Radial symmetric solutions of the Cahn-Hilliard equation with degenerate mobility

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Abstract. In this paper we study the radial symmetric solutions of the two-dimensional Cahn-Hilliard equation with degenerate mobility. We adopt the method of parabolic regularization. After establishing some necessary uniform estimates on the approximate solutions, we prove the existence and the nonnegativity of weak solutions.
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1 Introduction

This paper is devoted to the radial symmetric solutions of the Cahn-Hilliard equation with degenerate mobility

$$\frac{\partial u}{\partial t} + \operatorname{div}[m(u)(k\nabla\Delta u - \nabla A(u))] = 0,$$

with the boundary value conditions

$$\frac{\partial u}{\partial n}\Big|_{\partial B} = \vec{J} \cdot \vec{n}\,\Big|_{\partial B} = 0,$$

and initial value condition

$$u\Big|_{t=0} = u_0(x),$$

where B is the unit ball in $\mathbb{R}^2$, $\vec{n}$ is the outward unit normal to ∂B, $k > 0$,

$$\vec{J} = m(u)(k\nabla\Delta u - \nabla A(u)),$$

and $m(s), A(s)$ are appropriately smooth and satisfy the following structure conditions

$$(H_1) \qquad 0 \le m(s) \le C_1|s|^p,$$

$$(H_2) \qquad H(s) = \int_0^s A(r)dr \ge -\mu, \quad |A'(s)| \le C_2|s|^q + C_3,$$

for some positive constants p, q, μ, $C's$. We note that a reasonable choice of $A(s)$ is the cubic polynomial, namely

$$A(s) = \gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4, \quad \gamma_1 > 0,$$

which corresponds to the so called double-well potential

$$H(s) = \frac{1}{4}\gamma_1 s^4 + \frac{1}{3}\gamma_2 s^3 + \frac{1}{2}\gamma_3 s^2 + \gamma_4 s.$$

The Cahn-Hilliard equation was introduced to study several diffusion processes, such as phase separation in binary alloys, see [1, 2]. During the past years, such an equation has been paid extensive attention. In particular, there are vast literatures on the investigation of the Cahn-Hilliard equation with constant mobility, for an overview we refer to [3, 4]. However, there are only a few works devoted to the equation with degenerate mobility, see [5, 6, 7, 8, 9, 10, 11, 12], among which Elliott and Garcke [6] was the first who established the basic existence results of weak solutions for space dimensions large than one.

In this paper, we study the radial symmetric solutions of the Cahn-Hilliard equation. We will study the problem in two-dimensional case, which has particular physical derivation of modeling the oil film spreading over a solid surface, see [13]. After introducing the radial variable $r = |x|$, we see that the radial symmetric solution satisfies

$$\frac{\partial(ru)}{\partial t} + \frac{\partial}{\partial r}\left\{rm(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\right\} = 0, \quad rV = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{1.1}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=0} = \left.\frac{\partial u}{\partial r}\right|_{r=1} = \left.\widetilde{J}\right|_{r=0} = \left.\widetilde{J}\right|_{r=1} = 0, \tag{1.2}$$

$$\left.u\right|_{t=0} = u_0(r), \tag{1.3}$$

where

$$\widetilde{J} = m(u)\left(k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right).$$

It should be noticed that the equation (1.1) is degenerate at the points where $r = 0$ or $u = 0$, and hence the arguments for one-dimensional problem can not be applied directly. Because of the degeneracy, the problem does not admit classical solutions in general. So, we introduce the weak solutions in the following sense

Definition A function u is said to be a weak solution of the problem (1.1)–(1.3), if the following conditions are fulfilled:

(1) $ru(r,t)$ is continuous in $\overline{Q}_T$, where $Q_T = (0,1) \times (0,T)$;

(2) $\sqrt{rm(u)}u_{rrr} \in L^2(P)$, where $P = \overline{Q}_T \backslash (\{u=0\} \cup \{t=0\} \cup \{r=0\})$;

(3) For any $\varphi \in C^1(\overline{Q}_T)$, the following integral equality holds

$$-\int_0^1 ru(r,T)\varphi(r,T)dr + \int_0^1 ru_0(r)\varphi(r,0)dr + \iint_{Q_T} ru\frac{\partial\varphi}{\partial t}drdt$$
$$+ \iint_P rm(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\frac{\partial\varphi}{\partial r}drdt = 0;$$

(4) u satisfies the lateral boundary value condition (1.2) at the points where $u \neq 0$.

We first investigate the existence of weak solutions. Because of the degeneracy, we will first consider the regularized problem. Based on the uniform estimates for the approximate solutions, we obtain the existence. Owing to the background, we are much interested in the nonnegativity of the weak solutions. For this purpose, we construct a suitable test function and discuss such a property under some conditions on the data. This paper is arranged as follows. We first study the regularized problem in Section 2, and then establish the existence in Section 3. Subsequently, we discuss the nonnegativity of weak solutions in the last Section.

