Asymptotic behaviour of positive solutions of the model which describes cell differentiation

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1. Introduction

In this paper we will study the asymptotic behaviour of positive solutions to the system

$$(1) \qquad \left| \begin{array}{l} x_1'(t) = \frac{A(t)}{1+x_2^n(t)} - x_1(t) \\ x_2'(t) = \frac{B(t)}{1+x_1^n(t)} - x_2(t), \end{array} \right.$$

where A and B belong to $\mathcal{C}_+$ and $\mathcal{C}_+$ is the set of continuous functions $g : \mathcal{R} \longrightarrow \mathcal{R}$, which are bounded above and below by positive constants. n is fixed natural number. The system (1) describes cell differentiation, more precisely - its passes from one regime of work to other without loss of genetic information. The variables x_1 and x_2 make sense of concentration of specific metabolits. The parameters A and B reflect degree of development of base metabolism. The parameter n reflects the highest row of the repression's reactions. For more details on the interpretation of (1) one may see [1]. With $\mathcal{C}_\circ$ we denote the space of continuous and bounded functions $g : \mathcal{R} \longrightarrow \mathcal{R}$. For $g \in \mathcal{C}_\circ$ we define

$$g_L(\infty) = \lim\inf_{t\longrightarrow\infty} g(t), g_M(\infty) = \lim\sup_{t\longrightarrow\infty} g(t),$$

$$g_L = \inf\{g(t) : t \in \mathcal{R}\}, g_M = \sup\{g(t) : t \in \mathcal{R}\}.$$

2. Preliminary results

Here and further next lemmas will pay important role.

Lemma 1.[2]*Let* $g : (\alpha, \infty) \longrightarrow \mathcal{R}$ *be a bounded and differentiable function. Then there exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *such that* $t_n \longrightarrow_{n \longrightarrow \infty} \infty$, $g'(t_n) \longrightarrow_{n \longrightarrow \infty} 0$, $g(t_n) \longrightarrow_{n \longrightarrow \infty} g_M(\infty)$ *(resp.* $g(t_n) \longrightarrow_{n \longrightarrow \infty} g_L(\infty)$*).*

Lemma 2.[2]*Let* $g \in \mathcal{C}_\circ$ *be a differentiable function. Then there exists a sequence* $\{t_n\}_{n=1}^{\infty}$ *such that* $g'(t_n) \longrightarrow_{n \longrightarrow \infty} 0$, $g(t_n) \longrightarrow_{n \longrightarrow \infty} g_M$ *(resp.* $g(t_n) \longrightarrow_{n \longrightarrow \infty} g_L$*).*

Proposition 1.*Let* (x_1, x_2) *be a positive solution of (1) and* $A(t), B(t) \in \mathcal{C}_+$*. Then*

$$\frac{A_L(\infty)}{1 + B_M^n(\infty)} \leq x_{1L}(\infty) \leq x_{1M}(\infty) \leq A_M(\infty),$$
$$\frac{B_L(\infty)}{1 + A_M^n(\infty)} \leq x_{2L}(\infty) \leq x_{2M}(\infty) \leq B_M(\infty).$$

Proof. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty} \subset \mathcal{R}$ for which $t_m \longrightarrow_{m \longrightarrow \infty} \infty$, $x_1'(t_m) \longrightarrow_{m \longrightarrow \infty} 0$, $x_1(t_m) \longrightarrow_{m \longrightarrow \infty} x_{1M}(\infty)$. Then from

$$x_1'(t_m) = \tfrac{A(t_m)}{1 + x_2^n(t_m)} - x_1(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m \longrightarrow \infty} \frac{A(t_m)}{1 + x_2^n(t_m)} - x_{1M}(\infty) \leq A_M(\infty) - x_{1M}(\infty),$$

i. e.

