An existence theorem for parabolic equations on $\mathbf{R}^N$ with discontinuous nonlinearity

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Abstract

We prove existence of solutions for parabolic initial value problems $\partial_t u = \Delta u + f(u)$ on $\mathbf{R}^N$, where $f : \mathbf{R} \to \mathbf{R}$ is a bounded, but possibly discontinuous function.

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1 Introduction

We prove an existence theorem for the following parabolic initial value problem

$$\partial_t u = \Delta u + f(u) \tag{1}$$

$$u(x,0) = \alpha(x) \quad x \in \mathbf{R}^N \tag{2}$$

on $Q = \mathbf{R}^N \times (0,\infty)$, where $\alpha \in BUC(\mathbf{R}^N)$ is a bounded, uniformly continuous function and $f : \mathbf{R} \to \mathbf{R}$ is a bounded, measurable function. Throughout the paper Δ denotes the Laplacian, ∇ denotes gradient, $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbf{R}^N$ and 'measurable' means Borel-measurable.

Problems of the above form cover a wide range of models in applied sciences, e.g. in combustion theory and nerve conduction. Our main motivation is the model of best response dynamics arising in game theory [11]. In this model f is a differentiable function on $[0,1] \setminus \{a\}$ for some $a \in (0,1)$ and

$$f(0) = f(1) = 0, \qquad f(u) < 0, \ \text{ if } u \in (0,a); \qquad f(u) > 0, \ \text{ if } u \in (a,1).$$

(Outside the interval $[0,1]$ it can be extended as zero.) A typical special case is $f(u) = -u + H(u-a)$, where H is the Heaviside function.

Similar problems were investigated by several authors mainly on bounded domains. The equation is usually considered as a differential inclusion. One of the first results in this field was achieved by Rauch [14]. He proved the existence of a solution $u \in L^2([0,t^*], H_0^1(\Omega))$, where $\Omega \subset \mathbf{R}^N$ is a bounded domain and f is locally bounded. In [4] the existence of weak solutions is proved in a similar space. Bothe [2] extended the existence theorem for systems but also considered bounded domains. Terman [17] proved an existence theorem in the one dimensional case for the special nonlinearity $f(u) = -u + H(u-a)$. In his paper the solution is classical at those points (x,t) where $u(x,t) \neq a$. In [9] $f(u) = g(u) + H(u-1)$, g is nonnegative, nondecreasing and locally Lipschitz continuous, and the space domain is $[0,\pi]$. The existence of a $u \in C([0,t^*], H_0^1(0,\pi))$ solution is proved. The problem was studied in more general contexts on bounded domains. In [3] the problem is considered with a nonlinear elliptic operator, in [6] and [15] the case of functional partial differential equations is investigated. The results concerning the case of the whole space $\mathbf{R}^N$ are mainly for the elliptic case, see e.g. [1, 5]. For other results concerning existence and uniqueness questions for differential equations with discontinuous nonlinearity we refer to the monographs [10, 18] and the references therein.

Here we prove that there exists a continuous solution on $\mathbf{R}^N \times [0,\infty)$. We do not restrict ourselves to bounded domains and one space dimension. Moreover, our solutions are not even in $L^2(\mathbf{R}^N)$ (for fixed t), because we would like to treat spatially constant non zero solutions, and travelling waves connecting these, too. Hence none of the methods of the above papers works in itself. We have to combine several ideas to prove the existence theorem.

The usual way of introducing the corresponding differential inclusion is to define the semicontinuous functions

$$\underline{f}(u) = \lim_{\varepsilon\to 0} \inf\{f(s) : s \in (u-\varepsilon, u+\varepsilon)\} \quad ; \quad \overline{f}(u) = \lim_{\varepsilon\to 0} \sup\{f(s) : s \in (u-\varepsilon, u+\varepsilon)\} \ . \tag{3}$$

We note that if f is continuous in u, then $\underline{f}(u) = \overline{f}(u) = f(u)$. Now we can define the notion of a solution.

