FLOQUET THEORY FOR LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC SOLUTIONS

R. WEIKARD

ABSTRACT. If A is a ω-periodic matrix Floquet's theorem states that the differential equation $y' = Ay$ has a fundamental matrix $P(x)\exp(Jx)$ where J is constant and P is ω-periodic, i.e., $P(x) = P^*(\mathrm{e}^{2\pi ix/\omega})$. We prove here that P^* is rational if A is bounded at the ends of the period strip and if all solutions of $y' = Ay$ are meromorphic.

This version of Floquet's theorem is important in the study of certain integrable systems.

In the early 1880s Floquet established his celebrated theorem on the structure of solutions of periodic differential equations (see [4] and [5]). It is interesting to note that modern day versions of the theorem consider the case where the independent variable is real and the coefficients are, say, piecewise continuous (cf. Magnus and Winkler [13] or Eastham [3]) while in Floquet's original work, due to the influence of Fuchs [6], the independent variable is complex, the coefficients are analytic save for isolated points[1], and the general solution is explicitly required to be single-valued (it will then also be analytic except at isolated singularities.)

It is also interesting to realize that Floquet's theorem comes shortly after Hermite had established an analogue theorem for Lamé's equation (see [12]): for every value of z the solutions of $y'' - n(n+1)\wp(x)y = zy$ are doubly periodic of the second kind[2]. The proof of this theorem relied on the fact that, because of the particular coefficient $-n(n+1)$ in Lamé's equation, the general solution is single-valued. Shortly after this Picard extended this finding to other equations with doubly periodic coefficients and single-valued general solutions ([16], [17], [18]). It appears, however, that Floquet's work is independent of Hermite and Picard.

Another relative of Floquet's original theorem was discovered at about the same time by Halphen [11]: If the coefficients of a linear homogeneous differential expression are rational functions which are bounded at infinity, if the leading coefficient is one, and if the general solution is meromorphic, then there is a fundamental system of solutions whose elements are of the form $R(x)\exp(\lambda x)$ where R is a rational function and λ a certain complex number.

Date: December 6, 2000.

2000 *Mathematics Subject Classification.* 34M05, 34M15, 34C25.

Key words and phrases. Floquet theory.

Research supported in part by the US National Science Foundation under Grant No. DMS-9970299.

[1]Floquet asks for 'coefficients uniformes', i.e., single-valued coefficients. Since the solutions are analytic the coefficients must be, too. Since they are single-valued, branch points are excluded and hence the only possible singularities are isolated singularities.

[2]i.e., they satisfy $y(x + 2\omega) = y(x + 2\omega') = y(x)$ when ω and ω are fundamental periods of $\wp$

Halphen's theorem exhibits two differences when compared to Floquet's theorem: it requires the general solution to be meromorphic instead of only single-valued and it puts a condition on the behavior of the coefficients at infinity. Without either of these conditions the conclusion would be wrong as the examples $y'' + xy = 0$ and $y' + y/x^2 = 0$ show. In [7] Halphen's theorem is extended to the case of a first order system.

The goal of the present article is to present a version of Floquet's theorem analogous to Halphen's. It asks for the additional hypotheses that the matrix of coefficients is bounded at the end of the period strip and that it is such that all solutions are meromorphic. In return it gives more detailed information on the solution: there is a fundamental matrix $P(x)\exp(Jx)$ where J is constant and P is, in addition to being periodic, in fact a *rational* function of $\mathrm{e}^{2\pi ix/\omega}$ (see Theorem 1 below for a precise statement).

The interest in linear differential equations with meromorphic solutions has experienced a revival in recent years due to their connection with certain integrable systems, for example the KdV equation. The systems in question can be described using Lax pairs (P, L) of linear differential operators as $L_t = [P, L]$ (where the brackets denote a commutator). Please see, for instance, Novikov et al. [15] or Belokolos et al. [1] for more details and extensive discussions of applications.

