Global Solutions for a Nonlinear Wave Equation with the p-Laplacian Operator *

Hongjun Gao
Institute of Applied Physics and Computational Mathematics
100088 Beijing, China

To Fu Ma
Departamento de Matemática - Universidade Estadual de Maringá
87020-900 Maringá-PR, Brazil

Abstract

We study the existence and asymptotic behaviour of the global solutions of the nonlinear equation

$$u_{tt} - \Delta_p u + (-\Delta)^\alpha u_t + g(u) = f$$

where $0 < \alpha \leq 1$ and g does not satisfy the sign condition $g(u)u \geq 0$.

Key words: Quasilinear hyperbolic equation, asymptotic behaviour, small data.

AMS Subject Classification: 35B40, 35L70.

1. Introduction

The study of global existence and asymptotic behaviour for initial-boundary value problems involving nonlinear operators of the type

$$u_{tt} - \sum_{i=1}^{n} \{\sigma_i(u_{x_i})\}_{x_i} - \Delta u_t = f(t,x) \quad \text{in } (0,T) \times \Omega$$

goes back to Greenberg, MacCamy & Mizel [3], where they considered the one-dimensional case with smooth data. Later, several papers have appeared in that direction, and some of its important results can be found in, for

*Work done while the authors were visiting LNCC/CNPq, Brazil.

example, Ang & Pham Ngoc Dinh [1], Biazutti [2], Nakao [8], Webb [10] and Yamada [11]. In all of the above cited papers, the damping term $-\Delta u_t$ played an essential role in order to obtain global solutions. Our objective is to study this kind of equations under a weaker damping given by $(-\Delta)^\alpha u_t$ with $0 < \alpha \leq 1$. This approach was early considered by Medeiros and Milla Miranda [6] to Kirchhoff equations. We also consider the presence of a forcing term $g(x, u)$ that does not satisfy the sign condition $g(x, u)u \geq 0$. Our study is based on the pseudo Laplacian operator

$$-\Delta_p u = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right)$$

which is used as a model for several monotone hemicontinuous operators.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the nonlinear initial-boundary value problem

$$(1.1) \qquad \begin{cases} u_{tt} - \Delta_p u + (-\Delta)^\alpha u_t + g(x, u) = f(t, x) \quad \text{in} \quad (0, T) \times \Omega, \\ u = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \\ u(0, x) = u_0 \quad \text{and} \quad u_t(0, x) = u_1 \quad \text{in} \quad \Omega, \end{cases}$$

where $0 < \alpha \leq 1$ and $p \geq 2$. We prove that, depending on the growth of g, problem (1.1) has a global weak solution without assuming small initial data. In addition, we show the exponential decay of solutions when $p = 2$ and algebraic decay when $p > 2$. The global solutions are constructed by means of the Galerkin approximations and the asymptotic behaviour is obtained by using a difference inequality due to M. Nakao [7]. Here we only use standard notations. We often write $u(t)$ instead $u(t, x)$ and $u'(t)$ instead $u_t(t, x)$. The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$ and in $W_0^{1,p}(\Omega)$ we use the norm

$$\|u\|_{1,p}^p = \sum_{j=1}^{n} \|u_{x_j}\|_p^p.$$

For the reader's convenience, we recall some of the basic properties of the operators used here. The degenerate operator $-\Delta_p$ is bounded, monotone and hemicontinuous from $W_0^{1,p}(\Omega)$ to $W^{-1,q}(\Omega)$, where $p^{-1} + q^{-1} = 1$. The powers for the Laplacian operator is defined by

$$(-\Delta)^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha (u, \varphi_j)\varphi_j,$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ and $\varphi_1, \varphi_2, \varphi_3, \cdots$ are, respectively, the sequence of the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. Then

$$\|u\|_{D((-\Delta)^\alpha)} = \|(-\Delta)^\alpha u\|_2 \quad \forall\, u \in D((-\Delta)^\alpha)$$

and $D((-\Delta)^\alpha) \subset D((-\Delta)^\beta)$ compactly if $\alpha > \beta \geq 0$. In particular, for $p \geq 2$ and $0 < \alpha \leq 1$, $W_0^{1,p}(\Omega) \hookrightarrow D((-\Delta)^{\alpha/2}) \hookrightarrow L^2(\Omega)$.

