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Perturbed integral equations in modular function spaces

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Abstract. We focus our attention on a class of perturbed integral equations in modular spaces, by using fixed point Theorem I.1 (see [1]).

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1 Introduction

In the present work, we focus our attention on a class of perturbed integral equation which can be written as

$$u(t) = \exp(-tA)f_0 + \int_0^t \exp((s-t)A)Tu(s)ds \qquad (I)$$

in the modular space $C^\varphi = C([0,b], L^\varphi)$ (see [1]), where L^φ is the Musielak-Orlicz space, f_0 is a fixed element in L^φ, $A : L^\varphi \to L^\varphi$ is a linear operator and $T : L^\varphi \to L^\varphi$ is $\rho - c$-Lipschitz, i.e. there exists $k > 0$ such that $\rho(c(Tx - Ty)) \leq k\rho(x - y)$ for any x, y in L^φ (ρ being a modular). Since ρ is not subadditive, then the sum of these operators is not necessarily ρ-Lipschitz and the convexity of the integral presents a more delicate problem. Therefore, it is natural in our study to introduce c_0 constant c_0 and assume some hypotheses on A, T, and b.
For more details about the concepts of the above mentioned modular spaces, we refer the reader to the books by Musielak [4] and Kozlowski [3].
We begin by recalling the definition below.

Definition 1.1 *Let X be an arbitrary vector space over $K = (\mathbb{R}$ or $\mathbb{C})$*
a) A functional $\rho : X \to [0, +\infty]$ is called a pseudomodular if
i) $\rho(0) = 0$.
ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, $\forall x \in X$.
iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. If in place of iii) there holds also:

iii') $\rho(\alpha x+\beta y)\leq \alpha^s\rho(x)+\beta^s\rho(y)$ for $\alpha,\beta\geq 0$ and $\alpha^s+\beta^s=1$, with an $s\in(0,1[$, then the pseudomodular ρ is called s-convex. 1-convex pseudomodular are called convex. If besides i) there holds also.
i')$\rho(x)=0$ implies $x=0$, then ρ is called a modular.
b) If ρ is a pseudomodular in X, then .
$X_\rho=\{x\in X/\rho(\lambda x)\to 0 \ \text{ as } \ \lambda\to 0\}$ is called a modular space.
c) If ρ is a convex modular, then $\|x\|_\rho=\inf\{u>0,\rho(\frac{x}{u})\leq 1\}$ is called the Luxemburg norm.

Recall that ρ has the Fatou property if: $\rho(x-y)\leq \liminf\rho(x_n-y_n)$, whenever $x_n\overset{\rho}{\to}x$ and $y_n\overset{\rho}{\to}y$.
And we say that ρ satisfies the Δ_2-condition if:
$\rho(2x_n)\to 0$ as $n\to+\infty$ whenever $\rho(x_n)\to 0$ as $n\to+\infty$, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X_ρ.

2 Perturbed integral equation class

In this section, we will study the existence of solution of the following perturbed integral equation:

$$u(t)=\exp(-tA)f_0+\int_0^t\exp((s-t)A)\;Tu(s)ds \quad (I)$$

We present the general hypotheses of the equation (I).
H_1) Let ρ be a modular of the Musielak-Orlicz space L^φ, convex satisfying the Δ_2-condition and $\rho_a(u)=\sup\limits_{t\in[0,b]}\exp(-at)\rho(u(t))$ is a modular of $C([0,b],L^\varphi)$ with $a>0$ (see [1]).
H_2) Let $A:\ L^\varphi\to L^\varphi$ be a linear application, assume that there exist $\alpha_0>\max(e^{-1},eb^2)$ and $M>0$ such that $\rho(\alpha_0Ax)\leq M\rho(x)$ for any $x\in L^\varphi$.
H_3) Let $T:\ L^\varphi\to L^\varphi$ be $\rho-c$-Lipschitz with $c>0$, i.e there exists $k>0$ such that $\rho(c(Tx-Ty))\leq k\rho(x-y)$ for any $x,y\in L^\varphi$.
H_4) Let f_0 be fixed element in L^φ.