Definition 1 The function $u : \overline{Q} = \mathbf{R}^N \times [0,\infty) \to \mathbf{R}$ is called a *solution* of (1)–(2) if

(i) $u \in C^{1,0}(Q)$, that is u is continuous in $\overline{Q}$ and continuously differentiable w.r.t. x in Q

(ii) u satisfies the corresponding differential inclusion in the weak sense, that is there exists a bounded measurable function $h : Q \to \mathbf{R}$ such that

$$\int_Q (u\partial_t\varphi - \langle\nabla u, \nabla\varphi\rangle + h\varphi) = 0 \qquad \text{for all} \qquad \varphi \in C_0^\infty(Q) \tag{4}$$

and

$$\underline{f}(u(x,t)) \le h(x,t) \le \overline{f}(u(x,t)) \qquad \text{a.e. in} \quad Q \ . \tag{5}$$

The main result of this paper is the following theorem.

Theorem 1 *Let $\alpha \in BUC(\mathbf{R}^N)$ be a bounded, uniformly continuous function and $f : \mathbf{R} \to \mathbf{R}$ be a bounded, measurable function. Then (1)–(2) has a solution in Q.*

The proof of the Theorem in Section 3 consists of the following STEPs.

STEP1 Introduction of a sequence (f_n) of C^∞ functions approximating f.

STEP2 Solving the approximating equations $\partial_t u_n = \Delta u_n + f_n(u_n)$ with initial condition $u_n(x,0) = \alpha(x)$.

STEP3 Using the Arzelá–Ascoli theorem we get a uniformly convergent subsequence of (u_n). The solution u is defined as the limit of that subsequence.

STEP4 Constructing h as the limit of $f_n \circ u_n$.

STEP5 To prove that h satisfies (5).

STEP6 To prove that u is continuously differentiable w.r.t. x.

STEP7 To prove that u satisfies (4).

2 Preliminaries

We will consider our problem as an abstract evolution equation. Let $X = BUC(\mathbf{R}^N)$ be the space of bounded, uniformly continuous functions endowed with the supremum norm, $\|\cdot\|$. For $\psi \in X$ we define

$$(T(t)\psi)(x) = \int_{\mathbf{R}^N} K(x-y,t)\psi(y)\mathrm{d}y \tag{6}$$

where

$$K(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t}). \tag{7}$$

We will use the following properties of $\{T(t)\}_{t\ge 0}$.

Proposition 1 *1. T is an analytic semigroup of bounded linear operators on X.*

2. *$\|T(t)\| \leq 1$ for all $t \geq 0$.*

3. *The function $t \mapsto T(t)\psi$ is uniformly continuous on $[0, \infty)$ for all $\psi \in X$.*

4. *The function $u(x,t) = (T(t)\alpha)(x)$ is a solution of the homogeneous equation $\partial_t u = \Delta u$ with initial condition $u(x, 0) = \alpha(x)$. Moreover, $u(\cdot, t)$ is uniformly equicontinuous, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$, such that $|x_1 - x_2| < \delta$ and $t \geq 0$ imply $|u(x_1, t) - u(x_2, t)| < \varepsilon$.*

5. *There exists $a > 0$, such that for any $0 < \tau_1 < \tau_2$ there holds $\|T(\tau_2) - T(\tau_1)\| \leq a\sqrt{\tau_2 - \tau_1}/\sqrt{\tau_1}$.*

Proof. The first four statements are well-known, see e.g. [8, 12], the last one will be proved.
Let us denote the generator of the analytic semigroup $T(t)$ by A. Then by the analyticity (see [7] Ch. II. Theorem 4.6) there exists $a > 0$ such that

$$\|AT(t)\| \leq \frac{a}{t} \qquad \text{for} \quad t > 0 \ .$$

Let $\psi \in X$, $\|\psi\| = 1$. Using the above formula and (1.7) in [7] Ch. II. we obtain

$$\|T(\tau_2)\psi - T(\tau_1)\psi\| = \|(\psi + \int_0^{\tau_2} AT(s)\psi \mathrm{d}s) - (\psi + \int_0^{\tau_1} AT(s)\psi \mathrm{d}s)\| =$$

$$\|\int_{\tau_1}^{\tau_2} AT(s)\psi \mathrm{d}s\| \leq \int_{\tau_1}^{\tau_2} \frac{a}{s}\mathrm{d}s = a \ln\frac{\tau_2}{\tau_1} \leq a\sqrt{\frac{\tau_2 - \tau_1}{\tau_1}}$$