2. Existence of Global Solutions

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the growth condition

$$|g(x,u)| \leq a|u|^{\sigma-1} + b \quad \forall (x,u) \in \Omega \times \mathbb{R}, \tag{2.1}$$

where a, b are positive constants, $1 < \sigma < pn/(n-p)$ if $n > p$ and $1 < \sigma < \infty$ if $n \leq p$.

Theorem 2.1 *Let us assume condition* (2.1) *with $\sigma < p$. Then given $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$, there exists a function $u : (0,T) \times \Omega \to \mathbb{R}$ such that*

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{2.2}$$

$$u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;D((-\Delta)^{\frac{\alpha}{2}})), \tag{2.3}$$

$$u(0) = u_0 \quad \textit{and} \quad u'(0) = u_1 \quad \textit{a.e. in} \quad \Omega, \tag{2.4}$$

$$u_{tt} - \Delta_p u + (-\Delta)^\alpha u_t + g(x,u) = f \quad \textit{in} \quad L^2(0,T;W^{-1,q}(\Omega)), \tag{2.5}$$

where $p^{-1} + q^{-1} = 1$.

Next we consider an existence result when $\sigma \geq p$. In this case, the global solution is obtained with small initial data. For each $m \in \mathbb{N}$ we put

$$\gamma_m = \frac{1}{2}\|u_{1m}\|_2^2 + \frac{3}{2p}\|u_{0m}\|_{1,p}^p + \frac{aC_\sigma}{\sigma}\|u_{0m}\|_{1,p}^\sigma + \frac{2(\sqrt[p]{2}bC_1)^q}{q}$$

where $C_k > 0$ is the Sobolev constant for the inequality $\|u\|_k \leq C_k\|u\|_{1,p}$, when $W_0^{1,p}(\Omega) \hookrightarrow L^k(\Omega)$. We also define the polynomial Q by

$$Q(z) = \frac{1}{2p}z^p - \frac{aC_\sigma}{\sigma}z^\sigma,$$

which is increasing in $[0, z_0]$, where

$$z_0 = (2aC_\sigma)^{\frac{-1}{\sigma-p}}$$

is its unique local maximum. We will assume that

$$\|u_0\|_{1,p} < z_0 \tag{2.6}$$

and

$$\gamma + \frac{1}{4\lambda_1^\alpha} \int_0^T \|f(t)\|_2^2 dt < Q(z_0), \tag{2.7}$$

where $\gamma = \lim_{m\to\infty} \gamma_m$.

Theorem 2.2 *Suppose that condition* (2.1) *holds with* $\sigma > p$. *Suppose in addition that inital data satisfy* (2.6) *and* (2.7). *Then there exists a function* $u : (0,T) \times \Omega \to \mathbb{R}$ *satisfying* (2.2)-(2.5).

Proof of Theorem 2.1: Let r be an integer for which $H_0^r(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ is continuous. Then the eigenfunctions of $-\Delta^r w_j = \alpha_j w_j$ in $H_0^r(\Omega)$ yields a "Galerkin" basis for both $H_0^r(\Omega)$ and $L^2(\Omega)$. For each $m \in \mathbb{N}$, let us put $V_m = \text{Span}\{w_1, w_2, \cdots, w_m\}$. We search for a function

$$u_m(t) = \sum_{j=1}^m k_{jm}(t) w_j$$

such that for any $v \in V_m$, $u_m(t)$ satisfies the approximate equation

$$\int_\Omega \{u_m''(t) - \Delta_p u_m(t) + (-\Delta)^\alpha u_m'(t) + g(x, u_m(t)) - f(t,x)\} v\, dx = 0 \tag{2.8}$$

with the initial conditions

$$u_m(0) = u_{0m} \quad \text{and} \quad u_m'(0) = u_{1m},$$

where u_{0m} and u_{1m} are chosen in V_m so that

$$u_{0m} \to u_0 \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_{1m} \to u_1 \text{ in } L^2(\Omega). \tag{2.9}$$

Putting $v = w_j$, $j = 1, \cdots, m$, we observe that (2.8) is a system of ODEs in the variable t and has a local solution $u_m(t)$ in a interval $[0, t_m)$. In the next step we obtain the a priori estimates for the solution $u_m(t)$ so that it can be extended to the whole interval $[0,T]$.