$$x_{1M}(\infty) \leq A_M(\infty).$$

Let now $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ such that $t_m \longrightarrow_{m \longrightarrow \infty} \infty$, $x_2'(t_m) \longrightarrow_{m \longrightarrow \infty} 0$, $x_2(t_m) \longrightarrow_{m \longrightarrow \infty} x_{2L}(\infty)$. From

$$x_2'(t_m) = \frac{B(t_m)}{1 + x_1^n(t_m)} - x_2(t_m),$$

as $m \longrightarrow \infty$, we find that

$$0 = \lim_{m \longrightarrow \infty} \frac{B(t_m)}{1 + x_1^n(t_m)} - x_{2L}(\infty) \geq \frac{B_L(\infty)}{1 + A_M^n(\infty)} - x_{2L}(\infty)$$

or

$$x_{2L}(\infty) \geq \frac{B_L(\infty)}{1 + A_M^n(\infty)}.$$

Let $\{t_m\}_{m=1}^{\infty} \subset \mathcal{R}$ is susch that $t_m \longrightarrow_{m \longrightarrow \infty} \infty$, $x_2'(t_m) \longrightarrow_{m \longrightarrow \infty} 0$, $x_2(t_m) \longrightarrow_{m \longrightarrow \infty} x_{2M}(\infty)$. From

$$x_2'(t_m) = \frac{B(t_m)}{1 + x_1^n(t_m)} - x_2(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m \longrightarrow \infty} \frac{B(t_m)}{1 + x_1^n(t_m)} - x_{2M}(\infty) \leq B_M(\infty) - x_{2M}(\infty).$$

Consequently

$$x_{2M}(\infty) \leq B_M(\infty).$$

Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ such that $t_m \longrightarrow_{m \longrightarrow \infty} \infty$, $x_1'(t_m) \longrightarrow_{m \longrightarrow \infty} 0$, $x_1(t_m) \longrightarrow_{m \longrightarrow \infty} x_{1L}(\infty)$. From equality

$$x_1'(t_m) = \frac{A(t_m)}{1 + x_2^n(t_m)} - x_1(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m \longrightarrow \infty} \frac{A(t_m)}{1 + x_2^n(t_m)} - x_{1L}(\infty) \geq \frac{A_L(\infty)}{1 + B_M^n(\infty)} - x_{1L}(\infty)$$

or

$$x_{1L}(\infty) \geq \frac{A_L(\infty)}{1 + B_M^n(\infty)}.$$

This completes the proof.

Remark. Proposition 1 shows that (1) is permanent, i. e. there exist positive constants α and β such that

$$0 < \alpha \leq \liminf_{t \longrightarrow \infty} x_i(t) \leq \limsup_{t \longrightarrow \infty} x_i(t) \leq \beta < \infty, \quad i = 1, 2,$$

where $(x_1(t), x_2(t))$ is a positive solution of (1). In [3] was proved that permanence implies existence of positive periodic solutions of (1), when $A(t)$ and $B(t)$ are continuous positive periodic functions.

Let X_1 be a positive solution of the equation

$$x'(t) = A(t) - x(t),$$

and X_2 be a positive solution of the equation

$$x'(t) = B(t) - x(t).$$

Proposition 2. *Let X_1, X_2 be as above and $A(t), B(t) \in \mathcal{C}_+$. Then*

$$A_L(\infty) \leq X_{1L}(\infty) \leq X_{1M}(\infty) \leq A_M(\infty),$$

$$B_L(\infty) \leq X_{2L}(\infty) \leq X_{2M}(\infty) \leq B_M(\infty).$$

Proof. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty}$ of $\mathcal{R}$ for which $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $X_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $X_1(t_m) \longrightarrow_{m\longrightarrow\infty} X_{1L}(\infty)$. Then from

$$X_1'(t_m) = A(t_m) - X_1(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m\longrightarrow\infty} A(t_m) - X_{1L}(\infty) \geq A_L(\infty) - X_{1L}(\infty),$$

i. e.

$$X_{1L}(\infty) \geq A_L(\infty).$$

Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $X_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $X_1(t_m) \longrightarrow_{m\longrightarrow\infty} X_{1M}(\infty)$. From

$$X_1'(t_m) = A(t_m) - X_1(t_m),$$

as $m \longrightarrow \infty$, we find that

$$0 = \lim_{m\longrightarrow\infty} A(t_m) - X_{1M}(\infty) \leq A_M(\infty) - X_{1M}(\infty)$$

or

$$X_{1M}(\infty) \leq A_M(\infty).$$

In the same way we may prove other pair of inequalities.

