Periodic Perturbations of Non-Conservative Second Order Differential Equations

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Abstract

Consider the Lienard system $u'' + f(u)u' + g(u) = 0$ with an isolated periodic solution. This paper concerns the behavior of periodic solutions of Lienard system under small periodic perturbations.
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1 Introduction

Consider the second order differential equation of type

$$(E_\epsilon) \qquad\qquad x'' + g(t, x, x', \epsilon) = 0$$

where $\epsilon > 0$ is a small parameter, g is a T-periodic function in t and $g(t, x, 0, 0) = g(x)$ is independent of t.
In the case where g is independent of x' and is continuously differentiable in x the existence problem of non constant periodic solutions of (E_ϵ) has been studied by many authors.
Indeed, in the latter case certain among them proved existence of solutions

of $\quad x'' + g(t, x, \epsilon) = 0$. For a review see Chow-Hale [C-H] and Hale [H]. But many examples (as the one given by Hartman) proved non existence cases of that equation if we suppose g dependent on x'. We then cannot expect to generalize their results.

Let the following equation, which is a perturbed Lienard type

$$(1_\epsilon) \qquad x'' + f(x)x' + g(x) = \epsilon h(\frac{t}{T}, x, x', \epsilon)$$

where h is T-periodic in t, f and g are functions only dependent on x, satisfying conditions defined below. We look for periodic solutions of (1_ϵ) for ϵ small enough under some additional hypothesis. It is assumed that the unperturbed system has an isolated periodic solution. The perturbation is supposed to be *controllably periodic* in the Farkas sense [F2], i.e. it is periodic with a period which can be chosen appropriately.
We shall prove an existence theorem for this equation.

Loud [L] already proves for the case $f(x) \equiv c$, the existence of a periodic solution of the equation

$$x'' + cx' + g(x) = \epsilon h(t), \tag{1}$$

where the perturbation does not depend on the state. He uses for that a variant of the implicit function theorem. More exactly, he considers a function $\ g(x) = xk(x)\ $ where k is continuously differentiable and $k(x) > 0, \quad x \neq 0$. Moreover,

$$x\frac{d}{dx}k(x) > 0, \quad x \neq 0$$

or

$$x\frac{d}{dx}k(x) < 0, \quad x \neq 0$$

always holds with the possible exception of isolated points. Notice that the above conditions imply on one hand the monotonicity of the period function T for the system $x'' + g(x) = 0$. When g is in addition differentiable these conditions imply on the other hand $\ g''(0) = 0$.
Let $\ u(t)\ $ be a non-constant ω-periodic solution of the equation

$$x'' + g(x) = 0$$

and define

$$F(s) = \int_0^\infty u'(t+s)f(t)dt.$$

Also, [L] observes that if for some s_0, $F(s_0) = 0$ while $F'(s_0) \neq 0$ then for sufficiently small $\epsilon > 0$ there exists an ω-periodic solution $v(t, \epsilon)$ of the perturbed equation

$$x'' + g(x) = \epsilon f(t) = \epsilon f(t+\omega)$$

2 Existence and non-existence of periodic solutions of E_ϵ

2.1 A non existence result

According to P. Hartman ([H], p. 39), equation (1_ϵ) in general does not have a non constant periodic solution, even if $xg(t, x, x') > 0$.
The following example given by Moser proves the non existence of a non constant periodic solution of

$$x'' + \phi(t, x, x') = 0.$$

Let

$$\phi(t, x, y) = x + x^3 + \epsilon f(t, x, y), \qquad \epsilon > 0$$

satisfying the following conditions for $\phi \in C^1(R^3)$, $f(t+1, x, y) = f(t, x, y)$, with

$$f(0,0,0) = 0, \qquad f(t, x, y) = 0 \text{ } if \text{ } xy = 0$$

$$\frac{\phi}{x} \to \infty \quad when \quad x \to \infty$$

uniformly in $(t, y) \in R^2$,

$$\frac{\delta f}{\delta y} > 0 \text{ } if \text{ } xy > 0, \quad and \quad \frac{\delta f}{\delta y} = 0 \text{ } otherwise.$$

x, y verifying $\mid x \mid < \epsilon, \mid y \mid < \epsilon$.
In fact, we have $xf(t, x, y)$ and $yf(t, x, y)) > 0$ if $xy > 0, \mid x \mid < \epsilon, y$ arbitrary and $\phi = 0$ otherwise.

