Existence of stable periodic solutions of a semilinear parabolic problem under Hammerstein–type conditions *

Maria do Rosário GROSSINHO †
Departamento de Matemática,
ISEG, Universidade Técnica de Lisboa,
Rua do Quelhas 6, 1200 Lisboa, Portugal
and
CMAF, Universidade de Lisboa,
Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
E-mail: mrg@ptmat.lmc.fc.ul.pt

Pierpaolo OMARI ‡
Dipartimento di Scienze Matematiche,
Università di Trieste,
Piazzale Europa 1, I–34127 Trieste, Italia
E-mail: omari@univ.trieste.it

Abstract

We prove the solvability of the parabolic problem

$$\begin{cases} \partial_t u - \displaystyle\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j} u) + \sum_{i=1}^{N} b_i(x,t)\partial_{x_i} u = f(x,t,u) & \text{in } \Omega\times\mathbb{R}, \\ u(x,t) = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{in } \Omega\times\mathbb{R}, \end{cases}$$

assuming certain conditions on the asymptotic behaviour of the ratio $2\int_0^s f(x,t,\sigma)d\sigma/s^2$ with respect to the principal eigenvalue of the associated linear problem. The method of proof, which is based on the construction of upper and lower solutions, also yields information on the localization and the stability of the solution.

1991 Mathematics subject classification : 35K20, 35B10, 35B35.
Keywords : parabolic equation, periodic solution, Hammerstein's condition, upper and lower solutions, existence, localization, stability.

*Research supported by CNR–JNICT.
†Supported also by FCT, PRAXIS XXI, FEDER, under projects PRAXIS/PCEX/P/MAT/36/96 and PRAXIS/2/2.1/MAT/125/94.
‡Supported also by MURST 40% and 60% research funds.

1 Introduction and statements

Let $\Omega\,(\subset \mathbb{R}^N)$ be a bounded domain, with a boundary $\partial\Omega$ of class C^2, and let $T > 0$ be a fixed number. Set $Q = \Omega\times]0,T[$ and $\Sigma = \partial\Omega \times [0,T]$. Let us consider the parabolic problem

$$\left\{\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,u) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{array}\right. \tag{1.1}$$

We assume throughout that

$$A(x,t,\partial_x) = -\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}) + \sum_{i=1}^{N} b_i(x,t)\partial_{x_i},$$

where $a_{ij} \in C^0(\overline{Q})$, $a_{ij} = a_{ji}$, $a_{ij}(x,0) = a_{ij}(x,T)$ in Ω, $\partial_{x_k}a_{ij} \in L^\infty(Q)$, $b_i \in L^\infty(Q)$ and $\partial_{x_k}b_i \in L^\infty(Q)$ for $i,j,k = 1,\ldots,N$. We also suppose that the operator $\partial_t + A$ is uniformly parabolic, i.e. there exists a constant $\eta > 0$ such that, for all $(x,t) \in \overline{Q}$ and $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \eta|\xi|^2.$$

We further assume that $f : \Omega\times]0,T[\times\mathbb{R} \to \mathbb{R}$ satisfies the L^p-Carathéodory conditions, for some $p > N+2$, and there exist continuous functions $g_\pm : \mathbb{R} \to \mathbb{R}$ such that, for a.e. $(x,t) \in Q$

$$f(x,t,s) \leq g_+(s) \quad \text{for } s \geq 0 \qquad \text{and} \qquad f(x,t,s) \geq g_-(s) \quad \text{for } s \leq 0. \tag{1.2}$$

It is convenient, for the sequel, to suppose that all functions, which are defined on $\Omega\times]0,T[$, have been extended by T–periodicity on $\Omega \times \mathbb{R}$.

In this paper we are concerned with the solvability of (1.1) when the nonlinearity f lies in some sense to the left of the principal eigenvalue λ_1 of the linear problem

$$\left\{\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \lambda u & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{array}\right.$$

It was proven in [2] that the Dolph–type condition

$$\limsup_{s\to\pm\infty} \frac{g_\pm(s)}{s} < \lambda_1 \tag{1.3}$$

guarantees the existence of a solution of (1.1). On the other hand, it does not seem yet known whether the same conclusion holds under the more general Hammerstein–type condition

$$\limsup_{s\to\pm\infty} \frac{2G_\pm(s)}{s^2} < \lambda_1, \tag{1.4}$$

where $G_\pm(s) = \int_0^s g_\pm(\sigma)d\sigma$ for $s \in \mathbb{R}$. Our purpose here is to provide some partial answers to this question. Of course, the main difficulty, in order to use in this context conditions on the potential like (1.4), is due to the lack of variational structure of problem (1.1); whereas the only known proof of Hammerstein's result, for a selfadjoint elliptic problem in dimension $N \geq 2$, relies on the use of variational methods. Accordingly, we will employ a technique based on the construction of upper and lower solutions, which will be obtained as solutions of some related, possibly one–dimensional, problems. We stress that an important feature of the upper and lower solution method is that it also provides information about the localization and, to a certain extent, about the stability of the solutions. Yet, since we impose here rather weak regularity conditions on the coefficients of the operator A and on the domain Ω and we require no regularity at all on the function f, the classical results in [11], [1], [3], [10] do not apply. Therefore, we will use the following theorem recently proved in [4, Theorem 4.5]. Before stating it, we recall that a lower solution α of (1.1) is a function $\alpha \in W^{2,1}_p(Q)$ $(p > N+2)$ such that

$$\begin{cases} \partial_t\alpha + A(x,t,\partial_x)\alpha \leq f(x,t,\alpha) & \text{a.e. in } Q, \\ \alpha(x,t) \leq 0 & \text{on } \Sigma, \\ \alpha(x,0) \leq \alpha(x,T) & \text{in } \Omega. \end{cases}$$

Similarly, an upper solution β of (1.1) is defined by reversing all the above inequalities. A solution of (1.1) is a function u which is simultaneously a lower and an upper solution.

Lemma 1.1 *Assume that α is a lower solution and β is an upper solution of (1.1), satisfying $\alpha \leq \beta$ in Q. Then, there exist a minimum solution v*

and a maximum solution w of (1.1), with $\alpha \leq v \leq w \leq \beta$ in Q. Moreover, if $\alpha(\cdot,0) = 0 = \beta(\cdot,0)$ on $\partial\Omega$, then the following holds: for every $u_0 \in W_p^{2-2/p}(\Omega) \cap H_0^1(\Omega)$, with $\alpha(\cdot,0) \leq u_0 \leq v(\cdot,0)$ (resp. $w(\cdot,0) \leq u_0 \leq \beta(\cdot,0)$) in Ω, the set $\mathcal{S}_{u_0}$ of all functions $u : \overline{\Omega}\times[0,+\infty[\to \mathbb{R}$, with $u \in W_p^{2,1}(\Omega\times]0,\sigma[)$ for every $\sigma > 0$, satisfying

$$\begin{cases} \partial_t u + A(x,t,\partial_x)u = f(x,t,u) & a.e.\ in\ \Omega \times]0,+\infty[, \\ u(x,t) = 0 & on\ \partial\Omega \times]0,+\infty[, \\ u(x,0) = u_0(x) & in\ \Omega \end{cases} \tag{1.5}$$

and $\alpha \leq u \leq v$ (resp. $w \leq u \leq \beta$) in $\Omega\times]0,+\infty[$, is non–empty and every $u \in \mathcal{S}_{u_0}$ is such that $\lim_{t\to+\infty} |u(\cdot,t) - v(\cdot,t)|_\infty = 0$ (resp. $\lim_{t\to+\infty} |u(\cdot,t) - w(\cdot,t)|_\infty = 0$).

