Eigenvalue Approximations for Linear Periodic Differential Equations with a Singularity

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1 Introduction

We consider the second order, linear differential equation

$$y''(x) + (\lambda - q(x))y(x) = 0 \quad \text{for } x \in \mathbb{R} \tag{1.1}$$

where q is a real-valued, periodic function with period a. We further suppose that q has a single non-integrable singularity within $[0, a]$ which is repeated by periodicity. We take the singularity to be at the point $\frac{a}{2}$. More particularly, we suppose that $q \in L^1_{loc}[0, \frac{a}{2}) \cup (\frac{a}{2}, a]$. It is well known, see for example [4] and [6], that for certain types of singularity (1.1) may be "regularized" in the sense that it may be transformed to a differential equation all of whose coefficients belong to $L^1[0, a]$, and has spectral properties related to those of (1.1). This regularized form of (1.1) also gives rise to a Floquet theory similar to that described in [3] and [5]. Our object in this paper is to derive asymptotic estimates for the eigenvalues of (1.1) on $[0, a]$ with periodic and semi-periodic boundary conditions. Our approach to regularizing (1.1) follows that used in [1], [4] and [6]. We illustrate our methods by calculating asymptotic estimates for the periodic and semi-periodic eigenvalues of (1.1) in the case where $q(x) = 1/|x - 1|$ for $x \in [0, 2] \setminus \{1\}$ and repeats by periodicity.

2 The regularizing process

Our approach follows that of [4] and [6]. The differences are necessitated by the fact that the singularity at $\frac{a}{2}$ is repeated by periodicity and when we consider (1.1) on intervals of the form $[\tau, \tau + a]$ for $0 < \tau < a$ we have to take account of the three possibilities:

(a) $0 < \tau < \frac{a}{2}$ so the singularity is at $\frac{a}{2} \in [\tau, \tau + a]$

(b) $\frac{a}{2} < \tau < a$ so the singularity is at $\frac{3a}{2} \in [\tau, \tau + a]$

(c) $\tau = \frac{a}{2}$ so there are singularities at both endpoints of $[\tau, \tau + a]$.

Case a) and b) are similar so we will concentrate on case a) and describe the changes necessary for case b). Case c) is somewhat different and we consider this in §8 below.

We define the space

$$AC^*[0, a] := \{F : [0, \frac{a}{2}) \cup (\frac{a}{2}, a] \to \mathbb{R}, F \in AC[0, \frac{a}{2}) \cup (\frac{a}{2}, a]\} \tag{2.1}$$

where $AC[c, d]$ denotes the set of functions absolutely continuous on $[c, d]$. For $\alpha \in AC^*[0, a]$ to be chosen later we define quasi-derivatives, $y^{[i]}$ as follows.

$$\left.\begin{aligned} y^{[0]} &:= y \\ y^{[1]} &:= y' - \alpha y \\ y^{[2]} &:= (y^{[1]})' + \alpha y^{[1]} - (q - \alpha^2 - \alpha' - \lambda) y \end{aligned}\right\}. \tag{2.2}$$

It may readily be seen that if $y(\cdot, \lambda)$ is a solution of (1.1) then

$$\begin{pmatrix} y \\ y^{[1]} \end{pmatrix}' = \begin{pmatrix} \alpha & 1 \\ q - \alpha^2 - \alpha' - \lambda & -\alpha \end{pmatrix} \begin{pmatrix} y \\ y^{[1]} \end{pmatrix} \text{ almost everywhere.} \tag{2.3}$$

The object of the regularizing process is to choose $\alpha \in AC^*[0, a]$ in such a way that

$$\alpha \text{ and } q - \alpha^2 - \alpha' \in L^1[0,a]. \tag{2.4}$$

An account of how α may be chosen can be found in [4], [6] and the earlier references cited therein. We summarize the results of [4] which are relevant to our needs.

We set $\alpha := \sum_{r=1}^{n} f_r$ where

$$\left.\begin{aligned} f_1 &:= \int^x q(t)dt \\ f_r &:= -\int^x f(f_r + 2\sum_{s=1}^{r-1} f_s)dt \text{ for } 1 \le r \le n-1. \end{aligned}\right\} \tag{2.5}$$

The choice of n depends on the singularity of q at $\frac{a}{2}$. It is shown in [6] for example that if $q(x) = (x - \frac{a}{2})^{-k}$ then an n may be found so that (2.5) regularizes (1.1) for $1 \le k < 2$. An example of a choice of α is given in §9 below.

3 Floquet Theory

It is well known that (2.3) gives rise to a number of eigenvalue problems over $[0, a]$. We refer to [3], [5], and [7] for the details, but we summarize some of the results below.

The periodic problem with boundary conditions

$$y(a,\lambda) = y(0,\lambda), \quad y^{[1]}(0,\lambda) = y^{[1]}(a,\lambda)$$

has eigenvalues $\lambda_0 \le \lambda_1 \le \lambda_2 \le ...$ where $\lambda_n \to \infty$ as $n \to \infty$.

