

ON THE UNIFORMLY CONTINUITY OF THE SOLUTION MAP FOR TWO DIMENSIONAL WAVE MAPS

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Abstract

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Abstract

Abstract. The aim of this paper is to analyze the properties of the solution map to the Cauchy problem for the wave map equation with a source term, when the target is the hyperboloid \mathcal{H}^2 that is embedded in \mathcal{R}^3 . The initial data are in $\dot{H}^1 \times L^2$. We prove that the solution map is not uniformly continuous.

Abstract

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In this paper we study the properties of the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ to the Cauchy problem

$$(1) \quad u_{tt} - \Delta u - (|u_t|^2 - |\nabla_x u|^2)u = g(t, x),$$

$$(2) \quad u(0, x) = u_\circ(x) \in \dot{H}^1(\mathcal{R}^2), \quad u_t(0, x) = u_1(x) \in L^2(\mathcal{R}^2)$$

in the case when $x \in \mathcal{R}^2$ and the target is the hyperboloid $\mathcal{H}^2 : u_1^2 + u_2^2 - u_3^2 = -1$, $\mathcal{H}^2 \hookrightarrow \mathcal{R}^3$. Here

$$\begin{aligned} |u_t|^2 &= u_{1t}^2 + u_{2t}^2 - u_{3t}^2, \\ |\nabla_x u|^2 &= |\nabla_{x_1} u|^2 + |\nabla_{x_2} u|^2, \\ |\nabla_{x_i} u|^2 &= u_{1x_i}^2 + u_{2x_i}^2 - u_{3x_i}^2, \quad i = 1, 2. \end{aligned}$$

More precisely, we prove that the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ to the Cauchy problem (1), (2) is not uniformly continuous.

In [1] is proved that the solution map isn't uniformly continuous in the case when $g \equiv 0$.

When we say that the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ is uniformly continuous we understand: *for every positive constant ϵ there exist positive constants δ and R such that for any two solutions $u, v : \mathcal{R} \times \mathcal{R}^2 \longrightarrow \mathcal{H}^2$ of (1), (2), with right hands $g = g_1, g = g_2$ of (1), so that*

$$(3) \quad E(0, u - v) \leq \delta, \quad \|g_1\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R, \quad \|g_2\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R, \quad \|g_1 - g_2\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R,$$

the following inequality holds

$$(4) \quad E(t, u - v) \leq \epsilon \quad \text{for } \forall t \in [0, 1],$$

where

$$E(t, u) := \|\partial_t u(t, \cdot)\|_{L^2(\mathcal{R}^2)}^2 + \|\nabla_x u(t, \cdot)\|_{L^2(\mathcal{R}^2)}^2.$$

Here we prove

Theorem 1. *There exist constant $\epsilon > 0$ such that for every pair of positive constants δ and R there exists smooth solutions $u, v: \mathcal{R} \times \mathcal{R}^2 \longrightarrow \mathcal{H}^2$ of (1), (2), with right hands $g = g_1, g = g_2$ of (1), so that*

$$E(0, u - v) \leq \delta, \quad \|g_1\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R, \quad \|g_2\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R, \quad \|g_1 - g_2\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq \delta,$$

and

$$E(1, u - v) \geq \epsilon.$$

Proof. We suppose that the solution map $(u_\circ, u_1, g) \longrightarrow u(t, x)$ to the Cauchy problem (1), (2) is uniformly continuous. Then for every $\epsilon > 0$ there exist positive constants δ and R such that for any solution u of (1), (2) with right hand g of (1) for which

$$(5) \quad E(0, u) \leq \delta, \quad \|g\|_{L^1([0,1]L^2(\mathcal{R}^2))} \leq R$$

and the inequality

$$(6) \quad E(t, u) \leq \epsilon$$

holds for every $t \in [0, 1]$ (in this case $v = 0$, which is solution of (1) with right hand $g = 0$). Let

$$(\star) \quad \begin{aligned} u &= (u_1, u_2, u_3), \\ u_1 &= \sinh \chi \cos \phi_1, \\ u_2 &= \sinh \chi \sin \phi_1, \\ u_3 &= \cosh \chi, \quad \chi \geq 0, \quad \phi_1 \in [0, 2\pi], \end{aligned}$$

