Positive Solutions of Three-Point Nonlinear Second Order Boundary Value Problem

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Abstract

In this paper we apply a cone theoretic fixed point theorem and obtain conditions for the existence of positive solutions to the three-point nonlinear second order boundary value problem

$$u''(t) + \lambda a(t) f(u(t)) = 0, \quad t \in (0,1)$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $0 < \eta < 1$ *and* $0 < \alpha < \frac{1}{\eta}$.

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1 Introduction

In this paper, we are concerned with determining values for λ so that the three-point nonlinear second order boundary value problem

$$u''(t) + \lambda\, a(t) f(u(t)) = 0, \quad t \in (0,1) \tag{1.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1), \tag{1.2}$$

where $0 < \eta < 1$,

(A1) the function $f : [0,\infty) \to [0,\infty)$ is continuous,

(A2) $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval,

(L1) $\lim_{x \to 0} \frac{f(x)}{x} = \infty$,

(L2) $\lim_{x \to \infty} \frac{f(x)}{x} = \infty$,

(L3) $\lim_{x \to 0} \frac{f(x)}{x} = 0$, (L4) $\lim_{x\to\infty} \frac{f(x)}{x} = 0$,

(L5) $\lim_{x\to 0} \frac{f(x)}{x} = l$ with $0 < l < \infty$,
and

(L6) $\lim_{x\to\infty} \frac{f(x)}{x} = L$ with $0 < L < \infty$

has positive solutions. In the case $\lambda = 1$, Ruyun Ma [11] showed the existence of positive solutions of (1.1)-(1.2) when f is superlinear ($l = 0$ and $L = \infty$), or f is sublinear ($l = \infty$ and $L = 0$). In this research it is not required that f be either sublinear or superlinear. As in [8] and [11], the arguments that we present here in obtaining the existence of a positive solution of (1.1)-(1.2), rely on the fact that solutions are concave downward. In arriving at our results, we make use of Krasnosel'skii fixed point theorem [10]. The existence of positive periodic solutions of nonlinear functional differential equations have been studied extensively in recent years. For some appropriate references we refer the reader to [1], [2], [3], [4], [5], [6], [8], [9], [12], [13], [14], [15], [16] and the references therein.
In section 2, we state some known results and Krasnosel'skii fixed point theorem [10]. In section 3, we construct the cone of interest and present a lemma, four theorems and a corollary. In each of the theorems and the corollary, an open interval of eigenvalues is determined, which in return, imply the existence of a positive solution of (1.1)-(1.2) by appealing to Krasnosel'skii fixed point theorem.
We say that $u(t)$ is a solution of (1.1)-(1.2) if $u(t) \in C[0,1]$ and $u(t)$ satisfies (1.1)-(1.2).

2 Preliminaries

Theorem 2.1 (Krasnosel'skii) Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P}$ be a cone in $\mathcal{B}$. Suppose Ω_1 and Ω_2 are bounded open subsets of $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

is a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2$; or

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

In arriving at our results, we need to state four preliminary Lemmas. Consider the boundary value problem

$$u''(t) + y(t) = 0, \quad t \in (0,1), \qquad (I)$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1), \qquad (II)$$

Lemma 2.2 Let $\alpha\eta \neq 1$. Then, for $y \in C[0,1]$, the boundary value problem $(I)-(II)$ has the unique solution

$$\begin{aligned} u(t) &= \lambda\Big[-\int_0^t (t-s)y(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)y(s)ds \\ &+\frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)ds\Big]. \end{aligned} \quad (2.1)$$

The proof of (2.1) follows along the lines of the proof that is given in [7] in the case $\lambda = 1$, and hence we omit it.

The proofs of the next three lemmas can be found in [11].

Lemma 2.3 Let $0 < \alpha < \frac{1}{\eta}$ and assume $(A1)$ and $(A2)$ hold. Then, the unique solution of $(I)-(II)$ is non-negative for all $t \in (0,1)$.