The semi-periodic problem with boundary conditions

$$y(a,\lambda) = -y(0,\lambda), \; y^{[1]}(a,\lambda) = -y^{[1]}(0,\lambda)$$

has eigenvalues $\mu_0 \le \mu_1 \le \mu_2 \le \mu_3 \le ...$ where $\mu_n \to \infty$ as $n \to \infty$.

It is known [3 Theorem 4.4.1] and [5] that the numbers λ_n and μ_n occur in the order

$$\lambda_0 < \mu_0 \le \mu_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \le \lambda_4 < \cdots.$$

The intervals (λ_{2m}, μ_{2m}) and $(\mu_{2m+1}, \lambda_{2m+1})$ are known to correspond to the stability intervals of (2.3) on $\mathbb{R}$. Our primary objective is to investigate the location of these and to explore their relationship with the regularizing function α. We do this by deriving asymptotic estimates for the eigenvalues λ_n and μ_n of the periodic and semi-periodic problems. Our main tool is the following result due to Hochstadt [7]. For $0 < \tau < a$ we consider the problem of (2.3) on $[\tau, \tau + a]$ with Dirichelet boundary condition $y(\tau) = y(\tau + a) = 0$. (2.6) Let $\Lambda_{\tau,n}$ denote the eigenvalues.

Theorem A. The ranges of $\Lambda_{\tau,2m}$ and $\Lambda_{\tau,2m+1}$ as functions of τ are $[\mu_{2m}, \mu_{2m+1}]$ and $[\lambda_{2m+1}, \lambda_{2m+2}]$ respectively.

Our approach to estimating the λ_n and μ_n is to derive approximations for the $\Lambda_{\tau,n}$. These depend explicitly on τ and we are able to approximate their maximum and minimum values. In considering (2.3) over $[\tau, \tau + a]$, the value of τ, in the sense of which of a) b) or c) it satisfies is relevant to our argument in so far as it affects the position of the singularity.

4 Prüfer Transformation Formulae

We define functions $\rho(x, \lambda)$ and $\theta(x, \lambda)$ by the equations

$$\left.\begin{aligned} \lambda^{1/2} y(x,\lambda) &=: \rho(x,\lambda)\sin\theta(x,\lambda) \\ y^{[1]}(x,\lambda) &=: \rho(x,\lambda)\cos\theta(x,\lambda). \end{aligned}\right\} \tag{4.1}$$

It follows from (4.1) that $\tan\theta(x,\lambda) = \lambda^{1/2} y(x,\lambda) \big/ y^{[1]}(x,\lambda)$ and at zeros of $y^{[1]}(x,\lambda)$ is defined by continuity. The regularity of (2.3) ensures that θ is defined at the singularity of q and that $\theta(x, \lambda)$ is determined uniquely within integer multiples of π. It may be shown, as in [6], that θ satisfies the differential equation

$$\theta' = \lambda^{1/2} + \alpha\sin(2\theta) - \lambda^{-1/2}(q - \alpha^2 - \alpha')\sin^2\theta \text{ on } [\tau, \tau + a] \tag{4.2}$$

We normalize θ by the restriction that $0 \le \theta(\tau, \lambda) < \pi$.

It is well known that $\theta(\tau+a,\lambda)$ is a non-decreasing function of λ so eigenvalues of (2.3) with the boundary condition $y(\tau+a,\lambda)=y(\tau,\lambda)=0$ are characterized by

$$\theta(\tau,\Lambda_{\tau,n})=0, \quad \theta(\tau+a,\Lambda_{\tau,n})=(n+1)\pi. \tag{4.3}$$

It is immediate from (4.2) and (4.3) that $\Lambda_{n,\tau}^{1/2}=\dfrac{(n+1)\pi}{a}+O(1)$ as $n\to\infty$, but this estimate is too crude for our needs, in particular the dependence on τ is contained in the $O(1)$ term.

The detailed analysis of $\theta(\tau+a,\lambda)$ depends on which of the three cases of §2 we are in. We consider in some detail case a) and summarize the equivalent results for the other two.

5 The case $0<\tau<a/2$

We define for some natural number N to be chosen later, sequences of functions as follows for $\lambda>0$ and $t\in[\tau,\tau+a]$.

$$\begin{aligned}
&\phi_0(t,\lambda):=\theta(\frac{a}{2},\lambda)+\lambda^{1/2}(t-\frac{a}{2})\\
&\phi_j(\frac{a}{2},\lambda):=\theta(\frac{a}{2},\lambda)\\
&\phi_j'(t,\lambda):=\lambda^{1/2}-\alpha(t)\sin(2\phi_{j-1}(t,\lambda))+\lambda^{-1/2}\{q(t)-\alpha(t)^2-\alpha'(t)\}\sin^2(\phi_{j-1}(t,\lambda))
\end{aligned} \tag{5.1}$$

for $j=1,\cdots,N$.

$$\left.\begin{aligned}
\xi_1(t)&:=\Big|\int_{\frac{a}{2}}^{t}(\big|\alpha(x)\big|+\big|q(s))-\alpha(s)^2-\alpha'(s)\big|)ds\\
\xi_j(t)&=\Big|\int_{\frac{a}{2}}^{t}(\big|\alpha(s)\big|^2+\big|q(s)-\alpha(s)^2-\alpha'(s)\big|\xi_{j-1}(s))ds\Big|
\end{aligned}\right\} \tag{5.2}$$

for $j=1,\cdots,N-1$.