2 Regularized problem

To discuss the existence, we adopt the method of parabolic regularization, namely, the desired solution will be obtained as the limit of some subsequence of solutions of the following regularized problem

$$\frac{\partial(r_\varepsilon u)}{\partial t}+\frac{\partial}{\partial r}\left\{r_\varepsilon m_\varepsilon(u)\left[k\frac{\partial V}{\partial r}-A'(u)\frac{\partial u}{\partial r}\right]\right\}=0,\quad r_\varepsilon V=\frac{\partial}{\partial r}\left(r_\varepsilon\frac{\partial u}{\partial r}\right), \tag{2.1}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=0}=\left.\frac{\partial u}{\partial r}\right|_{r=1}=\left.\widetilde{J}_\varepsilon\right|_{r=0}=\left.\widetilde{J}_\varepsilon\right|_{r=1}=0, \tag{2.2}$$

$$\left.u\right|_{t=0}=u_0(r), \tag{2.3}$$

where $r_\varepsilon=r+\varepsilon$, $m_\varepsilon(s)=m(s)+\varepsilon$ and

$$\widetilde{J}_\varepsilon=m_\varepsilon(u)\left[k\frac{\partial V}{\partial r}-A(u)\frac{\partial u}{\partial r}\right].$$

Theorem 2.1 For each fixed $\varepsilon>0$ and suitably smooth u_0, under the assumptions (H_1), (H_2), the problem (2.1)–(2.3) admits a unique classical solution u in the space $C^{4+\alpha,1+\alpha/4}(\overline{Q}_T)$ for some $\alpha\in(0,1)$.

To prove the theorem, we need some a priori estimates on the solutions. We first have

Lemma 2.1 For any $\alpha\in(0,\frac{1}{2}]$ and $\beta<\alpha$, there is a constant M independent of ε such that

$$|r_\varepsilon^\alpha u(r,t)-s_\varepsilon^\alpha u(s,t)|\le M|r-s|^\beta$$

and

$$|r_\varepsilon u(r,t)-s_\varepsilon u(s,t)|\le M|r-s|^\beta$$

for all $r,s\in(0,1)$, where $s_\varepsilon=s+\varepsilon$.

Proof. We first introduce some notations. Let $I=(0,1)$ and for any fixed $\varepsilon\ge0$ denote by $W^{1,2}_{*,\varepsilon}(I)$ the class of all functions satisfying

$$\|u\|_{*,\varepsilon}=\left(\int_0^1(r+\varepsilon)|u'(r)|^2dr\right)^{1/2}+\left(\int_0^1(r+\varepsilon)|u(r)|^2dr\right)^{1/2}<+\infty.$$

It is obvious that $W^{1,2}(I) \subset W^{1,2}_{*,\varepsilon}(I)$, but the class $W^{1,2}_{*,\varepsilon}(I)$ is quite different from $W^{1,2}(I)$. In particular, we notice that the functions in $W^{1,2}_{*,\varepsilon}(I)$ may not be bounded. However, it is not difficult to prove that for $u \in W^{1,2}_{*,\varepsilon}(I)$, the following properties hold:

(1) If $0 < \alpha \le 1$, then

$$\sup_{0<r<1} ((r+\varepsilon)^\alpha |u(r)|) \le C\|u\|_{*,\varepsilon},$$

where C is a constant depending only on α;

(2) If $0 < \alpha \le \frac{1}{2}$, then for any $\beta < \alpha$

$$|(r_1+\varepsilon)^\alpha u(r_1) - (r_2+\varepsilon)^\alpha u(r_2)| \le C|r_1 - r_2|^\beta \|u\|_{*,\varepsilon},$$

where C is a constant depending only on α and β.

Now, we set

$$F_\varepsilon(t) = \int_0^1 \left[\frac{k}{2} r_\varepsilon \left(\frac{\partial u}{\partial r}\right)^2 + r_\varepsilon H(u) + r_\varepsilon \mu\right] dr$$

and get from the equation (2.1)

$$\begin{aligned}
&\frac{dF_\varepsilon(t)}{dt} \\
= &\int_0^1 \left[2\frac{k}{2} r_\varepsilon \frac{\partial u}{\partial r}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + r_\varepsilon A(u)\frac{\partial u}{\partial t}\right] dr \\
= &\int_0^1 \left[k r_\varepsilon \frac{\partial u}{\partial r}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + r_\varepsilon A(u)\frac{\partial u}{\partial t}\right] dr \\
= &\int_0^1 \left[-k\frac{\partial}{\partial r}\left(r_\varepsilon \frac{\partial u}{\partial r}\right) + r_\varepsilon A(u)\right] \cdot \\
&\qquad \cdot \left(-\frac{1}{r_\varepsilon}\frac{\partial}{\partial r}\left\{r_\varepsilon m_\varepsilon(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\right\}\right) dr \\
= &\int_0^1 \left[\frac{k}{r_\varepsilon}\frac{\partial}{\partial r}(r_\varepsilon \frac{\partial u}{\partial r}) - A(u)\right]\left(\frac{\partial}{\partial r}\left\{r_\varepsilon m_\varepsilon(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\right\}\right) dr \\
= &-\int_0^1 \frac{\partial}{\partial r}\left[kV - A(u)\right] r_\varepsilon m_\varepsilon(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right] dr \\
= &-\int_0^1 r m_\varepsilon \left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]^2 dr \\
\le &\ 0,
\end{aligned}$$

which implies that

$$F_\varepsilon(t) \le F_\varepsilon(0) \tag{2.4}$$

and

$$\int_0^1 r_\varepsilon \left(\frac{\partial u}{\partial r}\right)^2 dr \le C. \tag{2.5}$$

