

Eigenvalue Characterization for a Class of Boundary Value Problems

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Abstract

We consider the n th order ordinary differential equation $(-1)^{n-k}y^{(n)} = \lambda a(t)f(y)$, $t \in [0, 1]$, $n \geq 3$ together with boundary condition $y^{(i)}(0) = 0$, $0 \leq i \leq k - 1$, and $y^{(l)}(1) = 0$, $j \leq l \leq j + n - k - 1$, for $1 \leq j \leq k - 1$ fixed. Values of λ are characterized so that the boundary value problem has a positive solution.

1 Introduction

Let $n \geq 3$, $2 \leq k \leq n - 1$, and $1 \leq j \leq k - 1$ be given. In this paper we shall consider the n th order differential equation

$$(-1)^{n-k}y^{(n)} = \lambda a(t)f(y), \quad t \in [0, 1], \quad (1)$$

satisfying the boundary conditions

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \leq i \leq k - 1, \\ y^{(l)}(1) &= 0, \quad j \leq l \leq j + n - k - 1. \end{aligned} \quad (2)$$

Throughout, we assume the following hypotheses :

(H_1) $a(t)$ is a continuous nonnegative function on $[0, 1]$ and is not identically equal to zero on any subinterval of $[0, 1]$.

(H₂) $f : R \rightarrow [0, \infty)$ is continuous and nonnegative.

(H₃) The limits $f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u}$ exist in $[0, \infty)$.

We shall determine values of λ for which the boundary value problem (1), (2) has a positive solution. By a positive solution y of (1), (2), we mean $y \in C^{(n)}[0, 1]$ satisfies (1) on $[0, 1]$ and fulfills (2), and y is nonnegative and is not identically zero on $[0, 1]$. We let

$$Sp(a) = \{\lambda > 0 \mid (1), (2) \text{ has a positive solution}\}.$$

The motivation for the present work originates from many recent investigations. In the case $n = 2$ the boundary value problem (1), (2) describes a vast spectrum of scientific phenomena; we refer the reader to [1, 3, 5, 6, 14, 16]. It is noted that only positive solutions are meaningful in those models. Our results complement the work of many authors, see, e.g. [2, 4, 8, 9, 10, 11, 12, 13, 17, 18, 19]. In Section 2, we provide some definitions and background results, and state a fixed point theorem due to Krasnosel'skii [15]. Also, we present some properties of certain Green's function where needed. By defining an appropriate Banach space and cone, in Section 3, we characterize the set $Sp(a)$.

2 Background Notation and Definitions

We first present the definition of a cone in a Banach space and the Krasnosel'skii Fixed Point Theorem. **Definition 2.1.** Let \mathcal{B} be a Banach space over R . A nonempty closed convex set $\mathcal{P} \subset \mathcal{B}$ is said to be a cone provided the following are satisfied:

- (a) If $y \in \mathcal{P}$ and $\alpha \geq 0$, then $\alpha y \in \mathcal{P}$;
- (b) If $y \in \mathcal{P}$ and $-y \in \mathcal{P}$, then $y = 0$.

Theorem 2.1 *Let \mathcal{B} be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in \mathcal{B} . Assume Ω_1, Ω_2 are open subsets of \mathcal{B} with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow \mathcal{P}$$

be a completely continuous operator such that, either

- (i) $\|Tu\| \leq \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$;

(ii) $\|Tu\| \geq \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To obtain a solution for (1) and (2), we require a mapping whose kernel $G(t, s)$ is the Green's function of the boundary value problem

$$(-1)^{n-k}y^{(n)} = 0, \tag{3}$$

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \leq i \leq k-1, \\ y^{(l)}(1) &= 0, \quad j \leq l \leq j+n-k-1. \end{aligned}$$

Wong and Agarwal [20] have found that if y satisfies

$$(-1)^{n-p}y^{(n)} \geq 0, \tag{4}$$

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \leq i \leq p-1, \\ y^{(l)}(1) &= 0, \quad 0 \leq l \leq n-p-1, \end{aligned} \tag{5}$$

then, for $\delta \in (0, \frac{1}{2})$ and $t \in [\delta, 1 - \delta]$,

$$y(t) \geq \min\{b(p) \min\{c(p), c(n-p-1)\}, b(p-1) \min\{c(p-1), c(n-p)\}\} \|y\| \tag{6}$$

where the functions b and c are defined as

$$b(x) = \frac{(n-1)^{n-1}}{x^x(n-x-1)^{n-x-1}}, \quad c(x) = \delta^x(1-\delta)^{n-x-1}.$$

Aided by this, we have the following lemma.