We note from (5.2) that $\xi_j(t)\le C\xi_{j-1}(t)$ for $t\in[\tau,\tau+a]$ and $2\le j\le N+1$.

Lemma 1 *For $j = 1, \cdots, N+1$ and $x \in [\tau, \tau + a]$*

$$|\theta(x, \lambda) - \phi_{j-1}(x, \lambda)| \leq C\xi_{j-1}(x)$$

Lemma 2 *For any $g \in L[\tau, \tau + a]$ and $k > 0$*

(i) $\displaystyle\int_\tau^{\tau+a} g(t) \begin{matrix}\cos\\ \\ \sin\end{matrix} (k\theta(t, \lambda))dt = o(1)$

(ii) $\displaystyle\int_\tau^{\tau+a} g(t) \begin{matrix}\cos\\ \\ \sin\end{matrix} (k\phi_j(t, \lambda))dt = o(1)$

as $\lambda \to \infty$ where $0 \leq j \leq N$.

The proof of these results is similar to the proof of the corresponding results in [6]. The details may be found in [8].

Theorem 1 *If N is such that*

$$\left.\begin{array}{c} \alpha'\xi_{N+1}, \alpha^2\xi_N, \alpha(q - \alpha^2 - \alpha')\xi_N \in L[\tau, \tau + a] \\ \textit{and } \alpha(t)\xi_{N+1}(t) \to 0 \textit{ as } t \to \dfrac{a}{2} \end{array}\right\}. \tag{5.3}$$

Then, as $\lambda \to \infty$,

$$\begin{aligned} \theta(\tau + a, \lambda) - \theta(\tau, \lambda) \quad &= \quad \lambda^{1/2}a - \int_\tau^{\tau+a} \alpha(t)\sin(2\phi_N(t, \lambda))dt \\ &+\frac{1}{2}\lambda^{-1/2}\int_\tau^{\tau+a}[q(t) - \alpha(t)^2 - \alpha(t)^2 - \alpha'(t)]dt \\ &+\frac{1}{2}\lambda^{-1/2}[\alpha(t)(\sin^2\theta(t, \lambda) - \sin^2(\phi_N(t, \lambda))]_\tau^{\tau+a} + o(\lambda^{-1/2}). \end{aligned} \tag{5.4}$$

Proof. We integrate (4.2) from τ to $\tau + a$ and see that

$$\theta(\tau+a,\lambda)-\theta(\tau,\lambda)=\lambda^{1/2}a-\int_\tau^{\tau+a}\alpha(s)\sin(2\theta(s,\lambda))ds$$

$$+\frac{1}{2}\lambda^{-1/2}\int_\tau^{\tau+a}(q(s)-\alpha(s)^2-\alpha'(s))ds \tag{5.5}$$

$$-\frac{1}{2}\lambda^{-1/2}\int_\tau^{\tau+a}(q(s)-\alpha(s)^2-\alpha'(s))\cos(2\theta(s,\lambda))ds.$$

We note that the first and third terms of (5.5) occur in (5.4). The fourth term is $o(\lambda^{-1/2})$ by Lemma 2. The second term may be rewritten as

$$\int_\tau^{\tau+a}\alpha(t)\sin(2\theta(t,\lambda))dt=\int_\tau^{\tau+a}\alpha(t)\sin(2\phi_N(t,\lambda))dt$$
$$+\int_\tau^{\tau+a}\alpha(t)[\sin(2\theta(t,\lambda))-\sin(2\phi_N(t,\lambda))]dt. \tag{5.6}$$

The first term on the right hand side of (5.6) occurs in (5.4). We rewrite the second as

$$\int_\tau^{\tau+a}\alpha(t)\sin(2\theta(t,\lambda))\{\lambda^{-1/2}\theta'(t,\lambda)+\lambda^{-1/2}\alpha(t)\sin(2\theta(t,\lambda))$$

$$-\lambda^{-1}(q(t)-\alpha(t)^2-\alpha'(t))\sin^2\theta(t,\lambda)\}dt$$

$$-\int_\tau^{\tau+a}\alpha(t)\sin(2\phi_N(t,\lambda))\{\lambda^{-1/2}\phi_N'(t,\lambda)+\lambda^{-1/2}\alpha(t)\sin(2\phi_{N-1}(t,\lambda))$$

$$-\lambda^{-1}(q(t)-\alpha(t)^2-\alpha'(t)\sin^2\phi_{N-1}(t,\lambda)\}dt.$$

Integration by parts with the terms involving ϕ_N' and θ' yields the integrated term of (5.4). The remaining terms may be shown to be $o(\lambda^{-1/2})$ by Lemmas 1 and 2 and the inequalities

$$\Big|\sin(2\theta(t,\lambda))-\sin(2\phi_N(t,\lambda))\Big|\le C\Big|\theta(t,\lambda)-\phi_N(t,\lambda)\Big|\le C\xi_N(t).$$

