

Existence of global solution for a nonlocal parabolic problem

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Abstract

In this paper, we study a non-local initial boundary-value problem arising in Ohmic heating. By using a dynamical systems approach, some existence and uniqueness results are proved and the existence of a compact attractor is shown.

Mathematics Subject Classifications: 35K20, 35K35, 35K45, 35K60.

Key words: thermistor, a nonlocal, existence, attractor global.

1 Introduction

In this paper, we shall deal with the following nonlocal parabolic problem

$$\frac{\partial u}{\partial t} - \Delta u = \lambda \frac{f(u)}{\left(\int_{\Omega} f(u) dx\right)^2}, \text{ in } \Omega \times]0; T[, \quad (1.1)$$

$$u = 0 \quad \text{on } \partial\Omega \times]0; T[, \quad (1.2)$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \quad (1.3)$$

where $T > 0$, Ω is a regular open bounded subset of \mathbb{R}^N , $N \geq 1$, λ is a positive parameter and f a function from \mathbb{R} to \mathbb{R} satisfying the hypotheses $(H_1) - (H_2)$ below. Problem (1.1) – (1.3) represents, for example, static materiel such as thermistors [3, 6, 14, 15] and arises by reducing the system of two equations

$$u_t = \nabla \cdot (k(u) \nabla u) + \sigma(u) |\nabla \varphi|^2, \quad (1.4)$$

$$\nabla(\sigma(u) \nabla \varphi) = 0, \quad (1.5)$$

to a simple but realistic equation (see [8]). More precisely, u represents the temperature produced by an electric current flowing through a conductor, φ the electric potential, $\sigma(u)$ is the electrical conductivity and $k(u)$ is the thermal conductivity. Taking the latter to be constant, problem (1.4) – (1.5) can then be reduced to the single nonlocal equation(1.1), where $f(u) = \sigma(u)$ and $\lambda = \frac{I^2}{|\Omega|^2} \geq 0$, I is the electric current which is supposed to be constant and $|\Omega|$ is the measure of Ω .

Our goal here concerns the existence and uniqueness of weak solutions to (1.1)–(1.3). We shall also show existence of global attractor.

Let us first recall that problem (1.1) – (1.3) has been the subject of variety of investigation in the past decade. Particularly, some results have been obtained by many authors in the case where $N = 1$ and f taking particular forms: Montesinos and Gallego [11] proved the existence of weak solution under

$$0 < \sigma_1 \leq \sigma(s) \leq \sigma_2, \forall s \in \mathbb{R}. \quad (1.6)$$

Antontsev and Chipot [1] obtained also an existence and uniqueness results for (1.4)–(1.5) supposing that $\sigma \in C^0(\Omega)$ and (1.6); furthermore, a study of smoothness of solutions was treated in that paper under some assumptions on the conductivity and initial data.

In [8, 9, 13], major emphasis is placed on cases where the spatial dimension N is 1 or 2 and f is of the form $f(u) = \exp(u)$ or $\exp(-u)$. In these works, additional regularity assumptions are made on u_0 and a combination of usual Lyapounov functional and a comparison method is the main ingredient. Our purpose is to extend some of the results therein to problem (1.1) – (1.3), where here, the condition (1.6) is weakened to (H_2) below. Following the frame work of Fioas and Temam [12], we shall also deal with the asymptotic behaviour of the solutions of problem (1.1) – (1.3) via a dynamical systems approach. We start by proving the existence of absorbing sets in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$, which in turn paves the way for the existence of the global attractor. Cimatti [4] obtained similar results for particular cases, when $N = 1$, by constructing a Lyapounov functional. As a concluding result, we show that the attractor is bounded subset of $H^2(\Omega)$ under restrictive assumptions on data.

2 Existence and regularity of global attractor.

a) Existence and uniqueness.

We assume the following

(H1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitzian function .

(H2) There exist positive constants c_1, c_2 and α such that, for all $\xi \in \mathbb{R}$

$$\sigma \leq f(\xi) \leq c_1|\xi|^{\alpha+1} + c_2.$$

Let us denote by $\|\cdot\|_k$ the norm in $L^k(\Omega)$.