$\chi = Y^2$, where Y is solution to the Cauchy problem

$$(7) \quad Y_{tt} - \Delta Y = 0,$$

$$(8) \quad Y(0, x) = 0, \quad Y_t(0, x) = q(x),$$

$$q(x) = \int_{\mathcal{R}^2} \sin(x\xi) \phi(\xi) d\xi,$$

$$\phi(\xi) \equiv \phi_N(\xi) = H(A_N) \frac{1}{\sqrt{|\xi|}},$$

$H(\cdot)$ is the characteristic function of correspond set, $x\xi = x_1\xi_1 + x_2\xi_2$,

$$A_N = \{\xi \in \mathcal{R}^2, \xi_1 = r \cos \phi, \xi_2 = r \sin \phi, N_\circ \leq |\xi| \leq N, \phi \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)\},$$

$N > N_o > 0$ are fixed such that N_o is close enough to N , $\sin(\xi\eta) \geq a_1$, $\cos(|\eta|) \geq a_2$, $\sin(|\xi|) \geq a_4$ for $\xi \in A_N$, $\eta \in A_N$, where $0 < a_1 < 1$, $0 < a_2 < 1$, $0 < a_4 < 1$ (for instance $N_o = 1 - p$, $N = 1$, p is close enough to zero), $g = (g_1, g_2, g_3)$,

$$\begin{aligned} g_1 &= \cosh\chi \cos\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2} \sinh\chi \cos\phi_1 - \frac{2x_2}{r^2} \cosh\chi \sin\phi_1 \chi_{x_1} + \\ &\quad + \frac{2x_1}{r^2} \cosh\chi \chi_{x_2} \sin\phi_1 + \frac{\sinh^3\chi \cos\phi_1}{r^2}, \\ g_2 &= \cosh\chi \sin\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2} \sinh\chi \sin\phi_1 + \frac{2x_2}{r^2} \cosh\chi \cos\phi_1 \chi_{x_1} - \\ &\quad - \frac{2x_1}{r^2} \cosh\chi \chi_{x_2} \cos\phi_1 + \frac{\sinh^3\chi \sin\phi_1}{r^2}, \\ g_3 &= \sinh\chi f, \\ f &= 2Y_t^2 - 2Y_r^2 + \frac{\sinh(2Y^2)}{2r^2}, \end{aligned}$$

$x_1 = r \cos\phi_1$, $x_2 = r \sin\phi_1$, $r > 0$. Then the function u which is defined with (\star) is a solution to (1).

We can write the solution of the problem (7), (8) in the form

$$(9) \quad Y(t, x) = \int_{\mathcal{R}^2} \sin(t|\xi|) \sin(x\xi) \frac{\phi_N(\xi)}{|\xi|} d\xi.$$

For the function Y , which is defined with (9), we have the following estimates

$$(10) \quad \begin{aligned} \|Y\|_{L^2(\mathcal{R}^2)} &\leq \left\| \sin(t|x|) \frac{\phi(x)}{|x|} \right\|_{L^2(\mathcal{R}^2)} = \\ &= \left(\int_{A_N} \left(\sin(t|x|) \frac{\phi_N(x)}{|x|} \right)^2 dx \right)^{\frac{1}{2}} \leq \sqrt{\frac{\pi}{12}} \left(\int_{N_o}^N \frac{1}{\rho^2} d\rho \right)^{\frac{1}{2}} = \sqrt{\frac{\pi(N - N_o)}{12NN_o}}, \end{aligned}$$

$$(11) \quad \begin{aligned} |Y(t, x)| &\leq \int_{\mathcal{R}^2} \left| \sin(t|\xi|) \sin(x\xi) \frac{\phi_N(\xi)}{|\xi|} \right| d\xi \leq \int_{A_N} \frac{1}{|\xi|^{\frac{3}{2}}} d\xi = \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{N_o}^N \frac{\rho}{\rho^{\frac{3}{2}}} d\rho d\phi = \frac{\pi}{6} (\sqrt{N} - \sqrt{N_o}), \end{aligned}$$

$$(12) \quad |Y| \leq |x| \frac{\pi}{18} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}}).$$

$$(13) \quad \|Y_t\|_{L^2(\mathcal{R}^2)} \leq \|\cos(t|x|) \phi(x)\|_{L^2(\mathcal{R}^2)} \leq \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{N_o}^N \frac{1}{\rho} \rho d\rho d\phi \right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{12}} (N - N_o).$$

$$(14) \quad |Y_t(t, x)| = \left| \int_{\mathcal{R}^2} \cos(t|\xi|) \sin(x\xi) \phi_N(\xi) d\xi \right| \leq \int_{\mathcal{R}^2} \phi_N(\xi) d\xi = \frac{\pi}{18} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}}),$$