3.Asymptotic behaviour of positive solutions

The results which are formulated and proved below are connected to (1) and to

$$(1_*) \qquad \left| \begin{array}{l} x_1'(t) = \frac{A_*(t)}{1+x_2^n(t)} - x_1(t) \\ x_2'(t) = \frac{B_*(t)}{1+x_1^n(t)} - x_2(t), \end{array} \right.$$

where $A_*, B_* \in \mathcal{C}_+$ and $A(t) - A_*(t) \longrightarrow_{t\longrightarrow\infty} 0$, $B(t) - B_*(t) \longrightarrow_{t\longrightarrow\infty} 0$.

We notice that every solution to (1)(resp. (1_*)) with positive initial data $x(t_\circ) = (x_1(t_\circ), x_2(t_\circ)) > 0$ ($x_*(t_\circ) = (x_{1*}(t_\circ), x_{2*}(t_\circ)) > 0$) is defined and positive in $[t_\circ, \infty)$.

Theorem 1. *Let* $A, B, A_*, B_* \in \mathcal{C}_+$ *and*

$$A(t) - A_*(t) \longrightarrow_{t\longrightarrow\infty} 0, B(t) - B_*(t) \longrightarrow_{t\longrightarrow\infty} 0.$$

Let also

$$\frac{n^2 A_M^n(\infty) B_M^n(\infty)(1 + A_M^n(\infty))^{2n}(1 + B_M^n(\infty))^{2n}}{[A_L^n(\infty) + (1 + B_M^n(\infty))^n]^2 [B_L^n(\infty) + (1 + A_M^n(\infty))^n]^2} < 1.$$

If $(x_1(t), x_2(t))$ *and* $(x_{1*}(t), x_{2*}(t))$ *are positive solutions respectively of (1) and* (1_*), *then* $(x_1(t) - x_{1*}(t), x_2(t) - x_{2*}(t)) \longrightarrow_{t\longrightarrow\infty} (0, 0)$.

Proof. Let $h_1(t) = x_1(t) - x_{1*}(t)$, $h_2(t) = x_2(t) - x_{2*}(t)$. We have

$$h_1'(t) = x_1'(t) - x_{1*}'(t) =$$

$$= \frac{A(t)}{1 + x_2^n(t)} - x_1(t) - \frac{A_*(t)}{1 + x_{2*}^n(t)} + x_{1*}(t) =$$

$$= -A(t)\frac{(x_2(t) - x_{2*}(t))(x_2^{n-1}(t) + x_2^{n-2}(t)x_{2*}(t) + \cdots + x_{2*}^{n-1}(t))}{(1 + x_2^n(t))(1 + x_{2*}^n(t))} -$$

$$-h_1(t) + \frac{A(t) - A_*(t)}{(1 + x_{2*}^n(t))}.$$

Let

$$\alpha(t) = A(t)\frac{x_2^{n-1}(t) + x_2^{n-2}(t)x_{2*}(t) + \cdots + x_{2*}^{n-1}(t)}{(1 + x_2^n(t))(1 + x_{2*}^n(t))}, \quad \beta(t) = \frac{A(t) - A_*(t)}{(1 + x_{2*}^n(t))}.$$