Lemma 2.4 Let $\alpha\eta > 1$ and assume $(A1)$ and $(A2)$ hold. Then, $(I)-(II)$ has no positive solution.

Lemma 2.5 Let $0 < \alpha < \frac{1}{\eta}$ and assume $(A1)$ and $(A2)$ hold. Then, the unique solution of $(I)-(II)$ satisfies

$$\inf_{t\in[\eta,1]} u(t) \geq \gamma||u||,$$

where $\gamma = \min\{\alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \eta\}$.

The proofs of Lemmas 2.3, 2.4 and 2.5 depend on the fact that under conditions (A1) and (A2) the solution $u(t)$ concave downward for $t \in (0,1)$.

3 Main Results

Assuming (A1) and (A2), it follows from Lemmas 2.3 and 2.4, that (1.1)-(1.2) has a non-negative solution if and only if $\alpha < \frac{1}{\eta}$. Therefore, throughout this paper we assume that $\alpha < \frac{1}{\eta}$. Let $\mathcal{B} = C[0,1]$, with $||y|| = \sup_{t\in[0,1]} |y(t)|$.

Define a cone, $\mathcal{P}$, by

$$\mathcal{P} = \{y \in C[0,1] : y(t) \geq 0, t \in (0,1) \text{ and } \min_{t\in[\eta,1]} y(t) \geq \gamma\|y\|\}.$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$

$$\begin{aligned} Tu(t) &= \lambda\Big[-\int_0^t (t-s)a(s)f(u(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\ &+\frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds\Big]. \end{aligned} \quad (3.1)$$

By Lemma 2.2, (1.1)-(1.2) has a solution $u = u(t)$ if and only if u solves the operator defined by (3.1). Note that, for $0 < \alpha < 1/\eta$, the first two terms on the right of (3.1) are less than or equal to zero. We seek a fixed point of T in the cone $\mathcal{P}$.

For the sake of simplicity, we let

$$A = \frac{\int_0^1 (1-s)a(s)ds}{1-\alpha\eta}, \tag{3.2}$$

and

$$B = \frac{\eta\int_\eta^1 (1-s)a(s)ds}{1-\alpha\eta}. \tag{3.3}$$

Lemma 3.1 Assume that $(A1)$ and $(A2)$ hold. If T is given by (3.1), then $T : \mathcal{P} \to \mathcal{P}$ and is completely continuous.

Proof: Let $\phi, \psi \in C[0,1]$. In view of $A1$, given an $\epsilon > 0$ there exists a $\delta > 0$ such that for $||\phi - \psi|| < \delta$ we have

$$\sup_{t\in[0,1]} |f(\phi) - f(\psi)| < \frac{\epsilon}{A[2+\alpha(1-\eta)]}.$$

Using (3.1) we have for $t \in (0,1)$,

$$\begin{aligned} |(T\phi)(t) - (T\psi)(t)| &\leq \int_0^1 (1-s)a(s)|f(\phi(s)) - f(\phi(s))|ds \\ &+ \frac{\alpha}{1-\alpha\eta}\int_0^1 (1-s)a(s)|f(\phi(s)) - f(\phi(s))|ds \\ &+\frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)|f(\phi(s)) - f(\phi(s))|ds \\ &\leq [(1-\alpha\eta)A + \alpha A + A]|f(\phi(s)) - f(\phi(s))| \\ &\leq A[2+\alpha(1-\eta)] \sup_{t\in[0,1]} |f(\phi) - f(\psi)| < \epsilon. \end{aligned}$$