Similarly, we have

$$(15) \quad |Y_{x_i}| \leq \frac{\pi}{18}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}}),$$

$$(16) \quad \|Y_{x_i}\|_{L^2(\mathcal{R}^2)} \leq \sqrt{\frac{\pi}{12}(N - N_o)}.$$

On the other hand

$$(17) \quad \|f\|_{L^2(\mathcal{R}^2)} \leq \|2Y_t^2\|_{L^2(\mathcal{R}^2)} + \|2Y_r^2\|_{L^2(\mathcal{R}^2)} + \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\mathcal{R}^2)}.$$

Now we use (13), (14). Then

$$(18) \quad \|2Y_t^2\|_{L^2(\mathcal{R}^2)} \leq 2\frac{\pi}{18}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\|Y_t\|_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_o)}{12}}.$$

Similarly

$$(18') \quad \|2Y_r^2\|_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_o)}{12}}.$$

Let $\Omega = \{x \in \mathcal{R}^2 : |x| \leq 1\}$. Then

$$(19) \quad \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\mathcal{R}^2)} \leq \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\Omega)} + \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\mathcal{R}^2 \setminus \Omega)}$$

Since (12) holds, we have that there exists constant c_1 such that $|\sinh(2Y^2)| \leq c_1(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})^2|x|^2$ and

$$(20) \quad \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\Omega)} = \left(\int_{\Omega} \left(\frac{\sinh(2Y^2)}{2r^2} \right)^2 dx \right)^{\frac{1}{2}} \leq c_1(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})^2 \left(\int_{\Omega} \left(\frac{|x|^2}{2|x|^2} \right)^2 dx \right)^{\frac{1}{2}} = \sqrt{2\pi} \frac{c_1}{2} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}})^2.$$

On the other hand (here we use (11) and the fact that $\sinh x$ increases for every x)

$$(21) \quad \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\mathcal{R}^2 \setminus \Omega)} \leq \sinh\left(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_o})^2\right) \frac{\sqrt{\pi}}{2}.$$

From (19), (20), (21) we get

$$(22) \quad \left\| \frac{\sinh(2Y^2)}{2r^2} \right\|_{L^2(\mathcal{R}^2)} \leq \sinh\left(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_o})^2\right) \frac{\sqrt{\pi}}{2} + \sqrt{2\pi} \frac{c_1}{2} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}})^2$$

and from (17), (18), (18'), (22)

$$(22') \quad \|f\|_{L^2(\mathcal{R}^2)} \leq$$

$$\leq 2\frac{\pi}{9}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_o)}{12}} + \sinh\left(\frac{\pi^2}{18}(\sqrt{N} - \sqrt{N_o})^2\right)\frac{\sqrt{\pi}}{2} + \sqrt{2\pi}\frac{c_1}{2}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})^2,$$

$$(23) \quad \|g_3\|_{L^2(\mathcal{R}^2)} = \|\sinh\chi f\|_{L^2(\mathcal{R}^2)} \leq \sinh\left(\frac{\pi^2}{36}(\sqrt{N} - \sqrt{N_o})^2\right)\|f\|_{L^2(\mathcal{R}^2)}.$$

We note that when N_o is close enough to N $\|g_3\|_{L^1([0,1]L^2(\mathcal{R}^2))}$ is close enough to zero .
From third equation of (1) we get that χ is solution to the equation

$$\chi_{tt} - \Delta\chi + \frac{\sinh(2\chi)}{2r^2} = f,$$

i.e.