Corollary 1 *If (5.3) holds then*

$$\theta(\frac{a}{2},\lambda)-\theta(\tau,\lambda)=\lambda^{1/2}(\frac{a}{2}-\tau)-\int_{\tau}^{a/2}\alpha(t)\sin(2\phi_N(t,\lambda))dt$$

$$+\frac{1}{2}\lambda^{-1/2}\int_{\tau}^{a/2}[q(t)-\alpha(t)^2-\alpha'(t)]dt$$

$$+\lambda^{-1/2}\alpha(\tau)[\sin^2(\theta(\tau,\lambda))-\sin^2(\phi_N(\tau,\lambda))]+o(\lambda^{-1/2})$$

as $\lambda\to\infty$.

The proof is similar to the proof Theorem 1 and uses the fact that $\theta(\frac{a}{2},\lambda)=\phi_N(\frac{a}{2},\lambda)$.

A problem with the use of Theorem 1 to compute eigenvalue approximations lies in the fact that the computation of $\phi_N(t,\lambda)$ requires knowledge of $\theta(\frac{a}{2},\lambda)$. We require a secondary iteration, involving Corollary 1, to approximate $\theta(\frac{a}{2},\lambda)$ in terms of $\theta(\tau,\lambda)$.

6 Approximation to $\theta(\frac{a}{2},\lambda)$ when $0<\tau<\frac{a}{2}$.

We suppose from now on that there exists a function $E(\lambda)$ with $E(\lambda)\to 0$ as $\lambda\to\infty$ such that

$$\Big|\int_{\frac{a}{2}}^{t}\alpha(s)\ {\cos \atop \sin}\ (2\lambda^{1/2}s)ds\Big|\le E(\lambda)\text{ for all }t\in[\tau,\tau+a]. \tag{6.1}$$

For a demonstration of the circumstances where such a function exists we refer to ([6], Example 4.4). We further suppose that there exists an integer $M\ge 2$ so that, as $\lambda\to\infty$

$$\left.\begin{array}{c}\lambda^{-1/2}<CE(\lambda)^{M-1}\\ CE(\lambda)^{M}\le\lambda^{-1/2}\end{array}\right\}. \tag{6.2}$$

We define a sequence of constants $\{\Psi_k\}$ and functions $\{\psi_{i,j}(t,\lambda)\}$ for $k=1,\cdots,M$ and $j=0,\cdots,N$ as follows

$$
\left.\begin{aligned}
&\Psi_1 := \theta(\tau) + \lambda^{1/2}(\frac{a}{2} - \tau)\\
&\psi_{1,0}(t,\lambda) := \Psi_1 + \lambda^{1/2}(t - \frac{a}{2})\\
&\Psi_2 := \Psi_1 + \int_\tau^{a/2} \alpha(s)\sin(\psi_{1,0}(s,\lambda))ds\\
&\psi_{k,o}(t,\lambda) := \Psi_k + \lambda^{1/2}(t - \delta(t))\\
&\psi_{k,j}(t,\lambda) := \Psi_k + \lambda^{1/2}(t - \delta(t)) - \int_{\frac{a}{2}}^t \alpha(s)\sin(2\psi_{j,k-1}(s,\lambda))ds\\
&\Psi_{k+1} := \theta(\tau) + \lambda^{1/2}(\frac{a}{2} - \tau) - \int_\tau^{a/2} \alpha(s)\sin(2\psi_{k,n}(s))ds
\end{aligned}\right\} \tag{6.3}
$$

for $t \in [\tau, \tau + a]$ where $\delta(t) = \begin{cases} \frac{a}{2} & t \in [0,a] \\ \frac{3a}{2} & t \in (a, 2a] \end{cases}$.

These functions were introduced in [6]. The changes in the analysis of [6] are due to the appearance of $\delta(t)$ which takes into account the fact that q and α are extended beyond $[0,a]$ by periodicity. The proof of the following result is similar to the proof of ([6] Theorem 4.5), for the details we refer to [8].

Theorem 2 *If M and N are chosen as in (5.3) and (6.2) then*

$$
\theta(\tau + a) - \theta(\tau) = \lambda^{1/2}a - \int_\tau^{\tau+a} \alpha(t)\sin(2\psi_{M,N}(t,\lambda))dt
$$

$$
+\frac{1}{2}\lambda^{-1/2}\int_\tau^{\tau+a}[q(t) - \alpha(t)^2 - \alpha'(t)]dt
$$

$$
-\lambda^{-1/2}[\alpha(t)(\sin^2\theta(t,\lambda) - \sin^2\psi_{1,0}(t,\lambda))]_\tau^{\tau+a} + o(\lambda^{-1/2})
$$

as $\lambda \to \infty$.

