$$

c) Let $X=L^p=L^p(-\infty, \infty)$ $(1\leqq p\leqq\infty)$ be the Banach space of functions defined on $(-\infty, \infty)$. Let

$$\varrho(x) = \varrho_{\gamma,\delta}(x) = (1+|x|^{\gamma})^{\delta/2\gamma}e^{-|x|^{\gamma}/2} \quad (\gamma \geqq 2,\ \delta\geqq 0,\ -\infty < x <\infty).$$

We consider the following subspace of L^p:

$$L:= H_n:= \{\varrho(x)p_n(x)\colon\ p_n\in\Pi_n\} \quad (n = 1, 2, \ldots).$$

We have

$$L^{\perp} = H_n^{\perp}(L^p) = \left\{g\in L^q\colon\ \int_{-\infty}^{\infty} gp_n\varrho\,dx = 0,\quad \forall p_n\in\Pi_n\right\}$$

$$(1 \leqq p <\infty,\ 1/p+1/p = 1,\ n = 1, 2, \ldots)$$

and

$$H_n^{\perp}(L^{\infty}) \supset \left\{g\in L^1\colon\ \int_{-\infty}^{\infty} gp_n\varrho\,dx = 0 \quad \forall p_n\in\Pi_n\right\}:= \Omega.$$

For any $\varrho f\in L^p$ we define

$$E_n^p(\varrho, f) = \inf_{p_n\in\Pi_n} \|\varrho(f-p_n)\|_p \quad (n = 0, 1, 2, \ldots).$$

FREUD [3] proved the following inequality:

$$E_n^p(\varrho,f) \leqq \frac{c_6}{n^{1-1/\gamma}} \|\varrho f'\|_p \quad (1 \leqq p \leqq \infty,\ \varrho f \in M_p(\varrho),\quad n = 1, 2, \ldots) \tag{14}$$

where

$$M_p(\varrho) := \{\varrho f \in L^p\colon f \text{ is absolutely continuous on } (-\infty, \infty),\ \varrho f' \in L^p\}. \tag{15}$$

We define $T = T_p$ and $I = I_{q,n}$ as follows:

$$T(\varrho f) := \varrho f' \quad (\varrho f \in M_p(\varrho) := D(T_p)),$$

$$Ig(x) := \frac{1}{\varrho(x)} \int_{-\infty}^{x} \varrho(t) g(t)\, dt \quad (f \in D(I)),$$

where

$$D[I_{q,n}) := \left\{ \begin{matrix} g \in H_n^{\perp}(L^p) & (1 \leqq p < \infty) \\ g \in \Omega & (p = \infty) \end{matrix} \quad \begin{pmatrix} \text{and } g \\ \text{satisfies condition (16)} \end{pmatrix} \right\},$$

where

$$\int_{-\infty}^{x} \varrho(t) g(t)\, dt = O[|x|^{1/q} \varrho(x)] \quad (|x| \to \infty). \tag{16}$$

First we prove that T and I satisfy the conditions of Theorem 1. Let $f \in D(T_p)$ $(1 \leqq p \leqq \infty)$ and let $g \in D(I_{q,n})$ $(1/p + 1/q = 1)$,

$$G(x) := \int_{-\infty}^{x} g(t)\, \varrho(t)\, dt.$$

Using (16) we obtain

$$|f(x) G(x)| = |G(x)| \left| \int_0^x f'(t)\, dt + f(0) \right| = o[|x^{1/q}| \varrho(x)] + o\left[|x|^{1/q} \varrho(x) \left| \int_0^x |f'(t)|\, dt \right| \right] =$$

$$= o(1) + o\left[\left| \int_0^x |x|^{1/q} \varrho(x) |f'(t)\, dt| \right| \right] = o(1) + o\left[|x|^{1/q} \left| \int_0^x \varrho(t) |f'(t)| dt \right| \right] =$$

$$= o(1) + o\left[|x|^{1/q} \|\varrho f'\|_p \left| \left(\int_0^x dt \right)^{1/q} \right] = o(1) + o(1) = o(1) \quad (|x| \to \infty).$$

So we have by integration by part

$$\int_{-\infty}^{\infty} f(x) \varrho(x) g(x)\, dx = \int_{-\infty}^{\infty} f'(x) G(x)\, dx = \tag{17}$$

$$= \int_{-\infty}^{\infty} \varrho(x) f'(x) \frac{1}{\varrho(x)} G(x)\, dx = \int_{-\infty}^{\infty} T(\varrho f) Ig\, dx.$$

Since the integral (17) exists for every $T(\varrho f)\in L^p$ and $T[D(T)]=L^p$ we have $Ig\in L^q$ and (17) proves condition (6). Other properties of T and I follow from the definition.

We have by Theorem 1 and (14)

Theorem 4. *Let* $1\leqq q\leqq\infty$, $n=1,2,\ldots$. *For every* $g\in D(I_{q,n})$ *we have*

$$\left\|\frac{1}{\varrho(x)}\int_{-\infty}^{x}\varrho(t)g(t)\,dt\right\|_q\leqq\frac{c_7}{n^{1-1/\gamma}}\|g\|_q.$$

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Limit cases in the strong approximation of orthogonal series

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In honour of Professor K. Tandori on his 60th birthday

Introduction

Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We consider the orthogonal series

(1) $$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

By the Riesz—Fischer theorem series (1) converges in the metric L^2 to a square-integrable function $f(x)$. Denote $s_n(x)$ the n-th partial sum of (1).

In [1] we proved that *if* $0<\gamma<1$ *and*

(2) $$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty,$$

then

$$\frac{1}{n+1} \sum_{k=0}^{n} (s_k(x) - f(x)) = o_x(n^{-\gamma})$$

almost everywhere in (a, b).

G. Sunouchi [6] generalized our theorem to strong approximation in the following way: *If* $0<\gamma<1$ *and* (2) *holds, then*

(3) $$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x) - f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

also holds almost everywhere for any $\alpha>0$ *and* $0<p<\gamma^{-1}$, *where* $A_n^\alpha = \binom{n+\alpha}{n}$.

We generalized this result in [2] in the following ways:

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First we showed that *the assumptions of Sunouchi's theorem imply, for any increasing sequence* $\{\nu_k\}$ *of the natural numbers, that*

$$(4)\qquad C_n(f,\alpha,p,\{\nu_k\};x):=\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma})$$

also holds almost everywhere in (a, b). In the other words we proved that the conditions of Sunouchi's theorem imply the very strong approximation with the same order. Since we speak on *strong, very strong* and *extra strong* (or *mixed*) approximation according as in the investigated means the following partial sums $s_k(x)$, $s_{\nu_k}(x)$ $(\nu_k<\nu_{n+1})$ or $s_{\mu_k}(x)$ (where $\{\mu_k\}$ is a permutation of a subsequence of the natural numbers, or briefly a mixed sequence) appear, respectively.

Secondly we replaced the partial sums in (3) by (C,δ)-means, where δ would also take negative values.

Very recently in a joint paper with H. Schwinn [5] we have attained to the following four theorems:

Theorem A. *If* $0<p\gamma<\beta$ *then for any increasing* $\{\nu_k\}$ *of the natural numbers condition* (2) *implies that*

$$(5)\qquad h_n(f,\beta,p,\{\nu_k\};x):=\left\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma})$$

holds almost everywhere in (a, b).

Theorem B. *If* α *and* γ *are positive numbers,* $0<p\gamma<1$, *and* $\{\nu_k\}$ *is an increasing sequence, then condition* (2) *implies* (4) *almost everywhere in* (a, b).

The novelty of these theorems is that the restriction $\gamma<1$, which appeared in the previous theorems, is omitted. The following theorems, holding this advantage, extend these results to the case of extra strong approximation under a slight restriction of other type.

Theorem C. *Let* $\{\mu_k\}$ *be a fixed permutation of a subsequence of the natural numbers, moreover let* $\gamma>0$ *and* $0<p\gamma<\min(\beta,1)$. *Then condition* (2) *implies that*

$$(6)\qquad h_n(f,\beta,p,\{\mu_k\};x)=o_x(n^{-\gamma})$$

holds almost everywhere in (a, b).

Theorem D. *Let* $\{\mu_k\}$ *be a fixed permutation of some subsequence of the natural numbers, let* $\gamma>0$ *and* $0<p\gamma<\min(\alpha,1)$. *Then* (2) *yields that*

$$(7)\qquad C_n(f,\alpha,p,\{\mu_k\};x)=o_x(n^{-\gamma})$$

holds almost everywhere in (a, b).

In a recent paper [4] we started to investigate the order of approximation of the means h_n defined in (5) under the assumption $\beta=p\gamma$; i.e. we investigated the limit case of the restrictions of the parameters. We obtained, among others, that in the special case $p=2$, condition (2) with $\gamma=\beta/2$ implies only

$$h_n(f, \beta, 2, \{v_k\};\ x) = O_x(n^{-\gamma});$$

and in the case $p\neq 2$, condition (2) does not ensure even this order of approximation. In order to obtain the order $O_x(n^{-\gamma})$, new conditions were required instead of (2).

More precisely, we proved (Proposition 2 of [4])

Theorem E. *Let $\{v_k\}$ be an arbitrary sequence. Then for any positive β the following pairs of condition*

(8) $$0 < p \leqq 2 \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\beta-1}\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{p/2} < \infty$$

or

(9) $$p \geqq 2 \quad \text{and} \quad \sum_{n=1}^{\infty} n^{(2/p)(\beta-1)+1} c_n^2 < \infty$$

imply

(10) $$h_n(f, \beta, p, \{v_k\};\ x) = O_x(n^{-\beta/p}) \quad (\gamma = \beta/p)$$

almost everywhere in (a, b).

The aim of the present paper is to study whether Theorems B, C and D have extensions for the limit cases of the restrictions of the parameters similar to Theorem E. For Theorem E can be interpreted as an extension of Theorem A to the case $p\gamma=\beta$.

We shall also investigate what happens if we retain condition (2) but the parameter γ takes the limit value of those in the previous theorems. In these cases, as expected, the order of strong approximation will increase by a factor $(\log n)^{1/p}$.

Now we formulate our theorems:

Theorem 1. *For any positive α and for any increasing sequences $\{v_k\}$ of the natural numbers the following pairs of conditions*

(11) $$0 < p \leqq 2 \quad \text{and} \quad \sum_{n=1}^{\infty}\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{p/2} < \infty$$

or

(12) $$p \geqq 2 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n^2 n < \infty$$

imply

(13) $$C_n(f, \alpha, p;\ \{v_k\};\ x) = O_x(n^{-1/p})$$

almost everywhere in (a, b).

Theorem 2. *If* $\beta>0$, $\bar\beta=\min(\beta, 1)$ *and* $\{\mu_k\}$ *is an arbitrary permutation of some subsequence of the natural numbers, then each of the conditions* (8) *and* (9) *with* $\bar\beta$ *instead of* β *implies that*

(14) $$h_n(f, \beta, p, \{\mu_k\}; x) = O_x(n^{-\bar\beta/p})$$

holds almost everywhere in (a, b).

Theorem 3. *If* $\alpha>0$, $\bar\alpha=\min(\alpha, 1)$ *and* $\{\mu_k\}$ *is an arbitrary permutation of some subsequence of the natural numbers, then each of the conditions* (11) *and* (12) *implies that*

(15) $$C_n(f, \alpha, p, \{\mu_k\}; x) = O_x(n^{-\bar\alpha/p})$$

holds almost everywhere in (a, b).

In the following theorems the conditions on the coefficients will be of the same forms as condition (2). The results to be presented can be considered as extensions of Theorems A—D.

Theorem 4. *If* p *and* β *are positive numbers then for any increasing sequence* $\{\nu_k\}$ *condition*

(16) $$\sum_{n=1}^{\infty} c_n^2 n^{2\beta/p} < \infty \quad (\gamma = \beta/p)$$

implies that

(17) $$h_n(f, \beta, p, \{\nu_k\}; x) = o_x(n^{-\beta/p}(\log n)^{1/p})$$

holds almost everywhere in (a, b).

Theorem 5. *If* α *and* p *are positive numbers then for any increasing sequence* $\{\nu_k\}$ *condition*

(18) $$\sum_{n=1}^{\infty} c_n^2 n^{2/p} < \infty \quad (\gamma = 1/p)$$

implies that

(19) $$C_n(f, \alpha, p, \{\nu_k\}; x) = o_x(n^{-1/p}(\log n)^{1/p})$$

holds almost everywhere in (a, b).

Theorem 6. *If* p *and* β *are positive numbers, and* $\bar\beta=\min(\beta, 1)$, *then for any permutation* $\{\mu_k\}$ *of some subsequence of the natural numbers condition*

(20) $$\sum_{n=1}^{\infty} c_n^2 n^{2\bar\beta/p} < \infty \quad (\gamma = \bar\beta/p)$$

implies that

(21) $$h_n(f, \beta, p, \{\mu_k\}; x) = o_x(n^{-\bar\beta/p}(\log n)^{1/p})$$

holds almost everywhere in (a, b).

Theorem 7. *If* α *and* p *are positive numbers, and* $\bar{\alpha}=\min(\alpha, 1)$, *then for any sequence* $\{\mu_k\}$ *given in Theorem* 6 *condition*

(22) $$\sum_{n=1}^{\infty} c_n^2 n^{2\bar{\alpha}/p} < \infty \quad (\gamma = \bar{\alpha}/p)$$

implies that

(23) $$C_n(f, \alpha, p, \{\mu_k\};\ x) = o_x\big(n^{-\bar{\alpha}/p}(\log n)^{1/p}\big)$$

holds almost everywhere in (a, b).

§ 1. Lemmas

To prove the theorems we require the following lemmas:

Lemma 1 ([2, Lemma 5]). *Let* $\{\lambda_n\}$ *be a monotone sequence of positive numbers such that*

$$\sum_{n=1}^{m} \lambda_{2^n}^2 \leqq K\lambda_{2^m}^2\,{}^{*)}.$$

If

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^2 < \infty,$$

then we have

$$s_{2^n}(x) - f(x) = o_x(\lambda_{2^n}^{-1})$$

almost everywhere in (a, b)

Lemma 2 ([5, Lemma 4]). *Denote*

$$\sigma_n^*(x) = \begin{cases} c_0\varphi_0(x) & \textit{if} \quad n = 0, \\ \dfrac{1}{n-2^{m-1}} \displaystyle\sum_{k=2^m}^{n} (s_k(x) - s_{2^m}(x)) & \textit{if} \quad 2^m \leqq n < 2^{m+1};\ m = 0, 1, \ldots. \end{cases}$$

Then for any positive p *and* $m \geqq 1$

$$\int_a^b \Big\{\frac{1}{2^m} \sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p\Big\}^{2/p} dx \leqq K(p) \sum_{n=2^m+1}^{2^{m+1}} c_n^2.$$

Lemma 3 ([5, Lemma 5]). *Let* $\gamma > 0$ *and* $p \geqq 2$. *Then condition* (2) *implies that*

$$\sum_{m=0}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} k^{p\gamma-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p$$

is finite almost everywhere in (a, b).

---

*) $K, K_1, K_2, \ldots$ will denote positive constants not necessarily the same at each occurrence. Similarly $K(\alpha), K_1(\alpha), \ldots$ denote constants depending on the parameter α.

Lemma 4 ([5, Lemma 6]). *Under the assumptions of Lemma* 3 *we have*

$$\sum_{k=1}^{\infty} k^{p\gamma-1}|\sigma_k^*(x)|^p<\infty$$

almost everywhere in (a, b).

Lemma 5 ([5, Lemma 7]). *Condition* (2) *with any positive* γ *implies*

$$\sigma_n^*(x)=o_x(n^{-\gamma})$$

almost everywhere in (a, b).

Lemma 6 ([4, Lemma 3]). *Let* $\varkappa>0$ *and* $\{\lambda_n\}$ *be an arbitrary sequence of positive numbers. Assuming that condition*

$$\sum_{n=1}^{\infty}\lambda_n\Big\{\sum_{k=n}^{\infty}c_k^2\Big\}^{\varkappa}<\infty \tag{1.1}$$

implies a "certain property $T=(\{s_n(x)\})$" *of the partial sums* $s_n(x)$ *of* (1) *for any orthonormal system, then* (1.1) *implies that the partial sums* $s_{m_k}(x)$ *of* (1) *also have the same property* T *for any increasing sequence* $\{m_k\}$, *i.e. if*

$$(1.1)\Rightarrow T(\{s_n(x)\})\quad\textit{then}\quad(1.1)\Rightarrow T(\{s_{m_k}(x)\})$$

for any increasing sequence $\{m_k\}$.

Lemma 7. *Let* $\gamma>0$, $p\geqq2$ *and* $p\gamma\leqq1$. *For a given sequence* $\{\mu_k\}$ *of distinct positive integers we define another sequence* $\{m_k\}$ *as follows:* $m_k=2^m$ *if* $2^m\leqq\mu_k<2^{m+1}$. *Then* (2) *implies that the sum*

$$\mu_1(x):=\sum_{k=1}^{\infty}k^{p\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p$$

is finite almost everywhere in (a, b).

Proof. The case $p\gamma<1$ has been proved in [5] (see Lemma 8). If $p\gamma=1$ then

$$\mu_1(x)=\sum_{m=0}^{\infty}\sum_{i=2^m}^{2^{m+1}-1}|s_i(x)-s_{2^m}(x)-\sigma_i^*(x)|^p,$$

whence, by Lemma 2 and $p\geqq2$, we obtain that

$$\int_a^b(\mu_1(x))^{2/p}dx\leqq K\sum_{m=0}^{\infty}2^{2m/p}\sum_{n=2^m+1}^{2^{m+1}}c_n^2<\infty,$$

which prove our lemma with $\gamma=1/p$.

Lemma 8. *Let $\gamma>0$, $p\geqq 2$ and $p\gamma\leqq 1$. Then for any sequence $\{\mu_k\}$ of distinct positive integers the sum*

$$\mu_2(x):=\sum_{k=1}^{\infty} k^{p\gamma-1}|\sigma_{\mu_k}^*(x)|^p$$

is finite almost everywhere in (a, b) if (2) holds.

Proof. If $p\gamma<1$ then our lemma is proved in [5, Lemma 9]. The case $p\gamma=1$ follows from Lemma 4 with $\gamma=1/p$, and so the proof is complete.

Lemma 9 ([3, Lemma 2]). *Suppose that γ is a real number and that (2) holds. Then for any sequence $\{\mu_k\}$ of distinct positive integers we have the inequality*

$$\int_a^b \Big\{\sum_{k=0}^{\infty} \mu_k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}\,dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma},$$

where $m_k=2^m$ if $2^m\leqq\mu_k<2^{m+1}$.

Lemma 10. *Suppose that $\gamma>0$, $0<p\leqq 2$ and $\beta=p\gamma$, and that (2) holds. Using the notations of Lemma 9 we have that*

$$\int_a^b \Big\{\sup_{0\leqq n<\infty}\Big(\frac{(\log n)^{p/2}}{\log(n+2)}\sum_{k=1}^{n} k^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p\Big)^{1/p}\Big\}^2 dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} \tag{1.2}$$

holds if $\beta\leqq 1$; if $\beta>1$ then we only have

$$\int_a^b \Big\{\sup_{0\leqq n<\infty}\Big(\frac{(\log n)^{p/2}}{\log(n+2)}\sum_{k=1}^{n} k^{\beta-1}|s_k(x)-s_{m_k}(x)|^p\Big)^{1/p}\Big\}^2 dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma}. \tag{1.3}$$

Proof. First we prove (1.3). If $p=2$ then a simple integration gives (1.3). If $p<2$ we use the following form of Hölder's inequality

$$\sum_{k=1}^{n} k^{\beta-1}|s_k(x)-s_{m_k}(x)|^p \leqq \Big\{\sum_{k=1}^{n} k^{2(\beta/p)-1}|s_k(x)-s_{m_k}(x)|^2\Big\}^{p/2}\times \tag{1.4}$$
$$\times\Big\{\sum_{k=1}^{n} k^{(1-2(\beta/p))\,p/(2-p)+2(\beta-1)/(2-p)}\Big\}^{(2-p)/2}.$$

The sum in the second factor does not exceed $K\log n$, and so by (1.4)

$$\sum_{k=1}^{n} k^{\beta-1}|s_k(x)-s_{m_k}(x)|^p \leqq K_1(\log n)^{1-p/2}\Big\{\sum_{k=1}^{\infty} k^{2(\beta/p)-1}|s_k(x)-s_{m_k}(x)|^2\Big\}^{p/2},$$

whence by Lemma 9 with $\mu_k=k$ and $\gamma=\beta/p$ we obtain (1.3).

The proof of (1.2) for $p=2$ is also obtained by integration, but here we require that $\beta\leqq 1$. Namely,

$$\sum_{k=1}^{\infty} k^{\beta-1}\int_a^b |s_{\mu_k}(x)-s_{m_k}(x)|^2\,dx \leqq K\sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_k<2^{m+1}} k^{\beta-1}E_{2^m}^2 \leqq$$

$$\leqq K_1\sum_{m=0}^{\infty} E_{2^m}^2\cdot 2^{m\beta} = K_1\sum_{m=0}^{\infty} E_{2^m}^2 2^{m2\gamma} \leqq K_2\sum_{n=1}^{\infty} c_n^2 n^{2\gamma}.$$

In the case $p<2$ we distinguish two cases according as $\gamma\geqq 1/2$ or $0<\gamma<1/2$. If $\gamma\geqq 1/2$ then we use the Hölder's inequality in the following form:

$$\begin{aligned}(1.5)\qquad &\sum_{k=1}^{n} k^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p = \sum_{k=1}^{n} k^{\beta-1}\mu_k^{p(1/2-\gamma)}\mu_k^{p(\gamma-1/2)}|s_{\mu_k}(x)-s_{m_k}(x)|^p \leqq\\ &\leqq \Big\{\sum_{k=1}^{n} k^{2(\beta-1)/(2-p)}\mu_k^{p(1-2\gamma)/(2-p)}\Big\}^{1-p/2}\Big\{\sum_{k=1}^{n}\mu_k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}^{p/2}.\end{aligned}$$

Next we estimate the sum appearing in the first factor:

$$\sum\nolimits_1 = \sum_{k=1}^{n} k^{2(\beta-1)/(2-p)}\mu_k^{p(1-2\gamma)/(2-p)}.$$

If $\gamma=1/2$ then $\beta=p/2$, and so $\sum_1\leqq K\log n$. If $\gamma>1/2$ then $2(\beta-1)/(2-p)>-1$, and therefore with $2^{l-1}<n\leqq 2^l$

$$(1.6)\quad \sum\nolimits_1 \leqq \sum_{m=0}^{\infty}\sum_{\substack{2^m\leqq\mu_k<2^{m+1}\\ k\leqq n}} k^{2(\beta-1)/(2-p)}\mu_k^{p(1-2\gamma)/(2-p)} = \sum_{m=0}^{l}+\sum_{m=l+1}^{\infty} = \sum\nolimits_2+\sum\nolimits_3;$$

furthermore

$$\begin{aligned}(1.7)\qquad &\sum\nolimits_2 \leqq \sum_{m=0}^{l} 2^{mp(1-2\gamma)/(2-p)}\sum_{k=2}^{2^m} k^{2(\beta-1)/(2-p)} \leqq\\ &\leqq K_1\sum_{m=0}^{l} 2^{mp(1-2\gamma)/(2-p)}2^{m(1+2(\beta-1)/(2-p))} \leqq K_1\sum_{m=0}^{l} 1 \leqq K_2\log n\end{aligned}$$

and

(1.8)

$$\sum\nolimits_3 \leqq \sum_{m=l+1}^{\infty} 2^{mp(1-2\gamma)/(2-p)}\sum_{k=1}^{n} k^{2(\beta-1)/(2-p)} \leqq Kn^{1+2(\beta-1)/(2-p)}\cdot 2^{lp(1-2\gamma)(2-p)} \leqq K_1.$$

The estimates (1.5)—(1.8) and Lemma 9 give (1.2) for $\gamma\geqq 1/2$.

Finally we prove (1.2) for $0<\gamma<1/2$ and $p<2$. As in (1.5) we use again Hölder's inequality with k instead of μ_k. We obtain that

$$\sum_{k=1}^{n} k^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p \leqq$$

$$\text{(1.9)} \qquad \leqq \Big\{\sum_{k=1}^{n} k^{2(\beta-1)/(2-p)}k^{p(1-2\gamma)/(2-p)}\Big\}^{1-p/2}\Big\{\sum_{k=1}^{n} k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}^{p/2} \leqq$$

$$\leqq K(\log n)^{1-p/2}\Big\{\sum_{k=1}^{n} k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}^{p/2}.$$

If we can show that

$$\text{(1.10)} \qquad \int_a^b \Big\{\sum_{k=1}^{\infty} k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}\,dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma};$$

then (1.9) and (1.10) will yield (1.2) with $0<\gamma<1/2$, too.

Now we verify (1.10) as follows:

$$\sum_{k=1}^{\infty} k^{2\gamma-1}\int_a^b |s_{\mu_k}(x)-s_{m_k}(x)|^2\,dx = \sum_{k=1}^{\infty} k^{2\gamma-1}\sum_{n=m_k+1}^{\mu_k} c_n^2 \leqq$$

$$\leqq \sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_k<2^{m+1}} k^{2\gamma-1}\sum_{n=2^m+1}^{2^{m+1}} c_n^2 \leqq \sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} c_n^2\sum_{k=1}^{2^m} k^{2\gamma-1} \leqq K\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} c_n^2 n^{2\gamma}.$$

Herewith we completed the proof.

§ 2. Proof of the theorems

Proof of Theorem 1. If $\alpha=1$ then (13) follows from Theorem E with $\beta=1$, since $h_n(f, 1, p, \{v_k\}; x)=C_n(f, 1, p, \{v_k\}; x)$. On the other hand, in respect to the following elementary fact:

$$\text{(2.1)} \qquad \frac{A_{n-v}^{\alpha-1}}{A_n^\alpha} \leqq \frac{K}{n} \quad \text{for any} \quad \alpha \geqq 1,$$

we have for $\alpha>1$ that

$$\text{(2.2)} \qquad C_n(f, \alpha, p, \{v_k\}; x) \leqq KC_n(f, 1, p, \{v_k\}; x),$$

so (13) is proved for any $\alpha\geqq 1$.

Now let $0<\alpha<1$. We put $C_n(x):=C_n(f, \alpha, p, \{k\}; x)$ and $2^m\leqq n<2^{m+1}$ $(m\geqq 2)$. Then

$$\text{(2.3)} \qquad C_n(x) \leqq K\left(\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{2^{m-1}} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}+\right.$$

$$\left.+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}\right) = K\big(C_n^{(1)}(x)+C_n^{(2)}(x)\big).$$

Here the first term $C_n^{(1)}(x)$ is of the order $O_x(n^{-1/p})$, for after simplification it becomes a part of the mean $C_n(f, 1, p, \{k\}; x)$.

Now we estimate $C_n^{(2)}(x)$ as follows:

$$(2.4)\qquad C_n^{(2)}(x) \leqq K\Bigg(\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^p\Big\}^{1/p}+$$

$$+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_{2^{m-1}}(x)-f(x)|^p\Big\}^{1/p}+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p\Big\}^{1/p}+$$

$$+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_{2^m}(x)-f(x)|^p\Big\}^{1/p}+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|\sigma_k^*(x)|^p\Big\}^{1/p}\Bigg)=:K\sum_{i=1}^{5} D_n^{(i)}(x).$$

By Lemmas 1 and 5 we have

$$(2.5)\qquad D_n^{(2)}(x)+D_n^{(4)}(x)+D_n^{(5)}(x)=o_x(n^{-1/p}),$$

since it is almost trivial that conditions (11) and (12), separately, imply

$$(2.6)\qquad \sum_{n=1}^{\infty} c_n^2 n^{2/p}<\infty.$$

The implication (11)⇒(2.6) can be proved as follows: By $p/2\leqq 1$ we have

$$\sum_{n=1}^{\infty} c_n^2 n^{2/p} \leqq K(p)\sum_{m=1}^{\infty} m^{(2/p)-1}\sum_{n=m}^{\infty} c_n^2 \leqq K_1(p)\sum_{m=1}^{\infty} 2^{m2/p}\sum_{n=2^m+1}^{\infty} c_n^2 \leqq$$

$$\leqq K_2(p)\Big(\sum_{m=1}^{\infty} 2^m\Big\{\sum_{n=2^m+1}^{\infty} c_n^2\Big\}^{p/2}\Big)^{2/p} \leqq K_3(p)\Big(\sum_{n=1}^{\infty}\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{p/2}\Big)^{2/p}.$$

In order to estimate $D_n^{(1)}(x)$ and $D_n^{(3)}(x)$ we use the Hölder inequality with q being chosen such that $q>1$ and $(\alpha-1)q>-1$. Then

$$(2.7)\qquad \begin{aligned} D_n^{(1)}(x) &\leqq \frac{1}{(A_n^\alpha)^{1/p}}\Big\{\sum_{k=2^{m-1}+1}^{2^m-1}(A_{n-k}^{\alpha-1})^q\Big\}^{1/pq}\Big\{\sum_{k=2^{m-1}+1}^{2^m-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^{pq'}\Big\}^{1/pq'} \leqq \\ &\leqq K\Big\{\frac{1}{2^m}\sum_{k=2^{m-1}}^{2^m-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^{pq'}\Big\}^{1/pq'}=:D_m^*(x).\end{aligned}$$

Furthermore, by Lemma 2 and (2.6), we obtain that

$$\sum_{m=1}^{\infty}\int_a^b (2^{m/p}D_m^*(x))^2\,dx \leqq K_1\sum_{m=1}^{\infty} 2^{2m/p}\sum_{n=2^{m-1}+1}^{2^m} c_n^2<\infty,$$

which by (2.7) implies

$$D_n^{(1)}(x) \leqq D_m^*(x) = o_x(2^{-m/p}) = o_x(n^{-1/p}).$$

$D_n^{(3)}(x)$ can be estimated similarly by $D_{m+1}^*(x)$, and so $D_n^{(3)}(x) = o_x(n^{-1/p})$ also holds almost everywhere in (a, b).

Collecting the given estimates we obtain the following result

$$C_n(f, \alpha, p, \{k\}; x) = O_x(n^{-1/p})$$

almost everywhere in (a, b).

Hence, using Lemma 6 with $\varkappa=p/2$, $\lambda_n=1$ for $0<p\leqq 2$; and with $\varkappa=1$, $\lambda_n=1$ for $p\geqq 2$; furthermore with the property T given by

$$T(\{s_n(x)\}) := C_n(f, \alpha, p, \{k\}; x) = O_x(n^{-1/p}),$$

we obtain the statement of Theorem 1 immediately. The proof is complete.

Proof of Theorem 2. First we prove the case $\beta\leqq 1$, then $\bar{\beta}=\beta$, and so (14) means that

$$\sum\nolimits_1 := \sum_{k=1}^{\infty} k^{\beta-1} |s_{\mu_k}(x)-f(x)|^p \tag{2.8}$$

converges almost everywhere in (a, b).

To verify (2.8) we first consider the case $0<p<2$. Then, using the notation $E_n^2 = \sum_{k=n+1}^{\infty} c_k^2$, we have

$$\int_a^b \left(\sum\nolimits_1\right) dx = \sum_{m=0}^{\infty} \sum_{2^m\leqq\mu_k<2^{m+1}} k^{\beta-1} \int_a^b |s_{\mu_k}(x)-f(x)|^p\, dx \leqq$$

$$\leqq K \sum_{m=0}^{\infty} \sum_{2^m\leqq\mu_k<2^{m+1}} k^{\beta-1} \left(\int_a^b |s_{\mu_k}(x)-f(x)|^2\, dx\right)^{p/2} \leqq$$

$$\leqq K_1 \sum_{m=0}^{\infty} 2^{m\beta} E_{2^m}^p \leqq K_2 \sum_{n=1}^{\infty} n^{\beta-1} E_n^p < \infty,$$

whence, by the Beppo Levi theorem, the convergence of series (2.8) follows almost everywhere in (a, b).

If $p\geqq 2$, then the following obvious estimate

$$\sum\nolimits_1 \leqq \Big\{\sum_{k=1}^{\infty} k^{2(\beta-1)/p} |s_{\mu_k}(x)-f(x)|^2\Big\}^{p/2} := \left(\sum\nolimits_2\right)^{p/2} \tag{2.9}$$

shows that it is enough to prove that condition (9) with $\bar\beta$ implies the finiteness of $\sum_2$ almost everywhere. But, by $-1<2/p(\beta-1)<0$,

$$\int(\textstyle\sum_2)\,dx=\displaystyle\sum_{m=0}^{\infty}\sum_{2^m\le k<2^{m+1}}k^{2(\beta-1)/p}\int_a^b|s_{\mu_k}(x)-f(x)|^2\,dx\le$$

$$\le K\sum_{m=0}^{\infty}E^{m(1+2(\beta-1)/p)}E_{2^m}^2\le K_1\sum_{n=1}^{\infty}c_n^2n^{1+2(\beta-1)/p}<\infty,$$

so the series (2.9) converges, which completes the proof for $\beta\le 1$.

Since

$$h_n(f,\beta,p,\{\mu_k\};x)\le h_n(f,\bar\beta,p,\{\mu_k\};x) \tag{2.10}$$

always holds, thus the proof of Theorem 2 is complete.

Proof fo Theorem 3. On account of the following inequality

$$C_n(f,\alpha,p,\{\mu_k\};x)\le C(f,\bar\alpha,p,\{\mu_k\};x)$$

we may assume that $\alpha\le 1$, then $\bar\alpha=\alpha$. Furthermore the case $\alpha=1$ is the same as the case $\beta=1$ of Theorem 2, so we may assume that $0<\alpha<1$. In this case we can choose a number $q>1$ such that $(\alpha-1)q<-1$; and if we now use the Hölder's inequality with this q and $q'=q/(q-1)$ then

$$C_n(f,\alpha,p,\{\mu_k\};x)\le\left\{\frac{1}{(A_n^\alpha)^q}\sum_{k=0}^{n}(A_{n-k}^{\alpha-1})^q\right\}^{1/pq}\left\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{pq'}\right\}^{1/pq'}\le$$

$$\le Kn^{-\alpha/p}\left\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{pq'}\right\}^{1/pq'}$$

holds. This will prove our theorem if we can show that

$$\sum_{k=0}^{\infty}|s_{\mu_k}(x)-f(x)|^{pq'} \tag{2.11}$$

converges almost everywhere in (a,b). But, by the special case $\beta=1$ of Theorem 2, our assumptions (11) and (12) imply the convergence of the series

$$\sum_{k=0}^{\infty}|s_{\mu_k}(x)-f(x)|^{p}$$

almost everywhere, and so on account of $q'>1$ the series (2.11) converges almost everywhere, too. This has completed the proof.

Proof of Theorem 4. If $p=2$ then Theorem E yields a sharper estimate than (17). Thus we have to prove our theorem only for $p\ne 2$. First we prove (17) for $0<p<2$. By Lemma 6 it is also clear that it will be enough to prove (17) in the speical case $\nu_k=k$. Then

$$(2.12)\qquad h_n^p(f,\beta,p,\{k\};x)\leqq K\Big\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-s_{m_k}(x)|^p+$$

$$+\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{m_k}(x)-f(x)|^p\Big\}.$$

Here the second sum, by Lemma 1, does not exceed $o_x(n^{-\beta}\log n)$.

In the estimation of the first sum we can use the statement (1.3) of Lemma 10. So we obtain that this sum has the order $O_x\big(n^{-\beta}(\log n)^{1-p/2}\big)$ which is even better than the required $o_x(n^{-\beta}\log n)$.

These estimations and (2.12) obviously imply (17) for $0<p<2$ and $\nu_k=k$. If $p>2$ then we use the following estimation with the assumption $2^m\leqq n<2^{m+1}$:

$$h_n(f,\beta,p,\{k\};x)\leqq K\Big\{n^{-\beta}\sum_{\nu=0}^{m}\sum_{\nu=2^\nu}^{2^{\nu+1}-1}k^{\beta-1}|s_k(x)-f(x)|^p\Big\}^{1/p}\leqq$$

$$\leqq K_1\Big\{n^{-\beta}\sum_{\nu=0}^{m}\sum_{k=2^\nu}^{2^{\nu+1}-1}k^{\beta-1}|s_k(x)-s_{2^\nu}(x)-\sigma_k^*(x)|^p\Big\}^{1/p}+$$

$$+\Big\{n^{-\beta}\sum_{\nu=0}^{m}\sum_{k=2^\nu}^{2^{\nu+1}-1}k^{\beta-1}|s_{2^\nu}(x)-f(x)|^p\Big\}^{1/p}+\Big\{n^{-\beta}\sum_{k=1}^{2^{m+1}}k^{\beta-1}|\sigma_k^*(x)|^p\Big\}^{1/p}=K_1\sum_{i=1}^{3}D_n^{(i)}(x).$$

Using Lemma 3 and 4 with $\gamma=\beta/p$ we obtain that

$$D_n^{(1)}(x)=O_x(n^{-\beta/p})\quad\text{and}\quad D_n^{(3)}(x)=O_x(n^{-\beta/p}),$$

furthermore by Lemma 1

$$D_n^{(2)}(x)=o_x\big(n^{-\beta/p}(\log n)^{1/p}\big).$$

Summing up our partial estimations, we get that

$$h_n(f,\beta,p,\{k\};x)=o_x\big(n^{-\beta/p}(\log n)^{1/p}\big),$$

and this, by Lemma 6, conveys the assertion of Theorem 4.

Proof of Theorem 5. On account of Lemma 6 we have to prove (19) only for the special case $\nu_k=k$. In the special case $p=2$ Theorem 1 gives a better estimate than (19) does. Hence it is sufficient to consider the cases $p\neq 2$. We can follow the line of the proof of Theorem 1. Using the notations introduced there, we have

$$(2.13)\qquad C_n(f,\alpha,p,\{k\};x)\leqq K\big(C_n^{(1)}(x)+C_n^{(2)}(x)\big),$$

where $C_n^{(1)}(x)$ has the order $o_x\big(n^{-1/p}(\log n)^{1/p}\big)$ since

$$C_n^{(1)}(x)\leqq Kh_n(f,1,p,\{k\};x)$$

and so Theorem 4 conveys the order of approximation given above.

The sum $C_n^{(2)}(x)$ can be estimated exactly the same way as in the proof of Theorem 1, namely the condition (2.6) which was used during the estimation of $C_n^{(2)}(x)$ is the same as (18). So we have $C_n^{(2)}(x)=o_x(n^{-1/p})$. Collecting these estimations, by (2.13), we obtain (19) for $v_k=k$; and this was to be proved.

Proof of Theorem 6. On account of the obvious inequality (2.10) we may assume that $\beta\leqq 1$ and so $\bar\beta=\beta$. We may also omit the proof of the case $p=2$, for then Theorem 2 gives a sharper estimation than (21) claims. In the subsequent steps of the proof we distinguish two cases according to $0<p<2$ or $p>2$.

In the case $0<p<2$ we start with the following estimation

$$h_n^p(f,\beta,p,\{\mu_k\};x)\leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p+\right.$$

(2.14)

$$\left.+\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{m_k}(x)-f(x)|^p\right\}=\sum\nolimits_1+\sum\nolimits_2.$$

Here the first sum, by the statement (1.2) of Lemma 10, has the following order

$$\sum\nolimits_1=O_x\big(n^{-\beta}(\log n)^{1-p/2}\big). \tag{2.15}$$

To estimate $\sum_2$ we assume $2^{l-1}<n<2^l$. Then, by Lemma 1, we have

$$\sum\nolimits_2=\sum_{m=0}^{\infty}\Bigg(\sum_{\substack{2^m\leqq\mu_k<2^{m+1}\\ k\leqq n}}\frac{(k+1)^{\beta-1}}{(n+1)^\beta}\Bigg)o_x(2^{-m\beta})=\sum_{m=0}^{l}+\sum_{m=l+1}^{\infty}=\sum\nolimits_3+\sum\nolimits_4. \tag{2.16}$$

A simple consideration gives that

$$\sum\nolimits_3=n^{-\beta}\sum_{m=0}^{l}\Big(\sum_{k=1}^{2^m}k^{\beta-1}\Big)o_x(2^{-m\beta})=o_x(n^{-\beta}\log n) \tag{2.17}$$

and

$$\sum\nolimits_4=n^{-\beta}\sum_{m=l+1}^{\infty}\Big(\sum_{k=1}^{n}k^{\beta-1}\Big)o_x(2^{-m\beta})=o_x(2^{-l\beta})=o_x(n^{-\beta}). \tag{2.18}$$

Collecting the estimations (2.14)—(2.18) we obtain that

$$h_n^p(f,\beta,p,\{\mu_k\};x)=o_x(n^{-\beta}\log n)$$

holds almost everywhere in (a,b), and this proves (21) for $p<2$.

If $p>2$ then we use the following estimation:

$$h_n^p(f,\beta,p,\{\mu_k\};x)\leqq K\Bigg(\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p+$$

$$+\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|\sigma_{\mu_k}^*(x)|^p+\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta}|s_{m_k}(x)-f(x)|^p\Bigg)=$$

$$=K\big(\sum\nolimits_5+\sum\nolimits_6+\sum\nolimits_2\big).$$

Above we have verified that $\sum_2 = o_x(n^{-\beta}\log n)$.
To estimate $\sum_5$ we apply Lemma 7, whence

$$\sum\nolimits_5 = O_x(n^{-\beta})$$

follows. Similarly Lemma 8 gives that

$$\sum\nolimits_6 = O_x(n^{-\beta}).$$

Summing up these partial results, we again arrive at (21), and this completes the proof.

Proof of Theorem 7. Using the same arguments as we did at the beginning of the proof of Theorem 3 we may assume $0<\alpha<1$. Then we can use the Hölder inequality with $1/\alpha$ and $1/(1-\alpha)$ and obtain that

$$\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{\mu_k}(x)-f(x)|^p \leqq \Big\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\Big\}^{\alpha}\Big\{\sum_{k=0}^{n}(A_{n-k}^{\alpha-1})^{1/(1-\alpha)}\Big\}^{1-\alpha}.$$

Hence we obtain that

$$\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{\mu_k}(x)-f(x)|^p\right\}^{1/p} \leqq Kn^{-\alpha/p}(\log n)^{1/p-\alpha/p}\left\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\right\}^{\alpha/p}.$$

To prove (23) it suffices to verify that

$$(\log n)^{-\alpha/p}\Big\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\Big\}^{\alpha/p} = o_x(1) \tag{2.19}$$

holds almost everywhere in (a, b). If we apply Theorem 6 with $\beta=1$ and p/α (instead of p), then (21) gives that

$$\Big\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\Big\}^{\alpha/p} = o_x\big((\log n)^{\alpha/p}\big),$$

and so (2.19) is fulfilled, indeed. Theorem 7 is hereby proved.

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On strong approximation by logarithmic means of Fourier series

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Dedicated to Professor K. Tandori on his 60th birthday

Introduction. Let $L_{2\pi}^p$ $(1<p<\infty)$ be the class of all real-valued functions f, 2π-periodic, Lebesgue-integrable with p-th power over $\langle -\pi, \pi\rangle$.

Consider the Fourier series

$$S[f]=\frac{a_0(f)}{2}+\sum_{\nu=1}^{\infty}(a_\nu(f)\cos \nu x+b_\nu(f)\sin \nu x),$$

and denote by $S_k(x;f)$ and $\sigma_k^\alpha(x;f)$ the partial sums and (C,α)-means of $S[f]$, respectively, thus, e.g.,

$$\sigma_k^\alpha(x;\, f)=\frac{1}{A_k^\alpha}\sum_{\nu=0}^{k}A_{k-\nu}^{\alpha-1}S_\nu(x;\, f)\quad (\alpha>-1,\ k=0,1,2,\ldots),$$

where $A_k^\alpha=\binom{k+\alpha}{k}$.

Deokinandan [1] proved the following theorem: If $f\in L_{2\pi}^2$, and for a fixed δ, the condition

$$\int_t^\delta \frac{|f(x+u)+f(x-u)-2s|^2}{t}\,dt=o\left(\log\frac{1}{t}\right)\quad \text{as}\quad t\to 0+$$

holds, then

$$(*)\qquad \sum_{k=1}^{\infty}\frac{r^k}{k}|S_k(x;\, f)-s|^2=o\left(\log\frac{1}{1-r}\right)\quad \text{as}\quad r\to 1-.$$

In this paper we shall generalize and extend this theorem by taking the functions $f\in L_{2\pi}^p$ $(1<p<\infty)$ and replacing the partial sums $S_k(x;f)$ in $(*)$ by (C,α)-means with negative α. More precisely, we shall estimate the quantity

$$H_n^{\log}(x;\,\alpha,f)_q=\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}|\sigma_k^\alpha(x;\,f)-f(x)|^q\right\}^{1/q};$$

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as a measure of this deviation we introduce the function

$$w_x^{\log}(\delta; f)_{p,q} = \sup_{0<h\leqq\delta} \left\{\frac{1}{\log^{p/q}(h^{-1}+1)} \int_h^{\pi} \frac{1}{t} |\varphi_x(t)|^p \, dt\right\}^{1/p},$$

where $\varphi_x(t)=\varphi_x(t;f)=f(x+t)+f(x-t)-2f(x)$.

We shall show that our results cannot be improved for some classes of functions, too.

An analogous problem in the case of Riesz means was raised by LEINDLER [4] and solved by TOTIK [5].

By $C_j(\cdot)$ $(j=1, 2, 3, \ldots)$ we signify positive constants depending on the indicated parameters, only.

Statements of results. First, we give the estimate for $\alpha=0$.

Theorem 1. *If* $f\in L_{2\pi}^p$ $(1<p\leqq q<\infty)$, *then*

$$H_n^{\log}(x;\, 0, f)_{\bar{q}} \leqq C_1(p, q) w_x^{\log}\left(\frac{1}{n}; f\right)_{p,q}$$

for $n=1, 2, 3, \ldots$ *and* $\bar{q}\in(0, \pi\rangle$.

An interesting case is if $p=q=\bar{q}$. Then this result cannot be improved in the following sense.

Let $L^p(\Omega)$ be the subclass of $L_{2\pi}^p$, generated by a nonnegative and nondecreasing function Ω defined on $\langle 0, \pi\rangle$, with $\Omega(0+)=0$ and $\Omega(t)>0$ for any $t<\pi$, consisting of all functions $g\in L_{2\pi}^p$ such that

$$M_g = M_g(x) = \sup_{0<\delta\leqq\pi} \left\{ w_x^{\log}(\delta; g)_{p,p} \left(\frac{1}{\log(\delta^{-1}+1)} \int_\delta^{\pi} \frac{\Omega^p(t)}{t}\, dt\right)^{-1/p} \right\} < \infty,$$

and let

$$L_M^p(\Omega) = \{g\colon M_g < M,\ g\in L^p(\Omega),\ M=\text{constant}>0\}.$$

Theorem 2. *If* $t^{-1}\Omega(t)$ *is a nonincreasing function of* t, *then there exists an absolute constant* C_2 $(<C_1(p, p))$ *such that*

$$MC_2 \leqq \sup_{f\in L_M^p(\Omega)} \left\{ H_n^{\log}(x;\, 0, f)_p \left(\frac{1}{\log(n+1)} \int_{1/n}^{\pi} \frac{\Omega^p(t)}{t}\, dt\right)^{-1/p} \right\} \leqq MC_1(p, p)$$

for $n=1, 2, 3, \ldots$ *and* $1<p<\infty$.

For $\alpha\in\langle -1/2, 0)$, we have the following result.

Theorem 3. *Let* $f\in L^p_{2\pi}$, $1/(1+\alpha)\leqq p\leqq -1/\alpha$ *and* $-1/2\leqq\alpha<0$; *then*

$$H_n^{\log}(x;\ \alpha,f)_{\bar q} \leqq C_3(p,q,\alpha) \sup_{1/(1+\alpha)\leqq\bar p\leqq p} \left\{w_x^{\log}\left(\frac{1}{n};f\right)_{\bar p,q}\right\}$$

for $n=1,2,3,\dots$ *and* $p\leqq q\leqq -1/\alpha$, $\bar q\in(0,q\rangle$.

For $\alpha=-1/2$ Theorem 3 gives the following improvement of Deokinandan's result.

Corollary. *If* $f\in L^2_{2\pi}$, *then*

$$H_n^{\log}(x;\ -1/2,f)_{\bar q} \leqq C_3(2,2,-1/2)\, w_x^{\log}\left(\frac{1}{n};f\right)_{2,2}$$

for $n=1,2,3,\dots$ *and* $\bar q\in(0,2\rangle$.

In the special case $p=q=\bar q=1/(1+\alpha)$ $(-1/2\leqq\alpha<0)$ there holds a theorem of the same type as Theorem 2:

Theorem 4. *If* $t^{-1}\Omega(t)$ *is a nonincreasing function of* t, *then we can define the constant* $C_4(\alpha)$ *less than* $C_3(p,q,\alpha)$ *so that the following inequalities*

$$MC_4(\alpha) \leqq \sup_{f\in L^p_M(\Omega)} \left\{H_n^{\log}(x;\ \alpha,f)_p \left(\frac{1}{\log(n+1)} \int_{1/n}^{\pi} \frac{\Omega^p(t)}{t}\,dt\right)^{-1/p}\right\} \leqq MC_3(p,p,\alpha)$$

are true, whenever $-1/2\leqq\alpha<0$, $p=1/(1+\alpha)$ *and* $n=1,2,3,\dots$.

In connection with the above theorem we formulate a statement.

Remark. Using the method of Totik [6] we can prove analogously that the estimate obtained in Theorem 3, for $p=q=\bar q=1/(1+\alpha)$, is the best possible in the sense considered in [6], because the logarithmic method satisfies the desired condition.

Auxiliary results. Let us start with the following inequality of Hardy—Littlewood—Pólya (H—L—P).

Theorem A. *If* $q>1$ *and* $-1/2\leqq\alpha<0$ *such that* $-1\leqq\alpha q$, *then*

$$\left\{\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{n^{-\alpha-1/q}}{(m+n)^{1-\alpha}}\,|d_m|\right)^q\right\}^{1/q} \leqq \frac{\pi}{\sin(-\alpha\pi)}\left\{\sum_{m=1}^{\infty}\frac{|d_m|^q}{m}\right\}^{1/q}$$

for any sequence $\{d_n\}$ *of real numbers.*

This inequality can be deduced from the general inequality of H—L—P ([3] Theorem 318, p. 227) but, in our special case, it is easier to give the direct proof.

Namely,

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{n^{-\alpha-1/q}}{(m+n)^{1-\alpha}}\,|d_m|\right)^q \leqq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{n^{-\alpha-1/q}}{(m+n)m^{-\alpha}}\,|d_m|\right)^q=$$

$$=\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\left(\frac{1}{m+n}\right)^{1/q}\frac{|d_m|}{m^{1/q}}\left(\frac{m}{n}\right)^{\alpha-\beta+1/q}\left(\frac{1}{m+n}\right)^{1/q'}\left(\frac{n}{m}\right)^{-\beta}\right]^q,$$

where $\alpha<\beta<0$ and $1/q+1/q'=1$. Hence, by Hölder inequality, the left-hand side of our inequality does not exceed

$$\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\frac{|d_m|^q}{m(m+n)}\left(\frac{m}{n}\right)^{1+(\alpha-\beta)q}\left(\sum_{k=1}^{\infty}\frac{1}{k+n}\left(\frac{n}{k}\right)^{-\beta q'}\right)^{q/q'}\right\}=$$

$$=\sum_{m=1}^{\infty}\left\{\frac{|d_m|^q}{m}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{1+(\alpha-\beta)q}\left(\sum_{k=1}^{\infty}\frac{1}{k+n}\left(\frac{n}{k}\right)^{-\beta q'}\right)^{q/q'}\right\}\leqq$$

$$\leqq\sum_{m=1}^{\infty}\left\{\frac{|d_m|^q}{m}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{1+(\alpha-\beta)q}\left(\int_0^{\infty}\frac{x^{\beta q'}}{1+x}\,dx\right)^{q/q'}\right\}\leqq$$

$$\leqq\sum_{m=1}^{\infty}\frac{|d_m|^q}{m}\left(\int_0^{\infty}\frac{y^{-(1+\alpha q-\beta q)}}{1+y}\,dy\right)\left(\int_0^{\infty}\frac{x^{\beta q'}}{1+x}\,dx\right)^{q/q'}=$$

$$=\frac{\pi^{1+q/q'}}{\sin\big(\pi(1+\alpha q-\beta q)\big)\big(\sin(-\pi\beta q')\big)^{q/q'}}\sum_{m=1}^{\infty}\frac{|d_m|^q}{m},$$

if $0<1+\alpha q-\beta q<1$ and $0<-\beta q'<1$ (cf. [3], (9.2.1-2) p. 228).

Taking $\beta=\alpha/q'$, we obtain that $1+\alpha q-\beta q=1+\alpha$ and $\sin\big(\pi(1+\alpha)\big)=$ $=\sin(-\pi\alpha)$; whence we have our inequality with the desired constant.

We require the following inequality, too.

Theorem B ([3], Theorem 346, p. 255). *If* $\beta>1$ *and* $p>1$, *then*

$$\sum_{n=1}^{\infty}n^{-\beta}D_n^p\leqq C_5(\beta,p)\sum_{n=1}^{\infty}n^{-\beta}(nd_n)^p$$

for any positive sequence $\{d_n\}$ *and* $D_n=d_1+d_2+\ldots+d_n$.

The following two theorems of Hardy and Littlewood will be needed, too.

Theorem C ([7] Theorem 5.20, Ch. XII). *If* $h\in L_{2\pi}^r$ *and* $r\leqq s\leqq r'$ $(r'\geqq 2)$, *then*

$$\left\{\frac{|a_0(h)|^s}{2}+\sum_{n=1}^{\infty}(n+1)^{-\lambda s}\big(|a_n(h)|^s+|b_n(h)|^s\big)\right\}^{1/s}\leqq C_6(r)\left\{\int_{-\pi}^{\pi}|h(t)|^r\,dt\right\}^{1/r},$$

where $\lambda=1/s+1/r-1$ *and* $1/r+1/r'=1$.

Theorem D ([2] Theorem 10, p. 369). *If* $1<p\leqq q$ *and* $|t|^{-1}|\varphi_x(t)|^p\in L^1_{2\pi}$, *then*

$$\left\{\sum_{k=1}^{\infty}\frac{1}{k}|S_k(x;\,f)-f(x)|^q\right\}^{1/q}\leqq C_7(p,q)\left\{\int_0^{\pi}\frac{|\varphi_x(t)|^p}{t}\,dt\right\}^{1/p}.$$

Proof of Theorem 1. The relation

$$S_k(x;\,f)-f(x)=\frac{2}{\pi}\int_0^{\pi}\varphi_x(t)\,D_k(t)\,dt,$$

where $D_k(t)$ denote the Dirichlet's kernel, gives

$$\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}|S_k(x;\,f)-f(x)|^q\right\}^{1/q}\leqq$$

$$\leqq\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2}{\pi}\int_0^{1/n}\varphi_x(t)D_k(t)\,dt\right|^q\right\}^{1/q}+$$

$$+\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2}{\pi}\int_{1/n}^{\pi}\varphi_x(t)\,D_k(t)\,dt\right|^q\right\}^{1/q}=I_1+I_2.$$

Since

$$|D_k(t)|\leqq k+1/2<2k,$$

we have

$$I_1\leqq 2\frac{2}{\pi}\int_0^{1/n}|\varphi_x(t)|dt\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}k^{q-1}\right\}^{1/q}\leqq$$

$$\leqq\frac{2n}{\log^{1/q}(n+1)}\frac{2}{\pi}\int_0^{1/n}|\varphi_x(t)|\,dt\leqq\frac{4}{\pi}\left\{\frac{n}{\log^{p/q}(n+1)}\int_0^{1/n}|\varphi_x(t)|^p\,dt\right\}^{1/p}.$$

We observe

$$\left\{\frac{n}{2}\int_0^{2/n}|\varphi_x(t)|^p\,dt\right\}^{1/p}\leqq\left\{\frac{1}{2}n\int_0^{1/n}|\varphi_x(t)|^p\,dt\right\}^{1/p}+\left\{\int_{1/n}^{2/n}\frac{|\varphi_x(t)|^p}{t}\,dt\right\}^{1/p},$$

and, further

$$\sup_{0<u\leqq 1/n}\left\{\frac{1}{2u\log^{p/q}\left(\frac{1}{2u}+1\right)}\int_0^{2u}|\varphi_x(t)|^p\,dt\right\}^{1/p}\leqq$$

$$\leqq 2^{-1/p}\sup_{0<u\leqq 1/n}\left\{\frac{1}{u\log^{p/q}\left(\frac{1}{2u}+1\right)}\int_0^{u}|\varphi_x(t)|^p\,dt\right\}^{1/p}+$$

$$+\sup_{0<u\leqq 1/n}\left\{\frac{1}{\log^{p/q}\left(\frac{1}{2u}+1\right)}\int_u^{\pi}\frac{|\varphi_x(t)|^p}{t}\,dt\right\}^{1/p}.$$

Hence, because

$$\log\left(\frac{1}{2u}+1\right) \geqq \left(\frac{\log 3}{\log 2}-1\right)\log\left(\frac{1}{u}+1\right) \quad \text{if} \quad 0<u\leqq\frac{1}{n} \quad (n=1,2,3,\ldots),$$

we obtain

$$\sup_{0<u\leqq 1/n}\left\{\frac{1}{u\log^{p/q}\left(\frac{1}{2u}+1\right)}\int_0^u|\varphi_x(t)|^p\,dt\right\}^{1/p} \leqq$$

$$\leqq \sup_{0<v\leqq 2/n}\left\{\frac{1}{v\log^{p/q}\left(\frac{1}{v}+1\right)}\int_0^v|\varphi_x(t)|^p\,dt\right\}^{1/p} =$$

$$= \sup_{0<2u\leqq 2/n}\left\{\frac{1}{2u\log^{p/q}\left(\frac{1}{2u}+1\right)}\int_0^{2u}|\varphi_x(t)|^p\,dt\right\}^{1/p} \leqq$$

$$\leqq 2^{-1/p}\left(\frac{\log 3}{\log 2}-1\right)^{-1/q}\sup_{0<u\leqq 1/n}\left\{\frac{1}{u\log^{p/q}\left(\frac{1}{u}+1\right)}\int_0^u|\varphi_x(t)|^p\,dt\right\}^{1/p}+$$

$$+\left(\frac{\log 3}{\log 2}-1\right)^{-1/q}\sup_{0<u\leqq 1/n}\left\{\frac{1}{\log^{p/q}\left(\frac{1}{u}+1\right)}\int_u^\pi\frac{|\varphi_x(t)|^p}{t}\,dt\right\}^{1/p}.$$

Therefore

$$I_1 \leqq \frac{4}{\pi}\left\{\left(\frac{\log 3}{\log 2}-1\right)^{1/q}-2^{-1/p}\right\}^{-1} w_x^{\log}\left(\frac{1}{n};\, f\right)_{p,q} = C_8(p,q)\,w_x^{\log}\left(\frac{1}{n};\, f\right)_{p,q}.$$

To estimate of the second integral we apply Theorem D. Then

$$I_2 = \frac{2}{\pi}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^n\frac{1}{k}\left|\int_{1/n}^\pi\varphi_x(t)\,D_k(t)\,dt\right|^q\right\}^{1/q} =$$

$$= \left\{\frac{1}{\log(n+1)}\sum_{k=1}^n\frac{1}{k}\,|S_k(0;\, f^*)-f^*(0)|^q\right\}^{1/q} \leqq$$

$$\leqq C_7(p,q)\left\{\frac{1}{\log^{p/q}(n+1)}\int_0^\pi\frac{|\varphi_0(t;\, f^*)|^p}{t}\,dt\right\}^{1/p} =$$

$$= C_7(p,q)\left\{\frac{1}{\log^{p/q}(n+1)}\int_{1/n}^\pi\frac{|\varphi_x(t)|^p}{t}\,dt\right\}^{1/p},$$

where $f^*(t)=f(x)$ for $t\in(-1/n,\, 1/n)$ and $f^*(t)=f(x+t)$ otherwise.

Summing the above estimates we have the desired result with the constant $C_3(p, q)=C_8(p, q)+C_7(p, q)$.

Proof of Theorem 2. Since, for $f\in L_M^p(\Omega)$,

$$w_x^{\log}\left(\frac{1}{n};\ f\right)_{p,p} \leqq M\left\{\frac{1}{\log(n+1)}\int_{1/n}^{\pi}\frac{\Omega^p(t)}{t}\,dt\right\}^{1/p},$$

we have the first inequality, immediately.

To prove the second one, let us consider the function

$$f_x(t) = \frac{M}{4(8\pi+1)}\sum_{k=1}^{\infty}\left\{\Omega\left(\frac{1}{k}\right)-\Omega\left(\frac{1}{k+1}\right)\right\}\cos k(t-x).$$

By our assumption, $u^{-1}\Omega(u)\leqq(\lambda u)^{-1}\Omega(\lambda u)$ if $\lambda<1$. Arguing as in [6], we see that

$$|\varphi_x(t;f_x)| \leqq M\Omega(t),$$

and

$$w_x^{\log}\left(\frac{1}{n};\ f_x\right)_{p,q} \leqq M\left\{\frac{1}{\log^{p/q}(n+1)}\int_{1/n}^{\pi}\frac{\Omega^p(t)}{t}\,dt\right\}^{1/p},$$

which gives $f_x\in L_M^p(\Omega)$.

Hence, in view of

$$S_k(x;\ f_x)-f_x(x) = \frac{M}{4(8\pi+1)}\Omega\left(\frac{\pi}{k+1}\right),$$

we obtain

$$\sup_{f\in L_M^p(\Omega)} H_n^{\log}(x;\ 0,\ f)_p \geqq H_n^{\log}(x;\ 0, f_x)_p \geqq \frac{M}{4(8\pi+1)}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{\Omega^p\left(\frac{\pi}{k+1}\right)}{k}\right\}^{1/p} \geqq$$

$$\geqq \frac{M}{4(8\pi+1)}\left\{\frac{1}{\log(n+1)}\frac{1}{8}\int_1^{8n+1}\frac{\Omega^p\left(\frac{\pi}{u+1}\right)}{u+1}\,du\right\}^{1/p} \geqq C_2M\left\{\frac{1}{\log(n+1)}\int_{1/n}^{\pi}\frac{\Omega^p(t)}{t}\,dt\right\}^{1/p}.$$

Thus Theorem 2 is established.

Proof of Theorem 3. Since

$$\sigma_n^\alpha(x;\ f)-f(x) = \frac{2}{\pi}\int_0^{\pi}\varphi_x(t)K_n^\alpha(t)\,dt,$$

where $K_n^\alpha(t)$ denotes the (C, α)-kernel, and

$$|K_n^\alpha(t)| \leqq 2n,$$

the proof reduces to estimate the second term in the following expression

$$\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}|\sigma_k^\alpha(x;\ f)-f(x)|^q\right\}^{1/q} \leqq$$

$$\leqq \left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2}{\pi}\int_0^{1/n}\varphi_x(t)K_k^\alpha(t)\,dt\right|^q\right\}^{1/q}+$$

$$+\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2}{\pi}\int_{1/n}^{\pi}\varphi_x(t)K_k^\alpha(t)\,dt\right|^q\right\}^{1/q}=J_1+J_2.$$

Here

$$J_1 \leqq \frac{2}{\pi}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|2k\int_0^{1/n}\varphi_x(t)\,dt\right|^q\right\}^{1/q} \leqq C_8(p,\ q)\,w_x^{\log}\left(\frac{1}{n};\ f\right)_{p,q},$$

by the same argument as before.

Using the following form of the kernel

$$K_k^\alpha(t)=\frac{1}{A_k^\alpha}\,\frac{\sin\left\{(k+1/2+\alpha/2)t-\frac{\pi\alpha}{2}\right\}}{\left(2\sin\frac{t}{2}\right)^{1+\alpha}}-\frac{1}{2A_k^\alpha\sin\frac{t}{2}}\operatorname{Im}\left(e^{i(k+1/2)t}\sum_{\nu=k+1}^{\infty}A_\nu^{\alpha-1}e^{-i\nu t}\right)=$$

$$=\frac{\sin\left\{(k+1/2+\alpha/2)\,t-\frac{\pi\alpha}{2}\right\}}{A_k^\alpha\left(2\sin\frac{t}{2}\right)^{1+\alpha}}-\frac{1}{2A_k^\alpha\sin\frac{t}{2}}\sum_{\nu=k+1}^{\infty}A_\nu^{\alpha-1}\sin\left(k-\nu+\frac{1}{2}\right)t=$$

$$=\frac{\sin\left\{(k+1/2+\alpha/2)t-\frac{\pi\alpha}{2}\right\}}{A_k^\alpha\left(2\sin\frac{t}{2}\right)^{1+\alpha}}+\frac{1}{A_k^\alpha}\sum_{\mu=k}^{\infty}A_{\mu+1}^{\alpha-1}D_{\mu-k}(t),$$

we obtain

$$J_2\leqq\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2}{\pi}\int_{1/n}^{\pi}\frac{\sin\left((k+1/2+\alpha/2)\,t-\frac{\pi\alpha}{2}\right)}{A_k^\alpha\left(2\sin\frac{t}{2}\right)^{1+\alpha}}\varphi_x(t)\,dt\right|^q\right\}^{1/q}+$$

$$+\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2}{\pi}\int_{1/n}^{\pi}\frac{1}{A_k^\alpha}\sum_{\mu=k}^{\infty}A_{\mu+1}^{\alpha-1}D_{\mu-k}(t)\,\varphi_x(t)\,dt\right|^q\right\}^{1/q}\leqq$$

$$\leqq\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2/\pi}{A_k^\alpha}\int_{1/n}^{\pi}\frac{\varphi_x(t)\cos\left(\frac{1+\alpha}{2}\,t-\frac{\pi\alpha}{2}\right)}{\left(2\sin\frac{t}{2}\right)^{1+\alpha}}\sin kt\,dt\right|^q\right\}^{1/q}+$$

$$+\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{2/\pi}{A_k^\alpha}\int_{1/n}^{\pi}\frac{\varphi_x(t)\sin\left(\frac{1+\alpha}{2}t-\frac{\pi\alpha}{2}\right)}{\left(2\sin\frac{t}{2}\right)^{1+\alpha}}\cos kt\,dt\right|^q\right\}^{1/q}+$$

$$+\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{1}{A_k^\alpha}\sum_{\mu=k}^{\infty}A_{\mu+1}^{\alpha-1}\{S_{\mu-k}(0;\ f^*)-f^*(0)\}\right|^q\right\}^{1/q}=J_{21}+J_{22}+J_{23}.$$

The terms J_{21}, J_{22}, J_{23} will be estimated separately. First we consider the sum J_{23}. In view of

$$C_9(\alpha)k^\alpha\leqq A_k^\alpha\leqq C_{10}(\alpha)k^\alpha\quad(\alpha\neq-1,-2,\ldots,k=1,2,\ldots)$$

([7] (1.15) Ch. III) we have

$$J_{23}\leqq\frac{C_{10}(\alpha-1)}{C_9(\alpha)}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}k^{-\alpha q-1}\Big(\sum_{\nu=0}^{\infty}(\nu+k+1)^{\alpha-1}|S_\nu(0;\ f^*)-f^*(0)|\Big)^q\right\}^{1/q}\leqq$$

$$=\frac{C_{10}(\alpha-1)}{C_9(\alpha)}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\left(\sum_{\nu=1}^{\infty}\frac{k^{-\alpha-1/q}}{(\nu+k)^{1-\alpha}}|S_{\nu-1}(0;\ f^*)-f^*((0)|\right)^q\right\}^{1/q}.$$

Further, by Theorems A and D,

$$J_{23}\leqq\frac{C_{10}(\alpha-1)}{C_9(\alpha)}\frac{\pi}{\sin(-\pi\alpha)}\left\{\frac{1}{\log(n+1)}\sum_{\nu=1}^{\infty}\frac{1}{\nu}|S_{\nu-1}(0;\ f^*)-f^*(0)|^q\right\}^{1/q}\leqq$$

$$\leqq\frac{2^{1/q}\pi C_{10}(\alpha-1)}{C_9(\alpha)\sin(-\pi\alpha)}\left\{\left(\frac{1}{\log(n+1)}\sum_{\nu=1}^{\infty}\frac{|S_\nu(0;\ f^*)-f^*(0)|^q}{\nu}\right)^{1/q}+\frac{\left|\frac{1}{2}a_0(f^*)-f^*(0)\right|}{\log^{1/q}(n+1)}\right\}\leqq$$

$$\leqq\frac{2^{1/q}C_{10}(\alpha-1)}{C_9(\alpha)\sin(-\pi\alpha)}\left\{\left(\frac{1}{\log^{p/q}(n+1)}\int_{1/n}^{\pi}\frac{|\varphi_x(t)|^p}{t}dt\right)^{1/p}+\frac{1/\pi}{\log^{1/q}(n+1)}\int_{1/n}^{\pi}|\varphi_x(t)|\,dt\right\}\leqq$$

$$\leqq\frac{2^{1/q}C_{10}(\alpha-1)\pi\big(C_7(p,q)+1\big)}{C_9(\alpha)\sin(-\pi\alpha)}\left\{\frac{1}{\log^{p/q}(n+1)}\int_{1/n}^{\pi}\frac{|\varphi_x(t)|^p}{t}dt\right\}^{1/p}.$$

The estimates for J_{21} and J_{22} are similar, so we shall examine in detail the term J_{21} only. Then we can apply Theorem C with $\lambda=1/q+\alpha\geqq0$, $s=q$, $r=1/(1+\alpha)$, $r'=-1/\alpha$ and

$$h(t)=\begin{cases}\dfrac{\varphi_x(t)\sin\left(\frac{1+\alpha}{2}t-\frac{\alpha\pi}{2}\right)}{\left(2\sin\frac{t}{2}\right)^{1+\alpha}} & \text{if}\quad t\in\left(\frac{1}{n},\ \pi\right\rangle,\\ 0 & \text{otherwise,}\end{cases}$$

and obtain

$$J_{21} \leqq \frac{2}{C_9(\alpha)}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\left|k^{-\alpha-1/q}\frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\sin kt\,dt\right|^q\right\}^{1/q} =$$

$$= \frac{2}{C_9(\alpha)}\log^{-1/q}(n+1)\Big\{\sum_{k=1}^{n}|k^{-\alpha-1/q}b_k(h)|^q\Big\}^{1/q} \leqq$$

$$\leqq \frac{2C_6\left(\frac{1}{1+\alpha}\right)}{C_9(\alpha)}\log^{-1/q}(n+1)\Big\{\int_{-\pi}^{\pi}|h(t)|^{1/(1+\alpha)}\,dt\Big\}^{1+\alpha} \leqq$$

$$\leqq \frac{2C_6\left(\frac{1}{1+\alpha}\right)(\pi/2)^{1+\alpha}}{C_9(\alpha)}\left\{\frac{1}{\log^{1/q(1+\alpha)}(n+1)}\int_{1/n}^{\pi}\frac{|\varphi_x(t)|^{1/(1+\alpha)}}{t}\,dt\right\}^{1/\alpha}.$$

Thus our proof is completed, Theorem 3 holds with the

constant $C_3(p, q, \alpha) = C_8(p, q)+\dfrac{2^{1/q}C_{10}(\alpha-1)\pi(C_7(p, q)+1)}{C_9(\alpha)\sin(-\pi\alpha)}+\dfrac{C_6\left(\frac{1}{1+\alpha}\right)\pi^{1+\alpha}}{C_9(\alpha)2^{-1+\alpha}}$.

Proof of Theorem 4. The first estimate may be proved similarly as before. To prove the second one let us consider the function f_x, too. By the identity

$$\sigma_k^{\alpha+1}(x;\, f) = \frac{1}{A_n^{\alpha+1}}\sum_{\nu=0}^{k}A_\nu^\alpha\sigma_\nu^\alpha(x;\, f) \quad \text{(cf. [7] Theorem 1.21, Ch. III)},$$

applying Theorem B, we get

$$\sum_{k=1}^{n}\frac{1}{k}|\sigma_k^\alpha(x;\, f)-f(x)|^q \geqq$$

$$\geqq C_{10}^{-q}(\alpha)\sum_{k=1}^{n}k^{-(1+q(\alpha+1))}\left|kA_k^\alpha(\sigma_k^\alpha(x;\, f)-f(x))\right|^q \geqq$$

$$\geqq C_{10}^{-q}(\alpha)C_5^{-1}(1+q(\alpha+1), q)\sum_{k=1}^{n}k^{-(1+q(\alpha+1))}\Big(\sum_{\nu=0}^{k}\left|A_\nu^\alpha(\sigma_\nu^\alpha(x;\, f)-f(x))\right|\Big)^q \geqq$$

$$\geqq \frac{C_9^q(\alpha+1)}{C_{10}^q(\alpha)C_5(1+q(\alpha+1), q)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{1}{A_k^{\alpha+1}}\sum_{\nu=0}^{k}A_\nu^\alpha(\sigma_\nu^\alpha(x;\, f)-f(x))\right|^q =$$

$$= C_{11}(\alpha, q)\sum_{k=1}^{n}\frac{1}{k}|\sigma_k^{1+\alpha}(x;\, f)-f(x)|^q.$$

Hence, since

$$S_k(x;\, f_x)-f_x(x) = \frac{M}{4(8\pi+1)}\Omega\left(\frac{\pi}{k+1}\right),$$

we have

$$\sup_{f\in L^p_M(\Omega)} H_n^{\log}(x;\ \alpha, f)_{1/(1+\alpha)} \geqq H_n^{\log}(x;\ \alpha, f_x)_{1/(1+\alpha)} \geqq$$

$$\geqq C_{11}\left(\alpha, \frac{1}{1+\alpha}\right)\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\,|\sigma_k^{\alpha+1}(x;\ f_x)-f_x(x)|^{1/(1+\alpha)}\right\}^{1+\alpha} =$$

$$= C_{11}\left(\alpha, \frac{1}{1+\alpha}\right)\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{1}{A_k^{\alpha+1}}\sum_{\nu=0}^{k}A_{k-\nu}^{\alpha}\big(S_\nu(x;\ f_x)-f_x(x)\big)\right|^{1/(1+\alpha)}\right\}^{1+\alpha} =$$

$$= C_{11}\left(\alpha, \frac{1}{1+\alpha}\right)\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\left|\frac{1}{A_k^{\alpha+1}}\sum_{\nu=0}^{k}A_{k-\nu}^{\alpha}\,\frac{M}{4(8\pi+1)}\,\Omega\left(\frac{\pi}{\nu+1}\right)\right|^{1/(1+\alpha)}\right\}^{1+\alpha} \geqq$$

$$\geqq \frac{M}{4(8\pi+1)}\,C_{11}\left(\alpha, \frac{1}{1+\alpha}\right)\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{1}{k}\,\Omega^{1/(1+\alpha)}\left(\frac{1}{k+1}\right)\right\}^{1+\alpha}.$$

Hence, the desired inequality follows with the constant

$$C_4(\alpha) = C_2C_{11}\big(\alpha, 1/(1+\alpha)\big)$$

as in the proof of Theorem 2.

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Toeplitz-Kriterien für Matrizenklassen bei Räumen stark limitierbarer Folgen

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Kapitel 1. Einleitung, Bezeichnungen und grundlegende Ergebnisse. Im Satz von Toeplitz wird die Klasse (c, c) aller unendlichen Matrizen bestimmt, die den Raum c der konvergenten Folgen in sich abbilden; dabei wird (c, c) durch Bedingungen für die Matrixelemente charakterisiert, die sogenannten Toeplitz-Kriterien. Wir wollen Toeplitz-Kriterien für Klassen (X, Y) angeben, wo X oder Y Räume stark limitierbarer Folgen sind. Für einige interessante Fälle ist es in allgemeinen Sätzen gelungen, die Charakterisierung von (X, Y) auf die Bestimmung der zugehörigen Köthe-Toeplitz-Dualräume zurückzuführen. Für spezielle Wahl von X bzw. Y erhält man daraus Sätze, die unter anderem bekannte Ergebnisse von Maddox aus [1] verallgemeinern. Darüber hinaus werden Toeplitz-Kriterien für Matrixabbildungen zwischen Räumen absolut und stark limitierbarer Folgen angegeben.

Zunächst benötigen wir einige Bezeichnungen. Wir setzen die Begriffe „r-normierter Raum" und „Schauder-Basis" oder kurz „Basis" als bekannt voraus (s. [2], S. 94 und S. 87).

Mit A bezeichnen wir unendliche Matrizen $(a_{nk})_{n,k\in\mathrm{N}}$ komplexer Zahlen und mit A' die zu A transponierte Matrix.

Mit s bezeichnen wir die Menge aller komplexen Folgen $x=(x_k)_k$.

Wir benutzen die üblichen Bezeichnungen für die Folgenräume l_p $(0<p<\infty)$, l_∞, c_0 und c sowie $\|\cdot\|_p$ und $\|\cdot\|_\infty$ für die zugehörigen p-Normen $(0<p<1)$ bzw. Normen $(1\leqq p\leqq\infty)$.

Wir schreiben $e^{(m)}$ $(m\in\mathrm{N})$ für die Folge mit $e_k^{(m)}:=1$ für $k=m$ und $e_k^{(m)}:=0$ für $k\neq m$, e für die Folge mit $e_k:=1$ $(k=1, 2, \dots)$.

Sind X und Y Teilmengen von s, so schreiben wir:

(X, Y) für die Klasse aller Matrizen A, für die

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$A_n(x) := \sum_{k=1}^{\infty} a_{nk} x_k$ für alle $x \in X$ und für alle $n \in \mathbf{N}$ existiert und
$A(x) := (A_n(x))_n \in Y$ für alle $x \in X$;
$|X, Y| := \{A \in (X, Y) | \sum_{k=1}^{\infty} |a_{nk} x_k| < \infty$ für alle $n \in \mathbf{N}$ und für alle $x \in X\}$;
$(X, Y)' := \{A | A' \in (X, Y)\}$ und $|X, Y|' := \{A | A' \in |X, Y|\}$.
Ist X ein r-normierter Raum, so setzen wir

$$S_X := \{x \in X | \ \|x\| = 1\}.$$

Ist B ein beschränkter linearer Operator von einem r-normierten Raum X in einen normierten Raum Y, so definieren wir die Operatornorm von B wie üblich durch

$$\|B\| := \sup \{\|B(x)\| \ | x \in S_X\}.$$

Für Teilmengen X von s definieren wir die folgenden Dualräume von X

$$X^{\dagger} := \Big\{a \in s \Big| \sum_{k=1}^{\infty} a_k x_k \text{ konvergiert für alle } x \in X\Big\},$$

den Köthe—Toeplitz-Dualraum von X,

$$X^{|\dagger|} := \Big\{a \in s \Big| \sum_{k=1}^{\infty} |a_k x_k| < \infty \quad \text{für alle } x \in X\Big\}$$

und falls X ein r-normierter Teilraum von s ist

$$X^{\|\dagger\|} := \Big\{a \in s | \sup_{x \in S_X} \sum_{k=1}^{\infty} |a_k x_k| < \infty\Big\}$$

und weiter

$$\text{(1.1)} \qquad \|a\|^{\dagger} := \sup_{x \in S_X} \Big|\sum_{k=1}^{\infty} a_k x_k\Big| \quad \text{sowie} \quad \|a\|^{\|\dagger\|} := \sup_{x \in S_X} \Big(\sum_{k=1}^{\infty} |a_k x_k|\Big)$$

für alle $a \in s$, für die die Ausdrücke rechts existieren.

Für beliebige lineare metrische Räume X bezeichnen wir mit X^* den Raum der stetigen linearen Funktionale auf X. Sind zwei lineare metrische Räume X und Y isometrisch isomorph, so schreiben wir $X \cong Y$.

Im folgenden sei X stets eine Teilmenge von s.

Ist $a \in X$, so wird durch

$$\text{(1.2)} \qquad f_a(x) := \sum_{k=1}^{\infty} a_k x_k \quad \text{für alle } x \in X$$

ein lineares Funktional auf X definiert. Wir schreiben $X^{\dagger} \subset X^*$, wenn aus $a \in X^{\dagger}$ folgt $f_a \in X^*$; analog ist die Schreibweise $X^{|\dagger|} \subset X^*$ und $X^{\|\dagger\|} \subset X^*$ zu verstehen.

In [3] haben wir die folgenden Sätze bewiesen:

Satz I. *Sei X ein r-normierter Teilraum von s mit Basis $(e^{(k)})_k$. Gilt $X^\dagger \subset X^*$, so folgt $X^\dagger \cong X^*$.*

Satz II. *Sei X ein vollständiger r-normierter Teilraum von s. Gilt $X^\dagger \subset X^*$, so folgt*

$$A \in (X, l_\infty) \Leftrightarrow \sup_{n \in \mathbb{N}} \|(a_{nk})_k\|^\dagger < \infty.$$

Satz III. *Sei Y ein Teilraum von s und $(Y^\dagger, \|\cdot\|_\dagger)$ ein normierter Raum. Gilt $Y^{\dagger\|\dagger\|} := (Y^\dagger)^{\|\dagger\|} = Y$, so folgt*

$$A \in (l_\infty, Y) \Leftrightarrow \sup_{N \subset \mathbb{N}} \Big\|\Big(\sum_{k \in N} a_{nk}\Big)_n\Big\|_\dagger^{\|\dagger\|} < \infty.$$

Die Klasse (X, c) können wir durch den folgenden Satz bestimmen:

Satz 1.1. *Sei X ein vollständiger r-normierter Teilraum von s mit Basis $(e^{(k)})_k$. Dann gilt*

$$A \in (X, c) \Leftrightarrow \begin{cases} \text{(i)} & M := \sup_{n \in \mathbb{N}} \|(a_{nk})_k\|^\dagger < \infty \\ \text{(ii)} & \lim_{n \to \infty} a_{nk} = a_k \quad (k = 1, 2, \ldots). \end{cases}$$

Beweis. Es folgt, daß X ein FK-Raum ist (s. [8], Korollar 1, S. 208), und daher ist $X^\dagger \subset X^*$ (s. [8], Problem 1, S. 205).

Es gelte $A \in (X, c)$. Wegen $c \subset l_\infty$ folgt $A \in (Y, l_\infty)$ und daher (i) mit Satz II. Wegen $(A_n(x))_n \in c$ für alle $x \in X$ gibt es zu jedem $e^{(k)} \in X$ $(k = 1, 2, \ldots)$ ein $a_k \in \mathbb{C}$ mit $\lim_{n \to \infty} A_n(e^{(k)}) = \lim_{n \to \infty} a_{nk} = a_k$.