The last step follows from the simple inequality

$$\ln z \leq \sqrt{z - 1} \qquad \text{if} \quad z \geq 1 \ .$$

□

It is also well-known that the solution of the inhomogeneous problem

$$\partial_t v = \Delta v + h \tag{8}$$

$$v(x, 0) = \alpha(x) \quad x \in \mathbf{R}^N \tag{9}$$

can be expressed in the abstract framework. Let us assume that $h : \mathbf{R}^N \times [0, t^*]) \to \mathbf{R}$ is bounded and uniformly continuous for some $t^* > 0$, and let us introduce $H : [0, t^*] \to X$, $H(t)(x) = h(x, t)$. Then the solution of the inhomogeneous problem (8)-(9) takes the form $v(x, t) = V(t)(x)$, where

$$V(t) = T(t)\alpha + \int_0^t T(t - s)H(s)\mathrm{d}s. \tag{10}$$

The required regularity of V is proved in the next two Propositions. These statements can be proved in the abstract setting (see e.g. [13]), but here we prove them in our special case to make the paper self-contained. The uniform continuity of V follows from 3. of Proposition 1 and from the following statement.

Proposition 2 *Let $H : [0, t^*] \to X$ be continuous and $\|H(t)\| \leq M$ for all $t \in [0, t^*]$. Then for every $t_1, t_2 \in [0, t^*]$ we have*

$$\|\overline{V}(t_1) - \overline{V}(t_2)\| \leq M(2a + 1)\sqrt{t^*}\sqrt{|t_1 - t_2|} \ , \tag{11}$$

where

$$\overline{V}(t) = \int_0^t T(t - s)H(s)ds. \tag{12}$$

Proof. Let us assume $t_1 < t_2$. Then

$$\overline{V}(t_2) - \overline{V}(t_1) = \int\limits_0^{t_1} (T(t_2 - s) - T(t_1 - s))H(s)\mathrm{d}s + \int\limits_{t_1}^{t_2} T(t_2 - s)H(s)\mathrm{d}s. \tag{13}$$

For the norm of the second term we have

$$\|\int\limits_{t_1}^{t_2} T(t_2 - s)H(s)\mathrm{d}s\| \le M|t_2 - t_1|. \tag{14}$$

To estimate the norm of the first term we use 5. of Proposition 1

$$\|\int\limits_0^{t_1} (T(t_2 - s) - T(t_1 - s))H(s)\mathrm{d}s\| \le M \int\limits_0^{t_1} \frac{a\sqrt{t_2 - t_1}}{\sqrt{t_1 - s}}\mathrm{d}s = 2aM\sqrt{t_1}\sqrt{t_2 - t_1}. \tag{15}$$

Using (13), (14) and (15) we obtain the desired inequality. □

We will need that the boundedness of the function h implies the differentiability of v w.r.t. x. The notation ∂_k will be used for the partial derivative w.r.t. x_k.

Proposition 3 *Let $\alpha \in BUC(\mathbf{R}^N)$ and h be bounded and Borel-measurable in Q. Then the function*

$$v(x,t) = \int\limits_{\mathbf{R}^N} K(x-y,t)\alpha(y)\,dy + \int\limits_0^t \int\limits_{\mathbf{R}^N} K(x-y,t-s)h(y,s)\,dy\,ds \qquad (x,t) \in \overline{Q} \tag{16}$$

is continuously differentiable w.r.t. x, and for its partial derivatives we have

$$\partial_k v(x,t) = \int\limits_{\mathbf{R}^N} \partial_k K(x-y,t)\alpha(y)\,dy + \int\limits_0^t \int\limits_{\mathbf{R}^N} \partial_k K(x-y,t-s)h(y,s)\,dy\,ds \qquad (x,t) \in Q.$$