Lemma 2.2 *Let $n \geq 3$. Assume $u \in C^{(n)}[0, 1]$, $(-1)^{n-k}u^{(n)}(t) \geq 0$, $0 \leq t \leq 1$ and u satisfies (2).*

Then for $0 \leq t \leq 1$,

$$u^{(j)}(t) \geq 0$$

and for $t \in [\delta, 1 - \delta]$

$$u^{(j)}(t) \geq \sigma_1 |u^{(j)}|_\infty$$

where

$$\sigma_1 = \min\{b(k-j) \min\{c(k-j), c(n-k-1)\}, b(k-j-1) \min\{c(k-j-1), c(n-k)\}\}.$$

Proof: First, $u^{(j)} \in C^{(n-j)}[0, 1]$. Also $u^{(j)}$ satisfies

$$(-1)^{n-k} y^{(n-j)}(t) \geq 0.$$

Let the boundary condition (2) be partitioned into two parts:

$$\begin{aligned} y^{(i)}(0) &= 0, \quad j \leq i \leq k-1 \\ y^{(l)}(1) &= 0, \quad j \leq l \leq j+n-k-1 \end{aligned} \tag{7}$$

and

$$y^{(i)}(1) = 0, \quad 0 \leq i \leq j-1. \tag{8}$$

Now u satisfies (7), so $u^{(j)}$ satisfies $(k-j, n-k)$ homogeneous conjugate boundary conditions. The conclusion then follows from inequality (6).

Lemma 2.3 *Let $n \geq 3$. Assume $u \in C^{(n)}[0, 1]$, $(-1)^{n-k} y^{(n)}(t) \geq 0$, $0 \leq t \leq 1$, and u satisfies (2). Then for $0 \leq t \leq 1$,*

$$u(t) \geq 0$$

and for $t \in [\frac{1}{2}, 1 - \delta]$,

$$u(t) \geq \sigma_2 |u^{(j)}|_\infty$$

where $\sigma_2 = \frac{\sigma_1(\frac{1}{2}-\delta)^j}{j!}$ and $|u^{(j)}|_\infty = \max_{t \in [0,1]} |u^{(j)}(t)|$.

Proof: Since u satisfies (2), u satisfies (8) as well. Thus for $0 \leq t \leq 1$,

$$u(t) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds.$$

Aided by Lemma 2.2

$$\begin{aligned} u(t) &= \int_\delta^t \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds + \int_0^\delta \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds. \\ &\geq \int_\delta^t \frac{(t-s)^{j-1}}{(j-1)!} u^{(j)}(s) ds \\ &\geq \frac{\sigma_1(t-\delta)^j}{j!} |u^{(j)}|_\infty. \end{aligned}$$

Consequently, for $t \in [\frac{1}{2}, 1 - \delta]$,

$$u(t) \geq \frac{\sigma_1(\frac{1}{2}-\delta)^j}{j!} |u^{(j)}|_\infty.$$

The nonnegativity of u follows.

It is noted that Eloe [7] proved that $G^{(j)}(t, s) = \frac{\partial^j}{\partial t^j} G(t, s)$ is the Green's function of $y^{(n-j)} = 0$ subject to the boundary conditions

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \leq i \leq k - j - 1, \\ y^{(l)}(1) &= 0, \quad 0 \leq l \leq n - k - 1. \end{aligned} \tag{9}$$

The proof follows from the four properties of the Green's function. Consequently we have the following result, whose conclusion follows from Lemma 2.2.

Lemma 2.4 *For each $s \in (0, 1)$, and $t \in [\delta, 1 - \delta]$*

$$(-1)^{n-k} G^{(j)}(t, s) \geq \sigma_1 |G^{(j)}(\cdot, s)|_\infty$$

where $|G^{(j)}(\cdot, s)|_\infty = \max_{0 \leq t \leq 1} |G^{(j)}(t, s)|$.

3 Main Results

We are now in a position to give some characterization of $Sp(a)$. Define a Banach space, \mathcal{B} , by

$$\mathcal{B} = \{u \in C^{(j)}[0, 1] \mid u \text{ satisfies (8)}\}$$

with norm $\|u\| = \max_{0 \leq t \leq 1} |u^{(j)}(t)|$.