We adopt the following weak formulation for (1.1) – (1.3):

u is a solution of (1.1) – (1.3) if and only if

$$u \in L^\infty(\tau, +\infty, H_0^1(\Omega) \cap L^\infty(\Omega)) \text{ with } \frac{\partial u}{\partial t} \in L^2(\tau, +\infty, L^2(\Omega))$$

for any $\tau > 0$, and satisfying

$$\int_0^T \int_\Omega u \frac{\partial}{\partial t} \phi - \nabla u \nabla \phi \, dx dt = \int_0^T \left(\frac{\lambda}{\left(\int_\Omega f(u) \, dx \right)^2} \int_\Omega f(u) \phi \, dx \right) dt,$$

for any $\phi \in C^\infty((0, \infty), \Omega)$.

Now, we state our main result.

Theorem 2.1. *Let hypotheses (H_1) – (H_2) be satisfied. Assume that $u_0 \in L^{k_0+2}(\Omega)$ with k_0 such that*

$$k_0 \geq \max\left(0, \frac{\alpha N}{2} - 2\right). \tag{2.1}$$

Then, there exists $d_0 > 0$ such that if $\|u_0\|_{k_0+2} < d_0$, the problem (1.1) admits a solution u verifying for all $\tau > 0$

$$u \in L^\infty(\tau, +\infty, L^{k_0+2}(\Omega)), \quad |u|^\gamma u \in L^\infty(\tau, +\infty, H_0^1(\Omega)), \quad \text{with } \gamma = \frac{k_0}{2}.$$

Moreover, if $u_0 \in L^\infty(\Omega)$, then $u \in L^\infty(\tau, +\infty, L^\infty(\Omega))$ and is unique.

Remark. The value of d_0 will be given in the course of the proof.

Proof. We use a Faedo-Galerkin method see [10]. Let $u_m \subseteq D(\Omega)$ be such that $u_{0m} \rightarrow u_0$ in $H_0^1(\Omega)$ and let $(w_j)_j \subseteq H_0^1(\Omega)$ a special basis. We seek u to be the limit of a sequence $(u_m)_m$ such that

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j,$$

where g_{jm} is the solution of the following ordinary differential system

$$\begin{cases} \langle u'_m, w_j \rangle + (u_m, w_j) = \frac{\lambda}{\left(\int_\Omega f(u_m) \, dx \right)^2} \langle f(u_m), w_j \rangle, & 1 \leq j \leq m, \\ u_m(0) = u_{0m}. \end{cases} \tag{2.2}$$

It is easy to see that (2.2) has a unique solution u_m according to hypotheses (H_1) – (H_2) and Cartan's existence theorem concerning ordinary differential equations (see [5]). This solution is shown to exist on a maximal interval $[0; t_m[$. The following estimates enable us to assert that it can be continued on the whole interval $[0; T]$. We shall denote by C_i different positive constants, depending on data, but not on m .

Lemma 2.2. For any $\tau > 0$, there exists a constant $c_3(\tau), c_4(\tau)$ such that

$$\|u_m(t)\|_{k_0+2} \leq c_3(\tau), \forall t \geq \tau, \quad (2.3)$$

$$\|u_m(t)\|_\infty \leq c_4(\tau), \forall t \geq \tau. \quad (2.4)$$

Proof. (i) Multiplying the first equation of (2.2) by $|u_m|^k g_{jm}$, integrating on Ω , adding from $j = 1$ to m and using $(H_1) - (H_2)$, yields

$$\frac{1}{k+2} \frac{d}{dt} \|u_m\|_{k+2}^{k+2} + \frac{4}{(k+2)^2} \|\nabla |u_m|^{\frac{k}{2}} u_m\|_2^2 \leq c_5 \|u_m\|_{k+\alpha+2}^{k+\alpha+2} + c_6. \quad (2.5)$$

By using well-known Sobolev and Gagliardo-Nirenberg's inequalities, we have

$$\|u_m\|_{k_0+\alpha+2}^{k_0+\alpha+2} \leq c_7 \|u_m\|_{k_0+2}^\alpha \|\nabla |u_m|^\gamma u_m\|_2^2, \quad (2.6)$$

Thus, from (2.5) and (2.6), we obtain

$$\frac{1}{k_0+2} \frac{d}{dt} \|u_m\|_{k_0+2}^{k_0+2} \leq (c_8 \|u_m\|_{k_0+2}^\alpha - \frac{4}{(k_0+2)^2}) \|\nabla |u_m|^\gamma u_m\|_2^2 + c_6. \quad (2.7)$$