Integrating the equation (2.1) on $Q_t = (0,1) \times (0,t)$, we have

$$\int_0^1 r_\varepsilon u(r,t)dr = \int_0^1 r_\varepsilon u_0(r)dr. \tag{2.6}$$

For any $\rho \in (0,1)$,

$$\begin{aligned}
&\frac{1+2\varepsilon}{2}u(\rho,t) - \int_0^1 s_\varepsilon u(s,t)ds \\
= &\int_0^1 s_\varepsilon[u(\rho,t) - u(s,t)]ds \\
= &\int_0^1 \int_s^\rho s_\varepsilon \frac{\partial u}{\partial r}(r,t)drds \\
= &\int_0^\rho \int_s^\rho s_\varepsilon \frac{\partial u}{\partial r}(r,t)drds + \int_\rho^1 \int_s^\rho s_\varepsilon \frac{\partial u}{\partial r}(r,t)drds \\
= &\int_0^\rho \int_0^r s_\varepsilon \frac{\partial u}{\partial r}(r,t)dsdr + \int_\rho^1 \int_r^1 s_\varepsilon \frac{\partial u}{\partial r}(r,t)dsdr \\
= &\int_0^\rho \left(\frac{r^2}{2} + \varepsilon r\right) \frac{\partial u}{\partial r}(r,t)dr + \int_\rho^1 \left[\frac{1}{2}(1-r^2) + \varepsilon(1-r)\right] \frac{\partial u}{\partial r}(r,t)dr \\
\leq &\int_0^\rho r_\varepsilon \left|\frac{\partial u}{\partial r}(r,t)\right|dr + 2\int_\rho^1 \left|\frac{\partial u}{\partial r}(r,t)\right|dr.
\end{aligned}$$

Setting $\rho_\varepsilon = \rho + \varepsilon$ and multiplying the above inequality with $2\rho_\varepsilon^{1/2}$, we get

$$\begin{aligned}
&\left|(1+2\varepsilon)\rho_\varepsilon^{1/2}u(\rho,t) - 2\rho_\varepsilon^{1/2}\int_0^1 s_\varepsilon u(s,t)ds\right| \\
\leq \; &2\rho_\varepsilon^{1/2}\int_0^\rho r_\varepsilon \left|\frac{\partial u}{\partial r}(r,t)\right|dr + 4\rho_\varepsilon^{1/2}\int_\rho^1 \left|\frac{\partial u}{\partial r}(r,t)\right|dr \\
\leq \; &2\rho_\varepsilon^{1/2}\int_0^\rho r_\varepsilon \left|\frac{\partial u}{\partial r}(r,t)\right|dr + 4\int_\rho^1 r_\varepsilon^{1/2}\left|\frac{\partial u}{\partial r}(r,t)\right|dr \\
\leq \; &C\left(\int_0^1 r_\varepsilon \left|\frac{\partial u}{\partial r}(r,t)\right|^2 dr\right)^{1/2}.
\end{aligned} \tag{2.7}$$

From (2.5), (2.6) and (2.7), we see that $r_\varepsilon^{1/2}u(r,t)$ is uniformly bounded on Q_T. Furthermore $u(\cdot,t) \in W^{1,2}_{*,\varepsilon}(I)$ for any fixed $t \in (0,T)$, with $\|u(\cdot,t)\|_{*,\varepsilon}$ bounded by a constant C independent of ε. The desired estimates then follow from the properties of $W^{1,2}_{*,\varepsilon}(I)$ mentioned above. The proof is complete.

Lemma 2.2 For any $\alpha > 0$, there is a constant M independent of ε such that

$$r_\varepsilon^\alpha |u(r,t)| \leq M, \quad \|u\|_{*,\varepsilon} \leq M \tag{2.8}$$

and

$$|r_\varepsilon u(r,t_2) - r_\varepsilon u(r,t_1)| \leq M|t_2 - t_1|^{1/16} \tag{2.9}$$

for all $r \in (0,1)$, $t_1, t_2 \in (0,T)$.

Proof. The first two estimates have already been seen from the arguments in Lemma 2.1. To prove (2.9), we need an integral estimate first. Multiplying the equation (2.1) by V and integrating with respect to r over $(0,1)$, we get

$$\begin{aligned} 0 &= \int_0^1 \left\{ \frac{\partial r_\varepsilon u}{\partial t} V + \frac{\partial}{\partial r}\left[r_\varepsilon m_\varepsilon \left(k\frac{\partial V}{\partial r} - A'\frac{\partial u}{\partial r}\right)\right] V \right\} dr \\ &= -\int_0^1 \left\{ \frac{\partial}{\partial t}\left[r_\varepsilon \left(\frac{\partial u}{\partial r}\right)^2\right] + r_\varepsilon m_\varepsilon k \left(\frac{\partial V}{\partial r}\right)^2 \right\} dr + \int_0^1 r_\varepsilon m_\varepsilon A' \frac{\partial u}{\partial r}\frac{\partial V}{\partial r} dr \end{aligned}$$

i.e.