Umgekehrt seien (i) und (ii) erfüllt. Wegen (i) und (1.1) existiert $A_n(x) := \sum_{k=1}^\infty a_{nk} x_k$ für alle $x \in X$ und für alle $n \in \mathbb{N}$. Es ist $(a_k)_k \in X^\dagger$, denn: Sei $x = \sum_{k=1}^\infty x_k e^{(k)} \in X$ beliebig; sei $\varepsilon > 0$ gegeben. Wähle $k_0 \in \mathbb{N}$ so groß, daß für alle $l, m > k_0$ $(m > l)$ mit $x^{(l,m)} := \sum_{k=l}^m x_k e^{(k)}$ gilt: $\|x^{(l,m)}\| < (\varepsilon/M)^r$ $(M \neq 0$; für $M = 0$ ist die Behauptung klar). Dann gilt für alle $l, m > k_0$ und für alle $n \in \mathbb{N}$

$$\Big|\sum_{k=l}^m a_k x_k\Big| \leqq \Big|\sum_{k=l}^m (a_k - a_{nk}) x_k\Big| + \Big|\sum_{k=l}^m a_{nk} x_k\Big| \leqq$$

$$\leqq \Big|\sum_{k=l}^m (a_k - a_{nk}) x_k\Big| + \|(a_{nk})_k\|^\dagger \cdot \|x^{(l,m)}\|^{1/r} \leqq \Big|\sum_{k=l}^m (a_{nk} - a_k) x_k\Big| + \varepsilon.$$

Mit (ii) folgt daraus für $n \to \infty$: $\Big|\sum_{k=l}^m a_k x_k\Big| \leqq \varepsilon$ für alle $m, l > k_0$. Also konvergiert $\sum_{k=1}^\infty a_k x_k$ für alle $x \in X$, d. h. $(a_k)_k \in X^\dagger$. Wegen $X^\dagger \subset X^*$ gilt $f_a \in X^*$, wobei $f_a(x) := \sum_{k=1}^\infty a_k x_k$ für alle $x \in X$. Sei $x = \sum_{k=1}^\infty x_k e^{(k)} \in X$. Sei $\varepsilon > 0$ gegeben. Wähle $k_0 \in \mathbb{N}$

so groß, daß

$$\Big\|\sum_{k=k_0+1}^{\infty} x_k e^{(k)}\Big\|^{1/r} < \frac{\varepsilon}{2(M+\|f_a\|)}.$$

Wähle nun $N_0 \in \mathbf{N}$, so daß für alle $n > N_0$

$$\Big|\sum_{k=1}^{k_0} (a_{nk}-a_k)x_k\Big| < \varepsilon/2.$$

Dann gilt für alle $n > N_0$:

$$|A_n(x)-f_a(x)| = \Big|A_n\Big(\sum_{k=1}^{\infty} x_k e^{(k)}\Big)-f_a\Big(\sum_{k=1}^{\infty} x_k e^{(k)}\Big)\Big| \leqq$$

$$\leqq \Big|\sum_{k=1}^{k_0} (a_{nk}-a_k)\, x_k\Big| + \Big|A_n\Big(\sum_{k=k_0+1}^{\infty} x_k e^{(k)}\Big)\Big| + \Big|f_a\Big(\sum_{k=k_0+1}^{\infty} x_k e^{(k)}\Big)\Big| <$$

$$< \varepsilon/2+(M+\|f_a\|)\Big\|\sum_{k=k_0+1}^{\infty} x_k e^{(k)}\Big\|^{1/r} < \varepsilon.$$

Somit ist $(A_n(x))_n \in c$ für alle $x \in X$, und der Satz ist bewiesen.

Man erhält sofort:

Korollar 1.1. *Sei X ein vollständiger r-normierter Teilraum von s mit Basis $(e^{(k)})_k$. Dann gilt*

$$A \in (X, c_0) \Leftrightarrow \begin{cases} \text{(i)} & M < \infty \quad (M \text{ wie in Satz } 1.1) \\ \text{(ii)} & \lim\limits_{n\to\infty} a_{nk} = 0 \quad (k = 1, 2, \ldots) \end{cases}.$$

Im weiteren Verlauf der Arbeit benutzen wir noch die folgenden Bezeichnungen: Ist z eine beliebige komplexe Zahl, so schreiben wir:

$$\operatorname{sgn} z := \begin{cases} \dfrac{|z|}{z} & \text{für } z \neq 0 \\ 0 & \text{für } z = 0. \end{cases}$$

(1.3) $$\tilde{p} := \begin{cases} \max\{1, 1/p\} & \text{für } 0 < p < \infty \\ 1 & \text{für } p = \infty \end{cases} \qquad q := \begin{cases} \infty & \text{für } 0 < p \leqq 1 \\ \dfrac{p}{p-1} & \text{für } 1 < p < \infty \\ 1 & \text{für } p = \infty, \end{cases}$$

so daß $l_p^\dagger = l_q$ für alle p mit $0 < p \leqq \infty$, und wir erhalten damit für alle $x \in l_p$ und für alle $y \in l_q$

(1.4) $$\sum_{k=1}^{\infty} |x_k y_k| \leqq \|x\|_p^{\tilde{p}} \|y\|_q;$$

$N^{(\nu)} := \{l \in \mathbf{N} \mid 2^\nu \leqq l \leqq 2^{\nu+1}-1\}$ für alle $\nu \in \mathbf{N}_0$, Σ_ν bzw. $\max_\nu$ als die Summe bzw. das Maximum gebildet über alle Indizes $l \in N^{(\nu)}$.

Für alle $\alpha \in \mathbf{R}$ und für alle $n=0, 1, \ldots$ bezeichnen wir mit A_n^α den n-ten Cesaro-Koeffizienten der Ordnung α, also $A_n^\alpha := \binom{n+\alpha}{n}$. Ist $x \in s$ beliebig, so schreiben wir für alle $\alpha > 0$ und für alle $t \in \mathbf{R}$ $x(\alpha; t)$ für die Folge $(x_k(\alpha; t))_k$ mit

$$x_k(\alpha; t) := \left[\frac{A_{2^{\nu+1}-k}^{\alpha-1}}{A_{2^\nu-1}^\alpha}\right]^{1/t} \cdot x_k \quad \text{für} \quad k \in N^{(\nu)} \quad (\nu = 0, 1, \ldots),$$

und für alle $\nu \in \mathbf{N}_0$ schreiben wir $x^{(\nu)}(\alpha; t)$ für die Folge $(x_k^{(\nu)}(\alpha; t))_k$ mit

$$x_k^{(\nu)}(\alpha; t) := \begin{cases} x_k(\alpha; t) & \text{für} \quad k \in N^{(\nu)} \\ 0 & \text{für} \quad k \notin N^{(\nu)}. \end{cases}$$

Kapitel 2. Die Räume $[\tilde{C}_\alpha]^p$, $[\tilde{C}_\alpha]_0^p$ und $[\tilde{C}_\alpha]_\infty^p$. In diesem Kapitel beschäftigen wir uns mit bestimmten Räumen stark limitierbarer Folgen. Dazu definieren wir:

Definition 2.1. Für alle $\alpha > 0$ und für alle p mit $0 < p < \infty$ definieren wir die Mengen

$$[\tilde{C}_\alpha]^p := \{x \in s \mid \text{Es gibt ein } l \in \mathbf{C} \text{ mit } (\|(x-le)^{(\nu)}(\alpha; p)\|_p)_\nu \in c_0\},$$

$$[\tilde{C}_\alpha]_0^p := \{x \in s \mid (\|x^{(\nu)}(\alpha; p)\|_p)_\nu \in c_0\}$$

und

$$[\tilde{C}_\alpha]_\infty^p := \{x \in s \mid (\|x^{(\nu)}(\alpha; p)\|_p)_\nu \in l_\infty\}.$$

Bemerkungen. (1) $x \in [\tilde{C}_\alpha]^p$ bedeutet nach der von Borwein in [4] eingeführten Schreibweise für die starke Limitierbarkeit, daß die Folge x zu einem $l \in \mathbf{C}$ $[\tilde{C}_\alpha, I]^p$-limitierbar* ist, $x \in [\tilde{C}_\alpha]_0^p$ bedeutet, daß die Folge x zu 0 $[\tilde{C}_\alpha, I]^p$-limitierbar ist, und $x \in [\tilde{C}_\alpha]_\infty^p$ bedeutet, daß die Folge x $[\tilde{C}_\alpha, I]^p$-beschränkt ist; dabei bezeichnet $\tilde{C}_\alpha$ das modifizierte Cesaro-Verfahren der Ordnung α mit der zugehörigen Matrix $\tilde{C}_\alpha = (\tilde{C}_{nk}^\alpha)_{n,k}$ mit

$$\tilde{C}_{nk}^\alpha := \begin{cases} e_k(\alpha; 1) & \text{für} \quad k \in N^{(\nu)}, \quad n = 2^\nu \\ 0 & \text{sonst} \quad (\nu = 0, 1 \ldots). \end{cases}$$

(2) Ist $x \in [\tilde{C}_\alpha]^p$, dann ist das $l \in \mathbf{C}$ mit $\|(x-le)^{(\nu)}(\alpha; p)\|_p = o(1)$ $(\nu \to \infty)$ eindeutig bestimmt. (Das zeigt man leicht; oder s. [5], Satz 2.)

Um im folgenden die Bezeichnungen zu vereinfachen, schreiben wir $[\tilde{C}]$ für die Mengen $[\tilde{C}_\alpha]^p$, $[\tilde{C}_\alpha]_0^p$ und $[\tilde{C}_\alpha]_\infty^p$.

Man sieht sofort, daß die Mengen $[\tilde{C}]$ mit der üblichen Addition von Folgen und der üblichen Multiplikation von Folgen mit einem Skalar zu linearen Räumen werden.

Definition 2.2. Für alle $\alpha > 0$ und für alle p mit $0 < p < 1$ definieren wir die p-Norm und für alle p mit $1 \leqq p < \infty$ die Norm $\|\cdot\|_{[\tilde{C}]}$ für die Räume $[\tilde{C}]$ durch $\|x\|_{[\tilde{C}]} := \|(\|x^{(\nu)}(\alpha; p)\|_p)_\nu\|_\infty$ für alle $x \in [\tilde{C}]$.

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*) I bezeichnet die Einheitsmatrix bzw. des entsprechende Limitierungsverfahren.

Wir beweisen als erstes einen Satz über die Struktur der Räume $[\tilde{C}]$.

Satz 2.1.

(a) *Für alle* $\alpha>0$ *und für alle* p *mit* $0<p<\infty$ *sind die Räume* $[\tilde{C}]$ *mit* $\|\cdot\|_{[\tilde{C}]}$ *vollständig.*

(b) *Für alle* $\alpha>0$ *und für alle* p *mit* $0<p<\infty$ *ist* $(e^{(k)})_k$ *eine Basis für* $[\tilde{C}_\alpha]_0^p$, *und die Folgen* $e, e^{(k)}$ $(k\in\mathrm{N})$ *bilden eine Basis für* $[\tilde{C}_\alpha]^p$.

Beweis. (a) Die Vollständigkeit von $[\tilde{C}_\alpha]_0^p$ und von $[\tilde{C}_\alpha]_\infty^p$ bzgl. $\|\cdot\|_{[\tilde{C}]}$ folgt mit [6], S. 318, und die Vollständigkeit von $[\tilde{C}_\alpha]^p$ bzgl. $\|\cdot\|_{[\tilde{C}]}$ folgt mit Satz 5 (ii) aus [6].

(b) Man sieht leicht, daß für alle $k\in\mathrm{N}$ $e^{(k)}\in[\tilde{C}]$ und $e\in[\tilde{C}_\alpha]^p$. Sei $x=(x_k)_k\in[\tilde{C}_\alpha]_0^p$ beliebig. Dann ist für alle $m\in\mathrm{N}$

$$\sum_{k=1}^{m} x_k e^{(k)}\in[\tilde{C}_\alpha]_0^p.$$

Sei $\varepsilon>0$ beliebig. Wegen $x\in[\tilde{C}_\alpha]_0^p$ gibt es ein $\nu_0\in\mathrm{N}_0$, so daß für alle $\nu\geqq\nu_0$ $\|x^{(\nu)}(\alpha;p)\|_p<\varepsilon$. Dann gilt für alle $m\geqq 2^{\nu_0}$

$$\Big\|x-\sum_{k=1}^{m} x_k e^{(k)}\Big\|_{[\tilde{C}]}<\varepsilon.$$

Also hat x die Darstellung $x=\sum_{k=1}^{\infty} x_k e^{(k)}$, und man sieht leicht, daß diese Darstellung eindeutig ist.

Genauso zeigt man, daß jedes $x\in[\tilde{C}_\alpha]^p$ mit $\|(x-le)^{(\nu)}(\alpha;p)\|_p=o(1)$ $(\nu\to\infty)$ die eindeutige Darstellung $x=le+\sum_{k=1}^{\infty}(x_k-l)e^{(k)}$ hat.

Wir wollen nun einige Bemerkungen machen, die unter anderem die Verbindung der Räume $[\tilde{C}_\alpha]^p$ zu den bekannten Räumen w_p herstellen: In [1] sind die Räume w_p der $[C_1, I]^p$-limitierbaren Folgen für $0<p<\infty$ untersucht worden:

$$w_p:=\left\{x\in s|\text{Es gibt ein } l\in C \text{ mit } \left(\frac{1}{n}\sum_{k=1}^{n}|x_k-l|^p\right)_n\in c_0\right\}$$

mit den durch

$$\|x\|_{w_p}^{(1)}:=\sup_{n\in\mathrm{N}}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^p\right) \quad\text{und}\quad \|x\|_{w_p}:=\|x\|_{[\tilde{C}_1]_\infty^p}$$

für alle $x\in w_p$ definierten und auf w_p äquivalenten p-Normen $(0<p<1)$ bzw. Normen $(1\leqq p<\infty)$ $\|\cdot\|_{w_p}^{(1)}$ und $\|\cdot\|_{w_p}$ (s. [1], S. 286).

Wenn wir für alle $\alpha>0$ und für alle p mit $0<p<\infty$

$$[C_\alpha]^p:=\{x\in s|\text{Es gibt ein } l\in C \text{ mit } (\sigma_n^\alpha(|x-le|^p)_n\in c_0)\},$$

$$[C_\alpha]_0^p:=\{x\in s|(\sigma_n^\alpha(|x|^p))_n\in c_0\} \quad\text{und}\quad [C_\alpha]_\infty^p:=\{x\in s|(\sigma_n^\alpha(|x|^p))_n\in l_\infty\}$$

schreiben, wobei für alle $n=1,2,\ldots$ und für alle $x\in s$

$$\sigma_n^\alpha(x):=\frac{1}{A_{n-1}^\alpha}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}x_k \quad\text{und}\quad |x|^p \quad\text{für die Folge}\quad (|x_k|^p)_k$$

geschrieben wird, so ist für $\alpha:=1$ $[C_1]^p=w_p$. Wir schreiben kurz $[C]$ für $[C_\alpha]^p$, $[C_\alpha]_0^p$ und $[C_\alpha]_\infty^p$. Es gilt:

Satz 2.2. *Für alle* p *mit* $0<p<\infty$ *ist*
(a) $[C]\subset[\tilde{C}]$ *für alle* $\alpha>0$ *(d. h.* $[C_\alpha]^p\subset[\tilde{C}_\alpha]^p$ *usw.);*
(b) $[C]=[\tilde{C}]\Leftrightarrow\alpha=1$;
(c) $[C]$ *ist vollständig mit* $\|\cdot\|_{[\tilde{C}]}\Leftrightarrow\alpha=1$.

Beweis. (a) ergibt sich sofort aus der Definition der Mengen $[C]$ und $[\tilde{C}]$.

(b) Sei $\alpha=1$. Wegen Teil (a) müssen wir zeigen $[\tilde{C}]\subset[C]$. Wir zeigen die Behauptung nur im Fall $[\tilde{C}]=[\tilde{C}_1]_\infty^p$ und $[C]=[C_1]_\infty^p$; die anderen Fälle gehen vollkommen analog. Sei $x\in[\tilde{C}_1]_\infty^p$ beliebig. Zu jedem $n\in\mathbf{N}$ gibt es ein eindeutig bestimmtes $\nu(n)\in\mathbf{N}_0$ mit $n\in N^{(\nu(n))}$, und es gilt

$$\sigma_n^1(|x|^p)\leqq\sum_{j=0}^{\nu(n)}\frac{1}{n}\sum\nolimits_j|x_k|^p=\sum_{j=0}^{\nu(n)}\frac{2^j}{n}\left(\frac{1}{2^j}\sum\nolimits_j|x_k|^p\right)\leqq 2\|x\|_{[\tilde{C}]}^{\max\{1,p\}}.$$

Sei umgekehrt $\alpha\neq1$. Wir zeigen: $[C]\neq[\tilde{C}]$.
Sei zunächst $[C]=[C_\alpha]_\infty^p$ und $[\tilde{C}]=[\tilde{C}_\alpha]_\infty^p$. Wir definieren die Folge $x^{(\alpha)}$ durch:

für $\alpha<1$:

$$x_k^{(\alpha)}:=\begin{cases}2^{\nu/p} & \text{für } k=2^\nu+1\\ 0 & \text{für } k\neq2^\nu+1\end{cases}\qquad(\nu=0,1,\ldots)$$

und für $1<\alpha$:

$$x_k^{(\alpha)}:=\begin{cases}2^{\nu\alpha/p} & \text{für } k=2^{\nu+1}-1\\ 0 & \text{für } k\neq2^{\nu+1}-1\end{cases}\qquad(\nu=0,1,\ldots).$$

Sei nun $[C]=[C_\alpha]^p$ und $[\tilde{C}]=[\tilde{C}_\alpha]^p$ sowie $l\in\mathbf{C}$. Wir definieren die Folge $x^{(\alpha)}$ durch:

für $\alpha<1$:

$$x_k^{(\alpha)}:=\begin{cases}2^{\nu\alpha/p}+l & \text{für } k=2^\nu+1\\ l & \text{für } k\neq2^\nu+1\end{cases}\qquad(\nu=0,1,\ldots)$$

und für $1<\alpha$:

$$x_k^{(\alpha)}:=\begin{cases}2^{\nu/p}+l & \text{für } k=2^{\nu+1}-1\\ l & \text{für } k\neq2^{\nu+1}-1\end{cases}\qquad(\nu=0,1,\ldots).$$

($l=0$ ergibt den Fall $[C]=[C_\alpha]_0^p$ und $[\tilde{C}]=[\tilde{C}_\alpha]_0^p$.) Mit diesen Folgen gilt $x^{(\alpha)}\in[\tilde{C}]\setminus[C]$.

(c) Wir müssen zeigen, daß für $\alpha\neq1$ $[C]$ mit $\|\cdot\|_{[\tilde{C}]}$ nicht vollständig ist.

Dazu definieren wir für alle $m \in \mathbf{N}_0$ die Folge $(x_k^{(\alpha,\beta;m)})_k$ durch:

für $\alpha < \beta < 1$:

$$x_k^{(\alpha,\beta;m)} := \begin{cases} 0 & \text{für } k \geqq 2^{m+1} \\ 2^{\nu\beta/p} & \text{für } k = 2^\nu + 1 \\ 0 & \text{für } k \neq 2^\nu + 1 \end{cases} \quad (0 \leqq \nu \leqq m)$$

und für $1 < \beta < \alpha$:

$$x_k^{(\alpha,\beta;m)} := \begin{cases} 0 & \text{für } k \geqq 2^{m+1} \\ 2^{\nu\beta/p} & \text{für } k = 2^{\nu+1} - 1 \\ 0 & \text{für } k \neq 2^{\nu+1} - 1 \end{cases} \quad (0 \leqq \nu \leqq m).$$

Dann ist $(x^{(\alpha,\beta;m)})_m$ eine Cauchy-Folge in $[C_\alpha]_0^p \subset [C_\alpha]^p$, $[C_\alpha]_\infty^p$ bezüglich $\|\cdot\|_{[C]}$, jedoch konvergiert die Folge nicht in $[C_\alpha]_\infty^p$.

Wegen Satz 2.2 (b) haben wir für den Fall $\alpha = 1$ insbesondere $[\tilde{C}_1]^p = w_p$.

Kapitel 3. Die Dualräume der Räume $[\tilde{C}]$. Wir definieren nun weitere Mengen, die sich in Satz 3.1 als Köthe-Toeplitz-Dualräume von $[\tilde{C}]$ herausstellen werden.

Definition 3.1. Für alle $\alpha > 0$ und für alle p mit $0 < p < \infty$ definieren wir die Mengen $\mathscr{C}_p(\alpha)$ bzw. $\mathbf{C} \times \mathscr{C}_p(\alpha)$ durch $\mathscr{C}_p(\alpha) := \{a \in s \,|\, (\|a^{(\nu)}(\alpha; -p)\|_q)_\nu \in l_1\}$ bzw. durch $\mathbf{C} \times \mathscr{C}_p(\alpha) := \{\tilde{a} = (a_k)_{k \in \mathbf{N}_0} \in s \,|\, a_0 \in \mathbf{C},\ a = (a_k)_{k \in \mathbf{N}} \in \mathscr{C}_p(\alpha)\}$ und die Normen $\|\cdot\|_{\mathscr{C}_p(\alpha)}$ bzw. $\|\cdot\|_{\mathbf{C} \times \mathscr{C}_p(\alpha)}$ durch $\|a\|_{\mathscr{C}_p(\alpha)} := \|(\|a^{(\nu)}(\alpha; -p)\|_q)_\nu\|_1$ für alle $a \in \mathscr{C}_p(\alpha)$ bzw. durch $\|\tilde{a}\|_{\mathbf{C} \times \mathscr{C}_p(\alpha)} := |a_0| + \|a\|_{\mathscr{C}_p(\alpha)}$ für alle $\tilde{a} \in \mathbf{C} \times \mathscr{C}_p(\alpha)$.

Man sieht sofort, daß die Mengen $\mathscr{C}_p(\alpha)$ und $\mathbf{C} \times \mathscr{C}_p(\alpha)$ mit der üblichen Addition von Folgen und der üblichen Multiplikation von Folgen mit einem Skalar zu einem linearen Raum werden und daß $\|\cdot\|_{\mathscr{C}_p(\alpha)}$ und $\|\cdot\|_{\mathbf{C} \times \mathscr{C}_p(\alpha)}$ eine Norm für die Räume $\mathscr{C}_p(\alpha)$ und $\mathbf{C} \times \mathscr{C}_p(\alpha)$ sind.

Wir bestimmen nun die Dualräume der Räume $[\tilde{C}]$:

Satz 3.1. *Für alle* $\alpha > 0$ *gilt:*

(a) $[\tilde{C}]^\dagger = \mathscr{C}_p(\alpha)$ *falls* $0 < p < \infty$;

(b) $([\tilde{C}_\alpha]_0^p)^* \cong [\tilde{C}]^\dagger$ *falls* $0 < p < \infty$;

(c) $([\tilde{C}_\alpha]^p)^* \cong \mathbf{C} \times [\tilde{C}]^\dagger$ *(bzgl.* $\|\cdot\|_{\mathbf{C} \times [\tilde{C}]^\dagger}$*)* *falls* $0 < p < \infty$;

(d) $[\tilde{C}]^{\dagger\dagger} = [\tilde{C}_\alpha]_\infty^p$ *falls* $1 \leqq p < \infty$.

(Dabei wird $\mathbf{C} \times [\tilde{C}]^\dagger$ *anstelle von* $\mathbf{C} \times \mathscr{C}_p(\alpha)$ *geschrieben.)*

Beweis. (a)

(i) $\mathscr{C}_p(\alpha) \subset [\tilde{C}]^\dagger$. Sei $a \in \mathscr{C}_p(\alpha)$ beliebig, dann gilt für alle $x \in [\tilde{C}]$ mit $\tilde{p}$ und q wie in (1.3) mit zweimaliger Anwendung von (1.4)

$$(3.1)\quad \begin{cases} \sum_{k=1}^\infty |a_k x_k| = \sum_{\nu=0}^\infty \sum_\nu |a_k(\alpha; -p)|\,|x_k(\alpha; p)| \leqq \\ \leqq \sum_{\nu=0}^\infty \|a^{(\nu)}(\alpha; -p)\|_q \|x^{(\nu)}(\alpha; p)\|_p^{\tilde{p}} \leqq \\ \leqq \|(\|a^{(\nu)}(\alpha; -p)\|_q)_\nu\|_1 \|(\|x^{(\nu)}(\alpha; p)\|_p^{\tilde{p}})_\nu\|_\infty = \\ = \|a\|_{\mathscr{C}_p(\alpha)} \|x\|_{[C]}^{\tilde{p}} < \infty, \end{cases}$$

d. h. $a\in[\tilde{C}]^{\dagger}$. Wir bemerken noch, daß aus (3.1) für alle $a\in\mathscr{C}_p(\alpha)$ folgt

$$\|a\|^{\dagger} \leqq \|a\|_{\mathscr{C}_p(\alpha)} < \infty, \tag{3.2}$$

d. h. $\|a\|^{\dagger}$ ist auf ganz $\mathscr{C}_p(\alpha)$ definiert $(\|\cdot\|^{\dagger}$ wie in $(1.1))$.

(ii) $[\tilde{C}]^{\dagger}\subset\mathscr{C}_p(\alpha)$. Sei $a\in[\tilde{C}]^{\dagger}$ beliebig. Dann konvergiert die Reihe $A(x):=$ $:=\sum_{k=1}^{\infty} a_k x_k$ für alle $x\in[\tilde{C}]$. Für alle $\nu\in\mathbf{N}_0$ definieren wir die Abbildung $f_\nu:=[\tilde{C}]\to\mathbf{C}$ durch $f_\nu(x):=\Sigma_\nu a_k x_k$ für alle $x\in[\tilde{C}]$. Dann ist $f_\nu\in[\tilde{C}]^*$ für alle $\nu\in\mathbf{N}_0$, denn f_ν ist trivialerweise linear, und wie in (3.1) erhalten wir für alle $x\in[\tilde{C}]$

$$|f_\nu(x)| \leqq \|a^{(\nu)}(\alpha;\, -p)\|_q \|x\|_{[\tilde{C}]}^{\tilde{p}}. \tag{3.3}$$

Also gilt für die Abbildung A mit $A(x)=\lim_{m\to\infty}\sum_{\nu=0}^{m} f_\nu(x)$

$$A\in[\tilde{C}]^*,$$

d. h. es gibt eine Konstante $M\in\mathbf{R}$, so daß für alle $x\in[\tilde{C}]$

$$|A(x)| \leqq M\|x\|_{[\tilde{C}]}^{\tilde{p}}. \tag{3.4}$$

Sei nun $m\in\mathbf{N}_0$ beliebig. Wir definieren die Folge $x^{(m)}\in[\tilde{C}]$ durch:
für $0<p\leqq 1$

$$x_k^{(m)} := \begin{cases} 0 \quad \text{für} \quad k \geqq 2^{m+1} \\ e_{k(\nu)}(\alpha;\, -p)\operatorname{sgn} a_{k(\nu)} \quad \text{für} \quad k = k(\nu), \text{ wobei } k(\nu) \text{ die kleinste} \\ \quad \text{Zahl } k\in N^{(\nu)} \text{ ist, für die } |a_{k(\nu)}(\alpha;\, -p)| = \|a^{(\nu)}(\alpha;\, -p)\|_\infty \\ 0 \quad \text{für} \quad k \neq k(\nu) \qquad\qquad (0 \leqq \nu \leqq m) \end{cases} \tag{3.5}$$

und für $1<p<\infty$

$$x_k^{(m)} := \begin{cases} 0 \quad \text{für} \quad k \geqq 2^{m+1} \\ (e_k(\alpha;\, -p))^q |a_k|^{q/p} \|a(\alpha;\, -p)\|_q^{-q/p} \operatorname{sgn} a_k \text{ für alle } k\in N^{(\nu)} \text{ für} \\ \quad \text{diejenigen } \nu \leqq m, \text{ für die } \|a^{(\nu)}(\alpha;\, -p)\|_q \neq 0 \\ 0 \quad \text{sonst} \qquad\qquad (0 \leqq \nu \leqq m). \end{cases} \tag{3.6}$$

Dann gilt mit (3.4) für alle p mit $0<p<\infty$ mit (3.5) bzw. (3.6)

$$\sum_{\nu=0}^{m} \|a^{(\nu)}(\alpha;\, -p)\|_q = |A(x^{(m)})| \leqq M \quad (q \text{ wie in } (1.3)). \tag{3.7}$$

Da $m\in\mathbf{N}_0$ beliebig war, erhalten wir daraus $a\in\mathscr{C}_p(\alpha)$ für alle p mit $0<p<\infty$. Damit ist Teil (a) gezeigt.

Wir bemerken noch, daß für alle $m\in\mathbf{N}_0$ und für alle $a\in[\tilde{C}]$ für $0<p\leqq 1$ mit dem $x^{(m)}$ aus (3.5) und für $1<p<\infty$ mit dem $x^{(m)}$ aus (3.6) wegen $x^{(m)}\in S_{[\tilde{C}]}$ folgt

$$\sum_{\nu=0}^{m} \|a^{(\nu)}(\alpha;\, -p)\|_q \leqq \|a\|^{\dagger}$$

und da $m \in \mathbf{N}_0$ beliebig war $\|a\|_{\mathscr{C}_p(\alpha)} \leqq \|a\|^\dagger$. Im folgenden schreiben wir stets $\|\cdot\|_{[\tilde{C}]^\dagger}$ für $\|\cdot\|_{\mathscr{C}_p(\alpha)}$ und erhalten mit (3.2)

$$(3.8) \qquad \|a\|_{[\tilde{C}]^\dagger} = \|a\|^\dagger \quad \text{für alle } a \in [\tilde{C}]^\dagger.$$

(b) Die Folgen $e^{(k)}$ $(k \in \mathbf{N})$ bilden nach Satz 2.1 (b) eine Basis für $[\tilde{C}_\alpha]_0^p$. Wir haben in (a) Teil (ii) gesehen, daß $[\tilde{C}]^\dagger \subset [\tilde{C}]^*$. Daher folgt die Behaptung mit Satz I und (3.8).

(c) Seien $f \in ([\tilde{C}_\alpha]^p)^*$ und $x \in [\tilde{C}_\alpha]^p$ mit $\|(x-le)^{(\nu)}(\alpha; p)\|_p \to 0$ $(\nu \to \infty)$. Dann gilt nach der Bemerkung im Beweis von Satz 2.1 (b) für $x = (x_k)_k$

$$x = le + x_0 \quad \text{mit} \quad x_0 = \sum_{k=1}^{\infty} (x_k - l) e^{(k)} \in [\tilde{C}_\alpha]_0^p.$$

Wegen Teil (b) gibt es eine Folge $a = (a_k)_{k \in \mathbf{N}} \in [\tilde{C}]^\dagger$ mit

$$f(x_0) = \sum_{k=1}^{\infty} a_k (x_k - l).$$

Wegen $a \in [\tilde{C}]^\dagger$ und (3.1) konvergieren die Reihen $\sum_{k=1}^{\infty} a_k$ und $\sum_{k=1}^{\infty} a_k x_k$ absolut, und wir erhalten mit $a_0 := f(e) - \sum_{k=1}^{\infty} a_k$ wegen $f \in ([\tilde{C}_\alpha]^p)^*$

$$(3.9) \qquad f(x) = l a_0 + \sum_{k=1}^{\infty} a_k x_k.$$

Wegen $|l| \leqq \|x\|_{[\tilde{C}]}^{\tilde{p}}$ ($\tilde{p}$ aus (1.3)) erhalten wir

$$|f(x)| \leqq (|a_0| + \|a\|_{[\tilde{C}]^\dagger}) \|x\|_{[\tilde{C}]}^{\tilde{p}} \text{ d. h.}$$

$$\|f\| \leqq (|a_0| + \|a\|_{[\tilde{C}]^\dagger}) = \|\tilde{a}\|_{c \times [\tilde{C}]^\dagger} \text{ mit } \tilde{a} = (a_0, a_1, \ldots).$$

Für alle $m \in \mathbf{N}$ definieren wir die Folge $y^{(m)} := x^{(m)} + z^{(m)}$, wo $x^{(m)}$ wie in (3.5) bzw. (3.6) und

$$z_k^{(m)} := \begin{cases} 0 & \text{für } k \leqq 2^{m+1} - 1 \\ 0 & \text{für } k = 2^\nu \\ \operatorname{sgn} a_0 & \text{für } k \neq 2^\nu \end{cases} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \nu \geqq m+1.$$

Dann gilt: $y^{(m)} \in [\tilde{C}_\alpha]^p$, denn $\|(y^{(m)} - \operatorname{sgn} a_0 \cdot e)^{(\nu)}(\alpha; p)\|_p \to 0$ $(\nu \to \infty)$, $\|y^{(m)}\|_{[\tilde{C}]} \leqq 1$ und

$$|f(y^{(m)})| = \left| |a_0| + \sum_{v=0}^{m} \|a^{(\nu)}(\alpha; -p)\|_q + \sum_{\nu=m+1}^{\infty} \sum_{k=2^\nu+1}^{2^{\nu+1}-1} a_k \operatorname{sgn} a_0 \right| \leqq \|f\|.$$

Wegen der absoluten Konvergenz der Reihe $\sum_{k=1}^{\infty} a_k$ geht die letzte Summe auf der linken Seite für $m \to \infty$ gegen Null, und es folgt

$$|a_0| + \sum_{\nu=0}^{\infty} \|a^{(\nu)}(\alpha; -p)\|_q = |a_0| + \|a\|_{[\tilde{C}]^\dagger} = \|\tilde{a}\|_{C\times[\tilde{C}]^\dagger} \leqq \|f\|,$$

d. h. insgesamt $\|a\|_{C\times[\tilde{C}]^\dagger} = \|f\|$.

Es ist klar, daß umgekehrt die rechte Seite von (3.9) ein $f \in ([\tilde{C}_\alpha]^p)^*$ definiert, wenn $a_0 \in C$ und $a = (a_k)_{k\in\mathbf{N}} \in [\tilde{C}]^\dagger$ vorgegeben sind. Da die Darstellung von f in (3.9) eindeutig ist, definieren wir die Abbildung

$$T\colon ([\tilde{C}_\alpha]^p)^* \to C\times[\tilde{C}]^\dagger$$
$$f \mapsto \tilde{a}$$

mit $\tilde{a}$ aus (3.9). Man sieht leicht, daß T die gewünschten Eigenschaften hat.

(d) Sei $1 \leqq p < \infty$. Trivialerweise gilt $[\tilde{C}] \subset [\tilde{C}]^{\dagger\dagger}$. Sei $a \in [\tilde{C}]^{\dagger\dagger}$ beliebig. Dann konvergiert die Reihe $A(x) = \sum_{k=1}^{\infty} a_k x_k$ für alle $x \in [\tilde{C}]^\dagger$. Für alle $\nu \in \mathbf{N}_0$ definieren wir die Abbildung $f_\nu\colon [\tilde{C}]^\dagger \to \mathbf{C}$ wie in Teil (a). Dann ist $f_\nu \in ([\tilde{C}]^\dagger)^*$, denn f_ν ist trivialerweise linear, und für alle $x \in [\tilde{C}]^\dagger$ gilt wie in (3.3) mit Vertauschen der Rollen von a und x

$$|f_\nu(x)| \leqq \|a^{(\nu)}(\alpha; p)\|_p \|x\|_{[\tilde{C}]^\dagger}.$$

Da $A(x) = \lim_{m\to\infty} \sum_{\nu=0}^{m} f_\nu(x)$ für alle $x \in [\tilde{C}]^\dagger$ existiert und wegen Teil (b) $[\tilde{C}]^\dagger$ vollständig ist, ist $A \in ([\tilde{C}]^\dagger)^*$, d. h. es gibt eine Konstante $M \in \mathbf{R}$, so daß für alle $x \in [\tilde{C}^\dagger]$

$$|A(x)| \leqq M\|x\|_{[\tilde{C}]^\dagger}. \tag{3.10}$$

Sei nun $m \in \mathbf{N}_0$ beliebig. Wir definieren die Folge $x^{(m)} \in [\tilde{C}]^\dagger$ durch:
für $p=1$

$$x_k^{(m)} := \begin{cases} 0 \quad \text{für} \quad k \geqq 2^{m+1} \\ e_k(\alpha; 1)\operatorname{sgn} a_k \quad \text{für } k \in N^{(\nu_0)}, \text{ wobei } \nu_0 \text{ die kleinste} \\ \qquad \text{Zahl } \nu \leqq m \text{ ist, für die } \|a^{(\nu_0)}(\alpha; 1)\|_1 = \max_{\nu\leqq m} \|a^{(\nu)}(\alpha; 1)\|_1 \\ 0 \quad \text{sonst} \qquad\qquad\qquad\qquad\qquad (0 \leqq \nu \leqq m) \end{cases} \tag{3.11}$$

und für $1 < p < \infty$

$$x_k^{(m)} := \begin{cases} 0 \quad \text{für} \quad k \geqq 2^{m+1} \\ (e_k(\alpha; p))^p |a_k|^{p-1} \|a^{(\nu_0)}(\alpha; p)\|_p^{-p/q} \operatorname{sgn} a_k \text{ für } k \in N^{(\nu_0)}, \text{ wobei } \nu_0 \\ \qquad \text{die kleinste Zahl } \nu \leqq m \text{ ist, für die } \|a^{(\nu_0)}(\alpha; p)\|_p = \\ \qquad = \max_{\nu\leqq m} \|a^{(\nu)}(\alpha; p)\|_p, \text{ sofern } \|a^{(\nu_0)}(\alpha; p)\|_p \neq 0, \\ 0 \quad \text{sonst.} \end{cases} \tag{3.12}$$

Mit (3.10) erhalten wir für alle p mit $1 \leqq p < \infty$ $\max_{\nu\leqq m} \|a^{(\nu)}(\alpha; p)\|_p \leqq M$, und da $m \in \mathbf{N}_0$ beliebig war $a \in [\tilde{C}_\alpha]^p_\infty$.

Wir bemerken noch, daß aus (3.1) mit Vertauschen der Rollen von a und x für alle $a\in[\tilde{C}]^{\dagger\dagger}=[\tilde{C}_\alpha]^p_\infty$ und für alle $x\in S_{[\tilde{C}]^\dagger}$ folgt

$$\|a\|^\dagger \leqq \|a\|_{[\tilde{C}]} < \infty \quad (\|\cdot\|^\dagger \text{ aus } (1.1),\ X=[\tilde{C}]^\dagger),$$

d. h. $\|a\|^\dagger$ ist auf ganz $[\tilde{C}]^{\dagger\dagger}$ definiert. Andererseits folgt $\|a\|_{[\tilde{C}]}\leqq\|a\|^\dagger$ wie im Anschluß an Teil (a), so daß wir insgesamt

(3.13) $\qquad \|a\|^\dagger = \|a\|_{[\tilde{C}]}$ für alle $a\in[\tilde{C}]^{\dagger\dagger}$ $\quad(1\leqq p<\infty)$ erhalten.

Wir benötigen noch

Lemma 3.1. *Für alle* $\alpha>0$ *und für alle* p *mit* $1\leqq p<\infty$ *gilt:*

(a) $\qquad [\tilde{C}]^{\dagger\|\dagger\|} = [\tilde{C}_\alpha]^p_\infty$ *und* $\|x\|^{\|\dagger\|}_{[\tilde{C}]^\dagger} = \|x\|_{[\tilde{C}]}$ *für alle* $x\in[\tilde{C}_\alpha]^p_\infty$,

(b) $\qquad [\tilde{C}]^{\dagger\dagger\|\dagger\|} = [\tilde{C}]^\dagger$ *und* $\|x\|^{\|\dagger\|}_{[\tilde{C}]} = \|x\|_{[\tilde{C}]^\dagger}$ *für alle* $x\in[\tilde{C}]^\dagger$.

Beweis. (a) Sei $1\leqq p<\infty$. Wir schreiben kurz $\|\cdot\|$ für $\|\cdot\|^{\|\dagger\|}_{[\tilde{C}]^\dagger}$.
Aus (3.1) folgt für alle $x\in[\tilde{C}_\alpha]^p_\infty$ $\|x\|\leqq\|x\|_{[\tilde{C}]}$, d. h. $[\tilde{C}]^{\dagger\|\dagger\|}\subset[\tilde{C}_\alpha]^p_\infty$.
Wenn wir im Beweis von Satz 3.1 (d) die Rollen von x und a vertauschen, erhalten wir wegen $a^{(m)}\in S_{[\tilde{C}]^\dagger}$ für alle $m\in\mathbf{N}_0$:

$$\max_{\nu\leqq m}\|x^{(\nu)}(\alpha;p)\|_p = \sum_{k=1}^\infty |a_k^{(m)}x_k| \leqq \|x\|, \quad \text{d. h.} \quad \|x\|_{[\tilde{C}]}\leqq\|x\|,$$

d. h. insgesamt $\|x\|_{[\tilde{C}]}=\|x\|$ für alle $x\in[\tilde{C}_\alpha]^p_\infty$, und es ist $[\tilde{C}]^{\dagger\|\dagger\|}=[C_\alpha]^p_\infty$.
(b) Analog Teil (a).

Kapitel 4. Toeplitz-Kriterien für (X, Y), wo entweder $X=[\tilde{C}]$ oder $Y=[\tilde{C}]$. Wir sind nun in der Lage, Toeplitz-Kriterien für einige Matrizenklassen im Zusammenhang mit den Räumen $[\tilde{C}]$ herzuleiten. Für alle $\alpha>0$ gelten die folgenden Sätze:

Satz 4.1. $A\in([\tilde{C}], l_\infty)\Leftrightarrow\sup_{n\in\mathbf{N}}\|(a_{nk})_k\|_{[\tilde{C}]^\dagger}<\infty$ *für* $0<p<\infty$.

Satz 4.2. $A\in(l_\infty, [\tilde{C}_\alpha]^p_\infty)\Leftrightarrow\sup_{N\subset\mathbf{N}}\left\|\left(\sum_{k\in N}a_{nk}\right)_n\right\|_{[\tilde{C}]}<\infty$ *für* $1\leqq p<\infty$.

Satz 4.3. $A\in(l_1, [\tilde{C}_\alpha]^p_\infty)\Leftrightarrow\sup_{n\in\mathbf{N}}\|(a_{kn})_k\|_{[\tilde{C}]}<\infty$ *für* $1\leqq p<\infty$.

Satz 4.4. $A\in([\tilde{C}], l_1)\Leftrightarrow\sup_{N\subset\mathbf{N}}\left\|\left(\sum_{k\in N}a_{kn}\right)_n\right\|_{[\tilde{C}]^\dagger}<\infty$ *für* $0<p<\infty$.

Satz 4.5.

$$A\in([\tilde{C}_\alpha]^p_0, c)\Leftrightarrow\begin{cases}\text{(i)} & \sup_{n\in\mathbf{N}}\|(a_{nk})_k\|_{[\tilde{C}]^\dagger}<\infty\\ \text{(ii)} & a_{nk}\to a_k\quad(n\to\infty)\quad \textit{für alle}\quad k=1,2,\dots\end{cases}$$

für $0<p<\infty$.

Satz 4.6.

$$A\in([\tilde{C}_\alpha]^p, c) \Leftrightarrow \begin{cases} \text{(i)} & \sup_{n\in\mathbf{N}} \|(a_{nk})_k\|_{[\tilde{C}]^\dagger} < \infty. \\ \text{(ii)} & a_{nk} \to a_k \quad (n\to\infty) \quad \textit{für alle} \quad k=1,2,\dots \\ \text{(iii)} & \sum_{k=1}^{\infty} a_{nk} \to a \quad (n\to\infty) \end{cases}$$

für $0<p<\infty$.

Satz 4.7. $A\in(c, [\tilde{C}_\alpha]^p_\infty)=(l_\infty, [\tilde{C}_\alpha]^p_\infty)=(c_0, [\tilde{C}_\alpha]^p_\infty)$ *für* $1\leqq p<\infty$.

Beweis von Satz 4.1. Die Behauptung folgt wegen Satz 2.1 (a) und (3.8) aus Satz II.

Beweis von Satz 4.2. Satz 4.2 folgt wegen Lemma 3.1 (a) aus Satz III.

Beweis von Satz 4.3. Es ist bekannt, daß $|X, Y|=(X, Y)$, wenn X ein normaler Folgenraum ist (s. [7], S. 373). Nach einem bekannten Satz von ALLEN (s. [7], Satz 3) ist

$$|l_1, [\tilde{C}_\alpha]^p_\infty|' = |[\tilde{C}]^\dagger, l_\infty|,$$

da $l_1^{|\dagger|}=l_\infty$ und $[\tilde{C}]^\dagger=[\tilde{C}]^{|\dagger|}$, wie man sofort am Beweis von Satz 3.1 (a) sieht. Wegen der Normalität der Folgenräume l_1 und $[\tilde{C}]^\dagger$ erhalten wir damit $(l_1, [\tilde{C}_\alpha]^p_\infty)'=$ $=([\tilde{C}]^\dagger, l_\infty)$ und daher

$$(l_1, [\tilde{C}_\alpha]^p_\infty) = ([\tilde{C}]^\dagger, l_\infty)'. \tag{4.1}$$

Wegen Satz 3.1 (b) und (3.13) folgt mit Satz II

$$A'\in([\tilde{C}]^\dagger, l_\infty) \Leftrightarrow \sup_{n\in\mathbf{N}} \|(a_{kn})_k\|_{[\tilde{C}]}<\infty$$

und daher mit (4.1) die Behauptung des Satzes.

Beweis von Satz 4.4. Sei $0<p<\infty$ und $[\tilde{C}]=[\tilde{C}_\alpha]^p_\infty$ oder $[\tilde{C}]=[\tilde{C}_\alpha]^p_0$. Mit dem Satz von Allen erhalten wir wie oben

$$([\tilde{C}], l_1) = (l_\infty, [\tilde{C}]^\dagger)'. \tag{4.2}$$

(Hier wird nur $[\tilde{C}]^{\dagger\dagger}\supset[\tilde{C}]$ gebraucht. Da $[\tilde{C}_\alpha]^p$ im allgemeinen kein normaler Folgenraum ist, können wir den Satz von Allen für $[\tilde{C}_\alpha]^p$ nicht anwenden.)

Sei $1\leqq p<\infty$. Wir wenden Satz III mit $Y:=[\tilde{C}]^\dagger$ und $Y^\dagger=[\tilde{C}_\alpha]^p_\infty$ mit $\|\cdot\|_\dagger=$ $=\|\cdot\|_{[\tilde{C}]}$ an. Wegen $[\tilde{C}_\alpha]^p_0\subset[\tilde{C}_\alpha]^p_\infty$ ist $e^{(k)}\in[\tilde{C}_\alpha]^p_\infty$ für alle $k\in\mathbf{N}$. Satz 4.4 folgt für $1\leqq p<\infty$ aus Satz III mit Lemma 3.1 (b).

Sei nun $0<p<1$. Wenn wir den Beweis von Satz III (s. [3], S. 9—11) mit $Y=[\tilde{C}]^\dagger$ und $\|\cdot\|_{[\tilde{C}]^\dagger}$ anstelle von $\|\cdot\|_\dagger^{\|\dagger\|}$ durchführen, so erhalten wir die Behauptung des Satzes für $[\tilde{C}]=[\tilde{C}_\alpha]^p_\infty$ und $[\tilde{C}]=[\tilde{C}_\alpha]^p_0$ auch für $0<p<1$. Wegen

$([\tilde{C}_\alpha]_0^p, l_1)=([\tilde{C}_\alpha]_\infty^p, l_1)$ und $[\tilde{C}_\alpha]_0^p\subset[\tilde{C}_\alpha]^p\subset[\tilde{C}_\alpha]_\infty^p$ gilt

$$([\tilde{C}_\alpha]_0^p, l_1)\subset([\tilde{C}_\alpha]^p, l_1)\subset([\tilde{C}_\alpha]_0^p, l_1),$$

d. h. Satz 4.4 gilt auch für $[\tilde{C}_\alpha]^p$.

Beweis von Satz 4.5. Satz 4.5 folgt mit den Sätzen 2.1 (a), (b) und 3.1 (b) aus Satz 1.1.

Aus Satz 4.5 folgt sofort

Korollar 4.1.

$$A\in([\tilde{C}_\alpha]_0^p, c_0)\Leftrightarrow\begin{cases}\text{(i)} & \sup\limits_{n\in\mathbf{N}}\|(a_{nk})_k\|_{[\tilde{C}]^\dagger}<\infty\\ \text{(ii)} & \lim\limits_{n\to\infty} a_{nk}=0 \quad \textit{für alle } k=1,2,\ldots\end{cases}$$

für alle $\alpha>0$ *und für alle* p *mit* $0<p<\infty$.

Beweis von Satz 4.6. Es ist klar, daß aus $A\in([\tilde{C}_\alpha]^p, c)$ die Bedingungen (i), (ii) und (iii) folgen.

Umgekehrt seien (i), (ii) und (iii) erfüllt. Wir setzen

$$M:=\sup_{n\in\mathbf{N}}\|(a_{nk})_k\|_{[\tilde{C}]^\dagger}.$$

Es ist klar, daß

$$\|(a_k)_k\|_{[\tilde{C}]^\dagger}\leqq M. \tag{4.3}$$

Wir schreiben für alle $x\in[\tilde{C}_\alpha]^p$

$$A_n(x)=\sum_{k=1}^\infty a_kx_k+\sum_{k=1}^\infty(a_{nk}-a_k)(x_k-l)+\sum_{k=1}^\infty(a_{nk}-a_k)l$$

$$\left(\text{wo } l\in C \text{ mit } \lim_{\nu\to\infty}\|(x-le)^{(\nu)}(\alpha;p)\|_p=0\right).$$

Wegen (4.3) existieren $\sum_{k=1}^\infty a_kx_k$ für alle $x\in[\tilde{C}_\alpha]^p$, $\sum_{k=1}^\infty a_k$ (da $e\in[\tilde{C}_\alpha]^p$) und wegen (iii) $\sum_{k=1}^\infty a_{nk}$ für alle $n\in\mathbf{N}$, und es ist $a=\lim_{n\to\infty}\sum_{k=1}^\infty a_{nk}$. Sei $\tilde{A}:=(\tilde{a}_{nk})_{n,k}$ mit $\tilde{a}_{nk}:=a_{nk}-a_k$ für alle $n, k=1,2,\ldots$. Dann ist $\tilde{A}\in([\tilde{C}_\alpha]_0^p, c_0)$ nach Korollar 4.1. Wegen $x-le\in[\tilde{C}_\alpha]_0^p$ gilt also

$$\lim_{n\to\infty}\tilde{A}_n(x-le)=\lim_{n\to\infty}\sum_{k=1}^\infty(a_{nk}-a_k)(x_k-l)=0.$$

Daher ist für alle $x\in[\tilde{C}_\alpha]^p$: $\lim_{n\to\infty}A_n(x)=\sum_{k=1}^\infty a_kx_k+al-l\sum_{k=1}^\infty a_k$, d. h. $\left(A_n(x)\right)_n\in c$.

Aus Satz 4.6 folgt sofort:

Korollar 4.2.

$$A \in ([\tilde{C}_\alpha]^p, c_0) \Leftrightarrow \begin{cases} \text{(i)} & \sup_{n \in \mathbf{N}} \|(a_{nk})_k\|_{[\tilde{C}]^\dagger} < \infty \\ \text{(ii)} & \lim_{n \to \infty} a_{nk} = 0 \quad \textit{für alle} \quad k = 1, 2, \dots \\ \text{(iii)} & \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 0 \end{cases}$$

für alle $\alpha > 0$ *und für alle* p *mit* $0 < p < \infty$.

Beweis von Satz 4.7. Da l_∞ und c_0 normale Folgenräume sind, folgt Satz 4.7 mit Satz 3.1 (d) und einem bekannten Ergebnis von ALLEN (s. [7], Korollar 1 zu Satz 3, S. 375).

Zum Abschlluß dieses Kapitels bestimmen wir noch die Toeplitz-Kriterien für die Klasse $([\tilde{C}_\alpha]_\infty^p, c)$. Der Beweis des entsprechenden Satzes verläuft ähnlich wie der des Satzes 6 in [2], S. 169. Im folgenden schreiben wir $\|\cdot\|_\dagger$ für $\|\cdot\|_{[\tilde{C}]^\dagger}$. Zunächst benötigen wir

Lemma 4.1. *Für alle* $\alpha > 0$ *und für alle* p *mit* $0 < p < \infty$ *folgt aus* $\|b_{nk}\|_\dagger < \infty$ *für alle* $n \in \mathbf{N}$ *und aus* $\lim_{n \to \infty} \|(b_{nk})_k\|_\dagger = 0$ *die gleichmäßige Konvergenz von* $\|(b_{nk})_k\|_\dagger$ *in* n.

Beweis. Wegen $\|(b_{nk})_k\|_\dagger \to 0$ $(n \to \infty)$ gibt es zu jedem $\varepsilon > 0$ ein $n_0 := n_0(\varepsilon) \in \mathbf{N}$, so daß für alle $n > n_0$ gilt $\|(b_{nk})_k\|_\dagger < \varepsilon$. Wegen $\|(b_{nk})_k\|_\dagger < \infty$ gibt es für alle n mit $1 \leqq n \leqq n_0$ ein $m := m(n; \varepsilon)$, so daß $R_q^-(2^{m+1}, (b_{nk})_k) < \varepsilon$, wobei für alle $x \in [\tilde{C}]^\dagger$ und für alle $\mu \in \mathbf{N}_0$

$$R_q^-(\mu, x) := \sum_{\nu=\mu}^{\infty} \|x^{(\nu)}(\alpha; -p)\|_q \quad (q \text{ wie in } (1.3)).$$

Wähle $\nu_0 := 2^{m_0+1}$ mit $m_0 := \max\{m(n; \varepsilon) | 1 \leqq n \leqq n_0\}$. Dann gilt für alle $\mu \geqq \nu_0$

$$R_q^-(\mu, (b_{nk})_k) < \varepsilon \quad \text{für alle } n \in \mathbf{N}.$$

Satz 4.8. *Für alle* $\alpha > 0$ *und für alle* p *mit* $0 < p < \infty$ *gilt*

$$A \in ([\tilde{C}_\alpha]_\infty^p, c) \Leftrightarrow \begin{cases} \text{(i)} & \|(a_{nk})_k\|_\dagger \textit{ konvergiert gleichmäßig in } n \in \mathbf{N} \\ \text{(ii)} & \lim_{n \to \infty} a_{nk} = a_k \quad \textit{für alle} \quad k = 1, 2, \dots \end{cases}$$

Beweis. Die Bedingungen (i) und (ii) seien erfüllt. Aus (i) folgt, daß die Reihen $A_n(x) := \sum_{k=1}^{\infty} a_{nk} x_k$ für alle $x \in [\tilde{C}_\alpha]_\infty^p$ gleichmäßig in n und absolut konvergieren. Also existiert $\lim_{n \to \infty} A_n(x) = \sum_{k=1}^{\infty} a_k x_k$ für alle $x \in [\tilde{C}_\alpha]_\infty^p$ mit (ii).

Umgekehrt gelte $A \in ([\tilde{C}_\alpha]_\infty^p, c)$.

Dann folgt (ii) sofort. Wegen $c \subset l_\infty$ gilt weiter $([\tilde{C}_\alpha]_\infty^p, c) \subset ([\tilde{C}_\alpha]_\infty^p, l_\infty)$ und daher mit Satz 4.1

$$M := \sup_{n\in\mathbb{N}} \|(a_{nk})_k\|_\dagger < \infty. \tag{4.4}$$

Für alle $n, k \in \mathbb{N}$ setzen wir $b_{nk} := a_{nk} - a_k$ mit $a_k := \lim_{n\to\infty} a_{nk}$. Aus (4.4) und (ii) folgt leicht, daß $\|(a_k)_k\|_\dagger \leqq M$. Also gilt

$$\big(B_n(x)\big)_n := \Big(\sum_{k=1}^{\infty} b_{nk} x_k\Big)_n \in c \quad \text{für alle} \quad x \in [\tilde{C}_\alpha]^p_\infty. \tag{4.5}$$

Wir werden zeigen, daß daraus folgt

$$\lim_{n\to\infty} \|(b_{nk})_k\|_\dagger = 0. \tag{4.6}$$

Wenn wir (4.6) gezeigt haben, folgt mit Lemma 4.1 die gleichmäßige Konvergenz von $\|(b_{nk})_k\|_\dagger$ in n und daher nach der Definition von $B = (b_{nk})_{n,k}$ auch die gleichmäßige Konvergenz von $\|(a_{nk})_k\|_\dagger$ in n. Wir zeigen nun (4.6): Annahme, es wäre $\overline{\lim}_{n\to\infty} \|(b_{nk})_k\|_\dagger = c > 0$. Sei o.B.d.A. $\lim_{m\to\infty} \|(b_{mk})_k\|_\dagger = c$. Da $b_{mk} \to 0$ $(m \to \infty)$ für alle $k \in \mathbb{N}$, können wir ein $m(1) \in \mathbb{N}$ so wählen, daß

$$\Big\|\big(b^{(0)}_{m(1),k}(\alpha; -p)\big)_k\Big\|_q < \frac{c}{10} \quad \text{und} \quad \big|\|(b_{m(1),k})_k\|_\dagger - c\big| < \frac{c}{10}.$$

Wegen $\|(b_{m,k})_k\|_\dagger < \infty$ für alle $m \in \mathbb{N}$ gibt es ein $\nu(2) \geqq 1$ mit

$$R_q^-\big(\nu(2)+1, (b_{m(1),k})_k\big) < \frac{c}{10}.$$

Damit gilt $\Big|\sum_{\nu=1}^{\nu(2)} \|\big(b^{(\nu)}_{m(1),k}(\alpha; -p)\big)_k\|_q - c\Big| < 3c/10$. Zur Abkürzung schreiben wir für alle m, l_1, l_2

$$B(m, l_1, l_2) := \sum_{v=l_1}^{l_2} \Big\|\big(b^{(\nu)}_{mk}(\alpha; -p)\big)_k\Big\|_q.$$

Wir wählen nun $m(2) > m(1)$, so daß

$$\big|B\big(m(2), 1, \infty\big) - c\big| < \frac{c}{10} \quad \text{und} \quad B\big(m(2), 1, \nu(2)\big) < \frac{c}{10}$$

und weiter $\nu(3) > \nu(2)$, so daß $B\big(m(2), \nu(3)+1, \infty\big) < c/10$. Damit folgt

$$\big|B\big(m(2), \nu(2)+1, \nu(3)\big) - c\big| < \frac{3c}{10}.$$

Wenn wir so fortfahren, können wir rekursiv zwei Folgen natürlicher Zahlen $\big(m(r)\big)_r$ und $\big(\nu(r)\big)_r$ definieren mit $m(1) < m(2) < \dots$ und $0 = \nu(1) < \nu(2) < \dots$, für die für alle $r = 1, 2, \dots$ gilt: $B\big(m(r), 1, \nu(r)\big) < c/10$; $B\big(m(r), \nu(r+1)+1, \infty\big) < c/10$;

$|B(m(r), v(r)+1, v(r+1))-c|<3c/10$. Wir definieren nun eine Folge $x\in[\tilde{C}_\alpha]^p_\infty$ mit $\|x\|_{[\tilde{C}]}\leqq 1$ durch:

für $0<p\leqq 1$

$$x_k := \begin{cases} 0 \quad \text{für} \quad k = 1 \\ (-1)^r e_{k(v)}(\alpha; -p)\cdot \operatorname{sgn} b_{m(r),k(v)} \quad \text{für} \quad k = k(v), \text{ wobei } k(v) \text{ die} \\ \quad \text{kleinste Zahl } k\in N^{(v)} \text{ ist, für die } |b_{m(r),k(v)}(\alpha; -p)| = \left\|\left(b^{(v)}_{m(r),k}(\alpha; -p)\right)_k\right\|_\infty \\ 0 \quad \text{für} \quad k \neq k(v); \\ \qquad\qquad\qquad\qquad v(r) < v \leqq v(r+1); \quad r = 1, 2\ldots \end{cases}$$

und für $1<p<\infty$

$$x_k := \begin{cases} 0 \quad \text{für} \quad k = 1 \\ (-1)^r \left(e_k(\alpha; -p)\right)^q |b_{m(r),k}|^{q/p} \left\|\left(b^{(v)}_{m(r),k}(\alpha; -p)\right)_k\right\|_q^{-q/p} \operatorname{sgn} b_{m(r),k} \text{ für alle} \\ \quad k\in N^{(v)} \text{ für diejenigen } v, \text{ für die } \left\|\left(b^{(v)}_{m(r),k}(\alpha; -p)\right)_k\right\|_q \neq 0, \\ 0 \quad \text{sonst}; \\ \qquad\qquad\qquad\qquad v(r) < v \leqq v(r+1); \quad r = 1, 2, \ldots \end{cases}$$

Für diese Folgen gilt

$$\left|\sum_{k=1}^{\infty} b_{m(r),k} x_k - (-1)^r c\right| < \frac{c}{2} \quad (r = 1, 2, \ldots),$$

so daß $\left(B_n(x)\right)_n = \left(\sum_{k=1}^{\infty} b_{nk} x_k\right)_n$ keine Cauchy-Folge ist und daher nicht konvergiert. Gilt (4.5) nicht, dann gibt es also ein $x\in[\tilde{C}_\alpha]^p_\infty$ mit $\left(B_n(x)\right)_n \notin c$ im Widerspruch zu (4.5). Also muß (4.6) gelten, und der Satz ist bewiesen.

Für die Anregung und freundliche Unterstützung dieser Arbeit danke ich meinem Lehrer, Herrn Prof. Dr. K. Endl.

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The asymptotic distribution of generalized Rényi statistics

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In honor of Professor Károly Tandori on his sixtieth birthday

1. Introduction and preliminaries

For each integer $n \geqq 1$ let $U_1, \ldots, U_n$ be independent Uniform (0, 1) random variables, $U_{1,n} \leqq \ldots \leqq U_{n,n}$ be their corresponding order statistics and G_n the right continuous empirical distribution function based on these n independent uniform (0, 1) random variables. We shall begin by stating some results in the literature that motivated our present investigation.

DANIELS [5] showed that for any $-\infty < x < \infty$

$$P\left\{\sup_{0 \leqq s \leqq 1} \frac{G_n(s)-s}{s} \leqq x\right\} = F(x), \tag{1}$$

where

$$F(x) = \begin{cases} \dfrac{x}{x+1}, & \text{for } 0 \leqq x < \infty, \\ 0, & \text{for } x < 0. \end{cases}$$

Let $N(t)$, $0 \leqq t < \infty$, denote a right continuous Poisson process with parameter one. PYKE [10] proved that for any $-\infty < x < \infty$

$$P\left\{\sup_{0 \leqq t < \infty} \frac{N(t)-t}{t} \leqq x\right\} = F(x). \tag{2}$$

Combining statements (1) and (2), we have for each $n \geqq 1$

$$\sup_{0 \leqq s \leqq 1} \frac{G_n(s)-s}{s} \overset{\mathcal{D}}{=} \sup_{0 \leqq t < \infty} \frac{N(t)-t}{t},$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution. More generally, a slight modification of

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the techniques of Mason [8] establishes that for each $0 \leqq \nu < 1/2$, as $n \to \infty$,

$$n^{\nu} \sup_{0 \leqq s \leqq 1} \frac{G_n(s) - s}{s^{1-\nu}} \xrightarrow{\mathscr{D}} \sup_{0 \leqq t < \infty} \frac{N(t) - t}{t^{1-\nu}}. \tag{3}$$

(The symbol $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution.) A result closely related to (3) proven in MASON [8] is that for any $0 \leqq \nu < 1/2$, as $n \to \infty$,

$$n^{\nu} \sup_{0 \leqq s \leqq 1} \frac{|G_n(s) - s|}{(s(1-s))^{1-\nu}} \xrightarrow{\mathscr{D}} N_\nu \vee N'_\nu, \tag{4}$$

where

$$N_\nu := \sup_{0 \leqq t < \infty} \frac{|N(t) - t|}{t^{1-\nu}},$$

$N'_\nu \overset{\mathscr{D}}{=} N_\nu$, N_ν and N'_ν are independent random variables, and the symbol $\vee$ denotes maximum.

If a $1/2 \leqq \nu \leqq 1$ is chosen, the Poisson limit behavior in (3) and (4) breaks down. In particular, when $1/2 < \nu \leqq 1$ and n^ν is replaced by $n^{1/2}$, the limiting distribution of the left side of (4) is the same as that of

$$\sup_{0 \leqq s \leqq 1} \frac{|B(s)|}{(s(1-s))^{1-\nu}},$$

where $B(s)$, $0 \leqq s \leqq 1$, is a Brownian bridge defined on [0, 1]. When $\nu = 1/2$, with additional normalizing constants applied, the limiting distribution of the left side of (4) is an extreme value distribution. For details the reader is referred to O'REILLY [9], EICKER [6], JAESCHKE [7], the discussion in MASON [8], or to the exhaustive study in M. CSÖRGŐ, S. CSÖRGŐ, HORVÁTH, and MASON (Cs—Cs—H—M) [4].

When $\nu = 0$, the limiting Poisson behaviour of the left side of (3) can break down in another way, if the supremum is not taken over the entire interval [0, 1]. RÉNYI [11] (also see M. CSÖRGŐ [3]) showed that for any fixed $0 < a < 1$, as $n \to \infty$,

$$\left(\frac{a}{1-a}\right)^{1/2} \sup_{a \leqq s \leqq 1} n^{1/2} \frac{\{G_n(s) - s\}}{s} \xrightarrow{\mathscr{D}} \sup_{0 \leqq t \leqq 1} W(t), \tag{5}$$

and

$$\left(\frac{a}{1-a}\right)^{1/2} \sup_{a \leqq s \leqq 1} n^{1/2} \frac{|G_n(s) - s|}{s} \xrightarrow{\mathscr{D}} \sup_{0 \leqq t \leqq 1} |W(t)|, \tag{6}$$

where $W(t)$, $0 \leqq t \leqq 1$, denotes a standard Brownian motion defined on [0, 1]. CSÁKI [2] demonstrated that (5) remains true if a is replaced by any sequence of positive constants a_n such that as $n \to \infty$,

$$0 < a_n < 1, \quad a_n \to 0, \quad \text{and} \quad na_n \to \infty. \tag{7}$$

This suggests that if the supremum on the left side of (3) is restricted to $[a_n, 1]$, where the sequence a_n satisfies condition (7), when appropriately normalized, its limiting distribution should be the same as that of the supremum of a certain Gaussian process; and the same should be true if the supremum on the left side of (4) is restricted to an interval of the form $[a_n, 1-a_n]$. In the next section, we shall show that this is indeed the case. Such statistics will be called *generalized Rényi statistics.*

The main tool which we shall use to establish our results will be a new Brownian bridge approximation to the uniform empirical and quantile processes recently obtained by Cs—Cs—H—M [4]. We shall now describe some of its basic features.

In Cs—Cs—H—M [4] a probability space $(\Omega, \mathcal{A}, P)$ is constructed carrying a sequence $U_1, U_2, \ldots,$ of independent Uniform (0, 1) random variables and a sequence of Brownian bridges $B_n(s)$, $0 \leqq s \leqq 1$, $n=1, 2, \ldots,$ such that for the uniform empirical process

$$\alpha_n(s) = n^{1/2}\{G_n(s)-s\}, \quad 0 \leqq s \leqq 1,$$

and the uniform quantile process

$$\beta_n(s) = n^{1/2}\{s-U_n(s)\}, \quad 0 \leqq s \leqq 1,$$

where

$$U_n(s) = \begin{cases} U_{k,n}, & \text{if} \quad (k-1)/n < s \leqq k/n, \quad k = 1, \ldots, n \\ U_{1,n}, & \text{if} \quad s = 0, \end{cases}$$

we have

(8) $$\sup_{1/n \leqq s \leqq 1} \frac{|\alpha_n(s)-B_n(s)|}{s^{1/2-\delta_1}} = O_P(n^{-\delta_1}),$$

(9) $$\sup_{0 \leqq s \leqq 1-1/n} \frac{|\alpha_n(s)-B_n(s)|}{(1-s)^{1/2-\delta_1}} = O_P(n^{-\delta_1}),$$

(10) $$\sup_{1/(n+1) \leqq s \leqq 1} \frac{|\beta_n(s)-B_n(s)|}{s^{1/2-\delta_2}} = O_P(n^{-\delta_2}),$$

and

(11) $$\sup_{0 \leqq s \leqq 1-1/(n+1)} \frac{|\beta_n(s)-B_n(s)|}{(1-s)^{1/2-\delta_2}} = O_P(n^{-\delta_2}),$$

where δ_1 and δ_2 are any fixed numbers such that $0 \leqq \delta_1 < 1/4$ and $0 \leqq \delta_2 < 1/2$. Statements (8) and (9) are contained in Corollary 2.1, while statements (10) and (11) follow from Theorem 2.1 of the above paper. We shall also need the fact that statements (8) and (9) remain true on this probability space for any $0 < \delta_1 \leqq 1/2$ with the supremum taken over [0, 1] and the $O_P(n^{-\delta_1})$ replaced by $o_P(1)$. This follows from the general results on *q-metric convergence* in Cs—Cs—H—M [4]. *In the proofs of the next section it will be assumed without comment that we are on the probability space just described.*

2. The main results

For any $0 \leqq \nu < 1/2$, let

$$X_\nu := \sup_{0 \leqq t \leqq 1} \frac{W(t)}{t^\nu},$$

and

$$Y_\nu := \sup_{0 \leqq t \leqq 1} \frac{|W(t)|}{t^\nu}.$$

Since $0 \leqq \nu < 1/2$, a simple application of the law of the iterated logarithm for Brownian motion shows that X_ν and Y_ν are almost surely finite.

When $\nu = 0$, our first theorem contains the results of RÉNYI [11] and CSÁKI [2] quoted in the Introduction.

Theorem 1. *Let a_n be any sequence of positive contants such that for some $0 < \beta < 1$ we have $0 < a_n \leqq \beta$ for all large enough n, and $na_n \to \infty$. Then for every $0 \leqq \nu < 1/2$, as $n \to \infty$,*

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} \alpha_n(s)/(s^{1-\nu}(1-s)^\nu) \xrightarrow{\mathcal{D}} X_\nu; \tag{12}$$

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{0 \leqq s \leqq 1-a_n} \alpha_n(s)/((1-s)^{1-\nu} s^\nu) \xrightarrow{\mathcal{D}} X_\nu; \tag{13}$$

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} |\alpha_n(s)|/(s^{1-\nu}(1-s)^\nu) \xrightarrow{\mathcal{D}} Y_\nu; \tag{14}$$

and

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{0 \leqq s \leqq 1-a_n} |\alpha_n(s)|/((1-s)^{1-\nu} s^\nu) \xrightarrow{\mathcal{D}} Y_\nu. \tag{15}$$

Proof. First consider (12) and (14). Choose any $0 \leqq \nu < 1/2$. Observe that for all n sufficiently large

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} |\alpha_n(s) - B_n(s)|/(s^{1-\nu}(1-s)^\nu) \leqq$$
$$\leqq (1-\beta)^{-1/2} a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq \beta} |\alpha_n(s) - B_n(s)|/s^{1-\nu} +$$
$$+ \beta^{-1/2}(1-\beta)^{-1/2+\nu} \sup_{\beta \leqq s \leqq 1} |\alpha_n(s) - B_n(s)|/(1-s)^\nu := \Delta_{1,n} + \Delta_{2,n}.$$

Applying the version of statement (9) with the choice $\delta_1 = 1/2 - \nu$, where the supremum is taken over [0, 1], we see that

$$\Delta_{2,n} = o_P(1). \tag{16}$$

Also notice that for $0 < \delta_1 < 1/4$, not necessarily the same δ_1 as above,

$$\Delta_{1,n} \leqq (1-\beta)^{-1/2} a_n^{-\delta_1} \sup_{1/n \leqq s \leqq 1} |\alpha_n(s) - B_n(s)|/s^{1/2-\delta_1}. \tag{17}$$

Now applying (8) we see that the right side of inequality (17) equals

$$(1-\beta)^{-1/2}a_n^{-\delta_1}O_P(n^{-\delta_1}) = O_P((na_n)^{-\delta_1}), \tag{18}$$

which by the assumption that $na_n\to\infty$ as $n\to\infty$ equals $o_P(1)$.

Since for each $n\geqq 1$ such that $0<a_n<1$ the process

$$\left\{\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} B_n(s)/(s^{1-\nu}(1-s)^\nu)\colon a_n\leqq s\leqq 1\right\}$$

is equal in distribution to the process

$$\left\{\left(\frac{a_n}{1-a_n}\right)^{-\nu}\left(\frac{t}{1-t}\right)^\nu W\left(\left(\frac{a_n}{1-a_n}\right)\left(\frac{1-t}{t}\right)\right)\colon a_n\leqq t\leqq 1\right\},$$

we have for each $-\infty<x<\infty$ that

$$P\left\{\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu}\sup_{a_n\leqq s\leqq 1} B_n(s)/(s^{1-\nu}(1-s)^\nu)\leqq x\right\}$$

$$= P\left\{\left(\frac{a_n}{1-a_n}\right)^{-\nu}\sup_{a_n\leqq t\leqq 1}\left(\frac{t}{1-t}\right)^\nu W\left(\left(\frac{a_n}{1-a_n}\right)\left(\frac{1-t}{t}\right)\right)\leqq x\right\} =$$

$$= P\{\sup_{0\leqq t\leqq 1} W(t)/t^\nu \leqq x\}.$$

Obviously the same statement holds with $B_n(s)$ and $W(t)$ replaced by $|B_n(s)|$ and $|W(t)|$ respectively. Thus on account of (16), (17), and (18) we have (12) and (14). Assertions (13) and (15) follow from (12) and (14) respectively by symmetry considerations. This completes the proof of Theorem 1.

Theorem 2. *Let a_n be any sequence of positive constants such that $na_n\to\infty$ and $a_n\to 0$ as $n\to\infty$. Then for every $0\leqq\nu<1/2$, as $n\to\infty$,*

$$S_{n,\nu} := a_n^{1/2-\nu}\sup_{a_n\leqq s\leqq 1}\alpha_n(s)/s^{1-\nu}\xrightarrow{\mathscr{D}} X_\nu; \tag{19}$$

$$S'_{n,\nu} := a_n^{1/2-\nu}\sup_{0\leqq s\leqq 1-a_n}\alpha_n(s)/(1-s)^{1-\nu}\xrightarrow{\mathscr{D}} X_\nu; \tag{20}$$

$$T_{n,\nu} := a_n^{1/2-\nu}\sup_{a_n\leqq s\leqq 1}|\alpha_n(s)|/s^{1-\nu}\xrightarrow{\mathscr{D}} Y_\nu; \tag{21}$$

and

$$T'_{n,\nu} := a_n^{1/2-\nu}\sup_{0\leqq s\leqq 1-a_n}|\alpha_n(s)|/(1-s)^{1-\nu}\xrightarrow{\mathscr{D}} Y_\nu. \tag{22}$$

Moreover, the random variables $S_{n,\nu}$ and $S'_{n,\nu}$, respectively the random variables $T_{n,\nu}$ and $T'_{n,\nu}$, are asymptotically independent.

Proof. Choose any $0 \leqq \nu < 1/2$. Let b_n denote any sequence of positive constants such that (i) $nb_n \to \infty$, (ii) $b_n \to 0$, and (iii) $a_n/b_n \to 0$ as $n \to \infty$. Write

$$S_{n,\nu}(a_n, b_n) := a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq b_n} \alpha_n(s)/s^{1-\nu}$$

and

$$S'_{n,\nu}(a_n, b_n) := a_n^{1/2-\nu} \sup_{1-b_n \leqq s \leqq 1-a_n} \alpha_n(s)/(1-s)^{1-\nu}.$$

Notice that for all n sufficiently large

$$|S_{n,\nu} - S_{n,\nu}(a_n, b_n)| \leqq \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{b_n \leqq s \leqq 1} |\alpha_n(s)|/(s^{1-\nu}(1-s)^{\nu}). \tag{23}$$

Applying (14) and (iii), we see that the right side of inequality (23) equals

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \left(\frac{b_n}{1-b_n}\right)^{-1/2+\nu} O_P(1) = o_P(1).$$

Thus we have

$$|S_{n,\nu} - S_{n,\nu}(a_n, b_n)| = o_P(1). \tag{24}$$

In the same way we have

$$|S'_{n,\nu} - S'_{n,\nu}(a_n, b_n)| = o_P(1). \tag{25}$$

Hence to prove (19) and (20) it is sufficient to show that, as $n \to \infty$,

$$S_{n,\nu}(a_n, b_n) \xrightarrow{\mathcal{D}} X_\nu, \tag{26}$$

and

$$S'_{n,\nu}(a_n, b_n) \xrightarrow{\mathcal{D}} X_\nu. \tag{27}$$

Clearly

$$\sup_{a_n \leqq s \leqq b_n} \left| a_n^{1/2-\nu} \alpha_n(s)/s^{1-\nu} - \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \alpha_n(s)/(s^{1-\nu}(1-s)^{\nu}) \right| \leqq$$

$$\leqq \sup_{a_n \leqq s \leqq b_n} |(1-a_n)^{1/2-\nu}(1-s)^{\nu} - 1| \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} |\alpha_n(s)|/(s^{1-\nu}(1-s)^{\nu}),$$

which obviously equals

$$o(1)\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq b_n} |\alpha_n(s)|/(s^{1-\nu}(1-s)^{\nu}).$$

Statement (14) implies that this last expression equals $o(1)O_P(1) = o_P(1)$.

Notice that it was shown above that

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{b_n \leqq s \leqq 1} |\alpha_n(s)|/(s^{1-\nu}(1-s)^{\nu}) = o_P(1). \tag{28}$$

From (28) and (12) we have that

$$(29)\qquad \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \le s \le b_n} \alpha_n(s)/(s^{1-\nu}(1-s)^\nu) \xrightarrow{\mathscr{D}} X_\nu,$$

which by the preceding arguments implies (26). Since

$$S'_{n,\nu}(a_n, b_n) \stackrel{\mathscr{D}}{=} S_{n,\nu}(a_n, b_n),$$

assertion (27) follows from (26). Hence we have established (19) and (20). Statements (21) and (22) follow by almost the same argument as that just given.

We shall now demonstrate that the random variables $S_{n,\nu}$ and $S'_{n,\nu}$ are asymptotically independent. On account of (24) and (25) it suffices to show that the random variables $S_{n,\nu}(a_n, b_n)$ and $S'_{n,\nu}(a_n, b_n)$ are asymptotically independent.

Choose any sequence of positive integers $1 \le k_n \le n$ such that $k_n \to \infty$, $k_n/n \to 0$ and $nb_n/k_n \to 1/2$ as $n \to \infty$. Observe that the function

$$V_{n,\nu} := S_{n,\nu}(a_n, b_n) I(U_{k_n,n} > b_n)$$

$(I(x>y)=1$ or 0 according as $x>y$ or $x \le y)$ is almost surely a function only of the lower extreme order statistics $U_{1,n}, \ldots, U_{k_n,n}$ and the random variable

$$V'_{n,\nu} := S'_{n,\nu}(a_n, b_n) I(1-b_n > U_{n-k_n,n})$$

is almost surely a function only of the upper extreme order statistics $U_{n-k_n,n}, \ldots,$ $\ldots, U_{n,n}$. Since $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, we conclude by Satz 4 of ROSSBERG [12] that the random variables $V_{n,\nu}$ and $V'_{n,\nu}$ are asymptotically independent. Also an elementary argument shows that, as $n \to \infty$,

$$n U_{k_n,n}/k_n \xrightarrow{P} 1$$

and

$$n(1 - U_{n-k_n,n})/k_n \xrightarrow{P} 1.$$

(See page 18 of BALKEMA and DE HAAN [1].) Thus by our choice of k_n, we have, as $n \to \infty$,

$$P\{V_{n,\nu} = S_{n,\nu}(a_n, b_n) \quad \text{and} \quad V'_{n,\nu} = S'_{n,\nu}(a_n, b_n)\} \to 1,$$

which implies that the random variables $S_{n,\nu}(a_n, b_n)$ and $S'_{n,\nu}(a_n, b_n)$ are asymptotically independent. Subsequently the same is true for $S_{n,\nu}$ and $S'_{n,\nu}$. The proof of the assertion that $T_{n,\nu}$ and $T'_{n,\nu}$ are asymptotically independent is along the same lines, so the details are omitted. The proof of Theorem 2 is now complete.

The following theorem should be compared to the result stated in (4) in the Introduction. We see that the generalized Rényi statistic version of the statistic on the left side of (4) given in (31) below also exhibts asymptotic independence behavior due to the asymptotic independence of the suprema of the weighted empirical process in the upper and lower regions of (0, 1), except that now this behavior is Gaussian instead of Poisson.

Theorem 3. *Let a_n be any sequence of positive constants such that $na_n\to\infty$ and $a_n\to 0$ as $n\to\infty$. Then for every $0\leqq\nu<1/2$, as $n\to\infty$,*

$$a_n^{1/2-\nu}\sup_{a_n\leqq s\leqq 1-a_n}\alpha_n(s)/(s(1-s))^{1-\nu}\xrightarrow{\mathcal{D}} X_\nu\vee X'_\nu \tag{30}$$

and

$$a_n^{1/2-\nu}\sup_{a_n\leqq s\leqq 1-a_n}|\alpha_n(s)|/(s(1-s))^{1-\nu}\xrightarrow{\mathcal{D}} Y_\nu\vee Y'_\nu, \tag{31}$$

where $X'_\nu\overset{\mathcal{D}}{=}X_\nu$, respectively $Y'_\nu\overset{\mathcal{D}}{=}Y_\nu$, and X_ν and X'_ν, respectively Y_ν and Y'_ν, are independent random variables.

Proof. Choose any $0\leqq\nu<1/2$. Let b_n denote any sequence of positive constants satisfying conditions (i), (ii), and (iii) as stated in the proof of Theorem 2. Observe that

$$a_n^{1/2-\nu}\sup_{b_n\leqq s\leqq 1-b_n}|\alpha_n(s)|/(s(1-s))^{1-\nu}\leqq$$

$$2^{-1+\nu}a_n^{1/2-\nu}\sup_{b_n\leqq s\leqq 1/2}|\alpha_n(s)|/s^{1-\nu}+2^{-1+\nu}a_n^{1/2-\nu}\sup_{1/2\leqq s\leqq 1-b_n}|\alpha_n(s)|/(1-s)^{1-\nu},$$

which by Theorem 2 and (iii) equals

$$O_P\big((a_n/b_n)^{1/2-\nu}\big)=o_P(1).$$

Notice that

$$\Big|\sup_{a_n\leqq s\leqq b_n}\alpha_n(s)/(s(1-s))^{1-\nu}-S_{n,\nu}(a_n,b_n)\Big|\leqq\sup_{a_n\leqq s\leqq b_n}|(1-s)^{-1+\nu}-1|\,|\alpha_n(s)|/s^{1-\nu},$$

which by (21) and (ii) equals $o(1)O_P(1)=o_P(1)$. Similarly we have

$$\Big|\sup_{1-b_n\leqq s\leqq 1-a_n}\alpha_n(s)/(s(1-s))^{1-\nu}-S'_{n,\nu}(a_n,b_n)\Big|=o_P(1).$$

Therefore by (26), (27) and the asymptotic independence of $S_{n,\nu}(a_n,b_n)$ and $S'_{n,\nu}(a_n,b_n)$ established in the proof of Theorem 2, we have (30). Assertion (31) follows by essentially the same argument. Thus Theorem 3 is proven.

With very slight modification of the proofs of the foregoing theorems it can be shown that the statements of Theorems 2 and 3 remain true with α_n replaced by β_n. The statements of Theorem 1 with β_n substituted for α_n also remain true if the suprema in (12) and (14) are taken over the interval $[a_n, 1-1/(n+1)]$ and the suprema in (13) and (15) are taken over the interval $[1/(n+1), 1-a_n]$.

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On the $|C, \alpha>1/2, \beta>1/2|$-summability of double orthogonal series

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction

Let $(X, \mathscr{F}, \mu)$ be a positive measure space and $\{\varphi_{ik}(x)\colon i, k=0, 1, \ldots\}$ an orthonormal system (in abbreviation: ONS) on X. We will consider the double orthogonal series

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} \varphi_{ik}(x) \tag{1.1}$$

where $\{a_{ik}\colon i, k=0, 1, \ldots\}$ is a sequence of real numbers (coefficients).