Remark 1.1 We will say in the sequel that v (resp. w) is *relatively attractive from below* (resp. *from above*). Of course, this weak form of stability can be considerably strenghthened provided that more regularity is assumed in (1.1) (cf. [3]).

Remark 1.2 The condition $\alpha(\cdot,0) = 0$ on $\partial\Omega$ is not restrictive. Indeed, if it is not satisfied, we can replace α by the unique solution $\overline{\alpha}$, with $\alpha \leq \overline{\alpha} \leq v$ in Q, of

$$\begin{cases} \partial_t\overline{\alpha} + A(x,t,\partial_x)\overline{\alpha} = f(x,t,\alpha) + k_\rho(x,t,\alpha,\overline{\alpha}) & \text{in } Q, \\ \overline{\alpha}(x,t) = 0 & \text{on } \Sigma, \\ \overline{\alpha}(x,0) = \overline{\alpha}(x,T) & \text{in } \Omega, \end{cases}$$

where k_ρ is the function associated to f by Lemma 3.3 in [4] and corresponding to $\rho = \max\{|\alpha|_\infty, |\beta|_\infty\}$. A similar observation holds for β.

We start noting that Hammerstein's result can be easily extended to a special class of parabolic equations, which includes the heat equation.

Theorem 1.1 *Assume that $b_i = 0$, for $i = 1,\ldots,N$, and suppose that there exist constants c and q, with $c > 0$ and $q \in]1, \frac{2N}{N-2}[$ if $N \geq 3$, or $q \in]1,+\infty[$ if $N = 2$, such that*

$$|g_\pm(s)| \leq |s|^{q-1} + c \qquad \textit{for } s \in \mathbb{R}. \tag{1.6}$$

Moreover, assume that condition (1.4) holds. Then, problem (1.1) has a solution v and a solution w, satisfying $v \leq w$, such that v is relatively attractive from below and w is relatively attractive from above.

We stress that this theorem completes, for what concerns the stability information, the classical result of Hammerstein for the selfadjoint elliptic problem

$$\begin{cases} -\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As already pointed out, we do not know whether a statement similar to Theorem 1.1 holds for a general parabolic operator as that considered in (1.1). The next two results provide some contributions in this direction, although they do not give a complete answer to the posed question. In order to state the former, we need to settle some notation. For each $i = 1, \ldots, N$, denote by $]A_i, B_i[$ the projection of Ω onto the x_i–axis and set

$$\overline{a}_i = \min_{\overline{Q}} a_{ii} \qquad \text{and} \qquad \overline{b}_i = |b_i - \sum_{j=1}^{N} \partial_{x_j} a_{ji}|_\infty .$$

Then, define

$$\hat{\lambda}_1 = \max_{i=1,\ldots,N} \left\{ \left(\frac{\pi}{B_i - A_i} \right)^2 \overline{a}_i \, \exp\left(-\frac{\overline{b}_i}{\overline{a}_i}(B_i - A_i) \right) \right\} .$$

Theorem 1.2 *Assume*

$$\liminf_{s\to\pm\infty} \frac{2G_\pm(s)}{s^2} < \hat{\lambda}_1. \tag{1.7}$$

Then, the same conclusions of Theorem 1.1 hold.

The constant $\hat{\lambda}_1$ depends only on the coefficients of the operator A and on the domain Ω. It is strictly positive and generally smaller than the principal eigenvalue λ_1; therefore, it provides an explicitly computable lower estimate for λ_1. Moreover, $\hat{\lambda}_1$ coincides with λ_1 when $N = 1$, $a_{11} = 1$ and $b_1 = 0$, so that the equation in (1.1) is the one–dimensional heat equation. On the other hand, it must be stressed that the restriction from above on a limit superior required by (1.4) is replaced in (1.7) by a restriction from above on a limit inferior. Furthermore, in Theorem 1.2 the growth condition (1.6) is not needed anymore. We recall that conditions similar to (1.7) were first introduced in [6] for solving the one–dimensional two–point boundary value problem

$$\begin{cases} -u'' = f(x,u) & \text{in }]A,B[, \\ u(A) = u(B) = 0 \end{cases}$$

and were later used in [8] for studying the higher dimensional elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

It is worth noticing at this point that, if the coefficients of the operator A and the function f do not depend on t, then the same proof of Theorem 1.2 yields the solvability, under (1.7), of the, possibly non–selfadjoint, elliptic problem

$$\begin{cases} -\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u) + \sum_{i=1}^{N} b_i(x)\partial_{x_i}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

This observation provides an extension of the result in [8] to the more general problem (1.9), which could not be directly handled by the approach introduced in that paper. A preliminary version of Theorem 1.2 was announced in [9].

In our last result we show that the constant $\hat{\lambda}_1$ considered in Theorem 1.2 can be replaced by the principal eigenvalue λ_1, provided that a further control on the functions $g_\pm$ is assumed.

Theorem 1.3 *Assume*

$$\limsup_{s\to\pm\infty} \frac{g_\pm(s)}{s} \leq \lambda_1 \tag{1.10}$$

and

$$\liminf_{s\to\pm\infty} \frac{2G_\pm(s)}{s^2} < \lambda_1. \tag{1.11}$$

Then, the same conclusions of Theorem 1.1 hold.

We point out that the sole condition (1.10), which is a weakened form of (1.3), is not sufficient to yield the solvability of (1.1) (cf. [2]). Theorem 1.3 extends to the parabolic setting a previous result obtained in [5] for the selfadjoint elliptic problem (1.8). By the same proof one also obtains the solvability, under (1.10) and (1.11), of the, possibly non–selfadjoint, elliptic problem (1.9). We stress that, although the proof of Theorem 1.3 exploits some ideas borrowed from [5], nevertheless from the technical point of view it is much more delicate, due to the different regularity that solutions of (1.1) exhibit with respect to the space and the time variables.

2 Proofs

2.1 Preliminaries

In this subsection we state some results concerning the linear problem associated to (1.1), which apparently are not well–settled in the literature, when low regularity conditions are assumed on the coefficients of the operator A and on the domain Ω.