Moreover, for any $0 < t_1 < t_2$ the partial derivatives are bounded in $\mathbf{R}^N \times [t_1, t_2]$:

$$|\partial_k v(x,t)| \le \frac{\|\alpha\|}{\sqrt{\pi t_1}} + \frac{2\|h\|\sqrt{t_2}}{\sqrt{\pi}} \qquad \textit{for all} \qquad (x,t) \in \mathbf{R}^N \times [t_1, t_2]\ . \tag{17}$$

Proof. The first statement follows from the theorem on differentiation of parametric integrals. Since α and h are bounded, (17) follows easily from the formulas below:

$$\int\limits_{\mathbf{R}^N} |\partial_k K(x-y,t)|\mathrm{d}y = \frac{1}{\sqrt{\pi t}}$$

and

$$\int\limits_0^t \int\limits_{\mathbf{R}^N} |\partial_k K(x-y,t-s)|\mathrm{d}y\mathrm{d}s = \frac{\sqrt{t}}{\sqrt{\pi}}$$

which can be verified by direct integration, using that

$$\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \frac{|x|}{2t} \exp(-\frac{x^2}{4t})\mathrm{d}x = \frac{1}{\sqrt{\pi t}}\ .$$

□

Now we summarize the equicontinuity results concerning the solution of the inhomogeneous equation.

Proposition 4 *Let $\alpha \in BUC(\mathbf{R}^N)$, h be bounded and measurable in Q and v be defined in (16). Then for all $\varepsilon > 0$ and $t^* > 0$ there exists $\delta > 0$ (depending only on α and $\|h\|$), such that for all $t_1, t_2 \in [0, t^*]$ and $x_1, x_2 \in \mathbf{R}^N$,*

$$|(x_1, t_1) - (x_2, t_2)| < \delta \qquad \text{implies} \qquad |v(x_1, t_1) - v(x_2, t_2)| < \varepsilon.$$

(Here $|\cdot|$ is any norm in $\mathbf{R}^k$ for any $k \in \mathbf{N}$.)

Proof. We will prove that for all $\varepsilon > 0$ and $t^* > 0$ there exists $\delta > 0$, such that

$$|v(x, t_1) - v(x, t_2)| < \varepsilon \qquad \text{for all} \quad x \in \mathbf{R}^N, \quad |t_1 - t_2| < \delta \tag{18}$$

and

$$|v(x_1, t) - v(x_2, t)| < \varepsilon \qquad \text{for all} \quad |x_1 - x_2| < \delta, \quad t \in [0, t^*] \ . \tag{19}$$

Inequality (18) follows from Proposition 1 and Proposition 2. Namely, with $H(t)(x) = h(x, t)$, $V(t)(x) = v(x, t)$ and using (10) and (12) one obtains

$$\sup_{x \in \mathbf{R}^N} |v(x, t_1) - v(x, t_2)| = \|V(t_1) - V(t_2)\| \leq \|T(t_1)\alpha - T(t_2)\alpha\| + \|\overline{V}(t_1) - \overline{V}(t_2)\| \ . \tag{20}$$

According to 3. of Proposition 1 for any $\varepsilon > 0$ there exists $\delta_1 > 0$ (depending only on α) such that for $|t_1 - t_2| < \delta_1$ we have

$$\|T(t_1)\alpha - T(t_2)\alpha\| < \varepsilon/2 \ . \tag{21}$$

According to Proposition 2 there exists $\delta_2 > 0$ (depending only on $\|h\|$ and t^*) such that for $|t_1 - t_2| < \delta_2$ we have

$$\|\overline{V}(t_1) - \overline{V}(t_2)\| < \varepsilon/2 \ . \tag{22}$$

Thus (18) follows from (20)-(22) with $\delta = \min\{\delta_1, \delta_2\}$.