Let α and β be real numbers, $\alpha>-1$ and $\beta>-1$. We remind that the (C, α, β)-means of series (1.1) are defined by

$$\sigma_{mn}^{\alpha\beta}(x) = \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^\alpha A_{n-k}^\beta a_{ik} \varphi_{ik}(x)$$

where

$$A_m^\alpha = \binom{m+\alpha}{m} \quad (m, n = 0, 1, \ldots)$$

(for single series see, e.g. [13, p. 77]). Rhe case $\alpha=\beta=0$ gives back the rectangular partial sums:

$$\sigma_{mn}^{00}(x) = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik} \varphi_{ik}(x) = s_{mn}(x),$$

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while the case $\alpha=\beta=1$ provides the first arithmetic means of the rectangular partial sums:

$$\sigma_{mn}^{11}(x)=\sum_{i=0}^{m}\sum_{k=0}^{n}\left(1-\frac{i}{m+1}\right)\left(1-\frac{k}{n+1}\right)a_{ik}\varphi_{ik}(x)=$$

$$=\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}s_{ik}(x)\quad (m,n=0,1,\ldots).$$

2. Main results

Series (1.1) is said to be absolute (C,α,β)-summable (in abbreviation: $|C,\alpha,\beta|$-summable) at a point x if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\sigma_{mn}^{\alpha\beta}(x)-\sigma_{m-1,n}^{\alpha\beta}(x)-\sigma_{m,n-1}^{\alpha\beta}(x)+\sigma_{m-1,n-1}^{\alpha\beta}(x)|<\infty \tag{2.1}$$

where we agree on putting

$$\sigma_{-1,n}^{\alpha\beta}(x)=\sigma_{m,-1}^{\alpha\beta}(x)=\sigma_{-1,-1}^{\alpha\beta}(x)\equiv 0\quad (m,n=0,1,\ldots). \tag{2.2}$$

In other words, the series in (2.1) can be rewritten as

$$|\sigma_{00}^{\alpha\beta}(x)|+\sum_{m=1}^{\infty}|\sigma_{m0}^{\alpha\beta}(x)-\sigma_{m-1,0}^{\alpha\beta}(x)|+\sum_{n=1}^{\infty}|\sigma_{0n}^{\alpha\beta}(x)-\sigma_{0,n-1}^{\alpha\beta}(x)|$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\sigma_{mn}^{\alpha\beta}(x)-\sigma_{m-1,n}^{\alpha\beta}(x)-\sigma_{m,n-1}^{\alpha\beta}(x)+\sigma_{m-1,n-1}^{\alpha\beta}(x)|.$$

In the sequel, we will use the notations

$$\Delta_{mn}^{\alpha\beta}(x)=\sigma_{mn}^{\alpha\beta}(x)-\sigma_{m-1,n}^{\alpha\beta}(x)-\sigma_{m,n-1}^{\alpha\beta}(x)+\sigma_{m-1,n-1}^{\alpha\beta}(x)\quad (m,n=0,1,\ldots) \tag{2.3}$$

with agreement (2.2), and

$$\mathscr{A}_{pq}=\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1}a_{ik}^2\Big\}^{1/2}\quad (p,q=0,1,\ldots)$$

while identifying 2^{-1} with 0 in this paper.

Theorem 1. *If* $\alpha>1/2$, $\beta>1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\mathscr{A}_{pq}<\infty, \tag{2.4}$$

then series (1.1) *is* $|C,\alpha,\beta|$*-summable a.e. on* X.

Theorem 1 is the extension of the theorems of Tandori [11] ($\alpha=1$) and Leindler [5] ($\alpha>1/2$) from single orthogonal series to double ones. In the special case of two-dimensional trigonometric series, Theorem 1 was proved by PONOMARENKO and TIMAN [8].

Condition (2.4) is not only sufficient but also necessary for the a.e. $|C, \alpha, \beta|$-summability of series (1.1), for a fixed pair of $\alpha>1/2$ and $\beta>1/2$, if all ONS $\{\varphi_{ik}(x)\}$ are considered.

To be more specific, let $(X, \mathscr{F}, \mu)$ be the unit square $[0, 1]\times[0, 1]$ with the Borel measurable subsets as $\mathscr{F}$ and with the plane Lebesgue measure as μ. In the sequel, the unit interval $[0, 1]$ is denoted by I, the unit square $I\times I$ by S, and the Lebesgue measure by $|\cdot|$ (it will be clear from the context whether $|\cdot|$ means the linear or plane measure). We consider the two-dimensional Rademacher system $\{r_i(x_1)r_k(x_2)$: $i, k=0, 1, \ldots\}$ on S, where the

$$r_i(x_1) = \operatorname{sign} \sin (2^i \pi x_1) \quad (i = 0, 1, \ldots;\ x_1 \in I)$$

are the well-known Rademacher functions (see, e.g. [1, p. 51] or [13, p. 212]).

Theorem 2. *If* $\alpha>1/2$, $\beta>1/2$, *and condition* (2.4) *is not satisfied, then the two-dimensional Rademacher series*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik} r_i(x_1) r_k(x_2) \tag{2.5}$$

is not $|C, \alpha, \beta|$-*summable a.e. on* S.

This theorem is the extension of the corresponding results of BILLARD [2] $(\alpha=1)$ and GREPAČEVSKAJA [4] $(\alpha>1/2)$ from single orthogonal series to double ones.

Putting Theorems 1 and 2 together, we can draw the next

Corollary 1. *Let* $\alpha>1/2$ *and* $\beta>1/2$. *Series* (1.1) *is* $|C, \alpha, \beta|$-*summable a.e. for every double* ONS $\{\varphi_{ik}(x)$: $x=(x_1, x_2)\}$ *defined on S if and only if condition* (2.4) *is satisfied.*

This result for single ONS defined on I was proved by TANDORI [11] and LEINDLER [5] $(\alpha>1/2)$. Both authors constructed a new ONS in the counterexample, rather than using the Rademacher system.

Let $\{\lambda_{ik}$: $i, k=0, 1, \ldots\}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(i+1)(k+1)\lambda_{ik}} < \infty, \tag{2.6}$$

or equivalently,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{\lambda_{2^p, 2^q}} < \infty.$$

We note that $\{\lambda_{ik}\}$ is said to be nondecreasing if

$$\lambda_{ik} \leqq \min \{\lambda_{i+1,k}, \lambda_{i,k+1}\} \quad (i, k = 0, 1, \ldots).$$

Applying the Cauchy inequality to the series

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{\lambda_{2^{p}-1,2^{q}-1}^{1/2}} \lambda_{2^{p}-1,2^{q}-1}^{1/2} \Big\{ \sum_{i=2^{p}-1}^{2^{p}-1} \sum_{k=2^{q}-1}^{2^{q}-1} a_{ik}^{2} \Big\}^{1/2}$$

yields the following

Corollary 2. *If $\alpha > 1/2$, $\beta > 1/2$, and $\{\lambda_{ik}\}$ is a nondecreasing sequence of positive numbers satisfying condition* (2.6) *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^{2} \lambda_{ik} < \infty,$$

then series (1.1) *is $|C, \alpha, \beta|$-summable a.e.*

The corresponding result for single orthogonal series is due to UL'JANOV [12, pp. 46—47].

3. Generalizations and extensions

a) Let l be a real number, $l \geqq 1$. Following FLETT [3], series (1.1) is said to be $|C, \alpha, \beta|_{l}$-summable at x if

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [(m+1)(n+1)]^{l-1} |\Delta_{mn}^{\alpha\beta}(x)|^{l} < \infty.$$

The case $l=1$ gives back $|C, \alpha, \beta|$-summability.

Using the same arguments as in the proof of Theorems 1 and 2, one can derive the following three generalizations.

Theorem 1a. *If $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq l \leqq 2$, and*

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \mathcal{A}_{pq}^{l} < \infty, \tag{3.1}$$

then series (1.1) *is $|C, \alpha, \beta|_{l}$-summable a.e.*

Theorem 2a. *If $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq l \leqq 2$, and condition* (3.1) *is not satisfied, then series* (2.5) *is not $|C, \alpha, \beta|_{l}$-summable a.e.*

Corollary 2a. *If $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq l \leqq 2$, and $\{\lambda_{ik}\}$ is a nondecreasing sequence of positive numbers satisfying the conditions*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+1)(k+1)\lambda_{ik}^{l}} < \infty \tag{3.2}$$

and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 \lambda_{ik}^{2-l} < \infty,$$

then series (1.1) *is* $|C, \alpha, \beta|_l$*-summable a.e.*

In case $l=2$, condition (3.2) can be dropped.

Theorems 1a, 2a and Corollary 2a are the extensions of the corresponding theorems of SZALAY [9] and SPEVAKOV [10], respectively, from single orthogonal series to double ones.

b) Let $\varkappa=\{\varkappa_i\colon i=0, 1, \ldots\}$ and $\lambda=\{\lambda_k\colon k=0, 1, \ldots\}$ be two strictly increasing sequences of nonnegative numbers, both tending to ∞. Instead of (C, α, β)-means we can consider the $(R, \varkappa, \lambda)$-means of series (1.1) defined by

$$\sigma_{mn}(\varkappa, \lambda;\, x) = \sum_{i=0}^{m}\sum_{k=0}^{n}\left(1-\frac{\varkappa_i}{\varkappa_{m+1}}\right)\left(1-\frac{\lambda_k}{\lambda_{n+1}}\right) a_{ik}\varphi_{ik}(x) \quad (m, n = 0, 1, \ldots)$$

(for single series see, e.g. [1, p. 139]). Series (1.1) is said to be $|R, \varkappa, \lambda|$-summable at x if

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |\Delta_{mn}(\varkappa, \lambda;\, x)| < \infty,$$

where $\Delta_{mn}(\varkappa, \lambda;\, x)$ is defined similarly to (2.3).

Let $\varkappa(t)$ and $\lambda(t)$ be strictly increasing functions of the nonnegative variable t (constructed from the corresponding sequences, e.g. by means of linear interpolation), for which $\varkappa(i)=\varkappa_i$ and $\lambda(k)=\lambda_k$ for all nonnegative integers i and k. Denote by $\mathscr{K}(t)$ and $\Lambda(t)$ the uniquely determined inverse functions of $\varkappa(t)$ and $\lambda(t)$, respectively. For the sake of brevity, we write

$$\eta_p = [\mathscr{K}(2^p)], \quad \nu_q = [\Lambda(2^q)],$$

and

$$\tilde{\mathscr{A}}_{pq} = \Big\{\sum_{i=\eta_{p-1}}^{\eta_p-1}\ \sum_{k=\nu_{q-1}}^{\nu_q-1} a_{ik}^2\Big\}^{1/2} \quad (p, q = 0, 1, \ldots;\ \eta_{-1} = \nu_{-1} = 0),$$

where $[\cdot]$ denotes the integral part and in case $\eta_{p-1}=\eta_p$ or $\nu_{q-1}=\nu_q$ we take $\tilde{\mathscr{A}}_{pq}=0$.

The next two theorems can be also proved by using the methods of Sections 4 and 6.

Theorem 1b. *If*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \tilde{\mathscr{A}}_{pq} < \infty, \tag{3.3}$$

then series (1.1) *is* $|R, \varkappa, \lambda|$*-summable a.e.*

Theorem 2b. *If condition* (3.3) *is not satisfied, then series* (2.5) *is not* $|R, \varkappa, \lambda|$-*summable a.e.*

Both theorems are the extensions of those in [6] by the present author from single orthogonal series to double ones.

c) Finally, we point out that Theorems 1 and 2 as well as their variations mentioned above in (a) and (b) admit quite natural extensions to multiple orthogonal series, too.

4. Proof of Theorem 1

One important thing is to find a good representation for $\Delta_{mn}^{\alpha\beta}(x)$ defined by (2.3). Using the identities

$$A_{m-1}^{\alpha}=\frac{m}{\alpha+m}A_m^{\alpha},\quad A_{m-i}^{\alpha}=\frac{\alpha+m-i}{\alpha}A_{m-i}^{\alpha-1},$$

and

$$A_{m-i-1}^{\alpha}=\frac{m-i}{\alpha}A_{m-i}^{\alpha-1}\quad(\alpha>-1,\quad \alpha\neq 0),$$

one can obtain for $m\geqq 1$ and $n\geqq 1$

$$\Delta_{mn}^{\alpha\beta}(x)=\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\frac{ik}{mn}a_{ik}\varphi_{ik}(x), \tag{4.1}$$

for $m\geqq 1$ and $n=0$

$$\Delta_{m0}^{\alpha\beta}(x)=\sum_{i=1}^{m}\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\frac{i}{m}a_{i0}\varphi_{i0}(x), \tag{4.2}$$

for $m=0$ and $n\geqq 1$

$$\Delta_{0n}^{\alpha\beta}(x)=\sum_{k=1}^{n}\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\frac{k}{n}a_{0k}\varphi_{0k}(x), \tag{4.3}$$

while for $m=n=0$, $\Delta_{00}^{\alpha\beta}(x)=\sigma_{00}(x)=a_{00}\varphi_{00}(x)$. These representations are valid even in the cases where $\alpha=0$ or $\beta=0$, i.e. for all values of $\alpha>-1$ and $\beta>-1$.

We recall the inequality

$$\sum_{m=1}^{\infty}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2=O\left\{\frac{1}{i}\right\}\quad(i=1,2,\ldots;\ \alpha>1/2), \tag{4.4}$$

which is well-known in the literature (see, e.g. [1, p. 110]).

By Minkowski's inequality (keeping agreement (2.2) in mind),

$$\left\{\int_X \Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{mn}^{\alpha\beta}(x)|\Big]^2 d\mu(x)\right\}^{1/2} \leqq \left\{\int_X [\sigma_{00}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2}+ \tag{4.5}$$

$$+\sum_{m=1}^{\infty}\left\{\int_X [\Delta_{m0}^{\alpha\beta}(x)]^2 d\mu(x)\right\}+\sum_{n=1}^{\infty}\left\{\int_X [\Delta_{0n}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2}+$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\int_X [\Delta_{mn}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2} = S_1+S_2+S_3+S_4, \quad \text{say.}$$

Obviously, $S_1=|a_{00}|=\mathscr{A}_{00}$. We are going to show that S_2, S_3, and S_4 are also finite. We treat S_4 in more detail.

Applying the Cauchy inequality, then (4.1) and (4.4), we get that

$$S_4=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\left\{\int_X [\Delta_{mn}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2} \leqq \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{p/2}2^{q/2}\times$$

$$\times\left\{\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\int_X [\Delta_{mn}^{\alpha\beta}(x)]^2 d\mu(x)\right\}^{1/2} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{p/2}2^{q/2}\times$$

$$\times\left\{\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2\left[\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\right]^2\frac{i^2k^2}{m^2n^2}a_{ik}^2\right\}^{1/2} \leqq$$

$$\leqq \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p/2}2^{-q/2}\left\{\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2\left[\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\right]^2 i^2k^2a_{ik}^2\right\}^{1/2}=$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p/2}2^{-q/2}\left\{\sum_{i=1}^{2^{p+1}-1}\sum_{k=1}^{2^{q+1}-1}i^2k^2a_{ik}^2\sum_{m=\max\{2^p,i\}}^{2^{p+1}-1}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2\sum_{n=\max\{2^q,k\}}^{2^{q+1}-1}\left[\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\right]^2\right\}^{1/2}=$$

$$=O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p}2^{-q}\left\{\sum_{i=1}^{2^{p+1}-1}\sum_{k=1}^{2^{q+1}-1}i^2k^2a_{ik}^2\right\}^{1/2}.$$

Finally, using the elementary inequality

$$\{a+b+\dots\}^{1/2}\leqq a^{1/2}+b^{1/2}+\dots \quad (a\geqq 0,\quad b\geqq 0,\dots)$$

yields

$$\begin{aligned} S_4 &= O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p}2^{-q}\left\{\sum_{j=1}^{p+1}\sum_{l=1}^{q+1}2^{2j}2^{2l}\mathscr{A}_{jl}^2\right\}^{1/2}= \\ &= O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p}2^{-q}\sum_{j=1}^{p+1}\sum_{l=1}^{q+1}2^{j}2^{l}\mathscr{A}_{jl}= \\ &= O\{1\}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}2^{j}2^{l}\mathscr{A}_{jl}\sum_{p=j-1}^{\infty}2^{-p}\sum_{q=l-1}^{\infty}2^{-q}= \\ &= O\{1\}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\mathscr{A}_{jl}. \end{aligned} \tag{4.6}$$

Following the above pattern, one can show that

$$S_2 = O\{1\} \sum_{j=1}^{\infty} \mathscr{A}_{j0} \quad \text{and} \quad S_3 = O\{1\} \sum_{l=1}^{\infty} \mathscr{A}_{0l}. \tag{4.7}$$

Collecting (4.5)—(4.7) together, by (2.4) we can conclude

$$\Big\{\int_X \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}^{\alpha\beta}(x)|\Big]^2 d\mu(x)\Big\}^{1/2} = O\{1\} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \mathscr{A}_{jl} < \infty.$$

This means that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}^{\alpha\beta}(x)| \in L^2(X, \mathscr{F}, \mu)$$

and, in particular, (2.1) follows for almost every x.

5. Auxiliary results on finite sums in terms of two-dimensional Rademacher functions

We remind the following two results concerning finite sums in terms of one-dimensional Rademacher functions.

Lemma A. (see, e.g. [13, p. 213, Theorem (8.4)]). *For every $r>0$ there exists a constant D_r such that for every finite sum*

$$P(x_1) = \sum_{k=0}^{N} a_k r_k(x_1) \quad (N = 0, 1, \dots)$$

we have

$$\int_0^1 |P(x_1)|^r dx_1 \leqq D_r \Big\{\sum_{k=0}^{N} a_k^2\Big\}^{r/2}. \tag{5.1}$$

Lemma B. (Orlicz [7]). *Given any measurable set $E(\subset I)$ of positive measure, there exist an integer k_0 and a constant $C_1>0$ such that for every finite sum*

$$P(x_1) = \sum_{k=k_0}^{N} a_k r_k(x_1) \quad (N \geqq k_0)$$

we have

$$\int_E |P(x_1)| dx_1 \geqq C_1 \Big\{\sum_{k=k_0}^{N} a_k^2\Big\}^{1/2}.$$

Our goal is to extend these results to finite sums in terms of two-dimensional Rademacher functions. We note that the extension to higher dimensions runs in the same way.

Lemma 1. *For every finite sum*

$$P(x_1, x_2) = \sum_{i=0}^{M} \sum_{k=0}^{N} a_{ik} r_i(x_1) r_k(x_2) \quad (M, N = 0, 1, \dots)$$

we have

$$\int_0^1 \int_0^1 [P(x_1, x_2)]^4 dx_1 dx_2 \leqq D_4^2 [\sum_{i=0}^{M} \sum_{k=0}^{N} a_{ik}^2]^2. \tag{5.2}$$

Lemma 1 actually holds true for every $r>0$ instead of $r=4$ (cf. the proof of Lemma A in [13, pp. 213—214]). But inequality (5.2) is enough for our purpose.

Proof. By Fubini's theorem and (5.1) for the $r_i(x_1)$,

$$J = \int_0^1 \int_0^1 [P(x_1, x_2)]^4 dx_1 dx_2 = \int_0^1 \{\int_0^1 [\sum_{i=0}^{M} (\sum_{k=0}^{N} a_{ik} r_k(x_2)) r_i(x_1)]^4 dx_1\} dx_2 \leqq$$

$$\leqq D_4 \int_0^1 [\sum_{i=0}^{M} [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^2]^2 dx_2 = D_4 \sum_{i=0}^{M} \int_0^1 [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^4 dx_2 +$$

$$+ 2D_4 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} \int_0^1 [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^2 [\sum_{k=0}^{N} a_{jk} r_k(x_2)]^2 dx_2.$$

Applying the Cauchy—Schwarz inequality and (5.1) for the $r_k(x_2)$,

$$J \leqq D_4^2 \sum_{i=0}^{M} [\sum_{k=0}^{N} a_{ik}^2]^2 +$$

$$+ 2D_4 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} \{\int_0^1 [\sum_{k=0}^{N} a_{ik} r_k(x_2)]^4 dx_2 \int_0^1 [\sum_{k=0}^{N} a_{jk} r_k(x_2)]^4 dx_2\}^{1/2} \leqq$$

$$\leqq D_4^2 \sum_{i=0}^{M} [\sum_{k=0}^{N} a_{ik}^2]^2 + 2D_4^2 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} [\sum_{k=0}^{N} a_{ik}^2][\sum_{k=0}^{N} a_{jk}^2] = D_4^2 [\sum_{i=0}^{M} \sum_{k=0}^{N} a_{ik}^2]^2.$$

Lemma 2. *Given any measurable set $E(\subset S)$ of positive measure, there exist an integer n_0 and a constant $C_2>0$ such that for every finite sum*

$$P(x_1, x_2) = \sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik} r_i(x_1) r_k(x_2) \quad \textit{with} \quad \max\{m, n\} \geqq n_0 \tag{5.3}$$

$$(M \geqq m \geqq 0 \quad \textit{and} \quad N \geqq n \geqq 0),$$

we have

$$\iint_E |P(x_1, x_2)| dx_1 dx_2 \geqq C_2 \{\sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2\}^{1/2}. \tag{5.4}$$

Proof. We consider the 4-tuple system Π_4 defined by

$$\Pi_4 = \{r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\colon\ i_1, i_2, k_1, k_2 = 0, 1, \dots;\\ i_1 \leqq i_2,\ k_1 \leqq k_2, \quad \text{and} \quad (i_1, k_1) \neq (i_2, k_2)\}.$$

The proof of Lemma B (see [7]) is based on the fact that the 2-tuple system

$$\Pi_2 = \{r_{i_1}(x_1)r_{i_2}(x_1)\colon\ i_1, i_2 = 0, 1, \dots \quad \text{and} \quad i_1 < i_2\}$$

is an ONS on I. Hence it immediately follows that Π_4 is an ONS on S.

By Bessel's inequality, for any function $f(x_1, x_2) \in L^2(S)$,

$$\sum_{i_1=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{\substack{i_2=i_1\\(i_1,k_1)\neq(i_2,k_2)}}^{\infty}\sum_{k_2=k_1}^{\infty}\Big[\int_0^1\int_0^1 f(x_1, x_2)r_{i_1}(x_1)r_{k_1}(x_2)\,r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2\Big]^2 < \infty.$$

Letting $f=\chi_E$, the characteristic function of the set E, and $\varepsilon=|E|^2/16$ there exists an integer n_0 such that

$$\tag{5.5} \sum_{i_1=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{\substack{i_2=i_1\\(i_1,k_1)\neq(i_2,k_2)\\ \max\{i_1,k_1\}\geqq n_0}}^{\infty}\sum_{k_2=k_1}^{\infty}\Big[\iint_E r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2\Big]^2 \leqq \varepsilon.$$

We consider a finite sum $P(x_1, x_2)$ given by (5.3). Applying Hölder's inequality with the exponents 3/2 and 3, while representing 2 as the sum of 2/3 and 4/3, we get that

$$\iint_E [P(x_1, x_2)]^2\,dx_1\,dx_2 \leqq \Big\{\iint_E |P(x_1, x_2)|\,dx_1\,dx_2\Big\}^{2/3}\Big\{\iint_E [P(x_1, x_2)]\,dx_1\,dx_2\Big\}^{1/3},$$

whence

$$\tag{5.6} \iint_E |P(x_1, x_2)|\,dx_1\,dx_2 \geqq \Big\{\int_0^1\int_0^1 [P(x_1, x_2)]^4\,dx_1\,dx_2\Big\}^{-1/2}\Big\{\iint_E [P(x_1, x_2)]^2\,dx_1\,dx_2\Big\}^{3/2}.$$

We square out in $[P(x_1, x_2)]^2$ and use the Cauchy inequality and (5.5) to obtain

$$\iint_E [P(x_1, x_2)]^2\,dx_1\,dx_2 = |E|\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 +$$

$$+\sum_{i_1=m}^{M}\sum_{\substack{k_1=n\\(i_1,k_1)\neq(i_2,k_2)}}^{N}\sum_{i_2=m}^{M}\sum_{k_2=n}^{N} a_{i_1k_1}a_{i_2k_2}\iint_E r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2 \geqq$$

$$\geqq |E|\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 - \Big\{\sum_{i_1}\sum_{k_1}\sum_{i_2}\sum_{k_2} a_{i_1k_1}^2 a_{i_2k_2}^2 \times$$

$$\times \sum_{i_1}\sum_{k_1}\sum_{i_2}\sum_{k_2}\Big[\iint_E r_{i_1}(x_1)r_{k_1}(x_2)r_{i_2}(x_1)r_{k_2}(x_2)\,dx_1\,dx_2\Big]^2\Big\}^{1/2} \geqq$$

$$\geqq |E|\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 - \{4\varepsilon\}^{1/2}\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2 = \frac{|E|}{2}\sum_{i=m}^{M}\sum_{k=n}^{N} a_{ik}^2.$$

On the other hand, taking (5.2) into account, from (5.6) we get that

$$\iint_E |P(x_1, x_2)|\,dx_1 dx_2 \geqq \frac{|E|^{3/2}}{2^{3/2} D_4}\Big\{\sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2\Big\}^{1/2},$$

which is (5.3) to be proved.

6. Proof of Theorem 2

We need the following properties of the binomial coefficients.

(i) A_m^α is positive for $\alpha>-1$, and is increasing as a function of m for $\alpha>0$ (see, e.g. [13, p. 77, Theorem (1.17)]).

(ii) There exist two positive constants C_3 and C_4 depending only on α such that

$$C_3 \leqq \frac{A_m^\alpha}{m^\alpha} \leqq C_4 \quad (m = 1, 2, \ldots;\ \alpha > -1)$$

(see, e.g. [1, p. 69, formula (25)]). Hence one can derive that

$$\sum_{m=i}^{\infty} \frac{A_{m-i}^{\alpha-1}}{m A_m^\alpha} = O\{1\} \quad (i = 0, 1, \ldots;\ \alpha > 0). \tag{6.1}$$

(iii) There exists a positive constant C_5 depending on α such that

$$\frac{A_{2p}^\alpha}{A_{2p+1}^\alpha} \geqq C_5 \quad (p = 0, 1, \ldots;\ \alpha > 0). \tag{6.2}$$

We will prove that if series (2.5) is $|C, \alpha, \beta|$-summable on some subset of S with positive measure, for a certain pair of $\alpha>1/2$ and $\beta>1/2$, then (2.4) necessarily holds. Consequently, if (2.4) is not satisfied, then series (2.5) can be $|C, \alpha, \beta|$-summable only on a set of measure zero for any pair of $\alpha>1/2$ and $\beta>1/2$.

To begin with, by Egorov's theorem there exist a constant B and a set $E(\subset S)$ of positive measure such that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| \leqq B \quad \text{for every } (x_1, x_2) \in E. \tag{6.3}$$

We are going to apply Lemma 2. To this end, we must get rid of the Rademacher functions $r_i(x_1) r_k(x_2)$ in the definition of $\Delta_{mn}^{\alpha\beta}(x_1, x_2)$ for which $\max\{i, k\} < n_0$. This can be done in the following way. Set temporarily

$$\tilde{a}_{ik} = \begin{cases} a_{ik} & \text{if } \max\{i, k\} \geqq n_0, \\ 0 & \text{if } \max\{i, k\} < n_0; \end{cases}$$

and denote by $\tilde{\Delta}_{mn}^{\alpha\beta}(x_1, x_2)$ the corresponding differences. Then, by (6.1)

$$\sum_{\substack{m=1 \\ \max\{m,n\}\geq n_0}}^{\infty} \sum_{n=1}^{\infty} |\tilde{\Delta}_{mn}^{\alpha\beta}(x_1, x_2)| \leq \sum_{\substack{m=1 \\ \max\{m,n\}\geq n_0}}^{\infty} \sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| +$$

$$+ \sum_{\substack{m=1 \\ \max\{m,n\}\geq n_0}}^{\infty} \sum_{n=1}^{\infty} \sum_{i=1}^{\min\{m, n_0-1\}} \sum_{k=1}^{\min\{n, n_0-1\}} \frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}} \frac{A_{n-k}^{\beta-1}}{A_n^{\beta}} \frac{ik}{mn} |a_{ik}| =$$

$$= \sum_{\substack{m=1 \\ \max\{m,n\}\geq n_0}}^{\infty} \sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| + \sum_{i=1}^{n_0-1} \sum_{k=0}^{n_0-1} ik|a_{ik}| \Big\{ \sum_{m=i}^{n_0-1} \sum_{n=n_0}^{\infty} +$$

$$+ \sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} + \sum_{m=n_0}^{\infty} \sum_{n=k}^{n_0-1} \Big\} \frac{A_{m-i}^{\alpha-1}}{mA_m^{\alpha}} \frac{A_{n-k}^{\beta-1}}{nA_n^{\beta}} \leq$$

$$\leq \sum_{\substack{m=1 \\ \max\{m,n\}\geq n_0}}^{\infty} \sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| + 3 \sum_{i=1}^{n_0-1} \sum_{k=1}^{n_0-1} ik|a_{ik}| \times$$

$$\times \sum_{m=i}^{\infty} \frac{A_{n-k}^{\alpha-1}}{mA_m^{\alpha}} \sum_{n=k}^{\infty} \frac{A_{n-k}^{\beta-1}}{nA_n^{\beta}} = \sum_{\substack{m=1 \\ \max\{m,n\}\geq n_0}}^{\infty} \sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)| + O\{1\},$$

where $O\{1\}$ does not depend on (x_1, x_2). One can similarly obtain that

$$\sum_{m=n_0}^{\infty} |\tilde{\Delta}_{m0}^{\alpha\beta}(x_1, x_2)| = \sum_{m=n_0}^{\infty} |\Delta_{m0}^{\alpha\beta}(x_1, x_2)| + O\{1\}$$

and

$$\sum_{n=n_0}^{\infty} |\tilde{\Delta}_{0n}^{\alpha\beta}(x_1, x_2)| = \sum_{n=n_0}^{\infty} |\Delta_{0n}^{\alpha\beta}(x_1, x_2)| + O\{1\}.$$

So we may assume, without loss of generality, that $a_{ik}=0$ for $i, k=0, 1, \ldots, \ldots, n_0-1$ and use the notations a_{ik} and $\Delta_{mn}^{\alpha\beta}(x_1, x_2)$, rather than $\tilde{a}_{ik}$ and $\tilde{\Delta}_{mn}^{\alpha\beta}(x_1, x_2)$. Set

$$\delta_{pq}^{\alpha\beta}(x_1, x_2) = \sigma_{2^{p+1}-1, 2^{q+1}-1}^{\alpha\beta}(x_1, x_2) - \sigma_{2^{p-1}-1, 2^{q+1}-1}^{\alpha\beta}(x_1, x_2) -$$

$$- \sigma_{2^{p+1}-1, 2^{q-1}-1}^{\alpha\beta}(x_1, x_2) + \sigma_{2^{p-1}-1, 2^{q-1}-1}^{\alpha\beta}(x_1, x_2) \quad (p, q = 1, 2, \ldots).$$

On the one hand,

$$|\delta_{pq}^{\alpha\beta}(x_1, x_2)| \leq \sum_{m=2^{p-1}}^{2^{p+1}-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)|,$$

whence

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} |\delta_{pq}^{\alpha\beta}(x_1, x_2)| \leq 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\Delta_{mn}^{\alpha\beta}(x_1, x_2)|.$$

Consequently, by (6.3)

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\iint_E |\delta_{pq}^{\alpha\beta}(x_1, x_2)|\,dx_1dx_2 \leqq 4B|E|. \tag{6.4}$$

On the other hand, by (5.4)

$$\iint_E |\delta_{pq}^{\alpha\beta}(x_1, x_2)|\,dx_1dx_2 \geqq \tag{6.5}$$

$$\geqq C_2\left\{\sum_{i=2^{p-1}}^{2^{p+1}-1}\sum_{k=2^{q-1}}^{2^{q+1}-1}\left[\frac{A^{\alpha}_{2^{p+1}-1-i}}{A^{\alpha}_{2^{p+1}-1}}\right]^2\left[\frac{A^{\beta}_{2^{q+1}-1-k}}{A^{\beta}_{2^{q+1}-1}}\right]^2 a_{ik}^2\right\}^{1/2}.$$

On applying Lemma 2, we took into consideration only those terms in the representation of $\delta_{pq}^{\alpha\beta}(x_1, x_2)$ which contain a_{ik} with $2^{p-1}\leqq i\leqq 2^{p+1}-1$ and $2^{q-1}\leqq k\leqq 2^{q+1}-1$. Due to the monotony of A_m^{α} in m (see (ii) at the beginning at this Section) and (6.2) for $2^{p-1}\leqq i\leqq 2^p-1$ and $2^{q-1}\leqq k\leqq 2^q-1$,

$$\frac{A^{\alpha}_{2^{p+1}-1-i}}{A^{\alpha}_{2^{p+1}-1}}\,\frac{A^{\beta}_{2^{q+1}-1-k}}{A^{\beta}_{2^{q+1}-1}} \geqq \frac{A^{\alpha}_{2^p}}{A^{\alpha}_{2^{p+1}}}\,\frac{A^{\beta}_{2^q}}{A^{\beta}_{2^{q+1}}} \geqq C_5^2.$$

From here and (6.5) it follows that

(6.6)

$$\iint_E |\delta_{pq}^{\alpha\beta}(x_1, x_2)|\,dx_1dx_2 \geqq C_2C_5^2\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2} = C_2C_5^2\mathscr{A}_{pq} \quad (p, q = 1, 2, \ldots).$$

Combining (6.4) and (6.6) yields

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\mathscr{A}_{pq} \leqq \frac{4B|E|}{C_2C_5^2} < \infty. \tag{6.7}$$

Starting with

$$\delta_{p0}^{\alpha\beta}(x_1, x_2) = \sigma^{\alpha\beta}_{2^{p+1}-1,0}(x_1, x_2) - \sigma^{\alpha\beta}_{2^{p-1}-1,0}(x_1, x_2) \quad (p = 1, 2, \ldots)$$

and

$$\delta_{0q}^{\alpha\beta}(x_1, x_2) = \sigma^{\alpha\beta}_{0,2^{q+1}-1}(x_1, x_2) - \sigma^{\alpha\beta}_{0,2^{q-1}-1}(x_1, x_2) \quad (q = 1, 2, \ldots),$$

respectively, one can find in the same manner that

$$\sum_{p=1}^{\infty}\mathscr{A}_{p0} < \infty \quad \text{and} \quad \sum_{q=1}^{\infty}\mathscr{A}_{0q} < \infty. \tag{6.8}$$

The fulfilment of (6.7) and (6.8) is equivalent to that of (2.4).

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A local spectral theorem for closed operators

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

1. Introduction

Operators with spectral singularities were first studied in the particular case of ordinary differential operators by NAIMARK [10], SCHWARTZ [14], LJANCE [7], and PAVLOV [11], [12]. The first attempts at constructing a general theory for the case of a bounded operator have been made by BACALU [1] and NAGY [8]. This theory has been extended in the more general case of a closed operator by NAGY [9]. The aim of this paper is to present sufficient conditions for a closed operator (in a reflexive Banach space), having in its spectrum what may be loosely called an "exposed arc", to be K-scalar with a certain subset K of the spectrum (see the definitions below). These conditions are clearly of local character, and are satisfied e.g. by wide classes of ordinary differential operators. However, for reasons of space, the application of these results to the above-mentioned classes will be published elsewhere.

The main result (Theorem 1) may be regarded as an extension of a result of DUNFORD and SCHWARTZ [2; XVIII. 2.34]. Since the proof there does not seem to be adaptable to the local case, we follow completely different lines, and make essential use of a remarkable result of SUSSMANN [15]. We note that related results have been obtained by JONAS [5], [6] under certain assumptions on the global structure of the spectrum of the operator (which will not be made here).

Now we fix some notations and recall some definitions and results. In what follows X will denote a complex Banach space, and $C(X)$ and $B(X)$ will denote the set of closed and bounded linear operators in X, respectively. $\mathbf{C}$ and $\bar{\mathbf{C}}$ denote the complex plane and its one-point compactification, respectively. If $T\in C(X)$, then $\sigma(T)$ denotes its extended spectrum, i.e. its usual spectrum $s(T)$ if $T\in B(X)$, and

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$s(T)\cup\{\infty\}$ otherwise. If Y is a T-invariant linear manifold, then $T|Y=T_Y$ denotes the restriction of T to $Y\cap D(T)$. By a locally holomorphic (in general X-valued) function we mean a function that is holomorphic in each component of its domain.

If $H\subset\bar{\mathbf{C}}$, then H^c will mean $\bar{\mathbf{C}}\setminus H$, and $\bar{H}$ will denote the closure of H in the topology of $\bar{\mathbf{C}}$. Let $T\in C(X)$, and let $K=\bar{K}\subset(T)$. The σ-algebra B_K consists of those Borel sets in $\bar{\mathbf{C}}$ that either contain K or are contained in K^c. A K-resolution of the identity E for T is a Boolean algebra homomorphism of B_K into a Boolean algebra of projections in $B(X)$ with $E(\bar{\mathbf{C}})=I$ which is countably additive on B_K in the strong operator topology of $B(X)$ (i.e. a B_K-spectral measure), and satisfies

$$E(b)T\subset TE(b),\ \sigma\big(T|E(b)X\big)\subset\bar{b}\quad(b\in B_K).$$

An operator $T\in C(X)$ having a K-resolution of the identity E will be called a K-spectral operator. It can be shown, as in [8; Corollary 1 to Theorem 1] for the case of a bounded T, that the K-resolution of the identity for T is then unique. If E is the K-resolution of the identity for T, and $K_1=K\cap\sigma(T)$, then the definition

$$E_1(b)=\begin{cases}E(b\cup K) & \text{if } b\supset K_1\\ E(b\cap K^c) & \phantom{\text{if }} b\subset K_1^c\end{cases}\quad(b\in B_{K_1})$$

extends E to the K_1-resolution of the identity E_1 for T (cf. [8; p. 38]). So T is K-spectral if and only if T is K_1-spectral.

If E is the K-resolution of the identity for T, and the restriction $T|E(K^c)X$ is spectral of scalar type in the sense of Bade (cf. [2; XVIII.2.12]), then T will be called a K-scalar operator.

Note that an operator $T\in C(X)$ is spectral in the classical sense due to Bade (cf. [2; XVIII.2.1]) if and only if T is $\emptyset$-spectral with the $\emptyset$-resolution of the identity E for which $E(\{\infty\})=0$. By definition (cf. [9]), if an operator is K-spectral, then the spectral singularities of the operator are contained in the set K. So any statement establishing that an operator is K-spectral is an estimation of the set of the spectral singularities as well as a structure theorem for the operator.

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2. The results

Lemma 1. *Let X be a reflexive Banach space, and $T\in B(X)$. Let $a<b$, $c>0$ be real numbers, and*

$$J=\{z\in\mathbf{C}:\ a<\operatorname{Re} z<b,\ |\operatorname{Im}\ z|<c\}.$$

Let p be a one-to-one holomorphic $\mathbf{C}$*-valued mapping on a region containing* $\bar{J}$, $p(J)=H$, *and*

(1) $$G = H\cap\sigma(T)\subset p\big((a, b)\big) = G_1.$$

Assume that there are dense linear manifolds $X_0=X_0(H)$ *in* X *and* $X_0'=X_0'(H)$ *in* X^* *(the dual of* X*), respectively, such that with the notation* $R(z, T)=(z-T)^{-1}$

1° *for every* $(x_0, x_0^*)\in X_0\times X_0'$ *there exist in almost every point* $s\in(a, b)$

$$R^{\pm}\big(p(s), x_0^*, x_0\big) = \lim_{t\to 0\pm} x_0^* R\big(p(s-it), T\big)x_0,$$

2° *there is a positive number* $M=M(T, H)$ *such that for every* (x_0, x_0^*) *in* $X_0\times X_0'$

$$\int_G |R^+(z, x_0^*, x_0)-R^-(z, x_0^*, x_0)|\,|dz| \leqq M|x_0^*|\,|x_0|.$$

Assume further that there is a positive number $M_1=M_1(T, H)$ *such that for every* $z\in H\setminus G_1$ *(here* d *is the distance in* $\mathbf{C}$*)*

3° $$|R(z, T)| \leqq M_1 d(z, G_1)^{-1}.$$

With the notation

(2) $$K = H^c\cap\sigma(T)$$

the operator T *is then* K*-scalar with the* K*-resolution of the identity* E*, for which*

$$x_0^* E(b)x_0 = (2\pi i)^{-1} \int_b \big(R^+(z, x_0^*, x_0)-R^-(z, x_0^*, x_0)\big)\,dz \quad (b \textit{ Borel}, \quad b\subset G).$$

Remark. If X is an arbitrary Banach space, the spectrum of $T\in B(X)$ satisfies (1), with the notation (2) T is K-scalar, and H_1 is a compact set in H, then there is an $M_1=M_1(T, H_1)>0$ such that for every z in $H_1\setminus G_1$ the relation 3° holds. Indeed, if E denotes the K-resolution of the identity for T, then

(3) $$R(z, T) = R\big(z, T|E(G_1)X\big)E(G_1)+R\big(z, T|E(G_1^c)X\big)E(G_1^c).$$

Since $E(G_1^c)=E\big(G_1^c\cap\sigma(T)\big)=E(K)$, hence

$$\sigma\big(T|E(G_1^c)X\big) \subset K,$$

the second term on the right-hand side of (3) is bounded on the set H_1. The operator $T|E(G_1)X$ is spectral of scalar type, therefore there is an $L>0$ such that

$$|R\big(z, T|E(G_1)X\big)| = \Big|\int_{G_1} (z-v)^{-1}E(dv)|E(G_1)X\Big| \leqq Ld(z, G_1)^{-1}.$$

In view of (3) we obtain 3°.

The proof of Lemma 1. Let $C^r(\mathbf{C}, K)$ denote the algebra of those functions $f\colon \mathbf{C}\to\mathbf{C}$ which are of class C^r on $\mathbf{C}=\mathbf{R}^2$ and are locally holomorphic on K (i.e. on some neighbourhood $U=U(f)\subset\mathbf{C}$ of K; we may assume that U has a finite number of components.) Sussmann (see [15; Theorem 10 and Remark, p. 188]) has proved that 3° implies the existence of an algebra homomorphism

$$A\colon\ C^3(\mathbf{C},K)\to B(X)$$

such that $A(p_0)=I$ and $A(p_1)=T$ hold, where

$$p_k\colon z\mapsto z^k\quad (z\in\mathbf{C}).$$

Though Sussmann has stated this result for the case of a densely defined operator in Hilbert space, it is easy to check that it is valid (with the proof unchanged) under our conditions (cf. Vasilescu [16; V.5]).

Let $B_1=B_1(G,K)$ denote the algebra of those functions $f\colon \mathbf{C}\to\mathbf{C}$, for which the restriction $f|G$ is bounded Borel, and $f|K=0$. Let $f\in B_1$, $x_0\in X_0$, $x_0^*\in X_0'$, and define the bilinear form b_f on the set $X_0'\times X_0$ as follows:

$$b_f(x_0^*,x_0)=(2\pi i)^{-1}\int_G f(z)\big(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0)\big)\,dz. \tag{4}$$

By 2°, this form is continuous. Since it is densely defined, it can be uniquely extended to a continuous bilinear form $\overline{b_f}$ on $X^*\times X$. Since X is reflexive, there is a unique $Q(f)\in B(X)$ such that

$$x^*Q(f)x=\overline{b_f}(x^*,x)\quad (x^*\in X^*,\ x\in X).$$

The mapping $Q\colon B_1\to B(X)$ will be called a B_1 functional calculus. If $k(G)$ denotes the characteristic function of the set G, then clearly

$$Q(f)=Q\big(fk(G)\big)\quad (f\in B_1).$$

Let $\{f_n\}\subset B_1$ be a sequence, and $f_0\in B_1$. We shall write

$$\operatorname{Lim} f_n=f_0,$$

if $\lim_n f_n(z)=f_0(z)$ $(z\in G)$ pointwise, and

$$\sup\{|f_n(z)|\colon\ z\in G,\quad n=1,2,\ldots\}=L<\infty.$$

By 2°, we have then

$$|x^*Q(f_n)x|\leqq LM|x^*||x|\quad (x^*\in X^*,\ x\in X,\ n=0,1,\ldots).$$

Therefore, for every $x_0^*\in X_0'$, $x_0\in X_0$, $n=0,1,\ldots$

$$|x^*Q(f_n)x-x_0^*Q(f_n)x_0|\leqq LM(|x^*-x_0^*||x|+|x_0^*||x-x_0|).$$

Further we have

$$|x^*Q(f_n)x-x^*Q(f_0)x|\leqq|x^*Q(f_n)x-x_0^*Q(f_n)x_0|+|x_0^*Q(f_n)x_0-x_0^*Q(f_0)x_0|+$$
$$+|x_0^*Q(f_0)x_0-x^*Q(f_0)x|.$$

The first and third terms of the right-hand side are uniformly small in n, if $|x^*-x_0^*|$ and $|x-x_0|$ are small. By 2°, (4) and the Lebesgue convergence theorem, the second term is small, if n is large. Therefore, $\operatorname{Lim} f_n=f_0$ implies

(5) $$\lim_n x^*Q(f_n)x = x^*Q(f_0)x \quad (x^*\in X^*,\ x\in X).$$

Let $B_0=B_0(G, K)$ denote the subalgebra (in general without unit) of B_1 consisting of those f in B_1, the suppoıts of which satisfy $\operatorname{supp} f\cap K=\emptyset$. We show that

(6) $$A(f) = Q(f) \quad (f\in C^3(\mathbf{C}, K)\cap B_0).$$

Since $f\in B_0$, therefore

$$\operatorname{supp} f\cap\sigma(T) = \operatorname{supp} f\cap G$$

is a compact (in $\mathbf{C}$) subset of the analytic arc G_1. Sussmann [15; Lemma 6] shows that $A(g)=0$ for every $g\in C^3(\mathbf{C}, K)$ that vanishes in a neighbourhood of $\sigma(T)$. Hence for the function f there is $f_0\in C_0^3(H)\subset C^3(\mathbf{C}, K)\cap B_0$ such that

(7) $$A(f) = A(f_0), \quad Q(f) = Q(f_0).$$

(To obtain f_0 use a suitable partition of unity.) Further, the quoted proofs ([15; Theorem 10] and [16; V.5.2]) show that $f_0\in C_0^3(H)$ together with 1° and 2° imply

$$x_0^*A(f_0)x_0 = (2\pi i)^{-1}\int_G f_0(z)\big(R^+(z, x_0^*, x_0)-R^-(z, x_0^*, x_0)\big)dz = x_0^*Q(f_0)x_0$$

for every $x_0^*\in X_0'$, $x_0\in X_0$. From (7) we obtain that (6) is valid.

Now we show that the (clearly linear) mapping Q is an algebra homomorphism, i.e. for every $f, g\in B_1$

(8) $$Q(f)Q(g) = Q(fg).$$

At first let $f, g\in C^3(\mathbf{C}, K)\cap B_0$. Since A is an algebra homomorphism, (6) yields

$$Q(f)Q(g) = A(f)A(g) = A(fg) = Q(fg).$$

Now let $f\in C^0(\mathbf{C})\cap B_0$, g as above, then there is a sequence $\{f_n\}\subset C^3(\mathbf{C}, K)\cap B_0$ such that $\operatorname{Lim} f_n=f$. For every pair (f_n, g) the equality (8) holds, therefore (5) implies that (8) holds for the pair (f, g). Let g be as before, and let

$$D = \{f\in B_1\colon Q(f)Q(g) = Q(fg)\}.$$

Since $\{f_n\}\subset D$, $\operatorname{Lim} f_n=f$ imply, by (5), $f\in D$, we obtain that $D=B_1$. Fix now $f\in B_1$, and repeat this argument for the function g, then we obtain that (8) holds for every pair $f, g\in B_1$.

Let $B=B(G, K)$ denote the algebra of those functions $f\colon \mathbf{C}\to\mathbf{C}$ which are locally holomorphic on a neighbourhood $U=U(f)\subset\mathbf{C}$ of K, and for which $f|G$ is a bounded Borel function. Let $f\in B$, and let the open set U_1 contain K and be such that $f|U_1$ is locally holomorphic. Let U_0 have the same properties and let $\overline{U}_0\subset U_1$.

With the notation $U_2=C\setminus\overline{U_0}$, let (g_1, g_2) be a partition of unity subordinate to the open covering (U_1, U_2) of $\mathbf{C}$, and define the operator $F(f)\in B(X)$ as follows:

$$F(f)=A(fg_1)+Q(fg_2).$$

This operator is well-defined: clearly, $fg_1\in C^\infty(\mathbf{C}, K)$ and $fg_2\in B_0$; further, if (g_1^*, g_2^*) is a partition of unity subordinate to the open covering (U_1^*, U_2^*) of $\mathbf{C}$ (with the same properties), then $g_1-g_1^*=g_2^*-g_2$ implies

$$A(fg_1)-A(fg_1^*)=A\big(f(g_1-g_1^*)\big)=Q\big(f(g_2^*-g_2)\big)=Q(fg_2^*)-Q(fg_2),$$

since $f(g_1-g_1^*)\in C^\infty(\mathbf{C}, K)\cap B_0$ and, as (6) shows, the calculi A and Q coincide on this set. The mapping $F\colon B\to B(X)$ is clearly linear. We show that it is also multiplicative, i.e. it is an algebra homomorphism.

The mapping F is an extension of the calculi A and $Q_0=Q|B_0$. Indeed, let e.g., $f\in C^3(\mathbf{C}, K)$. Since $\overline{G_1}$ is compact in $\mathbf{C}$, we have $f\in B$. Let (U_1, U_2) and (g_1, g_2) be as above. Then $fg_2\in C^3(\mathbf{C}, K)\cap B_0$ and, by (6),

$$F(f)=A(fg_1)+Q(fg_2)=A(fg_1)+A(fg_2)=A(f).$$

The proof for the calculus Q_0 is similar.

If $f_1\in C^3(\mathbf{C}, K)$ and $f_2\in B_0$, then $f_1f_2\in B_0$. Further, we have

$$A(f_1)Q(f_2)=Q(f_2)A(f_1)=Q(f_1f_2). \tag{9}$$

Indeed, if, in addition, $f_2\in B_0\cap C^3(\mathbf{C}, K)$, then $f_1f_2\in C^3(\mathbf{C}, K)\cap B_0$. By (6) and by the multiplicativity of the calculus A, we obtain (9) for this case. Let

$$D_0=\{f_2\in B_0\colon \text{(9) holds with every } f_1\in C^3(\mathbf{C}, K)\}.$$

A reasoning similar to that in the proof of the multiplicativity of the calculus Q shows that $D_0=B_0$.

Now let $f_1, f_2\in B$, let $\overline{U_1}$ be contained in the intersection of the domains of local holomorphy of f_1 and f_2, otherwise let (U_1, U_2) and (g_1, g_2) be as above. By (9), we obtain

$$F(f_1)F(f_2)=\big(A(f_1g_1)+Q(f_1g_2)\big)\big(A(f_2g_1)+Q(f_2g_2)\big)=$$
$$=A(f_1f_2g_1^2)+2Q(f_1f_2g_1g_2)+Q(f_1f_2g_2^2).$$

Since F is an extension of the calculi A and Q_0, respectively, further $g_1+g_2\equiv 1$, we have

$$F(f_1)F(f_2)=F\big(f_1f_2(g_1+g_2)^2\big)=F(f_1f_2),$$

so F is multiplicative.

Applying our earlier notations, $p_i\in C^3(\mathbf{C}, K)$ $(i=0, 1)$ imply that $F(p_0)=A(p_0)=I$, and $F(p_1)=T$.

Let $k(b)=k(b;z)$ denote the characteristic function of the Borel set b in $\mathbf{C}$. If $k(b)\in B$, then define the operator $E(b)\in B(X)$ by

$$E(b)=F(k(b)). \tag{10}$$

Let S be a closed neighbourhood (in $\mathbf{C}$) of the set K. If the set b belongs to the σ-algebra B_S, then $k(b)\in B$. Since F is an algebra homomorphism, E is a homomorphism of the Boolean algebra B_S onto a Boolean algebra of projections in $B(X)$. If $\{b_n\}\subset B_S\cap S^c$ is a nondecreasing sequence of sets with the union b_0, then $E(b_n)=$ $=Q(k(b_n))$ and $\operatorname{Lim} k(b_n)=k(b_0)$. Hence, by (5),

$$\lim_n x^*E(b_n)x=x^*E(b_0)x \quad (x\in X,\, x^*\in X^*).$$

Therefore $E|B_S$ is countably additive in the weak and, by [2; IV. 10.1], in the strong operator topology of $B(X)$, i.e. it is a B_S-spectral measure. The multiplicativity of F implies that every $E(b)$ commutes with $T=F(p_1)$. Further, we show that

$$\sigma(T|E(b)X)\subset \bar{b} \quad (b\in B_S). \tag{11}$$

Let $z_0\notin\bar{b}$, and let $r\colon z\mapsto(z_0-z)^{-1}k(b;z)$ $(z\in\mathbf{C})$. Then $r\in B$, and $r(z)(z_0-z)=$ $=k(b;z)$. The multiplicativity of the calculus F yields that

$$F(r)(z_0-T)=(z_0-T)F(r)=E(b).$$

From this we see that (11) holds. Hence $E|B_S$ is the S-resolution of the identity of the operator T.

Let $b\in B_S\cap S^c$, $x_0^*\in X_0'$, $x_0\in X_0$. Since $E(b)=E(b\cap\sigma(T))$, by (4) we obtain

$$x_0^*E(b)x_0=x_0^*Q(k(b\cap\sigma(T)))x_0=(2\pi i)^{-1}\int_{b\cap G}(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0))\,dz.$$

Applying the notation $S_0=S^c\cap\sigma(T)=S^c\cap G$, the set S_0 is bounded, therefore

$$\int_{S_0} zE(dz)\in B(X).$$

Further, by (4),

$$\begin{aligned} x_0^*\int_{S_0} zE(dz)x_0 &= \int_{S_0} zx_0^*E(dz)x_0 = \\ &=(2\pi i)^{-1}\int_{S_0} z(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0))dz= \\ &= x_0^*Q(p_1k(S_0))x_0=x_0^*F(p_1k(S_0))x_0. \end{aligned}$$

By the multiplicativity of F, we have

$$\int_{S^c} zE(dz)=\int_{S_0} zE(dz)=F(p_1)F(k(S_0))=TE(S^c).$$

Hence for every $x_0=E(S^c)x_0$

$$Tx_0 = \int_{S^c} z(E|E(S^c)X)(dz)x_0,$$

i.e. the operator T is S-scalar for every closed neighbourhood S of the set K.

If $b\in B_K\cap K^c$, hence $k(b)\in B_1$, then let

$$E(b) = Q(k(b)), \quad E(\mathbf{C}\setminus b) = I-E(b).$$

This definition is clearly an extension of the definition of the mapping E in (10) to the σ-algebra B_K. If $\{b_n\}$ is a nondecreasing sequence in $B_K\cap K^c$, converging to b, further $x\in X$, $x^*\in X^*$, then $\operatorname{Lim} k(b_n)=k(b)$ implies

$$x^*E(b)x = x^*Q(k(b))x = \lim_n x^*Q(k(b_n))x = \lim_n x^*E(b_n)x.$$

Hence, as above, it follows that E is countably additive on B_K in the strong operator topology of $B(X)$. In particular, if $\{S_n\}$ is a sequence of closed neighbourhoods of K, converging nonincreasingly to K, and b is as above, then

$$E(b)x = \lim_n E(b\cap S_n^c)x \quad (x\in X)$$

in the norm topology of X. The technique of [8; Theorem 3] shows that the operator T is K-scalar.

Remark. It can be seen from the proof above that instead of 3° it is sufficient to have an estimation

3′ $$|R(z, T)| \leqq M_1 d(z, G_1)^{-r}$$

with some positive integer r. In this case the only necessary modification in the proof is that A will be an algebra homomorphism of $C^{r+2}(\mathbf{C}, K)$ into $B(X)$. The other parts of the proof remain unchanged.

In the following lemma we apply the notation p_k: $\overline{\mathbf{C}}\to\overline{\mathbf{C}}$, $p_k(z)=z^k$ $(z\in\overline{\mathbf{C}}$, k integer), further for $h\subset\overline{\mathbf{C}}$ we set $h^{-1}=p_{-1}(h)$.

Lemma 2. *If $T\in C(X)$ is K-scalar $(K=\overline{K}\subset\sigma(T))$, and there exists $T^{-1}\in C(X)$, then T^{-1} is K^{-1}-scalar.*

Proof. Let E denote the K-resolution of the identity for T. It is not hard to see that the projection-valued mapping E_1 defined by

$$E_1(b^{-1}) = E(b) \quad (b^{-1}\in B_{K^{-1}})$$

is the K^{-1}-resolution of the identity for T^{-1}. We shall show that the restriction $T^{-1}|E_1((K^{-1})^c)X$ is spectral of scalar type in the sense of Bade (cf. [2; XVIII.2.12]). Let

$$Y = E_1((K^{-1})^c)X = E(K^c)X.$$

Since T and E commute, and T is injective, $(T|Y)^{-1}=T^{-1}|Y$. Since $T|Y$ is spectral of scalar type, from [2; XVIII.2.11(h)] (and with the notations there) we obtain that

$$T^{-1}|Y=(T|Y)^{-1}=T_Y(p_1)^{-1}=T_Y(p_{-1}).$$

Thus for y in Y we have

$$T^{-1}y=\lim_{n\to\infty}\int_{|p_{-1}(z)|\leqq n} z^{-1}E(dz)y$$

in the sense that $y\in D(T^{-1}|Y)$ if and only if the right-hand side limit exists in the norm topology of Y. Hence, by [2; III.10.8. (f)],

$$T^{-1}y=\lim_{n\to\infty}\int_{|z|\leqq n} zE_1(dz)y,$$

again in the above sense. Therefore the operator T^{-1} is K^{-1}-scalar.

Theorem 1. *Let X be a reflexive Banach space, and $U\in C(X)$. Let $a<b$, $c>0$ be real numbers, and*

$$J=\{z\in\mathbf{C}:\ a<\operatorname{Re} z<b,\ |\operatorname{Im} z|<c\}.$$

Let p be a one-to-one conformal mapping of a region containing $\bar J$ into $\bar{\mathbf{C}}$ (hence p is meromorphic with at most one pole in this region), $p(J)=H\subset\bar{\mathbf{C}}$, and

$$G=H\cap\sigma(U)\subset p\big((a,b)\big)=G_1. \tag{1}$$

Assume that there are dense linear manifolds $X_0=X_0(H)$ in X and $X_0'=X_0'(H)$ in X^* such that

1° *for every (x_0, x_0^*) in $X_0\times X_0'$ there exists in almost every $s\in(a,b)$*

$$R^{\pm}(p(s), x_0^*, x_0)=\lim_{t\to 0\pm} x_0^*R\big(p(s-it), U\big)x_0,$$

2° *there is a positive number $M=M(T,H)$ such that for every (x_0, x_0^*) in $X_0\times X_0'$*

$$\int_G |R^+(z, x_0^*, x_0)-R^-(z, x_0^*, x_0)|\,|dz|\leqq M|x_0^*|\,|x_0|.$$

Assume further that there are a positive number $M_1=M_1(U,H)$ and a positive integer r such that for every $z\in H\setminus G_1$ (here $\bar d$ denotes the chordal distance in $\bar{\mathbf{C}}$)

3° $$|R(z,U)|\leqq M_1\bar d(z,G_1)^{-r}.$$

With the notation

$$K=H^c\cap\sigma(U)$$

the operator U is then K-scalar with the K-resolution of the identity E, for which

$$x_0^*E(b)x_0\doteq(2\pi i)^{-1}\int_b\big(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0)\big)dz\quad (b\ \text{Borel},\ b\subset G).$$

Proof. Due to (1), the resolvent set $\varrho(U)$ of the operator U is nonvoid, and there is a point $z\in\varrho(U)\cap G_1^c$. There is a positive number $c_1\leqq c$ such that (with understandable notations) the corresponding image set $H(c_1)=p(J(c_1))$ is a subset of $H=H(c)$, and $z\notin\overline{H(c_1)}$. With these notations then

$$K(c_1)=H(c_1)^c\cap\sigma(U)=H(c)^c\cap\sigma(U)=K(c)=K.$$

So we may assume that there is a point $z\in\varrho(U)\cap\overline{H}^c$. Hence the operator $T=(U-z)^{-1}$ belongs to $B(X)$. Without restricting the generality we may and will assume that $z=0$: i.e. that $0\in\varrho(U)\cap\overline{H}^c$ and $T=U^{-1}\in B(X)$.

With the notation p_k of the preceding lemma we have $\sigma(T)=p_{-1}(\sigma(U))$, and the function $p_{-1}\circ p$ is a one-to-one holomorphic mapping of a region containing $\bar{J}$ into $\mathbf{C}$. Further, $p_{-1}\circ p(J)=p_{-1}(H)$, and

$$p_{-1}(G)=p_{-1}(H)\cap\sigma(T)\subset p_{-1}\circ p((a,b))=p_{-1}(G_1).$$

So with the function $\bar{p}=p_{-1}\circ p$ replacing p and with the "reciprocals" of the sets occurring in condition (1), the bounded operator T satisfies condition (1) of Lemma 1. Now we show that it satisfies conditions 1° and 2° there. Since

$$R(z,T)=z^{-1}-z^{-2}R(z^{-1},U)\quad (z^{-1}\in\varrho(U)),$$

for every (x_0,x_0^*) in $X_0\times X_0'$ there exists in almost every point $s\in(a,b)$ the limit

$$R_1^{\pm}(\bar{p}(s),x_0^*,x_0)=\lim_{t\to 0\pm}x_0^*R(\bar{p}(s-it),T)x_0.$$

Further, for almost every point $z=\bar{p}(s)$ on $p_{-1}(G_1)$ $(s\in(a,b))$

$$R_1^+(z,x_0^*,x_0)-R_1^-(z,x_0^*,x_0)=-z^{-2}(R^+(z^{-1},x_0^*,x_0)-R^-(z^{-1},x_0^*,x_0)).$$

If the integral of a function f on G_1 exists, i.e.

$$\int_{G_1}|f(z)||dz|=\int_a^b|f(p(t))p'(t)|dt<\infty,$$

then (cf. [2; III.10.8])

$$\int_{G_1}|f(z)||dz|=\int_{p_{-1}(G_1)}|f(z^{-1})z^{-2}||dz|.$$

Hence

$$\int_{p_{-1}(G_1)}|R_1^+(z,x_0^*,x_0)-R_1^-(z,x_0^*,x_0)||dz|=\int_{G_1}|R^+(z,x_0^*,x_0)-$$
$$-R^-(z,x_0^*,x_0)||dz|\leqq M|x_0^*||x_0|\quad (x_0\in X_0,\ x_0^*\in X_0'),$$

so condition 2° of Lemma 1 is also satisfied.

Eschmeier [3; III.1.7. Korollar, p. 58] has shown that the growth condition 3°

on the set $H \setminus G_1$ implies

$$|R(z, T)| \leqq M_2 d(z, p_{-1}(G_1))^{-r-2} \quad (z \in p_{-1}(H \setminus G_1)).$$

(The exponent on the right-hand side can be $-r$ if $\infty \notin G_1$.) Thus Lemma 1 and the Remark after it yield that the bounded operator T is K^{-1}-scalar. Applying Lemma 2, we obtain that U is K-scalar.

For the K^{-1}-resolution of the identity E_1 of the operator T Lemma 1 gives that

$$x_0^* E_1(b^{-1}) x_0 = (2\pi i)^{-1} \int_{b^{-1}} \big(R_1^+(z, x_0^*, x_0) - R_1^-(z, x_0^*, x_0)\big) dz$$

$$(b^{-1} \ \text{Borel}, \ b^{-1} \subset G^{-1}).$$

If E denotes the K-resolution of the identity for U then, by Lemma 2, $E(b) = E_1(b^{-1})$ $(b \in B_K)$, and an integral transformation yields again

$$x_0^* E(b) x_0 = (2\pi i)^{-1} \int_b \big(R^+(z, x_0^*, x_0) - R^-(z, x_0^*, x_0)\big) dz \quad (b \ \ \text{Borel}, \quad b \subset G).$$

Remark. It is similarly seen as in the bounded case that if X is an arbitrary (not necessarily reflexive) Banach space, the spectrum of the operator $U \in C(X)$ satisfies (1), and with the notation above U is K-scalar, then for every $z_0 \in G_1 \cap \mathbf{C}$ there are a neighborhood $N = N(z_0)$ and a positive number $M_3 = M_3(U, N)$ such that

$$|R(z, U)| \leqq M_3 d(z, G_1)^{-1} \quad (z \in N \setminus G_1).$$

In particular cases of a spectrum of similar local structure several authors (cf. PAVLOV [13], GASYMOV and MAKSUDOV [4]) have considered the spectral singularities as those points of the curve, in a neighbourhood N of which the resolvent operator satisfies a growth condition of order larger than one, i.e. the set $\{d(z, G_1) R(z, U) \colon z \in N \setminus G_1\}$ is unbounded.

Corollary. *Under the conditions of Theorem 1 and with the notations there the set* $G \cap \mathbf{C}$ *is contained in the continuous spectrum of the operator* U.

Proof. Let $z \in G \cap \mathbf{C}$, and let E denote the K-resolution of the identity of the K-scalar operator U. By Theorem 1, we have $E(\{z\}) = 0$. Let $e \in B_K$. Since U is K-spectral, with the notation $U_e = U | E(e) X$ we have $\sigma(U_e) \subset \bar{e}$.

Let d be an open neighbourhood (in $\mathbf{C}$) of z such that $d \in B_K$. Then the set $e = d^c$ belongs to B_K, and $z \in \varrho(U_e)$. Hence

$$E(e) X = (z - U_e) E(e) X \subset (z - U) X,$$

where VY means $V\big(Y \cap D(V)\big)$ for any operator V and any set Y. Since E is countably additive, we obtain that $E(\{z\}^c) X \subset \overline{(z - U) X}$. Since $E(\{z\}) = 0$, we have

$$X = \overline{(z - U) X}. \tag{4}$$

Assume now that $(z-U)x=0$ for some x in X. Then, for every e as above,

$$(z-U_e)E(e)x = E(e)(z-U)x = 0.$$

Since $z\in\varrho(U_e)$, we obtain that $E(e)x=0$. By the countable additivity of E, we have $E(\{z\}^c)x=0$, hence $x=0$. So z is no eigenvalue, and (4) shows that it belongs to the continuous spectrum of U.

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Geodesics of a principal bundle with Kaluza—Klein metric

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction

Let be given a riemannian space-time manifold M, a principal fiber bundle $\pi\colon P\to M$ over M and a connection form φ on P. The structure group G of the bundle $\{P, \pi, M\}$ acting on P from right and the connection foım φ on P can be interpreted as internal symmetry group and potential of a given Yang—Mills field (cf. [2]). We suppose that the group G is equipped with a left invariant riemannian metric determined by a scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G. We call the riemannian metric defined by

$$(1)\qquad \langle X, Y\rangle_P := \langle \pi_* X, \pi_* Y\rangle_M + \langle \varphi(X), \varphi(Y)\rangle_{\mathfrak{g}},\quad (X, Y\in TP)$$

the Kaluza—Klein metric on P.

Usually, the bundle space metric $\langle\cdot,\cdot\rangle_P$ in Yang—Mills theory is defined with help of a biinvariant group metric. Now we generalize the construction of bundle metric using an arbitrary left invariant group metric instead of the biinvariant one. We shall investigate the geodesics in the bundle space P which can be interpreted as trajectories of "combined" classical motions of test particles in a given external Yang—Mills field φ and gravitational field $\langle\cdot,\cdot\rangle_M$.

The equation of geodesics in a principal bundle has been studied in [2], [3] under the assumption that the scalar product $\langle.,.\rangle_{\mathfrak{g}}$ is given by the Killing—Cartan form. For the description of geodesics in P with respect to the metric (1) we shall apply our results on geodesics in a riemannian submerison [5]. We shall use V. I. Arnold's approach of investigation of geodesics on a Lie group with left invariant metric interpreted as motion of a generalized rigid body [1].

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2. Geodesics on a Lie group with left invariant metric

Let be given a scalar product $\langle .,. \rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of the Lie group G. $\mathfrak{g}$ can be considered as the Lie algebra of left invariant vectorfield on G, thus $\langle .,. \rangle_{\mathfrak{g}}$ defines a left invariant riemannian metric on G. A scalar product on $T_gG(g\in G)$ defines a linear operator maping the tangent space T_gG onto its dual space T_g^*G. This maping gives a left invariant tensorfield I_g: $T_gG \to T_g^*G$ satisfying

$$\big(I_g(v), w\big) = \langle v, w \rangle_{\mathfrak{g}}, \quad v, w \in T_gG.$$

Proposition 1 (V. I. Arnold). *The curve $g(t)$ on the Lie group G is a geodesic with respect to the left invariant riemannian metric $\langle .,. \rangle_{\mathfrak{g}}$ if and only if the covector*

$$m(t) := R^*_{g(t)} \circ I_{g(t)} \dot{g}(t) \in \mathfrak{g}^* \tag{2}$$

is constant, where R_g: $G \to G$ is the right translation: $R_g(h) = hg$.

Proof. Let $e_1(g), \ldots, e_k(g) \in \mathfrak{g}$ be a left invariant orthonormal frame on G and $\omega^1, \ldots, \omega^k$ its dual coframe, that is $\omega^\lambda(e_\mu) = \delta^\lambda_\mu$ is satisfied (here and below λ, μ, ν run through the indices $1, \ldots, k$). If $c^\lambda_{\mu\nu}$ are the structure constants, that is

$$d\omega^\lambda = -c^\lambda_{\mu\nu} \omega^\mu \wedge \omega^\nu$$

the riemannian connection form $\varphi = \varphi^\lambda_\mu e_\lambda \otimes \omega^\mu$ can be expressed by

$$\varphi^\lambda_\mu = \frac{1}{2}(c^\lambda_{\nu\mu} - c^\mu_{\nu\lambda} + c^\nu_{\lambda\mu})\omega^\nu.$$

Indeed, $\varphi^\lambda_\mu = -\varphi^\mu_\lambda$ and $d\omega^\lambda = -\varphi^\lambda_\mu \wedge \omega^\mu$ are satisfied, and these properties determine the riemannian connection form uniquely. We get the equation of geodesics

$$\nabla_t \dot{g} = \nabla_t(\dot{g}^\lambda e_\lambda) = \left(\frac{d\dot{g}^\lambda}{dt} + c^\nu_{\lambda\mu}\dot{g}^\mu \dot{g}^\nu\right) e_\lambda.$$

The components of the tensor I_g are δ^λ_μ by the choice of frames, hence the equation of geodesics can be written as

$$\frac{d}{dt}(L^*_g \circ I_g \dot{g}) = \mathrm{ad}^*_v \circ L^*_g \circ I_g \dot{g}, \quad v = L_{g^{-1}*}\dot{g},$$

where L_g: $G \to G$ is the left translation and ad^*_v: $\mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation of $\mathfrak{g}$ that is $(\mathrm{ad}^*_v \zeta, w) = I([v, w])$, for any $w \in \mathfrak{g}$, $\zeta \in \mathfrak{g}^*$.

On the other hand we get using the fact that

$$L^*_g \circ I_g \dot{g} = L^*_g \circ R^*_{g^{-1}} m(t) = \mathrm{Ad}^*_g m(t)$$

the following equation

$$\frac{d}{dt}(L_g^* \circ I_g \dot{g}) = \frac{d}{dt}(\mathrm{Ad}_g^*)m + \mathrm{Ad}_g^* \frac{dm}{dt}. \tag{3}$$

We have

$$\frac{d}{dt}(\mathrm{Ad}_{g(t)})_{t_0} = \mathrm{Ad}_{g(t_0)} \frac{d}{dt}(\mathrm{Ad}_{g^{-1}(t_0)g(t)})_{t_0} = \mathrm{Ad}_{g(t_0)} \circ \mathrm{ad}_v,$$

where $v = L_{g^{-1}*}\dot{g} \in \mathfrak{g}$, thus we get

$$\frac{d}{dt}(\mathrm{Ad}_g^*) = \mathrm{ad}_v^* \circ \mathrm{Ad}_g^*$$

and consequently

$$\frac{d}{dt}(L_g^* \circ I_g \dot{g}) = \mathrm{ad}_v^* \circ \mathrm{Ad}_g^* \circ R_{g^{-1}}^* \circ I_g \dot{g} + \mathrm{Ad}_g^* \frac{dm}{dt} = \mathrm{ad}_v^* \circ L_g^* \circ I_g \dot{g} + \mathrm{Ad}_g^* \frac{dm}{dt}.$$

Comparing with (3) and using the fact that the operator Ad_g^* is invertible, the Proposition follows.

3. The Kaluza—Klein metric

We consider now the riemannian scalar product (1) on the total space P of the principal fiber bundle $\{P, \pi, M\}$. The riemannian manifolds P, M and the projection $\pi\colon P \to M$ yield a riemannian submersion, that is the map $\pi_*\colon TP \to TM$ preserves the length of horizontal vectors.

The principal fiber bundle $\{O_M(P), p, P\}$ of adapted frames on P of the submersion $\pi\colon P \to M$ is defined by the following: $\{O_M(P), p, P\}$ is a subbundle of the orthonormal frame bundle $\{O(P), p, P\}$ over P consisting of frames $(x; e_1, \dots, \dots, e_{n+k}) \in O(P)$ such that the vectors $e_1, \dots, e_n$ are horizontal (i.e. orthogonal to the fiber $\pi^{-1}(y)$, where $y = \pi(x)$) and the vectors $e_{n+1}, \dots, e_{n+k}$ are vertical (i.e. tangent to the fiber $\pi^{-1}(y)$). The structure group of the bundle of adapted frames is the product of orthogonal groups $O(n) \times O(k)$.

If ω and θ denote the $\mathbf{R}^{n+k}$-valued canonical form and the orthogonal Lie algebra $\mathfrak{o}(n+k)$-valued Riemannian connection form on $O(P)$, their components $\omega^a, \omega^\alpha, \theta^a_b, \theta^a_\beta, \theta^\alpha_b, \theta^\alpha_\beta$ satisfy the structure equations

$$d\omega^a = -\theta^a_c \wedge \omega^c - \theta^a_\gamma \wedge \omega^\gamma, \tag{4a}$$

$$d\omega^\alpha = -\theta^\alpha_c \wedge \omega^c - \theta^\alpha_\gamma \wedge \omega^\gamma, \tag{4b}$$

where the indices run the values: $a, b, c = 1, \dots, n$; $\alpha, \beta, \gamma = n+1, \dots, n+k$.

The form with the components θ^a_b, θ^α_β is a connection form on the adapted frame bundle $\{O_M(P), p, P\}$ resulted by the projection $\mathfrak{o}(n+k) \to \mathfrak{o}(n) \oplus \mathfrak{o}(k)$ of the values of the connection form θ. The difference form with components θ^a_β, θ^α_b is a tensorial form, thus they can be written as linear combinations of the components of the canonical form

$$\theta^a_\beta = \frac{1}{2} A^a_{\beta c}\omega^c + \frac{1}{2} T^a_{\beta\gamma}\omega^\gamma \quad \theta^\alpha_b = -\theta^b_\alpha, \tag{5}$$

where the tensors $A = A^a_{\beta c} e_a \otimes \omega^\beta \otimes \omega^c$ and $T = T^a_{\beta\gamma} e_a \otimes \omega^\beta \otimes \omega^\gamma$ are the fundamental invariants of the submersion (in detail see in [1]).

Let be given an orthonormal frame $E_1, \dots, E_k$ in the Lie algebra $\mathfrak{g}$. It defines a global orthonormal vertical frame field $\tilde{e}_{n+1}, \dots, \tilde{e}_{n+k}$ on P such that $\varphi(\tilde{e}_{n+\lambda}) = E_\lambda$. Thus the adapted frame bundle $\{O(P), p, P\}$ can be reduced to the subbundle $\{\tilde{O}_M(P), p, P\}$ consisting of the frames $\{e_1, \dots, e_n, \tilde{e}_{n+1}, \dots, \tilde{e}_{n+k}\}$, where $e_1, \dots, e_n$ are orthonormal horizontal tangent vectors. If we consider the $\mathbf{R}^{n+k}$-valued canonical form ω on this subbundle $\tilde{O}_M(P)$ and if we use the identification $\mathbf{R}^{n+k} = \mathbf{R}^n \oplus \mathfrak{g}$, the components $\omega^{n+1}, \dots, \omega^{n+k}$ of the canonical form ω are corresponding to the components $\varphi^1, \dots, \varphi^k$ of the connection form φ with respect to the frame $E_1, \dots, E_k$ in $\mathfrak{g}$. It follows that the forms $\omega^{n+1}, \dots, \omega^{n+k}$ satisfy the structure equations of the connection φ:

$$d\omega^{\lambda+n} = -\frac{1}{2} c^\lambda_{\mu\nu}\omega^{n+\mu} + \frac{1}{2} \Omega^\lambda_{bc}\omega^b \wedge \omega^c. \tag{6}$$

As compared (6) with (4b) and (5) we get

$$A^{n+\lambda}_{bc} = \Omega^\lambda_{bc}. \tag{7}$$

Since the parallel translation with help of the connection on the bundle $\{P, \pi, M\}$ commutes with the right action of G on P, it follows that the translation of the fibers is isometric. Indeed, if $y(t)$ is a curve in M, $v \in T_{x_0}P$ is a vertical tangent vector to the $\pi^{-1}(y(t_0))$ and $\varphi(v) = w \in \mathfrak{g}$, then $v = d/dt(R \exp tw)x_0|_0$. If τ_{t_0,t_1} denotes the translation $\pi^{-1}(y(t_0)) \to \pi^{-1}(y(t_1))$ along $y(t)$ we have

$$(\tau_{t_0,t_1})_* v = \frac{d}{dt}(R_{\exp tw} x_1)_0,$$

where $x(t)$ is a horizontal lift of the curve $y(t)$ such that $x(t_0) = x_0$, $x(t_1) = x_1$. It follows

$$\|v\|_P = \|\varphi(v)\|_{\mathfrak{g}} = \|w\|_{\mathfrak{g}} = \|(\tau_{t_0,t_1})_* v\|_P,$$

thus we get the isometry property. It follows that the normal variation of the metric tensor of the fibers vanishes i.e. the fibers are totally geodesic submanifolds:

$$T^\alpha_{\beta\gamma} \equiv 0. \tag{8}$$

We summarize our results (7), (8).

Theorem 1. *The Kaluza—Klein metric on P defines a riemannian submersion on the bundle* $\{P, \pi, M\}$. *The fibers of the bundle arre totally geodesic submanifolds. The fundamental tensors of the submersion satisfy*

$$\langle A(Z)Y, U\rangle_P = \langle \Omega(U, Y), \varphi(Z)\rangle_{\mathfrak{g}},$$
$$T(Z, U) = 0$$

for any tangent vectors $Y, Z, U \in T_x P$, $x \in P$.

4. Geodesics on P

We shall apply our recent result ([5], p. 353).

Theorem. *If* $\{P, \pi, M\}$ *is a riemannian submersion with totally geodesic fibers the curve* $x(\sigma)$ *is a geodesic of P if and only if the following conditions are satisfied*

(i) *the first vector of curvature of the projection curve* $y(\sigma) = \pi \circ x(\sigma)$ *is*

$$\tilde{\nabla}_\sigma y' = -\pi_* A(x')x',$$

where σ *is the arc-length parameter of* y, *prime denotes the derivative by* σ *and* $\tilde{\nabla}$ *is the riemannian covariant derivative on* M,

(ii) *the development* $z(\sigma) = \tau_{\sigma,\sigma_0} x(\sigma)$ *of* $x(\sigma)$ *in the fiber* $\pi^{-1}(y(\sigma_0))$ *is a geodesic, where the map* $\tau_{\sigma,\sigma_0}\colon \pi^{-1}(y(\sigma)) \to \pi^{-1}(y(\sigma_0))$ *is the parallel translation of the fibers along* $y(\sigma)$ *defined by the horizontal subspaces of* TP.

If $\{P, \pi, M\}$ is a principal fiber bundle with structure group G and the riemannian submersion on this bundle is defined by the Kaluza—Klein metric (1) we get the following description of geodesics.

Theorem 2. *The curve* $x(\sigma)$ *is a geodesic of P if and only if*

(i) *the curve* $z(\sigma) = \tau_{\sigma,\sigma_0} x(\sigma)$ *is a geodesic on the Lie group* $\pi^{-1}(y(\sigma_0)) \cong G$ *with respect to the induced fiber metric,*

(ii) *the first vector of curvature of the projection curve* $y(\sigma) = \pi \circ x(\sigma)$ *satisfies for any tangent vectorfield U of M along* $y(\sigma)$

$$\langle \tilde{\nabla}_\sigma y', U\rangle_M = \left(m, \Omega_{\bar{y}(\sigma)}(\bar{Y}, \bar{U})\right),$$

where $\bar{y}(\sigma)$ *is a horizontal lift of the curve* $y(\sigma)$, $\bar{Y}(\sigma)$ *and* $\bar{U}(\sigma)$ *are horizontal lifts of the vectorfields* $y'(\sigma)$ *and* $U(\sigma)$ *and* $m \in \mathfrak{g}^*$ *is a constant covector corresponding to the geodesic* $z(\sigma)$ *on* $\pi^{-1}(y(\sigma_0)) \cong G$

$$m(\sigma) = R^*_{g(\sigma)} \circ I_{g(\sigma)} z'$$

(cf. Proposition 1, $z(\sigma) = \bar{y}(\sigma) g(\sigma)$*).*

Proof. We have to prove the property (ii). We know that

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = -\langle\pi_* A(x')x', U\rangle_M = -\langle A(x')x', \bar{U}\rangle_P.$$

By Theorem 1 we get

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle\varphi(x'),\ \Omega_x(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}}.$$

If we identify the fibers $\pi^{-1}(y(\sigma))$ with the group G such that the unit of the group G is corresponding to the section $\bar{y}(\sigma)$ we can write

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle\varphi(z'),\ \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}},$$

where $x(\sigma)\in\pi^{-1}(y(\sigma))$ corresponds to the pair $\{y(\sigma), z(\sigma)\}$, $z(\sigma)\in\pi^{-1}(y(\sigma_0))$. But $\varphi(z')=(L_{g^{-1}})_*\ z'$ and the scalar product $\langle .,.\rangle_{\mathfrak{g}}$ is left invariant thus we have

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle(L_{g^{-1}})_* z',\ \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}} = \langle z', (L_g)_*\Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}} =$$
$$= \langle z', (R_g)_* \circ \mathrm{Ad}_g \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}}.$$

Since $\mathrm{Ad}_g\, \Omega_z(\bar{Y}, \bar{U}) = \Omega_{zg^{-1}}(\bar{Y}, \bar{U}) = \Omega_{\bar{y}}(\bar{Y}, \bar{U})$ we get

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle z', (R_g)_*(\Omega_{\bar{y}(\sigma)}(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}} = \left(R^*_{g(\sigma)} \circ I_{g(0)}\, z', \Omega_{\bar{y}(g)}(\bar{Y}, \bar{U})\right)$$

Thus the theorem is proved.

5. Geodesics on the orthonormal frame bundle

It is well known that the configuration space of a moving rigid body in the euclidean space E^3 is the orthonormal frame bundle $O(E^3)$ over E^3. This bundle is a trivial one that is the decomposition $O(E^3)=E^3\times O(3)$ is canonically defined and the action function of a free motion (the kinetic energy function) is the sum of the kinetic energy function of a moving masspoint in E^3 and of a rotating body. It means by the "Least Action Principle" that the trajectories of combined advancing and rotating motions will be geodesics on $O(E^3)$ with respect to the Kaluza—Klein metric corresponding to the trivial connection on the bundle $\{O(E^3), \pi, E^3\}$.

Analogously we can consider the orthonormal frame bundle $\{O(M), \pi, M\}$ over a riemannian manifold M as configuration space of "infinitesimal" rigid bodies in M. A scalar product $\langle .,.\rangle_{\mathfrak{o}(n)}$ on the orthogonal Lie algebra $\mathfrak{o}(n)$ corresponds to the kinetic energy function of a rotation, the Klauza—Klein metric (1) on $O(M)$ defined by the riemannian scalar product $\langle .,.\rangle_M$ on M, the riemannian connection form φ on $O(M)$ and the scalar product $\langle .,.\rangle_{\mathfrak{o}(n)}$ gives an action integral

$$\int \{\langle\pi_*\dot{x}, \pi_*\dot{x}\rangle_M + \langle\varphi(\dot{x}), \varphi(\dot{x})\rangle_{\mathfrak{o}(n)}\}\, dt$$

on the configuration space $O(M)$, the geodesics with respect to this metric will describe the inertial motion of an infinitesimal rigid body, corresponding to the least action principle. (The mass of the rigid body is supposed to be equal to 1.)