We shall make the following compatibility condition on u_0

$$\|u_0\|_{k_0+2} < \left(\frac{4}{c_8(k_0+2)^2} \right)^{\frac{1}{\alpha}} = d_0. \quad (2.8)$$

Then, there exists a small $\tau > 0$ such that

$$\|u_m(t)\|_{k_0+2} < d_0 \text{ for } t \in]0, \tau[. \quad (2.9)$$

Hence

$$\frac{1}{k_0+2} \frac{d}{dt} \|u_m\|_{k_0+2}^{k_0+2} + c_9 \|\nabla |u_m|^\gamma u_m\|_2^2 \leq c_6 \quad \forall \quad 0 < t < \tau. \quad (2.10)$$

By Poincaré's inequality and after integrating, it follows that

$$\|u_m(t)\|_{k_0+2} \leq c_{10}, \quad \forall \quad 0 < t < \tau,$$

Therefore, relation (2.3) is achieved by iterating successively the same process on intervals of periode τ such as $[0, \tau], [\tau, t + \tau], \dots$

(ii) By using Hölder's inequality, we get

$$\|u_m\|_{k+\alpha+2}^{k+\alpha+2} \leq c_{11} \|u_m\|_{k+2}^{\theta_1} \|u_m\|_{k_0+2}^{\theta_2} \|u_m\|_q^{\theta_3}, \quad (2.11)$$

with θ_1, θ_2 and θ_3 satisfying

$$\frac{\theta_1}{k+2} + \frac{\theta_2}{k_0+2} + \frac{\theta_3}{q} = 1 \quad \text{and} \quad \theta_1 + \theta_2 + \theta_3 = k + \alpha + 2.$$

We require moreover

$$\frac{\theta_1}{k+2} + \frac{\theta_3}{2(\gamma+1)} = 1.$$

Using the boundedness of $\|u_m\|_{k_0+2}$, the choice of q , Sobolev and Young's inequalities and relation (2.11), we derive that

$$\begin{aligned} c_5 \|u_m\|_{k+\alpha+2}^{k+\alpha+2} &\leq c_{12} \|u_m\|_{k+2}^{\theta_1} \|\nabla|u_m|^\gamma u_m\|_2^{\frac{\theta_3}{\gamma+1}} \\ &\leq c_{13} (k+2)^{\theta_4} \|u_m\|_{k+2}^{k+2} + \frac{2}{(k+2)^2} \|\nabla|u_m|^\gamma u_m\|_2^2, \end{aligned}$$

where θ_4 is some positive constant. Hence (2.5) becomes

$$\frac{1}{k+2} \frac{d}{dt} \|u_m\|_{k+2}^{k+2} + \frac{c_{14}}{(k+2)^2} \|\nabla|u_m|^\gamma u_m\|_2^2 \leq c_{15} (k+2)^{\theta_4} \|u_m\|_{k+2}^{k+2} + c_5.$$

Therefore, by applying lemma 4 ([7]) we conclude to (2.4).

Passage to the limit in (2.2) as $m \rightarrow \infty$. Multiplying the j th equation of system (2.2) by $g_{jm}(t)$, adding these equations for $j = 1, \dots, m$ and integrating with respect to the time variable, we deduce the existence of a subsequence of u_m such that

$$\begin{aligned} u_m &\rightarrow u \text{ weak star in } L^\infty(0, T; L^2(\Omega)), \\ u_m &\rightarrow u \text{ weak in } L^2(0, T; H_0^1(\Omega)), \\ u_{mt} &\rightarrow u_t \text{ weak in } L^2(0, T; H^{-1}(\Omega)), \\ u_m &\rightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)) \text{ and a.e in } Q_T. \end{aligned}$$

Straightforward standard compactness arguments allow us to assert that u is a solution of problem (1.1)

Uniqueness. Consider u_1 and u_2 two weak solutions of the problem (1.1) and define $w = u_1 - u_2$. Subtracting the equations verified by u_1 and u_2 , we obtain

$$\begin{aligned} \frac{dw}{dt} - \Delta w &= \frac{\lambda}{\left(\int_\Omega f(u_1) dx\right)^2} \left(f(u_1) - f(u_2)\right) \\ &\quad + \lambda \frac{\left(\int_\Omega f(u_2) - f(u_1) dx\right) \left(\int_\Omega f(u_2) + f(u_1) dx\right)}{\left(\int_\Omega f(u_1) dx\right)^2 \left(\int_\Omega f(u_2) dx\right)^2} f(u_2). \end{aligned} \quad (2.12)$$