The function $V = 2x^2 + x^4 + 2x'^2$ satisfies $V' = -4\epsilon x' f(t, x, x')$, so that $V' < 0$ if $xx' > 0, \mid x \mid < \epsilon$ and $V' = 0$ otherwise.
Thus x cannot be periodic unless $V' = 0$.

This example is significant because it shows kind of difficulties encountered to establish existence results of periodic solutions for Equation E_ϵ. To that end, the existence problem will be more convenient to study when additional hypotheses on the period are required. For example, the period of the perturbed Lienard equation $x'' + f(x)x' + g(x) = \epsilon h(t, x, x', \epsilon)$. has to be 'controlled' in order to state existence of periodic solution.

2.2 Case where g is independent of x'

Consider an equation of the type

$$x'' + \phi(t, x, \epsilon) = 0, \tag{2}$$

where $\epsilon > 0$ is a small parameter, ϕ is a continuous function, $T-$periodic in t such that $\phi(t, x, 0) = \tilde{g}(x)$.

More precisely, under the following hypotheses for the function g defined on $R \times (\alpha, \beta) \times]0, \epsilon_0]$. :

$$\begin{cases} (1) & \phi \ is \ T \ - \ periodic \ on \ t \\ (2) & \phi(t, x, 0) = \tilde{g}(x) \\ (3) & if \quad x \neq 0, we \ have \ \tilde{g}(x)x > 0. \end{cases} \tag{3}$$

That means for $\epsilon = 0$ the autonomous system

$$\begin{cases} x' = y \\ y' = -\tilde{g}(x) \end{cases} \tag{4}$$

has the origin $(0, 0)$ as a center.
This means the flow induced by the vector field of the Hamiltonian system (4) has a stationary point at the origin and is surrounded by a family of periodic orbits. Each orbit γ of this family lies on an energy level, say c, and $\gamma \equiv \gamma(c)$. The period function $T(c)$ depending on c is the minimal period of this orbit. We say T is monotone if the function $T(c)$ is monotone. The dependence of the period on the energy and the monotonicity conditions has been studied by a number of authors. For these questions, we refer to [C-C].

Using a version of the fixed point theorem due to W. Ding, P. Buttazzoni and A. Fonda [B-F] proved that there are periodic solutions of (2) provided that the period function $T = T(c)$ of the autonomous associated system is monotone and that ϵ is small enough. ϕ is assumed to be (only) continuous. More precisely, they show that if the function $\phi(t, x, \epsilon)$ is continuous, then the periodic solutions of such equations may be located near solutions of the autonomous equation, provided that periodic solutions of (4) exist and that the period function is strictly monotone. Moreover, there is a solution making exactly N rotations around the origin in the time kT.
Their results improve those of Loud [L], who assumed that the function ϕ had to be continuously differentiable.
Moreover, in light of the preceding example [H], it seems that the methods described above do not generalize if one supposes ϕ dependent on x' : $\phi \equiv \phi(t, x, x', \epsilon)$.
So, another condition on the period appears to be necessary to obtain existence of periodic solutions of the perturbed equation.
Nevertheless, one can show an analogous result to the preceding one under more restrictive hypotheses. By an appropriated choice of the period of a periodic solution of the perturbed Lienard equation.

3 A controllably periodic perturbation

One refers to a method due to Farkas inspired by the one of Poincaré. The determination of controllably periodic perturbed solution. This method proved to be itself very effective particularly for the perturbations of various autonomous systems. We know for example a good application for perturbed Van der Pol equations type [F2]. The perturbation is supposed to be 'controllably periodic', i.e., it is periodic with a period which can be chosen appropriately. Under very mild conditions it is proved that to each small enough amplitude of the perturbation there belongs a one parameter family of periods such that the perturbed system has a unique periodic solution with this period.
We logically may expect that the Farkas method can be again applied for perturbed Lienard equations. This has been considered and proved in the autonomous case by Farkas himself [F1] .
Our proof we give here made it more simple and contains some improvements

in using in particular methods of [F1] to estimate existence regions of periodic solutions for that equation.
Notice that the same problem has been considered before in the paper of Farkas and Abdel Karim [F-A] but our results are more general.