At the same time a detailed investigation of geodesics on $O(M)$ can serve as an application of the previously discussed equation of the classical motion of a Yang—Mills particle.

If $x(t)$ is a curve in $O(M)$ describing a combined motion of an infinitesimal rigid body (or particle) in the riemannian space M we call the projection curve $y(t)=\pi\circ x(t)$ the trajectory of the advancing motion and the curve $z(t)=\tau_{t,t_0}\circ x(t)$ on the fiber $\pi^{-1}(y(t_0))$ the trajectory of the rotation.

We know by Theorem 2 that the trajectory of the rotation of an inertial motion is a geodesic on the fiber $\pi^{-1}(y(t_0))$ and the trajectory of the advancing motion satisfies the differential equation

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \big(m, \Omega_{\bar{y}(\sigma)}(\bar{Y}, \bar{U})\big),$$

where m is a constant $\in \mathfrak{o}(n)^*$ defined by the initial values of the geodesic $x(t)$ and $\dot{x}(t)$. It means that the curve $y(t)$ is a trajectory of the motion of a masspoint in M under the action of the force $\big(m, \Omega_{\bar{y}}(\bar{Y}, \bar{U})\big)$. In our case $P=O(M)$ the Lie algebra valued curvature form Ω defines a tensorfield R on M called the curvature tensor and the constant $m\in\mathfrak{o}(n)^*$ defines a covariant constant tensorfield M_σ along the curve $y(\sigma)$ satisfying

$$R(X, Y)U = z\cdot\big(2\Omega(\bar{X}, \bar{Y})(z^{-1}U)\big) \quad \text{for} \quad X, Y, U\in T_yM$$

and

$$M_\sigma X = z\cdot\big(m^*(z^{-1}X)\big) \quad \text{for} \quad X\in T_{y(\sigma)}M,$$

where $z\in\pi^{-1}(y)$ is identified with the map $\mathbf{R}^n\to T_yM$ defined by the frame z and m^* denotes the vector from the Lie algebra $\mathfrak{o}(n)$ corresponding to the covector $m\in\mathfrak{o}(n)^*$ determined by the Killing—Cartan scalar product on $\mathfrak{o}(n)$, that is

$$m(v) = \operatorname{Trace} m^*v \quad \text{for} \quad v\in\mathfrak{o}(n).$$

Thus we get the following

Theorem 3. *The trajectory $y(\sigma)$ of the advancing motion of an infinitesimal rigid body in a riemannian space M satisfies the equation*

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \frac{1}{2}\operatorname{Trace}\big(M_\sigma\circ R(y', U)\big) \quad \textit{for} \quad U\in T_yM,$$

where $\tilde{\nabla}_\sigma M_\sigma=0$.

If the riemannian space M is of constant curvature k then the components R^a_{bcd} of the curvature tensor are

$$R^a_{bcd} = k\cdot(\delta^a_c g_{bd} - \delta^a_d g_{bc}\cdot)$$

Hence we can write

$$\frac{1}{2}\operatorname{Trace}\left(M_\sigma \circ R(y', U)\right) = \frac{1}{2}k \cdot M_a^b(\delta_c^a g_{bd} - \delta_d^a g_{bc})y'^c U^d =$$

$$= \frac{1}{2}k \cdot (M_c^b y'^c g_{bd} U^d - M_d^b g_{bc} y'^c U^d),$$

where M_a^b are the components of the tensor M_σ. Since the tensor M_σ is corresponding to the vector $m^* \in \mathfrak{o}(n)$ it is antisymmetric that is

$$\langle M_\sigma U, V\rangle = -\langle U, M_\sigma V\rangle \quad \text{for} \quad U, V \in TM,$$

or equivalently

$$M_c^b g_{bd} = -M_d^b g_{bc}.$$

Thus we get

$$\frac{1}{2}\operatorname{Trace}\left(M_\sigma \circ R(y', U)\right) = k \cdot \langle M_\sigma y', U\rangle_M.$$

It follows

Theorem 4. *The trajectory of the advancing motion of an infinitesimal rigid body in a riemannian space of constant curvature k satisfies the equation*

$$\tilde{\nabla}_\sigma y' = k \cdot M_\sigma y',$$

where $\tilde{\nabla}_\sigma M_\sigma = 0$.

If $\dim M = 3$, the action of antisymmetric tensors on the tangent space can be written in the form of cross product

$$M_\sigma y' = \mu_\sigma \times y',$$

where μ_σ is a uniquely determined tangent vectorfield along $y(\sigma)$. Since $\tilde{\nabla}_\sigma M_\sigma = 0$ we have $\tilde{\nabla}_\sigma \mu_\sigma = 0$. Thus we get

Corollary. *The trajectory of the advancing motion of an infinitesimal rigid body in a riemannian 3-space of constant curvature satisfies the equation*

$$\tilde{\nabla}_\sigma y' = k \cdot \mu_\sigma \times y',$$

where the vectorfield μ_σ *along* $y(\sigma)$ *is covariant constant.*

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Über Approximationseigenschaften differenzierter Hermitescher Interpolationspolynome mit Jacobischen Abszissen

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Herrn K. Tandori zum 60. Geburtstag gewidmet

1. Einleitung

Auf ihr Approximationsverhalten in bezug auf die m-te Ableitung einer in $[-1, 1]$ definierten und mindestens m-mal differenzierbaren reellen Funktion f werden Folgen von ebenfalls m-mal differenzierten f zugeordneten Hermiteschen Interpolationspolynomen $H_n[f]$ mit den Nullstellen $x_\nu = x_{\nu,n}(\alpha, \beta)$ der Jacobischen Orthogonalpolynome $P_n^{(\alpha,\beta)}(x)$ $(\alpha > -1, \beta > -1)$ als Abszissen untersucht. Dabei werde das Hermitesche Interpolationspolynom mit m paarweise disjunkten Abszissen x_ν gegeben durch

$$H_n[f;\ x] := \sum_{\nu=1}^{n} \left(h_\nu(x) f(x_\nu) + \mathfrak{h}_\nu(x) f'(x_\nu)\right) \tag{1.1}$$

mit

$$h_\nu(x) = h_{\nu,n}(x) := v_\nu(x) l_\nu^2(x), \tag{1.2}$$

$$v_\nu(x) = v_{\nu,n}(x) := 1 - \frac{\omega_n''(x_\nu)}{\omega_n'(x_\nu)}(x - x_\nu) = 1 + v_\nu'(x)(x - x_\nu) \tag{1.3}$$

und

$$\mathfrak{h}_\nu(x) = \mathfrak{h}_{\nu,n}(x) := (x - x_\nu) l_\nu^2(x) = \frac{1}{[\omega_n'(x_\nu)]^2} \frac{\omega_n^2(x)}{(x - x_\nu)}, \tag{1.4}$$

wobei l_ν die Lagrangeschen Grundpolynome

$$l_\nu(x) = l_{\nu,n}(x) := \begin{cases} \dfrac{\omega_n(x)}{(x - x_\nu)\omega_n'(x_\nu)} & (x \neq x_\nu) \\ 1 & (x = x_\nu) \end{cases} \tag{1.5}$$

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mit

$$\omega_n(x) = c \prod_{\nu=1}^{n} (x-x_\nu), \quad 0 \neq c \in \mathbf{R} \tag{1.6}$$

sind.

Im Sonderfall der Jacobischen Polynome $P_n^{(\alpha,\beta)}(x)=\omega_n(x)$ gilt für (1.3) (vgl. [10], S. 337 (14.5.1))

$$v_\nu(x) = 1 - \frac{\alpha-\beta+(\alpha+\beta+2)\,x_\nu}{1-x_\nu^2}(x-x_\nu). \tag{1.7}$$

In Satz 3 werden von α und β abhängende Bedingungen über f angegeben, unter denen die differenzierten Polynome $D^m H_n[f]=H_n^{(m)}[f]$ für $n\to\infty$ gleichmäßig im Intervall $[a, b]$ $(-1\leqq a<b\leqq 1)$ gegen $f^{(m)}$ konvergieren. Der Sonderfall $m=0$ sowie hiermit zusammenhängende Konvergenzuntersuchungen der verallgemeinerten Interpolationspolynome von Hermite—Fejér $\sum_{\nu=1}^{n} h_\nu(x) f(x_\nu)+\sum_{\nu=1}^{n} \mathfrak{h}_\nu(x) f_\nu'$ $(|f_\nu'|\leqq A<\infty)$ finden sich im wesentlichen (mit weiterer Literaturangabe) bei SZEGŐ [10], S. 338 f (man vergleiche hierzu auch SZABADOS [9]).

Entsprechende Untersuchungen über die m-mal differenzierten Lagrangeschen Interpolationspolynome wurden von den Verfassern in [5] durchgeführt. Die Approximationseigenschaften der differenzierten Lagrangeschen und Hermiteschen Interpolationspolynome werden am Schluß dieser Arbeit gegenübergestellt.

Um optimale Ergebnisse zu erhalten, verwenden wir analog zu [5] bei der Herleitung der Ergebnisse anstelle der Lebesguefunktionen $\sum_{\nu=1}^{n} |h_\nu(x)|$ und $\sum_{\nu=1}^{n} |\mathfrak{h}_\nu(x)|$ die von einem Hilfsparameter $\gamma\in\mathbf{R}$ abhängenden Funktionen

$$\mathscr{H}_n^m(x;\,\gamma) := \sum_{\nu=1}^{n} |D^m h_\nu(x)| \left(\frac{\sqrt{1-x_\nu^2}}{n}\right)^{\gamma} \quad (\gamma \geqq 0) \tag{1.8}$$

$$\mathfrak{H}_n^m(x;\,\gamma) := \sum_{\nu=1}^{n} |D^m \mathfrak{h}_\nu(x)| \left(\frac{\sqrt{1-x_\nu^2}}{n}\right)^{\gamma-1} \quad (\gamma \geqq 1) \tag{1.9}$$

Denn bei den Jacobischen Abszissen handelt es sich um Interpolationsstellen, für die $c_1/n\geqq|x_\nu-x_{\nu-1}|\geqq(c_2/n)\sqrt{1-x_\nu^2}>0$ $(\nu=2, \ldots, n;\ c_1, c_2\in\mathbf{R}^+$ unabhängig von $n)$ gilt. Mit Hilfe von $\mathscr{H}_n^m(x;\gamma)$ und $\mathfrak{H}_n^m(x;\gamma)$, deren asymptotisches Verhalten für $n\to\infty$ untersucht wird, und dem Approximationssatz von Jackson—Timan erhalten wir die Ergebnisse. (Man vergleiche hierzu [3], [6], [8].)

2. Hilfssätze über differenzierte Grundpolynome

In [5], Hilfssatz 1 wurden zwei verschiedene Darstellungen für die m-te Ableitung des n-ten Lagrangeschen Interpolationspolynoms $l_\nu(x)$ gegeben. Mit Rücksicht auf die bei der Hermiteschen Interpolation auftretenden Quadrate von $l_\nu(x)$ leiten wir in Analogie zu [5], Hilfssatz 1, entsprechende Formeln für $D^m l_\nu^2(x)$ sowie für $D^m \mathfrak{h}_\nu(x)$ und $D^m h_\nu(x)$ her. Es gilt

Hilfssatz 1.

$$D^m l_\nu^2(x) = \frac{(-1)^m m!}{[\omega_n'(x_\nu)]^2} \frac{{}^{(1)}\Omega_\nu{}^m(x)}{(x-x_\nu)^{m+2}} \quad (x \neq x_\nu,\ m \geqq 0) \tag{2.1}$$

mit

$$ {}^{(1)}\Omega_\nu{}^m(x) := \sum_{\mu=0}^{m} \frac{(-1)^\mu (m-\mu+1)}{\mu!} (x-x_\nu)^\mu D^\mu \omega_n^2(x), \tag{2.2}$$

$$D^m l_\nu^2(x_\nu) = \frac{1}{(m+1)(m+2)[\omega_n'(x_\nu)]^2} D^{m+2} \omega_n^2(x_\nu) \tag{2.3}$$

bzw.

$$D^m l_\nu^2(x) = \frac{\delta^m}{2[\omega_n'(x_\nu)]^2} D^{m+2} \omega_n^2(\bar{x}), \tag{2.4}$$

mit

$$\bar{x} := x_\nu + \delta(x-x_\nu), \quad |\delta| \leqq 1.$$

Beweis. Die Ableitungsformel (2.1) mit (2.2) ergibt sich mit Hilfe der Leibnizschen Regel unmittelbar aus (1.5). Wegen ${}^{(1)}\Omega_\nu{}^m(x_\nu) = 0$,

$$D\,{}^{(1)}\Omega_\nu{}^m(x) = \sum_{\mu=0}^{m} \frac{(-1)^\nu}{\mu!} (x-x_\nu)^\mu D^{\mu+1} \omega_n^2(x); \quad D\,{}^{(1)}\Omega_\nu{}^m(x_\nu) = 0$$

und

$$D^2\,{}^{(1)}\Omega_\nu{}^m(x) = \frac{(-1)^m}{m!} (x-x_\nu)^m D^{m+2} \omega_n^2(x), \tag{2.5}$$

was direkt aus (2.2) folgt, hat das Polynom ${}^{(1)}\Omega_\nu{}^m(x)$ in x_ν eine $(m+2)$-fache Nullstelle. Hiermit und mit Hilfe der l'Hospitalschen Regel ergibt sich (2.3) aus (2.1). Nach dem Taylorschen Satz folgt wegen ${}^{(1)}\Omega_\nu{}^m(x_\nu) = D\,{}^{(1)}\Omega_\nu{}^m(x_\nu) = 0$ aus (2.5)

$$ {}^{(1)}\Omega_\nu{}^m(x) = \frac{(-1)^m}{m!} \frac{(x-x_\nu)^2}{2} (\bar{x}-x_\nu)^m D^{m+2} \omega_n^2(\bar{x})$$

mit $\bar{x} = x_\nu + \delta(x-x_\nu)$, $|\delta| \leqq 1$, und hieraus (2.4).

Durch Übertragung des Beweises von [5], Hilfssatz 1, folgt für $\mathfrak{h}_\nu(x)$ aus (1.4)

Hilfssatz 2.

$$D^m \mathfrak{h}_\nu(x) = \frac{(-1)^m m!}{[\omega'_n(x_\nu)]^2} \frac{{}^{(2)}\Omega_\nu{}^m(x)}{(x-x_\nu)^{m+1}} \quad (x \neq x_\nu,\ m \geqq 0) \tag{2.6}$$

mit

$$(2.7)\qquad {}^{(2)}\Omega_\nu{}^m(x) := \sum_{\mu=0}^{m} \frac{(-1)^\mu}{\mu!}(x-x_\nu)^\mu D^\mu \omega_n{}^2(x),$$

$$(2.8)\qquad D^m \mathfrak{h}_\nu(x_\nu) = \frac{1}{(m+1)[\omega_n'(x_\nu)]^2} D^{m+1}\omega_n{}^2(x_\nu)$$

bzw.

$$(2.9)\qquad D^m \mathfrak{h}_\nu(x) = \frac{\delta^{*m}}{[\omega_n'(x_\nu)]^2} D^{m+1}\omega_n{}^2(x^*)$$

mit

$$x^* := x_\nu + \delta^*(x-x_\nu), \quad |\delta^*| \leqq 1.$$

Für $h_\nu(x)$ aus (1.2), (1.3) ergibt sich

Hilfssatz 3.

$$(2.10)\qquad D^m h_\nu(x) = v_\nu(x) D^m l_\nu{}^2(x) + m v_\nu'(x) D^{m-1} l_\nu{}^2(x) \quad (m \geqq 0).$$

3. Hilfssätze mit Jacobischen Polynomen $\omega_n(x) = P_n^{(\alpha,\beta)}(x)$

Von nun an sei $\omega_n(x)$ stets das Jacobische Orthogonalpolynom $P_n^{(\alpha,\beta)}(x)$ $(\alpha > -1, \beta > -1)$ vom Grad n mit den voneinander verschiedenen Nullstellen $x_\nu = x_{\nu,n}(\alpha,\beta)$. Ferner bezeichne ebenso wie in [5], (12) (mit $x = \cos\vartheta$)

$$(3.1)\qquad \mathfrak{N}_n := \{\vartheta_\nu \quad \text{mit} \quad P_n^{(\alpha,\beta)}(\cos\vartheta_\nu) = 0\}$$

die Menge der Nullstellen von $P_n^{(\alpha,\beta)}(\cos\vartheta)$, die der Größe nach geordnet seien. Zu vorgegebenem ϑ sei weiter wie in [5], (26) und (16)

$$(3.2)\qquad \mathfrak{M}_n = \mathfrak{M}_n(\vartheta, c) := \left\{\vartheta_\nu \in \mathfrak{N}_n \quad \text{mit} \quad |\vartheta - \vartheta_\nu| < \frac{c}{n}\right\} \subset \mathfrak{N}_n$$

(mit festem $c > 2\pi$) und

$$(3.3)\qquad \vartheta_j = \vartheta_{j(n,\vartheta)} := \min_i \{\vartheta_i \quad \text{mit} \quad |\vartheta - \vartheta_i| = \min_{\vartheta_\nu \in \mathfrak{M}_n} |\vartheta - \vartheta_\nu|\} \in \mathfrak{M}_n$$

die (gegebenenfalls kleinere) zu ϑ nächstbenachbarte Nullstelle aus $\mathfrak{M}_n$.

Hilfssatz 4. a) *Für* $0 \leqq \vartheta \leqq \pi/2$ *und* $n \to \infty$ *gilt*

$$(3.4)\qquad v_\nu(x) = \begin{cases} \vartheta_\nu^{-2}((1+\alpha)\vartheta_j^2 - \alpha\vartheta_\nu^2) + O\left(\vartheta_\nu^{-2}\left(\frac{\vartheta_j}{n} + \vartheta_j^4\right) + \vartheta_\nu^2 + \vartheta_j^2\right) & \left(0 < \vartheta_\nu \leqq \frac{\pi}{2}\right) \\ O((\pi - \vartheta_\nu)^{-2}) & \left(\frac{\pi}{2} \leqq \vartheta_\nu < \pi\right). \end{cases}$$

b) *für* $0 \leqq \vartheta \leqq \pi/2$ *und* $n \to \infty$ *gilt ferner*

$$(3.5) \qquad v_\nu'(x) = \begin{cases} -(2\alpha+2)\vartheta_\nu^{-2}(1+O(\vartheta_\nu^2)) & \left(0 < \vartheta_\nu \leqq \frac{\pi}{2}\right) \\ O\left((\pi-\vartheta_\nu)^{-2}\right) & \left(\frac{\pi}{2} \leqq \vartheta_\nu < \pi\right) \end{cases}$$

Beweis. Aus (1.7) folgt $v_\nu(x) = (1-x_\nu^2)^{-1}\{1+(2\alpha+1)x_\nu^2-(2\alpha+2)xx_\nu+$ $+(\beta-\alpha)(1-x_\nu)(x-x_\nu)\}$ bzw. $v_\nu'(x) = (1-x_\nu^2)^{-1}\{-(2+2\alpha)x_\nu+(\beta-\alpha)(1-x_\nu)\}$ und hieraus (3.4) und (3.5) mittels $x = 1-(1/2)\vartheta_j^2+O((\vartheta_j/n)+\vartheta_j^4)$ und $x_\nu = 1-(1/2)\vartheta_\nu^2+$ $+O(\vartheta_\nu^4)$ $(0 \leqq \vartheta, \vartheta_\nu \leqq \pi/2)$.

Weiter folgt aus [5], Hilfssätze 3 und 4 wegen

$$D^\mu \omega_n^2(x) = \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} D^\lambda \omega_n(x) D^{\mu-\lambda}\omega_n(x)$$

in der Bezeichnung von (3.3)

Hilfssatz 5. *Für* $n \to \infty$ *gilt mit* $0 \leqq \vartheta \leqq \pi/2$

$$(3.6) \qquad D_x^\mu [P_n^{(\alpha,\beta)}(\cos\vartheta)]^2 = O(n^{\mu-1})\vartheta_j^{-2\alpha-\mu-1}$$

$$(3.7) \qquad (D_x[P_n^{(\alpha,\beta)}(\cos\vartheta_\nu)])^{-2} = \begin{cases} O(\vartheta_\nu^{2\alpha+3}n^{-1}) & \left(0 < \vartheta_\nu \leqq \frac{\pi}{2}\right) \\ O\left((\pi-\vartheta_\nu)^{2\beta+3}n^{-1}\right) & \left(\frac{\pi}{2} \leqq \vartheta_\nu < \pi\right). \end{cases}$$

Für die Hilfsfunktion

$$(3.8) \qquad \psi_\nu^{\mu,m}(x;\, \gamma) := \frac{(1-x_\nu^2)^{\gamma/2} n^{-\gamma}}{[DP_n^{(\alpha,\beta)}(x_\nu)]^2} \frac{|D^\mu[P_n^{(\alpha,\beta)}(x)]^2|}{|x-x_\nu|^{m+2-\mu}} \qquad (0 \leqq x \leqq 1,\, x \neq x_\nu)$$

folgt aus diesem Hilfssatz 5 und der Ungleichung

$$(3.9) \qquad \left|\frac{\vartheta^2-\vartheta_\nu^2}{\cos\vartheta-\cos\vartheta_\nu}\right| < \frac{3\sqrt{2}}{4}\pi^2 \quad \left(0 \leqq \vartheta \leqq \frac{\pi}{2},\, 0 < \vartheta_\nu < \pi\right)$$

in der Bezeichnung von (3.3)

Hilfssatz 6. *Für* $0 \leqq \vartheta \leqq \pi/2$ *und* $0 < \vartheta_\nu < \pi$ *gilt, falls* $\vartheta_\nu \notin \mathfrak{M}_n$, *für* $n \to \infty$

(3.10)

$$\psi_\nu^{\mu,m}(x;\, \gamma) = O(1)\, n^{\mu-\gamma-2}\vartheta_j^{-2\alpha-\mu-1}|\vartheta_j^2-\vartheta_\nu^2|^{\mu-m-2} \cdot \begin{cases} \vartheta_\nu^{\gamma+2\alpha+3} & \left(0 < \vartheta_\nu \leqq \frac{\pi}{2}\right) \\ (\pi-\vartheta_\nu)^{\gamma+2\beta+3} & \left(\frac{\pi}{2} \leqq \vartheta_\nu < \pi\right) \end{cases}$$

mit einem nichtnegativen Hilfsparameter γ *und* $\mu = 0, 1, \ldots, m$.

4. Abschätzung von $\mathscr{H}_n^m(x;\gamma)$ mit $\omega_n(x)=P_n^{(\alpha,\beta)}(x)$ für $n\to\infty$

Zur Abschätzung der durch (1.8) und (1.9) mit $\omega_n(x)=P_n^{(\alpha,\beta)}(x)$ gegebenen Summen $\mathscr{H}_n^m(x;\gamma)$ und $\mathfrak{H}_n^m(x;\gamma)$ für hinreichend große Werte von n genügt es $0\leqq x=\cos\vartheta\leqq 1$ zu wählen. Zunächst wird das Verhalten von $\mathscr{H}_n^m(x;\gamma)$ für $n\to\infty$ untersucht, wobei sich das Abschätzungsverfahren an den Aufbau der Abschätzung von $\mathscr{L}_n^m(x;\gamma)$ in [5] anschließt. Nach (1.8), (1.2), (1.3) und Hilfssatz 3 gilt

$$\mathscr{H}_n^m(x;\ \gamma)\leqq\sum_{\nu=1}^{n}\Phi_\nu^m(x;\ \gamma) \tag{4.1}$$

mit

$$\Phi_\nu^m(x;\ \gamma):=\{|v_\nu(x)D^m l_\nu^2(x)|+m|v_\nu'(x)D^{m-1}l_\nu^2(x)|\}(1-x_\nu^2)^{\gamma/2}n^{-\gamma} \tag{4.2}$$

a) $\vartheta_\nu\in\mathfrak{M}_n$; $0\leqq\vartheta\leqq\pi/2$. Nach (3.2) und (3.3) gilt $\vartheta_\nu=O(\vartheta_j)$ und $\vartheta=O(\vartheta_j)$ für $n\to\infty$. Ferner ergibt sich aus (2.4) für $\omega_n(x)=P_n^{(\alpha,\beta)}(x)$ unter Heranziehung der Hilfssätze 4 und 5 für $n\to\infty$

$$|v_\nu(x)D^m l_\nu^2(x)|(1-x_\nu^2)^{\gamma/2}n^{-\gamma}=O(1)n^{m-\gamma}\vartheta_j^{\gamma-m}\quad(m\geqq 0) \tag{4.3}$$

$$|v_\nu'(x)D^{m-1}l_\nu^2(x)|(1-x_\nu^2)^{\gamma/2}n^{-\gamma}=O(1)n^{m-\gamma-1}\vartheta_j^{\gamma-m-1}\quad(m\geqq 1). \tag{4.4}$$

Da nach [5], Hilfssatz 2 $\big(1/(n\vartheta_j)\big)=O(1)$ für $n\to\infty$ gilt und die Anzahl der $\vartheta_\nu\in\mathfrak{M}_n$ gleichmäßig beschränkt ist, folgt aus (4.2—4)

$$\sum_{\vartheta_\nu\in\mathfrak{M}_n}\Phi_\nu^m(x;\ \gamma)=O(1)n^{m-\gamma}\vartheta_j^{\gamma-m}\quad\text{für}\quad n\to\infty. \tag{4.5}$$

b) $\vartheta_\nu\notin\mathfrak{M}_n$; $0\leqq\vartheta\leqq\pi/2$. Für $\vartheta_\nu\notin\mathfrak{M}_n$ schätzen wir aufgrund von (2.1) mit (2.2) jeden Summanden $\Phi_\nu^m(x;\gamma)$ aus (4.2) für $\omega_n(x)=P_n^{(\alpha,\beta)}(x)$ wie folgt weiter ab

$$\Phi_\nu^m(x;\ \gamma)\leqq\sum_{\mu=0}^{m}\varphi_\nu^\mu(x;\ \gamma)\quad(\vartheta_\nu\notin\mathfrak{M}_n) \tag{4.6}$$

mit

$$\varphi_\nu^\mu(x;\ \gamma):=\{(m-\mu+1)|v_\nu(x)|+(m-\mu)|(x-x_\nu)v_\nu'(x)|\}\frac{m!}{\mu!}\psi_\nu^{\mu,m}(x;\ \gamma) \tag{4.7}$$

und den in (3.8) definierten $\psi_\nu^{\mu,m}(x;\gamma)$. Hierfür gilt der

Hilfssatz 7. *Ist* $\vartheta_\nu \notin \mathfrak{M}_n$, $0 \leqq \vartheta \leqq \pi/2$, *so gelten für* $n \to \infty$

$$\varphi_\nu^\mu(x;\ \gamma) = O(1) n^{m-\gamma-1} \vartheta_j^{\gamma-m-1} \quad \left(0 < \vartheta_\nu \leqq \frac{2}{3}\vartheta_j\right) \tag{4.8}$$

$$\varphi_\nu^\mu(x;\ \gamma) = O(1) n^{\mu-\gamma-2} \vartheta_j^{\gamma-m} |\vartheta_j - \vartheta_\nu|^{\mu-m-2} \quad \left(\frac{2}{3}\vartheta_j \leqq \vartheta_\nu \leqq \frac{3}{2}\vartheta_j;\ \vartheta_\nu \notin \mathfrak{M}_n\right) \tag{4.9}$$

$$\varphi_\nu^\mu(x;\ \gamma) = O(1)[\vartheta_j^2 \vartheta_\nu^{-2} + \alpha + (m-\mu)] n^{\mu-\gamma-2} \vartheta_j^{-2\alpha-\mu-1} \vartheta_\nu^{\gamma-2m+2\mu+2\alpha-1} \quad \left(\frac{3}{2}\vartheta_j \leqq \vartheta_\nu \leqq \frac{\pi}{2}\right) \tag{4.10}$$

$$\varphi_\nu^\mu(x;\ \gamma) = O(1) n^{\mu-\gamma-2} \vartheta_j^{-2\alpha-\mu-1} (\pi - \vartheta_\nu)^{\gamma+2\beta+1} \quad \left(\frac{3}{2}\vartheta_j \leqq \vartheta_\nu \quad \textit{und} \quad \frac{\pi}{2} \leqq \vartheta_\nu < \pi\right). \tag{4.11}$$

Beweis. (4.8—11) folgen aus (4.7) und den Hilfssätzen 4 und 6, wobei für (4.8) $(n\vartheta_\lambda)^{-1} = O(1)$ $(\lambda = j, \nu)$ und für (4.10—11) die Beziehung $(\vartheta_\nu^2 - \vartheta_j^2)^{-1} \leqq 9/5\, \vartheta_\nu^{-2}$ $(3\vartheta_j/2 \leqq \vartheta_\nu < \pi)$ zu beachten sind.

Analog zu [5], Hilfssatz 6 erhalten wir weiter

Hilfssatz 8. *Für* $n \to \infty$ *gelten bei vorgegebenem* $\vartheta \in [0, \pi/2]$

$$\sum_{\vartheta_\nu \leqq \vartheta_j/2} \vartheta_\nu^\sigma = O(n \vartheta_j^{\sigma+1}) \quad (\sigma > -1),$$

$$\sum_{\substack{\vartheta_j/2 \leqq \vartheta_\nu \leqq 3\vartheta_j/2 \\ \vartheta_\nu \neq \vartheta_j}} |\vartheta_j - \vartheta_\nu|^\sigma = O(n) \cdot \begin{cases} \vartheta_j^{\sigma+1} & (\sigma < -1) \\ \log(n\vartheta_j) & (\sigma = -1) \end{cases}$$

$$\sum_{\vartheta_\nu \geqq 3\vartheta_j/2} \vartheta_\nu^\sigma = O(n \vartheta_j^{\sigma+1}) = O(n^{-\sigma}) \quad (\sigma < -1),$$

$$\sum_{\vartheta_\nu \geqq 3\vartheta_j/2} \vartheta_\nu^\sigma = O(n) \cdot \begin{cases} 1 & (\sigma > -1) \\ \log n & (\sigma = -1). \end{cases}$$

b.α) $0 < \vartheta_\nu \leqq 3\vartheta_j/2$ $(\leqq 3\pi/4)$. In diesem Fall ist $\vartheta_\nu = O(\vartheta_j)$ für $n \to \infty$, und es folgt aus den Hilfssätzen 7, (4.8—9) und 8 (unter Beachtung von $(\vartheta_j n)^{-1} = O(1)$) für $n \to \infty$

$$\sum_{\substack{0 < \vartheta_\nu \leqq 3\vartheta_j/2 \\ \vartheta_\nu \notin \mathfrak{M}_n}} \varphi_\nu^\mu(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{\gamma-m} \quad (\mu = 0, 1, \ldots, m)$$

und damit nach (4.6) für $n \to \infty$

$$\sum_{\substack{0 < \vartheta_\nu \leqq 3\vartheta_j/2 \\ \vartheta_\nu \notin \mathfrak{M}_n}} \Phi_\nu^m(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{\gamma-m}. \tag{4.12}$$

b.β) $3\vartheta_j/2 \leqq \vartheta_\nu < \pi$. Mittels $(n/\vartheta_j)^\mu \vartheta_\nu^{2\mu-2m} = O(1)(n/\vartheta_j)^m$ ($\mu=0, 1, \ldots, m$) ergibt sich im Fall $3\vartheta_j/2 \leqq \vartheta_\nu \leqq \pi/2$ aus Hilfssatz 7, (4.10) für $\alpha \neq 0$ und $n \to \infty$

$$\varphi_\nu{}^\mu(x;\ \gamma) = O(1) n^{m-\gamma-2} \vartheta_j^{-2\alpha-m-1} \vartheta_\nu^{\gamma+2\alpha-1} \quad (\mu = 0, 1, \ldots, m, \alpha \neq 0)$$

Analog folgt für $\alpha=0$ und $n \to \infty$

$$\varphi_\nu{}^\mu(x;\ \gamma) = O(1) n^{m-\gamma-2} \vartheta_j^{-m+1} \vartheta_\nu^{\gamma-3} \quad (\mu = 0, 1, \ldots, m, \alpha = 0).$$

Mit diesen Abschätzungen und mit (4.6) und Hilfssatz 8 ergibt sich für $n \to \infty$

(4.13)
$$\sum_{3\vartheta_j/2 \leqq \vartheta_\nu \leqq \pi/2} \Phi_\nu{}^m(x;\gamma) = O(1) n^{m-\gamma-1} \vartheta_j^{-2\alpha-m-1} \cdot \begin{cases} 1 & (2\alpha+\gamma > 0 \text{ bzw. } \alpha = \gamma = 0) \\ \log n & (2\alpha+\gamma = 0, \alpha \neq 0) \\ \vartheta_j^{2\alpha+\gamma} & (-2 < 2\alpha+\gamma < 0). \end{cases}$$

Im Fall $\vartheta_\nu > \pi/2$ dagegen folgt aus Hilfssatz 7, (4.11)

$$\varphi_\nu{}^\mu(x;\ \gamma) = O(1) n^{m-\gamma-2} \vartheta_j^{-2\alpha-m-1} (\pi - \vartheta_\nu)^{\gamma+2\beta+1} \quad (\mu = 0, 1, \ldots, m),$$

womit sich im Fall $\gamma+2\beta \geqq -1$ wegen $0 < \pi - \vartheta_\nu \leqq \pi/2$ und im Fall $-2 < \gamma+2\beta < -1$ vermöge Hilfssatz 8

(4.14)
$$\sum_{\pi/2 < \vartheta_\nu < \pi} \Phi_\nu{}^m(x;\ \gamma) = O(1) n^{m-\gamma-1} \vartheta_j^{-2\alpha-m-1}$$

für $n \to \infty$ ergibt.

Zusammen ergeben die Teilabschätzungen (4.5), (4.12—14) für die durch (1.8) erklärte Summe $\mathscr{H}_n^m(x;\gamma)$ nach (4.1) für $n \to \infty$ und $0 \leqq x \leqq 1$ die Abschätzung

(4.15)
$$\mathscr{H}_n^m(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{\gamma-m} \cdot \begin{cases} \max\{1, n^{-1} \vartheta_j^{-2\alpha-\gamma-1}\} & (2\alpha+\gamma > 0 \text{ bzw. } \alpha = \gamma = 0) \\ \max\{1, n^{-1} \vartheta_j^{-1} \log n\} & (2\alpha+\gamma = 0,\ \ \alpha \neq 0) \\ 1 & (-2 < 2\alpha+\gamma < 0). \end{cases}$$

Ebenso wie bei der Abschätzung von $\mathscr{L}_n^m(x;\gamma)$ in [5] läßt sich eine entsprechende Abschätzung auch für $-1 \leqq x \leqq 0$ bei Ersetzung von ϑ_j durch $(\pi - \vartheta_j)$ und von α durch β bzw. β durch α erhalten.

Im besonderen folgt aus (4.15)

Satz 1. *Bei vorgegebenem* $\alpha \geqq -1$ *und* $m \geqq 0$ *gilt für* $n \to \infty$

a) $$\mathscr{H}_n^m(x;\ \gamma) = O(1)$$

mit

1) $\gamma = m$ *für* $0 \leqq x \leqq 1$, *falls* $m+2\alpha < 0$ *und* $m = \alpha = 0$ *ist,*
und für $0 \leqq x \leqq 1-\delta$ $(0 < \delta < 1/2)$, *falls* $m+2\alpha > 0$ *ist,*

und mit

2) $\gamma=2m+2\alpha$ *für* $0\leqq x\leqq 1$, *falls* $m+2\alpha>0$ *ist*
bzw.

b)
$$\mathscr{H}_n^m(x;\ \gamma)=O(\log n)$$
mit $\gamma=m$ *für* $0\leqq x\leqq 1$, *falls* $m+2\alpha=0$, $\alpha\neq 0$ *ist.*

Zusatz. *Ein entsprechender Satz gilt bei Ersetzung von* α *durch* β *in* $-1\leqq x\leqq 0$.

5. Abschätzung von $\mathfrak{H}_n^m(x;\gamma)$ mit $\omega_n(x)=P_n^{(\alpha,\beta)}(x)$ für $n\to\infty$

Die Abschätzung der durch (1.9) erklärten Summen

$$\mathfrak{H}_n^m(x;\ \gamma)=\sum_{\nu=1}^{n}\Psi_\nu^m(x;\ \gamma) \tag{5.1}$$
mit
$$\Psi_\nu^m(x;\ \gamma):=|D^m\mathfrak{h}_\nu(x)|(1-x_\nu^2)^{(\gamma-1)/2}n^{-\gamma+1}\qquad(\gamma\geqq 1) \tag{5.2}$$
erfolgt ebenso wie die von $\mathscr{H}_n^m(x;\gamma)$

a)
$$\vartheta_\nu\in\mathfrak{M}_n;\ 0\leqq\vartheta\leqq\frac{\pi}{2}.$$

Ebenso wie bei der Herleitung von (4.5), nur mit Hilfe von (2.9) statt (2.4), ergibt sich für $n\to\infty$
$$\sum_{\vartheta_\nu\in\mathfrak{M}_n}\Psi_\nu^m(x;\ \gamma)=O(1)n^{m-\gamma}\vartheta_j^{\gamma-m} \tag{5.3}$$

b) $\vartheta_\nu\notin\mathfrak{M}_n$; $0\leqq\vartheta\leqq\pi/2$. In Analogie zu (4.6) schätzen wir im Fall $\vartheta_\nu\notin\mathfrak{M}_n$ aufgrund von (2.6) mit (2.7) jeden Summanden $\Psi_\nu^m(x;\gamma)$ wie folgt weiter ab

$$\Psi_\nu^m(x;\ \gamma)\leqq\sum_{\mu=0}^{m}\psi_\nu^\mu(x;\ \gamma) \tag{5.4}$$
mit
$$\psi_\nu^\mu(x;\ \gamma):=\frac{m!(1-x_\nu^2)^{(\gamma-1)/2}n^{-\gamma+1}}{\mu![DP_n^{(\alpha,\beta)}(x_\nu)]^2}|x-x_\nu|^{\mu-m-1}|D^\mu[P_n^{(\alpha,\beta)}(x)]^2|. \tag{5.5}$$

Für das Verhalten des Summanden $\psi_\nu^\mu(x;\gamma)$ folgt in Analogie zur Herleitung der Hilfssätze 6 und 7 für $0\leqq\vartheta\leqq\pi/2$

(5.6)
$$\psi_\nu^\mu(x;\ \gamma)=O(1)\,n^{\mu-\gamma-1}\vartheta_j^{-2\alpha-\mu-1}|\vartheta_j^2-\vartheta_\nu^2|^{\mu-m-1}\begin{cases}\vartheta_\nu^{2\alpha+2+\gamma} & \left(0<\vartheta_\nu\leqq\dfrac{\pi}{2}\right)\\ (\pi-\vartheta_\nu)^{2\beta+2+\gamma} & \left(\dfrac{\pi}{2}\leqq\vartheta_\nu<\pi\right)\end{cases}$$

sowie

Hilfssatz 9. *Im Fall* $\vartheta_\nu \notin \mathfrak{M}_n$, $0 \leqq \vartheta \leqq \pi/2$ *gelten für* $n \to \infty$

(5.7) $$\psi_\nu^\mu(x;\ \gamma) = O(1) n^{m-1-\gamma} \vartheta_j^{\gamma-1-m} \quad \left(0 < \vartheta_\nu \leqq \frac{2}{3}\vartheta_j\right)$$

(5.8) $$\psi_\nu^\mu(x;\ \gamma) = O(1) n^{\mu-1-\gamma} \vartheta_j^{\gamma-m} |\vartheta_j - \vartheta_\nu|^{\mu-m-1} \quad \left(\frac{2}{3}\vartheta_j \leqq \vartheta_\nu \leqq \frac{3}{2}\vartheta_j;\ \vartheta_\nu \notin \mathfrak{M}_n\right)$$

(5.9) $$\psi_\nu^\mu(x;\ \gamma) = O(1) n^{\mu-1-\gamma} \vartheta_j^{-2\alpha-\mu-1} \vartheta_\nu^{2\mu-2m+\gamma+2\alpha} \quad \left(\frac{3}{2}\vartheta_j \leqq \vartheta_\nu \leqq \frac{\pi}{2}\right)$$

(5.10)
$$\psi_\nu^\mu(x;\ \gamma) = O(1) n^{\mu-1-\gamma} \vartheta_j^{-2\alpha-\mu-1} (\pi - \vartheta_\nu)^{\gamma+2\beta+2} \quad \left(\frac{3}{2}\vartheta_j \leqq \vartheta_\nu \quad \textit{und} \quad \frac{\pi}{2} < \vartheta_\nu < \pi\right)$$

$$(\mu = 0, 1, \ldots, m)$$

b.α) $0 < \vartheta_\nu < 3\vartheta_j/2$ $(\leqq 3\pi/4)$. Mit $\vartheta_\nu = O(\vartheta_j)$ für $n \to \infty$ folgt nun mit Hilfe der Hifssätze 9, (5.7—8) und 8 (analog zur Herleitung von (4.12))

(5.11) $$\sum_{\substack{0 < \vartheta_\nu \leqq 3\vartheta_j/2 \\ \vartheta_\nu \notin \mathfrak{M}_n}} \Psi_\nu^m(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{\gamma-m} \log(n\vartheta_j).$$

b.β) $3\vartheta_j/2 \leqq \vartheta_\nu < \pi$. Ebenfalls mittels $(n/\vartheta_j)^\mu \vartheta_\nu^{2\mu-2m} = O(1)(n/\vartheta_j)^m$ $(\mu = 0, 1, \ldots, m)$ ergibt sich aus (5.9) für $n \to \infty$

$$\psi_\nu^\mu(x;\ \gamma) = O(1) n^{m-1-\gamma} \vartheta_j^{-m-2\alpha-1} \vartheta_\nu^{\gamma+2\alpha} \quad (\mu = 0, 1, \ldots, m)$$

und damit für $n \to \infty$

(5.12) $$\sum_{3\vartheta_j/2 < \vartheta_\nu \leqq \pi/2} \Psi_\nu^m(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{-2\alpha-m-1}$$

Im Fall $\vartheta_\nu > \pi/2$ folgt schließlich aus (5.10) entsprechend der Herleitung von (4.14) für $n \to \infty$

(5.13) $$\sum_{\pi/2 < \vartheta_\nu < \pi} \Psi_\nu^m(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{-2\alpha-m-1}.$$

Zusammen ergeben die Teilabschätzungen (5.3), (5.11—13) für $0 \leqq x \leqq 1$ und $n \to \infty$ die Abschätzung

(5.14) $$\mathfrak{H}_n^m(x;\ \gamma) = O(1) n^{m-\gamma} \vartheta_j^{\gamma-m} \max\{1, \log(n\vartheta_j), \vartheta_j^{-\gamma-2\alpha-1}\}.$$

Hieraus folgt

Satz 2. *Bei vorgegebenem* $\alpha > -1$ *und* $m \geqq 0$ *gilt für* $n \to \infty$.

a) $\mathfrak{H}_n^m(x;\gamma) = O(1)$ *mit* $\gamma = 2m + 2\alpha + 1$ *für* $0 \leqq x \leqq 1$ *und* $m \geqq 1$ *bzw.* $m = 0$, $\alpha > 0$

b) $\mathfrak{H}_n^m(x;\gamma)=O(\log n)$ *mit* $\gamma=m$ *für* $0\leqq x\leqq 1-\delta$ $(0<\delta<1/2)$ *und* $m\geqq 1$

c) $\mathfrak{H}_n^0(x;1)=O(n^{-1}\log n)$ *für* $0\leqq x\leqq 1-\delta$ $(0<\delta<1/2)$ *und* $\alpha>0$

$\mathfrak{H}_n^0(x;1)=O(\max\{n^{-1}\log n, n^{2\alpha}\})$ *für* $0\leqq x\leqq 1$ *und* $-1<\alpha\leqq 0$.

Zusatz. *Ein entsprechender Satz gilt bei Ersetzung von* α *durch* β *in* $-1\leqq x\leqq 0$.

6. Approximationsaussagen

Für die Abweichung des m-mal differenzierten Interpolationspolynoms von Hermite von $f^{(m)}$ gilt mit (1.1—4), (1.8—9), falls Q_{2n-1} ein Polynom vom Grad $\leqq 2n-1$ ist

(6.1) $$|H_n^{(m)}[f;\,x]-f^{(m)}(x)|\leqq |H_n^{(m)}[f;\,x]-Q_{2n-1}{}^{(m)}(x)|+|f^{(m)}(x)-Q_{2n-1}{}^{(m)}(x)|$$

und

(6.2) $$|H_n^{(m)}[f;\,x]-Q_{2n-1}{}^{(m)}(x)| = |H_n^{(m)}[f-Q_{2n-1};\,x]|\leqq$$
$$\leqq \sum_{\nu=1}^{n}|D^m h_\nu(x)|\left(\frac{\sqrt{1-x_\nu^2}}{n}\right)^{\gamma}\frac{|f(x_\nu)-Q_{2n-1}(x_\nu)|n^\gamma}{(\sqrt{1-x_\nu^2})^\gamma}+$$
$$+\sum_{\nu=1}^{n}|D^m \mathfrak{h}_\nu(x)|\left(\frac{\sqrt{1-x_\nu^2}}{n}\right)^{\gamma-1}\frac{|f'(x_\nu)-Q'_{2n-1}(x_\nu)|n^{\gamma-1}}{(\sqrt{1-x_\nu^2})^{\gamma-1}}$$
$$\leqq \mathscr{H}_n{}^m(x;\,\gamma)\max_{1\leqq\nu\leqq n}\frac{|f(x_\nu)-Q_{2n-1}(x_\nu)|n^\gamma}{(\sqrt{1-x_\nu^2})^\gamma}+\mathfrak{H}_n^m(x;\,\gamma)\max_{1\leqq\nu\leqq n}\frac{|f'(x_\nu)-Q'_{2n-1}(x_\nu)|n^{\gamma-1}}{(\sqrt{1-x_\nu^2})^{\gamma-1}}$$

(man beachte $x_\nu\neq\pm 1$, $\nu=1,\dots,n$). Für die weitere Abschätzung der Terme in (6.1—2) ziehen wir den Approximationssatz von Jackson—Timan für simultane Approximation heran: Für k-mal stetig differenzierbares f mit dem Stetigkeitsmodul $\omega(f^{(k)},\delta)$ existiert zu jedem $n\geqq k$ ein Polynom Q_n vom Grad $\leqq n$ mit der Eigenschaft

(6.3) $$|f^{(\varkappa)}(x)-Q_n^{(\varkappa)}(x)|\leqq c_\varkappa(\Delta_n(x))^{k-\varkappa}\omega(f^{(k)},\Delta_n(x));$$
$$\Delta_n(x)=\max\left\{\frac{\sqrt{1-x^2}}{n},\frac{1}{n^2}\right\},\quad x\in[-1,1]$$

$(\varkappa=0,1,\dots,k)$ mit (von f und n unabhängigen) positiven Konstante $c_\varkappa$ (vgl. z. Bsp. [11], [4], [7]).

In (6.1) ist $\varkappa=m(\leqq k)$ und in (6.2) $\varkappa=0$ und $\varkappa=1$ zu wählen. Gleichmäßige Konvergenz für die in Frage kommenden Intervalle I liegt dann vor, wenn die rechte Seite von (6.1) von der Ordnung $o(1)$ gleichmäßig bezüglich $x\in I$ ist. Aus den Sätzen 1 und 2 folgt sodann

Satz 3. *Es sei* f *in* $[-1,1]$ *mindestens m-mal stetig differenzierbar,* $H_n[f]$ *das zugehörige Hermitesche Interpolationspolynom mit Jacobischen Abszissen zu vorgege-*

benen Parameter $\alpha > -1$ *und* $\beta > -1$. *Die Folge* $H_n^{(m)}[f;\ x]$ *konvergiert für* $n \to \infty$ *gleichmäßig gegen* $f^{(m)}(x)$

a) *für* $-1+\delta \leqq x \leqq 1$ $(0<\delta<1/2)$ *im Fall*

i) $m \geqq 1$ *oder* $m=0$, $\alpha>0$ *unter der zusätzlichen Voraussetzung* $f^{(k)}$ *mit* $k=2m+[2\alpha+1]$ *ist für* $[-1, 1]$ *aus der Klasse* $\operatorname{lip}(2\alpha+1-[2\alpha+1])$*) *bzw. im Fall*

ii) $m=0$, $-1<\alpha\leqq 0$ *unter der zusätzlichen Voraussetzung* f' *stetig in* $[-1, 1]$

b) *für* $-1+\delta\leqq x\leqq 1-\delta$ $(0<\delta<1/2)$ *im Fall*

i) $m\geqq 1$ *unter der zusätzlichen Voraussetzung, daß* $f^{(m)}$ in $[-1, 1]$ *einer Dini-Lipschitzbedingung**) *genügt, bzw. im Fall*

ii) $m=0$, $\alpha>0$ *unter der zusätzlichen Voraussetzung* f' *stetig in* $[-1, 1]$.

Eine entsprechende Aussage gilt in $-1\leqq x\leqq 1-\delta$ *bei Vertauschung von* α *mit* β *und in* $-1\leqq x\leqq 1$ *bei Ersetzung von* α *durch* $\max(\alpha, \beta)$.

Zusatz a) *Im Fall* $m=0$, $-1<\alpha\leqq 0$ *folgt für die Konvergenzgeschwindigkeit* $H_n[f;\ x]-f(x)=o(\max\{n^{-1}\log n, n^{2\alpha}\})$ *(gleichmäßig für* $-1+\delta\leqq x\leqq 1$), *falls* f' *stetig in* $[-1, 1]$ *vorausgesetzt wird.*

Zusatz b) *Für* $-1<\alpha<0$ *konvergieren die verallgemeinerten Interpolationspolynome von Hermite—Fejér gleichmäßig für* $-1+\delta\leqq x\leqq 1$ $(0<\delta<1/2)$. *Für* $\alpha\geqq 0$ *liegt für* $-1+\delta\leqq x\leqq 1-\delta$ *gleichmäßige Konvergenz dieser Polynome vor.*

Bemerkungen **1.** Beim üblichen Abschätzungsverfahren (Lebesguefunktion und Jacksonsatz) erhalten wir statt (6.2) aus (4.15) und (5.14) im Fall $-1<\alpha\leqq 0$

$$(6.2') \quad \ldots \leqq \mathcal{H}_n^m(x;\ 0)n^{-\gamma}+\mathfrak{H}_n^m(x;\ 1)n^{-\gamma+1} = O(1)n^{2m-\gamma}+O(1)n^{2m+2\alpha+1-\gamma}.$$

Im Fall $-1<\alpha<-1/2$ erhalten wir hiermit gleichmäßige Konvergenz für $2m$-mal stetig differenzierbares f gegenüber $f^{(2m+[2\alpha+1])}\in\operatorname{lip}(2\alpha+1-[2\alpha+1])$ in Satz 3. Bezüglich des 1. Terms erhalten wir sogar eine Verbesserung um den Faktor $n^{2\alpha}$, falls $-1<\alpha<0$ gilt.

2. Der Vergleich der Konvergenzaussagen der $m(\geqq 1)$-mal differenzierten Interpolationspolynome von Lagrange und Hermite zeigt, daß für $-1<\alpha<1/2$ das m-mal differenzierte Hermitesche Interpolationspolynom in $-1+\delta\leqq x<1$ günstigere Abschätzungen liefert (es gilt $f^{(k)}\in\operatorname{lip} k'$ mit $k=2m+[2\alpha+1]$, $k'=2\alpha+1-[2\alpha+1]$ im Fall von Hermite und $k=2m+[\alpha+1/2]$, $k'=\alpha+1/2-[\alpha+1/2]$ im Fall von Lagrange).

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*) $g\in\operatorname{lip}\alpha :\Leftrightarrow \omega(g, \delta)=o(\delta^\alpha)$ $(0\leqq\alpha<1)$; g genügt der Dini—Lipschitz—Bedingung: $\Leftrightarrow$ $\Leftrightarrow \omega(g, \delta)=o((\log 1/\delta)^{-1})$ für $\delta\to 0$.

3. Mithilfe von (4.15) und (5.14) lassen sich wie in [15] die Approximationsaussagen von $H_n^{(m)}[f; x]$ für $x = \cos\vartheta(n)$, $\vartheta(n) = o(1)$ $(n \to \infty)$ verschärfen.

4. Sind die Strukturbedingungen der betrachteten Funktionen besser als in Satz 3 angegeben, so erhält man mithilfe von (4.15) und (5.14) sofort aus (6.2) und (6.3) sehr gute Abschätzungen über die Konvergenzgeschwindigkeit.

5. In den Arbeiten [1], [2] von Badkov betr. die Approximation von Funktionen durch Partialsummen der Jacobi—Fourier-Entwicklung werden ebenfalls feinere Methoden angewandt, die Approximationspolynome mit verbessertem Randverhalten benutzen.

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Embedding theorems and strong approximation

J. NÉMETH

Dedicated to Professor Károly Tandori on his 60th birthay

1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. We denote by $s_n = s_n(x) = s_n(f; x)$ the n-th partial sum of (1), and the usual supremum norm by $\|\cdot\|$. We define a class of functions in connection with the strong approximation:

$$S_p(\lambda) := \left\{f\colon \Big\| \sum_{n=0}^{\infty} \lambda_n |s_n - f|^p \Big\| < \infty\right\},$$

where $\lambda = \{\lambda_n\}$ is a monotonic sequence of positive numbers and $0 < p < \infty$.

It is well known that the classical de la Vallée Poussin means

$$\tau_n = \tau_n(f;\ x) := \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x), \quad n = 1, 2, \ldots$$

usually approximate the function f, in the supremum norm, better than the partial sums do. Thus it was reasonable that L. Leindler and A. Meir [2] introduced, in analogy to $S_p(\lambda)$, the following class of functions:

$$V_p(\lambda) := \left\{f\colon \Big\| \sum_{n=0}^{\infty} \lambda_n |\tau_n - f|^p \Big\| < \infty\right\}.$$

They proved the following result concerning the relation between $S_p(\lambda)$ and $V_p(\lambda)$.

Theorem ([2, Theorem 1]). *If $p \geqq 1$ and $\{\lambda_n\}$ is a monotonic sequence of positive numbers satisfying the restriction*

$$\lambda_n / \lambda_{2n} \leqq K \quad n = 1, 2, \ldots \tag{2}$$

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with fixed positive K, then

(3) $$S_p(\lambda) \subset V_p(\lambda)$$

holds.

Furthermore it is well known that in many cases the classical Fejér means

$$\sigma_n = \sigma_n(f; x) := \frac{1}{n+1} \sum_{k=0}^{n} s_k(x)$$

approximate the function better than the partial sums do, but worse than the τ_n means. Therefore we introduce a new class of functions in connection with the approximation by $\sigma_n(x)$:

$$F_p(\lambda) := \{f\colon \|\sum_{n=0}^{\infty} \lambda_n |\sigma_n - f|^p\| < \infty\}.$$

The aim of the present paper is to investigate some relations among this new class and the previous ones.

2. We shall establish the following results.

Theorem 1. *If $p > 1$ and $\{\lambda_n\}$ is a monotonic nonincerasing sequence of positive numbers, then*

(4) $$S_p(\lambda) \subset F_p(\lambda)$$

holds.

Our Remarks show that the assumptions $p > 1$ and $\lambda_n \downarrow$, in certain sense, are necessary. If we omit one or the other (4) does not hold any more.

Remark 1. *For every $p > 1$ there exist nondecreasing sequences $\{\lambda_n^*\}$ and $\{\lambda_n^{**}\}$ such that*

$$S_p(\lambda^*) \subset F_p(\lambda^*)$$

and

$(*)$ $$S_p(\lambda^{**}) \not\subset F_p(\lambda^{**}).$$

Remark 2. *For any $0 < p \leqq 1$ there exist a nonincreasing sequence $\{\lambda_n^*\}$ and a nondecreasing sequence $\{\lambda_n^{**}\}$ such that*

$$S_p(\lambda^*) \not\subset F_p(\lambda^*) \quad \textit{and} \quad S_p(\lambda^{**}) \not\subset F_p(\lambda^{**}).$$

The proof of these remarks is very elementary and simple, therefore we only present the suitable sequences $\{\lambda_n^*\}$ and $\{\lambda_n^{**}\}$, and the function f, furthermore we detail the proof of $(*)$.

Theorem 2. *If $0<p<\infty$ and $\{\lambda_n\}$ is a monotonic sequence of positive numbers satisfying* (2) *then*

$$F_p(\lambda)\subset V_p(\lambda) \tag{5}$$

holds.

Proof of Theorem 1. Since

$$\sum_{n=1}^{\infty}\lambda_n|\sigma_n-f|^p = \sum_{n=1}^{\infty}\lambda_n\left|\frac{1}{n+1}\sum_{k=0}^{n}(s_k-f)\right|^p \leqq \sum_{n=1}^{\infty}\lambda_n\frac{1}{n^p}\Big(\sum_{k=0}^{n}|s_k-f|\Big)^p = I_1, \tag{6}$$

we have to estimate I_1 from above. Using an inequality of Leindler (see inequality (8) of [1])

$$\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=1}^{n}a_k\Big)^p \leqq K_1\sum_{n=1}^{\infty}\lambda_n\left(\frac{a_n}{\lambda_n}\sum_{k=n}^{\infty}\lambda_k\right)^p \tag{7}$$

which holds for any $\lambda_n>0$, $a_n\geqq 0$ and $p\geqq 1$, we have

$$I_1 \leqq K_2\sum_{n=1}^{\infty}\frac{\lambda_n}{n^p}\left(\frac{\sum_{k=1}^{\infty}\frac{\lambda_k}{k^p}}{\frac{\lambda_n}{n^p}}\right)^p|s_n-f| \leqq K_3\sum_{n=1}^{\infty}\lambda_n|s_n-f|^p, \tag{8}$$

where the last inequality follows from the fact that $\{\lambda_n\}$ is monotonic nonincreasing sequence and that $p>1$. Inequalities (6) and (8) clearly imply (4), which proves Theorem 1.

The first part of Remark 1 instantly follows taking $\lambda_n^*=n^{(p-1)/2}$ (using inequality (7)). Now we prove the second part of Remark 1.

Let $\lambda_n^{**}=\dfrac{n^{p-1}}{\log^p(n+1)}$ and $f(x)=\sum_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$. We show that $f\in S_p(\lambda^{**})$ and $f\notin F_p(\lambda^{**})$. Since

$$\sum_{n=1}^{\infty}\lambda_n|s_n-f|^p = \sum_{n=1}^{\infty}\frac{n^{p-1}}{\log^p(n+1)}\left|\sum_{k=n+1}^{\infty}\frac{\cos kx}{k^2}\right|^p \leqq$$

$$\leqq \sum_{n=1}^{\infty}\frac{n^{p-1}}{\log^p(n+1)}\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)^p \leqq K\sum_{n=1}^{\infty}\frac{n^{p-1}}{\log^p(n+1)}\frac{1}{n^p} = \sum_{n=1}^{\infty}\frac{1}{n\log(n+1)}<\infty,$$

that is, $f\in S_p(\lambda^{**})$. Let $x=0$ and N be large enough. Then we get

$$\sum_{n=1}^{N}\lambda_n|\sigma_n(0)-f(0)|^p = \sum_{n=1}^{N}\frac{n^{p-1}}{\log^p(n+1)}\cdot\frac{1}{n^p}\left(\sum_{k=1}^{n}\sum_{l=k+1}^{\infty}\frac{1}{l^2}\right)^p \geqq$$

$$\geqq K\sum_{n=1}^{N}\frac{n^{p-1}}{\log^n(n+1)}\cdot\frac{1}{n^p}\log^p(n+1) = K\sum_{n=1}^{N}\frac{1}{n},$$

which gives that $f \notin F_p(\lambda^{**})$. Remark 2 can be obtained taking $\lambda_n^* = \frac{n^{p-1}}{(\log n)^{1+p}}$ and $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$; or $\lambda_n^{**} = \log n$ and $f(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{n^{1+1/p}(\log n)^{3/p}}$.

Proof of Theorem 2. If $0<p\leqq 1$, then using the trivial inequality

$$|a|^p \leqq |b|^p + |a+b|^p, \tag{9}$$

we get

$$\sum_{n=1}^{\infty} \lambda_n |\tau_n - f|^p = \sum_{n=1}^{\infty} \lambda_n \left| \frac{\sum_{k=n+1}^{2n} (s_k - f)}{n} \right|^p \leqq \tag{10}$$

$$\leqq \sum_{n=1}^{\infty} \lambda_n \left| \frac{\sum_{k=0}^{n} (s_k - f)}{n} \right|^p + \sum_{n=1}^{\infty} \lambda_n \left| \frac{\sum_{k=0}^{2n} (s_k - f)}{n} \right|^p = I_1.$$

Using condition (2) we obtain

$$I_1 \leqq \sum_{n=1}^{\infty} \lambda_n \left| \frac{\sum_{k=0}^{n} (s_k - f)}{n} \right|^p + K_1 \sum_{n=1}^{\infty} \lambda_{2n} \left| \frac{\sum_{k=0}^{2n} (s_k - f)}{2n} \right|^p \leqq \tag{11}$$

$$\leqq K_2 \sum_{n=1}^{\infty} \lambda_n \left| \frac{\sum_{k=0}^{n} (s_k - f)}{n} \right|^p \leqq K_3 \sum_{n=1}^{\infty} \lambda_n |\sigma_n - f|^p.$$

Estimates (10) and (11) give (5) in the case $0<p\leqq 1$.

In the case $p>1$ we use the inequality

$$|a|^p \leqq 2^{p-1}(|b|^p + |a+b|^p)$$

instead of (9) and we get

$$\sum_{n=1}^{\infty} \lambda_n |\tau_n - f|^p \leqq K \sum_{n=1}^{\infty} \lambda_n |\sigma_n - f|^p \tag{12}$$

similarly as before. Inequality (12) immediately gives assertion (5).

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On the comparison of multiplier processes in Banach spaces

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday, in friendship and high esteem

1. Introduction. This paper continues our previous investigations (cf. [3—5; 7]) on the comparison of (commutative) approximation processes in Banach spaces. Whereas the results of [3—5] were based upon rather restrictive global divisibility conditions, a local divisibility property was employed in [7] to estimate a given process in terms of the particular one of best approximation. The present paper now yields results on the general comparison of different processes, which in particular include classical inverse approximation theorems in the applications.

Since this paper, though essentially self-contained, may indeed be considered as a sequel to [7], we may be very brief concerning motivation for the approach and results. In fact, in [7] we followed the multiplier approach of [3—5] and employed global criteria for multipliers, based upon (radial) Riesz summability and corresponding global $BV_{j+1}[0, \infty]$-classes of functions. Section 2 now indicates how these concepts may be localized in order to formulate counterparts to those local conditions, important in the classical context of trigonometric analysis.

In Section 3 these localized concepts are then used to derive the general comparison Theorem 3.8. Here we are heavily influenced by work of H. S. Shapiro concerned with local divisibility within the Wiener ring of Fourier—Stieltjes transforms (see [13, Chapter 9], also Remark 3.11 for more detailed information). Indeed, standard "partition of unity" arguments are now available, even in the present abstract setting (cf. [2; 10; 12; 17] for similar arguments in the context of Besov spaces).

In Section 4 some first illustrating applications are given, emphasizing the unifying approach to the subject. In fact, we essentially confine ourselves to those concrete problems, already treated in [7], in order to point out the additional results now available.

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2. Local multipliers in Banach spaces. For a complex Hilbert space H let E be a (countably additive, selfadjoint, bounded linear) spectral measure in $\mathbf{R}^n$, the Euclidean n-space ($n\in\mathbf{N}$, the set of natural numbers) with inner product $\langle x, y\rangle := := \sum_{k=1}^{n} x_k y_k$ and norm $|x| := \langle x, x\rangle^{1/2}$. If $L^\infty(\mathbf{R}^n, E)$ is the space of complex-valued, E-essentially bounded functions, then for each $\tau\in L^\infty(\mathbf{R}^n, E)$ the integral

$$T(\tau) := \int_{\mathbf{R}^n} \tau(x)\, dE(x)$$

is a bounded linear operator of H into itself (for basic properties and further details see [9, pp. 900, 1930, 2186]).

For a given orthonormal structure (H, E) let X be a complex Banach space with norm $\|\cdot\|$ such that H and X are continuously embedded in some linear Hausdorff space (this hypothesis should be added in [5], see [17, p. 116]) and such that $H\cap X$ is dense in H and X, i.e.,

$$\overline{H\cap X}^{\|\cdot\|_H} = H, \quad \overline{H\cap X}^{\|\cdot\|} = X. \tag{2.1}$$

Then (cf. [5]) $\tau\in L^\infty(\mathbf{R}^n, E)$ is called a multiplier on X if for each $f\in H\cap X$

$$T(\tau)f := \int_{\mathbf{R}^n} \tau(x) dE(x) f\in H\cap X, \ \|T(\tau)f\| \leqq C\|f\| \tag{2.2}$$

(here and in the following C denotes a constant which may have different values at each occurrence). In view of (2.1, 2) the closure of $T(\tau)$ (represented by the same symbol) belongs to $[X]$, the space of bounded linear operators of X into itself. The set of all multipliers τ on X is denoted by $M=M(X)$, the corresponding set of multiplier operators $T(\tau)$ by $[X]_M$. With the natural vector operations, pointwise multiplication, and norm

$$\|\tau\|_M := \|T(\tau)\|_{[X]} := \sup\{\|T(\tau)f\| : f\in H\cap X, \ \|f\| \leqq 1\}$$

M is a commutative Banach algebra with unit, isometrically isomorphic (under T) to the subspace $[X]_M \subset [X]$.

To deal with multipliers, let us consider the Riesz factor ($u\in[0, \infty)$, $t\in(0, \infty)$, $j\in\mathbf{P} := \mathbf{N}\cup\{0\}$)

$$r_{j,t}(u) := \begin{cases} (1-u/t)^j, & 0 \leqq u \leqq t \\ 0, & u > t. \end{cases}$$

In the following $\mathscr{J}$ denotes an arbitrary index set. Moreover, $\alpha\circ\beta$ is the composition (in case it is defined) of the functions α and β: $(\alpha\circ\beta)(x) := \alpha(\beta(x))$, and α^{-1} the inverse function.

Definition 2.1. Let X be a Banach space satisfying (2.1) (with respect to a given orthonormal structure (H, E)) and consider a family $\psi:=\{\psi_\varrho\colon \varrho\in\mathcal{J}\}$ of functions $\psi_\varrho(x)$, defined on $\mathbf{R}^n$ with values in $[0, \infty)$. If $T(r_{j,t}\circ\psi_p)$ is X-measurable in $t>0$ (this condition should be added in [7], see [5]) and if for some $j\in\mathbf{P}$ the Riesz summability condition

$$(2.3)\qquad r_{j,t}\circ\psi_\varrho\in M \quad \text{with} \quad \|r_{j,t}\circ\psi_\varrho\|_M \leqq C<\infty$$

holds true, uniformly for $t>0$, $\varrho\in\mathcal{J}$, then X is called *R^j_ψ-bounded.*

Local multiplier criteria may then be derived in terms of the following classes of functions (see [11]).

Definition 2.2. For $0\leqq a<b\leqq\infty$ and $j\in\mathbf{P}$ *the space* $BV_{j+1}[a, b]$ is defined as the set of all complex-valued functions τ which are j-times differentiable on (a, b) such that $\tau^{(j)}$ is of bounded variation on each compact subinterval of (a, b) and

$$\int_{a+}^{b-} u^j\,|d\tau^{(j)}(u)|<\infty.$$

Obviously, $BV_{j+1}[c, d]\subset BV_{j+1}[a; b]$ for $0\leqq c\leqq a<b\leqq d\leqq\infty$. Moreover, $BV_{j+1}[a, b]$ is a Banach space under the norm

$$\|\tau\|_{BV_{j+1}[a,b]}:=\frac{1}{j!}\int_a^b u^j\,|d\tau^{(j)}(u)|+\sum_{k=0}^{j}\frac{1}{k!}\Big|\lim_{u\to b-} u^k\tau^{(k)}(u)\Big|.$$

Theorem 2.3. *Let X be R^j_ψ-bounded and τ a complex-valued function, defined on* $[0, \infty)$, *such that* $\tau\in BV_{j+1}[a, b]$ *for some* $0\leqq a<b\leqq\infty$. *Then for each* $\sigma_\varrho\in M(X)$ *satisfying*

$$(2.4)\qquad \sigma_\varrho(x)=0 \quad \textit{for} \quad x\in\mathbf{R}^n \quad \textit{with} \quad \psi_\varrho(x)\notin[a, b]$$

one has $\sigma_\varrho(\tau\circ\psi_\varrho)\in M(X)$. *In fact,*

$$(2.5)\qquad \|\sigma_\varrho(\tau\circ\psi_\varrho)\|_M\leqq C\|\sigma_\varrho\|_M\,\|\tau\|_{BV_{j+1}[a,b]}.$$

For a proof of this theorem as well as for further details concerning these localized concepts of BV_{j+1}-classes and multipliers see [11].

Remark 2.4. For $a=0$, $b=\infty$ condition (2.4) is empty so that the unit $\sigma_\varrho(x)=1$ for all $x\in\mathbf{R}^n$ is admissible. Thus Theorem 2.3 also includes our previous multiplier criterion $BV_{j+1}[0, \infty]\circ\psi\subset M(X)$, in particular, for every $\tau\in BV_{j+1}[0, \infty]$ (cf. Remark 2.8)

$$(2.6)\qquad \big\|T\big(\tau\circ(t\psi_\varrho)\big)\big\|_{[X]}=:\|\tau\circ(t\psi_\varrho)\|_M\leqq C\|\tau\|_{BV_{j+1}[0,\infty]},$$

uniformly for $t>0$, $\varrho\in\mathcal{J}$ (cf. [5; 15], also for fractional extensions).

To formulate some further results concerning $BV_{j+1}[a, b]$ (see [11; 16] for detailed proofs), let $C_{00}^{\infty}[0, \infty)$ be the set of realvalued functions on $[0, \infty)$, arbitrarily often differentiable with compact support (in notation: supp).

Proposition 2.5. *One has* $C_{00}^{\infty}[0, \infty)\subset BV_{j+1}[0, \infty]$. *Moreover, for* $\lambda\in C_{00}^{\infty}[0, \infty)$ *the family* $\{\lambda(tu)\colon t\in(0, \infty)\}$ *is continuous in t with respect to the topology of* $BV_{j+1}[0, \infty]$, *thus*

$$\lim_{s\to t}\|\lambda(su)-\lambda(tu)\|_{BV_{j+1}[0,\infty]}=0.$$

As an immediate consequence of (2.6) we conclude that for $\lambda\in C_{00}^{\infty}[0, \infty)$ and ψ, subject to (2.3), the family $\{\lambda(t\psi_{\varrho}(x))\colon t\in(0, \infty)\}$ is continuous in t with respect to the topology of M, thus for each $t\in(0, \infty)$

$$\lim_{s\to t}\left\|T(\lambda\circ(s\psi_{\varrho}))-T(\lambda\circ(t\psi_{\varrho}))\right\|_{[X]}=0, \tag{2.7}$$

uniformly for $\varrho\in\mathscr{I}$.

Theorem 2.6. *Consider families* $\{a_{\varrho}\}$, $\{b_{\varrho}\}$ *of numbers with* $0\leqq a_{\varrho}<b_{\varrho}\leqq\infty$ *for each* $\varrho\in\mathscr{I}$. *Suppose that the functions* $\tau_{\varrho}\in BV_{j+1}[a_{\varrho}, b_{\varrho}]$ *satisfy (cf. Definition 3.7)*

$$\sup_{\varrho\in\mathscr{I}}\|\tau_{\varrho}\|_{BV_{j+1}[a_{\varrho},b_{\varrho}]}<\infty, \tag{2.8}$$

$$\inf\{|\tau_{\varrho}(u)|\colon a_{\varrho}<u<b_{\varrho},\ \varrho\in\mathscr{I}\}>0. \tag{2.9}$$

Then $1/\tau_{\varrho}\in BV_{j+1}[a_{\varrho}, b_{\varrho}]$, *uniformly for* $\varrho\in\mathscr{I}$.

Remark 2.7. To illustrate condition (2.8), let $D^{(j)}$, $j\in\mathbf{N}$, be the set of realvalued, continuous, strictly increasing functions η on $[0, \infty)$ with $\eta(0)=0$, $\lim_{u\to\infty}\eta(u)=\infty$ which are $(j+1)$-times differentiable on $(0, \infty)$ such that

$$\text{(2.10)}\quad \begin{array}{ll}\text{(i)} & u^{k}|\eta^{(k+1)}(u)|\leqq C\eta'(u)\quad(0\leqq k\leqq j,\ u>0),\\ \text{(ii)} & \lim_{u\to 0+}u\eta'(u)=0.\end{array}$$

If $\varphi(\varrho)$ is a (real-valued) positive function on $\mathscr{I}$, then for every $\tau\in BV_{j+1}[a, b]$ and $\eta\in D^{(j)}$ the functions $\tau_{\varphi(\varrho)\eta}(u):=\tau(\varphi(\varrho)\eta(u))$ (of Hardy-type) belong to $BV_{j+1}[a_{\varrho}, b_{\varrho}]$ with $a_{\varrho}=\eta^{-1}(a/\varphi(\varrho))$, $b_{\varrho}=\eta^{-1}(b/\varphi(\varrho))$, and one has, uniformly for $\varrho\in\mathscr{I}$,

$$\|\tau_{\varphi(\varrho)\eta}\|_{BV_{j+1}[a_{\varrho},b_{\varrho}]}\leqq C\|\tau\|_{BV_{j+1}[a,b]}. \tag{2.11}$$

Remark 2.8. Obviously, $r_{j,1}\in BV_{j+1}[0, \infty]$, and therefore by (2.11) (take $\eta(u)=u$) $r_{j,t}\in BV_{j+1}[0, \infty]$, uniformly for $t>0$. Again by (2.11) it then follows that for every $\eta\in D^{(j)}$ and positive function $\varphi(\varrho)$ on $\mathscr{I}$

$$\|(r_{j,t})_{\varphi(\varrho)\eta}\|_{BV_{j+1}[0,\infty]}\leqq C\|r_{j,1}\|_{BV_{j+1}[0,\infty]},$$

uniformly for $t>0$, $\varrho\in\mathscr{I}$. In view of Remark 2.4 this implies that if X is R_{ψ}^{j}-bounded, then X is also $R_{\tilde{\psi}}^{j}$-bounded with $\tilde{\psi}_{\varrho}=\varphi(\varrho)(\eta\circ\psi_{\varrho})$.

3. General comparison theorems. Throughout X denotes an R_{ψ}^{j}-bounded Banach space.

Definition 3.1. A family $\{\tau_{\varrho}\}_{\varrho\in\mathscr{I}}$ of uniformly bounded multipliers is called *locally divisible (at the origin) of order* ψ if (cf. [7]) there exist some $\delta>0$ and a family $\{\theta_{\varrho}\}_{\varrho\in\mathscr{I}}$ of uniformly bounded multipliers such that

$$\tau_{\varrho}(x)=\psi_{\varrho}(x)\theta_{\varrho}(x) \quad \text{in case } \psi_{\varrho}(x)\leqq\delta. \tag{3.1}$$

If (3.1) holds true for all $x\in\mathbf{R}^{n}$, $\varrho\in\mathscr{I}$, then the family $\{\tau_{\varrho}\}$ is said to be *globally divisible*.

Proposition 3.2. *Local divisibility implies the global one of the same order.*

Proof. We proceed as in [7]. Let $\{\tau_{\varrho}\}$ satisfy (3.1) and $\lambda\in C_{00}^{\infty}[0,\infty)$ be such that $\lambda(t)=1$ for $0\leqq t\leqq\delta/2$ and $=0$ for $t\geqq\delta$. Since $1-\lambda(t)=0$ for $0\leqq t\leqq\delta/2$, the function $\sigma(t):=(1-\lambda(t))/t$ belongs to $BV_{j+1}[0,\infty]$. Thus $\{\sigma\circ\psi_{\varrho}\}$, $\{\lambda\circ\psi_{\varrho}\}\subset M$, uniformly for $\varrho\in\mathscr{I}$ (cf. (2.6)). Moreover, on $\mathbf{R}^{n}$

$$1-\lambda\circ\psi_{\varrho}=\psi_{\varrho}(\sigma\circ\psi_{\varrho}),\quad \tau_{\varrho}(\lambda\circ\psi_{\varrho})=\psi_{\varrho}\theta_{\varrho}(\lambda\circ\psi_{\varrho}),$$

and therefore $\tau_{\varrho}=\tau_{\varrho}(\lambda\circ\psi_{\varrho})+\tau_{\varrho}(1-\lambda\circ\psi_{\varrho})=\psi_{\varrho}[\theta_{\varrho}(\lambda\circ\psi_{\varrho})+\tau_{\varrho}(\sigma\circ\psi_{\varrho})]$. Hence the assertion follows since the terms in [...] are bounded in M, uniformly for $\varrho\in\mathscr{I}$.

Remark 3.3. Let τ be a function on $[0,\infty)$ satisfying $\{\tau\circ\psi_{\varrho}\}\subset M$, uniformly for $\varrho\in\mathscr{I}$. Let $\eta\in D^{(j)}$ be such that $\tau/\eta\in BV_{j+1}[0,\delta]$ for some $\delta>0$. If λ is given as in the previous proof, then again $\lambda\circ\psi_{\varrho}\in M$ and $(\lambda\circ\psi_{\varrho})(x)=0$ for all $x\in\mathbf{R}^{n}$ with $\psi_{\varrho}(x)>\delta$. Therefore $\theta_{\varrho}:=(\lambda\circ\psi_{\varrho})((\tau/\eta)\circ\psi_{\varrho})\in M$ by Theorem 2.3 with

$$\|\theta_{\varrho}\|_{M}\leqq C\|\lambda\circ\psi_{\varrho}\|_{M}\|\tau/\eta\|_{BV_{j+1}[0,\delta]}\leqq C\|\lambda\|_{BV_{j+1}[0,\infty]}\|\tau/\eta\|_{BV_{j+1}[0,\delta]},$$

uniformly for $\varrho\in\mathscr{I}$. But if $(\eta\circ\psi_{\varrho})(x)\leqq\eta(\delta/2)(=:\tilde{\delta})$, then

$$(\eta\circ\psi_{\varrho})(x)\theta_{\varrho}(x)=\lambda(\psi_{\varrho}(x))\tau(\psi_{\varrho}(x))=(\tau\circ\psi_{\varrho})(x)$$

so that the family $\{\tau\circ\psi_{\varrho}\}$ is locally divisible of order $\eta\circ\psi$ (cf. Remark 2.8). Thus, local BV_{j+1}-conditions (at the origin) ensure corresponding local (and therefore global) divisibility properties.

For $q>1$ let $p\in C_{00}^{\infty}[0,\infty)$ be such that (partition of unity)

$$0\leqq p(u)\leqq 1,\quad \operatorname{supp}(p)\subset[1,q],\quad \int_{0}^{\infty}p(u)\frac{du}{u}=1. \tag{3.2}$$

Since $\int_0^\infty p(us)u^{-1}du=1$ for every $s>0$, one has for the function

$$v(0):=1, \quad v(s):=\int_1^\infty p(us)\frac{du}{u}=\int_s^\infty p(u)\frac{du}{u} \tag{3.3}$$

that $v\in C_{00}^\infty[0,\infty)$ with $v(s)=1$ for $0\leqq s\leqq 1$ and $v(s)=0$ for $s\geqq q$.

Lemma 3.4. *For $s,t\in[0,\infty)$ there hold true the identities*

$$1-v(ts)=\int_0^t p(us)\frac{du}{u}, \tag{3.4}$$

$$p(s)=p(s)(1-v(qs)), \tag{3.5}$$

$$1-v(s)=\int_1^\infty [1-v(us)-p(us)]\frac{du}{u^2}. \tag{3.6}$$

Proof. (3.4,5) are immediate consequences of the definitions. Moreover,

$$\int_1^\infty [1-v(us)]\frac{du}{u^2}=\int_1^\infty \left[\left(\int_0^1+\int_1^u\right)p(rs)\frac{dr}{r}\right]\frac{du}{u^2}=$$

$$=\int_0^1 p(rs)\frac{dr}{r}+\int_1^\infty \left[\int_r^\infty \frac{du}{u^2}\right]p(rs)\frac{dr}{r}=1-v(s)+\int_1^\infty p(rs)\frac{dr}{r^2}.$$

Consider the operators $T(p\circ(t\psi_\varrho))$, $T(v\circ(t\psi_\varrho))$ which belong to $[X]_M$, uniformly for $t>0$, $\varrho\in\mathcal{J}$ (cf. (2.6)). By (2.7) terms like $T(p\circ(t\psi_\varrho))f$ are continuous in t with respect to the topology of X so that the following integrals are well-defined (in X).

Proposition 3.5. (a): *For each $f\in X$, $t>0$, $\varrho\in\mathcal{J}$*

$$\|T(p\circ(t\psi_\varrho))f\|\leqq C\|f-T(v\circ(qt\psi_\varrho))f\|, \tag{3.7}$$

$$\|f-T(v\circ\psi_\varrho)f\|\leqq\int_1^\infty [\|f-T(v\circ(u\psi_\varrho))f\|+\|T(p\circ(u\psi_\varrho))f\|]\frac{du}{u^2}, \tag{3.8}$$

$$\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2}\leqq C\int_1^\infty \|f-T(v\circ(u\psi_\varrho))f\|\frac{du}{u^2}. \tag{3.9}$$

(b): *If for each $f\in X$, $\varrho\in\mathcal{J}$ (theorem of Weierstrass-type)*

$$\lim_{t\to 0+}\|T(v\circ(t\psi_\varrho))f-f\|=0, \tag{3.10}$$

then one has additionally

$$\|f-T(v\circ(t\psi_\varrho))f\|\leqq\int_0^t \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u}. \tag{3.11}$$

Proof. In view of (2.6) assertion (3.7) is an immediate consequence of (3.5), whereas (3.8) follows by (3.6). Furthermore, (3.7) delivers

$$\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2} \leqq C\int_1^\infty \|f-T(v\circ(qu\psi_\varrho))f\|\frac{du}{u^2} =$$

$$= qC\int_q^\infty \|f-T(v\circ(u\psi_\varrho))f\|\frac{du}{u^2} \leqq qC\int_1^\infty \|f-T(v\circ(u\psi_\varrho))f\|\frac{du}{u^2},$$

thus (3.9). Concerning (3.11), the identity (3.4) implies that for $f\in X$, $0<\varepsilon<t$

$$f-T(v\circ(t\psi_\varrho))f = \int_\varepsilon^t T(p\circ(u\psi_\varrho))f\,\frac{du}{u}+[f-T(v\circ(\varepsilon\psi_\varrho))f].$$

In view of (3.10) this yields the assertion upon letting $\varepsilon\to 0+$.

The following result is to be compared with the Steckin-type estimate of [7] which now appears as an auxiliary result towards Theorem 3.8.