We start with some notation. Fixed t_1, t_2, with $t_1 \le t_2$, and given $u, v \in C^{1,0}(\overline{\Omega} \times [t_1, t_2])$, we write:

- $u \ge v$ if, for every $(x,t) \in \overline{\Omega} \times [t_1, t_2]$, $u(x,t) \ge v(x,t)$;
- $u \gg v$ if, for every $(x,t) \in \Omega \times [t_1, t_2]$, $u(x,t) > v(x,t)$ and, for every $(x,t) \in \partial\Omega \times [t_1, t_2]$, either $u(x,t) > v(x,t)$, or $u(x,t) = v(x,t)$ and $\partial_\nu u(x,t) < \partial_\nu v(x,t)$, where $\nu = (\nu_0, 0) \in \mathbb{R}^{N+1}$, $\nu_0 \in \mathbb{R}^N$ being the outer normal to Ω at $x \in \partial\Omega$.

Proposition 2.1 *There exist a number $\lambda_1 > 0$ and functions $\varphi_1, \varphi_1^* \in W_p^{2,1}(Q)$, for every p, satisfying, respectively,*

$$\begin{cases} \partial_t \varphi_1 - \sum_{i,j=1}^N \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}\varphi_1) + \sum_{i=1}^N b_i(x,t)\partial_{x_i}\varphi_1 = \lambda_1 \varphi_1 & \text{in } Q, \\ \varphi_1(x,t) = 0 & \text{on } \Sigma, \\ \varphi_1(x,0) = \varphi_1(x,T) & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} -\partial_t \varphi_1^* - \sum_{i,j=1}^N \partial_{x_j}(a_{ij}(x,t)\partial_{x_i}\varphi_1^*) - \sum_{i=1}^N \partial_{x_i}(b_i(x,t)\varphi_1^*) = \lambda_1 \varphi_1^* & \text{in } Q, \\ \varphi_1^*(x,t) = 0 & \text{on } \Sigma, \\ \varphi_1^*(x,0) = \varphi_1^*(x,T) & \text{in } \Omega. \end{cases}$$

Moreover, the following statements hold:
(i) $\varphi_1 \gg 0$ *and* $\varphi_1^* \gg 0$*;*
(ii) if ψ *is a solution of*

$$\begin{cases} \partial_t \psi - \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}\psi) + \sum_{i=1}^{N} b_i(x,t)\partial_{x_i}\psi = \lambda_1 \psi & \text{in } Q, \\ \psi(x,t) = 0 & \text{on } \Sigma, \\ \psi(x,0) = \psi(x,T) & \text{in } \Omega, \end{cases}$$

or, respectively, of

$$\begin{cases} -\partial_t \psi - \sum_{i,j=1}^{N} \partial_{x_j}(a_{ij}(x,t)\partial_{x_i}\psi) - \sum_{i=1}^{N} \partial_{x_i}(b_i(x,t)\psi) = \lambda_1 \psi & \text{in } Q, \\ \psi(x,t) = 0 & \text{on } \Sigma, \\ \psi(x,0) = \psi(x,T) & \text{in } \Omega, \end{cases}$$

then $\psi = c\varphi_1$*, or, respectively,* $\psi = c\varphi_1^*$*, for some* $c \in \mathbb{R}$*;*
(iii) λ_1 *is the smallest number* λ *for which the problems*

$$\begin{cases} \partial_t u - \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}u) + \sum_{i=1}^{N} b_i(x,t)\partial_{x_i}u = \lambda u & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} -\partial_t u - \sum_{i,j=1}^{N} \partial_{x_j}(a_{ij}(x,t)\partial_{x_i}u) - \sum_{i=1}^{N} \partial_{x_i}(b_i(x,t)u) = \lambda u & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{in } \Omega \end{cases}$$

have nontrivial solutions.

Proposition 2.1 is a immediate consequence of [4, Proposition 2.3].

Proposition 2.2 *Fix* $p > N + 2$*. Let* $q \in L_\infty(Q)$ *satisfy* $\operatorname{ess\,sup}_Q q < \lambda_1$*. Then, for every* $f \in L_p(Q)$ *the problem*

$$\begin{cases} \partial_t u + A(x,t,\partial_x)u = qu + f(x,t) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{in } \Omega \end{cases} \tag{2.1}$$

has a unique solution $u \in W_p^{2,1}(Q)$ *(which is asymptotically stable). Moreover, there exists a constant* $C > 0$*, independent of* f*, such that*

$$|u|_{W_p^{2,1}} \leq C\,|f|_p. \tag{2.2}$$

Finally, if $f \geq 0$ *a.e. in* Q*, with strict inequality on a set of positive measure, then* $u \gg 0$*.*

Proof. Fix a constant $k \geq 0$ such that

$$\text{ess inf}_Q(k-q) > (2\eta)^{-1}(\max_{i=1,\dots,N} |b_i|_\infty),$$

where η is the constant of uniform parabolicity of the operator $\partial_t + A$. Then, Proposition 2.1 in [4] guarantees that, for every $f \in L_p(Q)$, the problem

$$\begin{cases} \partial_t v + A(x,t,\partial_x)v + (k-q)v = f(x,t) & \text{in } Q, \\ v(x,t) = 0 & \text{on } \Sigma, \\ v(x,0) = v(x,T) & \text{in } \Omega. \end{cases} \tag{2.3}$$

has a unique solution $v \in W_p^{2,1}(Q)$ and, therefore, $v \in C^{1+\mu,\mu}(\overline{Q})$, for some $\mu > 0$. Let $f \in L_p(Q)$ be given and let v be the corresponding solution of (2.3). Set $\beta = v + s\varphi_1$, where $s > 0$ is such that $\beta \geq 0$ and $s(\lambda_1 - \text{ess sup}_Q q)\varphi_1 \geq kv$. We have

$$\partial_t \beta + A(x,t,\partial_x)\beta = q\beta + f + s(\lambda_1 - q)\varphi_1 - kv \geq q\beta + f \quad \text{a.e. in } Q,$$

that is β is an upper solution of (2.1). In a quite similar way we define a lower solution α of (2.1), with $\alpha \leq 0$. Therefore Lemma 1.1 yields the existence of a solution $u \in W_p^{2,1}(Q)$ of problem (2.1), with $\alpha \leq u \leq \beta$. The uniqueness of the solution is a direct consequence of the parabolic maximum principle (see e.g. [4, Proposition 2.2]) and its asymptotic stability follows from [4, Theorem 4.6]. Accordingly, the operator $\partial_t + A : W_p^{2,1}(Q) \to L_p(Q)$ is invertible and the open mapping theorem implies that its inverse is continuous, that is, (2.2) holds. Finally, the last statement follows from the parabolic strong maximum principle, as soon as one observes that if $f \geq 0$ a.e. in Q, then $\alpha = 0$ is a lower solution of (2.1). ∎

Proposition 2.3 *For $i = 1, 2$, let $q_i \in L_\infty(Q)$ be such that $q_1 \leq q_2$ a.e. in Q and let u_i be nontrivial solutions of*

$$\begin{cases} \partial_t u_i + A(x,t,\partial_x)u_i = q_i u_i & \text{in } Q, \\ u_i(x,t) = 0 & \text{on } \Sigma, \\ u_i(x,0) = u_i(x,T), & \text{in } \Omega, \end{cases}$$

respectively. If $u_2 \geq 0$, then $q_1 = q_2$ a.e. in Q and there exists a constant $c \in \mathbb{R}$ such that $u_1 = c\, u_2$.