A Priori Estimates: We replace v by $u_m'(t)$ in the approximate equation (2.8) and after integration we have

$$\frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{p}\|u_m(t)\|_{1,p}^p + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_m'(s)\|_2^2\, ds + \int_\Omega G(x, u_m(t))dx$$
$$\leq \int_0^t \|f(s)\|_2 \|u_m'(s)\|_2\, ds + \frac{1}{2}\|u_{1m}\|_2^2 + \frac{1}{p}\|u_{0m}\|_{1,p}^p + \int_\Omega G(x, u_0)dx,$$

where $G(x,u) = \int_0^u g(x,s)\,ds$. Now from growth condition (2.1) and the Sobolev embedding, we have that

$$\int_\Omega |G(x,u_m(t))|dx \leq \frac{a}{\sigma}C_\sigma \|u_m(t)\|_{1,p}^\sigma + bC_1\|u_m(t)\|_{1,p}. \tag{2.10}$$

But since $p > \sigma$, there exists a constant $\overline{C} > 0$ such that

$$\int_\Omega |G(x,u_m)|dx \leq \frac{1}{2p}\|u_m(t)\|_{1,p}^p + \overline{C},$$

and then we have

$$\frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2p}\|u_m(t)\|_{1,p}^p + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_m'(s)\|_2^2\,ds$$

$$\leq \int_0^t \|f(s)\|_2\|u_m'(s)\|_2\,ds + \frac{1}{2}\|u_{1m}\|_2^2 + \frac{3}{2p}\|u_{0m}\|_{1,p}^p + 2\overline{C}.$$

Using the convergence (2.9) and the Gronwall's lemma, there exists a constant $C > 0$ independent of t, m such that

$$\|u_m'(t)\|_2^2 + \|u_m(t)\|_{1,p}^p + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_m'(s)\|_2^2\,ds \leq C. \tag{2.11}$$

With this estimate we can extend the approximate solutions $u_m(t)$ to the interval $[0,T]$ and we have that

$$(u_m) \quad \text{is bounded in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{2.12}$$

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.13}$$

$$(u_m') \quad \text{is bounded in} \quad L^2(0,T;D((-\Delta)^{\frac{\alpha}{2}})), \tag{2.14}$$

and

$$(-\Delta_p u_m) \quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,q}(\Omega)). \tag{2.15}$$

Now we are going to obtain an estimate for (u_m''). Since our Galerkin basis was taken in the Hilbert space $H^r(\Omega) \subset W_0^{1,p}(\Omega)$, we can use the standard projection arguments as described in Lions [4]. Then from the approximate equation and the estimates (2.12)-(2.15) we get

$$(u_m'') \quad \text{is bounded in} \quad L^2(0,T;H^{-r}(\Omega)). \tag{2.16}$$

Passage to the Limit: From (2.12)-(2.14), going to a subsequence if necessary, there exists u such that

$$u_m \rightharpoonup u \quad \text{weakly star in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{2.17}$$

$$u'_m \rightharpoonup u' \quad \text{weakly star in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.18}$$

$$u'_m \rightharpoonup u' \quad \text{weakly in} \quad L^2(0,T;D((-\Delta)^{\frac{\alpha}{2}}), \tag{2.19}$$

and in view of (2.15) there exists χ such that

$$-\Delta_p u_m \rightharpoonup \chi \quad \text{weakly star in} \quad L^\infty(0,T;W^{-1,q}(\Omega)). \tag{2.20}$$