Theorem 3.6. *If $\{\tau_\varrho\}_{\varrho\in\mathscr{J}}$ is locally divisible of order ψ, then for each $f\in X$, $\varrho\in\mathscr{J}$*

$$\|T(\tau_\varrho)f\| \leqq C\int_1^\infty \|T(v\circ(u\psi_\varrho))f-f\|\frac{du}{u^2}. \tag{3.12}$$

Moreover, if (3.10) *holds true, then*

$$\|T(\tau_\varrho)f\| \leqq C\int_0^\infty \|T(p\circ(u\psi_\varrho))f\|\min\left\{1,\frac{1}{u}\right\}\frac{du}{u}. \tag{3.13}$$

Proof. Since $\chi(u):=up(u)=uv(u/q)p(u)$, $uv(u/q)\in C_{00}^\infty[0,\infty)\subset BV_{j+1}[0,\infty]$, one has the estimate (cf. (2.6))

$$\|T(\chi\circ(t\psi_\varrho))f\| \leqq C\|T(p\circ(t\psi_\varrho))f\| \tag{3.14}$$

as well as the identity (cf. (3.3))

$$\psi_\varrho(v\circ\psi_\varrho) = \int_1^\infty \psi_\varrho(p\circ(u\psi_\varrho))\frac{du}{u} = \int_1^\infty \chi\circ(u\psi_\varrho)\frac{du}{u^2},$$

the latter integral being absolutely convergent with respect to the topology of M (cf. (2.6, 7)). Consequently, since by Proposition 3.2 the family $\{\tau_\varrho\}$ is also globally divisible of order ψ, say $\tau_\varrho=\psi_\varrho\theta_\varrho$, one has the representation

$$\tau_\varrho(v\circ\psi_\varrho) = \Theta_\varrho\int_1^\infty \chi\circ(u\psi_\varrho)\frac{du}{u^2},$$

and therefore by (3.14)

$$\|T(\tau_\varrho)f\| \leqq \|T(\tau_\varrho)T(v\circ\psi_\varrho)f\| + \|T(\tau_\varrho)[f-T(v\circ\psi_\varrho)f]\| \leqq$$

$$\leqq C\Big[\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2} + \|f-T(v\circ\psi_\varrho)f\|\Big]. \tag{3.15}$$

Thus (3.12) follows by (3.8, 9). Finally, (3.13) is a consequence of (3.10, 11, 15) since

$$\|T(\tau_\varrho)f\| \leqq C\Big[\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2} + \int_0^1 \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u}\Big] =$$

$$= C\int_0^\infty \|T(p\circ(u\psi_\varrho))f\| \min\Big\{1, \frac{1}{u}\Big\}\frac{du}{u}.$$

To formulate the main result, let $\mathscr{A}$ be the set of functions α, continuously differentiable and positive on $(0, \infty)$ such that

$$\lim_{t\to 0+} \alpha(t) = 0, \quad \lim_{t\to\infty} \alpha(t) = \infty, \quad \alpha'(t) > 0 \quad (t > 0).$$

Obviously, $D^{(j)} \subset \mathscr{A}$ for every $j \in \mathbf{P}$. Moreover, if $\alpha, \beta \in \mathscr{A}$, then $\alpha\circ\beta, \alpha^{-1} \in \mathscr{A}$, too.

Definition 3.7. Let $\alpha \in \mathscr{A}$ and β be any function with $\alpha(t) < \beta(t)$ for each $t > 0$. A family $\{\sigma_t\}_{t>0}$ with $\sigma_t \in BV_{j+1}[\alpha(t), \beta(t)]$ is said to satisfy the *Tauberian condition of type* (α, β) if

$$\sup_{t>0} \|\sigma_t\|_{BV_{j+1}[\alpha(t),\beta(t)]} < \infty, \tag{3.16}$$

$$\inf\{|\sigma_t(u)| : \alpha(t) < u < \beta(t),\ t > 0\} > 0. \tag{3.17}$$

In view of Theorem 2.6 conditions (3.16, 17) are chosen in such a way that $1/\sigma_t \in BV_{j+1}[\alpha(t), \beta(t)]$, uniformly for $t > 0$.

Theorem 3.8. *Let $\gamma := \{\gamma_\varrho\}_{\varrho\in\mathscr{J}} \subset \mathscr{A}$ be such that X is (R^j_ψ-and) $R^j_{\gamma\circ\psi}$-bounded. Suppose that the family $\{\tau_\varrho\}_{\varrho\in\mathscr{J}}$ is locally divisible of order ψ and that the family $\{\sigma_t\}_{t>0}$ satisfies the Tauberian condition of type (α, β) such that for some $q > 1$*

$$\sup_{\varrho\in\mathscr{J}} \gamma_\varrho(q\delta_\varrho(t)) \leqq \beta(t) \quad (t > 0), \tag{3.18}$$

where $\delta_\varrho := \gamma_\varrho^{-1}\circ\alpha$. Let $\sigma_t\circ\gamma_\varrho\circ\psi_\varrho$ belong to $M(X)$ such that $\|T(\sigma_t\circ\gamma_\varrho\circ\psi_\varrho)f\|$ is measurable in t. If $\beta(t) = \infty$ for all $t > 0$ (i.e., (3.18) is trivial), then one has the comparison estimate ($f \in X$, $\varrho \in \mathscr{J}$)

$$\|T(\tau_\varrho)f\| \leqq C\int_0^{\delta_\varrho^{-1}(1)} \|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|\,\delta'_\varrho(u)\,du, \tag{3.19}$$

whereas in the general case $\beta(t)\leqq\infty$ *the additional assumption* (3.10) *implies*

$$\|T(\tau_\varrho)f\|\leqq C\int_0^\infty \|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|\min\{1,1/\delta_\varrho(u)\}\delta_\varrho'(u)\,du. \tag{3.20}$$

Proof. First of all, $\alpha,\gamma_\varrho\in\mathscr{A}$ imply $\gamma_\varrho^{-1},\delta_\varrho\in\mathscr{A}$ for each $\varrho\in\mathscr{J}$. Substituting $u=1/\delta_\varrho(t)$ it follows by (3.12, 13) that

$$\|T(\tau_\varrho)f\|\leqq C\int_0^{\delta_\varrho^{-1}(1)}\left\|T\left(v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f-f\right\|\delta_\varrho'(t)\,dt, \tag{3.21}$$

$$\|T(\tau_\varrho)f\|\leqq C\int_0^\infty\left\|T\left(p\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f\right\|\min\{1,1/\delta_\varrho(t)\}\delta_\varrho'(t)\,dt, \tag{3.22}$$

respectively. Let us first consider the case $\beta(t)=\infty$ for all $t>0$. Since X is R_ψ^j-bounded, the multipliers $1-v(\psi_\varrho(x)/\delta_\varrho(t))$ belong to M, uniformly for $t>0$, $\varrho\in\mathscr{J}$ (cf. (2.6)), and vanish (cf. (3.3)) for $(\gamma_\varrho\circ\psi_\varrho)(x)\leqq\alpha(t)$. Thus Theorem 2.3, 6 yield

$$\mu_{t,\varrho}:=\left[1-v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right]/\sigma_t\circ\gamma_\varrho\circ\psi_\varrho\in M,$$

$$\|\mu_{t,\varrho}\|_M\leqq C\left\|1-v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right\|_M\|1/\sigma_t\|_{BV_{j+1}[\alpha(t),\infty]}\leqq C,$$

since X is $R_{\gamma\circ\psi}^j$-bounded, too. Hence

$$\left\|f-T\left(v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f\right\|\leqq C\|T(\sigma_t\circ\gamma_\varrho\circ\psi_\varrho)f\| \tag{3.23}$$

which establishes (3.19) in view of (3.21). To prove (3.20), one has by (3.2, 18) that $p(\psi_\varrho(x)/\delta_\varrho(t))=0$ for $(\gamma_\varrho\circ\psi_\varrho)(x)\notin[\alpha(t),\beta(t)]$. Again Theorem 2.3, 6 yield

$$\mu_{t,\varrho}:=p\circ\frac{\psi_\varrho}{\delta_\varrho(t)}/\sigma_t\circ\gamma_\varrho\circ\psi_\varrho\in M$$

with $\|\mu_{t,\varrho}\|_M\leqq C$. Hence

$$\left\|T\left(p\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f\right\|\leqq C\|T(\sigma_t\circ\gamma_\varrho\circ\psi_\varrho)f\|,$$

giving (3.20) in view of (3.22).

Remark 3.9. Concerning the measurability of $\|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|$ with respect to u, assumed in Theorem 3.8, the proof indeed proceeds via the integrals on the right-hand side of (3.21, 22) (well-defined in view of (2.7)) plus a pointwise estimate of the integrands (cf. (3.23)). So, if the measurability of $\|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|$ cannot be assured in advance, one may replace the majorant $\|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|$ in the pointwise estimate (3.23) by some measurable one, e.g., by the monotone majorant $\sup\{\|T(\sigma_r\circ\gamma_\varrho\circ\psi_\varrho)f\|:\ r\geqq u\}$ (cf. [13, p. 219 ff], in particular the notion of a σ-modulus, which indeed generalizes the classical modulus of continuity (4.10)).

Remark 3.10. Obviously, (3.18) is satisfied if γ_ϱ is a homogeneous function (of some fixed positive degree) and $\tilde{q}\alpha(t)\leqq\beta(t)$ for some $\tilde{q}>1$. On the other hand, if, e.g., $\psi_\varrho(x)=\varrho^{-1}\log(1+|x|)$ and $\gamma_\varrho(t)=e^{\varrho t}-1$ so that $(\gamma_\varrho\circ\psi_\varrho)(x)=|x|$ (cf. (4.1)), then (3.18) reduces to $(1+\alpha(t))^q\leqq 1+\beta(t)$.

Remark 3.11. As already mentioned, the results of this section are extensions of corresponding ones known in the concrete situation of the (trigonometric) Fourier spectral measure (cf. Section 4). More specifically, the estimate (3.19) of Theorem 3.8 is to be compared with [1, Corollary 2.4], whereas (3.20) is related to [13, Theorem 9.4.4.5]. Of course, the present methods of proof need different tools (cf. Section 2), due to the abstract setting. Let us mention that one may now also formulate a counterpart to [13, Theorem 9.4.4.4], based upon local divisibility (at the origin) of two families of multipliers.

Without going into details, let us finally mention that, even in the present abstract frame, the sharpness of the estimates obtained may again be discussed along the lines outlined in [7] (see also the literature cited there).

4. Applications. Let us recall that the approach of Section 2 to a multiplier theory in abstract spaces subsumes many classical orthogonal expansions in the applications. Since this is already worked out in our previous papers (cf. [5; 15] and the literature cited there), we may here concentrate ourselves to a very important special situation, the (trigonometric) Fourier spectral measure over $\mathbf{R}^n$.

To this end, let X be one of the spaces $L^p(\mathbf{R}^n)$, $1\leqq p<\infty$, of functions f, pth power (Lebesgue) integrable over $\mathbf{R}^n$ with (finite) norm

$$\|f\|_p := \Big((2\pi)^{-n/2}\int_{\mathbf{R}^n}|f(x)|^p\,dx\Big)^{1/p}.$$

Let $\mathscr{F}$ be the Fourier—Plancherel transform on L^2 and $\mathscr{F}^{-1}$ the inverse transform. For a Borel measurable set $B\subset\mathbf{R}^n$ let $\mathscr{P}_B$ be the multiplication projection

$$(\mathscr{P}_B f)(x):=f(x)$$

for $x\in B$ and $=0$ for $x\notin B$. Then $E(B):=\mathscr{F}^{-1}\mathscr{P}_B\mathscr{F}$ is a spectral measure for $H=L^2$ (cf. [9, p. 1989]). Furthermore, for the spaces X mentioned above condition (2.1) is satisfied, and (2.2) coincides with the classical definition of Fourier multipliers $\tau\in M_p(\mathbf{R}^n):=M(L^p(\mathbf{R}^n))$ (cf. [14, p. 94]).

Concerning the Riesz summability condition (2.3) it is a classical result (cf. [14, p. 114]) that

(4.1) $$\psi_\varrho(x)=|x|,\quad \mathscr{J}=\{1\}\Rightarrow(2.3)\quad\text{for}\quad j>(n-1)|1/p-1/2|.$$

Other admissible choices of ψ_ϱ used in the following are based upon the fact that (in the Fourier spectral case) any surjective affine transformation A from $\mathbf{R}^n$ to $\mathbf{R}^m$ induces an isometry from $M_p(\mathbf{R}^m)$ to $M_p(\mathbf{R}^n)$ via $\sigma(Ax)$, $x\in\mathbf{R}^n$, $\sigma\in M_p(\mathbf{R}^m)$ (cf. [2; p. 15]). For example, take $m=1$, $0\neq\varrho\in\mathbf{R}^n$, and $Ax=\langle\varrho, x\rangle$. Then it follows by (4.1) (on $\mathbf{R}^1$) that

$$\psi_\varrho(x) = |\langle\varrho, x\rangle|, \quad \mathscr{J} = \mathbf{R}^n\setminus\{0\} \Rightarrow (2.3) \quad \text{for} \quad j > 0. \tag{4.2}$$

Note that in all these cases condition (3.10) is satisfied (theorem of Weierstrass-type).

In the following we revisit those applications, already mentioned in [7], and point out what kind of additional results are now available via the (localized) concepts of the previous sections.

4.1. Abel—Cartwright means. Let $\eta\in D^{(j)}$ for some $j>(n-1)|1/p-1/2|$ and $\varphi(t)>0$ for $t>0$. Consider the Abel—Cartwright means $W(\varphi(t)\eta)$, corresponding to the multiplier $w(\varphi(t)\eta(|x|))$, $w(u):=e^{-u}$. Since $w\in BV_{j+1}[0, \infty]$ for every $j\in\mathbf{P}$, the operators $W(\varphi(t)\eta)$ are well-defined in $[L^p(\mathbf{R}^n)]$ (cf. (2.6, 11), (4.1)). The results of Section 3 may now be used to compare means of different orders η.

Corollary 4.1. *Let $j>(n-1)|1/p-1/2|$ and $\eta_k\in D^{(j)}$, $k=1, 2$. Then for every $f\in L^p(\mathbf{R}^n)$, $t>0$*

$$\|W(\varphi(t)\eta_1)f-f\|_p \leqq C\varphi(t)\int_0^{1/\varphi(t)}\left\|W\left(\frac{\eta_2}{\eta_2(\eta_1^{-1}(u))}\right)f-f\right\|_p du. \tag{4.3}$$

Proof. Let $t\in\mathscr{J}=(0, \infty)$, $\tau_t(x)=1-w(\varphi(t)\eta_1(|x|))$. Since

$$(1-e^{-u})/u\in BV_{j+1}[0, \infty],$$

it follows that τ_t is globally divisible of order $\varphi(t)\eta_1(|x|)$ (cf. (2.6, 11), Remark 2.8, (4.1)). Setting $\sigma_s(u)=1-w(\eta_2(u)/\eta_2(\eta_1^{-1}(s)))$, one has $\sigma_s\in BV_{j+1}[0, \infty]$, uniformly for $s>0$ (cf. (2.11)), and $\sigma_s(u)\geqq 1-e^{-1}$ for $u\geqq\eta_1^{-1}(s)$. Hence it follows that $\{\sigma_s\}_{s>0}$ satisfies the Tauberian condition with $\alpha(s)=\eta_1^{-1}(s)$, $\beta(s)=\infty$. Moreover, for $\gamma_t(u)=$ $=\eta_1^{-1}(u/\varphi(t))$, thus $\delta_t(u)=\varphi(t)u$, one has $\sigma_s(\gamma_t(\varphi(t)\eta_1(|x|)))=\sigma_s(|x|)\in M$ (cf. (2.6), (4.1)). Therefore (3.19) implies (4.3) (note that the integrand depends continuously upon u, analogously to Proposition 2.5, (2.7)).

In particular, $\eta_\gamma(u)=\varphi(u)=u^\gamma$, $\gamma>0$, yields the standard Abel—Cartwright means $W_\gamma(t)(:=W(t^\gamma\eta_\gamma))$ which subsume for $\gamma=1$ the Abel—Poisson and for $\gamma=2$ the Gauss—Weierstrass means (cf. (4.12)). Corollary 4.1 then reduces to

Corollary 4.2. *For every $\gamma, \delta>0$ one has*

$$\|W_\gamma(t)f-f\|_p \leqq Ct^\gamma\int_t^\infty \|W_\delta(u)f-f\|_p u^{-\gamma-1}du. \tag{4.4}$$

Since $(1-\exp\{-u^\gamma\})/(1-\exp\{-u^\delta\})\in BV_{j+1}[0,\infty]$ for $0<\delta\leqq\gamma$ (cf. [15, p. 54 ff]), it follows by (2.6) that in these cases one has indeed the direct estimate (cf. [5])

$$\|W_\gamma(t)f-f\|_p\leqq C\|W_\delta(t)f-f\|_p, \tag{4.5}$$

which, of course, is stronger than (4.4).

On the other hand, concerning the sharpness of (4.4) for $\delta>\gamma$, it is shown in [6] that for each $0<\mu<1/2$, $0<\nu<1$ there exists an element $f_{\mu,\nu}$ such that for e.g. $\gamma=1$, $\delta=2$, $p=1$

$$\|W_2(t)f_{\mu,\nu}-f_{\mu,\nu}\|_1\begin{cases}=O(t^{2\mu})\\ \neq o(t^{2\mu})\end{cases}\quad(t\to 0+),$$

$$\limsup_{t\to 0+}\frac{\|W_1(t)f_{\mu,\nu}-f_{\mu,\nu}\|_1}{\|W_2(t)f_{\mu,\nu}-f_{\mu,\nu}\|_1}t^\nu>0. \tag{4.6}$$

Thus an estimate of type (4.5) is impossible for $\delta>\gamma$, even for nonsmooth elements.

4.2. Marchaud-type inequalities. For $h\in\mathbf{R}^n$ let symmetric differences of order $2r$, $r\in\mathbf{N}$, be given by

$$\Delta_h^{2r}f:=(\Delta_h^*)^r f,\quad (\Delta_h^* f)(x):=f(x+h)-2f(x)+f(x-h), \tag{4.7}$$

corresponding to the multipliers $(2(\cos\langle h,x\rangle-1))^r$. Let $S_{n-1}:=\{\omega\in\mathbf{R}^n\colon |\omega|=1\}$.

Corollary 4.3. *Given $r,s\in\mathbf{N}$ and $1\leqq p<\infty$, there exists a constant C such that for every $f\in L^p(\mathbf{R}^n)$, $\omega\in S_{n-1}$, $t>0$*

$$\|\Delta_{t\omega}^{2r}f\|_p\leqq\frac{C}{t}\int_0^\infty\|\Delta_{u\omega}^{2s}f\|_p\min\{1,(t/u)^{2r+1}\}\,du. \tag{4.8}$$

Proof. To apply Theorem 3.8, consider

$$d(u):=2(1-\cos u),\quad \sigma(u):=\int_0^1 d^s(uv)(1-v)\,dv. \tag{4.9}$$

Obviously, $d^s(u)=\sum_{j=0}^s a_{sj}\cos ju$ with $a_{s0}>0$, and therefore

$$\sigma(u)=\frac{a_{s0}}{2}+\sum_{j=1}^s a_{sj}\frac{1}{2}\frac{d(ju)}{(ju)^2}.$$

Now, $d(u)/u^2$ and consequently (cf. (2.11)) $d(ju)/(ju)^2$ belong to $BV_{j+1}[0,\infty]$ so that in view of $\lim_{u\to\infty}\sigma(u)=a_{s0}/2\neq 0$ there exists $a>0$ such that

$$\sigma(u)\geqq a_{s0}/4\neq 0\quad\text{for}\quad u\geqq a,\quad \sigma\in BV_{j+1}[a,\infty].$$

Again by (2.11) this implies that $\sigma_t(u):=\sigma(u/t)\in BV_{j+1}[at,\infty]$ with

$$\|\sigma_t\|_{BV_{j+1}[at,\infty]} \leqq C\|\sigma\|_{BV_{j+1}[a,\infty]} \leqq C\|\sigma\|_{BV_{j+1}[0,\infty]},$$

uniformly for $t>0$. Since also $|\sigma_t(u)|\geqq a_{s0}/4$ for $u\geqq\alpha(t):=at$, the family $\{\sigma_t\}_{t>0}$ satisfies the Tauberian condition with $\alpha(t)=at$ and $\beta(t)=\infty$.

Thus, in order to apply (3.19), set

$$(t,\omega)\in\mathcal{J}=(0,\infty)\times S_{n-1},\quad \psi_{t,\omega}(x)=[t|\langle\omega,x\rangle|/a]^{2r},$$
$$\gamma_{t,\omega}(u)=au^{1/2r}/t,\quad \tau_{t,\omega}(x)=d^r(t|\langle\omega,x\rangle|).$$

Since $d(u)/(u/a)^2$ and hence $[d(u)/(u/a)^2]^r$ belong to $BV_{j+1}[0,\infty]$, it follows that $\tau_{t,\omega}$ is globally divisible of order $\psi_{t,\omega}$ (cf. (2.6, 11), (4.2)). Moreover,

$$(\gamma_{t,\omega}\circ\psi_{t,\omega})(x)=|\langle\omega,x\rangle|,$$

and therefore $(\sigma_u\circ\gamma_{t,\omega}\circ\psi_{t,\omega})(x)=\sigma_u(|\langle\omega,x\rangle|)\in M_p$. Thus $X=L^p$ is R^j_ψ- and $R^j_{\gamma\circ\psi}$- bounded and

$$\|T(\sigma_u\circ\gamma_{t,\omega}\circ\psi_{t,\omega})f\|_p:=\Big\|\int_0^1 \Delta^{2s}_{v\omega/u}f(x)(1-v)\,dv\Big\|_p\leqq u\int_0^{1/u}\|\Delta^{2s}_{z\omega}f\|_p\,dz.$$

Moreover, $\delta_{t,\omega}(u)=\gamma^{-1}_{t,\omega}(au)=(tu)^{2r}$, $\delta'_{t,\omega}(u)=2rt^{2r}u^{2r-1}$, and $\delta^{-1}_{t,\omega}(1)=\gamma_{t,\omega}(1)=1/t$. Hence with (3.19)

$$\|\Delta^{2r}_{t\omega}f\|_p=:\|T(\tau_{t,\omega})f\|_p\leqq C\int_0^{1/t}u\int_0^{1/u}\|\Delta^{2s}_{z\omega}f\|_p\,dz\,2rt^{2r}u^{2r-1}\,du\leqq$$

$$\leqq Ct^{2r}\int_t^\infty v^{-2r-2}\Big(\int_0^t+\int_t^v\Big)\|\Delta^{2s}_{z\omega}f\|_p\,dz\,dv=$$

$$=Ct^{2r}\Big[\int_0^t\|\Delta^{2s}_{z\omega}f\|_p\,dz\int_t^\infty v^{-2r-2}dv+\int_t^\infty\|\Delta^{2s}_{z\omega}f\|_p\,dz\int_z^\infty v^{-2r-2}dv\Big]=$$

$$=\frac{c}{t}\int_0^t\|\Delta^{2s}_{z\omega}f\|_p\,dz+Ct^{2r}\int_t^\infty z^{-2r-1}\|\Delta^{2s}_{z\omega}f\|_p\,dz=$$

$$=\frac{c}{t}\int_0^\infty\|\Delta^{2s}_{z\omega}f\|_p\min\Big\{1,\Big(\frac{t}{z}\Big)^{2r+1}\Big\}\,dz.$$

Let the $2r$th modulus of continuity of $f\in L^p(\mathbf{R}^n)$ be defined for $r\in\mathbf{N}$, $h>0$ by (cf. (4.7))

(4.10) $$\omega_{2r}(h,f;\,L^p(\mathbf{R}^n)):=\sup\{\|\Delta^{2r}_{t\omega}f\|_p\colon\ \omega\in S_{n-1},\quad 0<t<h\}.$$

Then (4.8) implies the familiar Marchaud inequality (see also [1])

(4.11) $$\omega_{2r}(h,f;\,L^p(\mathbf{R}^n))\leqq Ch^{2r}\int_h^\infty\omega_{2s}(u,f;\,FL^p(\mathbf{R}^n))\,u^{-2r-1}\,du.$$

Indeed, for $\omega\in S_{n-1}$, $0<t\leqq h$

$$\|\Delta^{2r}_{t\omega}f\|_p \leqq C\Big[\omega_{2s}(t,f;L^p(\mathbf{R}^n))+t^{2r}\int_t^\infty \omega_{2s}(u,f;\ L^p(\mathbf{R}^n))u^{-2r-1}du\Big] \leqq$$
$$\leqq (2r+1)Ct^{2r}\int_t^\infty \omega_{2s}(u,f;\ L^p(\mathbf{R}^n))u^{-2r-1}du$$

so that (4.11) follows since $\omega_{2r}(t,f;\ L^p(\mathbf{R}^n))$ as well as the right-hand side are increasing functions of t.

4.3. A semidiscrete difference scheme for the heat equation. Let $n=1$. In order to approximate the exact solution of the heat equation $(x\in\mathbf{R},\ t>0)$

$$d/dt\,u(x,t)=d^2/dx^2\,u(x,t),\ \ u(x,0)=f(x)\in L^p(\mathbf{R}),$$

given by the Gauss—Weierstrass means (cf. Section 4.1)

$$W_2(t^{1/2})f(x)=(4\pi t)^{-1/2}\int_{-\infty}^{\infty} f(x-y)e^{-y^2/4t}dy, \tag{4.12}$$

consider the initial value problem for $h>0$

$$d/dt\,u_h(x,t)=h^{-2}[u_h(x+h,t)-2u_h(x,t)+u_h(x-h,t)],\quad u_h(x,0)=f(x).$$

This leads to the semidiscrete difference scheme (cf. [2, p. 69])

$$u_h(x,t)=D_h(t)f(x):=T(e^{-(t/h^2)d(hy)})f(x),$$

the function d being given by (4.9). Thus the multiplier $\varkappa_{h,t}(\in M_p(\mathbf{R})$, uniformly for $h,t>0)$ of the remainder $D_h(t)-W_2(t^{1/2})$ has the representation

$$\varkappa_{h,t}(x)=e^{-(t/h^2)d(hx)}-e^{-tx^2}\quad (h,\ t>0). \tag{4.13}$$

For example by the results obtained in [7] (see also [2, p. 72] for a concrete approach) it follows that for $t=h^2$

$$\|D_h(h^2)f-W_2(h)f\|_p\leqq C\omega_4(h,f;\ L^p(\mathbf{R})).$$

Theorem 3.8 now enables one to derive the following inverse estimate (cf. [2, p. 79]).

Corollary 4.4. *One has (cf.* (4.10) *with* $n=1$*)*

$$\omega_4(h,f;\ L^p(\mathbf{R}))\leqq C\int_0^\infty \|D_u(u^2)f-W_2(u)f\|_p\min\{1,(h/u)^4\}\frac{du}{u}. \tag{4.14}$$

Proof. With $h\in\mathscr{J}=(0,\infty)$, $\tau_h(x)=d^2(h|x|)$ (cf. (4.9)) it follows as in Section 4.2 that $\{\tau_h\}_{h>0}$ is globally divisible of order $(h|x|)^4$. Consider

$$\sigma(s):=e^{-d(s)}-e^{-s^2},\quad \sigma_u(s):=\sigma((2\pi-1)s/u).$$

Since $\sigma \in BV_{j+1}[2\pi-1, 2\pi+1]$ and for $|s-2\pi| \leqq 1$

$$\sigma(s) = e^{-d(s-2\pi)} - e^{-s^2} \geqq e^{-(s-2\pi)^2} - e^{-(2\pi-1)^2} \geqq e^{-1} - e^{-(2\pi-1)^2} > 0,$$

$\{\sigma_u\}_{u>0}$ satisfies the Tauberian condition with $\alpha(u)=u$, $\beta(u)=[(2\pi+1)/(2\pi-1)]u$ (cf. (2.11)). Moreover, for $\gamma_h(s)=s^{1/4}/h(2\pi-1)$, thus $\delta_h(s)=(sh(2\pi-1))^4$, condition (3.18) holds true for $q=[(2\pi+1)/(2\pi-1)]^4$, and one has (cf. (4.13))

$$(\sigma_u \circ \gamma_h \circ \psi_h)(x) = \sigma(|x|/u) = \varkappa_{u^{-1}, u^{-2}}(x) \in M_p(\mathbf{R}).$$

Therefore by (3.20)

$$\|\Delta_h^4 f\|_p \leqq C \int_0^\infty \|D_{u^{-1}}(u^{-2})f - W_2(u^{-1})f\|_p \min\{1, (uh(2\pi-1))^4\} \frac{du}{u}.$$

Substituting $1/u=z$, the result follows.

Let us finally mention that one may employ the analysis outlined in [8] in order to discuss the sharpness of (4.14) in a sense similar to (4.6).

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Об одной оценке ортогональных многочленов

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Профессору К. Тандори по случаю его 60-летия

Пусть

$$\sigma(e) = \int_e p(\Theta)\,d\Theta + \nu(e)$$

произвольная мера на $[0, 2\pi]$, имеющая бесконечное число точек роста. Здесь $p \in L_1[0, 2\pi]$, а ν-сумма скачков и сингулярной составляющей меры σ. Пусть

$$\Phi_n(z) = z^n + \dots$$

последовательность ортогональных по мере σ многочленов:

$$\int_0^{2\pi} \Phi_n(e^{i\Theta})\overline{\Phi_m(e^{i\Theta})}\,d\sigma(\Theta) = 0, \quad n \neq m.$$

Многочлены $\{\Phi_n(z)\}$ удовлетворяют рекуррентным соотношениям:

(1) $$\Phi_0 \equiv 1, \quad \Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n \Phi_n^*(z)$$

где $n = 0, 1, 2, \dots$ и

$$\Phi_n^*(z) = z^n \bar{\Phi}_n\left(\frac{1}{z}\right).$$

Числа $\{a_n\}_{n=0}^{\infty}$ удовлетворяют неравенствам:

(2) $$|a_n| < 1, \quad n = 0, 1, 2, \dots$$

и называются круговыми параметрами.

Обратно, если элементы последовательности $\{a_n\}_{n=0}^{\infty}$ удовлетворяют (2), то соотношения (1) определяют некоторую последовательность многочленов, которые будут ортогональными по некоторой мере σ, имеющей бесконечное число точек роста.

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Одна из задач теории ортогональных многочленов заключается в изучении их свойств по заданным круговым параметрам. Аналогичная задача возникает в отношении меры σ.

В монографии [1] Я. Л. Геронимус доказывает, что условие Г. Сегё

$$\int_0^{2\pi} \ln p(\Theta)\, d\Theta > -\infty \tag{3}$$

и неравенство

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty \tag{4}$$

эквивалентны. При выполнении неравенств (3), (4) нормировочные постоянные

$$\beta_n = \int_0^{2\pi} |\Phi_n(e^{i\theta})|^2 d\sigma(\theta)$$

удовлетворяют соотнощениям:

$$\beta_{n+1} \leqq \beta_n, \quad \lim_{n\to\infty} \beta_n = \beta > 0$$

В [1] доказывается также, что если

$$\sum_{n=0}^{\infty} |a_n| < \infty \tag{5}$$

то мера σ абсолютно непрерывна ($\nu=0$), $p(\Theta)>0$, $p\in C[0, 2\pi]$ и

$$\sup_n \|\Phi_n(e^{i\theta})\|_{C[0,2\pi]} < \infty$$

В работе [2] Г. Бакстер усилил этот результат, показав эквивалентность условия (5) и

$$p(\Theta) > 0, \quad p(\Theta)\in A[0, 2\pi].$$

Здесь A $[0, 2\pi]$-класс функций с абсолютно сходящимся рядом Фурье, а $C[0, 2\pi]$-непрерывные на всей оси, 2π-периодические функции с равномерной нормой. Укажем также на работу [3] в этом же направлении.

В настоящей работе мы продолжаем эти исследования. Пусть заданы круговые параметры $\{a_n\}_{n=0}^{\infty}$ и $|a_n|<1$,

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Положим

$$\varepsilon_{k,n}(\Theta) = \sum_{\nu=k}^{n} a_\nu e^{i\nu\Theta}, \quad 0 \leqq k \leqq n.$$

Справедлива

Теорема 1. *Для всех* $\Theta\in[0, 2\pi]$ *имеет место оценка:*

$$\left|\ln|\Phi_n(e^{i\theta})|\right| \leqq C_1+C_2\sum_{k=0}^{n}|a_k|\,|\varepsilon_{k,n}(\Theta)|$$

где постоянные C_1, C_2 *не зависят от* n *и* Θ.

Из теоремы 1 получается

Следствие 1. *Если круговые параметры действительны и*

$$a_n\downarrow 0 \quad \Big(\sum_{k=0}^{\infty}|a_k|^2<\infty\Big)$$

то мера σ *имеет вид:*

$$d\sigma = p(\Theta)d\Theta+\nu_0$$

где ν_0*-мера сосредоточенная в точке* 0.

Доказательство теоремы 1. Отправляясь от рекуррентного соотношения (1) легко получить равенство (см. [1])

$$\Phi_n^*(z) = \prod_{k=0}^{n}\left\{1-a_k z\frac{\Phi_k(z)}{\Phi_k^*(z)}\right\}.$$

Полагая $z=e^{i\theta}$ $(0\leqq\Theta\leqq 2\pi)$, получим

$$\Phi_n^*(e^{i\Phi}) = e^{in\theta}\overline{\Phi_n(e^{i\theta})}.$$

$$\overline{\Phi_n(e^{i\theta})}e^{in\theta} = \prod_{k=0}^{n-1}\left\{1-a_k e^{-i(k-1)\theta}\frac{\Phi_k(e^{i\theta})}{\overline{\Phi_k(e^{i\theta})}}\right\}.$$

Пусть

$$\Phi_k(e^{i\theta}) = |\Phi_k(e^{i\theta})|e^{i\lambda_k(\theta)}.$$

Функции $\lambda_k(\Theta)$ можно выбрать непрерывными на $[0, 2\pi]$. Все нули полинома $\Phi_k(z)$ лежат внутри единичного круга. Поэтому

$$\lambda_k(2\pi) = \lambda_k(0)+2\pi k.$$

Положим

$$g_k(\Theta) = k\Theta-\lambda_k(\Theta).$$

Тогда $g_k(\Theta)\in C[0, 2\pi]$ и $g_k(0)=g_k(2\pi)$. Имеем

$$|\Phi_n(e^{i\theta})|e^{ig_n(\theta)} = \prod_{k=0}^{n-1}\{1-a_k e^{-i(k-1)\theta}e^{2i\lambda_k(\theta)}\} = \prod_{k=0}^{n-1}\{1-a_k e^{i(k+1)\theta}e^{-2ig_k(\theta)}\}.$$

Из этого соотношения следует, что мы можем считать

(6) $$g_n(\Theta) = Im\sum_{k=0}^{n-1}\ln\{1-a_k e^{i(k+1)\theta}e^{-2ig_k(\theta)}\}$$

где

$$-\ln(1-z) = z+\frac{z^2}{2}+\ldots, \quad |z|<1.$$

Из (6)

$$|g_{n+1}(\Theta)-g_n(\Theta)| \leqq |\ln\{1-a_n e^{i(n+1)\Theta}e^{-2ig_n(\Theta)}\}| \leqq C_1|a_n|$$

где C_1 не зависит от n и Θ. Таким образом

(7) $$|g_{n+1}(\Theta)-g_n(\Theta)| \leqq c_1|a_n|.$$

Далее

$$\ln|\Phi_n(e^{i\Theta})| = \operatorname{Re}\sum_{k=0}^{n-1} a_k e^{i(k+1)\Theta}e^{-2ig_k(\Theta)}+O\Big(\sum_{k=0}^{n-1}|a_k|^2\Big)$$

где постоянная в O не зависит от n и Θ. Пусть $0\leqq k<n$, тогда

$$a_k e^{ik\Theta} = \varepsilon_{k,n}(\Theta)-\varepsilon_{k+1,n}(\Theta).$$

Имеем

$$\sum_{k=0}^{n-1}[\varepsilon_{k,n}-\varepsilon_{k+1,n}]e^{-2ig_k} = \sum_{k=k}^{n-1}\varepsilon_{k,n}[e^{-2ig_k}-e^{-2ig_{k-1}}]+\varepsilon_{0,n}-\varepsilon_{n,n}e^{-2ig_{n-1}}.$$

Отсюда, используя (7), получим

$$\big|\ln|\Phi_n(e^{i\Theta})|\big| \leqq C_2\sum_{k=1}^{n-1}|\varepsilon_{k,n}(\Theta)|\,|a_{k-1}|+|\varepsilon_{0,n}(\Theta)|+$$

$$+|\varepsilon_{n,n}(\Theta)|+C_3 \leqq C_4+C_5\sum_{k=0}^{n}|a_k|\,|\varepsilon_{k,n}(\Theta)|.$$

Теорема 1 доказана.

Доказательство Следствия 1. Имеем

$$|\varepsilon_{k,n}(\Theta)| \leqq A|a_k|\left(\left|\operatorname{ctg}\frac{\Theta}{2}\right|+1\right)$$

где A не зависит от n и Θ. По теореме 1

$$\big|\ln|\Phi_n(e^{i\Theta})|\big| \leqq C_1\left|\operatorname{ctg}\frac{\Theta}{2}\right|+C_2.$$

Отсюда

(8) $$\frac{1}{|\Phi_n^*(e^{i\Theta})|^2} \leqq e^{+2C_1|\operatorname{ctg}\Theta/2|+2C_2}.$$

Далее используем соотношение (см. [1])

$$\frac{\beta_n d\Theta}{|\Phi_n^*(e^{i\Theta})|^2} \to d\sigma(\Theta), \quad \beta_n = \|\Phi_n\|^2_{L_2,\sigma}$$

в смысле слабой сходимости мер. (Имеется в виду, что $v_n \to v$ в слабом смысле, если

$$\int_0^{2\pi} f dv_n \to \int_0^{2\pi} f dv$$

для любой $f \in C[0, 2\pi]$). Как указывалось выше, при выполнении условия Г. Сегё,

$$\lim_{n \to \infty} \beta_n = \beta, \quad 0 < \beta < \infty.$$

Поэтому в силу оценки (8) мера σ будет абсолютно непрерывна на интервале $(0, 2\pi)$. Следствие 1 доказано.

Замечание. Ясно, что утверждение следствия 1 остаётся справедливым, если $a_n \uparrow 0$. В обоих этих случаях на $[\delta, 2\pi - \delta]$ выполняются равномерные оценки:

$$\frac{1}{M_\delta} \leqq |\Phi_n(e^{i\theta})| \leqq M_\delta, \quad n = 0, 1, 2, \dots .$$

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Оценки для производных гармонических многочленов и сферических полиномов в L_p

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Введение

В работах А. Л. Шагиняна [1] и Е. Г. Гольштейна [2] были получены оценки производных гармонических многочленов в равномерной метрике и рассмотрены некоторые их приложения. В данной работе подобные оценки получены в L_p. Попутно доказываются некоторые оценки, относящиеся к полиномам

$$T_m(\mu) = \sum_{k=0}^{m} Y_k(\mu)$$

по сферическим гармоникам $Y_k(\mu)$ (μ обозначает точку на единичной сфере σ евклидова пространства $\mathbf{R}^n$). Основная трудность заключалась в доказательстве неравенства

$$\text{(1)} \qquad \|\operatorname{Grad} T_m\|_{L_p(\sigma)} \leqq \tau_p m \|T_m\|_{L_p(\sigma)}, \quad 1 \leqq p \leqq \infty.$$

В § 1 статьи приводятся обозначения, вспомогательные сведения и предложения. В § 2 дается оценка нормальной составляющей, в § 3 — тангенциальной составляющей градиента гармонического многочлена. Общие оценки производных гармонического многочлена получены в § 4, как следствие оценок § 2 и § 3. Заключительный § 5 посвящен оценкам простейшего псевдодифференциального оператора на сфере.

В последующей нашей работе будут получены неравенства разных метрик для гармонических многочленов и рассмотрены связанные с этими оценками вопросы теории приближений.

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§ 1. Вспомогательные сведения и предложения

Договоримся о следующих обозначениях: $\mathbf{R}^n$-евклидово пространство точек $x=(x_1, \ldots, x_n)$, $\sigma^{n-1}=\sigma$ — единичная сфера в $\mathbf{R}^n$ с центром в начале координат 0:

$$\sigma^{n-1} = \left\{x,\ x\in\mathbf{R},\ |x| = \sqrt{x_1^2+\ldots+x_n^2} = 1\right\}.$$

Точки на σ, как правило, будем обозначать через μ, μ', μ^0, μ^1 и т.д. $C(\sigma)=C$ — пространство непрерывных функций на σ с нормой

$$\|f\|_C = \max_{x\in\sigma} |f(x)|$$

$L_p(\sigma)=L_p$ — пространство суммируемых в степени p, $1\leqq p<\infty$, функций с нормой

$$\|f\|_{L_p} = \left\{\frac{1}{\Omega}\int_\sigma |f(\mu)|^p d\sigma\right\}^{1/p},$$

где $d\sigma$ — элемент объема на σ, а $\Omega=\Omega_{n-1}$ — объем сферы σ^{n-1} (при $p=\infty$ имеется ввиду обычная модификация). Пространство $L_p(\sigma)$ при $p=2$ превращается в гильбертово пространство $L_2(\sigma)$ со скалярным произведением

$$(f,\ g) = \frac{1}{\Omega}\int_\sigma f(\mu)\,g(\mu)\,d\sigma. \tag{2}$$

В дальнейшем символ X обозначает одно из пространств C или L_p, $1\leqq p<\infty$.

Для заданной на σ функции $f(\mu)$, $\operatorname{Grad} f$ обозначает градиент f, а Df — оператор Лапласа-Бельтрами от f. Оператор Лапласа-Бельтрами имеет в качестве собственных значений на сфере σ числа $\lambda_m=m(m+n-2)$, $m=0, 1, 2, \ldots$. Каждое из чисел λ_m является собственным значением конечной кратности, т. е. ему соответствует подпростанство из собственных функций размерности N_m

$$N_m = (2m+n-2)\,\frac{(m+n-3)!}{m!\,(n-2)!}.$$

Эти собственные функции именуются сферическими гармониками порядка m и обозначаются через $Y_m(\mu)$. Фиксированный ортонормированный базис в подпространстве сферических гармоник порядка m будем записывать в виде

$$Y_m^l(\mu),\quad l = 1, 2, \ldots, N_m.$$

В этих обозначениях произвольной суммируемой на сфере σ функции $f(\mu)$ можно поставить в соответствие ее ряд Лапласа

$$f(\mu) \sim \sum_{k=0}^{\infty} Y_k(f,\ \mu) = \sum_{k=0}^{\infty}\Big(\sum_{l=1}^{N_k} a_{k,l}Y_k^l(\mu)\Big) \tag{3}$$

где

(4)
$$a_{k,l} = (f, Y) = \frac{1}{\Omega} \int_\sigma f(\mu) Y_k^l(\mu) d\sigma,$$
$$Y_k(f, \mu) = \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_\sigma f(\mu^1) P_k^\lambda(\mu \cdot \mu^1) d\sigma(\mu^1), \quad \lambda = \frac{n-2}{2},$$

($\mu \cdot \mu^1$ обозначает скалярное произведение векторов 0μ, $0\mu^1$). Соответственно, произвольный многочлен $T_m(\mu)$ по сферическим гармоникам степени m запишется в виде

(5)
$$T_m(\mu) = \sum_{k=0}^{m} Y_k(\mu),$$

где T_m и Y_k связаны формулой (4). Многочлен $T_m(\mu)$ будем именовать сферическим многочленом степени m.

Напомним еще, что функции $P_k^\lambda(t)$, $-1 \leqq t \leqq 1$ в (4) представляют собой полиномы Гегенбауэра, возникшие в результате ортогонализации степеней $1, t, t^2, \ldots$ на отрезке $[-1, 1]$ с весом $(1-t^2)^{\lambda-1/2}$. Считается, что $P_k^\lambda(t)$ стандартизованы соотношением

$$P_k^\lambda(t) = \frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}, \quad 2\lambda = n-2.$$

(При $n=3$ речь идет о полиномах Лежандра: $P_k^{1/2}(t) = P_k(t)$.)

Перечисленные элементы гармонического анализа на сфере можно найти в любом из современных руководств, см., например, [4] (где полиномы P_k^λ обозначены через C_k^λ).

1.2. Сформулируем теперь результат, на который мы будем опираться. Он в существенном принадлемит Стейну [5] и Когбетлянцу [6] и в форме, эквивалентной нижеприведенной, содержится в статье Павелке [7].

Теорема А. *Пусть зафиксировано целое число $r > (n-2)/2$. Существует последовательность линейных ограниченных в X операторов $\{V_m^{(r)}\}_{m=1}^\infty$ со свойствами:*

1) *если $f \in X$, то $V_m^{(r)} f$ есть сферический полином степени не выше $(r+1)m-1$;*

2) *Найдется натуральное N, такое что если $m > N$ и T_m-сферический полином степени m, то*

(6)
$$V_m^{(r)} T_m(\mu) = T_m(\mu);$$

3) *Если для $f \in X$ ввести оператор $\sigma_k^{(r)}$ усреднения по Чезаро порядка r,*

определенный равенством

$$\sigma_k^{(r)}(f;\ x)=(A_k^r)^{-1}\sum_{j=0}^{k}A_{k-j}^{r}Y_j(f;\ x) \tag{7}$$

(где $A_i^r=\binom{r+i}{i}$ *— биномиальные коэффициенты), то*

$$V_m^r=a_{1,r}\sigma_{m-1}^{(r)}+a_{2,r}\sigma_{2m-1}^{(r)}+\ldots+a_{r+1,r}\sigma_{(r+1)m-1}^{(r)}, \tag{8}$$

где коэффициенты $a_{i,r}$ *зависят от* m *несущественно в том смысле, что* $|a_{i,r}|\leqq A$*;*

4) *Существуют постоянные* M_r *и* K_r*, не зависящие от* m*, такие что*

$$\|V_m^{(r)}f\|_X\leqq M_r\|f\|_X\quad (f\in X,\ m=1,2,\ldots) \tag{9}$$

$$\|f-V_m^{(r)}f\|_X\leqq K_rE_m(f)_X\quad (f\in X,\ m\geqq N) \tag{10}$$

где через $E_m(f)_X$ *обозначено наилучшее приближение функции* f *сферическими многочленами степени* m *по норме* X.

Отметим еще следующие моменты, связанные с теоремой А. Оператор $\sigma_m^{(r)}$ представляется сверткой

$$\sigma_m^{(r)}(f;\ x)=\frac{1}{\Omega}\int_\sigma K_m^r(\mu\cdot\mu^1)f(\mu^1)\,d\sigma(\mu^1)$$

с ядром

$$K_m^r(t)=(A_m^n)^{-1}\sum_{j=0}^{m}A_{m-j}^r\frac{j+\lambda}{\lambda}P_j^\lambda(t), \tag{11}$$

которое суммируемо по сфере σ, причем

$$\int_\sigma|K_m^r((\mu\cdot\mu^1)|\,d\sigma(\mu^1)\leqq k_r\quad\left(r>\lambda=\frac{n-2}{2},\ m=0,1,2,\ldots\right). \tag{12}$$

(Ясно, что написанный интеграл не зависит от μ.) С учетом сказанного неравенство (9) уточняется

$$\|V_m^{(r)}f\|_X\leqq(r+1)Ak_r\|f\|_X=M_r\|f\|_X. \tag{13}$$

1.3. В данном пункте будет доказана важная для дальнейшего вспомогательная оценка (неравенство (15)).

Обозначим через H_m подпространство сферических гармоник порядка m, а через $H_{\leqq m}$ — объединение подпространств H_k с $k\leqq m$: $H_{\leqq m}=\bigoplus_0^m H_k$. Для каждой фуксированной точки $\mu^0\in\sigma$ в H_k имеется единственная (с точностью до мультипликативной постоянной) сферическая гармоника, зависящая лишь от расстояния точки μ до μ^0 (или, по другому, от $\mu\cdot\mu^0$). Эта гармоника пропорциональна полиному $P_k^\lambda(\mu\cdot\mu^0)$. Линейная оболочка полиномов $P_k^\lambda(\mu\cdot\mu^0)$, $k=0,1,2,\ldots,$

..., m, образует подпростанство в $H_{\leqq m}$, которое мы обозначим через $Q_m(\mu^0)$. Объединение полиномов из $Q_m(\mu^0)$ при всевозможных $\mu^0 \in \sigma$ обозначим через Q_m ($\equiv$зональные полиномы степени $\leqq m$).

Лемма 1. *Пусть в линейном пространстве $H_{\leqq m}$ задан линейный коммутирующий со сдвигом оператор S, обладающий свойством*

(14) $$\|ST_m\|_{L_\infty} \leqq s_m \|T_m\|_{L_\infty} \quad \forall T_m \in H_{\leqq m}.$$

Тогда при $1 \leqq p \leqq \infty$ *справедливо неравенсто*

(15) $$\|SU_m\|_{L_p} \leqq s_m \|U_m\|_{L_p} \quad \forall U_m \in Q_m.$$

Доказательство. Поскольку при $p=\infty$ (15) является непосредственным следствием условия (14), можно считать в дальнейшем $1 \leqq p < \infty$. Возьмем произвольный элемент $U_m \in Q_m$ и рассмотрим свертку

(16) $$f(\mu) = \frac{1}{\Omega} \int\limits_\sigma U_m(\mu \cdot \mu') g(\mu') d\sigma(\mu'),$$

где $g \in L_{p'}$, $(1/p)+(1/p')=1$, $p \in [0, \infty)$. Ясно, что функция $f(\mu)$ в (16) — сферический многочлен из $H_{\leqq m}$. Применяя к (16) оператор S, можно, очевидно, написать

(17) $$Sf(\mu) = \frac{1}{\Omega} \int\limits_\sigma SU_m(\mu \cdot \mu') g(\mu') d\sigma(\mu').$$

В силу условия (14) леммы, имеем

$$\|Sf(\mu)\|_{L_\infty} \leqq s_m \|f\|_{L_\infty}.$$

Отсюда получим с помощью неравенстве Гёльдера

(18) $$|Sf(\mu)| \leqq \|Sf(\mu)\|_{L_\infty} \leqq s_m \|f\|_{L_\infty} \leqq$$
$$\leqq s_m \left(\frac{1}{\Omega} \int\limits_\sigma |U_m(\mu \cdot \mu')|^p d\sigma(\mu') \right)^{1/p} \|g\|_{L_{p'}} = s_m \|U_m\|_{L_p} \cdot \|g\|_{L_{p'}}.$$

С другой стороны, из (17) вытекает

(19) $$|Sf(\mu)| \leqq \|SU_m\|_{L_p} \|g\|_{L_{p'}}.$$

По свойствам, присущим неравенству Гёльдера, знак равенства в (19) при $1 < p < \infty$ достигается на элементе $g \in L_{p'}$, для которого $|g|^{p'}$ пропорционально $|SU_m|^p$. Выберем коэффициент пропорциональности так, чтобы $\|g\|_{L_{p'}}=1$, т. е. положим

(20) $$g(\mu') = \frac{|SU_m(\mu \cdot \mu')|^{p-1}}{\|SU_m(\mu \cdot \mu')\|_{L_p}^{p-1}} \cdot \operatorname{sign} SU_m(\mu \cdot \mu').$$

При таком выборе g равенство в (19) будет иметь место и при $p=1$ (причем норма определяемой соотношением (20) функции g в L_∞ равна 1). Таким образом при всех $1 \leqq p < \infty$ получим из (19) (при указанном выборе функции g)

$$|Sf(\mu)| = \|SU_m\|_{L_p}.$$

Но тогда из неравенства (19), поскольку $\|g\|_{L_{p'}} = 1$, вытекает

$$\|SU_m\|_{L_p} \leqq s_m \|U_m\|_{L_p}. \tag{21}$$

Следует помнить, конечно, что функции SU_m и U_m в (21) зависят от $\mu \cdot \mu'$ и при вычислении нормы интегрирование проводится по μ'. Однако, зависимость от μ исчезает при интегрировании, поскольку операторы S и E коммутируют с вращением. Таким образом лемма 1 доказана полностью.

§ 2. Оценка нормальной составляющей градиента гармонического полинома на сфере σ в метрике $L_p(\sigma)$, $1 \leqq p \leqq \infty$

Любой гармонический в $\mathbf{R}^n$ степени m полином w_m можно записать в виде

$$w_m(x) = \sum_{k=0}^{m} r^k Y_k\left(\frac{x}{|x|}\right), \tag{22}$$

где $r = |x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. Нормальная составляющая $\operatorname{grad} w_m$ запишется в виде $(\vec{n} = \vec{r})$

$$\left.\frac{\partial w}{\partial n}\right|_\sigma = \operatorname{grad} w_m|_\sigma \vec{n} = \left(\sum_{i=1}^{n} x_i \frac{\partial w_m}{\partial x_i}\right)_\sigma = \left.\frac{\partial w_m}{\partial r}\right|_{r=1}.$$

Следовательно, получим из (22)

$$\left.\frac{\partial w_m}{\partial r}\right|_{r=1} = \sum_{k=0}^{m} kY_k(\mu).$$

Теорема 1. *Существует постоянная v_p, не зависящая от w, такая что*

$$\Big\|\sum_{k=0}^{m} kY_k(\mu)\Big\|_{L_p} \leqq v_p m \Big\|\sum_{k=0}^{m} Y_k(\cdot)\Big\|_{L_p}, \quad 1 \leqq p \leqq \infty. \tag{23}$$

Замечание. Неравенство (23) получено Д. К. Угулавой в статье [3]. Однако соответствующее доказательство в [3] основывается на неопубликованном результатее автора. Мы воспроизводим здесь доказательство неравенства (23), опираясь на теорему А Стейна и др. (§ 1). Это доказательство само по себе несет в рамках данной статьи большую дополнительную нагрузку: препарируя его надлежащим образом, мы приходим к основной нашей оценке (1). Оценка (1) и (23) в совокупности решают задачи об оценке первых производных гармонического многочлена в L_p.

Доказательство. При $p=\infty$ это утверждение известно (см. [1], [2]). Воспользуемся этим. Именно, возьмем в качестве оператора S леммы 1 оператор

$$S\Big(\sum_{k=0}^{m} Y_k(\mu)\Big) = \sum_{k=0}^{m} kY_k(\mu). \tag{24}$$

Тогда из (23) (при $p=\infty$) и леммы 1 следует (см. (15)) неравенство

$$\|SU_m\|_{L_p} \leqq \nu_\infty m\| U_m\|_{L_p}, \quad 1 \leqq p \leqq \infty, \tag{25}$$

где U_m — полином порядка m по зональным гармоникам, а ν_∞ — постоянная, с которой выполняется неравенство (23) при $p=\infty$.

Далее, в силу утверждения 2) теоремы А существует оператор $V_m^{(r)}$, такой что $V_m^{(r)}T_m=T_m$ для $m>N$. Согласно (8) и (11) оператор $V_m^{(r)}$ представляется в виде свертки

$$V_m^{(r)} f = \frac{1}{\Omega} \int_\sigma G_m^{(r)}(\mu\cdot\mu') f(\mu')\, d\sigma(\mu'), \tag{26}$$

где (см. (8), (10), (11))

$$G_m^{(r)}(t) = a_{1,r} k_{m-1}^{(r)}(t) + a_{2,r} k_{2m-1}^{(r)} + \ldots + a_{r+1,r} k_{(r+1)n-1}^{(r)}. \tag{27}$$

Ядро $G_m^{(r)}(t)$ является полиномом (относительно $t=\mu\cdot\mu'$) по зональным гармоникам степени $\leqq[(r+1)m-1]$.

Применим к обеим частям равенства (26) с $f=T_m=\sum_{k=1}^{m} Y_k$ оператор S

$$S(V_m^{(r)} T_m) = \frac{1}{\Omega} \int_\sigma SG_m^{(r)}(\mu\cdot\mu') T_m(\mu')\, d\sigma. \tag{28}$$

Слева в (28) можно написать ST_m вместо $S(V_m^{(r)}T_m)$. Возьмем затем от обеих частей норму и справа воспользуемся неравенством Юнга. Получим

$$\begin{aligned} \|ST_m\|_{L_p} &\leqq \|SG_m^r(\mu\cdot\mu')\|_{L_1}\cdot\|T_m\| \leqq \nu_\infty[m(r+1)-1]\,\|G_m^r(\mu\cdot\mu'\|_{L_1}\cdot\|T_m\|_{L_p} \leqq \\ &\leqq \nu_\infty[m(r+1)-1](r+1)k_r\|T_m\|_{L_p} \leqq \nu_p\|T_m\|_{L_p}. \end{aligned} \tag{29}$$

Здесь на втором шаге использована оценка (25) для зонального многочлена $G_m^{(r)}$ степени $[m(r+1)-1]$, на третьем — представление (27) и оценка (12). Следует подчеркнуть еще, что норма $\|G_m^{(r)}(\mu\cdot\mu')\|_{L_p}$, взятая по переменному μ', не зависит от μ.

Таким образом, неравенство (23) доказано с константой

$$\nu_p \leqq \nu_\infty(r+1)^2 k_r.$$

§ 3. Оценка тангенциальной составляющей градиента гармонического полинома

Если w — гладкая функция в $\mathbf{R}^n$, то $\operatorname{grad} w$ на сфере σ можно разложить на две составляющие тангенциальную $\operatorname{grad}_\tau w|_\sigma$ и нормальную $\operatorname{grad}_n w$:

$$\operatorname{grad} w|_\sigma = \operatorname{grad}_\tau w|_\sigma + \operatorname{grad}_n w|_\sigma .$$

Очевидно, что

$$\operatorname{grad} w|_\sigma = \bar{r} \left.\frac{\partial w}{\partial r}\right|_\sigma .$$

Нормальная составляющая градиента гармонического многочлена w_m степени m оценена по норме L_p, $1 \leqq p \leqq \infty$, в предыдущем параграфе. В силу леммы 2 можно пытаться при оценке $\operatorname{grad}_\tau w_m$ воспользоваться той же схемой.

Лемма 2. *Пусть $w(x)$ — гармонический многочлен степени m и $\operatorname{grad}_\tau w|_\sigma$ — тангенциальная составляющая $\operatorname{grad} w$ на сфере σ. Тогда*

$$\|\operatorname{grad}_\tau w_m\|_{L_\infty(\sigma)} \leqq m \|w_m\|_{L_\infty(\sigma)} . \tag{30}$$

Доказательство. На сфере σ гармонический полином $w(x)$ превращается в сферический полином $T_m(\mu)$. Ясно, что составляющая $\operatorname{grad}_\tau w_m|_\sigma$ совпадает со сферическим градиентом $T_m(\mu)$

$$\operatorname{grad}_\tau w_m|_\sigma = \operatorname{Grad} T_m(\mu).$$

Зафиксируем точку $\mu^0 \in \sigma$. Проведем через нее по σ геодезическую окружность Γ в направлении $\operatorname{Grad} T_m(\mu^0)$. На окружности Γ полином $T_m(\mu)$ превращается в тригонометрический многочлен степени m, $\operatorname{Grad} T_m(\mu)$ совпадает с производной этого многочлена по дуге. Поэтому по неравенству Бернштейна

$$|\operatorname{Grad} T_m(\mu^0)| \leqq m \max_{\mu \in \Gamma} |T(\mu)|. \tag{31}$$

Заменяя в (31) $\max\limits_{\mu \in \Gamma} |T_m(\mu)|$ на $\max\limits_{\mu \in \sigma} |w_m(\mu)|$ и пользуясь произвольностью μ^0, получим

$$\|\operatorname{Grad} T_m\|_{L_\infty} \leqq m \|T_m\|_{L_\infty} . \tag{32}$$

Это неравенство записывается в виде (30) и лемма 2 доказана.

Наличие неравенства (30) не позвольяет, однако, непосредственно использовать лемму 1. Это вызвано тем, что рассматриваемый оператор $\operatorname{Grad} T_m$ — векторнозначен. Поэтому приходится несколько модифицировать рассуждения леммы 1. Будем пользоваться тем, что для любого поля единичных векторов $e(\mu)$ на сфере справедливо соотношение

$$|\operatorname{Grad} T_m(\mu) \cdot e(\mu)| \leqq |\operatorname{Grad} T_m(\mu)|.$$

Поэтому в силу (32)

$$\|\operatorname{Grad} T_\mu(\cdot)\cdot e(\cdot)\|_{L_\infty} \leqq m\|T_m\|_{L_\infty}. \tag{33}$$

Зафиксируем на сфере σ сферическую систему координат $\theta_1, \theta_2, \ldots, \theta_{n-2}, \varphi$, $0\leqq\theta_i\leqq\pi$, $i=1, \ldots, n-2$, $0\leqq\varphi<2\pi$. Тогда

$$\begin{aligned}
\mu_1 &= \cos\theta_1\\
\mu_2 &= \sin\theta_1\cos\theta_2\\
\mu_3 &= \sin\theta_1\sin\theta_2\cos\theta_3\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\mu_{n-1} &= \sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\varphi\\
\mu_n &= \sin\theta_1\sin\theta_2\ldots\sin\theta_{n-1}\sin\varphi
\end{aligned}$$

При этом с каждой точкой $\mu\in\sigma$ будет связан репер $e_{\theta_1}(\mu), e_{\theta_2}(\mu), \ldots, e_{\theta_{n-2}}(\mu)$, $e_\varphi(\mu)\equiv e_{\theta_{n-1}}$ ортонормированных векторов, направленных вдоль соответствующих координатных линий, проходящих через μ.

Действуя так же, как и при доказательстве леммы 1 (см. (16)), напишем

$$f(\mu) = \frac{1}{\Omega}\int_\sigma U_m(\mu\cdot\mu')\,g(\mu')\,d\sigma(\mu'). \tag{16'}$$

Рассмотрим проекцию $\operatorname{Grad} f(\mu)$ на один из базисных векторов $e_{\theta_j}(\mu)$. Очевидно, что

$$\operatorname{Grad} f(\mu)e_{\theta_j}(\mu) = \frac{1}{\Omega}\int_\sigma \operatorname{Grad} U(\mu\cdot\mu')e(\mu)g(\mu')\,d\sigma(\mu'). \tag{17'}$$

Отсюда с помощью (33) и неравенства Гёльдера получим (обозначив $\operatorname{Grad} f(\mu)\cdot e_{\theta_j}(\mu)=G_i(\mu)$)

$$\begin{aligned}
|G_i(\mu)| &\leqq \|G_i(\mu)\|_{L_\infty} \leqq m\,\|f(\mu)\|_{L_\infty}\\
&\leqq m\left(\frac{1}{\Omega}\int_\sigma |U_m(\mu\cdot\mu')|^p\,d\sigma(\mu')\right)^{1/p}\|g\|_{L_{p'}} = m\,\|U_m\|_{L_p}\|g\|_{L_p}.
\end{aligned} \tag{18'}$$

С другой стороны, из (17') вытекает

$$|G_i(\mu)| \leqq \left(\frac{1}{\Omega}\int_\sigma |\operatorname{Grad} U_m(\mu\cdot\mu')\cdot e_j(\mu)|^p\,d\sigma(\mu')\right)^{1/p}\|g\|_{L_{p'}}. \tag{19'}$$

Рассуждая, как и при доказательстве леммы 1, можно подобрать g так, что в (19') будет выполнятся равенство. При таком выборе g из (18'), (19') следует

$$\left\{\frac{1}{\Omega}\int_\sigma |\operatorname{Grad} U_m(\mu\cdot\mu')\cdot e_j(\mu)|^p\,d\sigma(\mu')\right\}^{1/p} \leqq m\|U_m\|_{L_p}, \quad j=1, \ldots, n-1. \tag{34}$$

Как известно, справедливо арифметическое неравенство

$$(35)\qquad \sum_{j=1}^{n-1} a_j^p \geqq c\Big(\sum_{j=1}^{n-1} a_j^2\Big)^{p/2}, \quad a_j \geqq 0, \quad p \geqq 1.$$

Поскольку

$$(36)\qquad \sum_{j=1}^{n-1} |\mathrm{Grad}\, U_m(\mu\cdot\mu')\cdot e_j(\mu)|^2 = |\mathrm{Grad}\, U_m(\mu\cdot\mu')|^2,$$

то из (34), (35) с учетом (36) следует неравенство

$$(37)\qquad \|\mathrm{Grad}\, U_m\|_{L_p} \leqq \tau_p^3 m \|U_m\|_{L_p}, \quad 1 \leqq p \leqq \infty.$$

Таким образом, доказано следующее утверждение:

Лемма 3. *Для зонального многочлена U_m степени $\leqq m$ справедливо неравенство типа Бернштейна* (37).

Теперь мы можем доказать интересующую нас оценку тангенциальной составляющей градиента гармонического полинома.

Теорема 2. *Пусть $w_m(x)$ — гармонический многочлен степени $\leqq m$. Тогда его сужение $T_m(\mu)$ на σ подчиняется оценке*

$$(38)\qquad \|\mathrm{Grad}\, T_m\|_{L_p} \leqq \tau_p m \|T_m\|_{L_p}, \quad 1 \leqq p \leqq \infty,$$

где постоянная $\tau_p > 0$ не зависит от m.

Замечание 1. Представляет интерес вычисление постоянной τ_p в (38). Согласно лемме 2 имеем $\tau_\infty = 1$. В неравнестве (37) постоянная τ_p^3 равна единице при $p=2, p=\infty$. Если бы удалось показать, что $\tau_p^3 = 1$ при всех $1 \leqq p \leqq \infty$, это открыло бы путь к получению хороших оценок для τ_p. Думается, не исключена возможность того, что $\tau_p = 1$.

Доказательство. Рассуждаем по схеме доказательства теоремы 1. Подставляя в (26) вместо f сферический полином T_m и применяя к обеим частям равенства оператор Grad, получим вместо (28)

$$\mathrm{Grad}\, T_m(\mu) = \frac{1}{\Omega} \int_\sigma \mathrm{Grad}_\mu G_m^{(r)}(\mu\cdot\mu') T_m(\mu')\, d\sigma(\mu').$$

Ясно, что

$$(39)\qquad |\mathrm{Grad}\, T_m(\mu)| \leqq \frac{1}{\Omega} \int_\sigma |\mathrm{Grad}_\mu G_m^{(r)}(\mu\cdot\mu')|\, |T_m(\mu')|\, d\sigma(\mu').$$

Возьмем от обеих частей неравенства L_p-норму и справа воспользуемся неравенством Юнга. По тем же соображениям, что и в выкладке (29), имеем

(40)

$$\|\operatorname{Grad} T_m\|_{L_p} \leqq \|\operatorname{Grad} G_m^{(r)}(\mu\cdot\mu')\|\cdot\|T_m\|_{L_p} \leqq \tau_1^3\,[m(r+1)-1]\,\|G_m^{(r)}(\mu\cdot\mu')\|_{L_1}\cdot$$
$$\cdot\|T_m\|_{L_p} \leqq \tau_1^3 k_r(r+1)^2 m\,\|T_m\|_{L_p}.$$

На втором шаге здесь мы воспользовались неравенством (37) при $p=1$. Неравенсто (38) (а следовательно и теорема 2) доказаны с постоянной $C_p \leqq \leqq \tau_1^3 k_r(r+1)^2$.

§ 4. Оценки производных гармонического многочлена

Мы подошли к основному результату данной статьи, который является следствием теорем 1 и 2:

Теорема 3. *Пусть $w_m(x)$ — произвольный гармонический многочлен степени $\leqq m$ от переменных $x_1, x_2, \ldots, x_n$. Тогда при $1 \leqq p \leqq \infty$ справедлива оценка*

(41) $$\|\operatorname{grad} w_m(x)\|_{L_p(\sigma)} \leqq c_p m \|w_m(x)\|_{L_p(\sigma)},$$

где постоянная C_p не зависит от m.

Из теоремы 3 можно извлечь ряд следствий.

Во многих вопросах приходится иметь дело не с $\operatorname{grad} w_m$, а с частными производными $\partial w_m/\partial x_j$, $j=1, \ldots, n$. Ясно, что из оценки (41) следует, что

(42) $$\left\|\frac{\partial w_m}{\partial x_j}\right\|_{L_p(\sigma)} \leqq c_p\|w_m\|_{L_p(\sigma)}, \quad j = 1, \ldots, n.$$

Более того, поскольку частные производные

$$D^\alpha w_m(x) = \frac{\partial^{|\alpha|} w_m(x)}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1+\ldots+\alpha_n,$$

сами являются гармоническими многочленами, из (42) по индукции получим

(43) $$\|D^\alpha w_m\|_{L_p(\sigma)} \leqq c_p^{|\alpha|}(m-|\alpha|)(m-|\alpha|+1)\ldots m\|w_m\|_{L_p(\sigma)} \leqq (c_p m)^{|\alpha|}\|w_m\|_{L_p(\sigma)}.$$

Оценки (41—43) можно обобщить на случай L_p-норм, вычисляемых по сфере σ_R произвольного радиуса R. Более того, можно перейти к L_p-нормам по шару $Ш_R$, ограниченному сферой σ_R. Это открывает путь для получения соответствующих оценок для произвольных гармонических многочленов в областях $\mathbf{R}^n$. На этом пути решаются некоторые вопросы теории приближений, которым посвящена наша статья, упомянутая во введении.

§ 5. О псевдодифференциальных операторах первого порядка на сфере. Оценки оператора Λ и его степеней

В предыдущих параграфах мы использовали дифференциальную операцию первого порядка на сфере — $\operatorname{Grad} f$. Эта операция имеет локальный характер (т. е. вычисления значения $\operatorname{Grad} f$ в точке μ требуется знать значения f в как угодно малой окрестности μ) и коммутирует с вращением. К сожалению она векторнозначна и это не всегда удобно.

Известно, что на сфере не существует скалярнозначных дифференциальных операторов первого порядка, коммутирующих с вращением. Единственным (с точностью до мультипликативной постоянной) дифференциальным оператором второго порядка, инвариантным относительно вращений, является оператор Лапласа—Бельтрами.

Рассмотрим некоторые псевдодифференциальные операторы первого порядка на σ, полезные в ряде вопросов. Прежде всего скажем еще несколько слов об операторе S из § 2.

Оператор S можно опеределить как мультипликаторный оператор, который функции

$$f(\mu) \sim \sum_{m=0}^{\infty} Y_m(f, \mu) \tag{44}$$

ставит в соотвествие функцию

$$Sf(\mu) \sim \sum_{m=0}^{\infty} mY_m(f, \mu). \tag{45}$$

Он возникает в результате таких действий. Сначала по функции f с разложением (45) строится гармоническая функция

$$u(\mu, r) \sim \sum_{m=0}^{\infty} r^m Y_m(f, \mu).$$

Вычисляя нормальную производную от u на сфере, получим

$$\left.\frac{\partial u}{\partial n}\right|_\sigma = \left.\frac{\partial u}{\partial r}\right|_\sigma = \sum_{m=0}^{\infty} mr^{m-1} Y_m(f, \mu)|_\sigma = Sf.$$

Мы видим, что оператор Sf связан с дифференциальной операцией первого поряадка — дифференцированием гармонического продолжения f по нормали. Ясно, что для вычисления значения Sf в точке $\mu \in \sigma$ требуется знать значения f на всей сфере.

Мультипликаторное определение S, выраженное соотношениями (44), (45), можно выразить равенством

$$Y_m(Sf, \mu) = mY_m(f, \mu), \quad m = 0, 1, 2, \ldots. \tag{46}$$

На этом языке оператор Лапласа—Бельтрами D характеризуется равенством

$$Y_m(Df, \mu) = m(m+n-2)Y_m(f, \mu),$$

а его степени

$$Y_m(D^\alpha f, \mu) = m^\alpha(m+n-2)^\alpha Y_m(f, \mu). \tag{47}$$

(Отметим, что определение (47) действует и при отрицательных α на функциях f с нулевым средним значением на сфере). При натуральных $\alpha=k$, $k=1, 2, \ldots$, равенство (47) определяет дифференциальный оператор D^k, для которого, конечно, выполняется свойство локальности. При нецелых $\alpha>0$ оператор D^α — псевдодифференциальный.

Рассмотрим, в частности, оператор $\Lambda=\sqrt{D}$. Из его определения

$$Y_m(\Lambda f, \mu) = \lambda_m Y_m(f, \mu), \quad \lambda_m = \sqrt{m(m+n-2)}$$

видно, что он близок оператору S в том смысле, что

$$\lambda_m \sim m \quad \text{при больших} \quad m.$$

Это обстоятельство позволяет перенести оценки, полученные в § 2 для оператора S, на оператор Λ. Именно, справедлива

Теорема 4. *Для любого сферического полинома $T_m(\mu)$ степени m справедлива оценка*

$$\|\Lambda T_m\|_{L_p} \leqq Cm\|T_m\|_{L_p}, \quad 1 \leqq p \leqq \infty,$$

где постоянная C не зависит от m.

Для доказательства теоремы 4 используем известный прием (см. [8], стр. 465), основанный на следующей лемме (в которой n обозначает размерность рассматриваемого пространства $\mathbf{R}^n$).

Лемма 4. *Пусть для функции $\omega(t)$, $t\leqq 0$, справедливо разложение по формуле Тейлора*

$$\omega(t) = 1+\omega'(0)t+\ldots+\frac{\omega^{(n-1)}(0)}{(n-1)!}t^{n-1}+\frac{\omega^{(n)}(\xi)}{n!}t^n, \quad 0<\xi<t, \tag{48}$$

причем $|\omega^{(n)}(t)|\leqq c_0$ при $0\leqq t\leqq n-2$, где c_0 — фиксированная постоянная. Тогда последовательность

$$\{\omega_m\}_{m=0}^\infty, \quad \omega_0 = 0, \quad \omega_m = \omega\left(\frac{n-2}{m}\right)$$

является p-мультипликатором рядов Лапласа, т. е. если f — произвольная функция из $L_p(\sigma)$, $1\leqq p\leqq\infty$, и

$$f(\mu) \sim \sum_{m=0}^\infty Y_m(f, \mu),$$

то выражение

$$\sum_{m=0}^{\infty} \omega_m Y_m(f, \mu)$$

представляет собой ряд Лапласа вполне определенной функции $g(\mu) \in L_p(\sigma)$ *такой, что*

$$\|g\|_{L_p(\sigma)} \leqq c\|f\|_{L_p(\sigma)}, \quad 1 \leqq p \leqq \infty,$$

где постоянная с не зависит от f.

Доказательство леммы. Подставим в (48) вместо t значение $(2\lambda)/m$ (где $\lambda=(n-2)/2$). Получим

$$\omega_m = 1+\frac{\omega'(0)2\lambda}{m}+\ldots+\frac{\omega^{(n-1)}(0)(2\lambda)^{n-1}}{(n-1)!}\cdot\frac{1}{m^{n-1}}+\frac{\omega^{(n)}(\xi)(2\lambda)^n}{n!}\cdot\frac{1}{m^n}.$$

Мы видим, что последовательность $\{w_m\}_1^\infty$ распадается в сумму последовательностей

$$\{1\}_1^\infty, \left\{\frac{c_1}{m}\right\}_1^\infty, \ldots, \left\{\frac{c_{n-1}}{m^{n-1}}\right\}_1^\infty, \left\{\frac{\omega^{(n)}(\xi)(2\lambda)^n}{n!}\cdot\frac{1}{m^n}\right\}_1^\infty, \tag{49}$$

каждая из которых является p-мультипликатором, $1 \leqq p < \infty$. Для последовательности из 1 это очевидно. Для последовательностей $\{c_j/m^j\}$, $j=1, \ldots, n-1$ подчеркнутое утверждение вытекает из результатов статьи [9]. Дело в том, что мультипликаторный оператор, соответствующий только что написанным последовательностям, изображается сферической сверткой, ядром которой является зональная функция

$$\sum_{m=1}^{\infty} \frac{m+\lambda}{\lambda} \frac{c_j}{m^j} P_m^\lambda(\cos\theta),$$

интегрируемая по σ (что и устанавливается в [9]). Таким образом, применение к упомянутой свертке неравенства Юнга решает вопрос при $j=1, \ldots, n-1$.

Осталось разобраться с действием последнего мультипликатора в (49). На этот раз дело сводится к грубым оценкам ядра

$$\sum_{m=1}^{\infty} \frac{\omega^{(n)}(\xi)(2\lambda)^n}{n!} \frac{1}{m^n} \frac{m+\lambda}{\lambda} P_m^\lambda(\cos\theta). \tag{50}$$

Легко проверяется, что написанный ряд сходится абсолютно и равномерно. В самом деле, в силу условий леммы, имеем: $|\omega^{(n)}(\xi)| \leqq c_0$ при $0 \leqq \xi \leqq n-2$ (это последнее неравенство выполнено при всех $m=1, 2, \ldots$). Кроме того (см. [4]),

$$|P_m^\lambda(\cos\theta)| \leqq P_m^\lambda(1)| = \binom{m+2\lambda-1}{m} = O(m^{2\lambda-1}).$$

Поэтому модуль общего члена ряда (50) мажорируется величиной $1/m^2$. Следовательно, сумма ряда (50) есть непрерывная зональная функция и в рассматриваемом случае речь идет об оценке сферической свертки, ядро которой непрерывно. В частности, такое ядро интегрируемо и дело опять сводится к применению неравенства Юнга. Лемма 4 доказана.

Доказательство теоремы 4. Запишем мультипликатор, соответствующий оператору Λ в виде

$$\sqrt{m(m+n-2)} = m\sqrt{1+\frac{n-2}{m}}, \quad m = 1, 2, \ldots.$$

Функция $\omega(t)=\sqrt{1+t}$ удовлетворяет условиям леммы 4. Поэтому последовательность

$$\omega_n = \sqrt{1+\frac{n-2}{m}}, \quad m = 1, 2, \ldots$$

задает ϱ-мультипликатор при $1 \leqq p \leqq \infty$. Обозначим соответсвующий этому мултипликатору оператор через Ω. Тогда имеем $\Lambda = S\Omega$ и следовательно

$$\|\Lambda T_m\|_{L_p} = \|S\Omega T_m\|_{L_p} \leqq c_m \|\Omega T_m\|_{L_p}. \tag{51}$$

На последнем шаге мы воспользовались тем, что мультипликаторный оператор Ω переводит T_m в сферический многочлен степени m и применили теорему 1. Осталось воспользоваться ограниченностью оператора Ω в $L_p(\sigma)$, чтобы из (51) получить неравенство, утверждаемое теоремой 4.

Замечание 2. Из теоремы 4 для натуральных степеней оператора Λ получаем неравенство

$$\|\Lambda^k T_m\|_{L_p(\sigma)} \leqq cm^k \|T_m\|_{L_p(\sigma)}, \quad k = 1, 2, \ldots. \tag{52}$$

При четных $k=2l$ неравенство (52) превращается в хорошо известное неравенство для степеней оператора D

$$\|D^l T_m\|_{L_p(\sigma)} \leqq cm^{2l} \|T_m\|_{L_p(\sigma)}, \quad l = 1, 2, \ldots.$$

Отметим, что в случае произвольных положительных k оценка (52) анонсирована в работе [10].

Замечание 3. По существу нашими рассуждениями доказано неравенство Бернштейна для оператора с мультипликатором $\{m\omega_m\}$, где ω_m берется из леммы 4.

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On the rate of approximation by orthogonal series

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Dedicated to Professor K. Tandori on his sixtieth birthday

1. Introduction

Let $\{\varphi_n(x)\}$ be a normalized system of orthogonal functions (ONS) with respect to the space $L^2[0, 1]$. We ask for additional conditions on coefficients $\{c_n\}$ with $\sum_{n=0}^{\infty} c_n^2 < \infty$ such that the partial sums $\{s_n(x)\}$ of the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x)$ are convergent to a limit function $f(x)$, uniquely a.e. determined by the Riesz—Fischer theorem, with a certain speed. K. TANDORI [10] proved the following basic result:

Theorem A. *Assume that $\{\lambda(n)\}$ is an increasing sequence tending to ∞. If $\sum_{n=2}^{\infty} c_n^2 \lambda^2(n) (\ln n)^2 < \infty$, then the estimate*

$$f(x) - s_n(x) = o_x\left(\frac{1}{\lambda(n)}\right) \quad a.e. \tag{1}$$

holds.

Asking for the finality of Theorem A as a consequence of a result of L. Leindler ([7], Hilfssatz 2) it follows that in case $\lambda(n+1) > C^* \lambda(n)$ $(C^* > 1)$ the factor $(\ln n)^2$ may be omitted. On the other side, for certain sequences increasing slowly enough, V. A. ANDRIENKO [2] proved the finality of Theorem A. Later on V. I. KOLYADA [6] proved the following result:

Theorem B. *Assume that the positive increasing sequence $\{\lambda(n)\}$ is such that*

$$\ln n = o(\lambda(n))$$

and that there exists a sequence $\{\nu_n\}$ with the properties:

$$\mu(n) = \nu_{n+1} - \nu_n \geqq 2, \quad 1 < \varrho \leqq \frac{\lambda(\nu_{n+1})}{\lambda(\nu_n)} \leqq r.$$

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If $\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) < \infty$, *then we have the estimate*

$$f(x) - s_n(x) = o_x\left(\frac{\ln \mu(q_{n+1})}{\lambda(n+1)}\right) \quad a.e.;$$

where q_n *is defined with the aid of the strictly increasing function* $v(t)$ *with* $v(n) = v_n$ *and its inverse* $v^{-1}(t)$ *by* $q_n = [v^{-1}(n)]$.

V. I. Kolyada also proved in [6] the finality of Theorem B in the following way: the speed $\ln \mu(q_{n+1})/\lambda(n+1)$ may not be replaced by a speed $(A(n))^{-1}$ tending faster to zero, i.e. if $A(n) \cdot \ln \mu(q_{n+1})/\lambda(n+1) \to \infty$.

In this paper we want to establish a general condition for estimations of type (1), which is also necessary for a special class of coefficients $\{c_n\}$. In the following let $\{\lambda(n)\}$ be a nondecreasing sequence tending to infinity. We consider in dependence of a fixed chosen constant $q > 1$ the uniquely determined sequence of increasing natural numbers $\{\mu_k\}$ with

(2) $$\lambda(\mu_{k+1}) \geqq q \cdot \lambda(\mu_k) \quad \text{and} \quad \lambda(\mu_{k+1} - 1) < q \cdot \lambda(\mu_k) \quad (k = 0, 1, \ldots).$$

Theorem 1. *Let*

$$\sum_{k=1}^{\infty} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2 \lambda^2(n) (\ln (n - \mu_k + 2))^2 < \infty$$

be fulfilled. Then the estimation

(3) $$f(x) - s_n(x) = o_x\left(\frac{1}{\lambda(n+1)}\right) \quad a.e.$$

holds.

We can extend this statement to partial sums $\{s_{n_i}(x)\}$, where $\{n_i\}$ is an increasing sequence of natural numbers. With respect to the above considered sequence $\{\mu_n\}$, let $\mathrm{I}(k)$ be defined by

(4) $$n_{\mathrm{I}(k)-1} < \mu_k - 1 \leqq n_{\mathrm{I}(k)} \quad (k = 1, 2, \ldots).$$

Then $\mathrm{I}(k+1) - \mathrm{I}(k)$ indicates the number out of $\{n_i\}$ between $\mu_k - 1$ and $\mu_{k+1} - 1$. The above definition also admits the case $\mathrm{I}(k) = \mathrm{I}(k+1)$; therefore let $\{k_j\}$ denote the sequence of those numbers when $\mathrm{I}(k_j + 1) - \mathrm{I}(k_j) > 0$. Putting

(5) $$C_i = \Big\{ \sum_{n=n_{i-1}+1}^{n_i} c_n^2 \Big\}^{1/2} \quad (i = 0, 1, \ldots; \quad n_{-1} = -1)$$

we prove

Theorem 2. *Let*

$$\sum_{j=1}^{\infty} \sum_{i=\mathrm{I}(k_j)+1}^{\mathrm{I}(k_j+1)} C_i^2 \lambda^2(n_{i-1}+1) (\ln (i - \mathrm{I}(k_j) + 2))^2 < \infty$$

be fulfilled. Then for $\{s_{n_i}(x)\}$ *the estimation*

$$f(x)-s_{n_i}(x)=o_x\left(\frac{1}{\lambda(n_i+1)}\right)\quad a.e.$$

holds.

Theorem 3. *Let*

$$\sum_{n=1}^{\infty} c_n^2\lambda^2(n)<\infty$$

be fulfilled and let $\alpha(n)$ *be defined by* $\alpha(n)=\ln(n-\mu_k+2)$ *if* $\mu_k\leqq n<\mu_{k+1}$. *Then the estimation*

$$f(x)-s_n(x)=o_x\left(\frac{\alpha(n)}{\lambda(n+1)}\right)\quad a.e.$$

holds.

It is possible to show that the conditions of these theorems are also necessary if the coefficients $\{c_n\}$ resp. $\{C_i\}$ are nonincreasing in a restricted sense. The following theorem is close to K. Tandori's theorem [9] on the necessity of the condition of coefficients in the Rademacher—Menchoff-theorem (cf. G. ALEXITS [1, p. 83]).