Theorem 2.1 *Under these conditions H_1-H_4 and for all $b>0$, the perturbed integral equation (I) has a solution $u\in C([0,b],L^\varphi)$.*

Remark.
If we restrict our attention to the Banach space $(L^\varphi,\|.\|_\rho)$. Then the equation (I) can be written as follows:

$$u'(t)+Au(t)=Tu(t) \quad (*).$$

Thus, if $A\equiv I$ then $(*)$ becomes

$$u'(t)+(I-T)u(t)=0.$$

But the latter equation has been treated before in [1] and [4]. This let us to reduce the study to the case $A \not\equiv I$ when $(*)$ can be written in the form below:

$$u'(t) + (I - [T + (I - A)])u(t) = 0.$$

Set $B = I - A$. It follows from the fact that ρ is not subadditive that $T + B$ is not necessarily ρ-Lipschitz contrary to the situation in [1] and [2].
We cite first the theorem below which we shall use in the proof of Theorem 2.1.

Theorem 2.2 . *(See [1])*
Let X_ρ be a ρ-complete modular space. Assume that ρ is an s-convex, satisfying the Δ_2-condition and having the Fatou property. Let B be a ρ-closed subset of X_ρ and $T: B \to B$ a mapping such that
$(*)$ $\exists c, k \in R^+ \;:\; c > \max(1, k), \quad \rho(c(Tx - Ty)) \leq k^s \rho(x - y)$ *for any* $x, y \in B$.
Then T has a fixed point.

Proof of Theorem 2.1.
1^{st}) step.
We use the following property. Under the hypotheses of Theorem 2.1, the operator A is continuous from $(L^\varphi, \|.\|_\rho)$ to itself. Indeed, we have $\rho(\alpha_0 Ax) \leq M\rho(x)$ for any $x \in L^\varphi$. Let $(x_n)_{n\in I\!N}$ be a sequence in L^φ such that $\|x_n\|_\rho \to 0$ as $n \to +\infty$. So $\rho(x_n) \to 0$ as $n \to +\infty$, which implies that $\rho(\alpha_0 Ax_n) \to 0$ as $n \to +\infty$. By Δ_2-condition, $\|\alpha_0 Ax_n\|_\rho \to 0$ as $n \to +\infty$. Hence $\|Ax_n\|_\rho \to 0$ as $n \to +\infty$. Thus, there exists a constant $c > 0$ such that $\|Ax\|_\rho \leq c\|x\|_\rho$, for any $x \in L^\varphi$.
Therefore, $\exp(A)(x) = \sum_{m=0}^{+\infty} \frac{A^m}{m!}(x)$ make a sense.
2^{end}) step.
We claim that $\frac{eb}{\alpha_0} < \frac{1}{b}$. Indeed, since $\alpha_0 > \max\{e^{-1}, eb^2\}$ we have:
a) If $e^{-1} \geq eb^2$ then $e^2b^2 \leq 1$ therefore $\frac{eb}{\alpha_0} < \frac{e^2b^2}{b} \leq \frac{1}{b}$.
b) If $eb^2 \geq e^{-1}$ then $e^2b^2 \geq 1$ therefore $\frac{eb}{\alpha_0} < \frac{eb}{eb^2} = \frac{1}{b}$.
Hence in both cases we have $\frac{eb}{\alpha_0} < \frac{1}{b}$, we choose c_0 such that $\frac{eb}{\alpha_0} \leq c_0 < \frac{1}{b}$ and $c = \frac{e}{c_0}$.
Then $c_0 b < 1$. Let $\lambda > 1$ such that $1 < \lambda < \frac{1}{c_0 b}$.
We consider $S: C([0, b], L^\varphi) \to C([0, b], L^\varphi)$ defined by.
$Su(t) = \exp(-tA) f_0 + \int_0^t \exp((s - t)A)\, Tu(s)ds$ for any $u \in C([0, b], L^\varphi)$. It is clear that $Su(t) \in L^\varphi$ for each $t \in [0, b]$. As Su is continuous from $[0, b]$ into $(L^\varphi, \|.\|_\rho)$, then, Su is ρ-continuous from $[0, b]$ into (L^φ, ρ). Let $u, v \in C([0, b], L^\varphi)$, we have
$\lambda(Su(t) - Sv(t)) = \int_0^t \lambda \exp((s - t)A)\, (Tu - Tv)(s)ds$. We put $Tu - Tv = x$.
Let $K = \{t_0, t_1, ...t_n\}$ be any subdivision of $[0, t]$. $\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp((t_i - t)A)x(t_i)$ is $\|.\|_\rho$-convergent, and consequently, ρ-convergent to $\int_0^t \lambda \exp((s - t)A)x(s)ds$ in L^φ when, $|K| = \sup\{|t_{i+1} - t_i|, i = 0, ..., n - 1\} \to 0$ as $n \to +\infty$. By Fatou property we have