In [8] Gesztesy and myself used Picard's theorem to prove that an elliptic function q is a stationary solution of some equation in the KdV hierarchy[3], if and only if for all $z \in \mathbb{C}$ all solutions of $Ly = y'' + qy = zy$ are meromorphic. In [9] we extended this result to the AKNS system. For a survey on these matters the reader may consult [10]. The corresponding results for rational and simply periodic (but meromorphic) coefficients have been obtained in [20] for the KdV hierarchy and in [21] for the Gelfand-Dikii hierarchy (for which L is a scalar ordinary differential expression of arbitrary order). A further generalization to expressions with matrix coefficients (which would cover, e.g., the AKNS system) is presented in [22]. It requires an extension of Halphen's theorem (as in [7]) and the algebraic Floquet theorem presented here to first order systems. It is a curious fact, that the periodic case appears to be a lot simpler than the rational case.

Before we begin let us remember a few basic facts from the theory of periodic functions (for more information see, e.g., Markushevich [14], Chapter III.4). For any 2π-periodic function f on $\mathbb{C}$ we define the function f^* on $\mathbb{C} - \{0\}$ by

$$f^*(t) = f(-i\log(t)) \text{ or } f^*(\mathrm{e}^{ix}) = f(x).$$

We will use subsequently the symbol $f^*(t)$ to refer to the function $f(\frac{\omega}{2\pi i}\log(t))$ whenever f is an ω-periodic function. Conversely, if a function f^* is given $f(x)$ will refer to $f^*(\mathrm{e}^{2\pi ix/\omega})$.

For any ω-periodic function f and any real number a the set $\{x \in \mathbb{C} : a \leq \Re(x/\omega) < a + 1\}$ is called a period strip of f. Of course a period strip of f is a fundamental domain for f.

[3]Recall that the famous soliton solutions of the KdV equation are stationary solutions of (higher order) equations in the KdV hierarchy. The solutions in question were therefore called elliptic solitons by Verdier [19].

If f is meromorphic on $\mathbb{C}$ then f^* is meromorphic on $\mathbb{C}-\{0\}$. If f has a definite limit (perhaps infinity) at the upper end of the period strip (i.e., as $\Im(x/\omega)$ tends to infinity) as well as at the lower end of the period strip (i.e., as $-\Im(x/\omega)$ tends to infinity), then it is of the form

$$f(x) = \frac{a_0 + a_1 \mathrm{e}^{2\pi ix/\omega} + ... + a_n \mathrm{e}^{2\pi inx/\omega}}{b_0 + b_1 \mathrm{e}^{2\pi ix/\omega} + ... + b_m \mathrm{e}^{2\pi imx/\omega}}, \tag{1}$$

i.e., f^* is a rational function. In particular, in this case f has only finitely many poles in any period strip. Note that, if f is a doubly periodic function one of whose periods is ω then f does not have finitely many poles in the strip $\{x \in \mathbb{C} : a \leq \Re(x/\omega) < a+1\}$ and hence does not have definite limits at the ends of this strip.

If f is bounded at the ends of the period strip, then f^* is bounded at zero and infinity and thus has finitely many poles. Hence zero and infinity are removable singularities of f^* so that f has definite limits at the end of the period strip. In this case we may choose $n = m$ in (1) and assume that b_0 and b_m are different from zero.

Definition 1. Let $\mathbb{P}_\omega$ denote the field of meromorphic functions with period ω. Two matrices $A, B \in \mathbb{P}_\omega^{n\times n}$ are called of the same kind (with respect to $\mathbb{P}_\omega$) if there exists an invertible matrix $T \in \mathbb{P}_\omega^{n\times n}$ such that

$$B = T^{-1}(AT - T').$$

The relation "of the same kind" is obviously an equivalence relation.

Theorem 1. *Suppose that A is an $n \times n$-matrix whose entries are meromorphic, ω-periodic functions which are bounded at the ends of the period strip. If the first-order system $y' = Ay$ has only meromorphic solutions, then there exists a constant $n \times n$-matrix J in Jordan normal form and an $n \times n$-matrix R^* whose entries are rational functions over $\mathbb{C}$ such that the following statements hold:*

(1) *The eigenvalues of $A^*(0)$ and J are the same modulo $i\mathbb{Z}$ if multiplicities are properly taken into account. More precisely, suppose that there are nonnegative integers ν_1, ..., ν_{r-1} such that λ, $\lambda + i\nu_1$, ..., $\lambda + i\nu_{r-1}$ are all the eigenvalues of $A^*(0)$ which are equal to λ modulo $i\mathbb{Z}$. Then λ is an eigenvalue of J with algebraic multiplicity r.*