$$\begin{aligned} & \frac{d}{dt}\int_0^1 r_\varepsilon \left(\frac{\partial u}{\partial r}\right)^2 dr + \int_0^1 r_\varepsilon m_\varepsilon k \left(\frac{\partial V}{\partial r}\right)^2 dr \\ \le\ & \frac{1}{2}\int_0^1 r_\varepsilon m_\varepsilon k \left(\frac{\partial V}{\partial r}\right)^2 dr + \frac{1}{2k}\int_0^1 r_\varepsilon m_\varepsilon (A')^2 \left(\frac{\partial u}{\partial r}\right)^2 dr \\ \le\ & \frac{k}{2}\int_0^1 r_\varepsilon m_\varepsilon k \left(\frac{\partial V}{\partial r}\right)^2 dr + \frac{1}{2k}\int_0^1 r_\varepsilon |u|^p (C_1|u|^q + C_2)^2 \left(\frac{\partial u}{\partial r}\right)^2 dr, \end{aligned}$$

which, together with the first two estimates in this lemma, implies that

$$\int_0^t \int_0^1 r_\varepsilon m_\varepsilon(r) \left(\frac{\partial V}{\partial r}\right)^2 drdt \le C. \tag{2.10}$$

Now, we begin to show (2.9). Without loss of generality, we assume that $t_1 < t_2$ and set $\Delta t = t_2 - t_1$. Integrating both sides of the equation (2.1) over $(t_1, t_2) \times (y, y + (\Delta t)^\alpha)$ and then integrating the resulting relation with respect to y over $(x, x + (\Delta t)^\alpha)$, we get

$$\begin{aligned} & (\Delta t)^\alpha \int_x^{x+(\Delta t)^\alpha} \int_0^1 (y + \theta(\Delta t)^\alpha + \varepsilon)\cdot \\ & \qquad \cdot\Big[u(y+\theta(\Delta t)^\alpha, t_2) - u(y+\theta(\Delta t)^\alpha, t_1)\Big] d\theta dy \\ = & -\int_x^{x+(\Delta t)^\alpha} \int_y^{y+(\Delta t)^\alpha} \int_{t_1}^{t_2} \frac{\partial}{\partial r}\left\{ r_\varepsilon m_\varepsilon \left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\right\} d\tau dr dy \\ = & -\int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} k\,[(y+(\Delta t)^\alpha+\varepsilon) m_\varepsilon(u(y+(\Delta t)^\alpha+\varepsilon,\tau))\cdot \\ & \cdot\frac{\partial}{\partial y}V(y+(\Delta t)^\alpha+\varepsilon,\tau) - (y+\varepsilon)m_\varepsilon(u(y+\varepsilon,\tau))\frac{\partial}{\partial y}V(y+\varepsilon,\tau)\Big] d\tau dy \\ & + \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \Big[-(y+\varepsilon)m_\varepsilon(u(y+\varepsilon,\tau)A'(u(y+\varepsilon,\tau))\frac{\partial}{\partial y}u(y+\varepsilon,\tau) \\ & \quad + (y+(\Delta t)^\alpha+\varepsilon)m_\varepsilon(u(y+(\Delta t)^\alpha+\varepsilon,\tau))A'(u(y+(\Delta t)^\alpha+\varepsilon,\tau))\cdot \\ & \quad \cdot \frac{\partial}{\partial y}u(y+(\Delta t)^\alpha+\varepsilon,\tau)\Big] d\tau dy. \end{aligned}$$

By the mean value theorem, there exists $x^* = y^* + \theta^*(\Delta t)^\alpha,\ y^* \in (x, x + (\Delta t)^\alpha), \theta^* \in (0,1)$ such that the left hand side of the above equality can be expressed by

$$
\begin{aligned}
& (\Delta t)^\alpha \int_x^{x+(\Delta t)^\alpha} \int_0^1 (y + \theta(\Delta t)^\alpha + \varepsilon) \cdot \\
& \qquad \cdot [u(y + \theta(\Delta t)^\alpha, t_2) - u(y + \theta(\Delta t)^\alpha, t_1)]\, d\theta dy \\
= \ & (\Delta t)^{2\alpha}(y^* + \theta^*(\Delta t)^\alpha + \varepsilon)\Big[u(y^* + \theta^*(\Delta t)^\alpha, t_2) - u(y^* + \theta^*(\Delta t)^\alpha, t_1)\Big].
\end{aligned}
$$