Proof. We can assume, without loss of generality, that $u_1^+ \neq 0$. Since

$$\partial_t u_2 + A(x,t,\partial_x)u_2 + q_2^- u_2 = q_2^+ u_2 \geq 0 \quad \text{a.e. in } Q,$$

we have $u_2 \gg 0$. If we set $c = \min\{d \in \mathbb{R} \,|\, d\, u_2 \geq u_1\}$ and $v = c\, u_2 - u_1$, we get, as $c > 0$ and $v \geq 0$,

$$\partial_t v + A(x,t,\partial_x)v + q_1^- v = q_1^+ v + c\,(q_2 - q_1)\, u_2 \geq 0 \quad \text{a.e. in } Q$$

and hence either $v \gg 0$, or $v = 0$. The minimality of c actually yields $v = 0$ and therefore $u_1 = cu_2$. This finally implies

$$0 = \partial_t v + A(x,t,\partial_x)v = (q_2 - q_1)\, c\, u_2 \quad \text{a.e. in } Q$$

and therefore $q_1 = q_2$ a.e. in Q. ∎

2.2 Proof of Theorem 1.1

We indicate how to build an upper solution β of (1.1), with $\beta \geq 0$; a lower solution α, with $\alpha \leq 0$, can be constructed in a similar way. If there exists a constant $\beta \geq 0$ such that $g_+(\beta) \leq 0$, β is an upper solution of (1.1). Therefore, suppose that $g_+(s) > 0$ for $s \geq 0$, and set

$$h(s) = \begin{cases} g_+(s) & \text{if } s \geq 0, \\ g_+(0) & \text{if } s < 0. \end{cases} \tag{2.4}$$

Let us consider the elliptic problem

$$\begin{cases} -\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

From conditions (1.4) and (1.6), it follows that (2.5) admits a solution $u \in H_0^1(\Omega)$. A bootstrap argument, like in [7], shows that $u \in W_p^2(\Omega)$, for all finite p, and the strong maximum principle implies that $u \gg 0$. The function β, defined by setting $\beta(x,t) = u(x)$ for $(x,t) \in \overline{Q}$, is by (1.2) an upper solution of (1.1). ∎

2.3 Proof of Theorem 1.2

Again we show how to construct an upper solution β of (1.1), with $\beta \geq 0$; a lower solution α, with $\alpha \leq 0$, being obtained similarly. Exactly as in the proof of Theorem 1.1, we can reduce ourselves to the case where $g_+(s) > 0$ for $s \geq 0$. Then, we define a function h as in (2.4). The remainder of the proof is divided in two steps: in the former, we study some simple properties of the solutions of a second order ordinary differential equation related to problem (1.1); in the latter, we use the facts established in the previous step for constructing an upper solution of the original parabolic problem.

Step 1. Let $A < B$ be given constants and let $p, q : [A,B] \to \mathbb{R}$ be functions, with p absolutely continuous and q continuous, satisfying

$$0 < p_0 := \min_{[A,B]} p(x) \leq \max_{[A,B]} p(x) =: p_\infty \tag{2.6}$$

and

$$0 < q_0 := \min_{[A,B]} q(x) \leq \max_{[A,B]} q(x) =: q_\infty. \tag{2.7}$$

Let also $h : \mathbb{R} \to \mathbb{R}$ be a continuous function and set $H(s) = \int_0^s h(\sigma)\, d\sigma$ for $s \in \mathbb{R}$. Consider the initial value problem

$$\begin{cases} -(pu')' = qh(u), \\ u(\frac{A+B}{2}) = d, \\ u'(\frac{A+B}{2}) = 0, \end{cases} \tag{2.8}$$

where d is a real parameter. By a solution of (2.8) we mean a function u of class C^1, with pu' of class C^1, defined on some interval $I \subset [A,B]$, with $\frac{A+B}{2} \in \overset{\circ}{I}$, which satisfies the equation on I and the initial conditions.

Claim. *Assume that there are constants* c, d, *with* $0 \le c < d$, *such that*

$$h(s) > 0 \qquad \text{for } s \in [c, d] \tag{2.9}$$

and

$$\int_c^d \frac{d\sigma}{\sqrt{H(d) - H(\sigma)}} \ge \left(\frac{\sqrt{2\, p_\infty q_\infty}}{p_0} \right) \frac{B - A}{2}. \tag{2.10}$$

Then, there exists a solution u *of (2.8), which is defined on* $[A, B]$ *and satisfies*

$$c \le u(x) \le d \quad \text{for } x \in [A, B],$$

$$u'(x) > 0 \ \text{ for } x \in [A, \frac{A+B}{2}[\quad \text{and} \quad u'(x) < 0 \ \text{ for } x \in]\frac{A+B}{2}, B].$$

Proof of the Claim. Let u be a maximal solution of (2.8). Note that, by (2.6), (2.7) and (2.9), u has a local maximum at the point $\frac{A+B}{2}$ and, if $]\omega_-, \omega_+[$ denotes the maximal interval included in $]A, B[$ where $u(x) \in]c, d]$, we have

$$u'(x) > 0 \ \text{ for } x \in]\omega_-, \tfrac{A+B}{2}[\quad \text{and} \quad u'(x) < 0 \ \text{ for } x \in]\tfrac{A+B}{2}, \omega_+[. \tag{2.11}$$

We want to prove that $\omega_- = A$ and $\omega_+ = B$. Assume, by contradiction, that

$$\omega_+ < B. \tag{2.12}$$

Similarly one should argue if $\omega_- > A$. From (2.11) we derive that u is decreasing on $[\frac{A+B}{2}, \omega_+[$ and, by the definition of ω_+, we have

$$\lim_{x \to \omega_+} u(x) = c =: u(\omega_+).$$

Now, pick $x \in [\frac{A+B}{2}, \omega_+[$, multiply the equation in (2.8) by $-pu'$ and integrate between $\frac{A+B}{2}$ and x. Taking into account that, by (2.9) and (2.11), $h(u)u' < 0$ on $]\frac{A+B}{2}, \omega_+[$, we obtain, using (2.6) and (2.7) as well,

$$\begin{aligned} \tfrac{1}{2}(p(x)u'(x))^2 &= -\int_{\frac{A+B}{2}}^{x} pqh(u)u'dt \\ &\le p_\infty q_\infty \left(H\left(u\left(\tfrac{A+B}{2}\right)\right) - H(u(x)) \right) \\ &= p_\infty q_\infty \left(H(d) - H(u(x)) \right). \end{aligned}$$

By (2.6) and (2.11), we have, for each $x \in]\frac{A+B}{2}, \omega_+[$,

$$(0 <)\ \ |u'(x)|^2 \leq 2 \left(\frac{p_\infty q_\infty}{{p_0}^2} \right) (H(d) - H(u(x)))$$

and hence

$$\frac{-u'(x)}{\sqrt{H(d) - H(u(x))}} \leq \frac{\sqrt{2\, p_\infty q_\infty}}{p_0}.$$