Thus, T is continuous. Notice from Lemma 2.3 that, for $u \in \mathcal{P}, Tu(t) \geq 0$ on $[0,1]$. Also, by Lemma 2.5, $T\mathcal{P} \subset \mathcal{P}$. Thus, we have shown that $T : \mathcal{P} \to \mathcal{P}$. Next, we show that f maps bonded sets into bounded sets. Let D be a positive constant and define the set

$$K = \{x \in C[0,1] : ||x|| \leq D\}.$$

Since $A1$ holds, for any $x, y \in K$, there exists a $\delta > 0$ such that if $||x - y|| < \delta$, implies

$$|f(x) - f(y)| < 1.$$

We choose a positive integer N so that $\delta > \frac{D}{N}$. For $x(t) \in C[0,1]$, define $x_j(t) = \frac{jx(t)}{N}$, for $j = 0, 1, 2, ..., N$. For $x \in K$,

$$\begin{aligned} ||x_j - x_{j-1}|| &= \sup_{t\in[0,1]}\left|\frac{jx(t)}{N} - \frac{(j-1)x(t)}{N}\right| \\ &\leq \frac{||x||}{N} \leq \frac{D}{N} < \delta. \end{aligned}$$

Thus, $|f(x_j) - f(x_j - 1)| < 1$. As a consequence, we have

$$f(x) - f(0) = \sum_{j=1}^{N} \Big(f(x_j) - f(x_{j-1}) \Big),$$

which implies that

$$\begin{aligned} |f(x)| &\leq \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})| + |f(0)| \\ &< N + |f(0)|. \end{aligned}$$

Thus, f maps bounded sets into bounded sets. It follows from the above inequality and (3.1), that

$$\begin{aligned} ||(Tx)(t)|| &\leq \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)|f(x(s))|ds \\ &\leq \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)(N + |f(0)|) \\ &\leq A(N + |f(0)|). \end{aligned}$$

Next, for $t \in (0,1)$, we have

$$\begin{aligned} (Tx)'(t) &= \lambda \Big[-\int_0^t a(s)f(u(s))ds - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - s)a(s)f(u(s))ds \\ &\quad + \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)f(u(s))ds \Big]. \end{aligned}$$

Hence,

$$\begin{aligned} |(Tx)'(t)| &\leq \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)|f(x(s))|ds \\ &\leq A(N + |f(0)|). \end{aligned}$$

Thus, the set

$$\{(Tx) : x \in \mathcal{P}, ||x|| \leq D\}$$

is a family of uniformly bounded and equicontinuous functions on the set $t \in [0,1]$. By Ascoli-Arzela Theorem, the map T is completely continuous. This completes the proof.

Theorem 3.2 Assume that $(A1), (A2), (L5)$ and $(L6)$ hold. Then, for each λ satisfying

$$\frac{1}{\gamma BL} < \lambda < \frac{1}{Al} \tag{3.4}$$

(1.1)-(1.2) has at least one positive solution.

Proof: We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be given as in (3.4), and choose $\epsilon > 0$ such that

$$\frac{1}{\gamma B(L-\epsilon)} \leq \lambda \leq \frac{1}{A(l+\epsilon)}.$$

By condition $(L5)$, there exists $H_1 > 0$ such that $f(y) \leq (l+\epsilon)y$, for $0 < y \leq H_1$. So, choosing $u \in \mathcal{P}$ with $||u|| = H_1$, we have

$$\begin{aligned}
(Tu)(t) &\leq \lambda \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\leq \lambda \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)(l+\epsilon)u(s)ds \\
&\leq \lambda \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)(l+\epsilon)||u||ds \\
&= \lambda \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)(l+\epsilon)H_1 ds \\
&\leq \lambda A(l+\epsilon)\|u\| \leq \|u\|.
\end{aligned}$$