For the right hand side, we have

$$
\begin{aligned}
& - \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} k\, [(y + (\Delta t)^\alpha + \varepsilon) m_\varepsilon(u(y + (\Delta t)^\alpha + \varepsilon, \tau) \\
& \qquad \frac{\partial}{\partial y} V(y + (\Delta t)^\alpha + \varepsilon, \tau) - (y + \varepsilon) m_\varepsilon(u(y + \varepsilon, \tau) \frac{\partial}{\partial y} V(y + \varepsilon, \tau)\Big]\, d\tau dy \\
& + \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \Big[-(y + \varepsilon) m_\varepsilon(u(y + \varepsilon, \tau) A'(u(y + \varepsilon, \tau)) \frac{\partial}{\partial y} u(y + \varepsilon, \tau) \\
& \qquad + (y + (\Delta t)^\alpha + \varepsilon) m_\varepsilon(u(y + (\Delta t)^\alpha + \varepsilon, \tau) A'(u(y + (\Delta t)^\alpha + \varepsilon, \tau)) \cdot \\
& \qquad\quad \cdot \frac{\partial}{\partial y} u(y + (\Delta t)^\alpha + \varepsilon, \tau)\Big]\, d\tau dy \\
= \ & - \int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} k\, \Big[(r + \varepsilon) m_\varepsilon(u(r + \varepsilon, \tau)) \frac{\partial}{\partial r} V(r + \varepsilon, \tau)\Big]\, d\tau dr \\
& + \int_x^{x+(\Delta t)^\alpha} (r + \varepsilon) m_\varepsilon(u(r + \varepsilon, \tau) \frac{\partial}{\partial r} V(r + \varepsilon, \tau) d\tau dr \\
& + \int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} (r + \varepsilon) m_\varepsilon(u(r + \varepsilon, \tau) A'(u(r + \varepsilon, \tau)) \frac{\partial}{\partial r} u(r + \varepsilon, \tau) d\tau dr \\
& - \int_x^{x+(\Delta t)^\alpha} (r + \varepsilon) m_\varepsilon(u(r + \varepsilon, \tau) A'(u(r + \varepsilon, \tau)) \frac{\partial}{\partial r} u(r + \varepsilon, \tau) d\tau dr \\
\leq \ & \int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} k r_\varepsilon m_\varepsilon(u(r + \varepsilon, \tau) \Big|\frac{\partial}{\partial r} V(r\tau)\Big| d\tau dr \\
& + \int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} r_\varepsilon m_\varepsilon(u(r + \varepsilon, \tau)) \Big|A'\Big| \Big|\frac{\partial u}{\partial r}\Big| d\tau dr \\
\leq \ & \left(\int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} k r_\varepsilon m_\varepsilon d\tau dr\right)^{1/2} \left(\int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} k r_\varepsilon m_\varepsilon (\frac{\partial V}{\partial r})^2 d\tau dr\right)^{1/2} \\
+ & \left(\int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} r_\varepsilon m_\varepsilon d\tau dr\right)^{1/2} \left(\int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} r_\varepsilon m_\varepsilon |A'_\varepsilon|^2 (\frac{\partial u}{\partial r})^2 d\tau dr\right)^{1/2}.
\end{aligned}
$$

By (2.8), (2.10) and the assumptions on $m(u), A(u)$, we see that

$$
|x_\varepsilon^* u(x^*, t_2) - x_\varepsilon^* u(x^*, t_1)| \leq C(\Delta t)^{\frac{1-3\alpha}{2}},
$$

which implies, by setting $\alpha = 1/4$ and using the properties of the functions in $W^{1,2}_{*,\varepsilon}(I)$, that

$$|r_\varepsilon u(r,t_2) - r_\varepsilon u(r,t_1)| \le C(\Delta t)^{1/16}.$$

The proof is complete.

Proof of Theorem 2.1. Using Lemma 2.1 and 2.2, we see that $r_\varepsilon u$ is uniformly bounded in $C^{1/4,1/16}(\overline{Q}_T)$-norm with the bound independent of ε. Similar to [14], we can further establish the estimates on the Hölder norm of Du. Then, using the classical Schauder theory, we may complete the proof of the remaining part in a standard way.

3 Existence

After the discussion of the regularized problem, we can now turn to the investigation of the existence of weak solutions of the problem (1.1)–(1.3). The main existence result is the following

Theorem 3.1 Under the assumptions (H_1), (H_2), the problem (1.1)–(1.3) admits at least one weak solution.

Proof. Let u_ε be the approximate solution of the problem (2.1)–(2.3) constructed in the previous section. Using the estimates in Lemma 2.1 and 2.2, for any $\beta < \frac{1}{2}$, and $(r_1,t_2),(r_2,t_1) \in Q_T$, we have

$$|r_{1\varepsilon} u_\varepsilon(r_1,t_2) - r_{2\varepsilon} u_\varepsilon(r_2,t_1)| \le C(|r_1 - r_2|^\beta + |t_1 - t_2|^{\beta/4})$$

with constant C independent of ε. So, we may extract a subsequence from $\{r_\varepsilon u_\varepsilon\}$, denoted also by $\{r_\varepsilon u_\varepsilon\}$, such that

$$r_\varepsilon u_\varepsilon(r,t) \to ru(r,t) \text{ uniformly in } \overline{Q}_T,$$

and the limiting function $ru \in C^{1/4,1/16}(\overline{Q}_T)$. By (2.8), we also have $r^\alpha u \in L^\infty(Q_T)$ with $\alpha > 0$ and for any $t \in (0,T)$, $u(\cdot,t) \in W^{1,2}_{*,0}(I)$ with the norm $\|u(\cdot,t)\|_{*,0}$ bounded by a constant independent of t.