$\rho(\int_0^t \lambda \exp((s-t)A)x(s)ds) \leq \liminf \rho(\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)\exp((t_i-t)A)x(t_i))$.

And $\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)\exp((t_i-t)A)x(t_i) = \sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0\frac{1}{c_0}\exp((t_i-t)A)x(t_i)$.

Moreover $\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0 \leq \lambda c_0 b \leq 1$

Then $\rho(\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)\exp((t_i-t)A)x(t_i)) \leq \sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0\rho(\frac{1}{c_0}\exp((t_i-t)A)x(t_i))$.

3^{rd} step. In this part, we show that

$$\rho(\frac{1}{c_0}\exp((t_i-t)A)x(t_i)) \leq \exp(M-1)\rho(\frac{e}{c_0}x(t_i))$$

We have $\frac{1}{c_0}\exp((t_i-t)A)x(t_i) = \sum_{m=0}^{+\infty} \frac{1}{c_0}\frac{(t-t_i)^m}{m!}A^m((-1)^m x(t_i))$.

And since $\sum_{m=0}^{+\infty} \frac{\exp(-1)}{m!} = 1$, then $\rho(\frac{1}{c_0}\exp((t-t_i)A)x(t_i)) \leq \sum_{m=0}^{+\infty} \frac{\exp(-1)}{m!}\rho(\frac{e}{c_0}b^m A^m x(t_i))$.

We have $\alpha_0 \geq \frac{eb}{c_0} > 0$, and since $\alpha_0 > \max(e^{-1}, eb^2)$, then $\alpha_0 > b$. Indeed,

i) if $e^{-1} \geq eb^2$, then $e^2b^2 \leq 1$ which implies that $eb \leq 1$. Therefore $b \leq e^{-1} < \alpha_0$.

ii) if $eb^2 \geq e^{-1}$, then $e^2b^2 \geq 1$ which implies that $eb \geq 1$. Therefore $eb^2 \geq b$ and $\alpha_0 > b$.

From the hypothesis $\rho(\alpha_0 Ax(t_i)) \leq M\rho(x(t_i))$,

we have

$$\begin{aligned}\rho(\alpha_0 bA^2x(t_i)) &\leq M\rho(bAx(t_i)) \\ &\leq M\rho(\alpha_0 Ax(t_i)) \\ &\leq M^2\rho(x(t_i))\end{aligned}$$

Which implies that $\rho(\frac{e}{c_0}b^m A^m x(t_i)) \leq M^m\rho(x(t_i)) \leq M^m\rho(\frac{e}{c_0}x(t_i))$ for any m in $I\!N^*$.