3.1 Basic hypotheses

Let us consider the (unperturbed) Lienard equation

$$(L) \qquad u'' + f(u)u' + g(u) = 0.$$

In order to have a unique periodic solution we suppose the functions f and g are of class C^2. The integrals

$$F(x) = \int_0^x f(t)dt, \qquad G(x) = \int_0^x g(t)dt$$

of f and g respectively are such that $lim_{x\to\infty} F(x) = \infty$, and $lim_{x\to\infty} G(x) = \infty$. It is assumed that F has a unique zero. Then it is known [C-L] that (L) has a stable non constant periodic solution $u_0(t)$ with period τ_0.
Equation (L) is usually studied by means of an equivalent plane system. The most used ones are :

$$\begin{cases} u' = v \\ v' = -g(u) - f(u)v \end{cases} \tag{5}$$

and also

$$\begin{cases} u' = v - F(u) \\ v' = -g(u) \end{cases} \tag{6}$$

In fact, they are equivalent to the 2-dimensional system

$$(S) \qquad \dot{x} = h(x)$$

after introducing the notations $x = col[x_1, x_2]$

$$\begin{cases} x_1 = -\dot{u}(t) - F(u(t)) \\ x_2 = u(t) \end{cases} \tag{7}$$

where $x = col[x_1, x_2]$ and $h(x) = col[g(x_2), -x_1 - F(x_2(t))$.
Suppose

$$u_0(0) = a, \qquad u_0'(0) = 0 > 0$$

so that the periodic solution of period τ_0 of the variational system

$$\dot{y} = h'_x(p(t))y$$

is

$$\dot{p}(t) = col[g(u_0(t)), \dot{u}_0(t)],$$

where

$$p(t) = col[-\dot{u}_0(t) - F(u_0(t)), u_0(t)].$$

So, the initial conditions are

$$p(0) = col[-F(a), a], \qquad \dot{p}(0) = col[g(a), 0].$$

Let us consider the following perturbed Lienard equation of the form

$$(L_R) \qquad \ddot{u} + f(u)\dot{u} + g(u) = \epsilon\gamma(\frac{t}{\tau}, u, \dot{u})$$

where $t \in R$, $\epsilon \in R$ is a small parameter, $\mid \epsilon \mid < \epsilon_0$, τ is a real parameter such that $\mid \tau - \tau_0 \mid < \tau_1$ for some $0 < \tau_1 < \frac{\tau_0}{2}$.
Moreover, the closed orbit

$$\{(u, v) \in R^2 : u(t) = u_0(t),\ v(t) = \dot{u}_0(t),\ t \in [0, \tau_0]\}$$

belongs to the region $\{(u, v) \in R^2 : u^2 + v^2 < r^2\}$.
In the same way as for (L), the 2-dimensional equivalent system for (L_R) is

$$(S_L) \qquad \dot{x} = h(x) + \epsilon q(\frac{t}{\tau}, x)$$

where $q = col[q_1, q_2]$,

$$\begin{cases} q_1 = -\gamma(\frac{t}{\tau}, x_2, -x_1 - F(x_2)) \\ q_2 = 0 \end{cases} \tag{8}$$

3.2 Existence of periodic solutions of (L_R)

Now we will use the Poincaré method for the determination of the approximate solution of the perturbed equation (L_R). The existence of the fundamental matrix solution of the first variational system of $\dot{x} = h(x)$ and the unique periodic solution $p(t)$ corresponding to $u_0(t)$ is assumed.
In order to get estimates for the existence of periodic solutions we have to calculate some constants. Following Farkas [F1], the Jacobi matrix J has the following form

$$J(\tau_0) = -I + \begin{pmatrix} g(a) & 0 \\ 0 & 0 \end{pmatrix} + Y(\tau_0)$$

$I = Id_2$ and $Y(t)$ is the fundamental solution matrix of the varational system with $Y(0) = I$

$$\dot{y} = \begin{pmatrix} O & g'(u_0(t)) \\ -1 & -f(u_0(t)) \end{pmatrix} y. \tag{9}$$

It is proved, that if $detJ(\tau_0) \neq 0$ then there exist uniquely determined functions $\tau(\epsilon, \phi)$ and $h(\epsilon, \phi)$ defined in the neighborhood of $(0,0)$ such that the function

$$u(t; \phi, p_0 + h(\epsilon, \phi), \epsilon, \tau(\epsilon, \phi))$$

is a periodic solution of system (S_L) with $\tau(0,0) = \tau_0$ and $h(0,0) = 0$. Moreover, an estimate is given for the region in which the the variables ϵ and ϕ may vary . For example in evaluating the norm of the difference of Jacobi matrices $J(\epsilon, \phi, \tau, h) - J(0, 0, \tau_0, 0)$.