Now, let $\delta > 0$ be fixed and set $P_\delta = \{(r,t);\ rm(u(r,t)) > \delta\}$. We choose $\varepsilon(\delta) > 0$, such that

$$r_\varepsilon m_\varepsilon(u_\varepsilon(r,t)) \ge \frac{\delta}{2}, \quad (r,t) \in P_\delta, 0 < \varepsilon < \varepsilon_0(\delta). \tag{3.1}$$

Then from (2.10)

$$\iint_{P_\delta} \left(\frac{\partial V_\varepsilon}{\partial r}\right)^2 drdt \le \frac{C}{\delta}. \tag{3.2}$$

To prove the integral equality in the definition of solutions, it suffices to pass the limit as $\varepsilon \to 0$ in

$$-\int_0^1 r_\varepsilon u_\varepsilon(r,T)\varphi(r,T)dr + \int_0^1 r_\varepsilon u_{0\varepsilon}\varphi(r,0)dr + \iint_{Q_T} r_\varepsilon u_\varepsilon \frac{\partial\varphi}{\partial t}drdt$$
$$+\iint_P r_\varepsilon m_\varepsilon(u_\varepsilon)\left[k\frac{\partial V_\varepsilon}{\partial r} - A'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial r}\right]\frac{\partial\varphi}{\partial r}drdt = 0.$$

The limits

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^1 r_\varepsilon u_\varepsilon(r,T)\varphi(r,T) = \int_0^1 ru(r,T)\varphi(r,T)dr,\\
&\lim_{\varepsilon\to 0}\int_0^1 r_\varepsilon u_{0\varepsilon}(r)\varphi(r,0)dr = \int_0^1 u_0(r)\varphi(r,0)dr,\\
&\lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon u_\varepsilon \frac{\partial\varphi}{\partial t}drdt = \iint_{Q_T} ru\frac{\partial\varphi}{\partial t}drdt
\end{aligned}$$

are obvious. It remains to show

$$\lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt = \iint_P rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt, \tag{3.3}$$

$$\lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon m_\varepsilon(u_\varepsilon)A'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt = \iint_P rm(u)A'(u)\frac{\partial u}{\partial r}\frac{\partial\varphi}{\partial r}drdt. \tag{3.4}$$

In fact, for any fixed $\delta > 0$,

$$\begin{aligned}
&\Big|\iint_{Q_T} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt - \iint_P rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big|\\
\le\ &\Big|\iint_{P_\delta} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt - \iint_{P_\delta} rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big|\\
&+\Big|\iint_{Q_T\setminus P_\delta} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big| + \Big|\iint_{P\setminus P_\delta} rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big|.
\end{aligned}$$

From the estimates (2.10), we have

$$\begin{aligned}
&\Big|\iint_{Q_T\setminus P_\delta} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big| \le C\delta\sup\Big|\frac{\partial\varphi}{\partial r}\Big|, \quad 0<\varepsilon<\varepsilon_0(\delta)\\
&\Big|\iint_{P\setminus P_\delta} rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big| \le C\delta\sup\Big|\frac{\partial\varphi}{\partial r}\Big|,
\end{aligned}$$

$$\begin{aligned}
&\Big|\iint_{P_\delta} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt - \iint_{P_\delta} rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big|\\
\le\ &\iint_{P_\delta}\Big|r_\varepsilon m_\varepsilon(u_\varepsilon) - rm(u)\Big|\Big|\frac{\partial V_\varepsilon}{\partial r}\Big|\Big|\frac{\partial\varphi}{\partial r}\Big|drdt\\
&\quad+\Big|\iint_{P_\delta} rm(u)\left(\frac{\partial V_\varepsilon}{\partial r} - \frac{\partial V}{\partial r}\right)\frac{\partial\varphi}{\partial r}drdt\Big|\\
\le\ &\sup|r_\varepsilon m_\varepsilon(u_\varepsilon) - rm(u)|\Big|\frac{\partial\varphi}{\partial r}\Big|\frac{C}{\sqrt{\delta}} + \Big|\iint_{P_\delta} rm(u)\left(\frac{\partial V_\varepsilon}{\partial r} - \frac{\partial V}{\partial r}\right)\frac{\partial\varphi}{\partial r}drdt\Big|
\end{aligned}$$

and hence

$$\overline{\lim_{\varepsilon\to 0}}\Big|\iint_{Q_T} r_\varepsilon m_\varepsilon(u_\varepsilon)k\frac{\partial V_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}drdt - \iint_P rm(u)k\frac{\partial V}{\partial r}\frac{\partial\varphi}{\partial r}drdt\Big| \le C\delta\sup\Big|\frac{\partial\varphi}{\partial r}\Big|.$$

By the arbitrariness of δ, we see that the limit (3.3) holds.

Finally, from the uniform convergence of $r_\varepsilon u_\varepsilon$ to ru, we immediately obtain

$$\begin{aligned}
& \lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon m_\varepsilon(u_\varepsilon)DA(u_\varepsilon)D\varphi drdt \\
= & \lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon DH(u_\varepsilon)D\varphi drdt \\
= & \lim_{\varepsilon\to 0}\int_0^T (\varepsilon+1)H(u_\varepsilon(1,t))D\varphi(1,t)drdt \\
& -\lim_{\varepsilon\to 0}\int_0^T \varepsilon H(u_\varepsilon(0,t))D\varphi(0,t)drdt \\
& -\lim_{\varepsilon\to 0}\int_0^T H(u_\varepsilon)D(r_\varepsilon D\varphi)drdt \\
= & \int_0^T H(u(1,t))D\varphi(1,t)drdt - \iint_{Q_T} H(u)D(rD\varphi)drdt \\
= & \iint_{Q_T} DH(u)rD\varphi drdt = \iint_{Q_T} rm(u)DA(u)D\varphi drdt.
\end{aligned}$$

The proof is complete.