$$\chi_{tt} - \Delta\chi = -\frac{\sinh(2\chi)}{2r^2} + f.$$

Then(here we use (11) and the fact that the functions $\sinh x$, $\cosh x$ are increasing for every $x \geq 0$)

$$(24) \quad \begin{aligned} \|g_1\|_{L^2(\mathcal{R}^2)} &\leq \cosh\left(\frac{\pi^2}{36}(\sqrt{N} - \sqrt{N_o})^2\right)\left\|f - \frac{\sinh 2\chi}{2r^2}\right\|_{L^2(\mathcal{R}^2)} + \\ &\cosh\left(\frac{\pi^2}{36}(\sqrt{N} - \sqrt{N_o})^2\right)\left\|\frac{2x_2}{r^2}\chi_{x_1}\right\|_{L^2(\mathcal{R}^2)} + \\ &\cosh\left(\frac{\pi^2}{36}(\sqrt{N} - \sqrt{N_o})^2\right)\left\|\frac{2x_1}{r^2}\chi_{x_2}\right\|_{L^2(\mathcal{R}^2)} + \\ &+ \left\|\frac{\sinh\chi}{r^2}\right\|_{L^2(\mathcal{R}^2)} + \sinh^2\left(\frac{\pi^2}{36}(\sqrt{N} - \sqrt{N_o})^2\right)\left\|\frac{\sinh\chi}{r^2}\right\|_{L^2(\mathcal{R}^2)}. \end{aligned}$$

Since $\chi_{x_1} = 2Y_{x_1}$,

$$\left|\frac{2x_2}{r^2}\chi_{x_1}\right| \leq 2\frac{|x_2|}{r^2}|Y||Y_{x_1}| \leq$$

(from (12))

$$\leq \frac{\pi}{9}\frac{|x_2||x_1|}{r^2}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})|Y_{x_i}|,$$

$$(25) \quad \left\|\frac{2x_2}{r^2}\chi_{x_1}\right\|_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\|Y_{x_1}\|_{L^2(\mathcal{R}^2)} \leq$$

(here we use (16))

$$\leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_o)}{12}}.$$

Similarly

$$(26) \quad \left\|\frac{2x_1}{r^2}\chi_{x_2}\right\|_{L^2(\mathcal{R}^2)} \leq \frac{\pi}{9}(N^{\frac{3}{2}} - N_o^{\frac{3}{2}})\sqrt{\frac{\pi(N - N_o)}{12}}$$

From (17), (22'), (22), (24), (25), (26) we get

$$(27) \quad \|g_1\|_{L^2(\mathcal{R}^2)} \leq C_1,$$

where C_1 is close enough to zero when N_o is close enough to N . Similarly,

$$(28) \quad \|g_2\|_{L^2(\mathcal{R}^2)} \leq C_2,$$

where C_2 is close enough to zero when N_o is close enough to N . From (23), (27), (28) we have

$$\|g\|_{L^2(\mathcal{R}^2)} \leq C,$$

where C is close enough to zero when N_o is close enough to N . From here the second inequality of (5) is hold for every $R > 0$ when N_o is close enough to N .

Since

$$\begin{aligned} Y(0, x) &= 0, \quad \chi(0, x) = 0, \quad \chi_t(0, x) = 2Y(0, x)Y_t(0, x) = 0, \\ \chi_{x_i}(0, x) &= 2Y(0, x)Y_{x_i}(0, x) = 0, \quad i = 1, 2, \\ u_{1t}(0, x) &= \cosh\chi(0, x)\cos\phi_1\chi_t(0, x) = 0, \\ u_{2t}(0, x) &= \cosh\chi(0, x)\sin\phi_1\chi_t(0, x) = 0, \\ u_{3t}(0, x) &= \sinh\chi\chi_t(0, x) = 0, \\ u_{1x_1}(0, x) &= \cosh\chi(0, x)\cos\phi_1\chi_{x_1}(0, x) + \sinh\chi(0, x)\sin\phi_1\frac{x_2}{r^2} = 0, \\ u_{2x_1}(0, x) &= \cosh\chi(0, x)\sin\phi_1\chi_{x_1}(0, x) - \sinh\chi(0, x)\cos\phi_1\frac{x_2}{r^2} = 0, \\ u_{3x_1}(0, x) &= \sinh\chi(0, x)\chi_{x_1}(0, x) = 0, \\ u_{1x_2}(0, x) &= \cosh\chi(0, x)\cos\phi_1\chi_{x_2}(0, x) - \sinh\chi(0, x)\sin\phi_1\frac{x_1}{r^2} = 0, \\ u_{2x_2}(0, x) &= \cosh\chi(0, x)\sin\phi_1\chi_{x_2}(0, x) + \sinh\chi(0, x)\cos\phi_1\frac{x_1}{r^2} = 0, \\ u_{3x_2}(0, x) &= \sinh\chi(0, x)\chi_{x_2}(0, x) = 0, \end{aligned}$$

we have

$$E(0, u) = 0,$$

i.e. the first inequality of (5) holds for every $\delta > 0$.