Let $\sigma = \sigma_2 = \frac{\sigma_1 (\frac{1}{2} - \delta)^j}{j!}$. Define a cone, $\mathcal{P}_\sigma \subset \mathcal{B}$, by

$$\mathcal{P}_\sigma = \{u \in \mathcal{B} \mid u^{(j)}(t) \geq 0 \text{ on } [0, 1], \text{ and } \min_{t \in [\delta, 1 - \delta]} u(t) \geq \sigma \|u\|\}.$$

Let

$$Tu(t) = (-1)^{n-k} \int_0^1 G(t, s) a(s) f(u(s)) ds, \quad 0 \leq t \leq 1, \quad u \in \mathcal{B}.$$

To obtain a solution of (1), (2), we shall seek a fixed point of the operator λT in the cone \mathcal{P}_σ . In order to apply the Krasnosel'skii Fixed Point Theorem, for $\lambda > 0$, we need the following.

Lemma 3.1 *For $\lambda > 0$, $\lambda T : \mathcal{P}_\sigma \rightarrow \mathcal{P}_\sigma$ and is a completely continuous operator.*

Proof: Let $u \in \mathcal{P}_\sigma$. It suffices to verify this lemma when $\lambda = 1$. By properties of $(-1)^{n-k}G^{(j)}(t, s)$, it is clear that $(Tu)^{(j)}(t) \geq 0$ and $(Tu)^{(j)}(t)$ is continuous on $[0, 1]$.

Furthermore, for any $0 \leq \tau \leq 1$

$$\begin{aligned} \min_{t \in [\delta, 1-\delta]} (Tu)^{(j)}(t) &\geq \int_0^1 \min_{t \in [\delta, 1-\delta]} (-1)^{n-k} G^{(j)}(t, s) a(s) f(u(s)) ds \\ &\geq \sigma \int_0^1 (-1)^{n-k} G^{(j)}(\tau, s) a(s) f(u(s)) ds \\ &\geq \sigma \int_0^1 |G^{(j)}(\cdot, s)|_\infty a(s) f(u(s)) ds \\ &\geq \sigma \|Tu\|. \end{aligned}$$

Also, the standard arguments yield that λT is completely continuous.

Theorem 3.2 *Assume (H_1) , (H_2) , and (H_3) with $f_0 < f_\infty < \infty$. Assume there exists a value of λ such that*

$$\lambda f_0 \int_0^1 \|G(\cdot, s)\| a(s) ds < 1, \tag{10}$$

and

$$\lambda \sigma^2 f_\infty \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds > 1. \tag{11}$$

Then the BVP (1),(2) has a positive solution in the cone \mathcal{P}_σ .

Proof: For each $\lambda > 0$ satisfying both of the conditions (10) and (11), let $\epsilon(\lambda) > 0$ be sufficiently small such that

$$\lambda(f_0 + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \leq 1, \tag{12}$$

and

$$\lambda \sigma^2 (f_\infty - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds \geq 1. \tag{13}$$

Consider f_0 first. There exists $H_1(\epsilon) > 0$ such that $f(u) \leq (f_0 + \epsilon)u$, for all $0 < u \leq H_1$. Let

$$\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < H_1\}.$$

For all $u \in \partial\Omega_1 \cap \mathcal{P}_\sigma$, $0 \leq u(s) \leq \|u\|$, and

$$\begin{aligned} \|\lambda Tu\| &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s) f(u(s)) ds \\ &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s) (f_0 + \epsilon) u(s) ds \\ &\leq \lambda (f_0 + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \cdot \|u\|. \end{aligned}$$

Hence, (12) implies that

$$\|\lambda Tu\| \leq \|u\|.$$

On the other hand, consider f_∞ . There exists $\bar{H}_2(\epsilon) > 0$ such that $f(u) \geq (f_\infty - \epsilon)u$, for all $u \geq \bar{H}_2$. Let

$$\begin{aligned} H_2 &= \max\{2H_1, \frac{1}{\sigma}\bar{H}_2\}, \\ \Omega_2 &= \{u \in \mathcal{B} \mid \|u\| < H_2\}. \end{aligned}$$