(2) *The equation $y' = Ay$ has a fundamental matrix Y given by*

$$Y(x) = R^*(\mathrm{e}^{2\pi ix/\omega}) \exp(2\pi Jx/\omega).$$

In particular every entry of Y has the form $f(\mathrm{e}^{2\pi ix/\omega}, x)\mathrm{e}^{\lambda x}$, where $\lambda + i\nu$ is an eigenvalue of $A^*(0)$ for some nonnegative integer ν and where f is a rational function in its first argument and a polynomial in its second argument.

Conversely, suppose R^* is an invertible $n \times n$-matrix whose entries are rational functions and that J is a constant $n \times n$-matrix. Then

$$Y(x) = R^*(\mathrm{e}^{2\pi ix/\omega}) \exp(2\pi Jx/\omega)$$

is a fundamental matrix of a system of first order linear differential equations $y' = Ay$ where A is in $\mathbb{P}_\omega^{n\times n}$ and is of the same kind as a matrix whose entries are bounded at both ends of the period strip.

Proof. For simplicity we assume in the proof that $\omega = 2\pi$. Perform the substitution $y(x) = w(\mathrm{e}^{ix})$ and $t = \mathrm{e}^{ix}$ to obtain the equation

$$tw'(t) = -iA^*(t)w(t).$$

Since, by assumption, A^* is analytic at zero, $t = 0$ is a simple singularity and hence a regular singular point of the differential equation $tw' = -iA^*w$. This implies that there exists a fundamental matrix $W(t) = P^*(t)t^T = P^*(t)\exp(T\log t)$ where P^* is analytic at zero and T is a constant matrix whose eigenvalues do not differ by nonzero integers for which we may assume without loss of generality that it is in Jordan normal form (see Coddington and Levinson [2], Theorem IV.4.2). The multi-valued function $W(t)$ represents a single-valued fundamental matrix $W(\mathrm{e}^{ix})$ of $y' = Ay$. This matrix, initially only defined in a half-plane of points with suitably large imaginary parts, may be analytically continued to a meromorphic function on the entire plane, which in turn means that P^* has been continued from a neighborhood of zero to a meromorphic function on the plane.

Next we study the lower end of the period strip by introducing the substitution $y = v(\mathrm{e}^{-ix})$ and $s = \mathrm{e}^{-ix}$ which leads to the equation $sv'(s) = iA^*(1/s)v(s)$ which again has a simple singularity at zero. Repeating the above argument one obtains a fundamental matrix $V(s) = Q^*(1/s)s^S = Q^*(1/s)\exp(S\log s)$ where Q^* is analytic at infinity and S is a constant matrix in Jordan normal form. Again $V(\mathrm{e}^{-ix})$ is a fundamental matrix of $y' = Ay$ and hence there exists a constant matrix C such that $W(\mathrm{e}^{ix}) = V(\mathrm{e}^{-ix})C$ or, equivalently,

$$Q^*(t)^{-1}P^*(t) = \exp(-S\log t)C\exp(-T\log t).$$

Since P^* and Q^* are single-valued near infinity the same must be true for the function

$$F(t) = \exp(-S\log t)C\exp(-T\log t).$$

Since $\exp(\mu\log t) = t^\mu$ the function F is a polynomial[4] in the variables $\log t$, t^{μ_1}, ..., t^{μ_N}, where the μ_j are eigenvalues of $-S$ and $-T$. Therefore, in order for F to have an isolated singularity at infinity it must not depend on $\log t$ or on any of the powers t^{μ_j} for which μ_j is not an integer. In other words F is a polynomial in those powers t^{μ_j} for which the μ_j are integers. Hence F has at most a pole at infinity and so does $P^*(t) = Q^*(t)F(t)$, i.e., P^* is a rational function.