4 Nonnegativity

Just as mentioned by several authors, it is much interesting to discuss the physical solutions. For the two-dimensional problem (1.1)–(1.3), a very typical example is the modeling of oil films spreading over an solid surface, where the unknown function u denotes the height from the surface of the oil film to the solid surface. Motivated by this idea, we devote this section to the discussion of the nonnegativity of solutions.

Theorem 4.1 The weak solution u obtained in Section 3 satisfy $u(x,t) \geq 0$, if $u_0(x) \geq 0$.

Proof. Suppose the contrary, that is, the set

$$E = \{(r,t) \in \overline{Q}_T; u(r,t) < 0\} \tag{4.1}$$

is nonempty.

For any fixed $\delta > 0$, choose a C^∞ function $H_\delta(s)$ such that $H_\delta(s) = -\sigma$ for $s \geq -\delta$, $H_\delta(s) = -1$, for $\delta \leq -2\delta$ and that $H_\delta(s)$ is nondecreasing for $-2\delta < s < -\delta$. Also, we extend the function $u(r,t)$ to be defined in the whole plane $\mathbb{R}^2$ such that the extension $\bar{u}(r,t) = 0$ for $t \geq T+1$ and $t \leq -1$. Let $\alpha(s)$ be the kernel of mollifier in one-dimension, that is, $\alpha(s) \in C^\infty(\mathbb{R})$, supp$\alpha = [-1,1]$, $\alpha(s) > 0$ in $(-1,1)$, and $\displaystyle\int_{-1}^1 \alpha(s)ds = 1$. For any fixed $k > 0, \delta > 0$, define

$$u^h(r,t) = \int_{\mathbb{R}} \bar{u}(s,r)\alpha_h(t-s)ds,$$

$$\beta_\delta(t) = \int_t^{+\infty} \alpha\left(\frac{s-\dfrac{T}{2}}{\dfrac{T}{2}-\delta}\right)\frac{1}{\dfrac{T}{2}-\delta}ds,$$

where $\alpha_h(s) = \frac{1}{h}\alpha(\frac{s}{h})$.

The function

$$\varphi_\delta^h(r,t) \equiv [\beta_\delta(t)H_\delta(u^h)]^h$$

is clearly an admissible test function, that is the following integral equality holds

$$\begin{aligned} -\int_0^1 ru(r,T)\varphi_\delta^h(T,r)dr + \int_0^1 ru_0(r)\varphi_\delta^h(r,0)dr + \iint_{Q_T} ru\frac{\partial\varphi_\delta^h}{\partial t}drdt \\ + \iint_P rm(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\frac{\partial\varphi_\delta^h}{\partial r}drdt = 0. \end{aligned} \tag{4.2}$$

To proceed further, we give an analysis on the properties of the test function $\varphi_\delta^h(r,t)$. The definition of $\beta_\delta(t)$ implies that

$$\varphi_\delta^h(r,t) = 0, \quad t \geq T - \frac{\delta}{2},\ h < \frac{\delta}{2}. \tag{4.3}$$

Since $\bar{u}(r,t)$ is continuous, for fixed δ, there exists $\eta_1(\delta) > 0$, such that

$$u^h(r,t) \geq -\frac{\delta}{2},\ t \leq \eta_1(\delta),\ 0 \leq r \leq 1,\ h < \eta_1(\delta), \tag{4.4}$$

which together with the definition of $\beta_\delta(t), H_\delta(s)$ imply that

$$H_\delta(u^h(r,t)) = -\delta,\ t \leq \eta_1(\delta),\ 0 \leq r \leq 1,\ h < \eta_1(\delta) \tag{4.5}$$

and hence

$$\varphi_\delta^h = -\delta,\ t \leq \frac{1}{2}\eta_1(\delta),\ 0 \leq r \leq 1,\ h < \frac{1}{2}\eta_1(\delta). \tag{4.6}$$

We note also that for any functions $f(t), g(t) \in L^2(R)$,

$$\begin{aligned} \int_{\mathbb{R}} f(t)g^h(t)dt &= \int_{\mathbb{R}} f(t)dt\int_{\mathbb{R}} g(s)\alpha_n(t-s)ds = \int_{\mathbb{R}} f(t)\int_{\mathbb{R}} g(s)\alpha_n(s-t)ds \\ &= \int_{\mathbb{R}} g(s)ds\int_{\mathbb{R}} f(t)\alpha_n(s-t)dt = \int_{\mathbb{R}} f^h(t)g(t)dt. \end{aligned}$$