We notice that $\beta(t) \longrightarrow_{t\longrightarrow\infty} 0$. For $h_1(t)$ we get the equation

$$h_1'(t) = -h_1(t) - \alpha(t)h_2(t) + \beta(t).$$

On the other hand

$$h_2'(t) = x_2'(t) - x_{2*}'(t) =$$

$$= \frac{B(t)}{1 + x_1^n(t)} - x_2(t) - \frac{B_*(t)}{1 + x_{1*}^n(t)} + x_{2*}(t) =$$

$$= -B(t)\frac{(x_1(t) - x_{1*}(t))(x_1^{n-1}(t) + x_1^{n-2}(t)x_{1*}(t) + \cdots + x_{1*}^{n-1}(t))}{(1 + x_1^n(t))(1 + x_{1*}^n(t))} -$$

$$-h_2(t) + \frac{B(t) - B_*(t)}{(1 + x_{1*}^n(t))}.$$

Let

$$\gamma(t) = B(t)\frac{x_1^{n-1}(t) + x_1^{n-2}(t)x_{1*}(t) + \cdots + x_{1*}^{n-1}(t)}{(1 + x_1^n(t))(1 + x_{1*}^n(t))}, \quad \delta(t) = \frac{B(t) - B_*(t)}{(1 + x_{1*}^n(t))},$$

$\delta(t) \longrightarrow_{t\longrightarrow\infty} 0$. Then

$$h_2'(t) = -\gamma(t)h_1(t) - h_2(t) + \delta(t).$$

For $h_1(t)$ and $h_2(t)$ we find the system

$$\left| \begin{array}{l} h_1'(t) = -h_1(t) - \alpha(t)h_2(t) + \beta(t) \\ h_2'(t) = -\gamma(t)h_1(t) - h_2(t) + \delta(t). \end{array} \right.$$

Let $h(t) = (h_1(t), h_2(t))$ and $|h|(t) = |h(t)|$. We assume that $|h_1|_M(\infty) > 0$. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty}$ of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $h_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $|h_1|(t_m) \longrightarrow_{m\longrightarrow\infty} |h_1|_M(\infty)$. From

$$|h_1'(t_m)| = |-h_1(t_m) - \alpha(t_m)h_2(t_m) + \beta(t_m)|,$$

as $m \longrightarrow \infty$, we have

$$0 \geq |h_1|_M(\infty) - \alpha_M(\infty)|h_2|_M(\infty),$$

i. e.

$$|h_1|_M(\infty) \leq \alpha_M(\infty)|h_2|_M(\infty). \tag{2}$$

Since $|h_1|_M(\infty) > 0$ then $|h_2|_M(\infty) > 0$. Let now $\{t_m\}_{m=1}^{\infty} \subset \mathcal{R}$ is such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $h_2'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $|h_2|(t_m) \longrightarrow_{m\longrightarrow\infty} |h_2|_M(\infty)$. As $m \longrightarrow \infty$, from

$$|h_2'(t_m)| = |-\gamma(t_m)h_1(t_m) - h_2(t_m) + \delta(t_m)|,$$

we get

$$0 \geq |h_2|_M(\infty) - \gamma_M(\infty)|h_1|_M(\infty)$$

or

$$|h_2|_M(\infty) \leq \gamma_M(\infty)|h_1|_M(\infty).$$

From last inequality and (2) we find that

$$|h_1|_M(\infty)|h_2|_M(\infty) \leq \alpha_M(\infty)\gamma_M(\infty)|h_1|_M(\infty)|h_2|_M(\infty),$$

from where

$$1 \leq \alpha_M(\infty)\gamma_M(\infty).$$

Since

$$\alpha_M(\infty) = \left(A(t)\frac{x_2^{n-1}(t) + x_2^{n-2}(t)x_{2*}(t) + \cdots + x_{2*}^{n-1}(t)}{(1 + x_2^n(t))(1 + x_{2*}^n(t))} \right)_M (\infty) \leq$$

$$\leq A_M(\infty).\frac{n.B_M^{n-1}(\infty)}{\left(1 + \frac{B_L^n(\infty)}{(1+A_M^n(\infty))^n}\right)^2} = \frac{n.A_M(\infty).B_M^{n-1}(\infty)(1 + A_M^n(\infty))^{2n}}{[(1 + A_M^n(\infty))^n + B_L^n(\infty)]^2},$$