Therefore,

$$\begin{aligned}\rho(\frac{1}{c_0}\exp((t_i-t)A)x(t_i)) &\leq \sum_{m=0}^{+\infty} \frac{\exp(-1)M^m}{m!}\rho(\frac{e}{c_0}x(t_i)) \\ &= \exp(M-1)\rho(\frac{e}{c_0}x(t_i)).\end{aligned}$$

4^{th} Step. We have

$$\begin{aligned}\rho(\lambda(Su(t)-Sv(t))) &\leq \liminf(\sum_{i=0}^{n-1} \lambda(t_{i+1}-t_i)c_0\exp(M-1)k\rho(u-v)(t_i)) \\ &\leq k\lambda\exp(M-1)\liminf(\sum_{i=0}^{n-1}(t_{i+1}-t_i)c_0\exp(at_i))\rho_a(u-v) \\ &= \lambda k\exp(M-1)\int_0^t c_0\exp(as)ds \quad \rho_a(u-v)\end{aligned}$$

therefore

$$\exp{(-at)}\rho(\lambda(Su(t)-Sv(t)))\leq k\lambda\exp{(M-1)}\int_0^t c_0\exp{(a(s-t))}ds\quad \rho_a(u-v)$$

Hence,

$$\rho_a(\lambda(su-sv))\leq k\lambda\exp{(M-1)}\frac{c_0}{a}(1-e^{-ab})\rho_a(u-v).$$

It suffices to take $a>ke^{M-1}c_0$, then we have $\lambda k\exp{(M-1)}\frac{c_0}{a}(1-e^{-ab})<\lambda$.
By Theorem 2.2, S has a fixed point which is a solution of the equation (I).

Remark

In third step, instead of the combination convex $\sum_{m=0}^{\infty}\frac{e^{-1}}{m!}=1$, we may choose the combination convex $\sum_{m=0}^{\infty}\frac{e^{-1}b^m}{m!}=1$, which gives the conclusion of theorem under the following hypotheses:
H_2' A : $L^\varphi\rightarrow L^\varphi$ is a linear application , and there exists $M>0$ such that : $\rho(Ax)\leq M\rho(x)$ for any $x\in L^\varphi$.
H_3' T : $L^\varphi\rightarrow L^\varphi$ is an application and for $\alpha_0=\frac{\exp{(b)}}{c_0}$ with $c_0b<1$ there exists $k>0$ such that: $\rho(\alpha_0(Tx-Ty))\leq k\rho(x-y)$.
Consider now the following perturbed integral equation.

$$u(t)=\exp{(-t)}\exp{(-tA)}f_0+\int_0^t\exp{(s-t)}\exp{((s-t)A)}\ Tu(s)ds\qquad (II).$$

The same techniques than in the proof of Theorem 2.1 are used to establish Theorem 2.3 below by taking care of the choose of λ in $(1,\frac{1}{1-e^{-b}}]$, which gives
$\rho(\int_0^t\lambda e^{s-t}e^{(s-t)A}x(s)ds)\leq\liminf(\sum_{i=0}^{n-1}\lambda(t_{i+1}-t_i)e^{t_i-t}\rho(e^{(t_i-t)A}x(t_i))$ and
$\sum_{i=0}^{n-1}\lambda(t_{i+1}-t_i)e^{t_i-t}\leq\lambda\int_0^t e^{s-t}ds\leq 1.$

Theorem 2.3 *Assume that for $\alpha_1\geq eb$, there exists $M>0$ such that $\rho(\alpha_1Ax)\leq M\rho(x)$ for any $x\in L^\varphi$ and there exists $k>0$ such that $\rho(e(Tx-Ty))\leq k\rho(x-y)$ for any x,y in L^φ. Then, the perturbed integral equation (II) has a solution $u\in C([0,b],L^\varphi)$.*

Remark.

By using the same technics as in the proof of Theorem 2.3, we can prove the existence of a solution of the equation below:

$$u(t)=e^{-t}f_0+\int_0^t\varphi(s-t)e^{(s-t)}Tu(s)ds,$$

where $\varphi:\mathbb{R}\rightarrow\mathbb{R}_*^+$ is a continuous function satisfying $\int_0^b\varphi(-s)ds<1$.

Conclusion

Concerning the equations (I) and (II), Theorem 2.1 and Theorem 2.3 give local solutions because of the constraint on b. In this frame, we notice that if A is ρ-Lipschitz i.e. if there exists $M > 0$ such that $\rho(Ax) \leq M\rho(x)$ for any $x \in L^\varphi$, then the equation (I) and the equation (II) have a solution in $[0, \frac{1}{e}]$.