By applying the Lions-Aubin compactness lemma we get from (2.12)-(2.13)

$$u_m \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \tag{2.21}$$

and since $D((-\Delta)^{\frac{\alpha}{2}}) \hookrightarrow L^2(\Omega)$ compactly, we get from (2.14) and (2.16)

$$u'_m \to u' \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{2.22}$$

Using the growth condition (2.1) and (2.21) we see that

$$\int_0^T \int_\Omega |g(x,u_m(t,x))|^{\frac{\sigma}{\sigma-1}}\,dxdt$$

is bounded and

$$g(x,u_m) \to g(x,u) \quad \text{a.e. in} \quad (0,T)\times\Omega.$$

Therefore from Lions [4] (Lemma 1.3) we infer that

$$g(x,u_m) \rightharpoonup g(x,u) \quad \text{weakly in} \quad L^{\frac{\sigma}{\sigma-1}}(0,T;L^{\frac{\sigma}{\sigma-1}}(\Omega)). \tag{2.23}$$

With these convergence we can pass to the limit in the approximate equation and then

$$\frac{d}{dt}(u'(t),v) + \langle\chi(t),v\rangle + ((-\Delta)^\alpha u'(t),v) + (g(x,u(t)),v) = (f(t),v) \tag{2.24}$$

for all $v \in W^{1,p}(\Omega)$, in the sense of distributions. This easily implies that (2.2)-(2.4) hold. Finally, since we have the strong convergence (2.22), we can use a standard monotonicity argument as done in Biazutti [2] or Ma & Soriano [5] to show that $\chi = -\Delta_p u$. This ends the proof. $\square$

Proof of Theorem 2.2: We only show how to obtain the estimate (2.11). The remainder of the proof follows as before. We apply an argument made by L. Tartar [9]. From (2.10) we have

$$\int_\Omega |G(x,u_m)|dx \le \frac{aC_\sigma}{\sigma}\|u_m\|_{1,p}^\sigma + \frac{1}{2p}\|u_m\|_{1,p}^p + \frac{(\sqrt[p]{2}bC_1)^q}{q}$$

and therefore

$$\frac{1}{2}\|u_m'(t)\|_2^2 + Q(\|u_m(t)\|_{1,p}) + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}}u_m'(s)\|_2^2\,ds$$
$$\le \gamma_m + \int_0^t \|f(s)\|_2\|u_m'(s)\|_2\,ds.$$

Since

$$\|(-\Delta)^{\frac{\alpha}{2}}u_m'(s)\|_2 \ge \lambda_1^{\frac{\alpha}{2}}\|u_m'(t)\|_2, \tag{2.25}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, this implies that

$$\frac{1}{2}\|u_m'(t)\|_2^2 + Q(\|u_m(t)\|_{1,p}) \le \gamma_m + \frac{1}{4\lambda_1^\alpha}\int_0^T \|f(t)\|_2^2\,dt. \tag{2.26}$$

We claim that there exists an integer N such that

$$\|u_m(t)\|_{1,p} < z_0 \quad \forall\, t\in[0,t_m) \quad m>N \tag{2.27}$$

Suppose the claim is proved. Then $Q(\|u_m(t)\|_{1,p}) \ge 0$ and from (2.7) and (2.26), $\|u_m'(t)\|_2$ is bounded and consequently (2.11) follows.

Proof of the Claim: Suppose (2.27) false. Then for each $m>N$, there exists $t\in[0,t_m)$ such that $\|u_m(t)\|_{1,p} \ge z_0$. We note that from (2.6) and (2.9) there exists N_0 such that

$$\|u_m(0)\|_{1,p} < z_0 \quad \forall\, m > N_0.$$

Then by continuity there exists a first $t_m^* \in (0,t_m)$ such that

$$\|u_m(t_m^*)\|_{1,p} = z_0, \tag{2.28}$$

from where

$$Q(\|u_m(t)\|_{1,p}) \ge 0 \quad \forall\, t\in[0,t_m^*].$$

Now from (2.7) and (2.26), there exist $N>N_0$ and $\beta\in(0,z_0)$ such that

$$0 \le \frac{1}{2}\|u_m'(t)\|_2^2 + Q(\|u_m(t)\|_{1,p}) \le Q(\beta) \quad \forall\, t\in[0,t_m^*] \quad \forall\, m>N.$$