$$\gamma_M(\infty) = \left(B(t) \frac{x_1^{n-1}(t) + x_1^{n-2}(t) x_{1*}(t) + \cdots + x_{1*}^{n-1}(t)}{(1 + x_1^n(t))(1 + x_{1*}^n(t))} \right)_M (\infty) \leq$$

$$\leq B_M(\infty). \frac{n.A_M^{n-1}(\infty)}{\left(1 + \frac{A_L^n(\infty)}{(1+B_M^n(\infty))^n}\right)^2} = \frac{n.B_M(\infty).A_M^{n-1}(\infty)(1 + B_M^n(\infty))^{2n}}{[(1 + B_M^n(\infty))^n + A_L^n(\infty)]^2}.$$

Therefore we get the contradiction

$$1 \leq \frac{n^2.A_M^n(\infty).B_M^n(\infty)(1 + A_M^n(\infty))^{2n}(1 + B_M^n(\infty))^{2n}}{[(1 + B_M^n(\infty))^n + A_L^n(\infty)]^2.[(1 + A_M^n(\infty))^n + B_L^n(\infty)]^2}.$$

The proof is complete.

Let

$$r_1 = \frac{A_M^2(\infty)}{A_L^2(\infty)}, \quad r_2 = \frac{B_M^2(\infty)}{B_L^2(\infty)},$$

$$p_1 = \frac{1}{r_1} \frac{1}{1 + B_M^n(\infty) r_2^n}, \quad p_2 = \frac{1}{r_2} \frac{1}{1 + A_M^n(\infty) r_1^n}.$$

Theorem 2. *Let $A, B, A_*, B_* \in \mathcal{C}_+$, $A(t) - A_*(t) \longrightarrow_{t \longrightarrow \infty} 0$, $B(t) - B_*(t) \longrightarrow_{t \longrightarrow \infty} 0$. If $(x_1(t), x_2(t))$ and $(x_{1*}(t), x_{2*}(t))$ are positive solutions respectively to (1) and (1_*) and*

$$\frac{n^2 r_1^n r_2^n A_M^n(\infty) B_M^n(\infty)}{(1 + A_L^n(\infty) p_1^n)^2 (1 + B_L^n(\infty) p_2^n)^2} < 1,$$

then $(x_1(t) - x_{1*}(t), x_2(t) - x_{2*}(t)) \longrightarrow_{t \longrightarrow \infty} (0, 0)$.

Proof. Let $x = \frac{x_1}{X_1}$, $y = \frac{x_2}{X_2}$, where X_1 and X_2 as in proposition 2. Then

$$x'(t) = \frac{1}{X_1(t)}.x_1'(t) - \frac{x_1(t)}{X_1^2(t)}.X_1'(t) =$$

$$= \frac{1}{X_1(t)}.\left[\frac{A(t)}{1 + x_2^n(t)} - x_1(t) \right] - \frac{x_1(t)}{X_1^2(t)} [A(t) - X_1(t)] =$$

$$= \frac{1}{X_1(t)}.\frac{A(t)}{1 + X_2^n(t) y^n(t)} - \frac{A(t)}{X_1(t)}.\frac{x_1(t)}{X_1(t)} = -\frac{A(t)}{X_1(t)}.x(t) + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]},$$

i. e.

$$x'(t) = -\frac{A(t)}{X_1(t)}.x(t) + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]}.$$

$$y'(t) = \frac{1}{X_2(t)}.x_2'(t) - \frac{x_2(t)}{X_2^2(t)}.X_2'(t) =$$

$$= \frac{1}{X_2(t)}.\left[\frac{B(t)}{1 + x_1^n(t)} - x_2(t) \right] - \frac{x_2(t)}{X_2^2(t)} [B(t) - X_2(t)] =$$