Integrating this relation between $\frac{A+B}{2}$ and ω_+ and changing variable, we get by (2.12)

$$\int_c^d \frac{d\sigma}{\sqrt{H(d) - H(\sigma)}} < \left(\frac{\sqrt{2 p_\infty q_\infty}}{p_0} \right) \frac{B - A}{2}.$$

Then, condition (2.10) yields a contradiction and the conclusions of the Claim follow. ■

Step 2. We prove now that problem (1.1) has an upper solution $\beta \in C^{2,1}(\overline{Q})$, with $\beta \gg 0$. Assume, without loss of generality, that

$$\max_{i=1,\ldots,N} \left\{ \left(\frac{\pi}{B_i - A_i} \right)^2 \overline{a}_i \exp\left(-\frac{\overline{b}_i}{\overline{a}_i}(B_i - A_i) \right) \right\}$$

is attained at $i = 1$ and set, for the sake of simplicity,

$$]A, B[:=]A_1, B_1[,$$

$$\overline{a} := \overline{a}_1 = \min_{\overline{Q}} a_{11}$$

and

$$\overline{b} := \overline{b}_1 = |b_1 - \sum_{j=1}^{N} \partial_{x_j} a_{j1}|_\infty.$$

Note that

$$\overline{a} > 0 \quad \text{and} \quad \overline{b} \geq 0. \tag{2.13}$$

Let us set, for $x \in [A, B]$,

$$p(x) := \exp\left(-\tfrac{\overline{b}}{\overline{a}} \left| x - \tfrac{A+B}{2} \right| \right) \qquad \text{and} \qquad q(x) := \tfrac{1}{\overline{a}} p(x)$$

and consider the ordinary differential equation

$$-(pu')' = q\,h(u), \tag{2.14}$$

where h is defined in (2.4). It is clear that p, q and h satisfy, respectively, (2.6), (2.7) and (2.9), for any fixed c, d, with $0 \le c < d$. Observe that (2.10) is also fulfilled, for $c = 0$ and for some $d > 0$. Indeed, since (1.7) implies that

$$\liminf_{s\to+\infty} (2\,H(s) - \hat{\lambda}_1 s^2) = -\infty,$$

we can find a sequence $(d_n)_n$, with $d_n \to +\infty$, such that, for each n,

$$(0 <)\ H(d_n) - H(s) < \frac{\hat{\lambda}_1}{2}(d_n^2 - s^2) \qquad \text{for } s \in [0, d_n[$$

and hence

$$\int_0^{d_n} \frac{d\sigma}{\sqrt{H(d_n) - H(\sigma)}} > \sqrt{\frac{2}{\hat{\lambda}_1}} \int_0^{d_n} \frac{d\sigma}{\sqrt{d_n^2 - \sigma^2}} = \sqrt{\frac{2}{\hat{\lambda}_1}}\,\frac{\pi}{2} = \left(\frac{\sqrt{2p_\infty q_\infty}}{p_0}\right)\frac{B-A}{2}.$$

Therefore, taking $d := d_n$, for some fixed n, we conclude that (2.10) holds. Accordingly, by the Claim, there exists a solution u of (2.14), which is defined on $[A, B]$ and satisfies

$$u(x) > 0 \quad \text{for } x \in]A, B[, \tag{2.15}$$

$$u'(x) > 0 \quad \text{for } x \in [A, \tfrac{A+B}{2}[\qquad \text{and} \qquad u'(x) < 0 \quad \text{for } x \in]\tfrac{A+B}{2}, B]. \tag{2.16}$$

From the definition of p it follows that u is of class C^2 on $[A, B] \setminus \left\{\frac{A+B}{2}\right\}$ and satisfies the equation

$$-\overline{a}\,u'' + \overline{b}\,\mathrm{sign}\left(x - \tfrac{A+B}{2}\right)\,u' = h(u), \tag{2.17}$$

everywhere on $[A, B] \setminus \left\{\frac{A+B}{2}\right\}$. Actually, since $u'(\frac{A+B}{2}) = 0$, a direct inspection of (2.17) shows that u is of class C^2 and satisfies equation (2.17) everywhere on $[A, B]$. Moreover, using (2.13), (2.15), (2.16) and (2.9), with $c = 0$ and any $d > 0$, we derive from (2.17)

$$u''(x) = \tfrac{1}{\overline{a}}\left(\overline{b}\,\mathrm{sign}\left(x - \tfrac{A+B}{2}\right) u'(x) - h(u(x))\right) < 0 \qquad \text{on } [A, B]. \tag{2.18}$$

Now, we set

$$\beta(x_1, \ldots, x_N, t) = u(x_1) \qquad \text{for } (x_1 \ldots x_N, t) \in \overline{Q}.$$

We have that $\beta \in C^{2,1}(\overline{Q})$ and $\beta \gg 0$. Let us check that β is an upper solution of problem (1.1). Indeed, using (2.13), (2.17), (2.18) and (1.2), as well as the definitions of $\overline{a}$ and $\overline{b}$, we have, for each $(x_1, \ldots, x_N, t) \in Q$,

$$\begin{aligned}
&\partial_t \beta(x_1, ..., x_N, t) - \sum_{i,j=1}^{N} a_{ij}(x_1, ..., x_N, t)\, \partial_{x_i x_j} \beta(x_1, ..., x_N, t) \; + \\
&\quad + \sum_{i=1}^{N} (b_i(x_1, ..., x_N, t) - \sum_{j=1}^{N} \partial_{x_j} a_{ji}(x_1, ..., x_N, t))\, \partial_{x_i} \beta(x_1, ..., x_N, t) \\
&= -a_{11}(x_1, ..., x_N, t) u''(x_1) + (b_1(x_1, ..., x_N, t) - \sum_{j=1}^{N} \partial_{x_j} a_{j1}(x_1, ..., x_N, t)) u'(x_1) \\
&\geq -\overline{a} u''(x_1) + \overline{b}\, \mathrm{sign}(x_1 - \tfrac{A+B}{2}) u'(x_1) = h(u(x_1)) = g_+(u(x_1)) \\
&\geq f(x_1, ..., x_N, t, \beta(x_1, ..., x_N, t)).
\end{aligned}$$

This concludes the proof of the Theorem 1.2. ∎

2.4 Proof of Theorem 1.3

Again we describe how to build an upper solution β of (1.1), with $\beta \geq 0$; a lower solution α, with $\alpha \leq 0$, being constructed in a similar way. As in the proof of Theorem 1.1, we can reduce ourselves to the case where $g_+(s) > 0$ for $s \geq 0$. Then, we define a function h as in (2.4), which by (1.10) and (1.11) satisfies

$$\limsup_{s \to +\infty} \frac{h(s)}{s} \leq \lambda_1 \tag{2.19}$$

and

$$\liminf_{s \to +\infty} \frac{2H(s)}{s^2} < \lambda_1. \tag{2.20}$$

Let us consider the problem

$$\begin{cases} \partial_t u + A(x, t, \partial_x) u = h(u) & \text{in } Q, \\ u(x, t) = 0 & \text{on } \Sigma, \\ u(x, 0) = u(x, T) & \text{in } \Omega \end{cases} \tag{2.21}$$

and let us prove that it admits at least one solution. Observe that, since $h(s) > 0$ for every s, any solution u of (2.21) is such that $u \gg 0$ and then, by condition (1.2), is an upper solution of (1.1).