A trivial τ_0-periodic solution of (9) is $col[g(u_0(t)), \dot{u}_0(t)]$. A trivial calculation gives the other linearly independent solution of (9)

$$col[g(u_0(t))v(t), \dot{u}0(t)v(t) + \frac{g(u_0(t))}{g'(u_0(t))}\dot{v}(t)]$$

where

$$v(t) = \int_0^t [g_0(s)]^{-2} g'(u_0(t)) exp[-\int_0^s f(u_0(\sigma))d\sigma] ds$$

for $t \in [0, \tau_0]$.

Then the fundamental solution matrix of (9) with $Y(0) = I$ is

$$Y(t) = \begin{pmatrix} \frac{g(u_0(t))}{g(a)} & g(a)g(u_0(t))v(t) \\ \frac{\dot{u}_0(t)}{g(a)} & g(a)\dot{u}_0(t)v(t) + g(a)\frac{g(u_0(t))}{g'(u_0(t))}\dot{v}(t) \end{pmatrix}.$$

According to Liouville's formula the Wronskian determinant $W(t)$ with $W(o) = 1$ is given by

$$W(t) = exp[-\int_0^t f(u_0(\tau))d\tau].$$

The characteristic multipliers of (9) are $\rho_1 = 1$ and

$$\rho_2 = W(\tau_0) = exp[-\int_0^{\tau_0} f(u_0(\tau))d\tau].$$

$\rho_2 < 1$ if and only if

$$\int_0^{\tau_0} f(u_0(\tau))d\tau > 0. \tag{10}$$

The initial conditions give

$$Y(\tau_0) = \begin{pmatrix} 1 & g^2(a)v(\tau_0) \\ 0 & \rho_2 \end{pmatrix}.$$

Thus, we get

$$J = \begin{pmatrix} g(a) & g^2(a)v(\tau_0) \\ 0 & \rho_2 - 1 \end{pmatrix},$$

$$J^{-1} = \begin{pmatrix} g^{-1}(a) & g^2(a)v(\tau_0)(1-\rho_2)^{-1} \\ 0 & -(1-\rho_2)^{-1} \end{pmatrix}.$$

Therefore,

$$\| J^{-1} \| = 2 \; max \; [g^{-1}(a), (1-\rho_2)^{-1}, g^2(a)v(\tau_0)(1-\rho_2)^{-1}].$$

The inverse matrix of $Y(t)$ is

$$Y^{-1}(t) = W(t) \begin{pmatrix} g(a)\dot{u}_0(t)v(t) + g(a)\frac{g(u_0(t))}{g'(u_0(t))}\dot{v}(t) & -g(a)g(u_0(t))v(t) \\ -\frac{\dot{u}_0(t)}{g(a)} & \frac{g(u_0(t))}{g(a)} \end{pmatrix}.$$

Now we have to determine the constants for system (L_R). Following [F1] let us denote

$$S = \{x = (x_1, x_2) \in R^2 \; / \; {x_2}^2 + [-x_1 - F(x_2(t))]^2 < r^2\}$$

$$\begin{cases} g_0 := max_{x\in S} \mid g(x_2) \mid, \\ g_1 := max_{x\in S} \mid g'(x_2) \mid, \\ g_2 = max_{x\in S} \mid g''(x_2) \mid . \end{cases} \tag{11}$$

$$f_1 := max_{x\in S} \mid f(x_2) \mid, \qquad f_2 := max_{x\in S} \mid f'(x_2) \mid$$

$$\begin{cases} q_0 := max_{x\in S, s\in R} \mid q(s,x) \mid, \\ q_1 := max_{x\in S, s\in R} \mid q'_x(s,x) \mid, \\ q_2 := max_{x\in S, s\in R} \mid q'_s(s,x) \mid . \end{cases} \tag{12}$$

$$K := max_{t\in[\frac{-\tau_0}{2},\tau_0]} \mid Y(t) \mid, \qquad K_{-1} := max_{t\in[\frac{-\tau_0}{2},\tau_0]} \mid Y^{-1}(t) \mid .$$