Theorem 4. *If* $c_n\geqq c_{n+1}$ *for* $\mu_k\leqq n\leqq\mu_{k+1}-2$; $k=1,2,\ldots$, *and*

$$\sum_{k=1}^{\infty}\sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2\lambda^2(n)\big(\ln(n-\mu_k+2)\big)^2=\infty,$$

then there exists an ONS $\{\varphi_n(x)\}$ *with*

$$\varlimsup_{n\to\infty}\lambda(n+1)|f(x)-s_n(x)|=\infty\quad(x\in[0,1]).\tag{6}$$

Remark. V. A. ANDRIENKO and L. V. G'RNEVSKA [3] have proved that $\sum_{n=0}^{\infty} c_n^2\lambda^2(n)<\infty$ implies estimation (3) if $\{\varphi_n(x)\}$ defines a convergence system (i.e. $\sum_{n=0}^{\infty} c_n^2<\infty$ implies the convergence a.e. of $\sum_{n=0}^{\infty} c_n\varphi_n(x)$); they further proved that in (3) $\{\lambda(n)\}$ must not be replaced by a sequence $\{\Lambda(n)\}$ tending faster to infinity. By Lemma 3 we can conclude that $\sum_{n=0}^{\infty} c_n^2\lambda^2(n)<\infty$ is also necessary, for in the case $\sum_{n=0}^{\infty} c_n^2\lambda^2(n)=\infty$ it always exists a convergence system such that estimation (3) fails.

In a way similiar to that of L. CSERNYÁK and L. LEINDLER [4] used to extend K. TANDORI's theorem [9] to subsequences $\{s_{n_i}(x)\}$, we can prove with terms (5)

Theorem 5. *If* $C_i \geqq C_{i+1}$ *for* $\mathrm{l}(k_j) \leqq i \leqq \mathrm{l}(k_j+1)-2$, $j=1,2,\ldots$, *and*

$$\sum_{j=1}^{\infty} \sum_{i=\mathrm{l}(k_j)}^{\mathrm{l}(k_{j+1})-1} C_i^2 \lambda^2(n_i+1)\big(\ln(i-l(k_j)+2)\big)^2 = \infty,$$

then there exists an ONS $\{\varphi_n(x)\}$ *with*

$$\overline{\lim_{i\to\infty}}\, \lambda(n_i+1)|f(x)-s_{n_i}(x)| = \infty \quad (x\in[0,1]).$$

Obviously Theorems 2 and 4 are generalizations of Theorems 1 and 3. But the result of Theorem 3 is a necessary step in the proof of Theorem 4 and the proof of Theorem 2 is based on Theorem 1. The finality of Theorem 3 follows finally with

Theorem 6. *If* $c_n \geqq c_{n+1}$ *for* $\mu_k \leqq n \leqq \mu_{k+1}-2$; $k=1,2,\ldots$, *and*

$$\sum_{n=1}^{\infty} c_n^2 \lambda^2(n) = \infty,$$

then there exists an ONS $\{\varphi_n(x)\}$ *with*

$$\overline{\lim_{n\to\infty}}\, \frac{\lambda(n+1)}{\alpha(n)} |f(x)-s_n(x)| = \infty \quad (x\in[0,1]).$$

2. Proof of Theorems 1, 2 and 3

The following result will be essential.

Lemma 1. *For any ONS* $\{\varphi_n(x)\}$ *the following estimation holds:*

$$\int_0^1 \Big\{\max_{1\leqq i\leqq j\leqq N} |c_i\varphi_i(x)+\ldots+c_j\varphi_j(x)|^2\Big\}dx \leqq K_1\Big(c_1^2+\sum_{n=2}^{N} c_n^2(\ln n)^2\Big).^{1)}$$

Proof: cf. K. Tandori [11, Satz VII]; see also A. Zygmund [13, p. 193].

Proof of Theorem 1. In the first step we prove the assertion for the partial sums $s_{\mu_k-1}(x)$, $k=1,2,\ldots$; namely by (2) we get

$$\sum_{k=1}^{\infty} \int_0^1 \lambda^2(\mu_k)\cdot(f(x)-s_{\mu_k-1}(x))^2 dx = \sum_{k=1}^{\infty} \lambda^2(\mu_k) \sum_{j=k}^{\infty} \sum_{n=\mu_j}^{\mu_{j+1}-1} c_n^2 =$$

$$= \sum_{j=1}^{\infty} \sum_{n=\mu_j}^{\mu_{j+1}-1} c_n^2 \sum_{k=1}^{j} \lambda^2(\mu_k) = O(1) \sum_{n=\mu_1}^{\infty} c_n^2\lambda^2(n) < \infty.$$

[1]) $K_1, K_2, \ldots$ denote absolute constants.

With the aid of B. Levi's theorem we conclude

$$f(x)-s_{\mu_k-1}(x) = o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e..} \tag{7}$$

For the remaining partial sums Lemma 1 leads us with

$$\delta_k(x) = \max_{\mu_k \leqq n < \mu_{k+1}-1} |s_n(x) - s_{\mu_{k+1}-1}(x)|$$

to

$$\sum_{k=1}^{\infty} \int_0^1 \lambda^2(\mu_k)\,\delta_k^2(x)\,dx \leqq K_2 \sum_{k=1}^{\infty} \lambda^2(\mu_k) \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2(\ln(n-\mu_k+2))^2 =$$

$$= O(1) \sum_{k=1}^{\infty} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2\lambda^2(n)\,(\ln(n-\mu_k+2))^2 < \infty.$$

This shows

$$s_n(x) - s_{\mu_{k+1}-1}(x) = o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e.} \quad (\mu_k \leqq n \leqq \mu_{k+1}-1);$$

by $\lambda(n+1) \leqq q \cdot \lambda(\mu_k)$ for $\mu_k \leqq n \leqq \mu_{k+1}-2$ (cf. (2)) together with (7) it follows

$$f(x)-s_n(x) = O(1)\{|f(x)-s_{\mu_{k+1}-1}(x)| + |s_{\mu_{k+1}-1}(x)-s_n(x)|\} = o_x\left(\frac{1}{\lambda(n+1)}\right) \quad \text{a.e.,}$$

thus Theorem 1 is proved.

Proof of Theorem 2. We represent $\{s_{n_i}(x)\}$ as (direct) partial sums $\{S_i(x)\}$ of an appropriate orthogonal series with coefficients (5); instead of $\{\lambda(n)\}$ the sequence $\{\Lambda(i)\}$ with $\Lambda(i)=\lambda(\mu_k)$ if $\mu_k \leqq n_i+1 < \mu_{k+1}$ is taken. Here with respect to (2) $\{\mathfrak{l}(k_j)\}$ assumes the role of $\{\mu_k\}$ (cf. (4)). Theorem 1 gives $f(x)-S_i(x)=f(x)-s_{n_i}(x)= =o_x\left(\frac{1}{\Lambda(i+1)}\right)$; noting that $\lambda(n_i+1)=O(\lambda(\mu_k))$ for $n_i+1<\mu_{k+1}$ the assertion follows immediately.

Proof of Theorem 3. By the proof of Theorem 1 it yields

$$f(x)-s_{\mu_k-1}(x) = o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e..} \tag{8}$$

Now, for the partial sums $s_n^*(x)$ of the series $\sum_{n=0}^{\infty} c_n^*\varphi_n(x)$ with $c_n^* = c_n \cdot (\ln(n-\mu_k+2))^{-1}$ if $\mu_k \leqq n < \mu_{k+1}$; $k=0, 1, \ldots,$ the proof of Theorem 1 has shown that

$$\hat{\delta}_k(x) = \max_{\mu_k \leqq n < \mu_{k+1}} |s_n^*(x) - s_{\mu_{k+1}-1}^*(x)| = o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e.}$$

and

$$\delta_k^*(x) = \max_{\mu_k \leqq n < \mu_{k+1}} |s_n^*(x) - s_{\mu_k-1}^*(x)| = o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e..}$$

By Abel's transformation (cf. G. ALEXITS [1; p. 68]) we get with $\mu_k \leqq n < \mu_{k+1}$

$$|s_n(x)-s_{\mu_k-1}(x)| = \Big|\sum_{\nu=\mu_k}^{n} c_\nu \varphi_\nu(x)\Big| = \Big|\sum_{\nu=\mu_k}^{n} \ln(\nu-\mu_k+2)c_\nu^* \varphi_\nu(x)\Big| =$$

$$= \Big|\sum_{\nu=\mu_k}^{n-1} \big(\ln(\nu-\mu_k+2)-\ln(\nu+1-\mu_k+2)\big)\big(s_\nu^*(x)-s_{\mu_k-1}^*(x)\big)+$$

$$+\ln(n-\mu_k+2)\big(s_n^*(x)-s_{\mu_k-1}^*(x)\big)\Big| \leqq 3\ln(n-\mu_k+2)\delta_k^*(x).$$

This proves $\lambda(\mu_k)(\ln(n-\mu_k+2))^{-1}(s_n(x)-s_{\mu_k-1}(x))\to 0$ a.e. $(n\to\infty)$, the assertion follows by (8) and (2).

3. Proof of Theorems 4, 5 and 6

To prove the necessity of the conditions stated in these theorems we need some auxiliary results. We use the following lemma of K. TANDORI [9] (cf. G. ALEXITS [1, p. 87]) which plays an important role in the proof of divergence phenomena of orthogonal series in general.

Lemma 2. *Let $\{a_n\}$ be a nonincreasing sequence of positive real numbers, and let $N_r = 2^{r+2}-4$, $r=0, 1, \dots$. Then, for every r, there exists a measurable set F_r with measure*

$$\mu(F_r) \geqq K_1^* \min\{1, N_{r+1}a_{N_{r+1}}^2(\ln N_{r+1})^2\} \quad (K_1^*>0),$$

and an ONS $\{\Phi_n(x)\}$ consisting of piecewise functions, such that

(a) *the sets $F_0, F_1, \dots$ are stochastically independent*[1]

(b) *for all $x\in F_r$ it exists a number $n_{r(x)} < 2^{r+2}$ such that $\Phi_{N_r}(x), \dots, \Phi_{N_r+n_{r(x)}}(x)$ are of the same sign and*

$$|\Phi_{N_r}(x)+\dots+\Phi_{N_r+n_{r(x)}}(x)| \geqq \frac{K_2^*}{a_{N_{r+1}}} \quad (K_2^*>0).$$

Remark. The proof of the lemma shows that F_r may be chosen as a simple set (i.e. consisting of a finite number of segments) and with the additional property: if $\Phi_0(x), \dots, \Phi_{N_{r+1}-1}(x)$ are constant in a segment I^*, then either $F_r \cap I^* = \emptyset$ or $I^* \subset F_r$.

To prove the necessity of the condition in Theorem 4 we first state

[1] $F_0, F_1, \dots$ are stochastically independent with respect to $[0, 1]$, if $k_1<k_2\dots<k_l$ then $\mu(F_{k_1}\cap F_{k_2}\cap \dots\cap F_{k_l}) = \mu(F_{k_1})\mu(F_{k_2})\dots\mu(F_{k_l})$.

Lemma 3. *Let $\{c_n\}$ be an arbitrary sequence of real numbers. If condition*

$$\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) = \infty$$

is fulfilled, then there exists an ONS $\{\varphi_n(x)\}$ consisting of piecewise constant functions which forms a convergence system with

$$\overline{\lim_{n\to\infty}}\, \lambda(n+1)|f(x)-s_n(x)| = \infty.$$

Proof. We can find a nonincreasing positive sequence $\{\varepsilon_n\}\to 0$ with

$$\sum_{n=1}^{\infty} c_n^2 \lambda^2(n)\, \varepsilon_n^2 = \infty. \tag{9}$$

We define the system of functions by induction. In the basic step we put with $r_1(x) = \operatorname{sign}(\sin 2\pi x)$ $(0 \leqq x \leqq 1)$

$$\varphi_0(x) = r_1(x).$$

Now let $\varphi_0(x), \ldots, \varphi_{m-1}(x)$ be defined. The segments where each of these functions are constant are denoted by $I_0^{(m)}, \ldots, I_{q_m}^{(m)}$ (with $\sum_{l=0}^{q_m} \mu(I_l^{(m)}) = 1$). Putting

$$\gamma(m) = \begin{cases} 1 & \text{if} \quad c_m = 0 \quad \text{or} \quad c_m^2 \lambda^2(m)\varepsilon_m^2 > 1, \\ c_m^2 \lambda^2(m)\varepsilon_m^2 & \text{elsewhere,} \end{cases}$$

we choose in each $I_l^{(m)} = (s_l^{(m)}, t_l^{(m)})$ a partial segment $J_l^{(m)} = (u_l^{(m)}, v_l^{(m)})$ with $\mu(J_l^{(m)}) = \gamma(m)\mu(I_l^{(m)})$ and with $u_l^{(m)} = s_l^{(m)}$. In general, for a segment $J = (u, v)$ and a function $f(x)$ defined on $[0, 1]$, let the denotation

$$f(J; x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right) & \text{if} \quad x \in J \\ 0 & \text{if} \quad x \notin J \end{cases} \tag{10}$$

be valid. Then we put

$$\varphi_m(x) = \frac{1}{\sqrt{\gamma(m)}} \sum_{l=0}^{q_m} r_1(J_l^{(m)}; x).$$

It is easy to verify that $\varphi_0(x), \ldots, \varphi_m(x)$ constitute a set of orthogonal and normalized functions.

The sets

$$G_m = \bigcup_{l=0}^{q_m} J_l^{(m)}, \quad m = 1, 2, \ldots,$$

are stochastically independent. Thus by the second Borel—Cantelli lemma (cf. W. Feller [5, p. 155]) we deduce that with $\mu(G_m) = \gamma(m)$ and $\sum_{n=1}^{\infty} \mu(G_m) = \infty$ (cf. (9)), for $\bar{G}=\overline{\lim}\, G_m$ $\mu(\bar{G})=1$ holds. Taking $x_0 \in \bar{G}$ we can find an infinite set of numbers m with

$$\lambda(m)|c_m \varphi_m(x)| = \frac{|c_m|\lambda(m)}{\sqrt{\gamma(m)}} \geqq \frac{1}{\varepsilon_m}.$$

Because of $\varepsilon_m \to 0$ and because of the estimate

$$|f(x)-s_{m-1}(x)| \geqq |c_m \varphi_m(x)| - |f(x)-s_m(x)|$$

the above stated equality contradicts the estimation $f(x)-s_{m-1}(x)=O_x\left(\frac{1}{\lambda(m)}\right)$ a.e.. Changing the values of $\{\varphi_n(x)\}$ in $[0, 1]-\bar{G}$ in an appropriate way, we get the assertion of Lemma 3.

To prove that $\{\varphi_n(x)\}$ is a convergence system, we mention a lemma of D. E. Menchoff ([8], Lemma 2) proving that the following conditions are sufficient for $\{\psi_n(x)\}$ to be a convergence system: Let the segments $I_l^{(n)}$, $l=1, \dots, p_n$, with $I_l^{(n)} \cap I_{l_0}^{(n)} = \emptyset$ if $l \neq l_0$ and $\sum_{l=1}^{p_n} \mu(I_l^{(n)})=1$, satisfy

(i) $\psi_n(x)$ has constant value in $I_l^{(n)}$, $l=1, \dots, p_n$;

(ii) $\int_{I_l^{(n)}} \psi_m(x)\, dx=0$ for $l=1, \dots, p_n$; $m=n+1, n+2, \dots$;

(iii) $\lim_{n\to\infty} \max_l \mu(I_l^{(n)})=0$;

(iv) if $m>n$ then for every $I_l^{(n)}=(s, t)$ $(1 \leqq l \leqq p_n)$ it exists an index $\sigma=\sigma(l, n, m)$ such that for $I_\sigma^{(m)}=(u, v)$ yields $u=s$.

It is obvious, that the above mentioned functions $\{\varphi_n(x)\}$ and the sets $J_l^{(m)}$ $(l=0, 1, \dots, q_m;\ m=1, 2, \dots)$ satisfy conditions (i)—(iv), which completes the proof.

Proof of Theorem 4. We first mention that the case $\sum_{n=1}^{\infty} c_n^2 \lambda^2(n)=\infty$ is treated in Lemma 3. Therefore, in the following, we may assume that $\sum_{n=1}^{\infty} c_n^2 \lambda^2(n) < \infty$. For the terms

$$\sigma_k = \begin{cases} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2 \lambda^2(n) (\ln (n-\mu_k+2))^2 & \text{if } \mu_{k+1}-\mu_k > 4 \\ 0 & \text{if } \mu_{k+1}-\mu_k \leqq 4 \end{cases}$$

where $\{\mu_k\}$ is given in (2), the series $\sum_{k=1}^{\infty} \sigma_k$ is divergent. Again we can find a nonincreasing sequence $\{\varepsilon_k\} \to 0$ with

$$\sum_{k=1}^{\infty} \sigma_k \varepsilon_k^2 = \infty. \tag{11}$$

On the basis of this series we define an ONS which satisfies (6). By the aid of the Rademacher functions $r_n(x)=\operatorname{sign}(\sin 2^n \pi x)$, $n=0, 1, \dots,$ we take

$$\varphi_n(x)=r_n(x), \quad n=0, 1, \dots, \mu_1-1.$$

Now let $\varphi_0(x), \dots, \varphi_{\mu_k-1}(x)$ be defined and let us denote by $I_l^{(k)}$, $l=0, \dots; q_k$; the segments in which each of these functions have constant value; we may assume that either $I_l^{(k)} \subset I_{l_0}^{(k-1)}$ or $I_l^{(k)} \cap I_{l_0}^{(k-1)} = \emptyset$. In the case of $\mu_{k+1}-\mu_k \leqq 4$ we put (by transformation (10))

$$\varphi_n(x) = \sum_{l=0}^{q_k} r_{n-\mu_k+1}(I_l^{(k)}; x), \quad n = \mu_k, \dots, \mu_{k+1}-1.$$

Otherwise if $\mu_{k+1}-\mu_k > 4$ we select $r_0 = r_0(k) \geqq 0$ such that with the numbers $\{N_r\}$ defined in Lemma 2 $N_{r_0+1} < \mu_{k+1}-\mu_k \leqq N_{r_0+2}$ is satisfied. Then we refer to Lemma 2 and set

$$a_n = c_{\mu_k+n}\varepsilon_k\lambda(\mu_k), \quad n = 0, 1, \dots, N_{r_0+1}. \tag{12}$$

Let I_l' and I_l'' denote the two halves of I_l, $l=0, 1, \dots, q_k$; then we put with the functions $\{\Phi_n(x)\}$ out of Lemma 2

$$\varphi_{\mu_k+n}(x) = \sum_{l=0}^{q_k} \{\Phi_n(I_l'; x) - \Phi_n(I_l''; x)\}, \quad n = 1, \dots, N_{r_0+1}.$$

In the case $N_{r_0+1}+1 < \mu_{k+1}-\mu_k$, we again select all segments I_l^*, $l=0, 1, \dots; q_k^*$; in which the already stated functions have constant value. We then put

$$\varphi_{\mu_k+N_{r_0+1}+n}(x) = \sum_{l=0}^{q_k^*} r_n(I_l^*; x), \quad n = 1, \dots, \mu_{k+1}-N_{r_0+1}-1.$$

With the transformation $(0, 1) \to I_l'$ resp. $(0, 1) \to I_l''$ if $\mu_{k+1}-\mu_k > 4$ the sets F_r, $r=0, 1, \dots, r_0(k)$, considered in Lemma 2 will be transformed into the sets $F_r(I_l')$ resp. $F_r(I_l'')$. Then we set

$$G_r^{(k)} = \bigcup_{l=0}^{q_k} \{F_r(I_l') \cup F_r(I_l'')\}.$$

Let $l_1, l_2, \dots$ denote those numbers with $\mu_{l_j+1}-\mu_{l_j} > 4$. Then the following is true:

(i) the sets $G_r^{(l_j)}$, $r=0, 1, \dots, r_0(l_j)$, are stochastically independent (which follows by the definition and by the Remark to Lemma 2); furthermore

$$\mu(G_r^{(l_j)}) = \sum_{l=0}^{q_{l_j}} \{\mu(F_r(I_l')) + \mu(F_r(I_l''))\} = \mu(F_r) \sum_{l=0}^{q_{l_j}} \{\mu(I_l') + \mu(I_l'')\} = \mu(F_r) \geqq$$
$$\geqq K^* \min\{1, N_{r+1} c^2_{\mu_{l_j}+N_{r+1}} \varepsilon^2_{l_j} \lambda^2(\mu_{l_j}) (\ln N_{r+1})^2\};$$

(ii) if $x \in G_r^{(l_j)}$, then there exists a number $n_r(x) = n_r(l_j; x) < 2^{r+2}$ with

$$|\varphi_{\mu_{l_j}+N_r}(x) + \dots + \varphi_{\mu_{l_j}+N_r+n_r(x)}(x)| \geqq \frac{B^*}{c_{\mu_{l_j}+N_{r+1}} \varepsilon_{l_j} \lambda(\mu_{l_j})}$$

and the considered functions have the same sign at x. By the second Borel—Cantelli lemma, using $c_n \geqq c_{n+1}$ and $\lambda(n) \leqq q\lambda(\mu_{l_j})$ if $\mu_{l_j} \leqq n \leqq \mu_{l_j+1}-1$, furthermore that

$$\sum_{j=1}^{\infty} \sum_{r=0}^{r_0(l_j)} \mu(G_r^{(l_j)}) = \infty, \tag{13}$$

(cf. (i), (11)) for $\bar{G} = \overline{\lim}\, G_r^{(l_j)}$ we have $\mu(\bar{G}) = 1$. On the other side we get by (ii) with $x \in G_r^{(l_j)}$, $0 \leqq r \leqq r_0(l_j)$; $j = 1, 2, \ldots$, that for a suitable $n_r(x)$ it holds

$$\lambda(\mu_{l_j} + N_r) \Big| \sum_{n=\mu_{l_j}+N_r}^{\mu_{l_j}+N_r+n_r(x)} c_n \varphi_n(x) \Big| \geqq \frac{B^*}{\varepsilon_{l_j}} \to \infty,$$

whence (6) obviously follows (changing the values of $\{\varphi_n(x)\}$ in a suitable set of measure 0, if necessary).

The proof of Theorem 5 is based on Theorem 4, the method being close to that of L. Csernyák and L. Leindler [4] where the following extension of K. Tandori's theorem [9] is proved: Let $\{C_i\}$ (cf. (5)) be a nonincreasing sequence. If $\sum_{i=2}^{\infty} C_i^2 (\ln i)^2 = \infty$, then it exists an ONS $\{\varphi_n(x)\}$ with $\overline{\lim}_{i\to\infty} |f(x) - s_{n_i}(x)| = \infty$. But at first we want to mention a result concerning the Rademacher functions $\{r_n(x)\}$. For any sequence $\{c_n\}$, $\sum_{n=0}^{\infty} c_n^2 < \infty$, and for $f(x) \sim \sum_{n=0}^{\infty} c_n r_n(x)$ given by the Riesz—Fischer theorem it holds

$$A\Big\{\sum_{n=0}^{\infty} c_n^2\Big\}^{1/2} \leqq \int_0^1 |f(x)|\,dx \leqq B\Big\{\sum_{n=0}^{\infty} c_n^2\Big\}^{1/2}. \tag{14}$$

(A, B absolutely constant; cf. A. Zygmund [12, p. 213]). On the basis of estimation (14) L. Csernyák and L. Leindler proved

Lemma 4. *For an arbitrary sequence* $\{c_n\}$ *let the sets* $E_{n,m}$ *be defined by*

$$E_{n,m} = \Big\{x \colon \Big|\sum_{\nu=n}^{n+m} c_\nu r_\nu(x)\Big| > \frac{1}{2}\Big\{\sum_{\nu=n}^{n+m} c_\nu^2\Big\}^{1/2}\Big\}.$$

Then the sets $E_{n,m}$ *are simple sets with*

$$\mu(E_{n,m}) \geqq \frac{A^2}{4},$$

A given by (14).

Proof of Theorem 5. (a) By Theorem 4 we can find for the sequence $\{C_i\}$ an ONS $\{\Phi_i(x)\}$ such that for the partial sums $\{S_i(x)\}$ of the series $\sum_{i=0}^{\infty} C_i \Phi_i(x)$ we have

$$\overline{\lim_{i\to\infty}}\, \lambda(n_i+1) |f(x) - S_i(x)| = \infty \quad (x \in [0, 1]).$$

As in [4] it will be proved that with the aid of $\{\Phi_i(x)\}$ we can set up an ONS $\{\varphi_n(x)\}$ for which the assertion of the theorem is true. We outline the proof and refer the reader to [4] for complete argumentation.

We distinguish two cases: $\mathrm{I}(k_j+1)-\mathrm{I}(k_j)=O(1)$ and $\mathrm{I}(k_j+1)-\mathrm{I}(k_j)\neq O(1)$ (for definition of $\mathrm{I}(k_j)$ cf. the preliminary remark regarding Theorem 2) and treat only the (more complicate) latter one.

When $\mathrm{I}(k_j+1)-\mathrm{I}(k_j)\neq O(1)$ let $l_1^*, l_2^*, \ldots$ denote those indices with

$$\mathrm{I}(l_j^*+1)-\mathrm{I}(l_j^*) > 4.$$

By the proof of Theorem 4 we can find some simple sets $G_r^{(l_j^*)}$, $r=0, 1, \ldots, r_0(l_j^*)$, with

$$\sum_{j=1}^{\infty}\sum_{r=0}^{r_0(l_j^*)} \mu(G_r^{(l_j^*)}) = \infty$$

(cf. (13)), and for $x\in G_r^{(l_j^*)}$ numbers $i_0=\mathrm{I}(l_j^*)+N_r$ and $n_r(x)$ such that

$$\lambda(n_{i_0}+1)\Big|\sum_{i=i_0}^{i_0+n_r(x)} C_i\Phi_i(x)\Big| > \frac{B^*}{\varepsilon_{l_j^*}} \qquad (\{\varepsilon_n\}\to 0), \tag{15}$$

and these functions are of the same sign on $G_r^{(l_j^*)}$.

(b) Next, corresponding to l_j^* and $r\leqq r_0(l_j^*)$ the number $\varkappa(j, r)$ is defined by

$$\varkappa(j, r) = \max\{(n_{i+1}-n_i)\colon \mathrm{I}(l_j^*)+N_r \leqq i < \mathrm{I}(l_j^*)+N_{r+1}\},$$
$$r = 0, 1, \ldots, r_0(l_j^*); \quad j = 1, 2, \ldots.$$

Now let us choose a fixed l_j^* and r. With the above value $\varkappa=\varkappa(j, r)$ we divide $[0, 1]$ $Q^*=2^{\varkappa+1}$ partial segments with equal length $I_q=(u_q, v_q)$, $1\leqq q\leqq Q^*$. With respect to some n_{i_0}, where $n_{\mathrm{I}(l_j^*)+N_r}\leqq n_{i_0}<n_{\mathrm{I}(l_j^*)+N_{r+1}}$, the number of segments I_q, where

$$\sum_{n=1}^{n_{i_0+1}-n_{i_0}} c_{n_{i_0}+n}r_n(x) > \frac{A}{2}\Big\{\sum_{n=1}^{n_{i_0+1}-n_{i_0}} c_{n_{i_0}+n}^2\Big\}^{1/2}, \tag{16}$$

is at least $p^*=2^{-3}A^2Q^*$ bearing in mind that $r_n((1/2)+x)=r_n((1/2)-x)$. Let us then change the segments I_q and simultaneously the corresponding values of the functions $r_n(x)$, $n=1, \ldots, n_{i_0+1}-n_{i_0}$, in such a way that (16) holds for the first p^* segments. The new functions are denoted by $r_{i_0,n}(x)$, $n=1, \ldots, n_{i_0+1}-n_{i_0}$, i.e. it yields

$$\sum_{n=1}^{n_{i_0+1}-n_{i_0}} c_{n_{i_0}+n}r_{i_0,n}(x) > \frac{A}{2}\Big\{\sum_{n=n_{i_0}+1}^{n_{i_0+1}} c_n^2\Big\}^{1/2},$$
$$x\in I_q; \quad q = 1, \ldots, p^*.$$

With the ONS $\{\Phi_i(x)\}$ mentioned in (a), we consider the functions (cf. (10))

$$g_{i_0 q}(x) = \Phi_{i_0}(I_q;\ x), \quad q = 1, \dots, Q^*,$$

and we put

$$\gamma_{n_{i_0}+n}(x) = \sum_{q=1}^{Q^*} r_{i_0,n}(x)\, g_{i_0 q}(x), \quad n = 1, \dots, n_{i_0+1} - n_{i_0}. \tag{17}$$

As in [4] we can prove that $\{\gamma_n(x)\}$ are orthogonal and normalized functions.

(c) For those n_i which are not covered by (b), namely when $i^* = \mathrm{I}(l_j^*) + N_{r_0(l_j^*)+1}$ and $n_{i^*} \leqq n_i < n_{\mathrm{I}(l_{j+1}^*)}$ we put $Q^* = 2^{n_{i+1} - n_i + 1}$ and define $\gamma_{n_i+1}(x), \dots, \gamma_{n_{i+1}}(x)$ analogously to (17) in (b).

(d) For fixed l_j^* and r with respect to the transformation $(0, 1) \to I_q$ in (b) the set $G_r^{(l_j^*)}$ will be transformed into the set $G_r^{(l_j^*)}(I_q)$. Taking

$$\bar{G}_r^{(l_j^*)} = \bigcup_{q=1}^{Q^*} G_r^{(l_j^*)}(I_q)$$

we get

$$\mu(\bar{G}_r^{(l_j^*)}) \geqq 2^{-4} A^3 \mu(G_r^{(l_j^*)}). \tag{18}$$

On the other side, with $i_0 = \mathrm{I}(l_j^*) + N_r$ we can find for $x \in \bar{G}_r^{(l_j^*)}$, e.g. $x \in G_r^{(l_j^*)}(I_{q_0})$, a number $n_r(x)$ with (cf. (15), (16))

$$\Big|\sum_{i=i_0}^{i_0+n_r(x)} \sum_{k=n_i+1}^{n_{i+1}} c_k \gamma_k(x)\Big| = \Big|\sum_{i=i_0}^{i_0+n_r(x)} g_{iq_0}(x) \sum_{k=n_i+1}^{n_{i+1}} c_k r_{i,k-n_i}(x)\Big| > \frac{A \cdot B^*}{2\varepsilon_{l_j} \lambda(\mu_{l_j^*})}. \tag{19}$$

(e) In the last step we have to give the ONS $\{\varphi_n(x)\}$ asked for. At first we put

$$\varphi_n(x) = r_n(x), \quad n = 0, 1, \dots, n_{\mathrm{I}(l_1^*)}.$$

Now let with $i_0 = \mathrm{I}(l_j^*) + N_\nu$ the functions $\varphi_0(x), \dots, \varphi_{n_{i_0}}(x)$ be defined. Denoting with $J_\sigma = J_\sigma^{(i_0)}$, $\sigma = 1, \dots, R(i_0)$ all those partial segments where these functions have constant value, we may assume that $J_\sigma^{(i_0)} \cap J_{\sigma'}^{(i_0-1)} = J_\sigma^{(i_0)}$ or that $J_\sigma^{(i_0)} \cap J_{\sigma'}^{(i_0-1)} = \emptyset$. The two halves of J_σ may be marked by J'_σ and J''_σ.

In the case $\nu = r_0(l_j^*) + 1$ and $\mathrm{I}(l_{j+1}^*) - \mathrm{I}(l_j^*) > N_{\nu+1}$, we put with $R = R(i_0)$ (cf. (10))

$$\varphi_k(x) = \sum_{\sigma=1}^{R} \big(\gamma_k(J'_\sigma;\ x) - \gamma_k(J''_\sigma;\ x)\big), \quad k = n_{i_0}+1, \dots, n_{\mathrm{I}(l_{j+1}^*)};$$

otherwise if $\nu \leqq r_0(l_j^*)$, then we take

$$\varphi_k(x) = \sum_{\sigma=1}^{R} \big(\gamma_k(J'_\sigma;\ x) - \gamma_k(J''_\sigma;\ x)\big), \quad k = n_{i_0}+1, \dots, n_{\mathrm{I}(l_j^*)+N_{\nu+1}}.$$

In the last case we also take up the set

$$H_\nu^{(j)} = \bigcup_{\sigma=1}^{R} \left(G_\nu^{(l_j^*)}(J_\sigma') \cup G_\nu^{(l_j^*)}(J_\sigma'')\right),$$

whereby $G_\nu^{(l_j^*)}$ is transformed into $G_\nu^{(l_j^*)}(J_\sigma')$ when $(0, 1)$ is transformed into J_σ'.

The sets $H_\nu^{(j)}$, $\nu=0, 1, \dots, r_0(l_j)$; $j=1, \dots$ are stochastically independent and

$$\sum_{j=1}^{\infty} \sum_{\nu=0}^{r_0(l_j^*)} \mu(H_\nu^{(j)}) = \infty$$

(cf. (18)), i.e.

$$\mu(\overline{\lim}\, H_\nu^{(j)}) = 1.$$

But with $x \in H_\nu^{(j)}$ we can find $n(x)$ such that for $i^* = \mathrm{I}(l_j^*) + N_\nu$ then e.g. for $x \in G_\nu^{(l_j^*)}(J_\sigma')$

$$\Big|\sum_{i=i^*}^{i^*+n(x)} \sum_{n=n_i+1}^{n_{i+1}} c_n \varphi_n(x)\Big| = \Big|\sum_{i=i^*}^{i^*+n(x)} \sum_{n=n_i+1}^{n_{i+1}} c_n \gamma_n(J_\sigma';\, x)\Big| \geqq \frac{AB^*}{2\varepsilon_{l_j^*} \lambda(\mu_{l_j^*})}$$

(cf. (19)) which is not bounded. But this contradicts the estimation $f(x) - s_{n_i}(x) = = O_x\left(\frac{1}{\lambda(n_i+1)}\right)$. Changing the values of $\{\varphi_n(x)\}$ on a set with measure 0 we get the assertion of Theorem 5.

(f) In the case $\mathrm{I}(k_j+1) - \mathrm{I}(k_j) = O(1)$ we proceed as in (c).

The proof of Theorem 6 is similiar to that of Theorem 4, taking

$$\sigma_k^2 = \begin{cases} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2 \lambda^2(n) & \text{if } \mu_{k+1} - \mu_k > 4, \\ 0 & \text{if } \mu_{k+1} - \mu_k \leqq 4, \end{cases}$$

in case $\mu_{k+1} - \mu_k \neq O(1)$ with

$$\sum_{k=1}^{\infty} \sigma_k^2 \varepsilon_k^2 = \infty \quad (\{\varepsilon_k\} \to 0)$$

and putting in (12)

$$a_n = c_{\mu_k+n} \varepsilon_k, \quad n = 0, 1, \dots, N_{r_0+1};$$

otherwise, if $\mu_{k+1} - \mu_k = O(1)$ the assertion follows by Lemma 3.

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Some Fourier multiplier criteria for anisotropic $H^p(\mathbf{R}^n)$-spaces

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Dedicated to Professor K. Tandori on the occassion of his 60th birthday

1. Introduction. In this paper we introduce anisotropic H^p-spaces along the pattern of COIFMAN and WEISS [7] and discuss the question when an operator T, given by its Fourier transform, is bounded on H^p. The multiplier criteria obtained partly improve, partly generalize results of MIYACHI [15], [16] and PERAL and TORCHINSKY [17]. Stress is laid on the practicableness of the multiplier criteria which are in the nature of best possible.

To fix ideas let us give some notations. By $L^p=L^p(\mathbf{R}^n)$, $0<p\leqq\infty$, we denote the standard Lebesgue spaces with (quasi-) norm $\|\cdot\|_p$, by S the set of all $C^\infty(\mathbf{R}^n)$-functions, rapidly decreasing at infinity, and by S' its dual, the set of all tempered distributions. As Fourier transformation F we define

$$F[f](\xi)=f^{\wedge}(\xi)=\int f(x)e^{-i\xi x}\,dx,\quad f\in S,$$

(when the integration domain is all of $\mathbf{R}^n$ we omit indicating it). By F^{-1} we denote the inverse Fourier transformation.

Let $A_t=t^P$ be a dilation matrix, $P=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$, $\nu=\operatorname{tr}P$, $\lambda_j>0$; we define the dilation operator δ_t by $\delta_t f(x)=f(A_t x)$. Following BESOV, IL'IN and LIZORKIN [2] (see also DAPPA [9]) we call $\varrho\in C(\mathbf{R}^n)$ an A_t-homogeneous distance function if $\varrho(x)>0$ for $x\neq0$ and $\varrho(A_t x)=t\varrho(x)$ for all $t>0$, $x\in\mathbf{R}^n$; all ϱ's are comparable with the typical distance function $\varrho_\varkappa(x)$ is the sense that

$$C\varrho(x)\leqq\varrho_\varkappa(x):=\Big(\sum_{j=1}^n|x_j|^{\varkappa/\lambda_j}\Big)^{1/\varkappa}\leqq C\varrho(x),\quad \varkappa>0 \tag{1.1}$$

(see [3], [20], [9]).

A p-atom a is a bounded function on $\mathbf{R}^n$ with the following properties:

i) there is a ϱ-ball $B_r(x_0)=\{x\in\mathbf{R}^n:\ \varrho(x-x_0)\leqq r\}$

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with supp $a \subset B_r(x_0)$,

ii) $\|a\|_\infty \leqq r^{-\nu/p}$,

iii) $\int x^\sigma a(x)\,dx = 0$ for $|\sigma| \leqq \left[\frac{\nu}{\lambda_{\min}}\left(\frac{1}{p}-1\right)\right] =: N,\ \lambda_{\min} = \min_j \lambda_j$.

Following COIFMAN and WEISS [7] we define $H^p = H^p(\mathbf{R}^n;\ P)$, $0<p\leqq 1$, as the set of all $f \in S'$ which can be represented in the form

$$f = \sum_{j=0}^{\infty} \mu_j a_j, \quad \sum |\mu_j|^p < \infty,$$

a_j being p-atoms for $j \geqq 0$, and

$$\|f\|_{H^p}^p = \inf\left\{\sum |\mu_j|^p : f = \sum \mu_j a_j\right\}.$$

If $\lambda_j \geqq 1$, $j=1, \dots, n$, then these H^p-spaces coincide with those in CALDERÓN and TORCHINSKY [5] (choose there A_t diagonal; see [13]). A bounded function m is said to be a Fourier multiplier for H^p if T_m, $T_m f = F^{-1}[m f^\wedge]$, maps H^p boundedly into H^p. The set of all multipliers m is normed by the operator (quasi-) norm of T_m:

$$\|m\|_{M(H^p)} = \sup\{\|T_m f\|_{H^p} : \|f\|_{H^p} \leqq 1\}.$$

Our aim is to give sufficient, nearly best possible multiplier criteria of Hörmander type for m to belong to $M(H^p)$, $0<p\leqq 1$. For this purpose we introduce function spaces $S(q, \gamma;\ B, D)$ as follows:

Let $\varphi \in C^\infty(\mathbf{R}_+)$ be a bump function with support in $[1/2, 2]$ and satisfy

$$\int_0^\infty \varphi^2\left(\frac{s}{t}\right)\frac{dt}{t} = 1, \quad s > 0.$$

Let $B(t)$ and $D(t)$ be positive continuous functions on $[0, \infty)$ with

$$0 < c \leqq \frac{B(st)}{B(t)},\ \frac{D(st)}{D(t)} \leqq C < \infty \tag{1.2}$$

for all s in a compact interval of $(0, \infty)$ and assume additionally that

$$B(t) \geqq c > 0, \quad t > 0. \tag{1.3}$$

Then $S(q, \gamma;\ B, D)$ consists of all $m \in L^1_{\mathrm{loc}}(\mathbf{R}^n_0)$ which have finite norm

$$\|m\|_{S(q,\gamma)} = \sup_{t>0} D(t)\{\|\varphi\delta_t m\|_q + B(t)^{-\gamma}\|D^\gamma(\varphi\delta_t m)\|_q\}, \quad 1 < q < \infty,$$

where $D^\gamma f = F^{-1}[|\xi|^\gamma f^\wedge]$ is the γ-th, n-dimensional Riesz derivative. Using Stein's Lemma [18; p. 133], an elementary calculation shows that

$$\sup_{t>0} D(t) B(t)^{-n/q} \|(\varphi\delta_t m)(\cdot / B(t))\|_{L^q_\gamma} \tag{1.4}$$

is an equivalent norm on $S(q, \gamma\colon B, D)$; here L^q_γ is the standard Bessel potential space [18; p. 135].

We have

$$\|m\|_\infty \leqq C\|m\|_{S(q,\gamma)}, \quad \gamma > n/q, \tag{1.5}$$

if $B(t)^{n/q}/D(t)$ is uniformly bounded in $t \geqq 0$, since by the imbedding properties of the Bessel potential spaces there holds

$$\|m\|_\infty \leqq C \sup_{t>0} \|\varphi(\xi/B(t))\delta_t m(\xi/B(t))\|_\infty \leqq C\big(\sup_{t>0} B(t)^{n/q}/D(t)\big)\|m\|_{S(q,\gamma)}.$$

Our results now read as follows.

Theorem 1. *Let* $0<p<1$, $m \in S(2,\gamma;\, B, D)$ *for* $\gamma > n(1/p-1/2)$ *and* $D(t) \geqq B(t)^{n(1/p-1/2)}$. *Then there holds*

$$\|T_m f\|_{H^p} \leqq C\|m\|_{S(2,\gamma)}\|f\|_{H^p}, \quad f \in H^p.$$

This will be proved in Sect. 2. Using Theorem 1 and interpolation of analytic families of operators acting on H^p-spaces we will derive in Sect. 3

Theorem 2. *Let* $1 \leqq p < 2$ *and* $D(t) \geqq B(t)^{n(1/p-1/2)}$. *If* $\gamma > n(1/p-1/2)$, $1/q < 1/p - 1/2$, *then*

$$\|T_m f\|_{H^p} \leqq C\|m\|_{S(q,\gamma)}\|f\|_{H^p}.$$

(Note that $H^p = L^p$ for $p>1$). In particular we deduce in Sect. 4 for quasi-radial multipliers $m(\xi) = m_0 \circ \varrho(\xi)$, m_0 defined on $\mathbf{R}_+$, the following

Corollary. *Let* $0<p<2$, $D(t) \geqq B(t)^{n(1/p-1/2)}$, $\gamma > n(1/p-1/2)$, *and* $\varrho \in C^{[\gamma]+1}(\mathbf{R}^n_0)$. *Then*

$$\|m_0 \circ \varrho\|_{M(H^p)} \leqq C \sum_{j=0}^{[\gamma]+1} \sup_{t>0} D(t) B(t)^{-j} \left(\int_t^{2t} |s^j m_0^{(j)}(s)|^q \frac{ds}{s}\right)^{1/q},$$

where $q=2$ *for* $0<p<1$ *and* $1/q < 1/p - 1/2$ *in the case* $1 \leqq p < 2$. *In particular, if* $B(t) = D(t) = 1$, *then we have also for fractional* $\gamma > n(1/p-1/2)$, $0 < p \leqq 1$, *that*

$$\|m_0 \circ \varrho\|_{M(H^p)} \leqq C\left\{\|m_0\|_\infty + \sup_{t>0}\left(\int_t^{2t} |s^\gamma m_0^{(\gamma)}(s)|^2 \frac{ds}{s}\right)^{1/2}\right\}. \tag{1.6}$$

Here the notion of a fractional derivative is that of Gasper and Trebels [12] (see also [6]).

Remarks. **1.** Theorem 1 is due to Miyachi [15] in the isotropic case (for $B=D=1$ see also [21]); Theorem 2 for $q=\infty$ is proved in Miyachi [16] (for the isotropic case).

2. It is not hard to generalize Theorem 1 in the sense that $F^{-1}[(1+|\xi|^2)^{\gamma/2}]$, $\gamma>n((1/p)-(1/2))$, $D(t)\geqq B(t)^{n(1/p-1/2)}$ is replaced by $F^{-1}[(1+\tilde{\varrho}(\xi))^{\beta}]$, $\beta>\tilde{\nu}((1/p)-(1/2))$, $D(t)\geqq B(t)^{\tilde{\nu}(1/p-1/2)}$, where $\tilde{\varrho}$ is a $C^{\infty}(\mathbf{R}_0^n)$-distance function homogeneous with respect to another dilation matrix $\tilde{A}_t=t^{\tilde{P}}$, the eigenvalues of $\tilde{P}$ have positive real parts, $\tilde{\nu}=\operatorname{tr}\tilde{P}$; thus we could partly regain a result of CALDERÓN and TORCHINSKY [5; II Theorem 4.6] in the case $\tilde{A}_t=A_t$.

3. Our results for $0<p\leqq 1$ are nearly optimal. As test multipliers consider the well discussed examples:

$$e^{i|\xi|^a}(1+|\xi|)^{-b}\in M(H^p),\quad 0<p\leqq 2,\quad a\neq 1, \tag{1.7}$$

if and only if $b\geqq an((1/p)-(1/2))$ (cf. [16]) and

$$(1-|\xi|)_+^a\in M(H^p),\quad 0<p\leqq\frac{2(n+1)}{n+3}, \tag{1.8}$$

if and only if $a>n((1/p)-(1/2))-1/2$ (see [19], [11], [9]).

It is not hard to verify the conditions of Corollary for the functions $e^{it^a}(1+t)^{-b}$ and $(1-t)_+^a$ so that Corollary gives the correct positive results for $0<p\leqq 1$, if we choose $A_t=\operatorname{diag}(t,\ldots,t)$, $\varrho(\xi)=|\xi|$.

4. The multiplier condition (1.6) is an essential improvement of a result of PERAL and TORCHINSKY [17; Theorem 1.4] in the case of diagonal dilation matrices with eigenvalues $\lambda\geqq 1$ since $\gamma>\nu((1/p)-(1/2))+1/2$, $\nu=\operatorname{tr}P\geqq n$ is assumed in [17] in comparison to our $\gamma>n((1/p)-(1/2))$.

5. The results of MADYCH [14] (see also DAPPA and LUERS [10] in the quasiradial case) suggest that Theorem 1 remains valid if the diagonal matrix A_t is replaced by t^{P^*}, P^* being a real $n\times n$ matrix whose eigenvalues have positive real parts.

We now give some applications of Corollary.

i) $$(1-\varrho(\xi))_+^a\in M(H^p),\quad a>n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2},\quad 0<p\leqq 1.$$

ii) Let $\Phi\in C^{\infty}(\mathbf{R}_+)$ be 1 for $t\geqq 2$ and 0 for $t\leqq 1$; choose

$$B_1(t)=\begin{cases} t^a\log^c(1+t), & t\geqq 1\\ \log^c 2, & t\leqq 1,\end{cases}\qquad D_1(t)=\begin{cases} t^b\log^d(1+t), & t\geqq 1\\ \log^d 2, & t\leqq 1,\end{cases}$$

where $a, b, c, d\geqq 0$, then

$$\Phi\circ\varrho(\xi)e^{iB_1\circ\varrho(\xi)}/D_1\circ\varrho(\xi)\in M(H^p),\quad 0<p<2,$$

if $d/c, b/a\geqq n((1/p)-(1/2))$ or $b/a>n((1/p)-(1/2))$ and $c, d\geqq 0$.

(iii) Let $B_1, D_1, a, b, c, d, \Phi$ be as in ii).

$$\Phi\circ\varrho(\xi)\big(\cos_+ B_1\circ\varrho(\xi)\big)^{\alpha+i\beta}/D_1\circ\varrho(\xi)\in M(H^p), \quad 0<p\leqq 1$$

for $\alpha>n((1/p)-(1/2))-1/2>0$; it is easy to verify the first condition in the Corollary for integer $\gamma>n((1/p)-(1/2))$; complex interpolation then gives the rest of the assertion.

2. Proof of Theorem 1. This is a modification of the corresponding proof of [15; Theorem 1] so that we will be quite concise at some part of the proof. We have only to prove

$$\|T_m a\|_p \leqq C\|m\|_{S(2,\gamma)}$$

for p-atoms a with C independent of a; for it is proved in [16; Theorem 3.4.] that this implies T_m to be bounded from H^p into H^p in the isotropic case; this argument can be generalized to the anisotropic case by a result of Triebel [22]. Since T_m is translation invariant we may assume that supp $a\subset\{x\colon \varrho(x)\leqq r\}$; further we choose $M>0$ so big that $\varrho(x)>Mr$ and $\varrho(y)\leqq r$, $0<s<1$, imply $\varrho(x)>2\varrho(sy)$. Then, by Hölder's inequality, the Parseval formula and (1.5),

$$\int_{\varrho(x)\leqq Mr} |T_m a(x)|^p\,dx \leqq C\|m\|^p_{S(2,\gamma)}. \tag{2.1}$$

If we set

$$\hat{K}_j(\xi) = \int_{2^j}^{2^{j+1}} \varphi^2(\varrho(\xi)/t)\,m(\xi)\,\frac{dt}{t}, \tag{2.2}$$

there remains to estimate

$$\int_{\varrho(x)\geqq Mr} |T_m a(x)|^p\,dx \leqq \sum_{j=-\infty}^{\infty}\int_{\varrho(x)\geqq Mr} |K_j * a(x)|^p\,dx. \tag{2.3}$$

Now observe that, by the properties of the p-atoms and by Taylor's formula

$$|K_j * a(x)| \leqq r^{-\nu/p}\int_{\varrho(y)\leqq r} |K_j(x-y)|\,dy, \tag{2.4}$$

$$|K_j * a(x)| \leqq Cr^{-\nu/p}\sum_{|\sigma|=N+1}\iint_{\Omega} |D^\sigma K_j(x-sy)|\,|y^\sigma|\,dy\,ds, \tag{2.5}$$

where $\Omega=\{(s,y)\colon 0<s<1,\ \varrho(y)<r\}$. In order to estimate the latter integral we use a covering argument for Ω. First observe that, by the triangle inequality and the boundedness of A_t, there is a $\delta=\delta(j,r)>0$ such that

$$\big||A_{2^j}(x-sy)|-|A_{2^j}(x-s'y')|\big| \leqq |A_{2^j}(sy-s'y')| \leqq \frac{1}{2}|A_{2^j}(x-sy)| \tag{2.6}$$

for $|sy-s'y'|<\delta$, $\varrho(x)\geqq Mr$. Now define a family of balls in $\mathbf{R}^{n+1}$ by

$$B_\varepsilon(s,y) = \{(s',y')\colon |s-s'|+|y-y'|\leqq\varepsilon\} \quad (s,y)\in\Omega;$$

choose $\delta'>0$ such that $|sy-s'y'|<\delta$ for $(s', y')\in B_{5\delta'}(s, y)$ and such that $|y-z|\leqq 5\delta'$, $\varrho(y)<r$ imply $\varrho(z)<2r$. Then (cf. [18; p. 9]) select a disjoint sequence of balls $B_{\delta'}(s_i, y_i)=B_i$ such that the expanded balls B_i^* (same center as B_i but with diameter five times as large) cover Ω. An elementary homogeneity consideration shows that at most K balls B_i^* overlap; here K does not depend on $\delta=\delta(j, r)$ (but only on the ratio $|B_i^*|/|B_i|$, $|B_i|$ the Lebesgue measure of B_i). We now have by (2.6)

$$\iint_\Omega |D^\sigma K_j(x-sy)||y^\sigma|\,dy\,ds \leqq C\sum_i (1+|A_{2^j}(x-s_iy_i)/B(2^j)|)^{-\gamma p}\iint_{B_i^*}|D^\sigma K_j(x-sy)|\times$$
$$\times(1+|A_{2^j}(x-sy)/B(2^j)|)^{\gamma p}|y^\sigma|\,dy\,ds$$

and therefore, by the Hölder and the integral Minkowski inequality,

$$\Big(\int_{\varrho(x)\geqq Mr}|r^{-\nu/p}\iint_\Omega |D^\sigma K_j(x-sy)|\,|y^\sigma|\,ds\,dy|^p\,dx\Big)^{1/p}\leqq$$
$$\leqq Cr^{-\nu/p}\sum_i \big\|(1+|A_{2^j}(x-s_iy_i)/B(2^j)|)^{-\gamma p}\big\|_{2/(2-p)}^{(2-p)/2p}\times$$
$$\times\iint_{B_i^*}\big\|(1+|A_{2^j}(x-sy)/B(2^j)|)^{\gamma}D^\sigma K_j(x-sy)\big\|_2|y^\sigma|\,ds\,dy\leqq \tag{2.7}$$
$$\leqq Cr^{-\nu/p}(2^j)^{-\nu(1/p-1/2)}B(2^j)^{n(1/p-1/2)}\int_{\varrho(y)\leqq 2r}\|\dots\|_2|y^\sigma|\,dy\leqq$$
$$\leqq Cr^{\nu(1-1/p)+\lambda\sigma}(2^j)^{-\nu(1/p-1/2)}B(2^j)^{n(1/p-1/2)}\|\dots\|_2,$$

where $\lambda\sigma=\sum_{j=1}^n\lambda_j\sigma_j$. Here the second inequality follows by the translation invariance of the L^2-norm and the fact that at most K of the B_i^* overlap, and the last inequality, on account of (1.1) with $\varkappa=1$, by

$$\int_{\varrho(y)\leqq 2r}|y^\sigma|\,dy\leqq C\prod_{j=1}^n\Big(\int_{|y_j|\leqq cr^{\lambda_j}}|y_j|^{\sigma_j}dy_j\Big).$$

Analogously,

$$\Big(\int_{\varrho(x)\geqq Mr}\Big|r^{-\nu/p}\int_{\varrho(y)\leqq r}|K_j(x-y)|\,dy\Big|^p\,dx\Big)^{1/p}\leqq \tag{2.8}$$
$$\leqq Cr^{\nu(1-1/p)}(2^j)^{-\nu(1/p-1/2)}B(2^j)^{n(1/p-1/2)}\|(1+|A_{2^j}x/B(2^j)|)^{\gamma}K_j(x)\|_2.$$

Thus there remains to estimate $\|\dots\|_2$ for $|\sigma|=0$ and for $|\sigma|=N+1$ (recall $N=\Big[\frac{\nu}{\lambda_{\min}}\Big(\frac{1}{p}-1\Big)\Big]\Big)$. Using the definition (2.2) of K_j, the fact that

$$|A_{2^j}x/B(2^j)|\approx|A_tx/B(t)|,\quad 2^j\leqq t\leqq 2^{j+1},$$

the integral Minkowski inequality and repeated substitutions, we see that $\|\ldots\|_2$ can be majorized by

$$C2^{j\nu/2}B(2^j)^{-n/2}\int_{2^j}^{2^{j+1}}\left\|F^{-1}[\varphi^2\circ\varrho(\xi/B(t))(A_t i\xi/B(t))^\sigma m(A_t\xi/B(t))](x)(1+|x|)^\gamma\right\|_2\frac{dt}{t}\leqq$$

$$\leqq C2^{j\nu/2}B(2^j)^{-n/2}\int_{2^j}^{2^{j+1}}\left\|\varphi^2\circ\varrho(\xi/B(t))(A_t\xi/B(t))^\sigma m(A_t\xi/B(t))\right\|_{L^2_\gamma}\frac{dt}{t}$$

by the Parseval formula. Now observe that

$$\|gf\|_{L^2_\gamma}\leqq C\sup_{|\tau|\leqq[\gamma]+1}\|D^\tau g\|_\infty\|f\|_{L^2_\gamma}=C\|g\|_{W^\infty_{[\gamma]+1}}\|f\|_{L^2_\gamma}$$

which is obvious for integer γ and hence, by interpolation, also for fractional γ. Thus we have for $2^j\leqq t\leqq 2^{j+1}$, on account of (1.2),

$$\|\ldots\|_2\leqq C\left\|\{\varphi\circ\varrho(\xi/B(t))(A_t\xi/B(t))^\sigma\}\right\|_{W^\infty_{[\gamma]+1}}(2^{j\nu/2}/D(2^j))\|m\|_{S(2,\gamma)}. \tag{2.9}$$

Now, by (1.3),

$$\left\|\varphi\circ\varrho(\xi/B(t))(A_t\xi/B(t))^\sigma\right\|_{W^\infty_{[\gamma]+1}}\leqq Ct^{\lambda\sigma}, \tag{2.10}$$

for $|\sigma|=0$ and $|\sigma|=N+1$. Combining (2.10), (2.9) with (2.7) and (2.8) we arrive at

$$\int_{\varrho(x)\geqq Mr}|K_j*a(x)|^p dx\leqq C(2^jr)^{\nu(p-1)}\min\{1,(2^jr)^{\lambda\sigma p}\}\|m\|^p_{S(2,\gamma)}$$

which clearly implies the convergence of the series in (2.3). Thus Theorem 1 is established.

3. Proof of Theorem 2. We essentially interpolate between $S(2,\gamma_0;B,D_0)\subset\subset M(H^{p_0})$, $p_0<1$ near 1, and $L^\infty\subset M(H^2)$. We use the following imbedding and interpolation properties of the Besov and Bessel potential spaces (see [18; p. 155], [1; p. 153]):

$$L^2_\gamma=B^{22}_\gamma;\ L^q_\gamma\subset B^{qq}_\gamma\quad\text{for}\quad q\geqq 2; \tag{3.1}$$

$$B^{\infty\infty}_\gamma\subset L^\infty\quad\text{for}\quad\gamma>0;\ [B^{22}_{\gamma_0},B^{\infty\infty}_{\gamma_1}]_\Theta=B^{qq}_\gamma,\quad\gamma_0\neq\gamma_1,$$

when

$$(\gamma,1/q)=(1-\Theta)\left(\gamma_0,\frac{1}{2}\right)+\Theta(\gamma_1,0),\quad 0<\Theta<1.$$

Now choose

$$\gamma_0>n\left(\frac{1}{p}-\frac{1}{2}\right),\quad\gamma_1>0(\text{small}),\quad D_0(t)=D(t)^{1/(1-\Theta)}$$

then $D_0(t)\geqq B(t)^{n(1/p_0-1/2)}$. With $V=L^q_\gamma$ or B^{qq}_γ we define the auxiliary function space $X_0(V;B,D)$ as consisting of those functions, locally integrable away from the origin, which satisfy

$$\sup_{k\in\mathbf{Z}} \|m\|_{X_{0,k}} < \infty, \quad \lim_{k\to\pm\infty} \|m\|_{X_{0,k}} = 0,$$

where

$$\|m\|_{X_{0,k}} = D(2^k)B(2^k)^{-n/q}\|\varphi\circ\varrho\delta_{2^k}m\|_V.$$

Observe that we have for $m\in X_0(L^q_\gamma;\ B, D)$

$$\sup_k \|m\|_{X_{0,k}} \approx \|m\|_{S(q,\gamma;\,B,\,D)}.$$

By the methods in [6] (see also [8]) one can show that

$$(V_i = L^{q_i}_{\gamma_i} \quad \text{or} \quad B^{q_i q_i}_{\gamma_i}, \quad q_0 = 2,\ q_1 = \infty)$$

$$[X_0(V_0;\ B, D_0),\ X_0(V_1;\ B, D_1)]_\Theta = X_0([V_0, V_1]_\Theta;\ B, D)$$

where $[,]_\Theta$ is Calderón's lower interpolation method [4]. By (3.1) we obtain

$$X_0(L^q_\gamma;\ B, D) \subset X_0(B^{qq}_\gamma;\ B, D) = [X_0(B^{22}_{\gamma_0};\ B, D_0), X_0(B^{\infty\infty}_{\gamma_1};\ B, 1)]_\Theta \subset$$
$$\subset [S(2, \gamma_0;\ B, D_0), L^\infty]_\Theta.$$

Now consider the dense subspace H^{p_0} of H^p-functions whose Fourier transforms have compact support away from the origin; for $f\in H^{p_0}$ let

$$\psi(\xi) = \int_\varepsilon^N \varphi^2(\varrho(\xi)/t)\,\frac{dt}{t}$$

such that $\psi=1$ on $\operatorname{supp} f\hat{}$ for suitable ε and N. Then, by the interpolation of analytic families of operators on H^p-spaces [5], [7], it finally follows that

$$\|F^{-1}[mf\hat{}]\|_{H^p} = \|F^{-1}[m\psi f\hat{}]\|_{H^p} \leqq C\|m\psi\|_{S(q,\gamma)}\|f\|_{H^p} \leqq C\|m\|_{\dot S(q,\gamma)}\|f\|_{H^p},$$

$1/p=(1-\Theta)(1/p_0)+(\Theta/2)$, thus the assertion.

4. Proof of Corollary. We observe that a further equivalent norm on $S(q,\gamma;$ $B, D)$ for integer γ is given by

$$\sup_{t>0} D(t)\,B(t)^{-n/q} \sum_{0\leqq|\sigma|\leqq\gamma} \|D^\sigma((\varphi\circ\varrho\delta_t m)(\xi/B(t)))\|_q,$$

which follows from (1.4) and the identification of the Bessel potential space with the Sobolev space [18; p. 135]. If we now consider quasi-radial functions $m=m_0\circ\varrho$, m_0 being defined on $\mathbf{R}_+$, then the first part of Corollary is an immediate consequence of

$$\left\|D^\sigma\{\varphi\circ\varrho(\cdot/B(t))\,m_0(t\varrho(\cdot/B(t)))\}\right\|_q \leqq \tag{4.1}$$
$$\leqq C\sum_{j=0}^{|\sigma|} t^{j-\nu/q}B(t)^{n/q-|\sigma|}\Big(\int_{t\leqq\varrho(\xi)\leqq 2t} |m_0^{(j)}(\varrho(\xi))|^q\,d\xi\Big)^{1/q};$$

for the introduction of polar coordinates $\varrho(\xi)\doteq s$, $d\xi\doteq s^{\nu-1}ds\,d\omega$ ($d\omega$ being a finite measure on $\{\xi:\ \varrho(\xi)=1\}$ leads at once to the assertion. To establish (4.1) we need the Leibniz rule

(4.2)

$$D^\sigma\{\varphi\circ\varrho(\xi/B(t))m_0(t\varrho(\xi/B(t)))\} = \sum_{\sigma=\sigma'+\sigma''} D^{\sigma'}(\varphi\circ\varrho(\xi/B(t)))D^{\sigma''}(m_0(t\varrho(\xi/B(t)))$$

and the following consequence of the chain rule:

$$D^\sigma(g\circ\varrho) = \sum_{j=1}^{|\sigma|} g^{(j)}(\varrho(\xi)) \sum \prod_{i=1}^{j} D^{\tau^{(i)}}\varrho(\xi), \tag{4.3}$$

where the sum is taken over all possible representations of $\sigma=\sum_{i=1}^{j}\tau^{(i)}$. Now

$$\left\|D^{\sigma'}(\varphi\circ\varrho(\xi/B(t)))\right\|_\infty \leqq \sum_{j=1}^{|\sigma'|}\left\|\varphi^{(j)}\circ\varrho(\xi/B(t)) \sum \prod_{i=1}^{j} D^{\tau^{(i)}}(\varrho(\xi/B(t)))\right\|_\infty$$

which is clearly bounded for $|\sigma'|\geqq 0$ on account of (1.3), since $\varphi\in C^\infty(\mathbf{R}_+)$ and, by definition of φ, we have only to consider

$$1/2 \leqq \varrho(\xi/B(t)) \leqq 2.$$

If $\sigma''=(0, \dots, 0)$, then

$$\Big(\int_{1/2\leqq\varrho(\xi/B(t))\leqq 2} \left|m_0(t\varrho(\xi/B(t)))\right|^q d\xi\Big)^{1/q} \leqq CB(t)^{n/q}t^{-\nu/q}\Big(\int_{t/2\leqq\varrho(\xi)\leqq 2t} |m_0\circ\varrho(\xi)|^q d\xi\Big)^{1/q}$$

which is of the desired type. Let $|\sigma''|\neq 0$; then

$$\Big(\int_{1/2\leqq\varrho(\xi/B(t))\leqq 2} \left|D^{\sigma''}\{m_0(t\varrho(\xi/B(t)))\}\right|^q d\xi\Big)^{1/q} \leqq$$

$$\leqq C\sum_{j=1}^{|\sigma''|}\Big(\int_{1/2\leqq\varrho(\xi/B(t))\leqq 2} \left|m_0^{(j)}(t\varrho(\xi/B(t)))t^jB(t)^{-|\sigma''|}\right|^q d\xi\Big)^{1/q}.$$

(4.1) in combination with the above estimates gives the assertion.

In the particular case $B(t)=D(t)=1$, $0<p<1$ the first condition in Corollary reduces to

$$\sup_{t>0}\sum_{j=0}^{\gamma}\Big(\int_t^{2t} |s^j m_0^{(j)}(s)|^2\frac{ds}{s}\Big)^{1/2}, \quad \gamma \text{ integer},$$

which is an equivalent norm on

$$S(2, \gamma;\ \mathbf{R}_+) = \{m\in L^2_{\mathrm{loc}}(\mathbf{R}_+):\ \|m\|_{S(2,\gamma;\mathbf{R}_+)}<\infty\},$$

$$\|m\|_{S(2,\gamma;\mathbf{R}_+)} = \sup_{t>0}\Big(\int_{\mathbf{R}}|F^{-1}[(1+\zeta^2)^{\gamma/2}[\varphi(\cdot)m_0(t\cdot)]^\wedge(\zeta)](s)|^2ds\Big)^{1/2}$$

(here $^\wedge$ and F^{-1} denote the one-dimensional Fourier transformation and its inverse, resp.).

Now define the operator

$$T_\varrho\colon S(2,\gamma;\ \mathbf{R}_+)\to S(2,\gamma;\ 1,1;\ \mathbf{R}^n),\quad T_\varrho m_0=m_0\circ\varrho.$$

Then the above estimates show that T_ϱ is bounded if $\gamma\in\mathbf{N}_0$. Therefore, for $\gamma_0,\gamma_1\in\mathbf{N}_0$, $\gamma_0\neq\gamma_1$,

$$T_\varrho\colon [S(2,\gamma_0;\ \mathbf{R}_+),\ S(2,\gamma_1;\ \mathbf{R}_+]^\Theta\to[S(2,\gamma_0;\ \mathbf{R}^n),\quad S(2,\gamma_1;\ \mathbf{R}^n)]^\Theta$$

is bounded, where $[,]^\Theta$ is Calderón's [4] upper interpolation method. But it is shown in [6] that

$$[S(2,\gamma_0;\ \mathbf{R}_+),\ S(2,\gamma_1;\ \mathbf{R}_+)]^\Theta=S(2,\gamma;\ \mathbf{R}_+),$$

where $\gamma=(1-\Theta)\gamma_0+\Theta\gamma_1$, $0<\Theta<1$; the same argument applies for the n-dimensional situation so that

$$T_\varrho\colon\ S(2,\gamma;\mathbf{R}_+)\to S(2,\gamma;\mathbf{R}^n)$$

is bounded, i.e.,

$$\|m_0\|_{S(2,\gamma;\,R_+)}\leqq C\|m_0\circ\varrho\|_{S(2,\gamma;\,1,1;\,\mathbf{R}^n)},\quad \gamma>0.$$

Further, in [6] it is shown that $\|m_0\|_{S(2,\gamma;\mathbf{R}_+)}$ for $\gamma>1/2$ is equivalent to the condition in Corollary, which hence is proved up to the case $p=1$. By the same reasoning we obtain for $p=1$

$$S(q,\gamma;\ \mathbf{R}_+)\subset M(H^1),\quad \gamma>n/2,$$

and $q>2$ arbitrarily close to 2. A slight increase of γ allows to take $q=2$ by the imbedding properties of the L^q_γ-spaces, thus the assertion holds.

Concluding let us observe that we have estimated the n-dimensional potential norm of the quasi-radial function $m=m_0\circ\varrho$ by a one-dimensional potential norm of m_0; i.e., loosely speaking, on the space of quasi-radial functions we have majorized an n-dimensional fractional differential operator by a tractable, one-dimensional fractional operator.

Added in proof: The authors realized that all the results of the paper remain valid if, in the definition of $S(q,\gamma,B,D)$, the diagonal matrix P is replaced by a real $n\times n$ matrix whose eigenvalues have positive real parts; then we have multiplier theorems on $H^p(\mathbf{R}^n,P^*)$. The key for this generalization is to be seen in a right application of Taylor's formula: replace (2.5) by

$$|K_j*a(x)|\leqq Cr^{-\nu/p}\iint_\Omega|(y\cdot\nabla)^{N+1}K_j(x-sy)|\,dyds$$

and modify appropriately the following estimates.

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On the almost everywhere divergence of Lagrange interpolation on infinite interval

J. SZABADOS and P. VÉRTESI

To Professor K. Tandori on his 60th birthday

1. Introduction

1.1. Let $X=\{x_{kn}\}$, $n=1, 2, \ldots$, $1 \leqq k \leqq n$, be an interpolatory matrix in $\boldsymbol{R}:=(-\infty, \infty)$, i.e.

$$(1.1)\qquad -\infty < x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n} < \infty, \quad n=1, 2, \ldots.$$

Restricting ourselves to nodes which are uniformly bounded with n, say $-1 \leqq x_{kn} \leqq 1$, $1 \leqq k \leqq n$, $n=1, 2, \ldots$, the behaviour of the corresponding Lagrange interpolatory polynomials

$$(1.2)\qquad L_n(f, X, x) = L_n(f, x) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x)$$

for $f \in C$ (i.e. f is continuous on $[-1, 1]$) were thoroughly investigated. The first result is due to FABER [1], proving that *for any interpolatory matrix* $X \subset [-1, 1]$ *there exists an* $f \in C$ *such that*

$$(1.3)\qquad \overline{\lim_{n\to\infty}} \|L_n(f, X, x)\| = \infty.$$

Here we used the notations

$$(1.4)\qquad l_{kn}(X, x) = l_{kn}(x) = \frac{\omega_n(x)}{\omega_n'(x_{kn})(x-x_{kn})}, \quad k=1, 2, \ldots, n, \quad n=1, 2, \ldots$$

(where $\omega_n(X, x)=\omega_n(x)=c_n \prod_{k=1}^{n}(x-x_{kn})$), $\|g(x)\|=\|g\|=\max_{-1\leqq x\leqq 1}|g(x)|$ for $g \in C$).

Almost seventy years later the second named author in a joint paper with P. Erdős proved the following conjecture of ERDŐS [2].

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Theorem 1.1 (Erdős and Vértesi [3]). *For any interpolatory matrix X in* $[-1, 1]$ *one can find a function* $F(x)\in C$ *such that*

$$\varlimsup_{n\to\infty} |L_n(F, X, x)| = \infty \quad \text{for almost all} \quad x\in[-1, 1]. \tag{1.5}$$

1.2. Some years ago the second of us completed the corresponding result for the unit circle ([4]).

In the same paper results for the trigonometric case were proven.

2. Infinite interval

2.1. Contrary to the case of the finite interval we know very little on the infinite ones. If the nodes are unbounded, as far as we know, even the analogue of Faber's result has not been proved yet.

Generally, the corresponding theorems deal with

a) special nodes (Laguerre or Hermite nodes, see e.g. [7]);

b) nodes where the maximal distance of the consecutive ones tends to zero (see e.g. [8]);

c) a *finite* interval $[a, b]\subset R$ (see e.g. [9]).

2.2. To illustrate these we quote the following statement of divergence type. Let $X^{(\alpha)}=\{x_{kn}^{(\alpha)}\}$, $1\leqq k\leqq n$, $n=1, 2, \ldots$, where $x_{kn}^{(\alpha)}$ is the k-th root of the n-th Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha>-1$.

Theorem 2.1 (Povchun [7]). *Let* $X=X^{(\alpha)}$, $\alpha>-1$. *Then there exists a function* $f(x)$ *continuous on* $[0, \infty)$ *such that*

$$\varlimsup_{n\to\infty} |L_n(f, X^{(\alpha)}, x)| = \infty \quad \text{a.e. on} \quad [0, \infty).$$

2.3. Dealing with infinite intervals the main problem may be formulated in the following way. The functions $F(x)$ serving as counter-example generally have the form $\sum_{k=1}^{\infty} c_k \varphi_k(x)$. Here $\sum |c_k| < \infty$ and $\varphi_k(x)$ are uniformly bounded polynomials, say, in $[-1, 1]$.

On the other hand, if $x\to\infty$, even the continuity of $F(x)$ is questionable. Of course, one can "cut" somehow $\varphi_k(x)$ but then we could not use the very useful property $L_N(\varphi_k, x)\equiv\varphi_k(x)$ if $N>\deg \varphi_k$.

In this paper we were able to overcome these problems and to generalize Theorem 1.1 to infinite intervals.

3. Results

3.1. First let $C(\mathbf{R}):=\{f(x); f(x)$ is continuous for any $x\in\mathbf{R}\}$. Our main statement is the following.

Theorem 3.1. *For any interpolatory matrix $X\subset\mathbf{R}$ one can find a function $F(x)\in C(\mathbf{R})$ such that*

$$\overline{\lim_{n\to\infty}}\, |L_n(F, x)| = \infty \quad \textit{for almost all} \quad x\in\mathbf{R}. \tag{3.1}$$

3.2. To prove this, first we use a special lemma which may deserve a separate formulation.

Lemma 3.2. *Let $\varepsilon>0$ be given. Then for any interpolatory matrix $X\subset\mathbf{R}$ and for any fixed $[a, b]\subset\mathbf{R}$ one can find a function $F_\varepsilon(x)=F(a, b, X, \varepsilon, x)\in C(\mathbf{R})$ such that*

$$\overline{\lim_{n\to\infty}}\, |L_n(F_\varepsilon, x)| = \infty \quad \textit{on} \quad S_\varepsilon \tag{3.2}$$

where $S_\varepsilon\subset[a, b]$ and $|S_\varepsilon|\geqq b-a-\varepsilon$.

3.3. Let us remark that the set of divergence in (3.1) *is dense and of second category on $\mathbf{R}$.* [1)]

4. Proof of Lemma 3.2

We shall use many ideas of [3], even our notations will be similar, too.

First a simple remark. If

$$\lambda_n(x) := \sum_{k=1}^{n} |l_k(x)|, \quad \lambda_n := \max_{x\in[x_{nn}, x_{1n}]} \lambda_n(x), \quad n = 1, 2, \ldots, \tag{4.1}$$

then

$$\lambda_n > \frac{\ln n}{8\sqrt{\pi}}, \quad n = 1, 2, \ldots. \tag{4.2}$$

---

1) Indeed, as Orlicz [6] proved:
If A is a topological space of second category, $D\subset A$ is a dense subset and $\{h_n\}$, $n=1, 2, \ldots$, are continuous functions on A with

$$\overline{\lim_{n\to\infty}}\, h_n(x) = \infty \quad \textit{whenever} \quad x\in D$$

then the set S, $D\subset S\subset A$,

$$S := \{x\in A;\ \overline{\lim_{n\to\infty}}\, h_n(x) = \infty\}$$

is dense and of second category in A.

Now let $P\subset\mathbf{R}$ be the set for which (3.1) holds. By $|\mathbf{R}\setminus P|=0$, P is dense in $\mathbf{R}$. If $h_n(x)=|L_n(f, x)|$ and $D=P$, we obtain statement 3.3.

This relation was proved by S. Bernstein and G. Faber (see e.g. NATANSON [5], Volume III, Ch. II, § 1). Actually, they supposed that the matrix X is contained in a *finite* interval but their proof does not use this fact.

4.1. First let us suppose that there exists a finite interval $[A, B]$ which contains X. It is enough to prove the lemma for $[a, b]\supseteq[A, B]$. Indeed, applying Theorem 1.1, we obtain an $F(x)$ which is continuous on $[a, b]$ and for which $\overline{\lim_{n\to\infty}}\,|L_n(F, x)|=\infty$ a.e. in $[a, b]$. If

$$F_\varepsilon(x) := \begin{cases} F(a) & \text{if} \quad x \leqq a \\ F(x) & \text{if} \quad a \leqq x \leqq b \\ F(b) & \text{if} \quad x \geqq b \end{cases}, \tag{4.3}$$

we obtain the lemma. (Remark that now $|S_\varepsilon|=b-a$, moreover F_ε and S_ε do not depend on ε.)

4.2. Now let us suppose that $\overline{\lim_{n\to\infty}}\,[\max(|x_{1n}|, |x_{nn}|)]=\infty$. For sake of simplicity we may assume that

$$\lim_{n\to\infty} x_{1n} = \infty \quad \text{and} \quad x_{11} < x_{12} < \dots \tag{4.4}$$

(otherwise we can select a subsequence having this property).

For an arbitrary sequence $A_n>0$, $A_n\nearrow\infty$, we can construct F_ε as follows. First, we cover all the points of X with an interval-system I_ε of total measure $\leqq\varepsilon$. We suppose that x_{kn}, say, is the middle point of its covering interval. Then obviously

$$\delta_n(\varepsilon) := \min_{x\in R\setminus I_\varepsilon} |l_{1n}(x)| > 0, \quad n = 1, 2, \dots. \tag{4.5}$$

I.e., if

$$F_\varepsilon(x) := \frac{A_1}{\delta_1(\varepsilon)} \quad \text{when} \quad x\in(-\infty, x_{11}], \tag{4.6}$$

then

$$|L_1(F_\varepsilon, x)| \geqq A_1 \quad \text{when} \quad x\in R\setminus I_\varepsilon. \tag{4.7}$$

4.3. Generally, if we defined F_ε on $(-\infty, x_{1,n-1}]$ $(n\geqq 2)$, then we define F_ε on $(-\infty, x_{1n}]$ as follows. Let

$$F_\varepsilon(x) := F_\varepsilon(x_{1,n-1}) \quad \text{if} \quad x\in[x_{1,n-1}, \alpha_n] \tag{4.8}$$

where $\alpha_n:=\max(x_{2n}, x_{1,n-1})$, $n=2, 3, \dots$. Considering (4.4), $\alpha_n<x_{1n}$. So we can define the function $F_\varepsilon(x)$ at x_{1n}. First we remark that

$$\Delta_n(\varepsilon) := \max_{x\in R\setminus I_\varepsilon} \left| \frac{\sum_{k=2}^{n} F_\varepsilon(x_{kn})\, l_{kn}(x)}{l_{1n}(x)} \right| < \infty, \tag{4.9}$$

because $\deg l_{1n} \geqq \deg\left(\sum_{k=1}^{n}\dots\right)$, moreover, we excluded a certain neighbourhood of the poles of the rational function appearing in (4.9). Now let

$$F_\varepsilon(x_{1n}) := \frac{A_n}{\delta_n(\varepsilon)} + \Delta_n(\varepsilon). \tag{4.10}$$

Finally, let F_ε be linear if $x \in [\alpha_n, x_{1n}]$. By these we completed the definition of F_ε on $(-\infty, x_{1n}]$. Using again (4.4), we can say that F_ε is defined for any $x \in \boldsymbol{R}$, moreover $F_\varepsilon \in C(-\infty, \infty)$. Then for any $x \in \boldsymbol{R} \setminus I_\varepsilon$ we have

$$\begin{aligned} |L_n(F_\varepsilon, x)| &= \Big|\sum_{k=1}^{n} F_\varepsilon(x_{kn})\, l_{kn}(x)\Big| = \Big|l_{1n}(x)\, F_\varepsilon(x_{1n}) + \sum_{k=2}^{n} \dots\Big| \geqq \\ &\geqq |l_{1n}(x)|\,[|F_\varepsilon(x_{1n})| - \Delta_n(\varepsilon)] \geqq \delta_n(\varepsilon)\left[\frac{A_n}{\delta_n(\varepsilon)} + \Delta_n(\varepsilon) - \Delta_n(\varepsilon)\right] = A_n \end{aligned} \tag{4.11}$$

(see (4.5), (4.9) and (4.10)), i.e. we proved

$$\overline{\lim_{n\to\infty}}\, |L_n(F_\varepsilon, x)| = \infty \quad \text{on} \quad \mathbf{R} \setminus I_\varepsilon$$

(see (4.11)) which is more than stated in Lemma 3.2.

5. A lemma

5.1. Later we need the next

Lemma 5.1. *If* $g_1, g_2, \ldots \in C(\boldsymbol{R})$ *and*

$$\overline{\lim_{n\to\infty}}\, g_n(x) = \infty \quad (x \in B,\ |B| < \infty), \tag{5.1}$$

then for arbitrary fixed A, ε *and* M $(A, \varepsilon > 0,\ M \geqq 1,$ *integer) there exist a set* $H \subseteqq B$ *and an index* $N \geqq M$ *such that* $|H| \leqq \varepsilon$ *and if* $x \in B \setminus H$ *then for a certain* $u(x)$ *we have*

$$g_{u(x)}(x) \geqq A \quad \textit{where} \quad M \leqq u(x) \leqq N. \tag{5.2}$$

(The proof of Lemma 5.1. is in [3, 4.4.13.].)

6. Proof of Theorem 3.1

6.1. For a fixed k, $k = 1, 2, \ldots$, applying Lemma 3.2 with the cast $\varepsilon := 2^{-k-1}$ and $[a, b] := I_k$ (I_k will be determined later), we get $F_k \in C(\boldsymbol{R})$, $\max\limits_{x \in I_k} |F_k(x)| \leqq 1$, for which

$$\overline{\lim_{n\to\infty}}\, |L_n(F_k, x)| = \infty \quad \text{if} \quad x \in T_k,$$

where $T_k \subset I_k$ and $|T_k| \geqq |I_k| - 2^{-k-1}$ ($F_k(x)$ and T_k correspond to $F_\varepsilon(x)[\max\limits_{a \leqq x \leqq b} |F_\varepsilon(x)|]^{-1}$ and S_ε, respectively).

Now let $k=1$ and $I_1:=[-1, 1]$. Using Lemma 5.1 with $g_n(x):=|L_n(F_1, x)|$, $A:=A_1:=2$, $\varepsilon:=\varepsilon_1:=2^{-2}$, $M:=m_1:=1$ and $B:=T_1$, we get the set $S_1 \subset T_1$ with $|S_1| \geqq 2-2^{-1}$ and the index n_1 such that

$$(6.1) \qquad |L_{u(x)}(F_1, x)| \geqq A_1 > 1^3 \mu_0 \quad \text{if} \quad x \in S_1,$$

moreover $m_1 \leqq u(x) \leqq n_1$. Here $\mu_0:=1$ (S_1, n_1 and $u(x)$ correspond to $B \setminus H, N$ and $u(x)$, respectively).

Now let $a_1:=\min(-1, \min_{1 \leqq n \leqq n_1} x_{nn})$, $b_1:=\max(1, \max_{1 \leqq n \leqq n_1} x_{1n})$, and let $I_2:=$ $:=[a_1-1, b_1+1]$. Clearly $[-2, 2] \subseteqq I_2$. We can construct a polynomial $\varphi_1(x)$ for which $\varphi_1(x_{in}):=F_1(x_{in})$ $(1 \leqq i \leqq n,\ 1 \leqq n \leqq n_1)$ and $|\varphi_1(x)-F_1(x)| \leqq 1$ if $x \in I_2$ (see [10, Part 3, Chapter 2, § 3, Lemma 3]). Then, by $|F_1(x)| \leqq 1$ $(x \in I_1)$,

$$(6.2) \qquad |\varphi_1(x)| \leqq 2 \quad \text{if} \quad x \in I_1.$$

By $\varphi_1(x_{in})=F_1(x_{in})$, (6.1) holds for φ_1, too. Finally, let $\deg \varphi_1 \leqq N_1$.