7 The case $a/2 < \tau < a$

We summarize the results in this case. We define for $t \in [\tau, \tau + a]$

$$\left.\begin{aligned} \phi_0(t,\lambda) &:= \theta(\frac{3a}{2}) + \lambda^{1/2}(t - \frac{3a}{2}) \\ \phi_j(\frac{3a}{2},\lambda) &:= \theta(\frac{3a}{2},\lambda) \\ \phi_j'(t,\lambda) &:= \lambda^{1/2} - \alpha(t)\sin(2\phi_{j-1}(t,\lambda)) \\ &+\lambda^{-1/2}[q(t) - \alpha(t)^2 - \alpha'(t)]\sin^2\phi_{j-1}(t,\lambda) \end{aligned}\right\} \tag{7.1}$$

$$\left.\begin{aligned} \xi_1(t) &:= \Big|\int_{\frac{3a}{2}}^t (\big|\alpha(s)\big| + \big|q(s) - \alpha(s)^2 - \alpha'(s)\big|)ds \\ \xi_j(t) &:= \Big|\int_{\frac{3a}{2}}^t (\big|\alpha(s)\big| + \big|q(s) - \alpha(s)^2 - \alpha'(s)\big|)ds \end{aligned}\right\} \tag{7.2}$$

for $j = 1, \cdots, N+1$ where N is chosen so that

$$\left.\begin{aligned} \alpha'\xi_{N+1}, \alpha^2\xi_N, \alpha(q - \alpha^2 - \alpha')\xi_N \in L[\tau, \tau + a] \\ \alpha(t)\xi_{N+1}(t) \to 0 \text{ as } t \to \frac{3a}{2} \end{aligned}\right\}. \tag{7.3}$$

We suppose the existence of $E(\lambda) \to 0$ as $\lambda \to \infty$ with

$$\Big|\int_{\frac{3a}{2}}^t \alpha(s) \begin{matrix}\cos\\ \sin\end{matrix} (2\lambda^{1/2}s)ds\Big| \le E(\lambda) \text{ for } t \in [\tau, \tau + a]$$

and choose M to satisfy (6.2). Finally we define

$$
\left.\begin{aligned}
\Psi_1 &:= \theta(\tau,\lambda) + \lambda^{1/2}(\delta(t)-\tau) \\
\psi_{1,0}(t,\lambda) &:= \Psi_1 - \int_\tau^{\frac{3a}{2}} \alpha(s)\sin(\psi_{1,0}(s,\lambda))ds \\
\Psi_2 &:= \Psi_1 - \int_\tau^{\frac{3a}{2}} \alpha(s)\sin(\psi_{1,0}(s,\lambda))ds \\
\psi_{k,o}(t,\lambda) &:= \Psi_k + \lambda^{1/2}(t-\frac{3a}{2}) \\
\psi_{k,j}(t,\lambda) &:= \Psi_k + \lambda^{1/2}(t-\frac{3a}{2}) - \int_{\frac{3a}{2}}^{t} \alpha(s)\sin(2\psi_{k,j-1}(s,\lambda))ds \\
\Psi_{k+1} &:= \theta(\tau,\lambda) + \lambda^{1/2}(\frac{3a}{2}-\tau) - \int_\tau^{\frac{3a}{2}} \alpha(s)\sin(2\psi_{k,N}(s,\lambda))ds
\end{aligned}\right\} \tag{7.4}
$$

for $k = 1, \cdots, M$ and $j = 0, \cdots, N$.

Analogously to Theorem 2 we may now prove

Theorem 3 *If (7.3) holds then, as $\lambda \to \infty$*

$$
\begin{aligned}
\theta(\tau+a,\lambda) - \theta(\tau,\lambda) = \lambda^{1/2}a - \int_\tau^{\tau+a} \alpha(t)\sin(2\psi_{M,N}(t,\lambda))dt \\
+\frac{1}{2}\lambda^{-1/2}\int_\tau^{\tau+a}(q(t)-\alpha(t)^2-\alpha'(t))dt \\
-\lambda^{-1/2}[\alpha(t)(\sin^2\theta(t,\lambda)-\sin^2\psi_{1,0}(t,\lambda)]_\tau^{\tau+a} + o(\lambda^{-1/2}).
\end{aligned} \tag{7.5}
$$

8 The case $\tau = \frac{a}{2}$.

The situation now is somewhat different from that considered above and also from that of [6] in as much as q has singularities at both endpoints of the interval $[\tau, \tau+a] = [\frac{a}{2}, \frac{3a}{2}]$. The idea is to approximate $\theta(t,\lambda)$ from within the interval. To this end we define two sequences of approximating functions as follows, for $t \in [\frac{a}{2}, \frac{3a}{2}]$.