Fix $p > N+2$ and associate to (2.21) the solution operator $\mathcal{S} : C^0(\overline{Q}) \to C^0(\overline{Q})$ which sends any function $u \in C^0(\overline{Q})$ onto the unique solution $v \in W_p^{2,1}(Q)$ of

$$\begin{cases} \partial_t v + A(x,t,\partial_x)v = h(u) & \text{in } Q, \\ v(x,t) = 0 & \text{on } \Sigma, \\ v(x,0) = v(x,T) & \text{in } \Omega. \end{cases}$$

It follows from Proposition 2.2 that $\mathcal{S}$ is completely continuous and its fixed points are precisely the solutions of (2.21). Let us consider the equation

$$u = \mu\,\mathcal{S}u, \tag{2.22}$$

with $\mu \in [0,1]$, which corresponds to

$$\begin{cases} \partial_t u + A(x,t,\partial_x)u = \mu h(u) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases} \tag{2.23}$$

By the Leray–Schauder degree theory, equation (2.22), with $\mu = 1$, and therefore problem (2.21), is solvable, if there exists an open bounded set $\mathcal{O}$ in $C^0(\overline{Q})$, with $0 \in \mathcal{O}$, such that no solution of (2.22), or equivalently of (2.23), for any $\mu \in [0,1]$, belongs to the boundary of $\mathcal{O}$. The remainder of this proof basically consists of building such a set $\mathcal{O}$.

Claim 1. *Let $(u_n)_n$ be a sequence of solutions of*

$$\begin{cases} \partial_t u_n + A(x,t,\partial_x)u_n = \mu_n h(u_n) & \text{in } Q, \\ u_n(x,t) = 0 & \text{on } \Sigma, \\ u_n(x,0) = u_n(x,T) & \text{in } \Omega, \end{cases} \tag{2.24}$$

with $\mu_n \in [0,1]$, such that $|u_n|_\infty \to +\infty$. Then, possibly passing to subsequences,

$$\frac{u_n}{|u_n|_\infty} \to v \quad \text{in } W_p^{2,1}(Q),$$

where $v = c\,\varphi_1$, for some $c > 0$, and

$$\frac{h(u_n)}{|u_n|_\infty} \to \lambda_1 v \quad \text{in } L_p(Q).$$

Proof of Claim 1. Let us write, for $s \in \mathbb{R}$,

$$h(s) = q(s)\, s + r(s),$$

with q, r continuous functions such that

$$0 \leq q(s) \leq \lambda_1 \tag{2.25}$$

and

$$\frac{r(s)}{s} \longrightarrow 0, \qquad \text{as } |s| \to +\infty. \tag{2.26}$$

Let us set, for each n,

$$v_n = \frac{u_n}{|u_n|_\infty},$$

where v_n satisfies

$$\begin{cases} \partial_t v_n + A(x,t,\partial_x) v_n = \mu_n q(u_n) v_n + \mu_n r(u_n)/|u_n|_\infty & \text{in } Q, \\ v_n(x,t) = 0 & \text{on } \Sigma, \\ v_n(x,0) = v_n(x,T) & \text{in } \Omega. \end{cases} \tag{2.27}$$

The sequence $(v_n)_n$ is bounded in $W_p^{2,1}(Q)$ and therefore, possibly passing to a subsequence, it converges weakly in $W_p^{2,1}(Q)$ and strongly in $C^{1+\alpha,\alpha}(\overline{Q})$, for some $\alpha > 0$, to a function $v \in W_p^{2,1}(Q)$, with $|v|_\infty = 1$. We can also suppose that $\mu_n \to \mu_0 \in [0,1]$ and $q(u_n)$ converges in $L_\infty(Q)$, with respect to the weak* topology, to a function $q_0 \in L_\infty(Q)$, satisfying by (2.25)

$$0 \leq q_0(x,t) \leq \lambda_1 \tag{2.28}$$

a.e. in Q. Moreover, by (2.26), we have

$$\frac{r(u_n(x,t))}{|u_n|_\infty} \longrightarrow 0 \tag{2.29}$$

uniformly a.e. in Q. The weak continuity of the operator $\partial_t + A : W_p^{2,1}(Q) \to L_p(Q)$ implies that v satisfies

$$\begin{cases} \partial_t v + A(x,t,\partial_x) v = \mu_0 q_0 v & \text{in } Q, \\ v(x,t) = 0 & \text{on } \Sigma, \\ v(x,0) = v(x,T) & \text{in } \Omega. \end{cases} \tag{2.30}$$

Now, if we set $q_1 = \mu_0 q_0$, $q_2 = \lambda_1$, $v_1 = v$ and $v_2 = \varphi_1$, Proposition 2.3 yields $\mu_0 q_0 = \lambda_1$ a.e. in Q, so that, by (2.28), $\mu_0 = 1$, and $v = c\varphi_1$, for some $c > 0$. We also have

$$\begin{aligned}\int_Q |\lambda_1 - q(u_n)|^p &\leq |\lambda_1 - q(u_n)|_\infty^{p-1} \int_Q |\lambda_1 - q(u_n)| \\ &\leq \lambda_1^{p-1} \int_Q (\lambda_1 - q(u_n)) \to 0,\end{aligned}$$

i.e. $q(u_n) \to \lambda_1$ in $L_p(Q)$, and therefore, by (2.29), $h(u_n)/|u_n|_\infty \to \lambda_1 v$ in $L_p(Q)$. Finally, Proposition 2.2 implies that $v_n \to v$ in $W_p^{2,1}(Q)$. ■

Claim 2. *There exists a sequence $(S_n)_n$, with $S_n \to +\infty$, such that, if u is a solution of (2.23), for some $\mu \in [0,1]$, then $\max_{\overline{Q}} u \neq S_n$, for every n.*