Now by induction, for any fixed $k \geqq 2$, using similar arguments and the notations

$$(6.3) \qquad \begin{cases} \mu_{k-1} := \max\limits_{1 \leqq i \leqq n_{k-1}} \max\limits_{x \in I_k} \lambda_i(x), \\ A_k := k^3 \mu_{k-1}+1, \quad \varepsilon_k := 2^{-k-1}, \quad m_k := N_{k-1}+1, \end{cases}$$

we get the set $S_k \subset I_k$, $|S_k| \geqq 2k-2^{-k}$, the index n_k and a polynomial $\varphi_k(x)$ of degree $\leqq N_k$ for which

$$(6.4) \qquad |\varphi_k(x)| \leqq 2 \quad \text{if} \quad x \in I_k,$$

moreover for a certain $u(x)$, $m_k \leqq u(x) \leqq n_k$,

$$(6.5) \qquad |L_{u(x)}(\varphi_k, x)| \geqq A_k > k^3 \mu_{k-1} \quad \text{if} \quad x \in S_k.$$

Let $a_k:=\min(-k, \min_{1 \leqq n \leqq n_k} x_{nn})$, $b_k:=\max(k, \max_{1 \leqq n \leqq n_k} x_{1n})$ and let $I_{k+1}:=[a_k-1, b_k+1]$. Clearly $[-(k+1), k+1] \subseteqq I_{k+1}$ and $I_k \subseteqq I_{k+1}$ $(k=2, 3, \ldots)$.

By construction we may suppose that

$$(6.6) \qquad m_1 < n_1 < N_1 < m_2 < n_2 < N_2 < \ldots,$$

while by (6.3)

$$(6.7) \qquad \mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \ldots.$$

6.2. Now let

$$(6.8) \qquad F(x) := \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{k^2 \mu_{k-1}}$$

and

$$(6.9) \qquad S := \bigcap_{k=1}^{\infty} \left(\bigcup_{i=k}^{\infty} S_i\right).$$

First we state that $F \in C(\boldsymbol{R})$. To this end, let $x \in \boldsymbol{R}$ be arbitrary. By $[-s, s] \subseteqq I_s$, one can find a fixed s for which $x\in I_s$. If we prove that F is bounded on (the closed) I_s, then, by the continuity of φ_k, F will be continuous on I_s, especially at x.

Let $F(x)=\left(\sum_{k=1}^{s-1}+\sum_{k=s}^{\infty}\right)c_k\varphi_k(x)$, where $c_k:=(k^2\mu_{k-1})^{-1}$. Here the first sum can be estimated by $\sum_{k=1}^{s-1} c_k \max_{x\in I_s}|\varphi_k(x)|:=\alpha_s<\infty$, the second one, using that $|\varphi_k(x)|\leqq 2$ because $x\in I_s\subseteqq I_k$ (see (6.4)), by $2\sum_{k=s}^{\infty}c_k<2\sum_{k=1}^{\infty}k^{-2}:=2E_2<\infty$ which was to be proven.

Now we prove that

$$\overline{\lim_{n\to\infty}}\,|L_n(F,x)|=\infty \quad \text{if} \quad x\in S. \tag{6.10}$$

Indeed, for any fixed $x\in S$ there exist infinitely many S_j, $j=r_1, r_2, \ldots$, for which $x\in S_j\subset I_j$. For each fixed j there exists an index $u_j=u(x)$ such that by (6.5)

$$|L_{u_j}(\varphi_j, x)|\geqq A_j, \quad x\in S_j,\ m_j\leqq u_j\leqq n_j. \tag{6.11}$$

For a fixed j $(j=r_1, r_2, \ldots)$ we write

$$L_{u_j}(F,x)=\left(\sum_{k=1}^{j-1}+\sum_{k=j}+\sum_{k=j+1}^{\infty}\right)c_kL_{u_j}(\varphi_k,x):=J_1+J_2+J_3.$$

In the first sum $L_{u_j}(\varphi_k,x)\equiv\varphi_k(x)$ because $\deg\varphi_k\leqq N_k\leqq N_{j-1}<m_j\leqq u_j$. I.e., $J_1=\left(\sum_{k=1}^{r_1-1}+\sum_{k=r_1}^{j-1}\right)c_k\varphi_k:=J_4+J_5$. Here $|J_4|\leqq\sum_{k=1}^{r_1-1}c_k\max_{x\in I_{r_1}}|\varphi_k(x)|=\alpha_{r_1}$ further $|J_5|\leqq$ $\leqq 2\sum_{k=r_1}^{j-1}c_k<2E_2$ by $|\varphi_k(x)|\leqq 2$ because $x\in I_{r_1}\subseteqq I_k$ (see (6.4)). Thus $|J_1|\leqq\alpha_{r_1}+2E_2$.

For J_2, by (6.11) and (6.5) we get that $|J_2|\geqq A_j j^{-2}\mu_{j-1}^{-1}>j$.

Finally we estimate $J_3=\sum_{k=j+1}^{\infty}c_k\left(\sum_{i=1}^{u_j}\varphi_k(x_{iu_j})l_{iu_j}(x)\right)$. Here all the values $|\varphi_k(x_{iu_j})|\leqq 2$ by $x_{iu_j}\in[a_j,b_j]\subset I_{j+1}\subseteqq I_k$, $k\geqq j+1$. I.e., $|J_3|\leqq 2\sum_{k=j+1}^{\infty}c_k\lambda_{u_j}(x)\leqq$ $\leqq 2\sum_{k=j+1}^{\infty}c_k\mu_j<2E_2$ (see (6.3) and (6.7)).

Summarizing, we get that

$$|L_{u_j}(F,x)|\geqq|J_2|-|J_1|-|J_3|\geqq j-\alpha_{r_1}-4E_2, \quad j=r_1, r_2, \ldots,$$

which is $\geqq j/2$ if j is big enough.

6.3. To complete the proof of the theorem we show that

$$|\boldsymbol{R}\setminus S|=0. \tag{6.12}$$

Indeed, by (6.9), $\boldsymbol{R}\setminus S=\bigcup_{k=1}^{\infty}(\boldsymbol{R}\setminus Q_k)=\bigcup_{k=1}^{\infty}P_k$ if $Q_k:=\bigcup_{i=k}^{\infty}S_i$ and $P_k:=\boldsymbol{R}\setminus Q_k$. Obviously $Q_1\supset Q_2\supset\ldots$ which means $P_1\subset P_2\subset\ldots$. Here $|P_k|\leqq 2^{-k+1}$. Indeed, considering that S_i overlaps $[-i, i]$ except a set H_i of measure $\leqq 2^{-i}$, we get that $Q_k=\bigcup_{i=k}^{\infty} S_i$ overlaps R except the set $P_k \subseteqq \bigcup_{i=k}^{\infty} H_i$. Hence $|P_k| \leqq \sum_{i=k}^{\infty} |H_i| \leqq \leqq \sum_{i=k}^{\infty} 2^{-i}=2^{-k+1}$. Using $P_1 \subset P_2 \subset \ldots$, we get that $|R \setminus S|=\left|\bigcup_{k=1}^{\infty} P_k\right|=\lim_{k\to\infty} |P_k|=0$ which was to be proven.

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On the generalized absolute Cesàro summability of double orthogonal series

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Dedicated to Professor Károly Tandori on his 60th birthday

Introduction. As usual we denote by σ_n^α the n-th Cesàro means of order α of a single numerical series $\sum a_n$. The following definition is due to FLETT [1]: A series $\sum a_n$ is said to be summable $|C, \alpha, \gamma|_\varkappa$ $(\alpha>-1, \gamma\geqq 0, \varkappa\geqq 1)$ if the series $\sum n^{\varkappa\gamma+\varkappa-1}|\sigma_n^\alpha-\sigma_{n-1}^\alpha|^\varkappa$ converges.

Very recently MÓRICZ [3] introduced a definition of $|C, (\alpha, \beta)|_\varkappa$ summability for a double series

(1) $$\sum_{i,k} a_{ik},$$

namely, series (1) is summable $|C, (\alpha, \beta)|_\varkappa$ $(\alpha>-1, \beta>-1, \varkappa\geqq 1)$ if

$$\sum_{m=1}^\infty \sum_{n=1}^\infty (mn)^{\varkappa-1}|\Delta_{mn}^{\alpha\beta}|^\varkappa<\infty,$$

where

(2) $$\Delta_{mn}^{\alpha\beta}=\sigma_{mn}^{\alpha\beta}-\sigma_{m-1,n}^{\alpha\beta}-\sigma_{m,n-1}^{\alpha\beta}+\sigma_{m-1,n-1}^{\alpha\beta} \quad (m, n=1, 2, \dots)$$

and

$$\sigma_{mn}^{\alpha\beta}=\frac{1}{A_m^\alpha}\frac{1}{A_n^\beta}\sum_{i=0}^m\sum_{k=0}^n A_{m-i}^\alpha A_{n-k}^\beta a_{ik} \quad (m, n=0, 1, \dots)$$

is the rectangular $(C, (\alpha, \beta))$ mean of double series (1) with the binomial coefficients

$$A_0^\alpha=1, \quad A_n^\alpha=\frac{(1+\alpha)(2+\alpha)\dots(n+\alpha)}{n!}, \quad (n=1, 2, \dots).$$

Considering the rectangular partial sum of series (1)

$$s_{mn}=\sum_{i=0}^m\sum_{k=0}^n a_{ik} \quad (m, n=0, 1, \dots)$$

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we have the identity

$$\sigma_{mn}^{\alpha\beta}=\frac{1}{A_m^\alpha}\frac{1}{A_n^\beta}\sum_{i=0}^{m}\sum_{k=0}^{m}A_{m-i}^{\alpha-1}A_{n-k}^{\beta-1}s_{ik}\quad (m,n=0,1\ldots)$$

which is the rectangular $(c,(\alpha,\beta))$ mean of the sequence $\{s_{ik}\}$. Denote by $\tau_{mn}^{\alpha\beta}$ the rectangular $(c,(\alpha,\beta))$ mean of the sequence $\{ika_{ik}\}$, that is,

$$\tau_{mn}^{\alpha\beta}=\frac{1}{A_m^\alpha}\frac{1}{A_n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}A_{m-1}^{\alpha-1}A_{n-k}^{\beta-1}ika_{ik}\quad (m,n=1,2,\ldots). \tag{3}$$

Now we introduce the definition of the $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$ summability as follows: Double series (1) is said to summable $|c,(\alpha,\beta),(\mu,\nu)|_\varkappa$ $(\alpha>-1,\beta>-1,0\leqq\mu<1,$ $0\leqq\nu<1,\ \varkappa\geqq 1)$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{\varkappa\mu-1}n^{\varkappa\nu-1}|\tau_{mn}^{\alpha\beta}|^\varkappa<\infty.$$

The identity

$$\tau_{mn}^{\alpha\beta}=mn(\sigma_{mn}^{\alpha\beta}-\sigma_{m-1,n}^{\alpha\beta}-\sigma_{m,n-1}^{\alpha\beta}+\sigma_{m-1,n-1}^{\alpha\beta})\quad (m,n=1,2,\ldots) \tag{4}$$

shows that our definition is an extension of the Flett's definition, and — by (2) and (4) — it is a generalisation of the Móricz's definition. Here we mention a very useful identity

$$\tau_{mn}^{\alpha\beta}=\alpha\beta(\sigma_{mn}^{\alpha\beta}-\sigma_{mn}^{\alpha,\beta-1}-\sigma_{mn}^{\alpha-1,\beta}+\sigma_{mn}^{\alpha-1,\beta-1})\quad (m,n=1,2,\ldots),$$

too.

Our first result extends a theorem of Flett ([1], Theorem 1) for summability $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$.

Theorem 1. *If* $\varrho\geqq\varkappa>1,\ \mu\geqq 0,\ \nu\geqq 0,\ \alpha>\mu-1,\ \beta>\nu-1$ *and* $\min(\delta,\bar\delta)>$ $>\varkappa^{-1}-\varrho^{-1}$ *then the inequality*

$$\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{\varrho\mu-1}n^{\varrho\nu-1}|\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}|^\varrho\Big\}^{1/\varrho}\leqq K\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{\varkappa\mu-1}n^{\varkappa\nu-1}|T_{mn}^{\alpha\beta}|^\varkappa\Big\}^{1/\varkappa} \tag{5}$$

*holds,**) *so the summability* $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$ *implies the summability*

$$|C,(\alpha+\delta,\beta+\bar\delta),(\mu,\nu)|_\varrho.$$

In 1960 Tandori [5] published the very interesting

Theorem A. *The condition*

$$\sum_{m=0}^{\infty}\Big(\sum_{n=2^m+1}^{2^{m+1}}a_n^2\Big)^{1/2}<\infty$$

*) K will denote a positive constants not necessarily the same at each occurrence.

is necessary and sufficient that for any orthonormal system $\{\varphi_n(x)\}$ *on the interval* $(0, 1)$, *the series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

be summable $|C, 1, 0|_1$ *almost everywhere in* $(0, 1)$.

This result has a lot of generalisations and extensions. For example, in the first step, it was extended for $|C, \alpha, 0|_1$ $(\alpha > -1)$ summability by LEINDLER [2], and using the Leindler's method, for $|C, \alpha, \gamma|_\varkappa$ $(\alpha > -1, 0 \leqq \gamma < 1, \varkappa \geqq 1)$ summability by the author [4].

Denote by I the two-dimensional unit interval $(0, 1)\times(0, 1)$ and let $\{\varphi_{ik}(x, y)\}$ be an orthonormal system on I. Very recently MÓRICZ [3] proved the following theorems.

Theorem B. *If* $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq \varkappa \leqq 2$ *and*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} a_{ik}^2\Big)^{\varkappa/2} < \infty \tag{6}$$

then the double orthogonal series

$$\sum_{i,k} a_{ik}\varphi_{ik}(x, y) \tag{7}$$

is $|C, (\alpha, \beta), (0, 0)|_\varkappa$ *summable almost everywhere on* I.

Theorem C. *If* $\alpha > 1/2, \beta > 1/2$ *and in the case* $\varkappa = 1$ *condition* (6) *is not satisfied, then the two-dimensional Rademacher series*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik} r_i(x) r_k(y) \quad (r_i(x) = \operatorname{sgn}\sin 2^i \pi x) \tag{8}$$

is not $|C, (\alpha, \beta), (0, 0)|_1$ *summable almost everywhere on* I.

Generalizing these results we have two theorems.

Theorem 2. *Let* $\alpha > 1/2$, $\beta > 1/2$, $0 \leqq \mu < 1$, $0 \leqq \nu < 1$, $1 \leqq \varkappa \leqq 2$. *If*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} i^{2\mu}k^{2\nu}a_{ik}^2\Big)^{\varkappa/2} < \infty, \tag{9}$$

then the inequality

(10)
$$\iint_I \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}|\tau_{mn}^{\alpha\beta}(x, y)|^\varkappa\Big)\,dx\,dy \leqq K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} i^{2\mu}k^{2\nu}a_{ik}^2\Big)^{\varkappa/2}$$

holds[**] *and double orthogonal series* (7) *is* $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$ *summable almost everywhere on* I.

Theorem 3. *If series* (8) *is summable* $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$ $(\alpha>1/2, \beta>1/2, 0\leqq\mu<1, 0\leqq\nu<1, 1\leqq\varkappa\leqq 2)$ *on a subset* $E\subset I$ *with positive measure then* (9) *holds.*

Proof of Theorem 1. The proof is based on the identity

$$\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}=\frac{1}{A_m^{\alpha+\delta}}\frac{1}{A_n^{\beta+\bar\delta}}\sum_{i=1}^{m}\sum_{k=1}^{n}A_{m-i}^{\delta-1}A_{n-k}^{\bar\delta-1}A_i^\alpha A_k^\beta\tau_{ik}^{\alpha\beta} \tag{11}$$

which can be proved by definition (3) and the elementary identity of binomial coefficients

$$\sum_{n=0}^{m}A_{m-n}^{\delta-1}A_n^{\alpha-1}=A_m^{\alpha+\delta-1}.$$

Let S denote the expression on the right of (5), λ and ω be numbers to be choosen later, but certainly such that $\lambda>(\varkappa')^{-1}$ $\left(\varkappa^{-1}+(\varkappa')^{-1}=1\right)$, $0<\omega<1$, $\min(\delta,\bar\delta)>1>1-\left(\varkappa'(1-\omega)\right)^{-1}$. Using (11) and applying Hölder's inequality with indices $\varrho, \varkappa'$ and $\varkappa\varrho/\varrho-\varkappa$ (if $\varrho=\varkappa$ then with indices ϱ and $\varkappa'$ only) we have

$$|\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}|\leqq Km^{-\alpha-\delta}n^{-\beta-\bar\delta}\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\delta-1}(n-k+1)^{\bar\delta-1}i^\alpha k^\beta|\tau_{ik}^{\alpha\beta}|=$$

$$=Km^{-\alpha-\delta}n^{-\beta-\bar\delta}\sum_{i=1}^{m}\sum_{k=1}^{n}\{(m-i+1)^{(\delta-1)\omega}(n-k+1)^{(\bar\delta-1)\omega}\times$$

$$\times i^{\alpha+\lambda-(\varkappa\mu-1)(\varrho-\varkappa)/\varkappa\varrho}k^{\beta+\lambda-(\varkappa\nu-1)(\varrho-\varkappa)/\varkappa\varrho}|\tau_{ik}^{\alpha\beta}|^{\varkappa/\varrho}\}\times$$

$$\times\{(m-i+1)^{(\delta-1)(1-\omega)}(n-k+1)^{(\bar\delta-1)(1-\omega)}i^{-\lambda}k^{-\lambda}\}\{i^{\varkappa\mu-1}k^{\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^\varkappa\}^{(\varrho-\varkappa)/\varkappa\varrho}\leqq$$

$$\leqq Km^{-\alpha-\delta}n^{-\beta-\bar\delta}\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\varrho(\delta-1)\omega}(n-k+1)^{\varrho(\bar\delta-1)\omega}\times \tag{12}$$

$$\times i^{\varrho\alpha+\varrho\lambda-(\varkappa\mu-1)(\varrho-\varkappa)/\varkappa}k^{\varrho\beta+\varrho\lambda-(\varkappa\nu-1)(\varrho-\varkappa)/\varkappa}|\tau_{ik}^{\alpha\beta}|^\varkappa\}^{1/\varrho}\times$$

$$\times\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{(\delta-1)(1-\omega)\varkappa'}(n-k+1)^{(\bar\delta-1)(1-\omega)\varkappa'}i^{-\lambda\varkappa'}k^{-\lambda\varkappa'}\}^{1/\varkappa'}\times$$

$$\times\{\sum_{i=1}^{m}\sum_{k=1}^{n}i^{\varkappa\mu-1}k^{\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^\varkappa\}^{(\varrho-\varkappa)/\varkappa\varrho}.$$

Having

$$\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{(\delta-1)(1-\omega)\varkappa'}(n-k+1)^{(\bar\delta-1)(1-\omega)\varkappa'}i^{-\lambda\varkappa'}k^{-\lambda\varkappa'}\leqq$$

$$\leqq Km^{(\delta-1)(1-\omega)\varkappa'+1-\lambda\varkappa'}n^{(\bar\delta-1)(1-\omega)\varkappa'+1-\lambda\varkappa'}$$

[**] $\tau_{mn}^{\alpha\beta}(x,y)$ is the rectangular $(C,(\alpha,\beta))$ mean of the sequence $\{ika_{ik}\varphi_{ik}(x,y)\}$.

and considering that the last factor in (12) is less than $S^{1-\varkappa/\varrho}$ we get

$$|\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}| \leqq Km^{-\alpha-1-\delta\omega+\omega-\lambda+(\varkappa')^{-1}} n^{-\beta-1-\bar\delta\omega+\omega-\lambda+(\varkappa')^{-1}} S^{1-\varkappa/\varrho} \times$$

$$\times \Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{c}(n-k+1)^{\bar c} i^{a+\varkappa\mu-1}k^{b+\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa}\Big\}^{1/\varrho},$$

where $a=\varrho(\alpha+\lambda+\varkappa^{-1}-\mu)$, $b=\varrho(\beta+\lambda+\varkappa^{-1}-\nu)$, $c=\varrho(\delta-1)\omega$ and $\bar c=\varrho(\bar\delta-1)\omega$. Since $\varrho\mu-1-\varrho(-\alpha-1-\delta\omega+\omega-\lambda+(\varkappa')^{-1})=-a-c-1$, and similarly $\varrho\nu-1-\varrho(-\beta-1-\bar\delta\omega+\omega-\lambda+(\varkappa')^{-1})=-b-\bar c-1$ for any $M\geqq 1$ and $N\leqq 1$ we obtain

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\varrho\mu-1}n^{\varrho\nu-1}|\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}|^{\varrho} \leqq$$

$$\leqq KS^{\varrho-\varkappa}\sum_{m=1}^{M}\sum_{n=1}^{N} m^{-a-c-1}n^{-b-\bar c-1}\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{c}(n-k+1)^{\bar c} i^{a+\varkappa\mu-1}k^{b+\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa} =$$

$$= KS^{\varrho-\varkappa}\sum_{m=1}^{M} m^{-a-c-1}\sum_{i=1}^{m}(m-i+1)^{c} i^{a+\varkappa\mu-1}\sum_{k=1}^{N}k^{b+\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa}\sum_{n=k}^{N}n^{-b-\bar c-1}(n-k+1)^{\bar c} \leqq$$

$$\leqq KS^{\varrho-\varkappa}\sum_{k=1}^{N}k^{\varkappa\nu-1}\sum_{m=1}^{M}m^{-a-c-1}\sum_{i=1}^{m}(m-i+1)^{c} i^{a+\varkappa\mu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa}$$

provided that $\bar c>-1$ and $b>0$, which is possible by choosing $\omega=\varkappa'/\varkappa'+\varrho$ and $(\varkappa')^{-1}-(1+\beta-\nu)<\lambda<(\varkappa')^{-1}$. With $\omega=\varkappa'/\varkappa'+\varrho$ and $(\varkappa')^{-1}-(1+\alpha-\mu)<\lambda<(\varkappa')^{-1}$ the inequalities $c>-1$ and $a>0$ are also fulfilled, so using the above method, we get

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\varrho\mu-1}n^{\varrho\nu-1}|\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}|^{\varrho} \leqq$$

$$\leqq KS^{\varrho-\varkappa}\sum_{k=1}^{N}k^{\varkappa\nu-1}\sum_{i=1}^{M}i^{\varkappa\mu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa} \leqq KS^{\varrho}$$

and this, assuming that M and N tend to infinity, proves inequality (5), moreover means the $|C,(\alpha+\delta,\beta+\bar\delta),(\mu,\nu)|_{\varrho}$ summability of double series (1).

Proof of Theorem 2. Applying Hölder's inequality and considering (3) we have the estimations:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^{\varkappa}\,dx\,dy \leqq$$

$$\leqq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}\Big(\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^{2}\,dx\,dy\Big)^{\varkappa/2} =$$

$$= K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1} m^{\varkappa\mu-1}n^{\varkappa\nu-1}\Big(\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^{2}\,dx\,dy\Big)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=2^p}^{2^{p+1}-1} m^{\varkappa\mu-1}2^{q\varkappa(\nu-1/2)}\Big(\sum_{n=2^q}^{2^{q+1}-1}\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^{2}\,dx\,dy\Big)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1/2)}2^{q\varkappa(\nu-1/2)}\Big(\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^{2}\,dx\,dy\Big)^{\varkappa/2} \equiv \sum\nolimits_1.$$

A routine calculation gives

$$\sum\nolimits_1 = K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1/2)}2^{q\varkappa(\nu-1/2)}\left(\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(\frac{A_{m-i}^{\alpha-1}A_{n-k}^{\beta-1}}{A_m^{\alpha}A_n^{\beta}}\right)^2 i^2k^2a_{ik}^2\right)^{\varkappa/2} =$$

$$= K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1/2)}2^{q\varkappa(\nu-1/2)}\left(\sum_{m=2^p}^{2^{p+1}-1}\sum_{i=1}^{m}\left(\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right)^2 i^2\sum_{k=1}^{2^{q+1}-1}k^2a_{ik}^2\sum_{n=\max(2^q,k)}^{2^{q+1}-1}\left(\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\right)^2\right)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{q=0}^{\infty}2^{q\varkappa(\nu-1)}\sum_{p=0}^{\infty}2^{p\varkappa(\mu-1/2)}\left(\sum_{k=1}^{2^{q+1}-1}k^2\sum_{i=1}^{2^{p+1}-1}i^2a_{ik}^2\sum_{m=\max(2^p,i)}^{2^{p+1}-1}\left(\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right)^2\right)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{q=0}^{\infty}2^{q\varkappa(\nu-1)}\sum_{p=0}^{\infty}2^{p\varkappa(\mu-1)}\Big(\sum_{k=1}^{2^{q+1}-1}\sum_{i=1}^{2^{p+1}-1}i^2k^2a_{ik}^2\Big)^{\varkappa/2} \equiv \sum\nolimits_2.$$

To approach the form of our condition we go on as follows:

$$\sum\nolimits_2 = K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{p\varkappa(\mu-1)}2^{q\varkappa(\nu-1)}\Big(\sum_{m=0}^{\infty}\sum_{n=0}^{q}\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}i^2k^2a_{ik}^2\Big)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{\varrho\varkappa(\mu-1)}2^{q\varkappa(\nu-1)}\sum_{m=0}^{p}\sum_{n=0}^{q}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}i^2k^2a_{ik}^2\Big)^{\varkappa/2} =$$

$$= K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}i^2k^2a_{ik}^2\Big)^{\varkappa/2}\sum_{p=m}^{\infty}\sum_{q=n}^{\infty}2^{q\varkappa(\nu-1)}2^{\varrho\varkappa(\nu-1)} \leqq$$

$$\leqq K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{m\varkappa(\mu-1)}2^{n\varkappa(\nu-1)}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}i^2k^2a_{ik}^2\Big)^{\varkappa/2}\Big).$$

Regarding condition (9) and inequality (10) we can apply the Levi's theorem, so our proof is completed.

Proof of Theorem 3. This proof is very similar to that of Theorem C, but certain modifications are required to obtain a wider range of parameters μ, ν and $\varkappa$. Since these are not quite obvious we give the proof here. We need a

Lemma (MÓRICZ [3], Lemma 2). *For every finite sum*

$$P(x, y) = \sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik} r_i(x) r_k(y) \quad (M \geqq m \geqq 0, \quad N \geqq n \geqq 0)$$

and for any set $E \subset I$ *of positive measure, there exist an integer* n_0 *and a constant* K_0 *such that if* $\max(m, n) \geqq n_0$ *then*

$$\iint_E |P(x, y)|\, dx\, dy \geqq K_0 \Big(\sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2\Big)^{1/2}.$$

To begin the proof of Theorem 3, without loss of generality, we may assume that $a_{ik} = 0$ for $i, k = 0, 1, \ldots, n_0 - 1$. By Egorov's theorem there exist a constant K^* and a set $E^* \subset E$ of positive measure such that for every $(x, y) \in E^*$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu+\varkappa-1} n^{\varkappa\nu+\varkappa-1} |\Delta_{mn}^{\alpha\beta}(x, y)|^{\varkappa} \leqq K^* \tag{13}$$

where $\Delta_{mn}^{\alpha\beta}(x, y)$ is the suitable difference (see (2)) in the case of series (8). Using Hölder's inequality we obtain

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu+\varkappa-1} n^{\varkappa\nu+\varkappa-1} \iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|^{\varkappa}\, dx\, dy \geqq$$

$$\geqq K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu+\varkappa-1} n^{\varkappa\nu+\varkappa-1} \Big(\iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy\Big)^{\varkappa} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=2^{p-1}}^{2^{p+1}-1} m^{\varkappa\mu+\varkappa-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} n^{\varkappa\nu+\varkappa-1} \Big(\iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy\Big)^{\varkappa} \geqq \tag{14}$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{q\varkappa\nu} \sum_{m=2^{p-1}}^{2^{p+1}-1} m^{\varkappa\mu+\varkappa-1} \Big(\sum_{n=2^{q-1}}^{2^{q+1}-1} \iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy\Big)^{\varkappa} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \Big(\sum_{m=2^{p-1}}^{2^{p+1}-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} \iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy\Big)^{\varkappa} \equiv \sum.$$

Setting

$$\delta_{pq}^{\alpha\beta}(x, y) = \sigma_{2^{p+1}-1,\, 2^{q+1}-1}^{\alpha\beta}(x, y) - \sigma_{2^{p-1}-1,\, 2^{q+1}-1}^{\alpha\beta}(x, y) -$$

$$- \sigma_{2^{p+1}-1,\, 2^{q-1}-1}^{\alpha\beta}(x, y) + \sigma_{2^{p-1}-1,\, 2^{q-1}-1}^{\alpha\beta}(x, y)$$

it is easy to see that

$$\delta_{pq}^{\alpha\beta}(x, y) = \sum_{m=2^{p-1}}^{2^{p+1}-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} \Delta_{mn}^{\alpha\beta}(x, y). \tag{15}$$

Applying now Lemma and using the monotonicity of the binomial coefficients we may write

$$\iint_{E^*} |\delta_{pq}^{\alpha\beta}(x, y)|\, dx\, dy \geqq K_0 \left(\sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} \left(\frac{A^{\alpha}_{2^{p+1}-1-i}}{A^{\alpha}_{2^{p+1}-1}} \right)^2 \left(\frac{A^{\beta}_{2^{q+1}-1-k}}{A^{\beta}_{2^{q+1}-1}} \right)^2 a_{ik}^2 \right)^{1/2}.$$

Using this fact and (15) we continue estimation (14) as follows

$$\sum \geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \left(\sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} \left(\frac{A^{\alpha}_{2^{p+1}-1-i}}{A^{\alpha}_{2^{p+1}-1}} \right)^2 \left(\frac{A^{\beta}_{2^{q+1}-1-k}}{A^{\beta}_{2^{q+1}-1}} \right)^2 a_{ik}^2 \right)^{\varkappa/2}.$$

Considering that there exists a constant $K_{p,q,\alpha,\beta}$ such that

$$\frac{A^{\alpha}_{2^{p+1}-1-i}}{A^{\alpha}_{2^{p+1}-1}} \frac{A^{\beta}_{2^{q+1}-1-k}}{A^{\beta}_{2^{q+1}-1}} \geqq \frac{A^{\alpha}_{2^p}}{A^{\alpha}_{2^{p+1}-1}} \frac{A^{\beta}_{2^q}}{A^{\beta}_{2^{q+1}-1}} \geqq K_{p,q,\alpha,\beta} > 0$$

we have

$$\sum \geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \Big(\sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} a_{ik}^2 \Big)^{\varkappa/2} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \Big(\sum_{i=2^{p}}^{2^{p+1}-1} \sum_{k=2^{q}}^{2^{q+1}-1} a_{ik}^2 \Big)^{\varkappa/2},$$

and by (13) and (14) inequality (9) holds.

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Holiday numbers: sequences resembling to the Stirling numbers of second kind

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction. The first appearance of the often rediscovered Stirling numbers seems to be Stirling's work Methodus differentialis in 1730, but some mathematicians attribute them to Euler without prima facie evidences. Although their importance was clear in that time, Ch. Jordan had to summarize their meaning in finite difference calculus in 1933 [6]. Combinatorial properties of Stirling numbers were exhibited by E. T. BELL [1], [2], [3], [4], but we must know that Dobinski's formula for the sum of Stirling numbers $\frac{1}{e}\sum_{j=0}^{\infty}\frac{j^n}{j!}$ was found as early as 1877 [5]!

The aim of the present paper is to investigate the analytic and combinatorial properties of two sequences introduced by Z. I. Szabó. Investigating Hilbert's fourth problem, in order to define a transformation on some cylindric functions whose domain is $\mathbf{R}^n$, Z. I. Szabó introduced the following transformation on continous real functions of one variable:

$$\overset{(n)}{f^{\wedge}}(x)=\int_0^{\pi/2}\cos^{n-2}\alpha f(x\sin\alpha)\,d\alpha,$$

and its inverse transformation for odd and even numbers as follows. We use the abbreviations $E=(1/t)(d/dt)$, $D=t(d/dt)$ and E^m, D^m for their powers. Now the inverses are

$$\overset{(2m+1)^{-1}}{\wedge}(f)=\frac{1}{2^{m-1}(m-1)!}\{E^{m-1}(t^{2m-1}\overset{(2m+1)}{f^{\wedge}}(t)\},$$

for $m\geqq 1$ and

$$\overset{(2m)^{-1}}{\wedge}(f)=\frac{1}{\pi(2m-3)!!}\{E^{m-1}(t^{2m-2}\overset{(2m)}{f^{\wedge}}(t))\}^{\overset{(2)}{\wedge}},$$

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for $m \geqq 2$. The inverses can be rewritten with some constants $a_{m,i}$ and $b_{m,i}$ as

$$\overset{(2m+1)^{-1}}{\wedge}(f) = \sum_{i=0}^{m} a_{m,i} t^i f^{\overset{(2m+1)}{\wedge}}(t)^{(i)}$$

and

$$\overset{(2m)^{-1}}{\wedge}(f) = \sum_{i=0}^{m-1} b_{m,i}\left\{t^{i+1} f^{\overset{(2m)}{\wedge}}(t)^{(i)}\right\}^{\overset{(2)}{\wedge}},$$

if $f^{\overset{(2m+1)}{\wedge}}$ (resp. $f^{\overset{(2m)}{\wedge}}$) is m times differentiable.

It is time to define the holiday numbers. We call $\psi(m, i)$ the *holiday numbers of the first kind* (resp. $\varphi(m, i)$ *of the second kind*) where

$$(0'') \qquad \Psi_m(y) = \{E^{m-1}(t^{2m-1}y)\}' = \sum_{i=0}^{m} \psi(m, i) t^i y^{(i)}$$

and

$$(0''') \qquad \Phi_m(y) = E^m(t^{2m}y) = \sum_{i=0}^{m} \varphi(m, i) t^i y^{(i)}.$$

The background of these names will be given in the fourth section.

Now we have by easy calculation that

$$a_{m,i} = \frac{1}{2^{m-1}(m-1)!}\psi(m, i) \quad \text{and} \quad b_{m,i} = \frac{2}{\pi(2m-3)!!}\varphi(m, i).$$

We note that Z. I. Szabó was interested only in the existence of $a_{m,i}$ and $b_{m,i}$ and not in their behaviour.

Our investigation is based on the substitution of the exponential function into Ψ_m and Φ_m, what is essentially the same as done by Bell [2], Rota [10], Rota, Kahaner, and Odlyzko [11], where the exponential function is substituted into the formula

$$(0') \qquad S_m(y) = D^m(y) = \sum_{i=1}^{m} S(m, i) t^i y^{(i)}.$$

The reason of the applicability of the same method is that in

$$x^{-\alpha_m}(\dots x^{-\alpha_2}(x^{-\alpha_1}(x^{\sum_{i=1}^{m}(\alpha_i+1)}y)')\dots)'$$

the coefficient of $y^{(i)}$ is a monomial of x.

If $\alpha_i = 0$, we have a trivial case.

If $\alpha_i = 1$, we have $(0''')$, if $\alpha_i = -1$, we have $(0')$. We mention the Lah numbers, which have properties similar to the Stirling and holiday numbers [7].

In the following we number the analogous formulae concerning with S, Ψ, Φ by (n'), (n''), (n'''). Even though the present paper does not contain any new result on Stirling numbers, we sketch proofs for them, since these proofs are carried over to the holiday numbers. All these results can be found either in RIORDAN [9] or in LOVÁSZ [8], in analytic and in combinatorial treatment. More references on Stirling numbers can be found in [10] and [11].

We are indebted to Z. I. Szabó and L. Lovász for the encouraging talks on holiday numbers.

2. Generating functions. We complete the definitions with

$$S(0,0)=\psi(0,0)=\varphi(0,0)=1.$$

Applying S_m, Ψ_m and Φ_m to t^k we have

$$S_m(t^k)=k^m t^k, \tag{1'}$$

$$\Psi_m(t^k)=(2m+k-1)(2m+k-3)\dots(k+1)t^k, \tag{1''}$$

$$\Phi_m(t^k)=(2m+k)(2m+k-2)\dots(k+2)t^k, \tag{1'''}$$

thus applying them to e^t we have

$$\sum_{n=0}^{\infty}\frac{n^m t^n}{n!}=S_m(e^t)=e^t\sum_{k=1}^{m}S(m,k)t^k, \tag{2'}$$

$$\sum_{n=0}^{\infty}\frac{(2m+n-1)(2m+n-3)\dots(n+1)}{n!}t^n=\Psi_m(e^t)=e^t\sum_{k=0}^{m}\Psi(m,k)t^k, \tag{2''}$$

$$\sum_{n=0}^{\infty}\frac{(2m+n)(2m+n-2)\dots(n+2)}{n!}t^n=\Phi_m(e^t)=e^t\sum_{k=0}^{m}\varphi(m,k)t^k. \tag{2'''}$$

On the one hand, dividing by e^t it gives explicit formulae

$$S(m,k)=\sum_{j=1}^{k}(-1)^{k+j}\binom{k}{j}\frac{j^m}{k!}, \tag{3'}$$

$$\psi(m,k)=\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\frac{(2m+j-1)(2m+j-3)\dots(j+1)}{k!}, \tag{3''}$$

$$\varphi(m,k)=\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\frac{(2m+j)(2m+j-2)\dots(j+2)}{k!}, \tag{3'''}$$

and so, the right sides of (3′), (3″) and (3‴) are zero for $k>m$. On the other hand, substituting $t=1$ into (2′), (2″), (2‴), we have

$$\sum_{k=1}^{m} S(m,k)=\frac{1}{e}\sum_{n=0}^{\infty}\frac{n^m}{n!},$$

$$\sum_{k=0}^{m}\psi(m,k)=\frac{1}{e}\sum_{n=0}^{\infty}\frac{(2m+n-1)(2m+n-3)\dots(n+1)}{n!},$$

$$\sum_{k=0}^{m}\varphi(m,k)=\frac{1}{e}\sum_{n=0}^{\infty}\frac{(2m+n)(2m+n-2)\dots(n+2)}{n!}.$$

Calculating from (2′), (2″), (2‴) the generating functions, we have

$$(4') \quad G_S(t,z)=1+\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{k=1}^{m}S(m,k)t^k=1+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(zn)^m}{m!}\frac{t^n}{n!}e^{-t}=e^{t(e^z-1)},$$

$$(4'') \quad \begin{aligned} G_\Psi(t,z)&=1+\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{k=0}^{m}\psi(m,k)t^k=\\ &=1+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2m+n-1)(2m+n-3)\dots(n+1)}{m!}z^m\frac{t}{n!}e^{-t}=\frac{1}{\sqrt{1-2z}}e^{t\left(\frac{1}{\sqrt{1-2z}}-1\right)},\end{aligned}$$

$$(4''') \quad \begin{aligned} G_\Phi(t,z)&=1+\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{k=0}^{m}\varphi(m,k)t^k=\\ &=1+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2m+n)(2m+n-2)\dots(n+2)}{m!}z^m\frac{t^n}{n!}e^{-t}=\frac{1}{1-2z}e^{t\left(\frac{1}{\sqrt{1-2z}}-1\right)}.\end{aligned}$$

3. Recurrences. Since $(ty)^{(n)}=\sum_i\binom{n}{i}t^{(i)}y^{(n-i)}=ty^{(n)}+ny^{(n-1)}$, we have from (1′), (1″), (1‴)

$$S_m(y)=S_{m-1}\big((ty)'\big)-S_{m-1}(y)$$

$$\Psi_m(y)=\Psi_{m-1}\big((ty')\big)+(2m-2)\Psi_{m-1}(y)$$

$$\Phi_m(y)=\Phi_{m-1}\big((ty)'\big)+(2m-1)\Phi_{m-1}(y)$$

and the following recurrences:

$$(5') \qquad S(m,k)=kS(m-1,k)+S(m-1,k-1)$$

$$(5'') \qquad \psi(m,k)=(2m+k-1)\psi(m-1,k)+\psi(m-1,k-1)$$

$$(5''') \qquad \varphi(m,k)=(2m+k)\varphi(m-1,k)+\varphi(m-1,k-1).$$

Now it is an easy task to tabulate some holiday numbers.

ψ m \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	3	5	1	0	0	0	0
3	15	33	12	1	0	0	0
4	105	279	141	22	1	0	0
5	945	2 895	1 830	405	35	1	0
6	10 395	35 685	26 685	7500	930	51	1

φ m \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
2	8	7	1	0	0	0	0
3	48	57	15	1	0	0	0
4	384	561	207	26	1	0	0
5	3 840	6 555	3 045	545	40	1	0
6	46 080	89 055	49 185	11 220	1185	57	1

We gain some more complicated recurrences comparing the genetating functions with their partial derivatives. Applying $\partial/\partial t$ to the generating functions we notice

$$\frac{\partial}{\partial t} G_S(t, z) = (e^z-1)\, G_S(t, z),$$

$$\frac{\partial}{\partial t} G_\Psi(t, z) = \left(\frac{1}{\sqrt{1-2z}}-1\right) G_\Psi(t, z),$$

$$\frac{\partial}{\partial t} G_\Phi(t, z) = \left(\frac{1}{\sqrt{1-2z}}-1\right) G_\Phi(t, z).$$

Comparing the coefficients in the previous identities we have the following recurrences.

(6′) $S(m, k+1) = \dfrac{1}{k+1}\sum_{j=1}^{m}\binom{m}{j} S(m-j, k)$ for $k \geqq 1$ and $S(m, 1) = 1$,

(6″) $\Psi(m, k+1) =$

$$= \frac{1}{k+1}\sum_{j=1}^{m}\binom{m}{j}(2j-1)!!\,\Psi(m-j, k) \quad \text{for } k \geqq 0 \text{ and } \Psi(m, 0) = (2m-1)!!$$

(6‴)

$$\varphi(m, k+1) = \frac{1}{k+1}\sum_{j=1}^{m}\binom{m}{j}(2j-1)!!\,\Psi(m-j, k) \quad \text{for } k \geqq 0 \text{ and } \varphi(m, 0) = 2^m m!$$

Applying $\partial/\partial z$ to the generating functions we notice

$$\frac{\partial}{\partial z} G_S(t, z) = te^z G_S(t, z),$$

$$\frac{\partial}{\partial z} G_\Psi(t, z) = \{(1-2z)^{-1}+t(1-2z)^{-3/2}\} G_\Psi(t, z),$$

$$\frac{\partial}{\partial z} G_\Phi(t, z) = \frac{2+t(1-2z)^{-1/2}}{1-2z} G_\Phi(t, z).$$

Comparing the coefficients (and using for (7‴) the easy identity

$$2^S s!\left\{1+\frac{(2t-1)!!}{2^t t!}\right\} = (2s+1)!!)),$$

we have

$$S(m+1, k) = \sum_{s=1}^{m} \binom{m}{s} S(m-s, k-1), \tag{7'}$$

$$\psi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} 2^s s!\,\psi(m-s, k) + \sum_{s=0}^{m} \binom{m}{s} (2s+1)!!\,\psi(m-s, k-1), \tag{7''}$$

$$\varphi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} 2^{s+1} s!\,\varphi(m-s, k) + \sum_{s=0}^{m} \binom{m}{s} (2s+1)!!\,\varphi(m-s, k-1). \tag{7'''}$$

From (7′), (7″), (7‴) we have recurrences analogous to the recurrence of Bell numbers (the sum of Stirling numbers of second kind):

$$\sum_{k=1}^{m+1} S(m+1, k) = \sum_{s=1}^{m} \binom{m}{s} \sum_{k=1}^{m-s} S(m-s, k),$$

$$\sum_{k=0}^{m+1} \psi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} (2^S s!+(2s+1)!!) \sum_{k=0}^{m-s} \psi(m-s, k),$$

$$\sum_{k=0}^{m+1} \varphi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} (2^{S+1} s!+(2s+1)!!) \sum_{k=0}^{m-s} \varphi(m-s, k).$$

4. The combinatorial meaning of the holiday numbers. The leader of the social department of a company is to make plans for m married couples for their holidays. We say that he is to compile a Ψ-plan, if his tasks were (i), (ii), (iii).

(i) He is to compile k nonempty, pairwise disjoint groups of married couples. It is possible that some couples do not belong to any group.

(ii) He is to make a complete matching in the groups made in (i). Every man or woman of the group can be matched with every man or woman of the same group.

(iii) He is to make a complete matching in the rest that may have been made in (i) on the way written in (ii). We say that the leader of the social department is to compile a Φ-plan, if (iii) were changed for (iv):

(iv) he is to order the married couples of the rest (for the next year holidays) and to write in his notebook the name of either the husband or the wife.

Let us denote by $\tilde{\psi}(m, k)$ (resp. $\tilde{\varphi}(m, k)$) the number of all possible Ψ-plans (resp. Φ-plans) for m married couples into k groups.

Theorem. *$\tilde{\psi}(m, k)=\psi(m, k)$ and $\tilde{\varphi}(m, k)=\varphi(m, k)$.*

Proof. We prove that $\tilde{\psi}$, ψ and $\tilde{\varphi}$, φ have the same initial values and obey the same recurrence. It is easy to see that $\psi(m, 0)=(2m-1)!!=\tilde{\psi}(m, 0)$ and $\varphi(m, 0)= =2^m m!=\tilde{\varphi}(m, 0)$. Now we prove that (6″) and (6‴) hold with $\tilde{\psi}$ and $\tilde{\varphi}$ instead of ψ and φ. In both formulae the right-hand side means the choice of j couples and a matching on them, and a plan for $m-j$ couples into groups. This way all the plans are enumerated $k+1$ times.

We note that (6′) has a combinatorial proof on a similar way. The reader can give alternative proofs on the theorem using other formulae rather than (6″), (6‴), e.g. using (5″) and (5‴) and distinguishing cases by the mth married couple (see [8] 1.6); using (3″) and (3‴) and sifting (see [8] 2.4); using (7″) and (7‴) and distinguishing cases whether the $(m+1)$th married couple travel or not and by the length of the alternating circle of couples and matched pairs containing the $(m+1)$th married couple in the latter case for Ψ (and by the position of the $(m+1)$th married couple in the notebook for Φ).

As a corollary of the theorem of the present section we give new explicit formulae for the holiday numbers:

$$\psi(m, k) = \sum_{\substack{x_1 \geqq 1 \; x_2 \geqq 1 \; \dots \; x_k \geqq 1 \\ x_1+\dots+x_k \leqq m}} \frac{1}{k!} \binom{m}{x_1} \binom{m-x_1}{x_2} \dots \binom{m-x_1 \dots - x_{k-1}}{x_k} \times$$

$$\times \Big(2m-1-2\sum_{i=1}^{k} x_i\Big)!! \prod_{i=1}^{k} (2x_i-1)!!$$

and

$$\varphi(m, k) = \sum_{\substack{x_1 \geqq 1 \; x_2 \geqq 1 \; \dots \; x_k \geqq 1 \\ x_1+\dots+x_k \geqq m}} \frac{1}{k!} \binom{m}{x_1} \binom{m-x_1}{x_2} \dots \binom{m-x_1-\dots-x_{k-1}}{x_k} \times$$

$$\times 2^{m-\sum_{i=1}^{k} x_i} \Big(m-\sum_{i=1}^{k} x_i\Big)! \prod_{i=1}^{k} (2x_i-1)!!.$$

5. Holiday transformation of sequences.

Theorem. *Suppose* $b_m = \sum_k \psi(m,k) a_k$ *and* $d_m = \sum_k \varphi(m,k) c_k$. *Then*

$$a_k = \sum_t s(k,t)(-1)^t \sum_i \binom{t}{i} 2^i \sum_j S(i,j) \frac{b_j}{(-2)^j},$$

and

$$c_k = \sum_t s(k,t)(-2)^t \sum_i \binom{t}{i} \sum_j S(i,j) \frac{d_j}{(-2)^j},$$

where $s(k,t)$ *is the Stirling number of the first kind.*

Proof. At first we state new explicit formulae for the holiday numbers in terms of the Stirling numbers of the first and the second kind

$$\psi(m,k) = 2^m \sum_{i,t} (-1)^{m+i} \binom{i}{t} 2^{-i} s(m,i) S(t,k), \tag{8''}$$

$$\varphi(m,k) = 2^m \sum_{i,t} (-1)^{m+i} \binom{i}{t} 2^{-t} s(m,i) S(t,k). \tag{8'''}$$

Having applied Ψ_m to t^x, (0″) and (1″) give

$$(2m-1+x)(2m-3+x)\dots(1+x) = \sum_k \psi(m,k) x(x-1)\dots(x-k+1).$$

But we have

$$(2m-1+x)(2m-3+x)\dots(x+1) = (-2)^m m! \binom{-\frac{x}{2}-\frac{1}{2}}{m}$$

$$= (-2)^m \sum_i s(m,i)\left(-\frac{x}{2}-\frac{1}{2}\right)^i = (-2)^m \sum_i s(m,i)\left(-\frac{1}{2}\right)^i \sum_t \binom{i}{t} \times$$

$$\times \sum_k S(t,k) x(x-1)\dots(x-k+1),$$

and comparing the coefficients of the linearly independent polynomials $x(x-1)\dots(x-k+1)$, we get (8″).

Analogously, having applied Φ_m to t^x, (0‴) and (1‴) give

$$(2m+x)(2m-2+x)\dots(2+x) = \sum_k \varphi(m,k) x(x-1)\dots(x-k+1),$$

and we have

$$(2m+x)(2m-2+x)\dots(2+x) = (-2)^m m! \binom{-\frac{x}{2}-1}{m} =$$

$$= (-2)^m \sum_i s(m,i)\left(-1-\frac{x}{2}\right)^i = (-2)^m \sum_i s(m,i)(-1)^i \times$$

$$\times \sum_t \binom{i}{t} 2^{-t} \sum_k S(t,k) x(x-1)\dots(x-k+1).$$

Comparing the coefficients we get (8‴). (8″) and (8‴) implies the theorem through a Stirling, a binomial and again a Stirling inversion as follows:

$$b_m = \sum_k \psi(m, k) a_k = \sum_k 2^m \sum_{i,t} (-1)^{m+1} \binom{i}{t} 2^{-i} s(m, i) S(t, k) a_k$$

is equivalent to

$$\sum_t \binom{i}{t} \sum_k S(t, k) a_k = (-2)^i \sum_j S(i, j) \frac{b_j}{(-2)^j}$$

which is equivalent to

$$\sum_k S(t, k) a_k = \sum_i (-1)^{t-i} \binom{t}{i} \sum_j (-2)^i S(i, j) \frac{b_j}{(-2)^j},$$

which is equivalent to the first part of the theorem. Analogously we find that

$$d_m = \sum_k \varphi(m, k) c_k = \sum_k 2^m \sum_{i,t} (-1)^{m+i} \binom{i}{t} 2^{-t} s(m, i) S(t, k) c_k$$

is equivalent to

$$\sum_t \binom{i}{t} 2^{-t} \sum_k S(t, k) c_k = \sum_j (-1)^i S(i, j) \frac{d_j}{(-2)^j},$$

which is equivalent to

$$\sum_k S(t, k) c_k = \sum_i (-1)^{t-i} \binom{t}{i} 2^t \sum_j (-1)^i S(i, j) \frac{d_j}{(-2)^j},$$

which is equivalent to the second statement of the theorem.

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Multivariable composition of Sobol'ev functions

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Dedicated to Prof. K. Tandori on his 60th birthday

1. Introduction

In this paper we shall prove some theorems on the composition of a multivariable outer function and inner functions of one variable belonging to certain Sobol'ev spaces. These theorems are based on a generalization (see [7] and Assertion 1) of the result of F. RIESZ [4] and the well-known Sobol'ev embedding theorems. Very interesting special cases are considered when $W_2^s(\Omega):=H^s(\Omega)$. Really, in this case the spaces $H^s(\Omega)$ can be characterized by Fourier coefficients, thus a relation can be proved between the convergency rate of the Fourier coefficients of the components and that of the composition.

2. Results

The following main results will be proved.

Theorem 1. *Let* $n\in\mathbf{N}$, $c, d\in\mathbf{R}$ $(c<d)$, $p, q, r, s\in\mathbf{R}$. *Suppose that* $p, q, r\in]1, \infty[$ *and that the equality*

$$(1)\qquad \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)=1-\frac{1}{r}$$

and the inequality

$$(2)\qquad s>1+\frac{n-1}{p}$$

are satisfied. Then, for each $f\in W_p^s(\mathbf{R}^n)$ *and monotonous functions* $g_1, g_2, \dots, g_n\in$ $\in W_q^1(c, d)$, *the composition* $f\circ g$ *belongs to* $W_r^1(c, d)$.

Theorem 1 and the Sobol'ev embedding theorems give us the following

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Theorem 2. *Let* $n\in\mathbf{N}$, $c, d\in\mathbf{R}$ $(c<d)$, $p, q, r, s_1, s_2, s_3\in\mathbf{R}$. *Suppose that* $p, q, r\in]1, \infty[$, $p, q\geqq r$, *and that the inequality*

$$s_0 := \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)+\frac{1}{r} < 1 \tag{3}$$

is satisfied. Let $s_3\in]s_0, 1]$,

$$s_1\in\left[1+\frac{n-1}{p},\ \frac{n}{p}+\left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{q}\right)^{-1}\right[, \tag{4}$$

$$s_2\in\left[1,\ \frac{1}{q}+\left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)^{-1}\right[\tag{5}$$

be numbers satisfying the inequality

$$\left(s_1-\frac{n}{p}\right)\left(s_2-\frac{1}{q}\right) > s_3-\frac{1}{r}. \tag{6}$$

Then, for each $f\in W_p^{s_1}(\mathbf{R}^n)$ *and monotonous functions* $g_1, g_2, \ldots, g_n\in W_q^{s_2}(c, d)$, *the composition* $f\circ(g_1, g_2, \ldots, g_n)$ *belongs to* $W_r^{s_3}(c, d)$.

Now, we mention the special case of Theorem 2, when $p, q, r:=2$. We can use the following characterization by the Fourier transform and series:

a) $f\in H^s(\mathbf{R}^n)$ if and only if $x\mapsto\hat{f}(x)(1+\|x\|^2)^{s/2}\in L_2(\mathbf{R}^n)$,

b) $g\in H^s(c, d)$ if and only if $\sum_{n=1}^{\infty}|\hat{g}(n)|^2 n^{2s}<\infty$

where $\{\hat{g}(n)\colon n\in\mathbf{N}\}$ are the Fourier coefficients of the function $g\in L_2(c, d)$ with respect to the system $\{\varphi_n\colon n\in\mathbf{N}\}$, of the eigenfunctions of the eigenvalue problem

$$(-1)^k x^{(2k)}+x = \lambda x,$$

$$x^{(k)}(c) = x^{(k+1)}(c) = \ldots = x^{(2k-1)}(c) = 0, \quad x^{(k)}(d) = x^{(k+1)}(d) = \ldots = x^{(2k-1)}(d) = 0$$

induced by the immersion $H^k(c, d)\subset L_2(c, d)$ and $s\leqq k$. If $k:=1$, then $\varphi_1(t)= =(d-c)^{-1}$,

$$\varphi_n(t) = 2(d-c)^{-1}\cos[(n-1)\pi(t-c)(d-c)^{-1}] \quad (n\geqq 2).$$

Theorem 3. *Let* $n\in\mathbf{N}$, $c, d\in\mathbf{R}$, $(c<d)$. *Suppose that the numbers*

$$s_3\in\left]\frac{3}{4}, 1\right[, \quad s_1\in\left]\frac{n+1}{2}, 2s_3+\frac{n-2}{2}\right[, \quad s_2\in\left[1, 2s_3-\frac{1}{2}\right[$$

satisfy the inequality

$$\left(s_1-\frac{n}{2}\right)\left(s_2-\frac{1}{2}\right) > s_3-\frac{1}{2}. \tag{7}$$

Then, if

$$x \mapsto \hat{f}(x)(1+\|x\|^2)^{s_1/2} \in L_2(\mathbf{R}^n), \tag{8}$$

$$\sum_{n=1}^{\infty} |\hat{g}_i(n)^2 n^{2s_2} < \infty \quad (i = 1, 2, \ldots, n) \tag{9}$$

and g_i $(i=1, 2, \ldots, n)$ *are monotonous, then*

$$\sum_{n=1}^{\infty} |\widehat{f\circ (g_1, \ldots, g_n)}(n)|^2 n^{2s_3} < \infty \tag{10}$$

also holds.

It is clear that in Theorem 2 the space $H^s(\mathbf{R}^n)$ can be replaced by $H^s(\Omega)$ for a bounded region $\Omega\subset\mathbf{R}^n$, such that the closure of the range R_g of $g=(g_1, \ldots, g_n)$ belongs to Ω. In this case for each $f\in H^s(\Omega)$ there exists a function $\tilde{f}\in H^s(\mathbf{R}^n)$ such that $\tilde{f}\circ(g_1, \ldots, g_n)=f\circ(g_1, \ldots, g_n)$ holds. For this it is enough to resctrict f to a region $R_g\subset\Omega_1\subset\Omega$ such that Ω_1 is a "good" region having the extension property (see [1], [2]). Thus the relation (8) can be replaced by

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^2 n^{2s_1} < \infty;$$

where $\{\hat{f}(n)\colon n\in\mathbf{N}\}$ are the Fourier coefficients with respect to an orthonormed system of eigenfunctions of an elliptic problem related to the embedding $H^k(\Omega)\subset L_2(\Omega)$ where $s\leqq k$ holds (see [3], [5]).

For the spaces $W_p^s(\mathbf{R}^n)$ and $W_p^2(c, d)$ we can prove Theorem 4 because the order of differentiability is high enough.

Theorem 4. *Let* $n\in\mathbf{N}$, $c, d\in\mathbf{R}$ $(c<d)$, $p\in\mathbf{R}$. *Suppose that* $1<p, s>2+(n-1)/p$. *If* $f\in W_p^s(\mathbf{R}^n)$ *and the monotonous functions* $g_1, g_2, \ldots, g_n\in W_p^2(c, d)$ *then the composition* $f\circ(g_1, g_2, \ldots, g_n)$ *belongs to the same space* $W_p^2(c, d)$.

In Theorems 1—4 the monotony of the inner functions play a very important role. Finally we show a theorem, in which the monotony of the inner functions can be omitted.

Theorem 5. *Let* $n\in\mathbf{N}$, $c, d\in\mathbf{R}$ $(c<d)$, $p, q\in]1, \infty[$, $s\in\mathbf{R}$. *Suppose that* $s>1+(n/p)$. *Then, if* $f\in W_p^s(\mathbf{R}^n)$, $g_i\in W_q^1(c, d)$ $(i=1, \ldots, n)$, *then* $f\circ(g_1, \ldots, g_n)\in W_q^1(c, d)$.

3. Outline on Sobol'ev spaces

In this section we survey some facts on the Sobol'ev spaces.

Let $k, n \in \mathbf{N}$, $p \in]1, \infty[$, $\Omega \subset \mathbf{R}^n$ be an open subset. The Sobol'ev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) := \{f\colon D^\alpha f \in L_p(\Omega),\ |\alpha| \leqq k\}$$

equipped with an appropriate norm (see [1, Ch. III]).

Let $W_p^0(\Omega) := L_p(\Omega)$.

If $s \in \mathbf{R}_+$, then the Sobol'ev space $W_p^s(\Omega)$ is defined by

$$W_p^s(\Omega) := \left\{f \in W_p^{[s]}(\Omega)\colon \int_{\Omega\times\Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+(s-[s])p}}\, dx\, dy < \infty, \quad |\alpha| = [s]\right\}$$

equipped again with an appropriate norm (see [1, Ch. VII]).

Here $[s]$ denotes the entire part of the real number s. The Sobol'ev embedding theorems will be used (see [1], [2]) in the next:

a) if $s > n/p$, then $W_p^s(\Omega) \subset C_B(\Omega)$ and the embedding is continuous and linear,

b) if $s_1, s_2 \in \mathbf{R}$, $p_1, p_2 \in]1, \infty[$, $s_1 \leqq s_2$, $p_1 \geqq p_2$ and $(n/p_1) - s_1 \geqq (n/p_2) - s_2$, then $W_{p_1}^{s_1}(\Omega) \supset W_{p_2}^{s_2}(\Omega)$ and the embedding is continuous and linear.

The mentioned result of F. Riesz [4] is the following: an absolutely continuous function $f\colon]a, b[\to \mathbf{R}$ (or $\mathbf{C}$) has its derivative $f' \in L_p[a, b]$ if and only if there exists a number $K \geqq 0$ such that, for any system $\{]a_i, b_i[\subset]a, b[\colon i \in I\}$ of nonoverlapping bounded subintervals, the inequality

$$\sum_i \frac{|f(b_i) - f(a_i)|^p}{|b_i - a_i|^{p-1}} \leqq K$$

holds, and the best constant is $\|f'\|_{L_p}^p$.

As it was mentioned in the introduction, the proofs are based on the Riesz theorem and

Assertion 1. (see [7]). *Let* $n \in \mathbf{N}$, $p, s \in \mathbf{R}$. *Suppose that* $p \in]1, \infty[$ *and* $s > 1 + ((n-1)/p)$. *If* $f \in W_p^s(\mathbf{R}^n)$, *then there exist real numbers* $K_i \geqq 0$ $(i = 1, 2, \ldots, n)$ *such that, for any integers* $I_i \in \mathbf{N}$, *systems*

$$\{]a_{ij}, b_{ij}[\subset \mathbf{R}\colon j = 1, 2, \ldots, I_i\}$$

of nonoverlapping bounded subintervals and sets $\{\xi_{ij} \in \mathbf{R}^{n-1}\colon j = 1, 2, \ldots, I_i\}$, *the inequalities*

$$\sum_{j=1}^{I_i} \frac{|f_{i,b_{ij}}(\xi_{ij}) - f_{i,a_{ij}}(\xi_{ij})|^p}{|b_{ij} - a_{ij}|^{p-1}} \leqq K_i$$

hold, where for any $i = 1, 2, \ldots, n$, $a \in \mathbf{R}$ *the function* $f_{i,a}\colon \mathbf{R}^{n-1} \to \mathbf{R}$ *(or* $\mathbf{C}$*) is defined by*

$$\xi \mapsto f(\xi_1, \ldots, \xi_{i-1}, a, \xi_i, \ldots, \xi_{n-1}).$$

We mention that the best constants K_i $(i=1, 2, ..., n)$ are

$$\int_{\mathbf{R}} \|(\partial_i f)_{i,t}\|^p_{W_p^{s-1}(\mathbf{R}^{n-1})}\, dt.$$

Next, we need the following

Assertion 2 (see [6]). *Let* $c, d \in \mathbf{R}$ $(c<d)$ $p \in]1, \infty[$. *If* $f, g \in W_p^1(a, b)$, *then* $fg \in W_p^1(a, b)$.

The proof can be made by the Riesz theorem.

4. Proofs and remarks

Proof of Theorem 1. Define the number $\alpha := r(1-(1/p))$. We can easily see that (1) is equivalent to the equality

$$\frac{r}{p}+\frac{r}{q}\left(1-\frac{1}{p}\right) = 1. \tag{11}$$

Our proof if based on the Riesz theorem and Assertion 1. For this, let $I \in \mathbf{N}$, $\{]c_i, d_i[\subset]c, d[: i=1, 2, ..., I\}$ be a system of nonoverlapping subintervals. Let

$$\xi_{ij} := \big(g_1(d_i), ..., g_{j-1}(d_i),\ g_{j+1}(c_i), ..., g_n(c_i)\big) \quad (\in \mathbf{R}^{n-1}),$$
$$d_{ij} := g_j(d_i), \quad c_{ij} := g_j(c_i)$$

$(j=1, 2, ..., n, i=1, 2, ..., I)$. Then by (11) we can estimate with the Hölder inequality:

$$\sum_{i=1}^{I} \frac{|f\circ(g_1, ..., g_n)(d_i) - f\circ(g_1, ..., g_n)(c_i)|^r}{|d_i-c_i|^{r-1}} \leqq$$
$$\leqq \sum_{i=1}^{I} n^{r-1} \sum_{j=1}^{n} \frac{|f_{j,d_{ij}}(\xi_{ij}) - f_{j,c_{ij}}(\xi_{ij})|^r}{|d_{ij}-c_{ij}|^{\alpha}} \cdot \frac{|g_j(d_i)-g_j(c_i)|^{\alpha}}{|d_i-c_i|^{r-1}} \leqq$$
$$\leqq n^{r-1} \sum_{j=1}^{n} \left(\sum_{i=1}^{I} \frac{|f_{j,d_{ij}}(\xi_{ij}) - f_{j,c_{ij}}(\xi_{ij})|^p}{|d_{ij}-c_{ij}|^{\alpha(p/r)}} \right)^{r/p} \left(\sum_{i=1}^{I} \frac{|g_j(d_i)-g_j(c_i)|^q}{|d_i-c_i|^{(r-1)(q/\alpha)}} \right)^{\alpha/q}.$$

We can check that $\alpha(p/r) = p-1$, $(r-1)(q/\alpha) = q-1$, $\alpha/q = (r/q)(1-(1/p))$ thus, by Assertion 1 and the Riesz theorem

$$\sum_{i=1}^{I} \frac{|f\circ(g_1, ..., g_n)(d_i) - f\circ(g_1, ..., g_n)(c_i)|^r}{|d_i-c_i|^{r-1}} \leqq n^{r-1}\Big(\sum_{j=1}^{n} \|\partial_j f\|^{p(r/p)}_{W_p^{s-1}} \|g_j'\|^{r(1-(1/p))}_{L_q} \Big),$$

because

$$\int_{\mathbf{R}} \|(\partial_j f)_{j,t}\|^p_{W_p^{s-1}(\mathbf{R}^{n-1})}\, dt \leqq \|\partial_j f\|^p_{W_p^{s-1}(\mathbf{R}^n)};$$

that is

$$\|(f\circ(g_1,\dots,g_n))'\|_{L_r} \leqq n^{(r-1)/r}\Big(\sum_{j=1}^{n}\|\partial_j f\|^r_{W_p^{s-1}}\|g_j'\|_{L_q}^{r(1-(1/p))}\Big)^{1/r} \leqq$$

$$\leqq n^{(r-1)/r}\|f\|_{W_p^s}\Big(\sum_{j=1}^{n}\|g_j'\|_{L_q}^{r(1-(1/p))}\Big)^{1/r}.$$

Proof of Theorem 2. From (3) the inequalities

$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) < s_3-\frac{1}{r} \leqq 1-\frac{1}{r}$$

follow immediately. Thus the inequalities

$$1-\frac{1}{p} \leqq s_1-\frac{n}{p} < \left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{q}\right)^{-1} \leqq \left(1-\frac{1}{r}\right)\left(1-\frac{1}{q}\right)^{-1} \leqq 1,$$

$$1-\frac{1}{q} \leqq s_2-\frac{1}{q} < \left(s_3-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)^{-1} \leqq \left(1-\frac{1}{r}\right)\left(1-\frac{1}{p}\right)^{-1} \leqq 1$$

hold, so the intervals (4), (5) are nonempty, and the inequality $1\leqq s_2<1+(1/q)$ is satisfied. Thus there exist numbers $p_0\geqq p$, $q_0\geqq q$, $r_0\leqq r$ such that for p_0, q_0, r_0

$$(12)\qquad s_1-\frac{n}{p}=:1-\frac{1}{p_0},\quad s_2-\frac{1}{q}=:1-\frac{1}{q_0},\quad \left(s_1-\frac{n}{p}\right)\left(s_2-\frac{1}{q}\right)=:1-\frac{1}{r_0}$$

hold. By theorem 1, if $f\in W_{p_0}^{s_1}(\mathbf{R}^n)$ and the monotonous functions $g_i\in W_{q_0}^1(c,d)$ $(i=1,\dots,n)$, then the composition $f\circ(g_1,\dots,g_n)$ belongs to $W_{r_0}^1(c,d)$. By (12) the $W_q^{s_2}(c,d)\subset W_{q_0}^1(c,d)$ and $W_r^{s_3}(c,d)\supset W_{r_0}^1(c,d)$ embeddings are continuous and linear, so, if $f\in W_p^{s_1}(\mathbf{R}^n)$ and the monotonous functions $g_i\in W_q^{s_2}(c,d)$ $(i=1,\dots,n)$, then $f\in W_{p_0}^{s_1}(\mathbf{R}^n)$ $(p=p_0)$, $g_i\in W_{q_0}^1(c,d)$, thus $f\circ(g_1,\dots,g_n)\in W_{r_0}^1(c,d)\subset W_r^{s_3}(c,d)$ too.

Theorem 3 follows from Theorem 2 and the characterizations a), b) (see Section 2).

Proof of Theorem 4. By assumption $f\in W_p^s(\mathbf{R}^n)$, and $g_i\in W_p^2(c,d)$ $(i=1,2,\dots,n)$, so by the chain rule

$$(14)\qquad (f\circ(g_1,g_2,\dots,g_n))'=\sum_{i=1}^{n}\partial_i f\circ(g_1,g_2,\dots,g_n)g_i'.$$

Again by assumption the functions $g_i\in W_p^2(c,d)$ are monotonous and clearly $\partial_i f\in W_p^{s-1}(\mathbf{R}^n)$ $s-1>1+((n-1)/p)$. Thus by Theorem 2 $\partial_i f\circ(g_1,g_2,\dots,g_n)\in$ $\in W_p^1(c,d)$. On the other hand $g_i'\in W_p^1(c,d)$ holds obviously, so by Assertion 2 the sum (14) of products belongs to $W_p^1(c, d)$. This means that the composition

$$f\circ(g_1, g_2, \ldots, g_n)\in W_p^2(c, d).$$

Remark 1. In the special case $p:=2$ we can use the characterization of the spaces of type H^s in terms of the Fourier coefficients. If $\Omega\subset\mathbf{R}^n$ and $(c, d)\subset\mathbf{R}$ are bounded, $f\in L_2(\Omega)$, the monotonous functions $g_i\in L_2(c, d)$ $(i=1, 2, \ldots, n)$ and there exists the composition $f\circ(g_1, \ldots, g_n)$, then from

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^2 n^{2s} < \infty,$$

$$\sum_{n=1}^{\infty} |\hat{g}_i(n)|^2 n^4 < \infty \quad (i = 1, \ldots, n)$$

follows that

$$\sum_{n=1}^{\infty} |\widehat{f\circ(g_1, \ldots, g_n)}(n)|^2 n^4 < \infty.$$

The Fourier coefficients relate to the above mentioned orthonormed systems of eigenfunctions.

Proof of Theorem 5. Apply the Sobol'eb embedding theorem for $W_p^s(\mathbf{R}^n)\subset\subset C^1(\mathbf{R}^n)$. Thus by the chain rule (14) is satisfied almost everywhere over (c, d). Let M denote the norm of the embedding $W_p^{s-1}(\mathbf{R}^n)\subset C_B(\mathbf{R}^n)$. Then

$$\|(f\circ(g_1, \ldots, g_n))'\|_{L_q} = \|\sum_{i=1}^{n} \partial_i f\circ(g_1, \ldots, g_n)g_i'\|_{L_q} \leqq$$

$$\leqq \sum_{i=1}^{n}\Big(\int_c^d |\partial_i f\circ(g_1, \ldots, g_n)|^q |g_i'|^q\Big)^{1/q} \leqq$$

$$\leqq \sum_{i=1}^{n} \|\partial_i f\|_{C_B(\mathbf{R}^n)}\|g_i'\|_{L_q} \leqq M \sum_{i=1}^{n} \|\partial_i f\|_{W_p^{s-1}}\|g_i'\|_{L_q} \leqq$$

$$\leqq M\|f\|_{W_p^s} \sum_{i=1}^{n} \|g_i'\|_{L_q}.$$

Thus $f\circ(g_1, \ldots, g_n)\in W_q^1(c, d)$.

Remark 2. The inequality

$$\|(f\circ(g_1, \ldots, g_n))'\|_{L_q} \leqq M\|f\|_{W_p^s} \sum_{i=1}^{n} \|g_i'\|_{L} \tag{15}$$

also is obtained.

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On G-finite W^*-algebras*

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Dedicated to Professor Károly Tandori on his 60th birthday

Let M be a W^*-algebra and G a group of *-automorphisms of M. In [3] we have proved that if there exists a faithful G-invariant *normal* state φ on M, then for every $t \in M$, the w^*-closure of the convex hull of the orbit of t under G contains a unique G-invariant element t^G. (In fact, we have proved this result under the more general assumption that the family of G-invariant normal states on M is faithful, i.e., M is G-finite. If M is σ-finite, for example, if M is an operator algebra in a separable Hilbert space, then this assumption obviously implies the existence of a faithful G-invariant normal state on M). In the present paper we shall prove that the assumption of normalcy of φ is superfluous in this theorem in case G contains all inner automorphisms of M. In fact, we shall prove the stronger result that the mapping $t \to t^G$ is normal, i.e., M is G-finite [3]. Under additional hypotheses, we shall also prove that φ is itself a normal state.

At the end of the paper we shall make two comments on our paper [4].

Proposition. *Let M be an Abelian W^*-algebra and G a group of *-automorphisms of M. If there exists a faithful G-invariant (not necessarily normal) positive linear form φ on M, then M is G-finite. (For the notion of G-finiteness, cf.* [3].*)*

Proof. Let e be the least upper bound of the supports of all G-invariant normal positive linear forms on M. According to [3], we have to prove that $e=1$. Assume on the contrary that $e \neq 1$. Since e is a G-invariant projection in M, the restrictions of the elements of G to the W^*-algebra $M(1-e)=(1-e)M(1-e)$ form a group G_{1-e} of *-automorphisms of $M(1-e)$. By the definition of e, the only G_{1-e}-invariant normal positive linear form on $M(1-e)$ is the zero functional. Consequently, to prove the theorem, i.e., to obtain a contradiction to the assumption $e \neq 1$, it is suf-

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ficient to show that there is a nonzero G_{1-e}-invariant normal positive linear form on $M(1-e)$. Since the restriction φ_{1-e} of φ to $M(1-e)$ is a faithful G_{1-e}-invariant normal positive linear form on $M(1-e)$, we may assume that $e=0$ and thus $M=M(1-e)$, $G_{1-e}=G$ and $\varphi_{1-e}=\varphi$. In other words, we have to prove that under the hypotheses of the theorem, there exists a nonzero G-invariant normal positive linear form on M.

Let S denote the family of those projections p in M for which $t\to\varphi(tp)=\varphi(ptp)$ is a *normal* positive linear form on M. We are going to show that

(1) if $p, q\in S$, then $p\vee q\in S$

(2) $\sup S=1$.

To prove (1), let $p, q\in S$. Then by the commutativity of M, we have $p\vee q=p+q-pq$ and thus the functional $t\to\varphi(t(p\vee q))=\varphi(tp)+\varphi(t(q-pq))$ is a normal positive linear form on M. (Because $p, q\in S$ and $q-pq\leqq q$.) This proves (1). To prove (2), we have to show that every nonzero projection p in M majorizes a nonzero projection belonging to S. This can be shown by using arguments of J. Dixmier [1], which originate from Lebesgue's work. Let p be a nonzero projection in M. Consider a *normal* positive linear form μ on M, such that $\mu(p)\geqq\varphi(p)$. Me are going to prove that there exists a nonzero projection q in M, such that $\varphi(r)\leqq\mu(r)$ for every projection r in M, such that $r\leqq q$. Then the spectral decomposition theorem will imply that $\varphi(t)\leqq\mu(t)$ for every $t\in Mq$, $t\geqq 0$. Since every positive linear form majorized by a *normal* positive linear form is normal [1], this will prove (2). Now assume on the contrary that every nonzero projection $q\leqq p$ in M majorizes a nonzero projection $r\in M$ such that $\varphi(r)>\mu(r)$. By Zorn's lemma, there exists a maximal family C of mutually orthogonal nonzero projections s in M such that $\varphi(s)>\mu(s)$ for $s\in C$. By the indirect hypothesis, $\sum_{s\in C}s=p$. Then $\varphi(p)=\varphi(\sum_{s\in C}s)\geqq\sum_{s\in C}\varphi(s)>\sum_{s\in C}\mu(s)=\mu(p)$, which contradicts the choice of μ (i.e., that $\mu(p)\geqq\varphi(p)$). Consequently, there is a nonzero projection $q\leqq p$ in M, such that $\varphi(r)\leqq\mu(r)$ for every projection $r\in M$ majorized by q. Hence (2) is proved.

Since S is an upward directed set, it may serve as an index set for generalized sequences. We shall prove that

$$(*)\qquad \psi(t)=\lim_{p\in S}\varphi(tp),\quad t\in M$$

exists (and is finite). First let $t\geqq 0$. If $p\leqq q$ and $p, q\in S$, then the equality $tq=tp+t(q-p)$ shows that $tq\geqq tp$. By the positivity of φ, the function $p\to\varphi(tp)$ is a nondecreasing nonnegative numerical-valued function on S and $\varphi(tp)\leqq\varphi(t)$ for $p\in S$. Consequently, the finite limit $\lim_{p\in S}\varphi(tp)$ exists and is equal to $\sup\{\varphi(tp)\colon p\in S\}$:

$$(**)\qquad \lim_{p\in S}\varphi(tp)=\sup\{\varphi(tp)\colon p\in S\},\quad t\geqq 0.$$

The existence of $\lim_{p\in S}\varphi(tp)$ for all $t\in M$ follows by linearity.

It is clear that ψ is a positive linear form on M. Moreover, ψ is normal. Indeed, it is an elementary observation that for $p \in S$, the functional $t \to \varphi(tp)$ is normal on M. The normalcy of ψ follows from $(**)$ by using the elementary result that the supremum of an upward directed family of normal positive linear forms is normal [1].

Now we are going to prove that ψ is G-invariant. First let us consider any element p_0 of S, g_0 of G and t_0 of M. Since φ is G-invariant and $p_0 \in S$, the linear form $t \to \varphi(tg_0(p_0))$ is normal. Consequently, $g_0(p_0) \in S$. We have

$$\psi(g_0(t_0)g_0(p_0)) = \lim_{p \in S} \varphi(g_0(t_0)g_0(p_0)p) = \lim_{p \geqq g_0(p_0),\, p \in S} \varphi(g_0(t_0)g_0(p_0)p) =$$
$$= \lim_{p \geqq g_0(p_0),\, p \in S} \varphi(g_0(t_0)g_0(p_0)) = \varphi(g_0(t_0)g_0(p_0)) = \varphi(g_0(t_0 p_0)) = \varphi(t_0 p_0) =$$
$$= \lim_{p \geqq p_0,\, p \in S} \varphi(t_0 p_0 p) = \lim_{p \in S} \varphi(t_0 p_0 p) = \psi(t_0 p_0).$$

So we have shown that $\psi(g_0(t_0)g_0(p_0)) = \psi(t_0 p_0)$. By using property (2) of S, proved above, we can let p_0 w^*-converge to 1 in this equality. Then relying on the normalcy of ψ and on the continuity properties of g_0, we obtain that $\psi(g_0(t_0)) = \psi(t_0)$. Since $g_0 \in G$ and $t_0 \in M$ have been chosen arbitrary, we have proved that ψ is G-invariant.

Finally, ψ is not identically zero. It is in fact faithful. Indeed, if $t \in M$, $t \geqq 0$ and $t \neq 0$, then $\psi(T) = \sup\{\varphi(tp) \colon p \in S\}$. By property (2) of S, we have $\varphi(tp) \neq 0$ for some $p \in S$. Therefore, $\psi(t) > 0$ and the proof of our proposition is complete.

Theorem. *Let M be a W^*-algebra and G a group of $*$-automorphisms of M containing the inner automorphism group. If there exists a faithful G-invariant (not necessarily normal) positive linear form on M, then M is G-finite.*

Proof. Since G contains the inner automorphism group, φ is central [1]. Let $t \in M$ be such that $t^* t = 1$. Then $\varphi(1 - tt^*) = \varphi(1) - \varphi(tt^*) = \varphi(t^* t) - \varphi(tt^*) = 0$. Since $1 - tt^*$ is a projection, the faithfulness of φ implies that $tt^* = 1$, i.e., M is a finite W^*-algebra. Let Z denote the center of M. It is easy to see that Z is invariant under the elements of G. Consequently, we can apply Proposition to Z, the restriction G_Z of G to Z and the restriction of φ to Z. We obtain that Z is G_Z-finite. The G-finiteness of M follows from (simple) results of [3].

Corollary. *Suppose that under the hypotheses of the theorem, for every element t of the center Z of M, the uniformly closed convex hull of the orbit of t under G contains at least one G-invariant element. If the restriction of φ to the algebra Z^G of G-invariant elements of Z is normal, then φ is normal on M.*

Proof. Let I be the inner automorphism group of M. We know [1] that for every $t \in M$, the norm closure of the convex hull of the orbit of t under I contains at least one element $t^\natural$ of Z. Moreover, by the hypotheses of the corollary, the norm closure of the convex hull of the orbit of $t^\natural$ under G contains a G-invariant element t_G. It is clear that t_G is a G-invariant element in the norm closure of the convex hull of the orbit of t under G. Now let $t \to t^G$ be the G-canonical mapping of M onto M^G [3]. Since t^G is the unique G-invariant element in the w^*-closure of the convex hull of the orbit of t under t^G, we have $t^G = t_G$ and t^G is in the *norm* closure of the convex hull of the orbit of t under G.

Since φ is norm-continuous and G-invariant, $\varphi(t) = \varphi(t^G)$, i.e., φ is the composition of the mappings $t \to t^G$ $(t \in M)$ and $s \to \varphi(s)$ $(s \in M^G)$. We know [3] that $t \to t^G$ is normal. If φ is normal on M^G, then this composite mapping, i.e., φ, is also normal on M.

Remark 1. We do not have to use generalized sequences in the proof of the Proposition if G, as a subset of the space of linear self-mappings of M is separable in the topology of pointwise w^*-convergence. In this case we can choose a dense countable subgroup $G_0 = \{g_1, g_2, \ldots\}$ of G, a non-zero projection q in M, such that φ is normal on Mq and take the ordinary limit $\psi(t) = \lim_{n \to \infty} \varphi(t[g_1(q) \vee \ldots \vee g_n(q)])$. It can be shown that ψ is a non-zero G-invariant normal positive linear form on M. It is easy to see that G is separable if the predual of M is separable. This will always be the case if we only consider W^*-algebras M of operators in a separable Hilbert space.

Remark 2. The assumption of Proposition that φ is faithful is essential. Indeed, let G be an abstract infinite Abelian group. Then G acts naturally on $M = l^\infty(G)$ as a group of $*$-automorphisms. A G-invariant state on $l^\infty(G)$ is nothing else but an invariant mean on G. We know that there are infinitely many invariant means on G, none of which are normal. In this situation the entire proof of the theorem is valid except that ψ will be identically zero. Indeed, S will consist of the characteristic functions of finite subsets of G. It is an elementary fact that every invariant mean is zero on such functions.

Finally, the author would like to make two comments on his paper [4]. The first comment is that in Proposition 2 and in its corollary the assumption that M is σ-finite should be replaced by the assumption that the predual of M is separable.

The second comment is that all the results of the above mentioned paper remain valid if G is only assumed to be an amenable group (instead of an Abelian one). Indeed, if $U_n \subset G$ is a summing sequence [2], then it is easy to prove that under the hypotheses of Lemma 1, the sequence $|U_n|^{-1} \sum_{g \in U_n} g(t)$ w^*-converges to t^G for every $t \in B^*$. The remaining results of the paper can be extended to amenable groups G without altering the proofs.

Added in proof: By adapting the method of proof of Proposition to the nonabelian case, Theorem can be broved withoit the assumption that G contains the inner automorphisms. As suggested by R. R. Smith, this can be done even in a simpler manner by appealing to the decomposition of a state into singular and normal parts.

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Representation of functionals via summability methods. I

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Dedicated to Professor K. Tandori on his 60th birthday

§ 1. Introduction

Summability theory has benefited from functional analysis: several of its fundamental results have source at the main principles of the latter. In this paper and in the continuation of it we show that conversely, some problems concerning functionals and measures can be solved by the aid of summability methods.

Let $C(K)$ be the sup-normed Banach space of real valued continuous functions defined on the compact Hausdorff space K. The representation problem of the bounded linear functionals on $C(K)$ has a long history. It was shown by HADAMARD [3] in 1903 that every $L\in C^*(K)$, where $K=[0, 1]$, has the form

$$Lf = \lim_{n\to\infty} \int_0^1 f(x)\, p_n(x)\, dx$$

where $\{p_n(x)\}$ is a suitable sequence of continuous functions. The so called Riesz representation theorem, which asserts that every $L\in C^*(K)$ has the form

(1.1) $$Lf = \int_K f\, d\mu$$

with a suitable signed Borel measure μ, was proved for $K=[0, 1]$ by F. RIESZ [5] in 1909, for metrizable K by BANACH and SAKS [1, 6] in 1937—38 and for every K by KAKUTANI [4] in 1941.