$$
\left.\begin{aligned}
&\phi_{\frac{a}{2},0}(t,\lambda) := \theta(\frac{a}{2},\lambda) + \lambda^{1/2}(t-\frac{a}{2})\\
&\phi_{\frac{a}{2},j}(\frac{a}{2},\lambda) := \theta(\frac{a}{2},\lambda)\\
&\phi'_{\frac{a}{2}}(t,\lambda) := \lambda^{1/2} - \alpha(t)\sin(2\phi_{\frac{a}{2},j-1}(t,\lambda))\\
&+\lambda^{-1/2}[q(t)-\alpha(t)^2-\alpha'(t)]\sin^2\phi_{\frac{a}{2},j-1}(t,\lambda)\\
&\phi_{\frac{3a}{2},0}(t,\lambda) := \theta(\frac{3a}{2},\lambda) - \lambda^{1/2}(\frac{3a}{2}-t)\\
&\phi_{\frac{3a}{2},j}(\frac{3a}{2},\lambda) := \theta(\frac{3a}{2},\lambda)\\
&\phi'_{\frac{3a}{2},j}(t,\lambda) := \lambda^{1/2} - \alpha(t)\sin(2\phi_{\frac{3a}{2},j-1}(t,\lambda))\\
&+\lambda^{-1/2}[q(t)-\alpha(t)^2-\alpha'(t)]\sin^2\phi_{\frac{3a}{2},j-1}(t,\lambda)
\end{aligned}\right\} \tag{8.1}
$$

for $j = 1, 2, ..., N$ where N is to be chosen later. We also define two sequences $\{\xi_{\frac{a}{2},j}(t)\}$ and $\{\xi_{\frac{3a}{2},j}(t)\}$ for $t \in \left[\frac{a}{2}, \frac{3a}{2}\right]$ by

$$
\begin{aligned}
\xi_{\frac{a}{2},1}(t) &:= \left|\int_{\frac{a}{2}}^{t} |\alpha(s)| + |q(s)-\alpha(s)^2-\alpha'(s)|ds\right|\\
\xi_{\frac{a}{2},j}(t) &:= \left|\int_{\frac{a}{2}}^{t} (|\alpha(s)| + |q(s)-\alpha(s)^2-\alpha'(s)|)\xi_{\frac{a}{2},j-1}(s)ds\right|\\
\xi_{\frac{3a}{2},j}(t) &:= \xi_{\frac{a}{2},j}(t) \text{ with } \frac{a}{2} \text{ replaced by } \frac{3a}{2}.
\end{aligned}
$$

We set Z to be an interior part of $\left[\frac{a}{2}, \frac{3a}{2}\right]$. Which interior point we take is immaterial for the theoretical results which follow, but an appropriate choice can simplify the computations, see §9 below.

Lemma 3 *Suppose that $N_{\frac{a}{2}}$ is chosen so that*

$$\alpha'\xi_{\frac{a}{2},N_{\frac{a}{2}}}, \alpha^2\xi_{\frac{a}{2},N_{\frac{a}{2}}},\ \alpha[q-\alpha^2-\alpha']\xi_{\frac{a}{2},N_{\frac{a}{2}}} \in L\left[\frac{a}{2},\frac{3a}{2}\right]$$

$$\alpha(t)\xi_{\frac{a}{2},N_{\frac{a}{2}+1}}(t) \to 0 \ \text{as}\ t \to \frac{a}{2}.$$

Then, as $\lambda \to \infty$

$$\begin{aligned}\theta(Z,\lambda)-\theta(\frac{a}{2},\lambda) &= \lambda^{1/2}(Z-\frac{a}{2}) - \int_{\frac{a}{2}}^{z}\alpha(t)\sin(2\phi_{\frac{a}{2},N_{\frac{a}{2}}}(t,\lambda))dt \\ &+\frac{1}{2}\lambda^{-1/2}\int_{\frac{a}{2}}^{z}[q(t)-\alpha(t)^2-\alpha'(t)]dt \\ &-\lambda^{-1/2}\alpha(Z)[\sin^2(\theta(Z,\lambda))-\sin^2(\phi_{\frac{a}{2},N_{\frac{a}{2}}}(Z,\lambda))]+o(\lambda^{-1/2}).\end{aligned}$$

The corresponding result for the right hand singularity is as follows.

Lemma 4 *Suppose $N_{\frac{3a}{2}}$ is such that*

$$\alpha'\xi_{\frac{3a}{2},N_{\frac{3a}{2}}},\ \alpha^2\xi_{\frac{3a}{2},N_{\frac{3a}{2}}},\ \alpha[q-\alpha^2-\alpha']\xi_{\frac{3a}{2},N_{\frac{3a}{2}}} \in L[\frac{a}{2},\frac{3a}{2}]$$

and $\alpha(t)\xi_{\frac{3a}{2},N_{\frac{3a}{2}}+1}(t) \to 0$ *as* $t \to \dfrac{3a}{2}$ *then*

$$\begin{aligned}\theta\left(\frac{3a}{2},\lambda\right)-\theta(Z,\lambda) &= \lambda^{1/2}\left(\frac{3a}{2}-Z\right) - \int_{Z}^{\frac{3a}{2}}\alpha(t)\sin(2\phi_{\frac{3a}{2},N_{\frac{3a}{2}}}(t,\lambda))dt \\ &+\frac{1}{2}\lambda^{-1/2}\int_{z}^{\frac{3}{2}}[q(t)-\alpha(t)^2-\alpha'(t)]dt \\ &+\lambda^{-1/2}\alpha(Z)\left[\sin^2(\theta(Z,\lambda))-\sin^2(\phi_{\frac{3a}{2},N_{\frac{3a}{2}}}(Z,\lambda))\right]+o(\lambda^{1/2})\end{aligned}$$

as $\lambda \to \infty$.