Now let us turn to the verification of (19). Let us introduce

$$v_1(x, t) = (T(t)\alpha)(x) \qquad v_2(x, t) = \overline{V}(t)(x) \ .$$

Then by 4. of Proposition 1 there exists $\delta_1 > 0$ (depending only on α) such that

$$|v_1(x_1, t) - v_1(x_2, t)| < \varepsilon \qquad \text{for all} \quad |x_1 - x_2| < \delta_1, \quad t \geq 0 \ . \tag{23}$$

According to Proposition 3

$$|\partial_k v_2(x, t)| \leq \frac{2\|h\|\sqrt{t^*}}{\sqrt{\pi}} \qquad \text{for all} \qquad (x, t) \in \mathbf{R}^N \times [0, t^*] \ .$$

Hence there exists $\delta_2 > 0$ (depending only on h and t^*) such that

$$|v_2(x_1, t) - v_2(x_2, t)| < \varepsilon \qquad \text{for all} \quad |x_1 - x_2| < \delta_2, \quad t \in [0, t^*] \ . \tag{24}$$

Thus (19) follows from (23) and (24) with $\delta = \min\{\delta_1, \delta_2\}$. □

Finally, we recall the results concerning the following parabolic equation with locally Lipschitzian nonlinearity g:

$$\partial_t u = \Delta u + g(u) \tag{25}$$

$$u(x, 0) = \alpha(x) \quad x \in \mathbf{R}^N \ . \tag{26}$$

In order to use the abstract framework let us introduce the function $G : X \to X$, $G(\psi) = g \circ \psi$. Then the solution of problem (25)-(26) takes the form $u(x, t) = U(t)(x)$, where

$$U(t) = T(t)\alpha + \int_0^t T(t - s)G(U(s))\mathrm{d}s \ . \tag{27}$$

It is easy to prove that the boundedness and local Lipschitz continuity of g implies the same properties for G. The following existence theorem is proved in [16] (Theorem 11.12.).

Proposition 5 *Let $\alpha \in X$ and G be bounded and locally Lipschitz continuous. Then there exists a continuous solution $U : [0, \infty) \to X$ of (27).*

3 Proof of Theorem 1

3.1 STEP1

We define a sequence (f_n) of C^∞ functions approximating f. Let $j : \mathbf{R} \to \mathbf{R}$ be a nonnegative C^∞ function, for which $\operatorname{supp} j \subset (-1,1)$ and $\int_{-\infty}^{\infty} j = 1$ hold. For every $n \in \mathbf{N}$ we define the C^∞ functions $j_n : \mathbf{R} \to \mathbf{R}$ as $j_n(u) = nj(nu)$. Then

$$\operatorname{supp} j_n \subset (-\frac{1}{n}, \frac{1}{n}) \qquad \text{and} \qquad \int\limits_{-\infty}^{\infty} j_n = 1 \ . \tag{28}$$

Since f is bounded and measurable, therefore we can define the C^∞ functions $f_n : \mathbf{R} \to \mathbf{R}$ as

$$f_n(u) = \int\limits_{-\infty}^{\infty} j_n(u-v) f(v) \mathrm{d}v \qquad u \in \mathbf{R} \ . \tag{29}$$

For these functions we have

Proposition 6 *(i)* $\sup |f_n| \le \sup |f|$ *for all* $n \in \mathbf{N}$.

(ii) For all $n \in \mathbf{N}$ *there exists* L_n*, such that* $|f_n(u) - f_n(v)| \le L_n |u - v|$ *for all* $u, v \in \mathbf{R}$.

Proof. The proof of (i) is obvious from (28) and (29). For the Lipschitz continuity (ii) we observe that f_n' is bounded:

$$|f_n'(u)| \le \int\limits_{-\infty}^{\infty} |j_n'(u-v) f(v)| \mathrm{d}v \le \int\limits_{-\infty}^{\infty} |j_n'(v)| \mathrm{d}v \ \sup |f| = n \int\limits_{-\infty}^{\infty} |j'(v)| \mathrm{d}v \ \sup |f| =: L_n \ .$$

□

3.2 STEP2

Now, we solve the following approximating equations:

$$\partial_t u_n = \Delta u_n + f_n(u_n) \tag{30}$$

$$u_n(x,0) = \alpha(x) \quad x \in \mathbf{R}^N \tag{31}$$

on $\mathbf{R}^N \times [0,\infty)$, where $\alpha \in BUC(\mathbf{R}^N)$ and f_n defined by (29) is bounded and (globally) Lipschitz continuous by Proposition 6.