Here we present another way for representing every bounded linear functional which, as it seems, have been overlooked so far. This is the form

(1.2) $$Lf = \lim_{n\to\infty} \frac{c_1 f(x_1) + \ldots + c_n f(x_n)}{n}$$

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with appropriate c_k's and x_k's. Naturally, (1.1) is a more convenient form than (1.2), nevertheless, (1.2) has some advantages: (1.2) may be exact up to the domain of L, (cf. Theorem 1 below), the c_k's and x_k's can be obtained, at least for positive L, in a constructive way, the representation (1.2) can be extended to larger spaces, finally a quite similar representation can be given for subadditive and homogeneous functionals: all we have to do is to replace lim by limsup.

The paper is organized as follows. In § 2 we give the representation (1.2) for $K=[0, 1]$ and treat the analogous problem with $c_k\equiv 1$. In § 3 we investigate the subadditive functionals and quasinorms and, finally, in § 4 the generalization to metrizable K's is given.

There will be a forthcoming paper with the following content: (1.2) can be extended to the space $Q[0, 1]$ of functions having discontinuities only of the first kind and $Q[0, 1]$ is maximal among certain "natural" spaces with this property; we shall determine those functionals of $R[0, 1]$, the space of Riemann-integrable functions, which have the form (1.2) and give an application to density measures and, finally, we also characterize those summability methods by which the $(C, 1)$-method in (1.2) can be replaced.

§ 2. Functionals in $C[0, 1]$

Let $c=\{c_k\}_{k=1}^{\infty}$ be a bounded sequence of real numbers and $X=\{x_k\}_{k=1}^{\infty}\subseteq[0, 1]$ a sequence from $[0, 1]$. For an $f\in C[0, 1]$ we define

$$L_{c,X}f = \lim_{n\to\infty}\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n} \tag{2.1}$$

if the limit on the right exists and let $D_{c,X}$ be the domain of $L_{c,X}$. Clearly, $D_{c,X}$ is a closed subspace of $C[0, 1]$ and $L_{c,X}$ is a bounded linear functional on $D_{c,X}$, $\|L_{c,X}\|\leq\sup_i|c_i|$.

Our first result states that every bounded linear functional has this form.

Theorem 1. *If $D\subseteq C[0, 1]$ is a closed subspace and $L:D\to\mathbf{R}$ is a bounded linear functional on D then there are sequences c and X such that $L=L_{c,X}$, $D=D_{c,X}$.*

Corollary 1. *If $L\in C^*[0, 1]$ then there are sequences $\{c_k\}$, $|c_k|\leq\|L\|$ and $\{x_k\}\subseteq[0, 1]$ such that*

$$Lf = \lim_{n\to\infty}\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n} \tag{2.2}$$

holds for every $f\in C[0, 1]$.

Corollary 2. *If $D\subseteq C[0, 1]$ is a closed subspace and L is a bounded linear* *functional on D then there is a sequence of polynomials* $\{p_n\}$ *such that*

$$\int_0^1 |p_n| = O(1) \quad (n = 1, 2, \ldots),$$

$$Pf \overset{\text{def}}{=} \lim_{n\to\infty} \int_0^1 fp_n$$

exists if and only if $f \in D$, *furthermore,* $Pf = Lf$ *for all* $f \in D$.

Let us call a functional of the form (2.1) partial weighted $(C, 1)$-functional, and a one with domain $C[0, 1]$ a weighted $(C, 1)$-functional. If for all k we have $c_k = 1$ then we call $L_{c,X} = L_X$ a partial $(C, 1)$-functional or $(C, 1)$-functional according as $\operatorname{Dom} L_X \subseteqq C[0, 1]$ or $\operatorname{Dom} L_X = C[0, 1]$, respectively. Thus, the (partial) $(C, 1)$-functionals have the form

$$L_X f = \lim_{n\to\infty} \frac{f(x_1) + \ldots + f(x_n)}{n} \quad (f \in \operatorname{Dom} L_X) \tag{2.3}$$

with a sequence $X = \{x_k\} \subseteqq [0, 1]$. It is clear that every such L_X is a positive linear (partial) functional of norm 1 $(L_X 1 = 1)$ which shall be abbreviated in the following as: L_X is a PL1 (partial) functional.

By Theorem 1 every bounded partial linear functional (i.e. a functional with domain $\subseteqq C[0, 1]$) is a partial weighted $(C, 1)$-functional. Now what about PL1 functionals? Does every partial PL1 functional have the form (2.3)? The answer is given in

Theorem 2. *Let* $D \subseteqq C[0, 1]$ *be a closed subspace and* L *a* PL1 *functional on* D. *The following assertions are equivalent to each other:*

(i) L *has the form* (2.3), *i.e. there exists a sequence* X *with* $L = L_X$, $D = \operatorname{Dom} L_X$,

(ii) *to every* $f \in C[0, 1] \setminus D$ *there are two* PL1 *extensions, say* $L_f^{(1)}$ *and* $L_f^{(2)}$, *of* L *to* $C[0, 1]$ *for which* $L_f^{(1)} f \neq L_f^{(2)} f$,

(iii) D *contains the constants, and if for an* $f \in C[0, 1]$ *we have*

$$\inf_{g\in D,\, g \geqq f} Lg = \sup_{g \in D,\, g \leqq f} Lg \tag{2.4}$$

then $f \in D$.

E.g. if $D = \{f \mid f(0) = f(1)\}$ and $Lf = f(1/2)$ for f in D, then there is no X with $(L, D) = (L_X, \operatorname{Dom} L_X)$. Indeed, for $f(x) = x$ (2.4) is satisfied, but $f \notin D$. In other words, the partial PL1 functionals of the form (2.3) are the ones which have no unique extension to any larger subspace of $C[0, 1]$.

Corollary 3. *If* L *is an arbitrary* PL1 *functional on* $C[0, 1]$ *then there exists* *a sequence* $\{x_k\}\subseteq[0,1]$ *with*

$$Lf=\lim_{n\to\infty}\frac{f(x_1)+\ldots+f(x_n)}{n} \tag{2.5}$$

for every $f\in C[0,1]$.

Corollary 4. *Let* $T=(t_{nk})_{n,k=1}^{\infty}$ *be a non-negative summation matrix,* $D\subseteq C[0,1]$ *a closed subspace containing the constants and* $L\colon D\to \boldsymbol{R}$ *a partial* PL1 *functional. If there is a sequence* $\{x_k\}\subseteq[0,1]$ *such that*

$$Lf=T-\lim_k f(x_k)\overset{\text{def}}{=}\lim_{n\to\infty}\sum_{k=1}^{\infty}t_{nk}f(x_k)\quad(f\in D)$$

and the limit on the right does not exist for any $f\notin D$, *then* L *is a partial* $(C,1)$*-functional.*

This corollary tells us that the $(C,1)$-method is the strongest one from the point of view of the representation of PL1 functionals.

In connection with the representation (2.3) the following very natural questions arise: when do we have $\operatorname{Dom} L_X=C[0,1]$, and in this case for which other sequences $Y=\{y_k\}\subseteq[0,1]$ do we have $L_X=L_Y$? The answers are given by

Proposition. (i) *The limit*

$$\lim_{n\to\infty}\frac{1}{n}(f(x_1)+\ldots+f(x_n)) \tag{2.6}$$

exists for every $f\in C[0,1]$ *if and only if there is a sequence* $\{z_m\}\subseteq[0,1]$ *dense in* $[0,1]$ *such that* $\{x_k\}$ *has density in every interval* $[0,z_m]$.

(ii) *Two sequences* X *and* Y *determine the same* PL1 *functional (via* (2.3)*) if and only if there is a dense sequence* $\{z_m\}$ *in* $[0,1]$ *such that* X *and* Y *have the same density in every interval* $[0,z_m]$.

Remark. If we allow the sequence $\{c_k\}$ in (2.1) to be unbounded then (2.1) still defines a (possibly unbounded) linear functional L on some linear subspace of $C[0,1]$. However, if the domain of L is $C[0,1]$ (or any closed subspace of it) then, by the uniform boundedness principle, the obtained L is bounded, so we have lost very little in assuming $\{c_k\}$ to be bounded.

Proofs. In the proofs of the above statements the following lemma will be useful.

Lemma 1. *Let* $g_1,g_2,\ldots$ *be arbitrary functions from* $C[0,1]$ *and* L *a partial linear functional with* $g_j\in\operatorname{Dom} L$ *for* $j=1,2,\ldots$. *If there are partial* $(C,1)$*-functionals* $L_1,L_2,\ldots$ *with* $g_j\in\operatorname{Dom} L_n$ *for* $n,j=1,2,\ldots$ *such that*

$$\lim_{n\to\infty}L_ng_j=Lg_j\quad(j=1,2,\ldots)$$

then there exists a partial $(C, 1)$*-functional* L' *with*

$$L'g_j = Lg_j \quad (j = 1, 2, \ldots).$$

Proof. For the sake of brevity we introduce the notation

$$\sigma_n(\{x_k\}, f) = \frac{1}{n}(f(x_1)+\ldots+f(x_n)).$$

Let L_n be represented in the form (2.3) by the sequence $\{x_k^{(n)}\}_{k=1}^{\infty}$ i.e. let

$$\lim_{m\to\infty} \sigma_m(\{x_k^{(n)}\}, f) = L_n f \quad (f\in \operatorname{Dom} L_n).$$

We define the increasing sequence $\{m_i\}$, $\{M_i\}$ and $\{N_i\}$ in succession so that the following conditions be satisfied:

$|L_n g_j - Lg_j| < 1/i$ for $1 \leqq j \leqq i$ and $n \geqq m_i$, $|\sigma_n(\{x_k^{(m_i)}\}, g_j) - Lg_j| < 1/i$ for $1 \leqq j \leqq i$ and $n \geqq M_i$, $M_{i+1}/N_i < 1/i$, $(\sum_{j=1}^{i-1} N_j)/N_i < 1/i$ for $i=1, 2, \ldots$, and finally we put $K_0=0$ and $K_i = \sum_{j=1}^{i} N_j$ for $i=1, 2, \ldots$. Let $x_n = x_{n-K_{i-1}}^{(m_i)}$ for $K_{i-1} < n \leqq K_i$. We claim that the partial $(C, 1)$-functional L' represented by the sequence $\{x_k\}_{k=1}^{\infty}$ is suitable for us.

Indeed, let j be an arbitrary but fixed natural number. For $i>j$, $K_i < n \leqq K_{i+1}$ we distinguish two cases according as $n-K_i$ is less than M_{i+1} or not.

1) $n-K_i < M_{i+1}$. By the definitions

$$|\sigma_n(\{x_k\}, g_j) - Lg_j| \leqq |\sigma_n(\{x_k\}, g_j) - \sigma_n(\{x_k^{(m_i)}\}, g_j)| +$$

$$+ |\sigma_n(\{x_k^{(m_i)}\}, g_j) - Lg_j| \leqq \frac{1}{n}\Big(\sum_{r=1}^{K_{i-1}} \max|g_j| + \sum_{r=K_i+1}^{n} \max|g_j| +$$

$$+ \sum_{n=N_i+1}^{n} \max|g_j|\Big) + \frac{1}{i} = \frac{1}{n}\max|g_j|\big(K_{i-1} + (n-K_i) + n - N_i\big) + \frac{1}{i} \leqq$$

$$\leqq \frac{1}{i}(1 + 4\max|g_j|).$$

2) $n-K_i \geqq M_{i+1}$. We obtain similarly

$$|\sigma_n(\{x_k\}, g_j) - Lg_j| = \Bigg|\frac{1}{n}\sum_{r=1}^{K_{i-1}} (g_j(x_r) - Lg_j) +$$

$$+ \frac{N_i}{n}(\sigma_{N_i}(\{x_k^{(m_i)}\}, g_j) - Lg_j) + \frac{n-K_i}{n}(\sigma_{n-K_i}(\{x_k^{(m_{i+1})}\}, g_j) - Lg_j)\Bigg| \leqq$$

$$\leqq \frac{K_{i-1}}{n}(|Lg_j| + \max|g_j|) + \frac{N_i}{in} + \frac{n-K_i}{n(i+1)} \leqq \frac{1}{i}(|Lg_j| + \max|g_j| + 1)$$

and the proof is over.

We shall prove our theorems and their corollaries in the following order: Corollary 3, Theorem 2, Corollary 1, Theorem 1, Corollaries 2, 4, Proposition.

Proof of Corollary 3. For a natural number n let $g_0^{(n)}\equiv 1$ and for $i=1, 2, \ldots, 2^n$

$$g_i^{(n)}(x) = \begin{cases} 0 & \text{if } 0 \leqq x \leqq (i-1)/2^n \\ 1 & \text{if } i/2^n \leqq x \leqq 1 \\ \text{linear on } [(i-1)/2^n, i/2^n]. \end{cases} \tag{2.7}$$

Since L is positive with unit norm we have $L1=1$ and

$$0 \leqq Lg_{2^n}^{(n)} \leqq \ldots \leqq Lg_1^{(n)} \leqq Lg_0^{(n)} = 1.$$

To every $\varepsilon>0$ there are integers $0<m_{2^n}<\ldots<m_1<m_0$ such that

$$\left|\frac{m_i}{m_0} - Lg_i^{(n)}\right| < \varepsilon \quad (0 \leqq i \leqq 2^n) \tag{2.8}$$

be satisfied. Let $x_1=x_2=\ldots=x_{m_{2^n}}=1$, $x_{m_{2^n}+1}=\ldots=x_{m_{2^n-1}}=1-(1/2^n)$, $\ldots$, $x_{m_2+1}=\ldots$ $\ldots=x_{m_1}=1/2^n$, $x_{m_1+1}=\ldots=x_{m_0}=0$. Clearly for every $0\leqq i\leqq 2^n$ we have

$$\sum_{j=1}^{m_0} g_i^{(n)}(x_j) = m_i$$

i.e., by (2.8),

$$\left|\frac{1}{m_0}\sum_{j=1}^{m_0} g_i^{(n)}(x_j) - Lg_i^{(n)}\right| < \varepsilon \quad (0 \leqq i \leqq 2^n).$$

The sequence $x_1, x_2, \ldots, x_{m_0}, x_1, \ldots, x_{m_0}, x_1, \ldots$ represents a $(C, 1)$-functional $L_\varepsilon^{(n)}$ with

$$|L_\varepsilon^{(n)} g_i^{(n)} - Lg_i^{(n)}| < \varepsilon \quad (0 \leqq i \leqq 2^n).$$

Putting here $\varepsilon=1, 1/2, \ldots$, Lemma 1 yields a partial $(C, 1)$-functional L_n with

$$L_n g_i^{(n)} = Lg_i^{(n)} \quad (0 \leqq i \leqq 2^n).$$

But then the same equality holds for the linear combinations of the $g_i^{(n)}$'s and among them there is any $g_j^{(m)}$ with $m\leqq n$. Thus,

$$\lim_{n\to\infty} L_n g_i^{(m)} = Lg_i^{(m)}$$

for all m and $0\leqq i\leqq 2^m$ and another application of Lemma 1 yields a partial $(C, 1)$-functional L' with

$$L' g_i^{(m)} = Lg_i^{(m)} \quad (m = 1, 2, \ldots, 0 \leqq i \leqq 2^m). \tag{2.9}$$

Since the linear combinations of the $g_i^{(m)}$'s are dense in $C[0, 1]$ and both L and L' have norm one, the equality $L=L'$ readily follows from (2.9).

Proof of Theorem 2. (i) $\Rightarrow$ (ii). If $f\notin D$ then, by assumption, there are two subsequences $\{n_k^{(1)}\}$ and $\{n_k^{(2)}\}$ of the natural numbers such that

$$\lim_{k\to\infty}\sigma_{n_k^{(1)}}(X,f)\neq\lim_{k\to\infty}\sigma_{n_k^{(2)}}(X,f) \tag{2.10}$$

and both of these limits exist. Let us define the partial functional L' and L'' by

$$L'g=\lim_{k\to\infty}\sigma_{n_k^{(1)}}(X,g),\quad L''g=\lim_{k\to\infty}\sigma_{n_k^{(2)}}(X,g).$$

Since

$$L'g\leqq\sup g\,(g\in\operatorname{Dom}L'),\quad L''g\leqq\sup g\ (g\in\operatorname{Dom}L''),$$

L' and L'' can be extended by the Hahn—Banach theorem to $C(0, 1)$ so that the previous inequalities remain valid for all $g\in C[0,1]$. The obtained functionals $L_f^{(1)}$ and $L_f^{(2)}$ are clearly PL1 functionals and, by (2.10), $L_f^{(1)}f=L'f\neq L''f=L_f^{(2)}f$.

(ii)$\Rightarrow$(i). By assumption to every $f\notin D$ there are two PL1 extensions $L_f^{(1)}$ and $L_f^{(2)}$ of L with $L_f^{(1)}f\neq L_f^{(2)}f$, say $L_f^{(1)}f<L_f^{(2)}f$. Then there is a neighbourhood U_f of f and an $\varepsilon_f>0$ such that

$$L_f^{(1)}g\leqq L_f^{(2)}g-\varepsilon_f\quad\text{for all}\quad g\in U_f.$$

Since $C[0,1]\setminus D$ is a separable metric space it satisfies the Lindelöf property, so that

$$C[0,1]\setminus D=\bigcup_{m=1}^{\infty}U_{f_m}$$

for some sequence $\{f_m\}_1^\infty\subseteqq C[0,1]\setminus D$. Let $\{L_n\}$ be a sequence of the functionals $\{L_{f_m}^{(1)},L_{f_m}^{(2)}\}_{m=1}^\infty$ which contains every $L_{f_m}^{(1)}$ and $L_{f_m}^{(2)}$.

By the above proved Corollary 3 there are sequences $\{x_k^{(n)}\}_{k=1}^\infty$ representing L_n in the sense (2.5). Now let $\{x_n\}$ be any sequence guaranteed by the following lemma:

Lemma 2. *If $\{x_k^{(n)}\}_{k=1}^\infty$, $n=1,2,\dots$ are the just introduced sequences then there is a union $\{x_n\}_{n=1}^\infty=\bigcup_{n=1}^{\infty}\{x_k^{(n)}\}_{k=1}^\infty$ of these sequences such that*

(i) *every $\{x_k^{(n)}\}_{k=1}^\infty$ is a subsequence, say $\{x_{j_k^{(n)}}\}_k$, of $\{x_n\}$ and it has upper density 1 in $\{x_n\}$, i.e.*

$$\limsup_{k\to\infty}k/j_k^{(n)}=1,$$

(ii) *for every m there are four indices $n_1(m)$, $n_2(m)$, $k_1(m)$ and $k_2(m)$ such that*

(α) $m-(k_1(m)+k_2(m))=o(m)\quad(m\to\infty)$

(β) *the finite sequences $\{x_k^{(n_1(m))}\}_{k=1}^{k_1(m)}$ and $\{x_k^{(n_2(m))}\}_{k=1}^{k_2(m)}$ form two disjoint subsequences of $\{x_k\}_{k=1}^m$ and*

(γ) *for a dense countable subset $D'\subset D$ and for every $f\in D'$ we have*

$$\sigma_{k_1(m)}(\{x_k^{(n_1(m))}\},f)=L_{n_1(m)}f+o(1)\quad(k_1(m)\to\infty)$$
$$\sigma_{k_2(m)}(\{x_k^{(n_2(m))}\},f)=L_{n_2(m)}f+o(1)\quad(k_2(m)\to\infty).$$

(ii) (α) and (β) say that for every m the sequence $\{x_k\}_1^m$ is essentially formed from two initial segments $\{x_k^{(n_1(m))}\}_{k=1}^{k_1(m)}$ and $\{x_k^{(n_2(m))}\}_{k=1}^{k_2(m)}$. The proof of this lemma is straightforward, we omit it.

Returning to the proof of (ii)$\Rightarrow$(i) we claim that the partial $(C, 1)$-functional L' respresented by the sequence $\{x_n\}$ satisfies $L'=L$, $\mathrm{Dom}\, L'=D$. Indeed, for $f\in D'$ (cf. (ii) γ, in the preceding lemma) we have $L_n f=Lf$ for every n, hence, by Lemma 2, (ii)

$$\sigma_m(\{x_n\}, f) = o_m(1)+\frac{k_1(m)}{m}\sigma_{k_1(m)}(\{x_k^{(n_1(m))}\}, f)+$$
$$+\frac{k_2(m)}{m}\sigma_{k_2(m)}(\{x_k^{(n_2(m))}\}, f) = o_m(1)+\frac{k_1(m)}{m}(Lf+o_{k_1(m)}(1))+$$
$$+\frac{k_2(m)}{m}(Lf+o_{k_2(m)}(1)) = Lf+o_m(1)$$

where $o_m(1)$ denotes a quantity that tends to zero together with m. The relation above shows $f\in\mathrm{Dom}\, L'$ and $L'f=Lf$. Since this holds for every $f\in D'$ and D' is dense in D we can conclude that $D\subset\mathrm{Dom}\, L'$ and L' agrees with L on D. On the other hand, if $f\notin D$ then $f\in U_{f_n}$ for some n and thus, by our construction and Lemma 2, (i)

$$\liminf_{m\to\infty}\sigma_m(\{x_k\}, f) \leqq \limsup_{m\to\infty}\sigma_m(\{x_k\}, f)-\varepsilon_n$$

i.e. $f\notin\mathrm{Dom}\, L'$ and so $L=L'$ has been verified.

The equivalence of (ii) and (iii) is clear from any standard proof of the Hahn—Banach extension theorem.

We have completed the proof of Theorem 2.

Proof of Corollary 1. By Riesz' decomposition theorem $L=\alpha L_1-\beta L_2$ where $\alpha\geqq 0$, $\beta\geqq 0$, $\alpha+\beta=\|L\|$ and L_1 and L_2 are PL1 functionals. If L_1 and L_2 are represented by the sequences $\{x_k^{(1)}\}$ and $\{x_k^{(2)}\}$ via (2.5) and $\{n_k^{(1)}\}$ and $\{n_k^{(2)}\}$ are disjoint subsequences of the natural numbers N with density $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$, respectively, furthermore $N=\{n_k^{(1)}\}\cup\{n_k^{(2)}\}$ then the sequences

$$x_n=\begin{cases} x_k^{(1)} & \text{if } n=n_k^{(1)} \\ x_k^{(2)} & \text{if } n=n_k^{(2)}, \end{cases} \quad c_n=\begin{cases} \|L\| & \text{if } n=n_k^{(1)} \\ -\|L\| & \text{if } n=n_k^{(2)} \end{cases}$$

clearly satisfy the requirements of Corollary 1.

The Proof of Theorem 1 is similar to that of Theorem 2 if we notice that to every $f\notin D$ there are two extensions of L, say $L_f^{(1)}$ and $L_f^{(2)}$, for which $L_f^{(1)}f\neq L_f^{(2)}f$, $\|L_f^{(1)}\|$, $\|L_f^{(2)}\|\leqq\|L\|+1$ and if we apply Corollary 1 instead of Corollary 3.

Corollary 2 follows easily from Theorem 1 since the Dirac measures δ_{x_i} can be approximated by polynomials p_i^* satisfying $p_i^* \geqq 0$, $\int_0^1 p_i^* = 1$ and $\max_{|x-x_i| \geqq 1/i} p_i^*(x) < 1/i$ in the sense $\int_0^1 f p_i^* = f(x_i) + o(1)$ $(f \in C[0, 1])$.

In the Proof of Corollary 4 the fact that to every $f \notin D$ there are two PL1 extensions $L_f^{(1)}$ and $L_f^{(2)}$ of L with $L_f^{(1)} f \neq L_f^{(2)} f$ can be proved exactly as in the case of the $(C, 1)$-method in Theorem 2 (use that by $1 \in D$, $L1 = 1$), and we have to apply only Theorem 2.

Proof of the Proposition. (i). Let

$$\tau(z) = \liminf_{n \to \infty} \sigma_n(\{x_k\}, \chi_{[0,z]})$$

and

$$\mu(z) = \limsup_{n \to \infty} \sigma_n(\{x_k\}, \chi_{[0,z]}),$$

$$\chi_{[0,z]}(x) = \begin{cases} 1 & \text{if } 0 \leqq x \leqq z \\ 0 & \text{if } z < x \leqq 1 \end{cases}$$

be the lower and upper density of $\{x_k\}$ in $[0, z]$. τ and μ are increasing functions, so they are continuous everywhere but a denumerable set. If $\varepsilon > 0$ and

$$g_{z,\varepsilon}(x) = \begin{cases} 1 & \text{if } x \leqq z \\ 0 & \text{if } x \geqq z + \varepsilon \\ \text{linear on } [z, z+\varepsilon] \end{cases}$$

then

$$\tau(z) \leqq \mu(z) \leqq \lim_{n \to \infty} \sigma_n(\{x_k\}, g_{z,\varepsilon}) \leqq \tau(z+\varepsilon)$$

and so $\tau(z) = \mu(z)$ at every point z where τ is continuous, and this proves the necessity of the condition.

Conversely, if $\tau(z_n) = \mu(z_n)$ for every z_n in a dense set then the limit (2.3) exists for every f which is the linear combination of the characteristic funtions of the intervals $[0, z_n]$ and every continuous function can be approximated uniformly by such f's.

(ii) can be proved similarly.

We have completed our proofs.

§ 3. Subadditive functions and quasinorms

In this paragraph we present some representation theorems for subadditive functionals which are very close in spirit to the results of the previous chapter.

Recall that a functional τ: $C[0, 1] \to \mathbf{R}$ is called subadditive if

$$\tau(f+g) \leqq \tau(f) + \tau(g) \tag{3.1}$$

is satisfied for all $f, g\in C(0, 1)$. It is positive homogeneous if $\tau(\lambda f)=\lambda\tau(f)$ for all $f\in C[0, 1]$ and $\lambda\geqq 0$. If τ is both subadditive and positive homogenenous then we call it convex functional. Quasinorms are the non-negative convex even functionals, i.e. besides (3.1) they satisfy $\tau(f)\geqq 0$, $\tau(\lambda f)=|\lambda|\tau(f)$ for all f and λ.

If $\{c_k\}\subseteqq\mathbf{R}$, $|c_k|=O(1)$ and $\{x_k\}\subseteqq[0, 1]$ are two sequences then each of the following defines a bounded convex functional on $C[0, 1]$

$$\tau(f) = \limsup_{n\to\infty}\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n}, \tag{3.2}$$

$$\tau(f) = \limsup_{n\to\infty}\left|\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n}\right|, \tag{3.3}$$

$$\tau(f) = \limsup_{n\to\infty}\frac{|c_1||f(x_1)|+\ldots+|c_n||f(x_n)|}{n}. \tag{3.4}$$

Obviously the τ in (3.3) and (3.4) is a quasinorm, furthermore $|f|\leqq|g|$ implies $\tau(f)\leqq\tau(g)$ in (3.4). Now all of these statements have converses:

Theorem 3. *Every bounded convex functional τ on $C[0, 1]$ has the form* (3.2) *with suitable sequences $\{c_k\}\subseteqq R$, $|c_k|=O(1)$ and $\{x_k\}\subseteqq[0, 1]$.*

Theorem 4. *Every bounded quasinorm on $C[0, 1]$ has the form* (3.3).

Theorem 5. *Every bounded quasinorm τ on $C[0, 1]$ with the property*

$$\tau(f)\leqq\tau(g)\quad\text{whenever}\quad |f|\leqq|g|$$

has the form (3.4).

E.g. every L^p-norm $(1\leqq p<\infty)$

$$\tau_p(f)=\left\{\int_0^1|f|^p\right\}^{1/p}$$

has the form (3.4) with suitable $\{c_k\}$ and $\{x_k\}$.

From our results one can deduce other representation theorems, e.g. Theorem 4 implies that every bounded quasinorm τ on $C[0, 1]$ has the form

$$\tau(f)=\limsup_{n\to\infty}\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n}-\liminf_{n\to\infty}\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n}.$$

We mention also that, as can be seen easily from the proofs, the sequences $\{c_i\}$ in Theorems 3—5 can be chosen so that they also satisfy $|c_i|\leqq\|\tau\|$.

We also give the characterization of those convex functionals which can be obtained from (3.2)—(3.4) with $c_k \equiv 1$ i.e. which have the forms

$$\tau(f) = \limsup_{n\to\infty} \frac{f(x_1)+\ldots+f(x_n)}{n}, \tag{3.5}$$

$$\tau(f) = \limsup_{n\to\infty} \left| \frac{f(x_1)+\ldots+f(x_n)}{n} \right|, \tag{3.6}$$

$$\tau(f) = \limsup_{n\to\infty} \frac{|f(x_1)|+\ldots+|f(x_n)|}{n}, \tag{3.7}$$

respectively.

Clearly, we have $\|\tau\|=1$ in these cases.

Theorem 6. *For a convex functional τ with norm* 1 *the following assertions are equivalent:*

(i) τ *has the form* (3.5),
(ii) $\tau(1)=-\tau(-1)=1$,
(iii) $\tau(f+c)=\tau(f)+c$ *for* $f\in C[0, 1]$, $c\in\mathbf{R}$,
(iv) $\tau(f)\leqq \max f$ for $f\in C[0, 1]$,
(v) *if* $L\leqq\tau$ *is a linear functional then L is positive and has norm* 1 *(i.e. L is* PL1 *functional).*

Theorem 7. *For a quasinorm τ with norm* 1 *the following assertions are equivalent:*

(i) τ *has the form* (3.6),
(ii) (α) $\max\big(\tau(f+c), \tau(f-c)\big)=\tau(f)+c$ *for all f and* $c\geqq 0$ *and*
(β) $|f|\leqq g$ *implies* $\tau(f)\leqq\tau(g)$ *for all f and g,*
(iii) $\tau(f)=\max\big(\mu(f), \mu(-f)\big)$ $(f\in C[0, 1])$ *with a μ satisfying any of the conditions of Theorem 6.*

Theorem 8. *For a quasinorm τ with norm* 1 *the following assertions are equivalent:*

(i) τ *has the form* (3.7),
(ii) (α) $\tau(f)\leqq\tau(g)$ *whenever* $|f|\leqq|g|$ *and*
(β) $\tau(f+c)=\tau(f)+c$ *for all* $f\geqq 0$ *and* $c\geqq 0$,
(iii) $\tau(f)=\mu(|f|)$ *with a μ satisfying any of the conditions of Theorem 6.*

Remarks. (1) Most of our results have analogues for superadditive functionals i.e. for functionals satisfying

$$\tau(f+g) \geqq \tau(f)+\tau(g),$$

naturally we have to use lim inf instead of lim sup. We do not go into the details.

(2) Here, again, we might restrict ourselves to bounded sequences $\{c_k\}$ because of the uniform boundedness principle.

Proofs. First we verify Theorem 6.

(i)⇒(ii) is obvious.

(ii)⇒(iii). By the subadditivity we have

$$\tau(f)+c = \tau(f)-\tau(-c) \leqq \tau(f+c) \leqq \tau(f)+\tau(c) = \tau(f)+c.$$

(iii)⇒(iv). Since τ has norm 1 we obtain

$$\tau(f) = \tau(f-\min f)+\min f \leqq \|\tau\|\,\|f-\min f\| +\min f =$$
$$= \max (f-\min f)+\min f = \max f.$$

(iv)⇒(v). If $f \geqq 0$ then we have

$$Lf = -L(-f) \geqq -\tau(-f) \geqq -\max(-f) \geqq 0,$$

i.e. L is positive, furthermore

$$1 = \|\tau\| \geqq \tau(1) \geqq L1 = -L(-1) \geqq -\tau(-1) \geqq -\max(-1) = 1$$

i.e. $L1=1$ which, together with the positivity of L prove that $\|L\|=1$.

(v)⇒(i). By the Hahn—Banach theorem and (v)

$$\tau(f) = \sup_{\substack{L \leqq \tau \\ \|L\|=1,\ L \text{ positive}}} Lf \qquad (f \in C[0, 1]).$$

Since $C[0, 1]$ is separable and any convex functional τ with norm 1 satisfies

$$\tau(f)-\varepsilon \leqq \tau(f)-\tau(f-g) \leqq \tau(g) \leqq \tau(f)+\tau(g-f) \leqq \tau(f)+\varepsilon$$

provided $\|g-f\| \leqq \varepsilon$, we obtain at once that there is a sequence L_n of PL1 functionals for which $L_n \leqq \tau$ and

$$\tau(f) = \sup_n L_n f \quad (f \in C[0, 1]).$$

If L_n is represented by the sequence $\{x_k^{(n)}\}_{k=1}^\infty$ in the sense of Corollary 3 and if $\{x_n\}$ the sequence associated with $\{x_k^{(n)}\}_{k=1}^\infty$, $n=1, 2, \ldots$ by Lemma 2 (i) then an easy calculation gives (3.5).

We have completed our proof.

The Proof of Theorem 3 is much the same as that of (v)⇒(i) above if we use Corollary 1, the formula

$$\tau(f) = \sup_{L \leqq \tau} Lf \quad (f \in C[0, 1])$$

and the fact that $L \leqq \tau$ implies $-\|\tau\|\,\|f\| \leqq -\tau(-f) \leqq -L(-f) = Lf \leqq \tau(f) \leqq \|\tau\|\,\|f\|$, i.e. $\|L\| \leqq \|\tau\|$.

Proof of Theorem 7. (i)⇒(ii) is obvious because $\tau(f+c)=\tau(f)+c$ or $\tau(f-c)=\tau(f)+c$ according as

$$\tau(f) = \limsup_{n\to\infty} \sigma_n(\{x_k\}, f)$$

or

$$\tau(f) = \limsup_{n\to\infty} \sigma_n(\{x_k\}, -f)$$

respectively.

(ii)⇒(iii). First of all we notice that for $f\geqq 0$ and $c\geqq 0$ we have

$$\tau(f+c) = \max\big(\tau(f+c),\ \tau(f-c)\big) = \tau(f)+c \tag{3.8}$$

because $|f-c|\leqq f+c$ implies $\tau(f-c)\leqq\tau(f+c)$.

Now let us define μ by $\mu(f)=\tau(f+c)-c$ where $f\in C[0, 1]$ and c is a constant with $f+c\geqq 0$. By (3.8) μ is uniquely defined and an easy consideration yields that μ is a convex functional with $\mu(1)=-\mu(-1)=1$. Since for large $c>0$

$$-\tau(f) = -\tau(-f) \leqq -\tau(-f)+\tau(f+c)+\tau(-f)-c =$$
$$= \mu(f) \leqq \tau(f)+\tau(c)-c = \tau(f),$$

μ also has norm 1. Thus, μ satisfies the condition of Theorem 6. Applying the previous inequality also to $-f$ we obtain

$$\max\big(\mu(f),\ \mu(-f)\big) \leqq \tau(f)$$

and here the equality sign holds for all f because of (ii), α, which proves (ii)⇒(iii).

(iii)⇒(i). If μ is represented by $\{x_k\}$ in the sense of (3.5) (see Theorem 6), then we have (3.6) for this $\{x_k\}$ because

$$\limsup_{n\to\infty} |s_n| = \max\big(\limsup_{n\to\infty} s_n,\ \limsup_{n\to\infty} (-s_n)\big)$$

for every sequence $\{s_n\}$.

The proof is complete.

The proof of Theorem 4 is easy on the ground of Theorem 3. By Theorem 3 there are sequences $\{c_k\}$, $\{x_k\}$ for which

$$\tau(f) = \tau(\pm f) = \limsup_{n\to\infty} \pm \frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n}$$

and this immediately gives (3.3).

Proof of Theorem 8. Again, (i)⇒(ii) is obvious.

(ii)⇒(iii). Arguing as in the proof of (ii)⇒(iii) in Theorem 7 we obtain that $\tau(f)=\mu(f)$ for all non-negative f with a μ satisfying the conditions of Theorem 6. This also proves our assertion because for every $f\in C[0, 1]$

$$\tau(f) = \tau(|f|) = \mu(|f|).$$

(iii)$\Rightarrow$(i). It is clear, that if μ has the form (3.5) then $\tau(f)=\mu(|f|)$ has the form (3.7) with the same sequence $\{x_k\}$.

Proof of Theorem 5. Let us consider the positive cone $C_+=\{f\in C[0,1] \mid f\geqq 0\}$. For every $f\in C_+$ there is, by the Hahn—Banach theorem, a linear functional L_f with $L_f f=\tau(f)$, $|L_f g|\leqq\tau(g)$ $(g\in C[0,1])$. Let

$$L_f^+ g=\sup_{0\leqq h\leqq g} L_f h, \quad g\in C_+$$

and

$$L_f^+ g \stackrel{\text{def}}{=} L_f^+ g^+ - L_f^+ g^-, \quad g\in C[0,1]$$

where $g=g^+-g^-$ is the decomposition of g into its positive and negative parts. Then L_f^+ is a positive linear functional on $C[0,1]$ (the positive part of L_f) with the properties

$$L_f^+ g = L_f^+(g^+-g^-)\leqq L_f^+ g^+ = \sup_{0\leqq h\leqq g^+} L_f h \leqq$$

$$\leqq \sup_{0\leqq h\leqq g^+} \tau(h)\leqq\tau(g^+)\leqq\tau(|g|)=\tau(g) \quad (g\in C[0,1]),$$

$$\tau(f)\geqq L_f^+ f\geqq L_f f=\tau(f), \quad \|L_f^+\|\leqq\|\tau\|.$$

Thus, for all $f\in C_+$

$$\tau(f)=\sup_{\substack{\|L\|\leqq\|\tau\|,\ L \text{ positive}\\ L\leqq\tau}} Lf$$

and this yields again a sequence $\{L_n\}$ of positive linear functionals such that $\|L_n\|\leqq$ $\leqq\|\tau\|$, $L_n\leqq\tau$ and

$$\tau(f)=\sup_n L_n f \quad (f\in C_+).$$

By Corollary 3 every L_n has the form

$$L_n f=\lim_{m\to\infty}\frac{\|L_n\| f(x_1^{(n)})+\ldots+\|L_n\| f(x_m^{(n)})}{m}$$

with a suitable sequence $\{x_k^{(n)}\}_{k=1}^{\infty}$. Now Lemma 2 (i) gives a sequence $\{x_n\}$ and also a corresponding sequence $\{c_n\}$ (every c_n is some $\|L_k\|$) with

$$\tau(f)=\tau(|f|)=\sup_n L_n(|f|)=\limsup_{m\to\infty}\frac{c_1|f(x_1)|+\ldots+c_m|f(x_m)|}{m}$$

and we are done.

We have completed our proofs.

§ 4. Extension to compact metric spaces

Theorem 9. *All of the results of* §§2 *and* 3 *hold if we replace in them* $C[0, 1]$ *by* $C(K)$ *where* K *is an arbitrary compact metric space.*

Naturally, in Corollary 2 the term "polynomial" must be replaced by "generalized polynomial" corresponding to a system satisfying the assumptions of the Stone—Weierstrass approximation theorem.

If K is a compact Hausdorff space then the metrizability of K is equivalent to the separability of $C(K)$. Now what about nonseparable spaces? Does Theorem 9 hold without the metrizability assumption? The answer is no: if K is a non-separable compact topological group with Hausdorff topology and μ $(\mu(K)=1)$ is the left invariant Haar-measure on K then

$$Lf = \int_K f\,d\mu$$

is not a $(C, 1)$-functional. Indeed, if $\{x_k\}_{k=1}^{\infty}$ is any sequence from K then there is a function $f\in C(K), f\geqq 0, f\not\equiv 0$ such that f is zero on the closure of $\{x_k\}$, but, by the properties of μ, $Lf>0$.

Now at this point one might suspect that the metrizability of K is necessary in Theorem 9. However, this again turns out to be false: if K is the one point (so called Alexandroff) compactifications of a non-countable discrete space, i.e.

$$K = \{x_\alpha\}_{\alpha\in A}\cup\{w\}, \quad |A| > \aleph_0$$

then every continuous function is constant on $K\setminus$\{a countable set\} and hence for every complex Borel measure μ

$$\int f\,d\mu = \sum_{\alpha\in A} f(x_\alpha)\mu(\{x_\alpha\}) + f(w)\big(\mu(K) - \sum_{\alpha\in A}\mu(\{x_\alpha\})\big)$$

(take into account that in the sums we have $\mu(\{x_\alpha\})\neq 0$ for at most a countable set of the α's), and it is obvious that the functional on the right hand side is a $(C, 1)$-functional.

We were not able to give necessary and sufficient conditions for a compact Hausdorff space K that every $L\in C^*(K)$ be a weighted $(C, 1)$-functional.

Proof of Theorem 9. Since $C(K)$ is separable, all of our considerations remain valid for $C(K)$ if we can prove the analogue of Corollary 3. Examining the proof of Corollary 3 we can see that it is enough to show that every PL1 functional L is the weak*-limit of a sequence of functionals of the form

$$L^n f = \frac{1}{n}(f(x_1)+\dots+f(x_n)).$$

Since the extremal points of the weakly compact and convex set of all PL1 functionals are exactly the point evaluations ($\equiv$ functionals corresponding to point masses), the required statement follows from the Krein—Milman theorem [2, p. 440]: if M is a compact closed subset of a locally convex linear topological space then M is the closure of the convex hull of its extremal points. We have completed our proof.

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Bibliographie

M. A. Armstrong, Basic Topology, (Undegraduate Texts in Mathematics), XII+251 pages, Springer-Verlag, New York—Heidelberg—Berlin—Tokyo, 1983.

This is a topology book for undergraduates familiar with a first course in analysis, together with a knowledge of elementary group theory and linear algebra. The author shows several approaches in several branches of topology rather than too deep results in any articular area. The book deals with general topology, geometric and algebraic topology.

The clear geometric motivation is supplied by 132 figures. The author cares the delicate equilibrium of lengthy theories and applications, what helps the beginner reader.

The first chapter is an introductory one, the following three chapters contain a basic knowledge in general topology (compactness, connectedness, product spaces, glueing, topological groups). In the fifth chapter the fundamental group is introduced and is applied to prove the Brouwer fixed point theorem for a disc and the Jordan curve theorem. The following two chapters are devoted to triangulations, complexes, barycentric subdivision and simplicial approximation, the classification of closed surfaces. Two chapters deal with simplicial homology and its applications (degree of maps, the Euler-Poincaré formula, the Borsuk-Ulam theorem, the Lefschetz fixed point theorem, the invariance of dimension), but the author misses to give any systematic method for calculating homology groups, since the beginner may meet this trouble later as well. The last chapter is devoted to knots, an appendix on generators and relations is also included.

The book is highly recommended to undergraduate and first year graduate students. I have to emphasize its wide coverage, you have really an unusual introduction to topology. The previous edition of the book was published by McGraw-Hill (Great Britain) in 1979.

L. A. Székely (Szeged)

Differential Geometric Methods in Mathematical Physics, Proceedings, Clausthal, Germany, 1978. Edited by H. D. Doebner (Lecture Notes in Physics, 139), VIII+330 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

This volume contains the text of 16 lectures presented at the conference "Differential Geometric Methods in Mathematical Physics" held at the Technical University of Clausthal, Germany, July 1978.

The lectures have been arranged into the following four groups according to the subject they are dealing with.

1. Quantization Methods and Special Quantum Systems: geometric quantization, vectorfield quantization, quantization of stochastic phase spaces, dynamics of magnetic monopoles, spectrum generating groups. 2. Gauge Theories: phase space of the classical Yang—Mills equation, nonlinear σ-models, gauging geometrodynamics, exceptional gauge groups. 3. Elliptic Operators, Spectral Theory and Applications: the Atiyah—Singer theorem applied to quantum-filed theory, spectral theory applied to phase transitions. 4. Geometric Methods and Global Analysis: systems of non-

Hausdorff spaces and non-Euclidean spaces, Weyl geometry, Lorentz manifolds, manifolds of embeddings.

The excellent papers communicated in this book are worth studying for mathematicians and physicists interested in any of the four mentioned research fields.

L. Gy. Fehér (Szeged)

Equadiff 82, Proceedings, Würzburg, 1982. Edited by H. W. Knobloch and K. Schmitt (Lecture Notes in Mathematics, 1017), XIII+666 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

The international conference EQUADIFF 82 held at the University of Würzburg during the week August 23 to August 28, 1982 was the fourth in a sequence of international conferences, with focus on the subject of differential equations, which were started in 1970 in Marseille.

What is the use of conferences? Besides to meet old friends and to get acquainted with new ones, it is to obtain a cross section of current research in the field. These proceedings consisting of 59 papers, give such a cross section also to those experts of the theory of differential equations not being able to participate the conference. It focussed on the branches of ordinary, functional and stochastic differential equations, partial differential equations of evolution type, and difference equations. Unfortunately, there is no room in such a review to list all the lectures, here are some topics special attention was paid to: infinite dimensional dynamical systems, semigroups of operators in Banach spaces, stability and bifurcation theory, Hamiltonian systems, functional differential equations with infinite delay, epidemic models, diffusion reaction model, numerical methods and applications in physics, engineering and biology.

The volume will give a very useful panoramic vision to every expert in the theory of differential equations and its applications.

L. Hatvani (Szeged)

P. Erdős, A. Hajnal, A. Máté, R. Rado, Combinatorial Set Theory: Partition Relations for Cardinals (Disquisitiones Mathematicae Hungaricae, 13 and Studies in Logic and the Foundations of Mathematics, Vol. 106), 347 pages, Jointly published by Akadémiai Kiadó, Budapest and North-Holland Publ. Co. Amstedam—New York—Oxford, 1984.

Starting from the familiar result known as Ramsey's theorem, a large portion of set theory, called the partition calculus, is developed in a considerable pace. The main interest of this calculus is in generalizing Ramsey's theorem for large cardinals. By now, the subject has achieved a stage when a systematic synthesis is possible (and desirable for further research). This book is devoted to such a systematic treatment. As the authors write: "we want to give a discussion of the ordinary partition relation for cardinals without the assumption of the generalized continuum hypothesis; we tried to make this latter as complete as possible".

The first two chapters, entitled Introduction and Preliminaries, respectively, provide the necessary backgrounds from classical set theory; for example, the basics on Zermelo—Fraenkel set theory, including axioms, Mostowski's Collapsing Lemma, equivalents for the Axiom of Choice, stationary sets, Fodor's and Solovay's theorems and the like, are considered briefly. Partition calculus proper begins in Chapter III, where Ramsey's result and its first important generalization, the Erdős—Dushnik—Miller theorem are proved. A separate section is included here in order to describe the main partition symbols used in the literature. In the next chapter (infinite) trees are treated in details, thus obtaining a powerful tool (the stepping-up lemma) for deriving positive ordinary partition relations. Chapter V is devoted to negative ordinary partition relations, while the next one develops important auxiliary results, called Canonization Lemmas, which are used later for establishing some positive partition relations for singular cardinals. Partition relations on large cardinals are investigated in Chapter VII, in particular a theorem due to Hanf and Tarski is proved. The next two chapters consider ordinary partition relations with "superscript" two and greater than two, respectively. Finally, the last two chapters give some applications of combinatorial methods, including Arhangel'skiĭ's result on the cardinality of the first countable compact Hausdorff space, the set-mappings theorems due to Fodor and Hajnal, the effect of using Suslin, Kurepa or Aronszajn trees in obtaining results without the (generalized) continuum hypothesis, and some positive results on the existence of (infinitary) Jónsson algebras.

The volume is clearly written; its complete understanding requires little familiarity with other branches of set theory and mathematics, only; and so it will certainly be a useful reading for anyone interested in infinite combinatorial methods.

P. Ecsedi—Tóth (Szeged)

Evaluating Mathematical Programming Techniques Proceedings, Boulder, Colorado, 1981. Edited by John M. Mulvey (Lecture Notes in Economics and Mathematical Systems, 199), XI+379 pages, Springer-Verlag Berlin, Heidelberg—New York.

This book contains approximately 30 lectures given at a twoday conference in Boulder. The main topic of this meeting was to consider how mathematical programming techniques ought to be evaluated.

The papers of the first two sections deal with several test problems and their computational experiments. The reader can find several comparisons of differently generated test problems for linear and non-linear problems. There are statistical reviews and methodological approaches as well. The next part of the book contains those examinations which consider computational comparisons for such integer programming and combinatorial optimization problems as the Euclidean travelling salesman — and the multidimensional knapsack problem. There is a description of an interesting algorithm, named SLIP, to choose the options from several algorithm factors. The next section is considered with identifying ideas that, perhaps, will guide future studies on comparisons of algorithms and codes. Three methods were selected for critical review: The Sandgreen—Ragsdell's, the Schittkowski's and the Miele—Gonzalez's studies. The remaining part of the book contains such approaches to software testing which use other disciplines, for example statistical methods.

These proceedings give a good overview of the recent research on this field of mathematics.

G. Galambos (Szeged)

E. Fried, Abstrakte Algebra. Eine elementare Einführung, IV+340 Seiten, Akadémiai Kiadó, Budapest, 1983.

Dieses Buch ist die Übersetzung des im Jahre 1972 erschienenen ungarischen Originals. Das Buch hat das Ziel, die Methoden darzulegen, die in der abstrakten Algebra auftreten. Gleichzeitig wird auch gezeigt, daß man diese Methoden zu Lösungen von Aufgaben welchen Typs verwenden kann. Dieses Ziel wird allerdings auf elementarem Weg erreicht. Die Kapitel beschäftigen sich mit Gruppen und Halbgruppen; Ringen, Körpern und Vektorräumen; Verbänden, Booleschen Algebren; universellen Algebren und Kategorien. Ein großer Vorzug des Buches ist, daß es die sich erhebenen algebraischen Begriffe durch interessante Beispiele klarstellt. Die Aufgaben sind außerordentlich nützlich und anschaulich. Dieses Buch populären Charakters ist für alle vorzuschlagen, die sich für die abstrakte Algebra interessieren.

L. Megyesi (Szeged)

James Glimm—Arthur Jaffe, Quantum Physics: A Functional Integral Point of View, XX+417 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

It is a basic "experimental fact" that physical problems of the most diverse origin can be dealt with be using common mathematical techniques. For classical physics the mathematical method of primary importance was provided by the theory of differential equations. The favourite candidates for this role in the case of modern physics are probability theory and analysis over function spaces. The authors, leading constructive field theorists, apply this fascinatingly uniform approach to three of the main branches of modern physics: quantum mechanics, statistical mechanics and quantum field theory.

The book consists of three parts of different style and intention. Part I is an introduction to the conceptual structure of quantum and statistical physics. Here the authors' purpose was to make the treatment of physics self-contained as far as it is possible. This is a big help for mathematicians but physicists and students also will find this survey useful. Among others there is a clear explanation of the famous Feynman—Kac formula here from the view-point of Wiener integrals. Part II is devoted to the main subject of the book: quantization of nonlinear fields. It gives a mathematically self-contained development of the theory of certain non-Gaussian measures on function spaces. Complete construction of boson fields with polynomial interaction in two spacetime dimensions has been presented. The compatibility of relativistic quantum mechanics and the constructed nonlinear quantum field theory is proved. This and other examples (all of them exist in two or three dimensions) answer the long standing question about mathematical implementability of quantization defined by renormalized perturbation theory in the positive. But it has been an open question up till now for the four dimensional case. Scattering theory, bound states, phase transitons and critical pints, the method of cluster expansion, reconstruction of quantum mechanics from path integrals form the theme of Part III. This part of the book is written at a more advanced level and provides an introduction to the literature.

The present book is highly recommended to all who are interested in the mathematical structure or applications of statistical and quantum physics.

L. Gy. Fehér (Szeged)

Victor Guillemin and Shlomo Sternberg, Symplectic techniques in physics, XI+468 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sidney, 1984.

This clearly written, excellent book contains extraordinarily wealthy material on pure symplectic geometry and on its extensive physical applications. Symplectic geometry has appeared as the modern version of the "old theory" of canonical transformations. Now it is one of the most active reserach areas where the frontiers of mathematics and physics are connected along questions of fundamental importance for both closely related sciences. Many of the results presented in the book were available only in journal articles so far and some of them are new. The authors' purpose is twofold: to provide an introduction to the subject and to present the central results from a modern point of view.

In the first chapter they give a general survey of the mathematical and physical ideas involved in the development of symplectic geometry. Among them are the optical analogies of classical and quantum mehcanics, the problem of quantizations, particle motion in electromagnetic and gravity fields. The style is rather elementary but almost all the questions detailed further on with more sophisticated techniques are sketched here. Chapter II is devoted to the description of the key mathematical results about symplectic manifolds and homogeneous spaces, Hamiltonian group actions and their moment maps, foliations and reduction procedures. Applications concerning the problem of collective motions — collective Hamiltonians and an outline of the geometric quantization theory are given. The main points of the third chapter are the motion of particles in Yang—Mills and gravity fields, the principle of general covariance. A new local normal form description is presented for Hamiltonian actions of compact Lie groups. Chapter IV deals with the use of group theoretical methods in the investigation of complete integrability, with questions from the theory of solitop and the higher-order calculus of variations. The final part contains several standard results on Lie algebras ang highly non-standard ones about the deformation theory of Lie algebras and the associated symplectic homogeneous spaces. This is of central interest in the limiting process from a general physical theory to one of its special cases according to Bohr's principle of correspondence. The main results are taken from Coppersmith's unpublished thesis and generalize for example the widely known contraction of the Poincaré algebra to the Galilean one. Besides the mentioned themes many important physical examples and mathematical theorems are treated.

In conclusion this book is warmly recommended to everyone interested in symplectic geometry and its applications. It can be used as an up-to-date textbook for graduated students and will have durable significance for mathematicians and theoretical physicists.

L. Gy. Fehér (Szeged)

L. Henkin, J. D. Monk, A. Tarski, H Andréka, and I. Németi, Cylindric Set Algebras (Lectur, Notes in Mathematics, 883), VII+323 pages, Springer-Verlag, Berlin—Heidelberg—New York 1981.

Henkin, Monk and Tarski published a book in 1971, entitled Cylindric Algebras, Part I, which soon became a basic reference to algebraists and logicians. Their intention in writing a second part on the topic was clearly put in the title; and, indeed, they noted there that a preliminary chapter of the continuation was available in mimeographed copies. The first paper of the present volume is a revised and considerably extended version of that chapter, and also, is considered by the authors as a starting piece of a series of papers "which would form the bulk of Part II of their earlier work". The paper is (almost) self-contained and presents the basic definitions and properties of several different kinds of cylindric set algebras. In particular, applications of such general algebraic concepts as subalgebras, homomorphisms, direct and ultraproducts, relativization, reducts and the like are treated in details. The results obtained in this way are nice and deep.

The volume contains a second study due to Andréka and Németi. The first three authors write in the introduction (referring to their paper): "As their writing proceeded, they learned of the closely related results obtained by Andréka and Németi, and invited the latter to publish jointly with themselves". In the second paper "certain aspects of the theory are investigated more thoroughly; in particular, many results which are merely formulated in the first paper are provided with proofs in the second one".

Central to the discussion of Andréka and Németi are the so called regular generalized cylindric set algebras which play a role analogous to that of played by Boolean set algebras in the theory of general Boolean algebras.

These two long papers are extremely clearly written, contain several new, nice and deep results and so, no doubt, will become a basic reference for anyone interested in algebraization of logics. We warmly recommend to model theorists and algebraists to have this book on their bookshelf.

P. Ecsedi-Tóth (Szeged)

Paul van den Heuvel, The Stability of a Macroeconomic System with Quantity Constraints (Lecture Notes in Economics and Mathematical Systems, 211), VII+169 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1983.

One of the central questions of mathematical economics is the study of the existence, uniqueness and stability of equilibrium of economical systems. The classical walrasian equilibrium concept was investigated already in the 19th century. This equilibrium is characterized by the equality of demand and supply for all goods by the assumption that the prices are fully flexible. In the keynesian economical models the prices are not completely flexible and the rigid prices cause inequalities between demand and supply. These inequalities are called quantity constraints in the mathematical model, and the corresponding equilibrium concept is called non-walrasian.

Barro—Grossman (1971) and Malinvaud (1977) defined a neokeynesian model where the economics consist of two sectors: consumption and production, and three commodities: labour, consumption good and money. This book is devoted to the detailed mathematical study of this economical model, the corresponding equilibrium concept and the related stability questions.

The book is written in a very clear style. The only prerequisite for its reading is some basic knowledge in convex analysis and ordinary differential equations. This book is of interest to specialists engaged in equilibrium theory of economical systems. Moreover it is warmly recommended to everyone who is willing to get acquainted with the background of mathematical economics.

Péter T. Nagy (Szeged)

Paul Kelly—Gordon Matthews, The Non-Euclidean, Hiperbolic Plane, Its Structure and Consistency (Universitext), XIII+333 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

The purpose of this book is to give an introduction to axiomatic Bolyai—Lobatchevsky plane geometry accessible to anyone with a good background in high school mathematics. The authors present a strong "metrical" axiom system which makes possible to derive the basic structure of hyperbolic plane geometry and of its euclidean models without difficulties concerning the order and congruence relations and their consequencies. They say "The development... is especially directed to college students who may become secondary teachers. For that reason, the treatment is designed to emphasize those aspects of hyperbolic plane geometry which contribute to the skills, knowledge, and insights to teach euclidean geometry with some mastery". Chapter I outlines the history of the "parallel-postulate" problem. Chapter II is a review of George Birkhoff' axiom system of absolute geometry. Chapter III gives a syntetic development of central theorems in hyperbolic plane geometry. Finally, a short introduction to "distance geometries" is given in an appendix.

The excellent textbook is warmly recommended to students and to everyone who is interested in teaching of geometry.

Péter T. Nagy (Szeged)

D. V. Lindley—W. F. Scott, New Cambridge Elementary Statistical Tables, 80 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1984.

The concept of what constitutes a familiar or elementary statistical procedure has changed in the last 30 years and the rapid progress of pocket calculators made some statistical tables unnecessary. The reason for the publication of this set of tables is to replace the Cambridge Elementary Statistical Tables (Lindely and Miller, 1953). As the authors explain in the Preface they wanted to provide a convenient set of tables for the teaching and study of statistics in schools and university.

The binomial, Poisson, normal, χ^2 and t distributions have been fully tabulated and the book contains the percentage points of these distributions as well. The percentage points of the Behrens', F, Spearman's S, Kendall's K, Wilcoxon's signed — rank and Mann — Whitney distributions and the upper percentage points of the one — and two — sample Kolmogorov—Smirnov, Friedman's and Kruskal—Wallis statistics are also given. There are also tables of sums of squares of normal scores, random sampling numbers, random normal deviates and expected values of normal order statistics. The gerat experience of the Cambridge University Press guarantees that there are not misprints in the tables.

Lajos Horváth (Szeged)

J. Macki—A. Strauss, Introduction to Optimal Control Theory (Undergraduate Texts in Mathematics), XIII+165 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

Perhaps one of the most characteristic feature of this book can be gathered already from the sentences of the first chapter: "In control theory, one is interested in governing the state of a system by using controls. The best way to understand these three concepts is through examples." Then we find examples: A national economy, Water storage and supply and the example which is used throughout the monograph: The rocket car. Because of the generality of the concepts of systems, state and control the authors could just as well have chosen examples from biology, space flight and other fields. It is worth enumerating some of the exercises of Chapter 1.: A model for the optimal harvesting of fish; A model for the control of epidemics; The moon landing problem; Neoclassical economic growth model.

The book is written in a nice style, emphasizing motivation and explanation, avoiding the ponderous "definition — axiom — theorem — proof" approach.

In proving theorems the authors often just prove a relatively simple case. The general case is omitted or is given in the appendices. At the end of the chapters one finds several notes, references and examples.

The book is in some sense self-contained, the prerequisities are only advanced calculus (including Lebesgue integration), basic course in ordinary differential equation and linear algebra.

Chapter headings are: Introduction and motivation; Controllability (with an appendix containing the proof of the bang-bang principle); Linear autonomous time-optimal control problems; Existence theorems for optimal control problems; Necessary conditions for optimal controls — the Pontryagin Maximum principle; Appendix to the previous chapter: a proof of the Pontryagin Maximum principle.

Summarizing, this excellent concise introduction is useful not only for advanced undergraduates in mathematics, but also for economists, engineers and applied scientists because the authors find the ideal balance between the theory and application of mathematics.

L. Pintér (Szeged)

Measure Theory and its Applications, Proceedings of a Conference held at Sherbrooke, Québec, Canada, June 7—18, 1982. Edited by J. M. Belley, J. Dubois and P. Morales (Lecture Notes in Mathematics, 1033), XV+317 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

This volume contains 28 papers on several branches of measure theory. The topics are ergodic theory, vector measures, measure theory and topology. Choquet, G., Representation integrale, Convexes et cones convexes non localement compacts, Formes lineaires positives et mesures; Oxtoby, J., Transitive points in a family of minimal sets; Hida, T. and Streit, L., White noise analysis and its application to Feynman integral.

Seven papers are in French. The conference proceedings are completed with a selection of open problems of the problem section. In order to call your attention to the contained open problems, I quote a beautiful problem of P. Erdős (cited by D. Mauldin in the book).

Let K be a compact subset of $\mathbf{R}^2$, with Lebesgue measure $\lambda(K)>0$. Does there exist a point x such that $\{\|y-x\|:\ y\in K\}$ contains an open interval?

L. Á. Székely (Szeged)

Angelo B. Mingarelli, Volterra—Stieltjes Integral Equations and Generalized Ordinary Differential Expressions (Lecture Notes in Mathematics, 989), XIV+317 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

The author's aim is to present a qualitative and spectral theory of Volterra—Stieltjes integral equations giving special interest to applications in ordinary differential equations as well as in difference equations.

In Chapter 1. the author studies the extension of the classical Sturmian results — comparison and separation theorems — on Stieltjes integro—differential equations. In Chapter 2. oscillation and non-oscillation theorems are given on Volterra—Stieltjes integral equations with applications to second order differential and difference equations. In Chapter 3. the author uses a natural extension of a method introduced by I. S. Kac. This method shows that the integral equation investigated before can be thought of as defining generalized differential operators. Here is treated the famous Weyl classification (limit point, limit circle) of singular generalized differential operators.

In the final two chapters one finds Sturm—Liouville difference and differential equations with an indefinite weight-function, and the spectral theory of generalized differential operators.

The appendices containing a part of the necessary mathematical background make the book in some sense self-contained.

This thoughtful work on a vividly developing field is warmly recommended for everybody interested in differential and integral equations.

L. Pintér (Szeged)

Model Theory of Algebra and Arithmetic (Proceedings, Karpacz, Poland, 1979). Edited by L Pacholski, J. Wierzejewski, and A. J. Wilkie (Lecture Notes in Mathematics, 834), VI+410 pages Springer-Verlag, Berlin—Heidelberg—New York, 1980.

The volume consists mostly of papers presented by invited lecturers at the Conference on Applications of Logic to Algebra and Arithmetic held at Bierutowice—Karpacz in Poland, September 1—7 1979. Some invited papers not presented personally and a few contributed ones are included, too. The 21 titles of the book are as follows: J. Becker, J. Denef and L. Lipshitz, Further remarks on the elementary theory of formal power series rings; C. Berline, Elimination of quantifiers for nonsemi-simple rings of characteristic p; M. Boffa, A. Macintyre and F. Point, The quantifier elimination problem for rings without nilpotent elements and for semi-simple rings; E. Bouscaren, Existentially closed modules: types and prime models; G. Cherlin, Rings of continuous functions: decision problems; P. Clote, Weak partition relations, finite games, and independence results in Peano arithmetic; F. Delon, Hensel fields in equal characteristic $p>0$; M. A. Dickmann, On polynomials over real closed rings; J. Duret, Les corps faiblement algébriquement clos non séparablement clos ont la propriété d'indépendance; U. Felgner, Horn-Theories of abelian groups; P. Hájek and P. Pudlák, Two orderings of the class of all countable models of Peano arithmetic; A. Macintyre, Ramsey quantifiers in arithmetic; K. L. Manders, Computational complexity of decision problems in elementary number theory; K. McKenna, Some diophantine Nullstellensätze; G. Mills, A tree analysis of unprovable combinatorial statements; J. B. Paris, A hierarchy of cuts in models of arithmetic; C. Smorynski and J. Stavi, Cofinal extension preserves recursive saturation; L. von Den Dries, Some model theory and number theory for models of weak systems of arithmetic; A. J. Wilkie, Applications of complexity theory to Σ_0-definability problems in arithmetic; G. Wilmers, Minimally saturated models; B. I. Zilber, Totally categorial theories: structural properties and the non-finite axiomatizability.

The book gives a good overview on the present state of the arts (of course, at the date of edition) and so it is recommended to experts as well as to graduate students intrested in the subject.

P. Ecsedi—Tóth (Szeged)

E. E. Moise, Introductory Problem Courses in Analysis and Topology (Universitext), VII+94 pages, Springer-Verlag, New York—Heidelberg—Belin, 1982.

This book consists of two parts: Analysis and Topology. In each chapter there are given definitions and theorems guaranteed to be true. The first job for the reader is to prove the proposed theorems. In the problems stated after one finds true and false propositions the reader's task is to give the right answer. This requires to focus his/her attention on the heart of the matter which is not easy for a student, but the result is an unusually rapid progress.

Let us mention some examples: After defining continuous and Lipschitzian functions we have: "Theorem (?). Every continuous function is Lipshitzian. Theorem (?). If f is a continuous function $[a, b]\to\mathbf{R}$, then f is Lipschitzian. Theorem (?). Every Lipschitzian function is continuous." The following example is taken from the second part (Topology): „A set $M\subset\mathbf{R}$ has the Heine—Borel property if for each collection G of open intervals covering M, there is a finite subcollection G' of G which also covers M. Theorem (?). If M has the Heine—Borel property, then M is closed and bounded."

These problem courses are useful for the major part of students because as the author says: "Some have supposed that problem courses are advantageous only for students of real brilliancy, but my own experience over many years indicates the contrary. The time that is 'wasted' while students grope their way makes the pace of a problem course very slow. (It often happens that a whole hour is spent analyzing a 'proof' which turns out to be quite worthless.) This means that a competent student is able to keep track, and finds at the end that he understands the course completely. This is a valuable experience, and for many students it is new."

L. Pintér (Szeged)

Roe Nottrot, Optimal Processes on Manifolds; an Application of Stokes' Theorem (Lecture Notes in Mathematics, 963), VI+124 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

The optimal control theory developed extremely rapidly during the last twenty years. The central topic of investigations was the generalizations of Pontryagin's "maximum principle" which has to be satisfied by an "optimal process". This book gives a unified treatment of variational theory of optimal processes described by a set of ordinary or partial differential equations. The general maximum principle is proved by an application of Stokes' theorem.

Chapter I contains a brief introduction to alternating differential forms, integration, Stokes' theorem on a smooth manifold. Chapter II is devoted to the proof of the maximum principle. Chapter III—V deal with applications of the general theory to processes described by ordinary, first and second order partial differential equations.

This book is highly recommended to all who are interested in control theory.

Péter T. Nagy (Szeged)

David R. Owen, A First Course in the Mathematical Foundations of Thermodynamics (Undergraduate Texts in Mathematics), XVII+178 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

There are two approaches to the foundations of thermodynamics. One of them, beeing in close contact with statistical physics, postulates the existence of the state functions energy and entropy. The other, more traditional treatment, having its roots in the works of Carnot, Joule and Clausius, is based on the concepts of work, heat and temperature.

This book is a modern and a mathematically precise version of the traditional approach, operating of course on more general and more abstract notions. The truth of the classical theory has not been questioned since the end of the last century, nevertheless when reading a standard textbook on thermodynanics, often a kind of dissatisfaction may rise in the reader, wanting a more coherent expansion of the subject. The feeling of frustration is resolved by D. R. Owen. Unsophisticated mathematics and appropriate amount of reference to physical content make this book interesting and useful. It will be certainly appreciated by everyone, who is familiar with the empirical facts of heat theory, but who wishes to see the strict logical order of precisely defined quantities, fundamental assumptions and exact theorems of thermodynamics. After an overview of the properties of homogeneous fluid bodies, the concept of a system with perfect accessibility and the general notion of a thermodynamical system, containing both the states and the possible processes of them, are introduced. They make possible to state the first and second laws, and to prove the existence of energy and entropy in a simple manner.

In the course of development the role of the ideal gas — the classical object of the theory — is sometimes felt to be exaggerated, but the reader is compensated at the end of the book by the treatment of elastic filaments and viscous bodies, usually falling out of the scope of traditional applications.

M. G. Benedict (Szeged)

L. C. Piccinini, G. Stampacchia, G. Vidossich, Ordinary Differential Equations in R^n. Problems and Methods (Applied Mathematical Sciences, 39), XII+385 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

The authors write in the Preface: "Our text ... aims to give a simple and rapid introduction to the various themes, problems, and methods of the theory of ordinary differential equations. The book has been conceived in such a way so that even the reader who has merely had a first course in calculus may be able to study it and to obtain a panoramic vision of the theory". The book answers this not easy purpose in an excellent way.

The purpose is not easy because just the basic theory of differential equation requires knowledges not included in the first courses in calculus. One of the advantages of the book making it a very good introductory lecture notes is the way as it acquaints the reader with these things. We can find short (but completed with proofs where it is necessary) surveys on these topics in bodies of the chapters where they are applied (e.g. review of metric spaces, review of Banach spaces, elements of linear algebra, the topological degree). At the end of the sections exercises are proposed illustrating the results and giving possible applications and complementary material.

The book is not only a good text-book but also a very useful monograph for the experts in differential equations. Reading the book they can get acquainted with new aspects of the basic theory. The most original chapter is the third one, titled Existence and Uniqueness for the Cauchy problem under the condition of continuity. It gives a "panoramic vision" of the results obtained by the Italian school in this field (the Peano-phenomenon, G-convergence, ...). Separate chapter is devoted to boundary value problems, which is written in a modern way but so that it is understandable for beginners, too. The chapters are concluded by bibliographical notes which serve as a guide for further studies.

L. Hatvani (Szeged)

Probability Measures on Groups VII. Proceedings, Oberwolfach, 24—30 April 1983. Edited by H. Heyer (Lecture Notes in Mathematics, 1064), X+588 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

The main aim of this series of meetings on Probability Measures on Groups is to cover the recent advances in this field of probability theory and to present new results. This volume contains four survey articles written by H. Heyer, A. Janssen, R. Schott, M. E. Walter and 31 research papers. The authors have raised also some open problems and possible extensions of the presented material. The editor of this collection has classified the contributions under the following topics: (i) probability measures on locally compact groups (decomposition, infinite divisibility), (ii) random walks on groups and homogeneous spaces (reccurence, polynomial growth, dichotomy theorems), (iii) Markov processes on hypergroups (transience, Lévy—Khinchin formulae, central limit theorems), (iv) Non-commutative probability theory (subadditive ergodic theorems, Gaussian functionals), (v) Random matrices and operators (law of large numbers, random Schrödinger operators, characteristic exponents).

Lajos Horváth (Szeged)

W. T. Reid, Sturmian Theory for Ordinary Differential Equations, XII+559 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

On the first page one finds: "Dedicated to Dr. Hyman J. Ettlinger. Inspiring teacher, who introduced the author as a graduate student to the wonderful world of differential equations."

In this book the reader is introduced by the author into one of the most interesting branches of ordinary differential equations in such a way, that he can hardly get rid of attractive problems.

The famous 1836 paper of Sturm, dealing with oscillation and comparison theorems for linear homogeneous second order ordinary differential equations, is one of the most important starting points of the investigations of the qualitative theory of solutions of differential equations.

The basic works of M. Bocher, D. Hilbert, G. D. Birkhoff, G. A. Bliss, M. Morse are fundamental for the development of the theory.

"The prime purpose of the present monograph is the presentation of a historical and comprehensive survey of the Sturmian theory for self-adjoint differential systems, and for this purpose the classical Sturmian theory is but an important special instance."

The organization of the chapters seems to be ideal to the reviewer. Every chapter contains a body of material which presents concepts and methods central for the investigated theme. This is followed by a section with more detailed comments and references to pertinent literature and — frequently — up to date problems. Finally — the most interesting for many readers — we have a section on Topics and Exercises. The well chosen problems characterizing the author's interest, make the typical feature of the book.

The titles of chapters are: Historical prologue; Sturmian theory for real linear homogeneous second order ordinary differential equations on a compact interval; Self-adjoint boundary problems associated with second order linear differential equations; Oscillation theory on a non-compact interval; Sturmian theory for differential systems; Self-adjoint boundary problems; A class of definite boundary problems; Generalizations of Sturmian theory.

This book is warmly recommended to everyone who is interested in differential equations, and also to other mathematicians — working in various branches of mathematics — who after having read this book will have a survey of this theme and perhaps will find pleasure in making research in this field.

L. Pintér (Szeged)

Séminaire de Probabilités XVIII, 1982/83. Edited by J. Azéma and M. Yor (Lecture Notes in Mathematics, 1059), IV+518 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

This 18th of the now Paris-based Seminaires de Probabilités consists of original research papers in diverse areas of the theory of stochastic processes. The presented 33 papers provide up-to-date overviews of the research activities of the French school of probability. We do not have space to list the titles of the papers published in this volume but we hope that the following list of authors will help to form an opinion of the content of this book. The authors of these proceedings: M. T. Barlow, E. Perkins, R. F. Bass, L. C. G. Rogers, F. B. Kninght, W. S. Kendall, R. F. Gundy, F. Bronner, J. Jacod, M. Liao, H. Rost, C. Stricker, W. A. Zheng, P. A. Meyer, S. W. He, A. Erhard, D. Bakry, C. S. Chou, J. Ruiz de Chavez, J. G. Wang, L. Schwartz, M. Talagrand, Ph. Novelis, F. Russo, R. C. Dalang, J. Neveu, R. Azencott and M. Emery.

Lajos Horváth (Szeged)

K. T. Smith, Primer of Modern Analysis (Directions for Knowing All Dark Things, Rhind Papyrus, 1800 B. C.), (Undergraduate Texts in Mathematics), XV+446 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

Seeing a new introduction to analysis, the reader is to look for those topics the author can show from a new point of view, topics, which are unusual on the level of introduction. You will find these topics, so Smith's book is worth reading.

The author has dealt with the crucial points of analysis from the notion of limit up to research level. The book consists of three parts. The first one is a usual first course in calculus, the rest of the book is of greater importance. The second part contains metric spaces, algebra and geometry in $\mathbf{R}^n$, the calculus in $\mathbf{R}^n$ and surfaces. It is a fruitful approach to treat algebraic and geometric structures of $\mathbf{R}^n$ in a textbook of analysis, since just the connection of several structures belongs to the "set of dark things" of a part of students.

The third part of the book is devoted to some advanced topics of analysis, e.g. Lebesgue measure (included Sard's theorem on regular values), differentiability of regular Borel measures and of Lipschitz functions a.e. surface area, degree of maps (through degree of C^∞ maps by the approach of Milnor), and extensions of differentiable functions.

It is impossible to write a book introducing to all chapters of analysis. The present book omits e.g. complex analysis and Fourier analysis. I think the author is right, a too long book would terrify the possible readers.

I recommend the book to under graduates and graduates as a reference. It is the second edition of the original book published by Bodgen and Quigley, 1971, with substantial revisions.

L. A. Székely (Szeged)

J. Stoer and R. Bulirsch, Introduction to Numerical Analysis, IX+609 pages, Springer-Verlag, New York—Berlin—Heidelberg, 1980.

Since the publication of its original German edition this book has found very wide acceptance. The table of contents reads as follows.

1. Error Analysis 2. Interpolation 3. Topics in Integration 4. Systems of Linear Equations 5. Finding Zeros and Minimum Points by Iterative Methods 6. Eigenvalue Problems 7. Ordinary Differential Equations 8. Iterative Methods for the Solution of Large Systems of Linear Equations. Some Further Methods.

This introduction contains all standard topics and much more. To name some extras, the sections on minimization problems, extrapolation methods and for example, the multiple shooting method are not considered in all customary introductory courses.

The presentation of the material meets the highest mathematical standards. Most of the results including theorems about convergence rates and error bounds appear with full proofs. Special attention is paid to the practical implementation and the comparison of different methods. Numerous algorithmic descriptions are more or less formally provided in ALGOL 60. Each chapter ends with a list of references covering a significant amount of research articles and textbooks published between 1960 and 1974. Some new items up to 1978 are contained, too.

Nearly 200 exercises referring to interesting generalizations and additional results help the reader in the deeper understanding of the ideas presented in the main text. The readability of the book has considerably been increased by illuminating figures, comprehensive informal descriptions and fully worked out numerical examples.

Because of these outstanding features we can warmly recommend the book to all students of mathematics or computer science at the advanced undergraduate or graduate level. Even the more experienced lecturer can utilize some useful ideas about teaching this topic.

J. Virágh (Szeged)

K. R. Stromberg, An Introduction to Classical Real Analysis, IX+575 pages, Wadsworth International Group, Belmont, California, 1981.

This volume is the outgrowth of lectures held by the author during the past twenty years". The emphasis is on "twenty years". This characterizes the whole work, which seems to be a masterpiece among the various books discussing classical analysis.

Chapter headings are: Preliminaries; Numbers; Sequences and series; Limits and continuity; Differentiation; The elementary transcendental functions; Integration; Infinite series and infinite products; Trigonometric series.

This is real analysis in the sense that we do not have the theory of analytic functions but complex numbers and complex functions appear throughout the book. This text contains the preparatory topics which are necessary for learning complex and abstract analysis. The book is recommended first of all for advanced undergraduate and beginning graduate students, but also an experienced teacher will find something new in every chapter: a natural introduction of a notion, a simpler proof of a well-known theorem etc. Here in Szeged the sixth chapter is especially interesting, containing the elementary development of the theory of the Lebesgue integral due to F. Riesz (who spent his most fruitful years at Szeged University).

The excellent, attractive examples and exercises mobilize the reader's activity, sometimes one cannot get away from them without having the solution. The author says in the Preface: "I spent at least three times as much effort in preparing the exercises as I did on the main text itself".

Having read the book one comprehends a former reviewer's opinion (taken from a prepublication review): "This is the book I wish that I had written."

L. Pintér (Szeged)

J. Szép—F. Forgó, Einführung in die Spieltheorie XXIX+292 Seiten, Akadémiai Kiadó, Budapest, 1983.

Dieses Buch ist die deutsche Übersetzung der im Jahre 1974 erschienenen ungarischen Ausgabe.

Das Buch gewhärt einen Überblick über die Ergebnisse der Spieltheorie. Die ersten Kapitel enthalten die ganz allgemeinen Begriffe und Sätze der Spieltheorie. Das Buch beschäftigt sich am ausführlichsten mit dem 2-Personen-Spiel — das ist der am besten verwendbare Zweig der Spieltheorie, aber es behandelt auch einige spezielle n-Personen-Spiel und wirft mehrere neue Probleme auf.

L. Megyesi (Szeged)

Theory and Applications of Singular Perturbations. Proceedings. Edited by W. Eckhaus and E. M. de Jager (Lecture Notes in Mathematics, 942), V+363 pages, Springer-Verlag, Berlin—Heildeberg—New York, 1982.

This volume contains 22 papers delivered at the conference on "Theory and Applications of Singular Perturbations" held at Oberwolfach, August 16—22, 1981. The papers deal with recent specialized research in the theory of singular perturbations which is an important area of the theory of ordinary and partial differential equations. Roughly speaking a differential equation is singularly perturbed if the differential operator in the equation is multiplied by a small parameter.

The first part of these notes consists of primarily pure analytic considerations on free boundary problems, nonselfadjoint and nonlinear elliptic eigenvalue problems, coercive singular perturbations singular-singularly perturbed equations, etc. Classical, functional, nonstandard mathematical techniques and numerical methods are applied.

The second part is devoted to the applications concerning two-dimensinal viscous flows, incompressible flows at high Reynolds number, swirling flow between rotating coaxial disks, wave pattern, combustion theory the physics of ionized gases, Kramers' diffusion problem and the kinetic theory of enzymes.

T. Krisztin (Szeged)

B. L. van der Waerden, Geometry and Algebra in Ancient Civilizations, X+233 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

Until quite recently, in almost all books and papers on the early history of mathematics, we could read that the history of mathematics begins with the Babylonian and Egyptian arithmetic, algebra and geometry. However, this picture has been changed by three recent discoveries.

The first of them was made by A. Seidenberg, who studied the altar constructions and other ritual applications of mathematics in the Indian Śulvasūtras. In these relatively ancient texts squares are constructed equal in area to a given rectangle. In the constructions the "Theorem of Pythagoras" was used similarly to that of Euclid.

Secondly, the author compared the ancient Chinese collection "Nine Chapters of the Arithmetical Art" with Babylonian collections of mathematical problems and found so many similarities that the existence of a common origin seemed to him unavoidable. According to his views the "Theorem od Pythagoras" must have played a central role in this source.

The third discovery was made by A. Thom and A. S. Thom. They found that in the construction of megalithic monuments in Southern England and Scotland "Pythagorean Triangels", right-angled triangles whose sides are integral multiples of a fundamental unit of length, have been used.

Combining these three discoveries the author has ventured a tentative reconstruction of a mathematical science which must have existed in the Neolitic Age, between 3000 and 2500 B.C., and spread from Central Europe to the British Islands, to the Near and Far East.

In Chapters 1 and 2 we can read the author's ideas on this ancient science analyzing written sources, archeological evidences, and comparing Chinese and Babylonian mathematical texts as well. One of the fundamental evidences of his hypothesis on the common origin is that we not only meet "Pythagorean Triples" like $(3, 4, 5)$ in Babylonian, Indian, Chinese and Greek texts, but their methods of calculating such triples are very similar, and there are equivalent methods as well.

In the subsequent chapters there is a very interescting comparative analysis of some fields of the Greek, Babylonian, Indian and Chinese mathematics and astronomy.

In Chapters 3, 4 and 6 the several traces of the pre-Babylonian geometry and algebra, which can be discerned in the work of Euclid and Diophantos and in popular Greek mathematics are discussed. In this discussion the four newly found books of Diophantos' Arithmetica (the Books IV to VII) are concerned, too.

In Chapter 5 the different methods in solving linear Diphantine equations and Pell's equations due to the Indian and Chinese mathematicians are compared with the Greek science. We can conclude here to the fact that the Euclidean Algorithm played a central role in their solutions.

In Chapter 7 the author points out that the work of the excellent Chinese geometer Liu Hui (third century A. D.) and some mathematical passages in the works of the great Indian astronomer Āryabhata (sixth century A. D.) are influenced by the work of Greek geometers and astronomers like Archimedes and Apollonios.

This very interesting, informative and enjoyable book — the first volume of the author's "History of Algebra" — is highly recommended to anyone, who is interested in the ancient history of mathematics, especially in the origin of mathematics. We hope that the further volume (or volumes) will appear soon.

Lajos Klukovits (Szeged)

M. J. Wygodski, Höhere Mathematik griffbereit, XI+832 Seiten, Akademie-Verlag, Berlin, 1982.

Dieses Buch ist die (verarbeitete, erweiterte) deutsche Übersetzung des russischen Originals. Über das Ziel des Buches vermittelt das Vorwort folgendes:

"Das Buch ... hat eine zweifache Bestimmung. Erstens übermittelt das Buch Auskünfte über sachgemäße Fragen: Was ist ein Vektorprodukt? Wie bestimmt man die Fläche eines Drehkörpers? Wie entwickelt man eine Funktion in eine trigonometrische Reihe? usw. Die entsprechenden Definitionen, Theoreme, Regeln und Formeln, begleitet von Beispielen und Hinweisen, findet man schnell.

Zweitens ist das Buch für eine systematische Lektüre bestimmt. Es beansprucht nicht die Rolle eines Lehrbuches. Beweise werden daher nur in Ausnahmefällen vollständig gegeben. Jedoch kann das Buch als Hilfsmittel für eine erste Auseinandersetzung mit dem Gegenstand dienen".

Der überwiegende Teil des Buchinhalts fällt auf das Gebiet der Geometrie und der mathematischen Analysis.

L. Megyesi (Szeged)

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