Thus, we may deduce that

$$P := max_{t\in[\frac{-\tau_0}{2},\tau_0]} \mid \dot{p}(t) \mid \leq \frac{K}{2}.$$

The initial phase ϕ and the period τ have to verify the following, which can be easily obtained from the above estimates, see [F1]

$$\phi < \frac{\tau_0}{2}, \qquad \mid \tau - \tau_0 \mid < \frac{\tau_0}{2}.$$

If in addition we suppose ϵ and h are such that

$$\frac{3}{2} g_0 \mid \epsilon \mid + \mid h \mid < \sigma exp(-\frac{3}{2} g_1 \tau_0)$$

(here σ is the distance between the path of the periodic solution and the boundary of S) then a solution of (L_R) exists.
Then, we have proved the following

Theorem 1 *If 1 is a simple characteristic multiplier of (9) (that means inequality (10) holds) then there are two functions* $\tau, h : U \to R$ *and a constant* $\tau_1 < \frac{\tau_0}{2}$ *such that the periodic solution* $u(t,\phi,a+h(\epsilon,\phi),\epsilon,\tau(\epsilon))$ *of equation*

$$(L_R) \qquad \ddot{u} + f(u)\dot{u} + g(u) = \epsilon\gamma(\frac{t}{\tau(\epsilon)}, u, \dot{u})$$

exists for $(\epsilon,\phi) \in U$, *and* $\mid \tau - \tau_0 \mid < \tau_1,\ \tau(0,0) = \tau_0,\ h(0,0) = 0.$

3.3 Special cases

As a corollary we may deduce from the above some results about autonomous perturbations of the Lienard system

$$(L_{R_A}) \qquad \ddot{u} + f(u)\dot{u} + g(u) = \epsilon\gamma$$

where the perturbation is independent on the time variable $\gamma \equiv \gamma(u, \dot{u}, \epsilon, \tau)$. The equivalent plane system is of the form

$$\dot{x} = h(x) + \epsilon q(\frac{t}{\tau}, x)$$

where $q = col[q_1, q_2]$.
Consider the following autonomous perturbed Lienard equation of the form

$$(L_{R_A}) \qquad \ddot{u} + f(u)\dot{u} + g(u) = \epsilon\gamma \equiv \gamma(u, \dot{u}, \epsilon, \tau)$$

where $t \in R$, $\epsilon \in R$ is a small parameter, $\mid \epsilon \mid < \epsilon_0$, τ is a real parameter such that $\mid \tau - \tau_0 \mid < \tau_1$ for some $0 < \tau_1 < \frac{\tau_0}{2}$.
Moreover, the closed orbit

$$\{(u, v) \in R^2 : u(t) = u_0(t),\ v(t) = \dot{u}_0(t),\ t \in [0, \tau_0]\}$$

belongs to the region $S = \{(u, v) \in R^2 : u^2 + v^2 < r^2\}$. γ is a function of class C^2.
In this case the perturbation is independent on the initial phase ϕ.
We then have the following

Corollary 2 *Suppose inequality (10) holds, then there are constants* ϵ_0 *and* $\tau_1 < \frac{\tau_0}{2}$ *such that to each* $\epsilon \in [-\epsilon_0, \epsilon_0]$ *there exist two functions* τ, h *only dependent on* $\epsilon : \tau \equiv \tau(\epsilon),\ h \equiv h(\epsilon)$ *such that the problem*

$$\begin{cases} L_{R_A}) \qquad \ddot{u} + f(u)\dot{u} + g(u) = \epsilon\gamma \equiv \epsilon\gamma(u, \dot{u}, \epsilon, \tau) \\ u(0) = a + h(\epsilon), \quad \dot{u}(0) = 0 \end{cases} \tag{13}$$

has a unique periodic non constant solution $u(t, \epsilon)$ *with period* $\tau(\epsilon)$. *Moreover,* $\tau(0) = \tau_0$, *and* $\mid \tau - \tau_0 \mid < \tau_1,\ \tau(0) = \tau_0,\ h(0) = 0$.

On other hand, a second special case may occur when the perturbation does not depend on the state of the system. That means the perturbation is independent on u

$$\gamma \equiv \gamma(\frac{t}{\tau}).$$

Then the above estimates can be easely calculated.

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