Consequently, $||Tu|| \leq ||u||$. So, if we set

$$\Omega_1 = \{y \in \mathcal{P} : \|y\| < H_1\},$$

then

$$||Tu|| \leq ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.5}$$

Next we construct the set Ω_2. Considering $(L6)$ there exists $\overline{H_2}$ such that $f(y) \geq (L-\epsilon)y$, for all $y \geq \overline{H_2}$. Let $H_2 = \max\{2H_1, \frac{\overline{H_2}}{\gamma}\}$ and set

$$\Omega_2 = \{y \in \mathcal{P} : \|y\| < H_2\}.$$

If $u \in \mathcal{P}$ with $||u|| = H_2$, then

$$\min_{t\in[\eta,1]} y(t) \geq \gamma||y|| \geq \overline{H_2}.$$

Thus, by a similar argument as in [11], we have

$$\begin{aligned}
(Tu)(\eta) &\geq \lambda \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(u(s))ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)(L-\epsilon)u(s)ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)(L-\epsilon)\gamma||u||ds \\
&= \lambda \frac{\gamma\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)(L-\epsilon)H_2 ds \\
&\geq \lambda B\gamma(L-\epsilon)\|u\| \\
&\geq \|u\|.
\end{aligned}$$

Thus, $||Tu\| \geq ||u||$. Hence

$$||Tu\| \geq ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.6}$$

Applying (i) of Theorem 2.1 to (3.5) and (3.6) yields that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete.

Theorem 3.3 Assume that $(A1), (A2), (L5)$ and $(L6)$ hold. Then, for each λ satisfying

$$\frac{1}{\gamma Bl} < \lambda < \frac{1}{AL} \tag{3.7}$$

(1.1)-(1.2) has at least one positive solution.

Proof: We construct the sets Ω_1 and Ω_2 in order to apply Theorem 2.1. Let λ be given as in (3.7), and choose $\epsilon > 0$ such that

$$\frac{1}{\gamma B(l-\epsilon)} \leq \lambda \leq \frac{1}{A(L+\epsilon)}.$$

By condition $(L5)$, there exists $H_1 > 0$ such that $f(y) \leq (l-\epsilon)y$, for $0 < y \leq H_1$. So, choosing $u \in \mathcal{P}$ with $||u|| = H_1$, we have

$$\begin{aligned}
(Tu)(\eta) &\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)f(u(s))ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)(l-\epsilon)u(s)ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)(l-\epsilon)\gamma||u||ds \\
&= \lambda \frac{\gamma\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)(l-\epsilon)H_1 ds \\
&\geq \lambda B\gamma(l-\epsilon)\|u\| \\
&\geq \|u\|.
\end{aligned}$$

Thus, $||Tu\| \geq ||u||$. So, if we let

$$\Omega_1 = \{y \in \mathcal{P} : \|y\| < H_1\},$$

then

$$||Tu|| \geq ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.8}$$

Next we construct the set Ω_2. Considering $(L6)$ there exists $\overline{H_2}$ such that $f(y) \leq (L+\epsilon)y$, for all $y \geq \overline{H_2}$.
We consider two cases; f is bounded and f is unbounded. The case where f is bounded is straight forward. If $f(y)$ is bounded by $Q > 0$, set

$$H_2 = \max\{2H_1, \lambda QA\}.$$

Then if $u \in \mathcal{P}$ and $||u|| = H_2$, we have

$$\begin{aligned}
(Tu)(t) &\le \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)f(u(s))ds \\
&\le \lambda \frac{Q}{1-\alpha\eta} \int_0^1 (1-s)a(s)ds \\
&= \lambda AQ \\
&\le H_2 \\
&= \|u\|.
\end{aligned}$$

Consequently, $||Tu|| \le ||u||$. So, if we set

$$\Omega_2 = \{y \in \mathcal{P} : \|y\| < H_2\},$$

then

$$||Tu|| \le ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.9}$$

When f is unbounded, we let $H_2 > \max\{2H_1, \overline{H_2}\}$ be such that $f(y) \le f(H_2)$, for $0 < y \le H_2$. For $u \in \mathcal{P}$ with $||u|| = H_2$,

$$\begin{aligned}
(Tu)(t) &\le \lambda \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)f(u(s))ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)f(H_2)ds \\
&\le \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)(L+\epsilon)H_2 ds \\
&= \lambda \frac{1}{1-\alpha\eta} \int_0^1 (1-s)a(s)(L+\epsilon)||u||ds \\
&= \lambda A(L+\epsilon)||u|| \\
&\le \|u\|.
\end{aligned}$$

Consequently, $||Tu|| \le ||u||$. So, if we set

$$\Omega_2 = \{y \in \mathcal{P} : \|y\| < H_2\},$$

then

$$||Tu|| \le ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.10}$$

Applying (ii) of Theorem 2.1 to (3.8) and (3.9) yields that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Also, applying (ii) of Theorem 2.1 to (3.8) and (3.10) yields that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete.