Then the monotonicity of Q in $[0, z_0]$ implies that

$$0 \le \|u_m(t)\|_{1,p} \le \beta < z_0 \quad \forall t \in [0, t_m^*],$$

and in particular, $\|u_m(t_m^*)\|_{1,p} < z_0$, which is a contradiction to (2.28). □

Remarks: From the above proof we have the following immediate conclusion: The smallness of initial data can be dropped if either condition (2.1) holds with $\sigma = p$ and coefficient a is sufficiently small, or $\sigma > p$ and the sign condition $g(x,u)u \ge 0$ is satisfied. □

3. Asymptotic Behaviour

Theorem 3.1 *Let u be a solution of Problem* (1.1) *given by*
(a) *either Theorem 2.1 with the additional assumption: there exists $\rho > 0$ such that*

$$g(x,u)u \ge \rho G(x,u) \ge 0, \tag{3.1}$$

(b) *or Theorem 2.2.*
Then there exists positive constants C and θ such that

$$\|u'(t)\|_2^2 + \|u(t)\|_{1,p}^p \le C \exp(-\theta t) \quad if \quad p = 2,$$

or

$$\|u'(t)\|_2^2 + \|u(t)\|_{1,p}^p \le C(1+t)^{\frac{-p}{p-2}} \quad if \quad p > 2.$$

The proof of Theorem 3.1 is based on the following difference inequality of M. Nakao [7].

Lemma 3.1 (Nakao) *Let $\phi : \mathbf{R}^+ \to \mathbf{R}$ be a bounded nonnegative function for which there exist constants $\beta > 0$ and $\gamma \ge 0$ such that*

$$\sup_{t \le s \le t+1} (\phi(s))^{1+\gamma} \le \beta(\phi(t) - \phi(t+1)) \quad \forall\, t \ge 0.$$

Then
(i) *If $\gamma = 0$, there exist positive constants C and θ such that*

$$\phi(t) \le C \exp(-\theta t) \quad \forall\, t \ge 0.$$

(ii) *If $\gamma > 0$, there exists a positive constant C such that*

$$\phi(t) \le C(1+t)^{\frac{-1}{\gamma}} \quad \forall\, t \ge 0.$$

Let us define the approximate energy of the system (1.1) by

$$E_m(t) = \frac{1}{2}\|u'_m(t)\|_2^2 + \frac{1}{p}\|u_m(t)\|_{1,p}^p + \int_\Omega G(x, u_m(t))dx. \tag{3.2}$$

Then the following two lemmas hold.

Lemma 3.2 *There exists a constant $k > 0$ such that*

$$kE_m(t) \le \|u_m(t)\|_{1,p}^p + \int_\Omega g(x, u_m(t))u_m(t)dx. \tag{3.3}$$

Proof. In the case (a), condition (3.3) is a direct consequence of (3.1). In the case (b) the result follows from (2.27). In fact, from assumption (2.1) with $b = 0$ we have

$$\int_\Omega |G(x, u_m)|dx \le \frac{aC_\sigma}{\sigma}\|u_m\|_{1,p}^\sigma \quad \text{and} \quad \int_\Omega |g(x, u_m)u_m|dx \le aC_\sigma\|u_m\|_{1,p}^\sigma.$$

Then given $\delta > 0$,

$$\|u_m\|_{1,p}^p + \int_\Omega g(x, u_m)u_m dx = \frac{\delta}{p}\|u_m\|_{1,p}^p + \delta\int_\Omega G(x, u_m)dx$$

$$-\delta\int_\Omega G(x, u_m)dx + (1 - \frac{\delta}{p})\|u_m\|_{1,p}^p + \int_\Omega g(x, u_m)u_m\, dx$$

implies that

$$\|u_m(t)\|_{1,p}^p + \int_\Omega g(x, u_m(t))u_m(t)dx \ge \frac{\delta}{p}\|u_m(t)\|_{1,p}^p + \delta\int_\Omega G(x, u_m(t))dx$$

$$+(1 - \frac{\delta}{p})\|u_m(t)\|_{1,p}^p - (1 + \frac{\delta}{\sigma})aC_\sigma\|u_m(t)\|_{1,p}^\sigma.$$