Let us introduce $F_n : X \to X$ as

$$F_n(\psi) = f_n \circ \psi \qquad \psi \in X \ .$$

Then F_n is bounded and (globally) Lipschitz continuous. Hence applying Proposition 5 we obtain the existence of a continuous function $U_n : [0,\infty) \to X$, such that

$$U_n(t) = T(t)\alpha + \int\limits_0^t T(t-s) F_n(U_n(s)) \mathrm{d}s. \tag{32}$$

Let $u_n : \mathbf{R}^N \times [0,\infty) \to \mathbf{R}$ be defined as $u_n(x,t) = U_n(t)(x)$, then we have

$$u_n(x,t) = \int\limits_{\mathbf{R}^N} K(x-y,t)\alpha(y) \mathrm{d}y + \int\limits_0^t \int\limits_{\mathbf{R}^N} K(x-y,t-s) f_n(u_n(y,s)) \mathrm{d}y \mathrm{d}s \ . \tag{33}$$

3.3 STEP3

Now we prove the existence of a solution u. Let us fix an arbitrary positive number $t^* > 0$. Since $\sup|f_n| \le \sup|f|$ by Proposition 6, (33) implies

$$|u_n(x,t)| \le \|\alpha\| + t^* \sup|f| \qquad \text{for all } n \in \mathbf{N},\ x \in \mathbf{R}^N,\ t \in [0,t^*]\ .$$

According to Proposition 4 the sequence (u_n) is equicontinuous in $\mathbf{R}^N \times [0,t^*]$, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $n \in \mathbf{N}$, $t_1, t_2 \in [0,t^*]$ and $x_1, x_2 \in \mathbf{R}^N$,

$$|(x_1,t_1) - (x_2,t_2)| < \delta \qquad \text{implies} \qquad |u_n(x_1,t_1) - u_n(x_2,t_2)| < \varepsilon\ .$$

Thus by the Arzelá–Ascoli theorem (u_n) has a uniformly convergent subsequence on $C \times [0,t^*]$ for any compact subset $C \subset \mathbf{R}^N$. Hence there exists a continuous function $u : \mathbf{R}^N \times [0,\infty) \to \mathbf{R}$ and a subsequence denoted also by (u_n), which tends uniformly to u on compact subsets of $\mathbf{R}^N \times [0,\infty)$.

3.4 STEP4

We show the existence of h satisfying

$$u(x,t) = \int_{\mathbf{R}^N} K(x-y,t)\alpha(y)\mathrm{d}y + \int_0^t \int_{\mathbf{R}^N} K(x-y,t-s)h(y,s)\mathrm{d}y\mathrm{d}s \qquad (x,t) \in Q\ . \tag{34}$$

Since $|f_n(x)| \le \sup|f|$ for all $x \in \mathbf{R}$, $f_n \circ u_n$ is a bounded sequence in $L^\infty(Q)$. Hence it has a weak-$*$ convergent subsequence (denoted also by $f_n \circ u_n$), because $L^\infty(Q)$ is the dual space of $L^1(Q)$, and therefore the Banach–Alaoglu theorem can be applied. Let us denote the weak-$*$ limit of this subsequence by $h \in L^\infty(Q)$. Since $(y,s) \mapsto K(x-y,t-s)$ is in $L^1(Q)$, we get

$$\int_0^t \int_{\mathbf{R}^N} K(x-y,t-s)f_n(u_n(y,s))\mathrm{d}y\mathrm{d}s \to \int_0^t \int_{\mathbf{R}^N} K(x-y,t-s)h(y,s)\mathrm{d}y\mathrm{d}s \qquad \text{as } n \to \infty.$$

Thus passing to the limit $n \to \infty$ in (33) we obtain (34).