The proofs of these results are similar to the proof of Theorem 1. The details may again be found in [8].

We combine Lemma 3 and 4 to obtain

Theorem 4 *Under the hypotheses of Lemmas 3 and 4*

$$\begin{aligned}\theta(\tfrac{3a}{2},\lambda)-\theta(\tfrac{a}{2},\lambda) &= a\lambda^{1/2}-\int_{\frac{a}{2}}^{z}\alpha(t)\sin(2\phi_{\frac{a}{2},N_{\frac{a}{2}}}(t,\lambda))dt\\ &-\int_{z}^{\frac{3a}{2}}\alpha(t)\sin(2\phi_{\frac{3a}{2},N_{\frac{3a}{2}}}(t,\lambda))dt\\ &+\frac{1}{2}\lambda^{-1/2}\int_{\frac{a}{2}}^{\frac{3a}{2}}[q(t)-\alpha'(t)-\alpha(t)^2]dt\\ &+\lambda^{1/2}\alpha(Z)[\sin^2\phi_{\frac{a}{2},N_{\frac{a}{2}}}(Z,\lambda)-\sin^2\phi_{\frac{3a}{2},N_{\frac{3a}{2}}}(Z,\lambda)]\\ &+o(\lambda^{-1/2})\ \textit{as}\ \lambda\to\infty.\end{aligned}$$

We remark that unlike cases a) and b), there is no need to use a secondary iteration to approximate θ at the singularities since $\theta(\frac{a}{2},\lambda)$ and $\theta(\frac{3a}{2},\lambda)$ for $\lambda=\Lambda_{n,\frac{a}{2}}$ are known from the boundary conditions.

9 An Example

We consider the case when

$$q(x)=\frac{1}{|x-1|}\text{ for }x\in[0,2]\backslash\{1\} \tag{9.1}$$

and is extended to $\mathbb{R}$ by periodicity.

We first approximate the eigenvalues of (1.1) with the boundary values

$$y(\tau)=y(\tau+2)=0. \tag{9.2}$$

In order to follow the preceeding analysis we identify the 3 cases: a) $0<\tau<1$, b) $1<\tau<2$, c) $\tau=1$.

We use the regularization $\alpha=\int^x q(t)dt$ and choose the constant of integration to be zero so that

$$\alpha(x)=ln|x-1|\text{ for }x\in[0,2]\backslash\{1\}. \tag{9.3}$$

whence

$$q(x)-\alpha(x)^2-\alpha'(x)=-(ln|x-1|)^2\text{ for }x\in[0,2]\backslash\{1\}. \tag{9.4}$$

The regularizing function is then extended to $\mathbb{R}$ by periodicity.

Case a

We define

$$\xi_1(x) := \left|\int_1^x (|\alpha(t)| + |q(t) - \alpha(t)^2 - \alpha'(t)|)dt\right|$$

$$\xi_j(x) := \left|\int_1^x (|\alpha(t)| + |q(t) - \alpha(t)^2 - \alpha'(t)|)\xi_{j-1}(t)dt\right|$$

for $x \in [0, 2]$ and extend them to $\mathbb{R}$ by periodicity. It may be shown that

$$\left.\begin{aligned} \xi_1(x) &= O(|x-1|^{1/2}) \\ \xi_2(x) &= O(|x-1|) \\ \alpha'(x) &= O(|x-1|^{-3/2}) \end{aligned}\right\}. \tag{9.5}$$

Combining (9.3)-(9.5) we see that

$$\alpha'\xi_2,\ \alpha^2\xi_1,\ \alpha[q - \alpha^2 - \alpha'] \in L[\tau, \tau+2]$$

and, as $t \to 1$, $\alpha(t)\xi_2(t) \to 0$. Hence we take $N := 1$ in Theorem 1. It may also be shown that the $E(\lambda)$ of (6.1) may be taken to be $C\lambda^{-1/4}$ so, from (6.2), $M := 2$. The relevant functions from (6.3) are

$$\left.\begin{aligned} &\Psi_1 = \lambda^{1/2}(1-\tau) \\ &\psi_{1,0}(t,\lambda) = \lambda^{1/2}(1-\tau) + \lambda^{1/2}(t-1) \\ &\Psi_2 = \lambda^{1/2} - \int_\tau^1 \alpha(s)\sin(2\lambda^{1/2}(1-\tau) + 2\lambda^{1/2}(s-1))ds \\ &\psi_{2,0}(t,\lambda) = \Psi_2 + \lambda^{1/2}(t - \delta(t)) \\ &\psi_{2,1}(t,\lambda) = \Psi_2 + \lambda^{1/2}(t-\delta(t)) - \int_1^t \alpha(s)\sin(2\Psi_2 + 2\lambda^{1/2}(s - \delta(s)))ds \end{aligned}\right\} \tag{9.6}$$

where

$$\delta(t) = \begin{cases} 1 & \text{for } t \in [0,2] \\ 3 & \text{for } t \in (2,3]. \end{cases}$$