Now, since $\|u_m(t)\|_{1,p} \le (2aC_\sigma)^{\frac{-1}{\sigma-p}} = z_0$ uniformly in t, m, we have that $\|u_m(t)\|_{1,p}^{\sigma-p} \le (2aC_\sigma)^{-1}$. Then taking $\delta \le (\sigma p)(p + 2\sigma)^{-1}$, we conclude that

$$(1 - \frac{\delta}{p})\|u_m(t)\|_{1,p}^p - (1 + \frac{\delta}{\sigma})aC_\sigma\|u_m(t)\|_{1,p}^\sigma$$

$$= \left[(1 - \frac{\delta}{p}) - (1 + \frac{\delta}{\sigma})aC_\sigma\|u_m(t)\|_{1,p}^{\sigma-p}\right]\|u_m(t)\|_{1,p}^p \ge 0.$$

This implies that

$$\|u_m(t)\|_{1,p}^p + \int_\Omega g(x,u_m(t))u_m(t)dx \geq \frac{\delta}{p}\|u_m(t)\|_{1,p}^p + \delta\int_\Omega G(x,u_m(t))dx$$

and therefore (3.3) holds. □

Lemma 3.3 *For any $t > 0$,*

$$E_m(t) \geq \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2p}\|u_m(t)\|_{1,p}^p.$$

Proof. In the case (a), the Lemma is a consequence of (3.1). In the case (b), we use again the smallness of the approximate solutions.

$$\begin{aligned} E_m(t) &\geq \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{p}\|u_m(t)\|_{1,p}^p - \frac{aC_\sigma}{\sigma}\|u_m(t)\|_{1,p}^\sigma \\ &= \frac{1}{2}\|u_m'(t)\|_2^2 + \frac{1}{2p}\|u_m(t)\|_{1,p}^p - Q(\|u_m(t)\|_{1,p}). \end{aligned}$$

Since $Q(\|u_m(t)\|_{1,p}) \geq 0$, the result follows. □

Proof of Theorem 3.1: We first obtain uniform estimates for the approximate energy (3.2). Fix an arbitrary $t > 0$, we get from the approximate problem (2.8) with $f = 0$ and $v = u_m'(t)$

$$E_m(t) + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}}u_m'(s)\|_2^2 ds = E_m(0),$$

from where

$$E_m'(t) + \|(-\Delta)^{\frac{\alpha}{2}}u_m'(t)\|_2^2 = 0. \tag{3.4}$$

Integrating (3.4) from t to $t+1$ and putting

$$D_m^2(t) = E_m(t) - E_m(t+1)$$

we have in view of (2.25)

$$D_m^2(t) = \int_t^{t+1} \|(-\Delta)^{\frac{\alpha}{2}}u_m'(s)\|_2^2\, ds \geq \lambda_1^\alpha \int_t^{t+1} \|u_m'(s)\|_2^2\, ds. \tag{3.5}$$

By applying the Mean Value Theorem in (3.5), there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_m'(t_i)\|_2 \leq \sqrt{2}\lambda_1^{-\frac{\alpha}{2}} D_m(t) \quad i = 1, 2. \tag{3.6}$$

Now, integrating the approximate equation with $v = u_m(t)$ over $[t_1, t_2]$, we infer from Lemma 3.2 that

$$k \int_{t_1}^{t_2} E_m(s)\,ds \leq (u'_m(t_1), u_m(t_1)) - (u'_m(t_2), u_m(t_2))$$

$$+ \int_{t_1}^{t_2} \|u'_m(s)\|_2^2\,ds - \int_{t_1}^{t_2} ((-\Delta)^{\frac{\alpha}{2}} u'_m(s), (-\Delta)^{\frac{\alpha}{2}} u_m(s))ds.$$

Then from Lemma 3.3, Hölder's inequality and Sobolev embeddings, we have in view of (3.5) and (3.6), there exist positive constants C_1 and C_2 such that

$$\int_{t_1}^{t_2} E_m(s)ds \leq C_1 D_m^2(t) + C_2 D_m(t) E_m^{\frac{1}{p}}(t).$$