For all $u \in \partial\Omega_2 \cap \mathcal{P}_\sigma$, $u(s) \geq \sigma\|u\|$, $\frac{1}{2} \leq s \leq 1 - \delta$, and

$$\begin{aligned} \|\lambda Tu\| &\geq \min_{t \in [\delta, 1-\delta]} \lambda Tu(t) \\ &\geq \int_0^1 \min_{t \in [\delta, 1-\delta]} (-1)^{n-k} G(t, s) a(s) f(u(s)) ds \\ &\geq \lambda \int_0^1 \sigma \|G(\cdot, s)\| a(s) f(u(s)) ds \\ &\geq \lambda \sigma \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) f(u(s)) ds \\ &\geq \lambda \sigma \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) (f_\infty - \epsilon) u(s) ds \\ &\geq \lambda \sigma \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) (f_\infty - \epsilon) \sigma \|u\| ds \\ &\geq \lambda \sigma^2 (f_\infty - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds \|u\|. \end{aligned}$$

Hence, (13) implies that

$$\|\lambda Tu\| \geq \|u\|.$$

Finally, we apply part (i) of Krasnosel'skii's Fixed Point Theorem and obtain a fixed point u_1 of λT in $\mathcal{P}_\sigma \cap (\overline{\Omega_2} \setminus \Omega_1)$. Note that for $\frac{1}{2} \leq t \leq 1 - \delta$,

$$u_1(t) \geq \sigma \|u_1\| \geq \sigma H_1 > 0.$$

Hence, u_1 is a nontrivial solution of (1),(2). Successive applications of Rolle's theorem imply that u_1 does not vanish on $(0, 1)$ and so u_1 is a positive solution.

This completes the proof.

Corollary 3.3 *Assume all the conditions for Theorem 3.2 hold. Then*

(i) *For $f_0 = 0$ and $f_\infty = \infty$ (superlinear), $Sp(a) = (0, \infty)$.*

(ii) *For $f_0 = 0$ and $f_\infty < \infty$, $((\sigma^2 f_\infty \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, \infty) \subseteq Sp(a)$.*

(iii) *For $f_0 > 0$ and $f_\infty = \infty$, $(0, (f_0 \int_0^1 \|G(\cdot, s)\| a(s) ds)^{-1}) \subseteq Sp(a)$.*

(iv) *For $0 < f_0 < f_\infty < \infty$,*

$((\sigma^2 f_\infty \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, (f_0 \int_0^1 \|G(\cdot, s)\| a(s) ds)^{-1}) \subseteq Sp(a)$.

Theorem 3.4 *Assume $(H_1), (H_2)$, and (H_3) with $f_\infty < f_0 < \infty$. Assume there exists a value of λ such that*

$$\lambda \sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds > 1. \quad (14)$$

In addition, if f is not bounded, assume also that

$$\lambda f_\infty \int_0^1 \|G(\cdot, s)\| a(s) ds < 1. \quad (15)$$

Then the BVP (1),(2) has a positive solution in the cone \mathcal{P}_σ .

Proof: For each $\lambda > 0$ satisfying the condition (14), let $\epsilon(\lambda) > 0$ be sufficiently small such that

$$\lambda \sigma^2 (f_0 - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds \geq 1. \quad (16)$$

Consider $f_0 \in \mathcal{R}^+$ first. There exists $H_1(\epsilon) > 0$ such that $f(u) \geq (f_0 - \epsilon)u$, for $0 < u \leq H_1$.

Let

$$\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < H_1\}.$$

For all $u \in \partial\Omega_1 \cap \mathcal{P}_\sigma$, $u(s) \geq \sigma\|u\|$, $\frac{1}{2} \leq s \leq 1 - \delta$, and so

$$\begin{aligned}
 \|\lambda Tu\| &\geq \min_{t \in [\delta, 1-\delta]} \lambda Tu(t) \\
 &\geq \lambda \int_0^1 \min_{t \in [\delta, 1-\delta]} (-1)^{n-k} G(t, s) a(s) f(u(s)) ds \\
 &\geq \lambda \int_0^1 \sigma \|G(\cdot, s)\| a(s) f(u(s)) ds \\
 &\geq \lambda \sigma \int_0^1 \|G(\cdot, s)\| a(s) (f_0 - \epsilon) u(s) ds \\
 &\geq \lambda \sigma (f_0 - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) u(s) ds \\
 &\geq \lambda \sigma (f_0 - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) \sigma \|u\| ds \\
 &\geq \lambda \sigma^2 (f_0 - \epsilon) \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds \|u\|.
 \end{aligned}$$