A calculation using the functions of (9.6) in Theorem 2 yields the fact that the eigenvalues $\Lambda_{\tau,n}$ for $0 < \tau < 1$ satisfy

$$(n+1)\pi = 2\Lambda_{\tau,n}^{1/2} + \Big\{ \cos 2\Psi_2 \Big[\int_0^1 u^{-1}(1-\cos u)du + ln(2\Lambda_{\tau,n}^{1/2}) - \int_1^\infty u^{-1}\cos(u)du \Big] + ln(1-\tau)\sin^2 2\Lambda_{\tau,n}^{1/2} - 4 \Big\} \Lambda_{\tau,n}^{-1/2} + o(\Lambda_{\tau,n}^{-1/2}) \text{ as } n \to \infty.$$

Upon reversion this gives,

$$\Lambda_{\tau,n}^{1/2} = \frac{(n+1)\pi}{2} + \frac{(-1)^{n+1}}{(n+1)\pi} \Big\{ \int_0^1 u^{-1}(1-\cos u)du - \int_1^\infty u^{-1}\cos(u)du + ln[(n+1)\pi] - 4 \Big\} + o(n^{-1}) \text{ as } n \to \infty. \tag{9.7}$$

Repeating the analysis using Theorem 3 gives the estimate of (9.7) again in the case where $1 < \tau < 2$.

Case c $\tau = 1$.

The interval is now [1,3] with a singularity of q at both end points. It is convenient to exploit the symmetry of q by choosing Z to be 2. It may be shown that the functions of §8 satisfy.

$$\alpha'\xi_{1,2},\ \alpha^2\xi_{1,1},\ \alpha[q-\alpha^2-\alpha']\xi_{1,1} \in L[1,2]$$

$$\alpha'\xi_{3,2},\ \alpha^2\xi_{3,1},\ \alpha[q-\alpha^2-\alpha^2]\xi_{3,1} \in L[2,3].$$

So we take the $N_{\frac{a}{2}}$ and $N_{\frac{3a}{2}}$ of Theorem 4 to be 1 and

$$\left.\begin{aligned} \phi_{1,0}(t,\lambda) &= \theta(1,\lambda) + \lambda^{1/2}(t-1) \\ \phi_{1,1}(1,\lambda) &= \theta(1,\lambda) \\ \phi'_{11}(t,\lambda) &= \lambda^{1/2} - \alpha(t)\sin(2\phi_{1,0}(t,\lambda)) \\ &\quad + \lambda^{-1/2}[q(t)-\alpha(t)^2-\alpha'(t)]\sin^2\phi_{1,0}(t,\lambda) \end{aligned}\right\} \tag{9.8}$$

for $t \in [1,2]$.

We also take, for $t \in [2,3]$

$$\begin{aligned}
\phi_{3,0}(t,\lambda) &= \theta(3,\lambda) - (3-t) \\
\phi_{3,1}(3,\lambda) &= \theta(3,\lambda) \\
\phi'_{3,1}(t,\lambda) &= \lambda^{1/2} - \alpha(t)\sin(2\phi_{3,0}(t,\lambda)) \\
&\qquad +\lambda^{-1/2}[q(t)-\alpha(t)^2-\alpha'(t)]\sin^2\phi_{3,0}(t,\lambda).
\end{aligned} \tag{9.9}$$

Theorem 4, (9.8) and (9.9) give, after some calculation that

$$\Lambda_{1,n}^{1/2} = \frac{(n+1)\pi}{2} - \frac{1}{(n+1)\pi}\left[\int_0^1 u^{-1}(1-\cos u) - \int_1^\infty u^{-1}\cos(u) + ln[(n+1)\pi] - 4\right]$$
$$+o(n^{-1}) \text{ as } n\to\infty. \tag{9.10}$$

We note that in both (9.7) and (9.10) the dependence on τ is confined to the error term. We also note that when $n = 2m$, (9.7) and (9.10) are the same to within the error term so, by Theorem A

$$\left.\begin{array}{l}\mu_{2m}\\ \\ \\ \mu_{2m+1}\end{array}\right\} = \frac{(2m+1)\pi}{2} - \frac{1}{(2m+1)\pi}\left[\int_0^1 u^{-1}(1-\cos u) - \int_1^\infty u^{-1}\cos u + ln((2m+1)\pi) - 4\right] + o(m^{-1}).$$

The signs of the second terms of (9.7) and (9.10) differ when $n = 2m+1$ so using, the fact that

$$\int_0^1 u^{-1}(1-\cos u)du - \int_1^\infty u^{-1}\cos u du + ln[2(m+1)\pi] - 4 > 0$$

for m sufficiently large, we see by Theorem A that

$$\left.\begin{array}{l}\lambda_{2m+2}\\ \\ \\ \lambda_{2m+1}\end{array}\right\} = (m+1)\pi \begin{array}{c}+\\ \\ \\ -\end{array} \left[\int_0^1 u^{-1}(1-\cos u)du - \int_1^\infty u^{-1}\cos u du + ln[2(m+1)\pi] - 4\right] + o(n^{-1}).$$

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