3.5 STEP5

Proposition 7 *For every $\varepsilon > 0$ and every compact set $C \subset \overline{Q}$ there exists $N \in \mathbf{N}$, such that for $n > N$*

$$\underline{f}_\varepsilon(u(y,s)) \le f_n(u_n(y,s)) \le \overline{f}_\varepsilon(u(y,s)) \quad \textit{for all } (y,s) \in C\ , \tag{35}$$

where

$$\underline{f}_\varepsilon(u) = \inf\{f(s) : s \in (u-\varepsilon, u+\varepsilon)\} \quad ; \quad \overline{f}_\varepsilon(u) = \sup\{f(s) : s \in (u-\varepsilon, u+\varepsilon)\}\ .$$

Proof. Since u_n tends uniformly to u in C, there exists $N \in \mathbf{N}$ (not depending on the point $(y,s) \in C$), such that for $n > N$ we have

$$|u_n(y,s) - u(y,s)| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \frac{1}{n} < \frac{\varepsilon}{2}. \tag{36}$$

If $a \in (u_n(y,s) - 1/n, u_n(y,s) + 1/n)$ and $n > N$ then by (36) $a \in (u(y,s) - \varepsilon, u(y,s) + \varepsilon)$ and hence $\underline{f}_\varepsilon(u(y,s)) \le f(a) \le \overline{f}_\varepsilon(u(y,s))$. Therefore

$$f_n(u_n(y,s)) = \int_{-\infty}^{\infty} j_n(u_n(y,s) - a)f(a)\mathrm{d}a = \int_{u_n(y,s)-1/n}^{u_n(y,s)+1/n} j_n(u_n(y,s) - a)f(a)\mathrm{d}a \le$$

$$\le \overline{f}_\varepsilon(u(y,s)) \int_{-\infty}^{\infty} j_n(u_n(y,s) - a)\mathrm{d}a = \overline{f}_\varepsilon(u(y,s)).$$

Similarly, we obtain $f_n(u_n(y,s)) \ge \underline{f}_\varepsilon(u(y,s))$, which proves the statement. □

Proposition 8

$$\underline{f}(u(x,t)) \le h(x,t) \le \overline{f}(u(x,t)) \qquad a.e.\ in\ Q\ . \tag{37}$$

Proof. Let $\beta \in L^1(Q)$ be an arbitrary nonnegative function with compact support C. Multiplying (35) by β and integrating over Q we obtain

$$\int_Q (\underline{f}_\varepsilon \circ u)\beta \le \int_Q (f_n \circ u_n)\beta \le \int_Q (\overline{f}_\varepsilon \circ u)\beta$$

for $n > N$. Since $f_n \circ u_n \to h$ weak-$*$ in $L^\infty(Q)$ and $\beta \in L^1(Q)$, passing to the limit $n \to \infty$ we get

$$\int_Q (\underline{f}_\varepsilon \circ u)\beta \le \int_Q h\beta \le \int_Q (\overline{f}_\varepsilon \circ u)\beta.$$

By Lebesgue's dominated convergence theorem we obtain as $\varepsilon \to 0$

$$\int_Q (\underline{f} \circ u)\beta \le \int_Q h\beta \le \int_Q (\overline{f} \circ u)\beta.$$

By choosing appropriate indicator functions for β, this proves the statement. □

3.6 STEP6

Since u satisfies (34) and h is bounded, Proposition 3 shows that u is continuously differentiable w.r.t. x, that is $u \in C^{1,0}(Q)$.

3.7 STEP7

Since f_n is differentiable, it follows from (33) that u_n is a classical solution, i.e.,

$$\partial_t u_n = \Delta u_n + f_n(u_n)$$

Multiplying this equation by a test function $\varphi \in C_0^\infty(Q)$ and integrating over Q one obtains

$$\int_Q (u_n \partial_t \varphi - \langle \nabla u_n, \nabla \varphi \rangle + f_n(u_n)\varphi) = 0 \tag{38}$$

Since u_n tends to u uniformly on compact subsets of Q, we have

$$\int_Q u_n \partial_t \varphi \to \int_Q u \partial_t \varphi$$

and, since u is continuously differentiable by STEP6,

$$\int_Q \langle \nabla u_n, \nabla \varphi \rangle = -\int_Q u_n \Delta \varphi \to -\int_Q u \Delta \varphi = \int_Q \langle \nabla u, \nabla \varphi \rangle$$

as n goes to infinity. Moreover, we have

$$\int_Q f_n(u_n)\varphi \to \int_Q h\varphi$$

because $\varphi \in L^1(Q)$ and $f_n(u_n)$ tends weakly–$*$ to h in $L^\infty(Q)$. Hence passing to the limit in (38) one obtains (4). Thus the proof of the Theorem is complete.

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