Taking the inner product of (2.12) by w and using (H_1) and (2.4), we get

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 \leq c_{16} \|w(t)\|_2^2,$$

which implies that $w = 0$. Hence the solution is unique. \square

b) We denote by $\{T(t), t \geq 0\}$ the continuous semi-group generated by (1.1) and defined by

$$\begin{aligned} T(t) : L^\infty(\Omega) &\rightarrow L^\infty(\Omega) \\ u_0 &\rightarrow T(t)u_0 = u(t, \cdot). \end{aligned}$$

In this part, we refer to [12] for used concepts.

Theorem 2.3. *Assume that (H_1) – (H_2) are satisfied, Then $T(t)$ possesses a maximal attractor which is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$, compact and connected in $L^\infty(\Omega)$.*

Proof.

(i) (2.4) imply that there exists an absorbing set in $L^k(\Omega)$, $1 \leq k \leq \infty$.

(ii) To obtain existence of absorbing sets in $H_0^1(\Omega)$ and the uniform compactness of $T(t)$, multiply (2.2) by $g'_{jm}(t)$, add from $j = 1$ to m and integrate on Ω by using Young's inequality, it follows therefore that, for any $t \geq \tau > 0$

$$\int_{\Omega} \left(\frac{\partial u_m}{dt}\right)^2 dx + \frac{d}{dt} \|\nabla u_m\|_2^2 \leq c_{17}(\tau), \quad (2.13)$$

which gives

$$\frac{d}{dt} \|\nabla u_m\|_2^2 \leq c_{17}(\tau), \forall t \geq \tau > 0. \quad (2.14)$$

On the other hand, multiplying (2.2) by g_{jm} , adding and integrating on $\Omega \times [t, t + \tau]$ we get

$$\int_t^{t+\tau} \|\nabla u_m(s)\|_2^2 ds \leq c_{18}(\tau), \forall t \geq \tau > 0. \quad (2.15)$$

Then, by the uniform Gronwall's lemma (see [12], p.89) and the lower semi-continuity of the norm, we have

$$\|\nabla u(t)\|_2^2 \leq c_{19}(\tau), \forall t \geq \tau. \quad (2.16)$$

Therefore, the open ball $B(0, c_{19}(\tau))$ is an absorbing set in $H_0^1(\Omega)$.

Hence, by theorem (1.1) ([12], p.23), we conclude to the results of theorem (2.3).

Theorem 2.4. *We suppose (H_1) – (H_2) and*

$(H3)$ $f \in C^1(\mathbb{R})$.

Then, we have

$$y(t) \equiv \|u_t\|^2 \leq c_{20}(\tau), \quad \text{for any } t \geq \tau > 0.$$

Proof. Differentiating equation (1.1) with respect to time (the justification of the formal derivatives can be done as in [5]), we get

$$u_{tt} - \Delta u_t = \frac{\lambda f'(u) u_t}{\left(\int_{\Omega} f(u) dx\right)^2} - 2\lambda f(u) \frac{\int_{\Omega} f'(u) u_t dx}{\left(\int_{\Omega} f(u) dx\right)^3}. \quad (2.17)$$

Multiplying (2.17) by u_t , integrating over Ω and using the L^∞ estimate of u and Hölder's inequality, yields

$$\frac{1}{2} y'(t) \leq c_{21}(\tau) y(t). \quad (2.18)$$

On the other hand, taking the scalar product of (1.1) with u_t , using Young's inequality, integrating on $[t, t + \tau]$ and using estimate (2.16), then gives

$$\int_t^{t+\tau} y(s) ds \leq c_{23}(\tau), \quad \text{for any } t \geq \tau. \quad (2.19)$$

From (2.18) and the uniform Gronwall's lemma, we have

$$y(t) \leq c_{23}(\tau), \quad \text{for any } t \geq \tau.$$

Therefore,

$$u_t \in L^\infty(\tau, \infty, L^2(\Omega)).$$

By (1.1), we then get

$$-\Delta u \in L^\infty(\tau, \infty, L^2(\Omega)),$$

that is,

$$u(t) \quad \text{is in a bounded subset of } H^2(\Omega).$$

Hence the existence of an absorbing set in $H^2(\Omega)$ is shown.

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(Received October 27, 2003; Revised version received January 16, 2005)