Theorem 3.4 Assume that $(A1)$, $(A2)$, $(L1)$ and $(L6)$ hold. Then, for each λ satisfying

$$0 < \lambda < \frac{1}{AL} \tag{3.11}$$

(1.1)-(1.2) has at least one positive solution.

Proof: Apply $(L1)$ and choose $H_1 > 0$ such that if $0 < y < H_1$, then

$$f(y) \geq \frac{y}{\lambda\gamma B}.$$

Define

$$\Omega_1 = \{y \in \mathcal{P} : \|y\| < H_1\}.$$

If $y \in \mathcal{P} \cap \partial\Omega_1$, then

$$\begin{aligned}
(Tu)(\eta) &\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(u(s))ds \\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)\frac{u(s)}{\lambda\gamma B}ds \\
&\geq \lambda\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)\frac{\gamma||u||}{\lambda\gamma B}ds \\
&\geq \|u\|.
\end{aligned}$$

In particular, $||Tu\| \geq ||u||$, for all $u \in \mathcal{P} \cap \partial\Omega_1$. In order to construct Ω_2, we let λ be given as in (3.11), and choose $\epsilon > 0$ such that

$$0 \leq \lambda \leq \frac{1}{A(L+\epsilon)}.$$

The construction of Ω_2 follows along the lines of the construction of Ω_2 in Theorem 3.3, and hence we omit it. Thus, by (ii) of Theorem 2.1, (1.1)-(1.2) has at least one positive solution.

Theorem 3.5 Assume that $(A1)$, $(A2)$, $(L2)$ and $(L5)$ hold. Then, for each λ satisfying

$$0 < \lambda < \frac{1}{Al} \tag{3.12}$$

(1.1)-(1.2) has at least one positive solution.

Proof: Assume $(L5)$ holds. Then, we may take the set Ω_1 to be the one obtained for Theorem 3.1. That is,

$$\Omega_1 = \{y \in \mathcal{P} : \|y\| < H_1\}.$$

Hence, we have

$$||Tu|| \leq ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_1.$$

Next, we assume $(L2)$. Choose $\overline{H_2} > 0$ such that $f(y) \geq \frac{y}{\lambda\gamma B}$, for $y \geq \overline{H_2}$. Let $H_2 = \max\{2H_1, \frac{\overline{H_2}}{\gamma}\}$ and set

$$\Omega_2 = \{y \in \mathcal{P} : \|y\| < H_2\}.$$

If $u \in \mathcal{P}$ with $||u|| = H_2$,

$$\begin{aligned}
(Tu)(\eta) &\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)f(u(s))ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)\frac{u(s)}{\lambda\gamma B}ds \\
&\geq \lambda \frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)a(s)\frac{\gamma||u||}{\lambda\gamma B}ds \\
&\geq \|u\|.
\end{aligned}$$

Consequently,

$$||Tu|| \geq ||u||, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2.$$

Applying (i) of Theorem 2.1 yields that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.
We state the next results as corollary, because by now, its proof can be easily obtained from the proofs of the previous results.

Corollary 3.6 Assume that $(A1)$ and $(A2)$ hold. Also, if either $(L3)$ and $(L6)$ hold, or, $(L4)$ and $(L5)$ hold, then (1.1)-(1.2) has at least one positive solution if λ satisfies either $1/(\gamma BL) < \lambda$, or, $1/(\gamma Bl) < \lambda$.

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