Example of the equation (I).

Let φ be a Musielak-Orlicz function on a measurable space $([0,1], \mathcal{A}, \mu)$, ρ_φ be a modular defined by

$$\rho_\varphi(u) = \int_0^1 \varphi(s, |u(s)|)ds,$$

for any $u \in L^\varphi$ and $\alpha_0 > \max(e^{-1}, eb^2)$, $\quad c_0 \in [\frac{eb}{\alpha_0}, \frac{1}{b}[$. Assume that ρ_φ is convex satisfying the Δ_2-condition.

In this example, we study the existence of a solution of the following integral equation

$$u(t) = \exp(-tA)f_0 + \int_0^t \exp[(s-t)A](\int_0^1 K_1(\xi, u(s))d\xi)ds \qquad (I'),$$

where $K_1 : [0,1] \times L^\varphi \to L^\varphi$ is a measurable function satisfying

1) $\lim\limits_{\lambda \to 0^+} \int_0^1 \varphi(\xi, \lambda|(\int_0^1 K_1(s,u)ds).\xi|)d\xi = 0$ for any $u \in L^\varphi$.

2) $|(\int_0^1 (K_1(\xi, u(s)) - K_1(\xi, v(s)))d\xi)|) \leq k|(u-v)(s)|$, for any u, v in L^φ, with $k \in]0,1[$.

f_0 is a fixed element in L^φ and the operator A is equal to $k_0 I$, where I is the identity function of L^φ with $k_0 \leq \frac{1}{\alpha_0}$.

Let T be a mapping from L^φ into L^φ defined by

$$Tu = \int_0^1 \frac{c_0}{e} K_1(s,u)ds.$$

Hence, we have $\rho_\varphi(\alpha_0 k_0 x) \leq \alpha_0 k_0 \rho_\varphi(x)$ for any $x \in L^\varphi$, .i.e. $\rho(\alpha_0 Ax) \leq \alpha_0 k_0 \rho(x)$ for any $x \in L^\varphi$.

Now, we show that T is $\rho - \frac{e}{c_0}$-Lipschitz.

At first, by 1), we have $\int_0^1 \varphi(\xi, \lambda|Tu(\xi)|)d\xi \to 0$ as $\lambda \to 0^+$. Hence, by the definition of L^φ, $\;Tu \in L^\varphi$ for any $u \in L^\varphi$.

On the other hand, let $x, y \in L^\varphi$

$$\begin{aligned}
\rho_\varphi(\frac{e}{c_0}(Tx - Ty)) &= \int_0^1 \varphi(s, \frac{e}{c_0}|(Tx-Ty).(s)|)ds \\
&= \int_0^1 \varphi(s, |\int_0^1 (K_1(\xi, x(s)) - K_1(\xi, y(s)))d\xi)|)ds
\end{aligned}$$

Therefore, by 2)

$$\begin{aligned}
\rho_\varphi(\frac{e}{c_0}(Tx - Ty)) &\leq \int_0^1 \varphi(s, k|(x-y)(s)|)ds \\
&= \rho_\varphi(k(u-v)) \\
&= k\rho_\varphi(u-v).
\end{aligned}$$

Hence T is $\rho - \frac{e}{c_0}$-Lipschitz. So by Theorem 2.1 the equation (I') has a solution in $C([0,b], L^{\varphi})$.

References

[1] Ait Taleb.A, Hanebaly.E. A fixed point theorem and its application to integral equations in modular function spaces.Proc. Amer. Math. Soc.127, no 8, 2335-2342 (1999) 128 , no. 2, 419-426 (2000).

[2] Khamsi, M.A. Nonlinear Semigroups in Modular Function Spaces. Thèse d'état. Département de Mathématique, Rabat (1994).

[3] Kozlowski.W.M. Modular Function Spaces. Dekker New-York (1988).

[4] Musielak.J. Orlicz Spaces and Modular Spaces. L.N vol. 1034, S.P. (1983).

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