Proof of Claim 2. By (2.20), we can find a sequence $(s_n)_n$, with $s_n \to +\infty$, and a constant $\varepsilon > 0$, such that

$$\lambda_1 - \frac{2H(s_n)}{s_n^2} > \varepsilon \tag{2.31}$$

for every n. Assume, by contradiction, that there exist a subsequence of $(s_n)_n$, which we still denote by $(s_n)_n$, and a sequence $(u_n)_n$ of solutions of (2.24) such that

$$\max_{\overline{Q}} u_n = u_n(x_n, t_n) = s_n,$$

where $(x_n, t_n) \in \Omega \times [0, T]$. Since $|u_n|_\infty \to +\infty$, we can suppose by Claim 1 that $v_n = u_n/|u_n|_\infty \to v$ in $W_p^{2,1}(Q)$, and therefore in $C^{1,0}(\overline{Q})$, with $v = c\varphi_1$, for some $c > 0$, and

$$|u_n|_\infty^{-1} |\lambda_1 u_n - h(u_n)|_p \to 0. \tag{2.32}$$

There is also a constant $K > 0$ such that

$$|u_n|_\infty^{-1} |\lambda_1 u_n - h(u_n)|_\infty \leq K \tag{2.33}$$

and

$$|\nabla_x v_n|_\infty \leq K, \tag{2.34}$$

for every n. Moreover, we have

$$|\partial_t v_n - \partial_t v|_p \to 0 \tag{2.35}$$

and, possibly for a subsequence,

$$x_n \to x_0 \qquad \text{and} \qquad t_n \to t_0,$$

with $(x_0, t_0) \in \Omega \times [0, T]$, because (x_0, t_0) is a maximum point of v. Using Fubini's theorem and possibly passing to subsequences, we also obtain from (2.32) and (2.35), respectively,

$$|u_n|_\infty^{-1}(\lambda_1 u_n(\cdot, \overline{t}) - h(u_n(\cdot, \overline{t}))) \to 0, \tag{2.36}$$

$$\partial_t v_n(\cdot, \overline{t}) - \partial_t v(\cdot, \overline{t}) \to 0 \tag{2.37}$$

in $L_p(\Omega)$, for a.e. $\overline{t} \in [0, T]$, and

$$|u_n|_\infty^{-1}(\lambda_1 u_n(\overline{x}, \cdot) - h(u_n(\overline{x}, \cdot))) \to 0, \tag{2.38}$$

$$\partial_t v_n(\overline{x}, \cdot) - \partial_t v(\overline{x}, \cdot) \to 0 \tag{2.39}$$

in $L_p(0, T)$, for a.e. $\overline{x} \in \Omega$. Moreover, we have that

$$\int_0^T |\partial_t v(\overline{x}, \tau)|^2 \, d\tau \quad \text{is finite} \tag{2.40}$$

for a.e. $\overline{x} \in \Omega$. Let us write

$$\begin{aligned}
&\frac{\lambda_1}{2} s_n^2 - H(s_n) = \frac{\lambda_1}{2} u_n^2(x_n, t_n) - H(u_n(x_n, t_n)) \\
= \;& \left[\frac{\lambda_1}{2} \left(u_n^2(x_n, t_n) - u_n^2(\overline{x}, t_n) \right) - H(u_n(x_n, t_n)) + H(u_n(\overline{x}, t_n)) \right] \\
+ \;& \left[\frac{\lambda_1}{2} \left(u_n^2(\overline{x}, t_n) - u_n^2(\overline{x}, \overline{t}) \right) - H(u_n(\overline{x}, t_n)) + H(u_n(\overline{x}, \overline{t})) \right] \\
+ \;& \left[\frac{\lambda_1}{2} \left(u_n^2(\overline{x}, \overline{t}) - u_n^2(x^*, \overline{t}) \right) - H(u_n(\overline{x}, \overline{t})) + H(u_n(x^*, \overline{t})) \right],
\end{aligned} \tag{2.41}$$

where the choices of points $\overline{t} \in [0, T]$, such that (2.36) and (2.37) hold, $\overline{x} \in \Omega$, such that (2.38), (2.39) and (2.40) hold, and $x^* \in \partial\Omega$ will be specified later.

Let us observe that, for each n, we can find a sequence $(w_k^{(n)})_k$ in $C^1(\overline{Q})$ such that $w_k^{(n)} \to u_n$ in $W_p^1(Q)$ and therefore in $C^0(\overline{Q})$, since $p > N+2$. This implies in particular that $w_k^{(n)} \to u_n$ in $L_\infty(Q)$ and $\partial_t w_k^{(n)} \to \partial_t u_n$ in $L_p(Q)$. Hence, using Fubini's theorem and possibly passing to a subsequence, we get

$\partial_t w_k^{(n)}(x,\cdot) \to \partial_t u_n(x,.)$ in $L_p(0,T)$ for a.e. $x \in \Omega$. Hence, it follows that, for each n,

$$\begin{aligned}\left(\lambda_1 w_k^{(n)}(\overline{x},\cdot) - h(w_k^{(n)}(\overline{x},\cdot))\right) \partial_t w_k^{(n)}(\overline{x},\cdot) \\ \to (\lambda_1 u_n(\overline{x},\cdot) - h(u_n(\overline{x},\cdot)))\, \partial_t u_n(\overline{x},\cdot)\end{aligned}$$

in $L_p(0,T)$, for a.e. $\overline{x} \in \Omega$, and therefore, for a.e. $\overline{t} \in [0,T]$,

$$\begin{aligned}
&\frac{\lambda_1}{2}\left(u_n^2(\overline{x},t_n) - u_n^2(\overline{x},t)\right) - (H(u_n(\overline{x},t_n)) - H(u_n(\overline{x},\overline{t}))) \\
&= \lim_{k\to+\infty}\left[\frac{\lambda_1}{2}\left({w_k^{(n)}}^2(\overline{x},t_n) - {w_k^{(n)}}^2(\overline{x},\overline{t})\right)\right. \\
&\qquad \left. - \left(H(w_k^{(n)}(\overline{x},t_n)) - H(w_k^{(n)}(\overline{x},\overline{t}))\right)\right] \\
&= \lim_{k\to+\infty}\int_{\overline{t}}^{t_n}\left(\lambda_1 w_k^{(n)}(\overline{x},\tau) - h(w_k^{(n)}(\overline{x},\tau))\right)\partial_t w_k^{(n)}(\overline{x},\tau)\,d\tau \\
&= \int_{\overline{t}}^{t_n}(\lambda_1 u_n(\overline{x},\tau) - h(u_n(\overline{x},\tau)))\,\partial_t u_n(\overline{x},\tau)\,d\tau. \qquad (2.42)
\end{aligned}$$

Moreover, for each n, we have $H(u_n(\cdot,t)) \in C^1(\overline{\Omega})$ for every $t \in [0,T]$ and hence, by (2.33) and (2.34), we obtain, for every $\overline{x} \in \Omega$,

$$\begin{aligned}
&|u_n|_\infty^{-2}\left|\frac{\lambda_1}{2}\left(u_n^2(x_n,t_n) - u_n^2(\overline{x},t_n)\right) - (H(u_n(x_n,t_n)) - H(u_n(\overline{x},t_n)))\right| \\
&\leq \int_0^1 \left|\lambda_1 v_n(\sigma_n(\tau),t_n) - |u_n|_\infty^{-1} h(u_n(\sigma_n(\tau),t_n))\right| \times \\
&\qquad \times |\nabla_x v_n(\sigma_n(\tau),t_n)|\,|\sigma_n'(\tau)|\,d\tau \\
&\leq K^2\,\ell(\sigma_n), \qquad (2.43)
\end{aligned}$$

where σ_n is a path, joining x_n to $\overline{x}$ and having range contained in Ω, and $\ell(\sigma_n)$ denotes its length. Because $x_n \to x_0$, with $x_n, x_0 \in \Omega$, and $\overline{x}$ can be chosen in a dense subset of Ω, we can suppose that