Hence, (16) implies that

$$\|\lambda Tu\| \geq \|u\|.$$

On the other hand, consider $f_\infty \in \mathcal{R}^+$. Given $f_0 > f_\infty$, there are two subcases for us to consider:

Case 1: f is bounded. Let $\lambda > 0$ satisfying condition (14) be given throughout this case. Let $N > 0$ be large enough so that

$$f(u) \leq N, \text{ for all } u \geq 0,$$

and

$$\lambda N \int_0^1 \|G(\cdot, s)\| a(s) ds > H_1.$$

Let

$$H_2 = \lambda N \int_0^1 \|G(\cdot, s)\| a(s) ds,$$

and

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}.$$

Then, for all $u \in \partial\Omega_2 \cap \mathcal{P}_\sigma$,

$$\begin{aligned}
 \|\lambda Tu\| &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s) f(u(s)) ds \\
 &\leq \lambda N \int_0^1 \|G(\cdot, s)\| a(s) ds \\
 &= \|u\|.
 \end{aligned}$$

Coupled with condition (14), we apply part (ii) of Krasnosel'skii's Fixed Point Theorem and obtain a fixed point of λT in $\mathcal{P}_\sigma \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Case 2: f is not bounded. Assume now that $\lambda > 0$ also satisfies the condition (15). Without loss of generality, we let the preceding ϵ also satisfy

$$\lambda(f_\infty + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \leq 1. \quad (17)$$

There exists $\bar{H}_2 > 0$ such that for all $u \geq \bar{H}_2$, $f(u) \leq (f_\infty + \epsilon)u$. Since f is continuous at $u = 0$, it is unbounded on $(0, \infty)$ as u approaches $+\infty$. Let

$$H_2 > \max\{2H_1, \bar{H}_2\}$$

be such that

$$f(u) \leq f(H_2)$$

for all $0 \leq u \leq H_2$. Let

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < H_2\}.$$

For all $u \in \partial\Omega_2 \cap \mathcal{P}_\sigma$, $0 \leq s \leq 1$,

$$\begin{aligned} f(u(s)) &\leq f(H_2) \\ &\leq (f_\infty + \epsilon)H_2, \end{aligned}$$

and so,

$$\begin{aligned} \|\lambda Tu\| &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s) f(u(s)) ds \\ &\leq \lambda \int_0^1 \|G(\cdot, s)\| a(s) (f_\infty + \epsilon) H_2 ds \\ &\leq \lambda (f_\infty + \epsilon) \int_0^1 \|G(\cdot, s)\| a(s) ds \cdot \|u\|. \end{aligned}$$

Hence, (17) implies that

$$\|\lambda Tu\| \leq \|u\|.$$

Finally, we apply part (ii) of Krasnosel'skii's Fixed Point Theorem and obtain a fixed point u_1 of λT in $\mathcal{P}_\sigma \cap \overline{\Omega_2} \setminus \Omega_1$.

By an argument similar to that in the proof of Theorem 3.2 there is a positive solution, u_1 , of (1), (2).

Corollary 3.5 (Case 1) Assume all the conditions for Theorem 3.4 hold and in addition that f is bounded. Then

(i) For $f_0 = 0$, $Sp(a) = (0, \infty)$.

(ii) For $f_0 < \infty$, $((\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, \infty) \subseteq Sp(a)$.

Corollary 3.6 (Case 2) Assume all the conditions for Theorem 3.4 hold. Then

(i) For $f_0 = \infty$ and $f_\infty = 0$ (Sublinear), $Sp(a) = (0, \infty)$.

(ii) For $f_0 = \infty$ and $f_\infty > 0$, $(0, (f_\infty \int_0^1 \|G(\cdot, s)\| a(s) ds)^{-1}) \subseteq Sp(a)$.

(iii) For $0 < f_0 < \infty$ and $f_\infty = 0$, $((\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, \infty) \subseteq Sp(a)$.

(iv) For $0 < f_\infty < f_0 < \infty$,

$((\sigma^2 f_0 \int_{\frac{1}{2}}^{1-\delta} \|G(\cdot, s)\| a(s) ds)^{-1}, (f_\infty \int_0^1 \|G(\cdot, s)\| a(s) ds)^{-1}) \subseteq Sp(a)$.

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