Taking this into account and using (4.3), (4.5), (4.6), we have

$$\begin{aligned} &\iint_{Q_T} ru\frac{\partial}{\partial t}\varphi_\delta^h drdt \\ = &\int_{-\infty}^{+\infty} dt\int_0^1 ru\left[\frac{\partial}{\partial t}\left(\beta_\delta(t)H_\delta(u^h)\right)\right]^h dr \\ = &\iint_{Q_T} (ru)^h\frac{\partial}{\partial t}\left(\beta_\delta(t)H_\delta(u^h)\right)drdt \end{aligned}$$

and hence by integrating by parts

$$
\begin{aligned}
&\iint_{Q_T} (ru)^h \frac{\partial}{\partial t}\left(\beta_\delta(t) H_\delta(u^h)\right) drdt \\
= &\int_0^1 (ru)^h(r,T)\beta_\delta(T)H_\delta(u^h(r,T))dr - \int_0^1 (ru)^h(r,0)\beta_\delta(0)H_\delta(u^h(r,0))dr \\
&- \iint_{Q_T} \beta_\delta(t)H_\delta(u^h)\frac{\partial (ru)^h}{\partial t} drdt \\
= &\ \delta\int_0^1 (ru)^h(r,0)dr - \iint_{Q_T} r\beta_\delta(t)\frac{\partial}{\partial t}F_\delta(u^h)drdt,
\end{aligned}
$$

where $F_\delta(s) = \displaystyle\int_0^s H_\delta(\sigma)d\sigma$.

Again by (4.5)

$$
\begin{aligned}
F_\delta(u^h(r,0)) &= \int_0^{u^h(r,0)} H_\delta(\sigma)d\sigma \\
&= \int_0^1 H_\delta(\lambda u^h(r,0))d\lambda \cdot u^h(r,0) \\
&= -\delta u^h(r,0)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\iint_{Q_T} (ru)^h \frac{\partial}{\partial t}(\beta_\delta(t)H_\delta(u^h))drdt \\
= &\ \delta\int_0^1 (ru)^h(r,0)dr + \int_0^1 r\beta_\delta(0)F_\delta(u^h(r,0))dr + \iint_{Q_T} rF_\delta(u^h)\beta_\delta'(t)drdt \\
= &\ -\frac{1}{\frac{T}{2}-\delta}\iint_{Q_T} rF_\delta(u^h)\alpha\left(\frac{t-\frac{T}{2}}{\frac{T}{2}-\delta}\right)drdt.
\end{aligned} \tag{4.7}
$$

From (4.3), (4.6) it is clear that

$$
-\int_0^1 ru(r,T)\varphi_\delta^h(T,r)dr = 0,\ 0 < h < \frac{1}{2}\eta_1(\delta), \tag{4.8}
$$

$$
\int_0^1 ru_0(r)\varphi_\delta^h(r,0)dr = -\delta\int_0^1 ru_0(r)dr. \tag{4.9}
$$

Substituting (4.7), (4.8) and (4.9) into (4.2), we have

$$
\begin{aligned}
&-\frac{2}{T-2\delta}\iint_{Q_T} rF_\delta(u^h)\alpha\left(\frac{t-\frac{T}{2}}{\frac{T}{2}-\delta}\right)drdt - \delta\int_0^1 ru_0(r)dr \\
+ &\iint_P rm(u)\left[k\frac{\partial V}{\partial r} - A'(u)\frac{\partial u}{\partial r}\right]\frac{\partial \varphi_\delta^h}{\partial r}drdt = 0.
\end{aligned} \tag{4.10}
$$

By the uniform continuity of $u(r,t)$ in $\overline{Q}_T$, there exist $\eta_2(\delta) > 0$, such that

$$u(r,t) \geq -\frac{\delta}{2} \quad \forall\ (r,t) \in P^\delta, \tag{4.11}$$

where $P^\delta = \{(r,t); \mathrm{dist}((r,t),P) < \eta_2(\delta)\}$. Here we have used the fact that $u(r,t) > 0$ in P. Thus

$$H_\delta(u^h(r,t)) = -\delta, \quad \forall\ (r,t) \in P^{\delta/2},\ 0 < h < \frac{1}{2}\eta_2(\delta)$$

where $P^{\delta/2} = \{(r,t); \mathrm{dist}((r,t),P) < \frac{1}{2}\eta_2(\delta)\}$.

This and the definition of $u^h, H_\delta(s)$ show that the function $\varphi_\delta^h(r,t)$ is only a function of t in P, whenever $h < \frac{1}{2}\eta_2(\delta)$. Therefore

$$D\varphi_\delta^h(r,t) = 0,\ \forall\ (r,t) \in P,\ 0 < h < \frac{1}{2}\eta_2(\delta) \tag{4.12}$$

and so (4.10) becomes

$$-\delta \int_0^1 r u_0(r) dr - \frac{2}{T-2\delta} \iint_{Q_T} r F_\delta(u^h) \alpha \left(\frac{2t-T}{T-2\delta} \right) drdt = 0, \tag{4.13}$$

where $\eta(\delta) = \min(\eta_1(\delta), \eta_2(\delta))$. Letting h tend to zero, we have

$$-\delta \int_0^1 r u_0(r) dr - \frac{2}{T-2\delta} \iint_{Q_T} r F_\delta(u) \alpha \left(\frac{2t-T}{T-2\delta} \right) drdt = 0. \tag{4.14}$$

From the definition of $F_\delta(s), H_\delta(s)$, it is easily seen that

$$F_\delta(u(r,t)) \to -\chi_E(r,t) u(r,t) \quad (\delta \to 0)$$

and so by letting δ tend to zero in (4.14), we have

$$\iint_E |u(r,t)| \alpha \left(\frac{2t-T}{T} \right) drdt = 0,$$

which contradicts the fact that $\alpha \left(\frac{2t-T}{T} \right) > 0$ for $0 < t < T$. We have thus proved the theorem.

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