$$K^2\ell(\sigma_n) < \frac{\varepsilon}{4}, \qquad (2.44)$$

for all large n. Fix $\overline{x} \in \Omega$ such that (2.38), (2.39), (2.40), (2.42) and (2.44) hold. For every $\overline{t} \in [0,T]$, we derive from (2.33), (2.39) and (2.42)

$$|u_n|_\infty^{-2}\left|\frac{\lambda_1}{2}\left(u_n^2(\overline{x},t_n) - u_n^2(\overline{x},\overline{t})\right) - (H(u_n(\overline{x},t_n)) - H(u_n(\overline{x},\overline{t})))\right|$$

$$
\begin{aligned}
&\leq \left| \int_{\overline{t}}^{t_n} \left| \lambda_1 v_n(\overline{x},\tau) - |u_n|_\infty^{-1} h(u_n(\overline{x},\tau)) \right| |\partial_t v_n(\overline{x},\tau)| \, d\tau \right| \\
&\leq K \int_{\overline{t}}^{t_n} |\partial_t v_n(\overline{x},\tau)| \, d\tau \\
&\leq K \int_0^T |\partial_t v_n(\overline{x},\tau) - \partial_t v(\overline{x},\tau)| \, d\tau + K \int_{\overline{t}}^{t_n} |\partial_t v(\overline{x},\tau)| \, d\tau \\
&\leq \frac{\varepsilon}{4} + K \, |t_n - \overline{t}|^{1/2} \left(\int_0^T |\partial_t v(\overline{x},\tau)|^2 \, d\tau \right)^{1/2} \qquad (2.45)
\end{aligned}
$$

for all large n. Since $t_n \to t_0$ and $\overline{t}$ can be chosen in a dense subset of $[0,T]$, we can pick $\overline{t}$ such that

$$
K \left(\int_0^T |\partial_t v(\overline{x},\tau)|^2 \, d\tau \right)^{1/2} |t_n - \overline{t}|^{1/2} < \frac{\varepsilon}{4}, \qquad (2.46)
$$

for all large n. Notice that, at this point, $\overline{x} \in \Omega$ and $\overline{t} \in [0,T]$ have been fixed. Next, let B be a ball of radius R, centered at $\overline{x}$ and containing $\overline{\Omega}$, and set, for each n

$$
\gamma_n(x) = \begin{cases} |u_n|_\infty^{-1} \left| \lambda_1 u_n(x,\overline{t}) - h(u_n(x,\overline{t})) \right| & \text{if } x \in \Omega, \\ |u_n|_\infty^{-1} h(0) & \text{if } x \in B \setminus \Omega \, . \end{cases}
$$

From (2.38), it follows that $\gamma_n \to 0$ in $L_1(B)$. We now assume $N \geq 2$; the case where $N = 1$ can be dealt with in a similar (and even simpler) way. We introduce spherical coordinates in $\mathbb{R}^N$ centered at $\overline{x}$. Denoting by $(\rho, \phi_1, \ldots, \phi_{N-2}, \psi)$ with $\rho \in [0,R]$, $(\phi_1, \ldots, \phi_{N-2}) \in [0,\pi]^{N-2}$, $\psi \in [0,2\pi]$ the spherical coordinates of a point $x \in B$ and by Φ this change of coordinates, we get

$$
\begin{aligned}
\int_{[0,\pi]^{N-2} \times [0,2\pi]} &\left(\int_0^R \gamma_n(\Phi) |\det \Phi'| d\rho \right) d\phi_1 \ldots d\phi_{N-2} d\psi \\
&= \int_B \gamma_n dx \to 0 \, ,
\end{aligned}
$$

where $|\det \Phi'| = \rho^{N-1} (\sin \phi_1)^{N-2} (\sin \phi_2)^{N-3} \cdot \ldots \cdot \sin \phi_{N-2}$. Hence, possibly passing to a subsequence, we have

$$
\int_0^R |\gamma_n(\Phi)| \, \rho^{N-1} d\rho \to 0
$$

for a.e. $(\phi_1, \ldots, \phi_{N-2}, \psi) \in [0, \pi]^{N-2} \times [0, 2\pi]$. Passing to a further subsequence, we also have that, for a.e. fixed $(\phi_1, \ldots, \phi_{N-2}, \psi)$,

$$\gamma_n(\Phi) \to 0$$

for a.e. $\rho \in [0, R]$. On the other hand, the functions γ_n are continuous and uniformly bounded, by the linear growth of h, and therefore, by Lebesgue's theorem,

$$\int_0^R \gamma_n(\Phi) d\rho \to 0$$

for a.e. $(\phi_1, \ldots, \phi_{N-2}, \psi)$. This means that

$$\int_{[\overline{x}, y]} \gamma_n \to 0$$

for a.e. $y \in \partial B$. Denoting by x^* the first intersection point of $[\overline{x}, y]$ with $\partial\Omega$, we obtain

$$\int_0^1 |u_n|_\infty^{-1} \left|\lambda_1 u_n(x^* + \tau(\overline{x} - x^*), \overline{t}) - h(u_n(x^* + \tau(\overline{x} - x^*), \overline{t}))\right| d\tau \to 0\,.$$

Hence, using (2.34) and (2.36), we have

$$\begin{aligned}
& |u_n|_\infty^{-2} \left| \frac{\lambda_1}{2} \left(u_n^2(\overline{x}, \overline{t}) - u_n^2(x^*, \overline{t}) \right) - (H(u_n(\overline{x}, \overline{t})) - H(u_n(x^*, \overline{t}))) \right| \\
= & \int_0^1 \left| \lambda_1 v_n(x^* + \tau(\overline{x} - x^*), \overline{t}) - |u_n|_\infty^{-1} h(u_n(x^* + \tau(\overline{x} - x^*), \overline{t})) \right| \times \\
& \qquad \times |\nabla_x v_n(x^* + \tau(\overline{x} - x^*), \overline{t}) \cdot (\overline{x} - x^*)| \, d\tau \\
\leq & \; K|\,\overline{x} - x^*| \int_0^1 |\lambda_1 v_n(x^* + \tau(\overline{x} - x^*), \overline{t}) + \\
& -|u_n|_\infty^{-1} h(u_n(x^* + \tau(\overline{x} - x^*), \overline{t}))\Big| \, d\tau \quad < \quad \frac{\varepsilon}{4}
\end{aligned} \tag{2.47}$$

for all large n. Combining the above estimates (from (2.41) to (2.47)), we get a contradiction with (2.31). Accordingly, we take as $(S_n)_n$ a tail–end of $(s_n)_n$. ∎

We are now ready to prove the existence of a solution of (2.21). Let us define the following open bounded set in $C^0(\overline{Q})$, with $0 \in \mathcal{O}$,

$$\mathcal{O} = \left\{ u \in C^0(\overline{Q}) \mid -S_n < u(x,t) < S_n \quad \text{for every } (x,t) \in \overline{Q} \right\},$$