It remains to find the relationship between the eigenvalues of T and those of $A^*(0)$. Suppose that the characteristic polynomial of $A^*(0)$ is given by

$$\prod_{j=1}^{s}\prod_{k=1}^{r_j}(\lambda - (\lambda_j + i\nu_{j,k}))$$

[4]If the $n \times n$-matrix J is a Jordan block (i.e., all superdiagonal entries are equal to one) with eigenvalue λ, then

$$\exp(Jz) = \begin{pmatrix} 1 & z & \frac{z^2}{2} & \cdots\cdots & \frac{z^{n-1}}{(n-1)!} \\ 0 & 1 & z & \cdots\cdots & \frac{z^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots\cdots & 1 \end{pmatrix} \exp(\lambda z).$$

where the λ_j are pairwise distinct modulo $i\mathbb{Z}$ and where the $\nu_{j,k}$ are nonnegative integers. The lemma preceding Theorem IV.4.2 in [2] shows that the characteristic polynomial of T is then given by

$$\prod_{j=1}^{s}(\lambda + i\lambda_j)^{r_j}.$$

The first part of the theorem follows now upon letting $J = iT$ and $R^* = P^*$.

To prove the converse note that $Y(x) = R(x)\exp(Jx)$ satisfies $Y' = AY$ where

$$A = RJR^{-1} + R'R^{-1}.$$

Choose $T = R^{-1}$. Then $T' = -R^{-1}R'R^{-1}$ and hence

$$A = T^{-1}(JT - T'),$$

i.e., A is of the same kind as the constant matrix J. □

Acknowledgment: It is a pleasure to thank Jeremy Gray for discussions on the history of the subject.

References

[1] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its & V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[2] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, Krieger, Malabar, 1985.

[3] M. S. P. Eastham, *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh and London, 1973.

[4] G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques, C. R. Acad. Sci. Paris **91** (1880), 880–882.

[5] ———, Sur les équations différentielles linéaires à coefficients périodiques, Ann. Sci. École Norm. Sup. **12** (1883), 47–88.

[6] J. L. Fuchs, Zur Theorie der linearen Differentialgleichungen mit veränderlichen Koeffizienten, J. Reine Angew. Math. **66** (1866), 121–160.

[7] F. Gesztesy, K. Unterkofler, and R. Weikard, On a theorem of Halphen and its application to integrable systems, J. Math. Anal. Appl. **251** (2000), 504–526.

[8] F. Gesztesy and R. Weikard, Picard potentials and Hill's equation on a torus, Acta Math. **176** (1996), 73–107.

[9] ———, A characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy, Acta Math. **181** (1998), 63–108.

[10] ———, Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—an analytic approach, Bull. Amer. Math. Soc. (N.S.) **35** (1998), 271–317.

[11] G.-H. Halphen, Sur une nouvelle classe d'équations différentielles linéaires intégrables, C. R. Acad. Sci. Paris **101** (1885), 1238–1240.

[12] C. Hermite, Sur quelques applications des fonctions elliptiques, C. R. Acad. Sci. Paris **85** (1877), 689–695, 728–732, 821–826.

[13] W. Magnus and S. Winkler, *Hill's Equation*, Dover, New York, 1979.

[14] A. I. Markushevich, *Theory of Functions in a Complex Variable* (three volumes in one), Chelsea 1965.

[15] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York, 1984.

[16] E. Picard, Sur une généralisation des fonctions périodiques et sur certaines équations différentielles linéaires, C. R. Acad. Sci. Paris **89** (1879), 140–144.

[17] ———, Sur une classe d'équations différentielles linéaires, C. R. Acad. Sci. Paris **90** (1880), 128–131.

[18] ______, Sur les équations différentielles linéaires à coefficients doublement périodiques, J. reine angew. Math. **90** (1881), 281–302.
[19] J.-L. Verdier, New elliptic solitons, Algebraic Analysis (ed. by M. Kashiwara and T. Kawai), Academic Press, Boston, 1988, 901–910.
[20] R. Weikard, On rational and periodic solutions of stationary KdV equations, Doc. Math. J.DMV **4** (1999), 109-126.
[21] ______, On Commuting Differential Operators, Electron. J. Differential Equations 2000 (2000), No. 19, 1-11.
[22] ______, On Commuting Matrix Differential Operators, preprint.

Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35294-1170, USA
E-mail address: rudi@math.uab.edu