From (6) we get that

$$(29) \quad \|\partial_t u\|_{L^2(\mathcal{R}^2)}^2 \leq \epsilon \quad \forall \quad t \in [0, 1].$$

On the other hand

$$\begin{aligned} \|\partial_t u\|_{L^2(\mathcal{R}^2)}^2 &= \|\partial_t u_1\|_{L^2(\mathcal{R}^2)}^2 + \|\partial_t u_2\|_{L^2(\mathcal{R}^2)}^2 - \|\partial_t u_3\|_{L^2(\mathcal{R}^2)}^2 = \\ &= \|\cosh\chi\cos\phi_1\chi_t\|_{L^2(\mathcal{R}^2)}^2 + \|\cosh\chi\sin\phi_1\chi_t\|_{L^2(\mathcal{R}^2)}^2 - \|\sinh\chi\chi_t\|_{L^2(\mathcal{R}^2)}^2 = \\ &= \int_{\mathcal{R}^2} \chi_t^2 (\cosh^2\chi - \sinh^2\chi) dx = \int_{\mathcal{R}^2} \chi_t^2 dx = \|\chi_t\|_{L^2(\mathcal{R}^2)}^2. \end{aligned}$$

Therefore, using (29), we get

$$\|\chi_t\|_{L^2(\mathcal{R}^2)} \leq \epsilon^{\frac{1}{2}}$$

or

$$2\|YY_t\|_{L^2(\mathcal{R}^2)} \leq \epsilon^{\frac{1}{2}}.$$

From here

$$(30) \quad 2 \int_{\mathcal{R}^2} \psi Y Y_t dx \leq \|\psi\|_{L^2(\mathcal{R}^2)} \epsilon^{\frac{1}{2}}$$

for any function $\psi \in L^2(\mathcal{R}^2)$. Let

$$(31) \quad B := a_1^4 a_2^2 a_4^2 \frac{\pi^5}{18 \cdot 126^2} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}}) (N^{\frac{7}{4}} - N_o^{\frac{7}{4}})^2 (\sqrt{N} - \sqrt{N_o})^2$$

and $\epsilon = \frac{B}{2}$,

$$\psi \equiv \psi_N(\xi) = H(A_N) \frac{1}{|\xi|^{\frac{1}{4}}}.$$

For $t = 1$ and $x \in A_N$ we have

$$(32) \quad Y \geq a_1 a_4 \frac{\pi}{6} (\sqrt{N} - \sqrt{N_o}) > 0,$$

$$(33) \quad Y_t \geq a_1 a_2 \frac{\pi}{18} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}}) > 0.$$

$$(34) \quad \int_{A_N} \psi dx = \frac{\pi}{21} (N^{\frac{7}{4}} - N_o^{\frac{7}{4}}),$$

$$(35) \quad \|\psi\|_{L^2(A_N)} = \sqrt{\frac{\pi}{18}} (\sqrt{N^{\frac{3}{2}} - N_o^{\frac{3}{2}}}).$$

From (30), (32), (33), (34), (35) we have for $t = 1$

$$a_1^2 a_2 a_4 \frac{\pi^{\frac{5}{2}}}{126 \sqrt{18}} (N^{\frac{3}{2}} - N_o^{\frac{3}{2}})^{\frac{1}{2}} (N^{\frac{7}{4}} - N_o^{\frac{7}{4}}) (\sqrt{N} - \sqrt{N_o}) \leq \epsilon^{\frac{1}{2}},$$

i. e. $\epsilon \geq B$ which is contradiction with $\epsilon = \frac{B}{2}$. Therefore the solution map is not uniformly continuous.

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