By the Mean Value Theorem, there exists $t^* \in [t_1, t_2]$ such that

$$E_m(t^*) \leq C_1 D_m^2(t) + C_2 D_m(t) E_m^{\frac{1}{p}}(t).$$

The monotonicity of E_m implies that

$$E_m(t+1) \leq C_1 D_m^2(t) + C_2 D_m(t) E_m^{\frac{1}{p}}(t).$$

Since $E_m(t+1) = E_m(t) - D_m^2(t)$, we conclude that

$$E_m(t) \leq (C_1 + 1) D_m^2(t) + C_2 D_m(t) E_m^{\frac{1}{p}}(t).$$

Using Young's inequality, there exist positive constants C_3 and C_4 such that

$$E_m(t) \leq C_3 D_m^2(t) + C_4 D_m^{\frac{p}{p-1}}(t). \tag{3.7}$$

If $p = 2$ then

$$E_m(t) \leq (C_3 + C_4) D_m^2(t)$$

and since E_m is decreasing, then from Lemma 3.1 there exist positive constants C and θ (independent of m) such that

$$E_m(t) \leq C \exp(-\theta t) \quad \forall\, t > 0. \tag{3.8}$$

If $p > 2$, then relation (3.7) and the boundedness of $D_m(t)$ show the existence of $C_5 > 0$ such that

$$E_m(t) \leq C_5 D_m^{\frac{p}{p-1}}(t),$$

and then

$$E_m^{\frac{2(p-1)}{p}}(t) \leq C_5^{\frac{2(p-1)}{p}} D_m^2(t).$$

Applying Lemma 3.1, with $\gamma = (p-2)/p$, there exists a constant $C > 0$ (independent of m) such that

$$E_m(t) \leq C(1+t)^{\frac{-p}{p-2}} \quad \forall\, t \geq 0. \tag{3.9}$$

Finally we pass to the limit (3.8) and (3.9) and the proof is complete in view of Lemma 3.3. □

Acknowledgement: The authors thank Professor J. E. Muñoz Rivera for his hospitality during their stay in LNCC/CNPq.

References

[1] D. D. Ang & A. Pham Ngoc Dinh; Strong solutions of a quasilinear wave equation with nonlinear damping, *SIAM J. Math. Anal.* **19**, (1988), 337-347 .

[2] A. C. Biazutti; On a nonlinear evolution equation and its applications, *Nonlinear Anal. T. M. A.* **24**, (1995), 1221-1234 .

[3] J. Greenberg, R. C. MacCamy & V. J. Mizel; On the existence, uniqueness and stability of solutions of the equation $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$, *J. Math. Mech.* **17**, (1968), 707-728 .

[4] J. L. Lions; *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier Villars, Paris, 1969.

[5] T. F. Ma & J. A. Soriano; On weak solutions for an evolution equation with exponential nonlinearities, *Nonlinear Anal. T. M. A.* **37**, (1999), 1029-1038.

[6] L. A. Medeiros & M. Milla Miranda; On a nonlinear wave equation with damping, *Rev. Mat. Univ. Complutense de Madrid* **3**, (1990), 213-231.

[7] M. Nakao; A difference inequality and its applications to nonlinear evolution equations, *J. Math. Soc. Japan* **30**, (1978), 747-762 .

[8] M. Nakao; Energy decay for the quasilinear wave equation with viscosity, *Math Z.* **219**, (1992), 289-299 .

[9] L. Tartar; *Topics in Nonlinear Analysis*, Publications Mathématiques d'Orsay, Université Paris-Sud, Orsay, 1978.

[10] G. F. Webb; Existence and asymptotic for a strongly damped nonlinear wave equation, *Canadian J. Math.* **32**, (1980), 631-643 .

[11] Y. Yamada; Some remarks on the equation $y_{tt} - \sigma(y_x)y_{xx} - y_{xtx} = f$, *Osaka J. Math.* **17**, (1980), 303-323 .